Analysis oriented S-procedure

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Abstract – *Variants of the S-procedure provides an important tool in robust stability and robust performance design. This paper presents an analysis oriented analog of the full block S-procedure (extended KYP lemma), also extending its applicability, by relaxing the usual compactness assumption, allowing unbounded domains defined using quadratic multipliers.*

The proof of this result reveals the role of the more elementary variant of the S-procedure and gives us the opportunity to emphasize the role played by the theory of indefinite spaces and those constructions that reveals the "linear" aspects of different feedback control problems formulated in the linear fractional framework.

Keywords: S-procedure, KYP lemma, robust control, robust performance

1. Introduction and motivation

Robust stability and robust performance analysis and synthesis of control systems with parameter uncertainties and parameter variations is one of the fundamental issues in system theory. In the most common framework models are augmented with performance specifications and uncertainties. Weighting functions are applied to the performance signals to meet performance specifications and guarantee a tradeoff between performances. The uncertainties are modeled by both un-modeled dynamics and parametric uncertainties. As a result of this construction a linear fractional transformation (LFT) interconnection structure, which is the basis of control design, is achieved.

As a common structure, these algorithms have an analysis phase and a synthesis phase. The analysis phase consists of solving a set of linear matrix inequalities (LMIs) that are obtained by using some variant of the S-procedure and usually involves a relaxation of an infinite number of conditions to a set of finite number of constraints. The synthesis phase consists of obtaining the the controller, e.g., [8], [13]. The main theoretical tools in this respect are the full-block S-procedure, a variant of the Elimination lemma and some variant of the classical S-procedure.

KYP lemma and the full block S-procedure (extended KYP lemma), [11], [13], performs a separation step in the analysis where the task of finding the controller is formulated as an inequality containing fix matrices: as source of the analysis conditions by using the Elimination lemma, [7], [13]. The classical S-procedure is a relaxation method, see [23], [14]: it tries to solve a system of quadratic inequalities via a LMI relaxation.

Let us consider the sets $\mathcal{D}_A = \{I - A\delta \text{ } is\ nonsingular\}$ and $\mathcal{M}_A = \{P \mid \Delta_P \subset \mathcal{D}_A\}$, with

$$\Delta_P = \{\delta \mid \begin{pmatrix}\delta\\ I\end{pmatrix}^* P \begin{pmatrix}\delta\\ I\end{pmatrix} > 0\}.$$

Lemma 1 (Extended KYP lemma, [12]) *For a given compact set Δ we have*

$$\begin{pmatrix}I\\ F(\delta)\end{pmatrix}^* P_p \begin{pmatrix}I\\ F(\delta)\end{pmatrix} < 0, \quad \forall\delta \in \Delta \tag{1}$$

where $F(\delta) = D + C\delta(I - A\delta)^{-1}B$, if and only if there exists a symmetric (Hermitian) multiplier $P \in \mathcal{M}_A$[1] which satisfies

C-1 $\begin{pmatrix}I & 0\\ A & B\end{pmatrix}^* P \begin{pmatrix}I & 0\\ A & B\end{pmatrix} + \begin{pmatrix}0 & I\\ C & D\end{pmatrix}^* P_p \begin{pmatrix}0 & I\\ C & D\end{pmatrix} < 0,$

C-2 $\begin{pmatrix}\delta\\ I\end{pmatrix}^* P \begin{pmatrix}\delta\\ I\end{pmatrix} > 0, \quad \forall\delta \in \Delta.$

This result can also be viewed as a generalization of the Finsler's lemma, see [8]. For the motivation of labeling this statement as an extended KYP lemma see [20]. These inequalities play a crucial role in the design of robust controllers. Much effort has been done in the lossless parametrization of the multiplier P for a given

