MIMO Decoupling Control Design for Lotka-Volterra Systems

K. M. Hangos [1,2], L. Keviczky [1,3]

[1] Institute of Computer Science and Automation
Hungarian Academy of Sciences
H-1111 Budapest, Kende u 13-17, HUNGARY
[2] Department of Electrical Engineering and Information Systems,
University of Pannonia, Veszprém

Abstract – *The underlying dynamically similar linear MIMO LTI model, that can be associated to a Lotka-Volterra system model with a positive equilibrium point, is used in this paper to design a decoupling controller based on Youla-parametrization. The method is illustrated on a simple nonlinear fermentation model, where the structure of the designed feedback can be related to the structure of the original open-loop system.*

Keywords: MIMO processes, Lotka-Volterra systems, Youla-parametrization, decoupling control

1 Introduction

A wide range of systems can only be tackled using nonlinear techniques [6], that are applicable only for a narrow class of nonlinear systems, while the more generally applicable methods suffer from computational complexity problems. One possible way of balancing between general applicability and computational feasibility is to find nonlinear system classes with good descriptive power but well characterized structure, and utilize this structure when developing control design methods.

The class of quasi-polynomial (QP) systems is one of the candidates for this purposes, because nonlinear systems with smooth nonlinearities can be transformed into quasi-polynomial form [5]. This means, that any applicable method for quasi-polynomial systems can be regarded as a general technique for nonlinear systems. QP-systems are invariant under quasi-monomial transformation [3], this enables to partition them into equivalence classes represented by a Lotka-Volterra (LV) system, and use LV models as canonical forms.

Recently, improved methods have been developed for stabilizing feedback design of QP systems based on control Lyapunov functions [7] and also using the underlying reduced linear dynamics [8]. All of the above attempts, however, have used polynomial nonlinear feedback to achieve their control goals.

The aim of this paper is to apply a *linear output feedback controller* for a special subset of Lotka-Volterra systems using a generalization of well-established methods based on Youla-parametrization [1], [2]. The structure of the controller, namely the inputs and outputs to be used will also be related to the structure of the open-loop system model.

2 Lotka-Volterra models and their underlying linear dynamics

The so-called quasi-polynomial (QP) model is a set of nonlinear ODEs in the form

$$\dot{z}_j = z_j\left(L_j + \sum_{i=1}^{n} K_{ji} \prod_{k=1}^{m} z_k^{Y_{ki}}\right), j = 1, \dots, m. \qquad (1)$$

where $K \in \mathcal{R}^{m\times n}$, $Y \in \mathcal{R}^{m\times n}$ are constant parameter matrices (coefficient matrix and exponent matrix, respectively), $L \in \mathcal{R}^m$ is a vector. The monomial-like terms in (1) of the form

$$x_i = \prod_{k=1}^{m} z_k^{Y_{ki}}, \quad i = 1, \dots, n \qquad (2)$$

are the so-called quasi-monomials of the system, and usually $n \geq m$.

It is known (e.g. [5]) that the set of quasi-polynomial systems can be split into classes of equivalence according to the matrix invariant

$$M = Y \cdot K. \qquad (3)$$

A unique, descriptive element of such QP equivalence class is the Lotka-Volterra model

$$\dot{x}_i = x_i\left(\lambda_i + \sum_{j=1}^{n} M_{ij} x_j\right), \quad i = 1, \dots, n \qquad (4)$$

where $M \in \mathcal{R}^{n\times n}$, $\Lambda = [\lambda_1, \dots, \lambda_n]^T \in \mathcal{R}^n$. It is important to note that the state variables of the Lotka-Volterra form are the quasi-monomials of the original QP model (1).

Because of its descriptive nature, the values of the Lotka-Volterra parameter matrices can be computed from the QP parameter matrices of any quasi-polynomial model belonging to the same class of equivalence as $M = Y \cdot K$ and $\Lambda = Y \cdot L$. The model (4) can also be written in a more compact matrix-vector notation as

$$\dot{x} = diag(x)\, M\, (x - x^*), \qquad (5)$$

where $diag(x)$ is a diagonal matrix with x_i in its ith diagonal entry, and x^* is a unique positive equilibrium point of the system, which is the (nonzero) solution of the steady state version of (4):

$$0 = \Lambda + Mx^* \quad (6)$$

2.1 The translated X-factorable phase space transformation and the underlying linear dynamics

Assume that the following set of ordinary differential equations (ODEs)

$$\frac{dX}{dt} = F(X) \quad (7)$$

is defined on the positive orthant $\mathcal{P}^n$. The singular solutions of Eq. (7) are defined by $F(X) = 0$. Consider the following nonlinear translated X-factorable transformation of Eq. (7)

$$\frac{dX}{dt} = \mathrm{diag}(X)F(X - c) \quad (8)$$

where the elements of $c = [c_1, \dots, c_n]^T$ are positive real numbers, and $X = [X_1, \dots, X_n]^T$.