[1] Condition (1) implicitly implies that $\Delta \subset \mathcal{D}_A$. With a general P conditions C-1 and C-2 does not ensure this property, which explains the necessity of the additional constraint $P \in \mathcal{M}_A$. According to the Separation Lemma, however, see, e.g., [8], we have that $P \in \mathcal{M}_A$ if

$$\begin{pmatrix}I\\ A\end{pmatrix}^* P \begin{pmatrix}I\\ A\end{pmatrix} < 0.$$

In practical applications usually

$$\begin{pmatrix}0\\ I\end{pmatrix}^* P_p \begin{pmatrix}0\\ I\end{pmatrix} \geq 0,$$

and thus for a P satisfying C-1 and C-2 we automatically have that $P \in \mathcal{M}_A$. Moreover often the performance multiplier P_p is nonsingular and the corresponding graph subspace is a maximal negative subspace.

uncertainty set Δ. It can be shown that the set Δ_a for which the actual performance, i.e., inequality (1), holds is always larger, i.e., $\Delta_P \subset \Delta_a$. This means that every design that is based on Lemma 1 is necessarily conservative regardless, whether the relaxation method used for the multiplier search was lossless or not. This topic is not our concern here.

In this paper we show that an analysis oriented analog of Lemma 1 also holds, i.e., for every multiplier $P \in \mathcal{M}_A$ for which we have the performance assessment (1) the LMI $C-1$, allowing also equality, holds for αP with a suitable $\alpha > 0$. As a consequence one can release the compactness assumption on Δ in Lemma 1, allowing also unbounded domains defined using quadratic multipliers through inequality $C-2$.

The proof of this result gives us the opportunity to emphasize the role played by the theory of indefinite spaces and those constructions that reveals the "linear" aspects of different feedback control problems formulated in the LFT framework. Section 2 gives an overview of the topic and provides the background that makes possible to relate a quadratic performance problem to negative subspaces of certain indefinite spaces. The main result of the paper, i.e., the analysis oriented KYP lemma, is formulated in Section 3 while the proof of the result is given in Section 4. A possible application of the result is sketched in Section 5.

2. Linear relations and LFTs

The common tool in formulating robust feedback control problems is to use system interconnections that can be described as linear fractional transforms (LFTs), as a general framework to include the rational dependencies that occur. Not only the performance criteria is expressed by an LFT of the so called generalized plant and that of the controller, but also the most fundamental object, the state space form of a system is actually an LFT: the feedback connection of a memoryless operator and that of a special system, an integrator.

If P is partitioned as $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ then a lower and an upper LFT is defined as

$$\mathfrak{F}_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}, \qquad (2)$$

$$\mathfrak{F}_u(P,\Delta) = P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}, \qquad (3)$$

provided that the correspondig inverse, $(I - P_{22}K)^{-1}$ and $(I - P_{11}\Delta)^{-1}$, respectively, exists, see Figure 1. Then P is called the coefficient matrix of the LFT.

There is an intimate relationship between linear relations and LFTs, revealed by the concept of transformers, introduced in [15], [17]. If X and Y are two sets, a relation $T \subset X \times Y$ is defined as a set of pairs $(x,y) \in T$, where $x \in X, y \in Y$. If X and Y are linear spaces $(X \oplus Y = X \times Y)$ a linear relation T is a linear subspace of $X \oplus Y$.

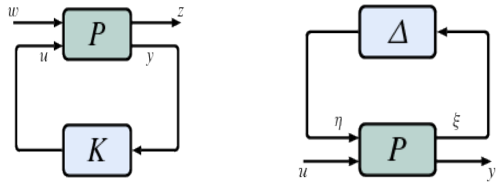

Figure 1: Linear fractional transformation

Recall that:

$$dom(T) = \{x \in X : (x,y) \in T \quad for\ some \quad y \in Y\};$$

$$ran(T) = \{y \in Y : (x,y) \in T \quad for\ some \quad x \in X\};$$

$$ker(T) = \{x \in X : (x,0) \in T \quad forsome \quad y \in Y\};$$

$$mul(T) = \{y \in Y : (0,y) \in T \quad for\ some \quad x \in X\};$$

$$T^{-1} = \{(y,x) : (x,y) \in T\}.$$

If $x \in dom(T)$ then $T(x) = \{y \in Y : (x,y) \in T\}$ and if $y \in ran(T)$, then $T^{-1}(y) = \{x \in X : (x,y) \in T\}$.

Addition of linear relations $T_1, T_2 \subset X \times Y$ is defined as:

$$T_1 + T_2 = \{\{x, y_1 + y_2\} : \{x, y_1\} \in T_1, \{x, y_2\} \in T_2\}.$$

This sum is direct, when $T_1 \cap T_2 = \{\{0,0\}\}$. Accordingly, $\lambda T = \{\{x, \lambda y\} : \{x,y\} \in T\}$. Let $T \subset X \times Y$ and $R \subset Y \times Z$ be linear relations. Then the product $RT \subset X \times Z$ is the linear relation defined by

$$RT = \{\{x,z\} \in X \times Z : \{x,y\} \in T, \{y,z\} \in R\}.$$

The product of relations is clearly associative. These definitions agree with the usual ones that correspond to operators.

A linear operator $P: X \mapsto Y$ is equivalent to a special relation defined by a graph subspace $\mathcal{G}_P = \mathrm{Im}\begin{pmatrix} I \\ P \end{pmatrix}$, i.e., the graph of the operator. For details see, e.g., [1].

Möbius transformations, which are defined as

$$Z' = \mathfrak{M}_S(Z) = (C + DZ)(A + BZ)^{-1}, \qquad (4)$$

relates two graph subspaces, $\mathcal{G}_Z$ and $\mathcal{G}_{Z'}$, through the invertible linear operator $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, i.e., $\mathcal{G}_{Z'} = S\mathcal{G}_Z$. Moreover, it turns out that the Möbius transformation inherits the group structure of the linear operators, i.e.,

$$\mathfrak{M}_P \circ \mathfrak{M}_Q = \mathfrak{M}_{PQ},$$

for details see, e.g., [20]. It turns out that LFTs can be obtained in the same way as the Möbius transformations, by performing some interchange in the signal spaces and by considering linear relations instead of the linear operators.

Given the linear spaces $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ consider $\mathcal{L} = \mathcal{X} \oplus \mathcal{Y}$ and $\tilde{\mathcal{L}} = (\mathcal{X}_2 \oplus \mathcal{Y}_1) \oplus (\mathcal{X}_1 \oplus \mathcal{Y}_2)$. Observe that we have $\tilde{\mathcal{L}} = S_p\mathcal{L}$ with the permutation matrix $S_p = \begin{pmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$. Thus every linear operator $T: (\mathcal{X}_2 \oplus \mathcal{Y}_1) \mapsto (\mathcal{X}_1 \oplus \mathcal{Y}_2)$ induces a relation $\mathcal{R}_T \subset \mathcal{L}$ through its graph subspace, i.e.,

$$\mathcal{R}_T = S_p \mathcal{G}_T \sim \begin{pmatrix} T_{11} & T_{12} \\ I & 0 \\ 0 & I \\ T_{21} & T_{22} \end{pmatrix}. \quad (5)$$

It turns out that evaluating this relation on the graph subspaces $\mathcal{G}_Z$, i.e., on the linear operators $Z: \mathcal{X}_1 \mapsto \mathcal{X}_2$, we obtain a graph subspace $\mathcal{G}_{Z'} = \mathcal{R}_T \mathcal{G}_Z$, corresponding to the linear operator $Z': \mathcal{Y}_1 \mapsto \mathcal{Y}_2$, provided that we have $Z \in \{Z \mid (I - T_{11}Z) \text{ is boundedly invertible}\}$.