Assume that $F(X)$ is composed of polynomial-type functions with a finite number of singular solutions. It can be shown ([9]) that the above transformation can move the singular solutions into the positive orthant, and leaves the geometry of the state (or phase) space unchanged within it (but not at or near the boundary). Therefore, the dynamics of the solutions of Eqs. (7) and (8) are *structurally similar*.

It is easy to see that a LV model has polynomial right-hand sides, so one can associate a structurally similar linear ODE model

$$\dot{x} = M\,(x - x^*), \quad (9)$$

to the model (5), that is called the *underlying linear dynamic* model of it.

3 MIMO decoupling control for square stable LTI systems

Let us consider a square MIMO LTI system that has the same number (p) of input and output variables in the form

$$\begin{aligned} \dot{x} &= \frac{dx}{dt} = Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (10)$$

where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times p}$, $C \in \mathcal{R}^{p\times n}$ and $D \in \mathcal{R}^{p\times p}$ are the constant coefficient matrices of the model. The transfer function matrix (TFM) is then in the following form

$$\mathbb{P}(s) = C(sI - A)^{-1}B + D = C\Psi(s)B + D = \frac{\mathbb{B}(s)}{a(s)} \quad (11)$$

where $a(s)$ is the characteristic polynomial of matrix A, and the TFM $\Psi(s)$ is in the form

$$\Psi(s) = \frac{\mathbf{adj}(sI - A)}{a(s)} \quad (12)$$

The above model form is called the "naive" model [1]. Assume that the system is inverse stable, and the reference model $\mathbb{R}_n(s)$ is also given in its naive form in such a way, that it reflects both the decoupling and the speeding up controller design goals, i.e.

$$\mathbb{R}_n(s) = \frac{\mathbb{B}_n(s)}{a_n(s)} = \frac{I}{a_n(s)} \quad (13)$$

where I is the unit matrix and $a_n(s)$ is a low degree stable polynomial with real eigenvalues. Then the Youla-parametrized MIMO regulator $\mathbb{C}$ becomes

$$\mathbb{C}(s) = a(s)\mathbb{B}^{-1}(s)\mathbb{R}_n(s)\left(I - \mathbb{R}_n(s)\right)^{-1} =$$
$$= \frac{a(s)}{a_n(s) - 1}\mathbb{B}^{-1}(s) \quad (14)$$

4 Decoupling output feedback controller design for Lotka-Volterra systems

The input extension of the system (4) is assumed to be in the form

$$\dot{x}_i = x_i\left(\lambda_i + \sum_{j=1}^{n} M_{ij}x_j + \sum_{j=1}^{p} \Gamma_{ij}u_j\right), \quad i = 1, \dots, n \quad (15)$$

This means that $x_i u_j$ terms are appearing in the i-th state equation.

The motivation behind this simple structure is twofold:
(i) the design will be based on the underlying linear dynamics that enables to have a linear static full state feedback for stabilizing,
(ii) this structure corresponds to the model structure of lumped process models when the flow-rates are chosen as manipulable input variables (see in [4]).

The design is based on the fact, that nonlinear X-factorable transformation described in subsection 2.1 enables us to examine a dynamically similar LTI state space model instead of the original one that can be characterized by the LTI matrix pair (M, Γ).

4.1 The transfer function matrix of the underlying LTI model

The design principle implies, that the state and input matrices of the MIMO LTI state space model (10) are

$$A = M, B = \Gamma \quad (16)$$

In order to achieve a suitable output set for the special needs of Lotka-Volterra systems, one needs to recognize, that usually the number of quasi-monomials in a QP model (1) is larger than the number of its variables, while the embedding of the model into LV-form (4) requires an increase of the state variables to be equal to n. Therefore, it is natural to select the original QP variables with physical meaning to be in the set of output variables, and thus choose the matrices in the output equation in (10) to be

$$C = [I|0], D = 0 \tag{17}$$

where $I \in \mathcal{R}^{p\times p}$ is a unit matrix.

The above mentioned variable extension implies the appearance of zero eigenvalues in the state matrix A. However, the corresponding integrator elements will not appear in the transfer function, as we shall see it through the example in section 5.

4.1.1 The Youla-parametrized MIMO regulator

An important speciality of designing an output feedback for a stable Lotka-Volterra system is, that it has a positive equilibrium point x^* for a positive steady state point of the input vector u^* such that x^* is the solution of (6), i.e.

$$0 = (\Lambda + u^*) + Mx^*.$$

This implies that the centred version of the physical input variable should be used as the input u, and the centred version of the physical output variable is used as the output y when performing the design.

This is depicted in figure 1 in the case of the nonlinear fermentation process example that is described next in section 5. The nonlinear model is given as $\mathbb{P}'$, while the dynamically equivalent centred linear system is shown by the box in point-boundaries denoted by $\mathbb{P}$ in the figure.