This map is given by the (upper) LFT

$$Z' = \mathfrak{F}_u(T, Z) = T_{22} + T_{21} Z (I - T_{11} Z)^{-1} T_{12}.$$

In the special case of state space representation we have $\mathcal{Y}_1 = \mathcal{U}$, i.e., the input space, $\mathcal{Y}_2 = \mathcal{Y}$, i.e., the output space and the state space $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$, while

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{and} \quad Zx(t) = \int_0^t x.$$

Analogously, by a slight modification of the permutation matrix S_p, i.e., by considering

$$\tilde{\mathcal{L}} = (\mathcal{Y}_1 \oplus \mathcal{X}_2) \oplus (\mathcal{Y}_2 \oplus \mathcal{X}_1)$$

one can obtain the expression of the (lower) LFT $Z' = \mathfrak{F}_l(T, Z) = T_{11} + T_{12} Z (I - T_{22} Z)^{-1} T_{21}$, too.

This construction extends the linearization "trick" already encountered for the Möbius transforms to the LFTs: on the level of equivalence classes, i.e., subspaces (operator graphs, relations), the map is linear while on the level of the representants the map is rational (Möbius, LFT). Moreover, the group structure on the representants is also present, however, the familiar matrix product should be changed to the less understood Redheffer (star) product.

From the definition it is clear that the composition of the LFTs are LFTs, provided that the connection is consistent, i.e., the signal dimensions are compatible and the related inverses exist, see Figure 2. Nested LFTs corresponds to the composition of the associated linear relations. Performing the computations, one can find the expression of the Redheffer product A and B as the corresponding operation on the level of the linear operators:

$$B \star A = \begin{pmatrix} \mathfrak{F}_l(A, B_{11}) & A_{12}(I - B_{11}A_{22})^{-1} B_{12} \\ B_{21}(I - A_{22}B_{11})^{-1} A_{21} & \mathfrak{F}_u(B, A_{22}) \end{pmatrix},$$

i.e., $\mathfrak{F}_u(B, \mathfrak{F}_u(A, C)) = \mathfrak{F}_u(B \star A, C)$.

Since the state space representation is itself an LFT and the computational rules for state space connections obey to this star product, see [24].

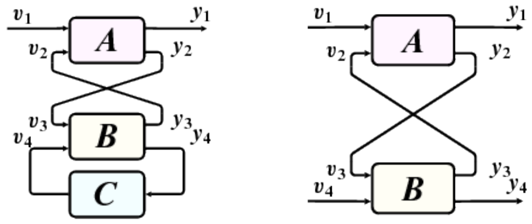

Figure 2: Composition of LFTs

If invertibility conditions holds for the matrix $\begin{pmatrix} T_{11} & T_{12} \\ I & 0 \end{pmatrix}$ then one has $\mathfrak{F}_u(T, Z) = \mathfrak{M}_{\hat{T}}(Z)$ with

$$\hat{T} = \begin{pmatrix} T_{12}^{-1} & -T_{12}^{-1} T_{11} \\ T_{22} T_{12}^{-1} & T_{21} - T_{22} T_{12}^{-1} T_{11} \end{pmatrix}.$$

The transformation $T \mapsto \hat{T}$ is called Potapov-Ginsbourg transformation. This relation between an LFT and a Möbius transformations has the advantage to use a more accessible operation (matrix product) instead of the star product. This fact was widely exploited in the solution of the robust control problems, see, e.g., the factorization approach of [3] or in the so called chain scattering-approach of [9].

Concerning the topic of this paper the main motivation of introducing this construction, however, is the fact that it provides a natural framework to introduce indefinite spaces, see, e.g., [4], [2] for infinite dimension and [6] for the matrix setting.

To illustrate the idea let us consider the linear spaces $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ as indefinite spaces with inner products $[\cdot,\cdot]_X = \langle \mathcal{J}_X \cdot, \cdot \rangle$ and $[\cdot,\cdot]_Y = \langle \mathcal{J}_Y \cdot, \cdot \rangle$. If we endow the space $\mathcal{L} = \mathcal{X} \oplus \mathcal{Y}$ with the inner product $[\cdot,\cdot] = -[\cdot,\cdot]_X + [\cdot,\cdot]_Y$ then for the space $\tilde{\mathcal{L}} = (\mathcal{X}_2 \oplus \mathcal{Y}_1) \oplus (\mathcal{X}_1 \oplus \mathcal{Y}_2)$ the corresponding inner product can be expressed as $[\cdot,\cdot] = \langle \mathcal{J}_{\tilde{\mathcal{L}}} \cdot, \cdot \rangle$, where $\mathcal{J}_H = \begin{pmatrix} -I_{H_1} & 0 \\ 0 & I_{H_2} \end{pmatrix}$ for $H = H_1 \oplus H_2$.

Maximal negative subspaces of $\tilde{\mathcal{L}}$ are obviously maximal negative subspaces of $\mathcal{L}$. On one hand side these subspaces are parametrized by contractions T, see, e.g., [6], on the other hand we have

$$(\star)^*(-\mathcal{J}_X)\begin{pmatrix} T_{11} & T_{12} \\ I & 0 \end{pmatrix} + (\star)^* \mathcal{J}_{\mathcal{Y}} \begin{pmatrix} 0 & I \\ T_{21} & T_{22} \end{pmatrix} < 0.$$

Thus if T is a contraction then $Z' = \mathfrak{F}_u(T, Z)$ maps the contractive ball to the contractive ball. It turns out, that conversely, if $Z' = \mathfrak{F}_u(T, Z)$ has this property, then the matrix $T_\alpha = \begin{pmatrix} T_{11} & \alpha T_{12} \\ \alpha^{-1} T_{21} & T_{22} \end{pmatrix}$ is a contraction for a suitable $\alpha > 0$, see [16]. Observe that

$$Z' = \mathfrak{F}_u(T, Z) = \mathfrak{F}_u(T_\alpha, Z), \quad (6)$$

thus we can put this result in a slightly modified form: if $\| \mathfrak{F}_u(T, Z) \| < 1$ holds for $\| Z \| < 1$ then there exists

$\alpha > 0$ such that

$$(\star)^*(-\alpha \mathcal{J}_X)\begin{pmatrix} T_{11} & T_{12} \\ I & 0 \end{pmatrix} + (\star)^* \mathcal{J}_Y \begin{pmatrix} 0 & I \\ T_{21} & T_{22} \end{pmatrix} < 0,$$

i.e., the graph subspace $\mathcal{G}_{T_\alpha}$ is a maximal negative graph subspace in $\tilde{\mathcal{L}}$.

The assertion states that nothing is "lost" in the description of the corresponding maximal negative subspaces by considering the parametrization $\{T_\alpha \mid \alpha > 0\}$, i.e., for the maximal negative subspace that corresponds to the given performance claim for $\mathfrak{F}_u(T, Z)$, necessarily exists a maximal negative graph subspace in $\tilde{\mathcal{L}}$ with symbol T_α. It seems that actually this parametrization of the symbols translates to the parametrization $\{P_\alpha = \alpha P \mid \alpha > 0\}$ of the relevant multiplier set.

It is instructive to compare this fact with the role of the set of D-scales, or $D - G$ scales, encountered in the theory of μ analysis and synthesis.

3. Analysis oriented KYP lemma

After a slight modification, i.e., replacing the J-spaces with the indefinite spaces defined by $[\cdot,\cdot]_X = \langle \bar{P} \cdot, \cdot \rangle$ and $[\cdot,\cdot]_Y = \langle P_p \cdot, \cdot \rangle$, where

$$\bar{P} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^* (-P) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad (7)$$

and keeping in mind that there is a one-to-one correspondence between maximal negative subspaces of $\tilde{\mathcal{L}}$ and those of $\mathcal{L}$, one expect that the following assertion, the main result of this paper, is also valid:

Theorem 1 (Analysis oriented extended KYP lemma) *Consider the set* Δ_a *defined by the inequality*

$$\begin{pmatrix} \delta \\ I \end{pmatrix}^* P \begin{pmatrix} \delta \\ I \end{pmatrix} > 0,$$

where $P \in \mathcal{M}_A$. Then

$$\begin{pmatrix} I \\ F(\delta) \end{pmatrix}^* P_p \begin{pmatrix} I \\ F(\delta) \end{pmatrix} < 0, \quad \forall \delta \in \Delta_a$$

where $F(\delta) = D + C\delta(I - A\delta)^{-1}B$ if and only if

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* (\alpha P) \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \leq 0,$$

for some $\alpha > 0$.

In the rest of the paper we will provide a proof of this assertion and highlight its deep relation with the classical S-procedure.

Remark 2 *The presence of the non-strict inequality, in contrast to C-1, reflects the analysis nature of the assertion. In Lemma 1 the entire set on which the performance inequality is true cannot be caught, in general, by a multiplier P only for very special plants and if equality is allowed in C-1.*

To illustrate this point let $\mathfrak{F}_u(T, \delta) = \delta(I - A\delta)^{-1}$, i.e., $T = \begin{pmatrix} A & I \\ I & 0 \end{pmatrix}$, take an arbitrary $P \in \mathcal{M}_A$ and consider the performance $P_p = \begin{pmatrix} 0 & I \\ I & A \end{pmatrix}^* (-P) \begin{pmatrix} 0 & I \\ I & A \end{pmatrix}$.

Then, in Theorem 1, the only possibility is $\alpha = 1$, i.e., equality.

4. An extended S-procedure

When the LFT can be reduced to a Möbius transform, i.e., $\mathfrak{F}_u(T, \delta) = \mathfrak{M}_{\hat{T}}(\delta)$, than the implication

$$\begin{pmatrix} I \\ \delta \end{pmatrix}^* \bar{P} \begin{pmatrix} I \\ \delta \end{pmatrix} < 0 \quad \Rightarrow \quad \begin{pmatrix} I \\ \mathfrak{F}_u(T, \delta) \end{pmatrix}^* P_p \begin{pmatrix} I \\ \mathfrak{F}_u(T, \delta) \end{pmatrix} < 0$$

is equivalent to

$$\begin{pmatrix} I \\ \delta \end{pmatrix}^* \bar{P} \begin{pmatrix} I \\ \delta \end{pmatrix} < 0 \Rightarrow (\star)^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ \delta \end{pmatrix} < 0,$$

where $\bar{P}$ is defined by (7). Since $P \in \mathcal{M}_A$ we do not miss any δ on the left hand side, thus we can apply directly the multivariate S-procedure, see Lemma 6 in the Appendix, to infer that there exists an $\alpha > 0$ such that

$$(\star)^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}^{-1} \leq \alpha \bar{P},$$

which is the desired inequality.

While not every LFT can be expresses as a Möbius transform, every LFT is a composition of two Möbius transforms and a special LFT, an affine one. Indeed

$$\mathfrak{F}_u(T, \delta) = \mathfrak{F}_u(T_D, \mathfrak{F}_u(T_{BC}, \mathfrak{F}_u(T_A, \delta))),$$

where

$$T_A = \begin{pmatrix} A & I \\ I & 0 \end{pmatrix}, \qquad T_{BC} = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \qquad T_D = \begin{pmatrix} 0 & I \\ I & D \end{pmatrix}.$$

Observe that

$$\begin{pmatrix} I & 0 \\ A & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & 0 \\ A & B \end{pmatrix}, \begin{pmatrix} I & 0 \\ D & I \end{pmatrix}\begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ C & D \end{pmatrix},$$

thus by considering

$$P \mapsto \begin{pmatrix} I & 0 \\ A & I \end{pmatrix}^* P \begin{pmatrix} I & 0 \\ A & I \end{pmatrix}, \quad P_p \mapsto \begin{pmatrix} I & 0 \\ D & I \end{pmatrix}^* P_p \begin{pmatrix} I & 0 \\ D & I \end{pmatrix}$$

that corresponds to the Möbius transforms $\mathfrak{F}_u(T_A, \cdot)$ and $\mathfrak{F}_u(T_D, \cdot)$, respectively, the problem can be reduced to the class $F(\delta) = C\delta B$. This reduction, however, is not completely trivial. Let us suppose for a while that this assertion is true, and we will return to its proof later.

It is not hard to figure out that considering singular value decompositions for B and C, respectively, we can consider, without restricting generality, only the special case

$$B = \begin{pmatrix} I_B & 0 \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} I_C & 0 \\ 0 & 0 \end{pmatrix},$$

where I_B, I_C are identity matrices of a specific size.

Thus, in what follows we prove the assertion for this case. Consider the partitioning $\delta = \begin{pmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{pmatrix}$ induced by I_B and I_C. Then, the original implication, i.e.,

$$\binom{I}{\delta}^* \bar{P} \binom{I}{\delta} < 0 \quad \Rightarrow \quad \binom{I}{F(\delta)}^* P_p \binom{I}{F(\delta)} < 0$$

reads as

$$(\star)^* \bar{P} \begin{pmatrix} 0 & 0 & I_B & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix} \binom{\eta}{w} < 0 \quad \Rightarrow$$

$$(\star)^* P_p \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I_C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \binom{\eta}{w} < 0 \qquad (8)$$

on the set

$$\{(\eta, w) \mid \exists \delta : (\xi, \eta) \in \mathcal{G}_\delta, \xi = \begin{pmatrix} I_B & 0 \\ 0 & 0 \end{pmatrix} w\} = \{\begin{pmatrix} \delta_{11} \\ \delta_{21} \\ I \\ 0 \end{pmatrix} \xi_1\}.$$

One can observe that after a suitable permutation of the blocks this subspace is a graph subspace and we are in a position to apply apply the multivariate S-procedure, Lemma 6, in order to conclude the existence of an $\alpha > 0$ such that

$$(\star)^* P_p \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ I_C & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \leq \alpha (\star)^* \bar{P} \begin{pmatrix} 0 & 0 & I_B & 0 \\ 0 & 0 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix},$$

which is exactly the desired result.

Finally, it remains to prove that we can reduce the original problem to the affine one. The only nontrivial implication is that if

$$\binom{I}{\delta}^* \bar{P} \binom{I}{\delta} < 0 \quad \Rightarrow \quad \binom{I}{\mathfrak{F}_u(T,\delta)}^* P_p \binom{I}{\mathfrak{F}_u(T,\delta)} < 0$$

then

$$\binom{I}{\bar{\delta}}^* \tilde{P} \binom{I}{\bar{\delta}} < 0 \quad \Rightarrow \quad \binom{I}{\bar{\delta}B}^* P_p \binom{I}{\bar{\delta}B} < 0,$$

where

$$\tilde{P} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}^* \bar{P} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix},$$

and we take, without restricting generality, $C = I$ and $D = 0$. Observe that while for the first implication $\delta \in \mathcal{D}_A$ by assumption, for the second case we do not have $\bar{\delta} \in \mathcal{D}_{-A}$ for all $\bar{\delta}$. Thus, an extension of the validity of the right hand side is needed for the case $\bar{\delta} \not\in \mathcal{D}_{-A}$.

By using the ideas from Section 2 we have to prove that the implication

$$(\star)^* \tilde{P} \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \binom{\eta}{\xi} < 0 \quad \Rightarrow \quad \binom{\eta}{\xi}^* P_p \binom{\eta}{\xi} < 0,$$

that holds for all (η, ξ) that satisfy

$$\begin{pmatrix} I & -A \\ 0 & I \end{pmatrix} \binom{I}{\delta} \zeta = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \binom{\eta}{\xi}$$

for suitable δ and ζ also holds for pairs for which

$$\binom{I}{\bar{\delta}} \bar{\zeta} = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} \binom{\bar{\eta}}{\bar{\xi}}$$

for suitable $\bar{\delta}$ and $\bar{\zeta}$. This follows from the fact that for every $\bar{\delta} \not\in \mathcal{D}_{-A}$ and every $\varepsilon > 0$ there is a $\bar{\delta}_\varepsilon \not\in \mathcal{D}_{-A}$ such that $\| \bar{\delta} - \bar{\delta}_\varepsilon \| < 0$. Since every pair $(\bar{\eta}, \bar{\xi})$ that corresponds to a $\bar{\delta}$ can be approximated arbitrary closed by a pair $(\bar{\eta}_\varepsilon, \bar{\xi}_\varepsilon)$ that corresponds to a $\bar{\delta}_\varepsilon$, and by using the continuity of the quadratic map defined by P_p, the assertion follows.

Thus at the heart of the result we have the fact that the implication

$$(\star)^* \bar{P} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \binom{\eta}{w} < 0 \quad \Rightarrow (\star)^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \binom{\eta}{w} < 0,$$

which holds on a set $\{(\eta, w)\}$ constrained by the feedback connection (LFT), can be lifted (extended) to the unconstrained set, for which the common S-procedure can be applied.

This proof of the Theorem 1 makes explicit the relation of the extended KYP lemma (full-block S-procedure) with the well-known S-procedure, which in the original proofs, based on the idea of the Finsler's lemma, remain hidden. Moreover, the new interpretation eliminates the necessity to restrict the set Δ to the compact ones.

The applied technique makes possible to extend the result to the infinite dimensional (operator) setting, however, due to space limitations, that case will be presented elsewhere.

5. A possible application

While at a formal level the analysis oriented version of the extended KYP lemma bears essentially the same information that the classical version, it represents a different view on the topic. To illustrate a potential application of this new viewpoint let us consider the nontrivial task of finding a common solution X for the finite set of LMIs

$$\binom{I}{D_i + C_i X B_i}^T Q_i \binom{I}{D_i + C_i X B_i} < 0,$$

where the matrices Q_i, D_i, B_i, C_i are given. For a motivation of this problem see, e.g., [10], [22].

It is known that the solution sets of the individual inequalities are either empty or a set obtained as an image of the contractive ball through a Möbius transform, see [19]. Thus, if the problem is solvable, there always exists a matrix ellipsoid

$$\binom{X}{I}^T P \binom{X}{I} > 0$$

formed entirely by solutions of the inequality. It follows that applying Theorem 1 we can formulate the following result:

Lemma 2 *The given set of inequalities has a common solution if and only if there is a multiplier P and constants $\beta_i > 0$ such that*

$$\begin{pmatrix} I & 0 \\ 0 & B_i \end{pmatrix}^* P \begin{pmatrix} I & 0 \\ 0 & B_i \end{pmatrix} + \begin{pmatrix} 0 & I \\ C_i & D_i \end{pmatrix}^* (\beta_i Q_i) \begin{pmatrix} 0 & I \\ C_i & D_i \end{pmatrix} \leq 0.$$

By considering matrix ellipsoids, i.e., by imposing suitable sign constraints on the block diagonal matrices of P one can relax the implicit nonlinear condition for the inertia for a multiplier P imposed by the solvability of

$$\begin{pmatrix} X \\ I \end{pmatrix}^T P \begin{pmatrix} X \\ I \end{pmatrix} > 0.$$

Thus we can reduce the problem to a set of LMIs that can be efficiently handled.

6. Conclusion

The paper presents a construction that reveals the "linear" aspects of different feedback control problems formulated in the LFT framework and makes possible to relate these problems to the elements of the theory of indefinite spaces.

By using these techniques we have shown that an analysis oriented analog of extended KYP Lemma also holds. As a consequence one can release the compactness assumption on Δ, allowing also unbounded domains defined using quadratic multipliers.

The presented techniques are suitable to extend these results to the operator (infinite dimensional) setting, which will be reported in a forthcoming paper.

7. References

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