5 A nonlinear fermentation process example

The example is a fermentation process system where the substrate and biomass is fed to the reactor with a constant flow-rate S_F^0 and X_F^0, and they can be outlet from the reactor with manipulable flow-rates F_1 and F_2, respectively. Then the state equations of the dynamic model originating from component mass balances can be written in the following form

$$\begin{aligned}\dot{z}_1 &= \mu_{max} z_1 z_2 + X_F^0 - F_1 z_1 \\ \dot{z}_2 &= -\frac{\mu_{max}}{Y} z_1 z_2 + S_F^0 - F_2 z_2\end{aligned} \tag{18}$$

where z_1 is the concentration of the biomass, and z_2 is that of the substrate, respectively. The values of the system parameters are listed in Table 1.

5.1 The Lotka-Volterra open-loop model

The two inputs two outputs case of the model (18) is a QP system model, it is apparent if it is written in the form

z_1	biomass concentration	$[\frac{g}{l}]$
z_2	substrate concentration	$[\frac{g}{l}]$
S_F^0	substrate inlet feed flow-rate	2 $[\frac{g}{l\cdot h}]$
X_F^0	biomass inlet feed flow-rate	1 $[\frac{g}{l\cdot h}]$
F_1	biomass outlet flow-rate	$[\frac{l}{h}]$
F_2	substrate outlet flow-rate	$[\frac{l}{h}]$
Y	yield coefficient	1 -
μ_{max}	kinetic parameter	1 $[\frac{1}{h}]$

Table 1: Variables and parameters of the fermentation process model (18)

analogous to (1) as:

$$\begin{aligned}\dot{z}_1 &= z_1(-F_1 + \mu_{max} z_2 + X_F^0 z_1^{-1}) \\ \dot{z}_2 &= z_2\left(-F_2 - \frac{\mu_{max}}{Y} z_1 + S_F^0 z_2^{-1}\right)\end{aligned} \tag{19}$$

Let us choose the reference values for the flow-rates F_1 and F_2 to be

$$F_1^* = 2, F_2^* = 1$$

then the model (19) has an equilibrium (steady-state) point at $z^* = [1,1]^T$.

Let us choose the output variables to be the centred state variables, while the input variables are centred flow-rates

$$y = [z_1 - z_1^*, z_2 - z_2^*]^T, u = [F_1 - F_1^*, F_2 - F_2^*]^T. \tag{20}$$

In order to apply the results of Section 4, the process model must be embedded into Lotka-Volterra form (for the details of the embedding procedure, see [5]), that is

$$\begin{aligned}\dot{x}_1 &= x_1\left(-F_2 - \frac{\mu_{max}}{Y} x_2 + S_F^0 x_4\right) \\ \dot{x}_2 &= x_2(-F_1 + \mu_{max} x_1 + X_F^0 x_3) \\ \dot{x}_3 &= x_3(F_1 - \mu_{max} x_1 - X_F^0 x_3) \\ \dot{x}_4 &= x_4\left(F_2 + \frac{\mu_{max}}{Y} x_2 - S_F^0 x_4\right)\end{aligned} \tag{21}$$

where the new state variables are the quasi-monomials in the QP-model (19)

$$x_1 = z_2, x_2 = z_1, x_3 = z_1^{-1}, x_4 = z_2^{-1}.$$

5.1.1 MIMO decoupling output feedback

The decoupling output feedback structure is based on the open-loop Lotka-Volterra model of the system described in subsection 5.1 with centred input and output variables.

It is depicted in figure 1.

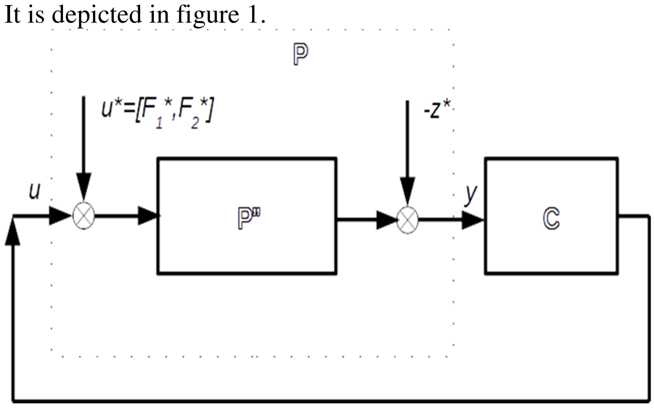

Figure 1: The MIMO closed control loop of the example

The corresponding MIMO LTI system matrices derived from the underlying dynamically similar system of the Lotka-Volterra model are as follows.

$$A = M = \begin{bmatrix} 0 & -1 & 0 & 2 \\ 1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}, \quad B = \Gamma = \begin{bmatrix} 0 & -1 \\ -1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (22)$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad D = 0 \ . \quad (23)$$

It is easy to check that the eigenvalues of M are stable $[0, 0, -1.5 - 0.866i, -1.5 + 0.866i]$, so the MIMO decoupling design can be used.

Now the TFM model of the system will be developed using the nominator TFM

$$\mathbb{B}(s) = s^2 \begin{bmatrix} -(s+2) & -1 \\ 1 & -(s+1) \end{bmatrix} \quad (24)$$

that leads to its naive form

$$\mathbb{P}(s) = \frac{1}{(s^2+3s+3)} \begin{bmatrix} -(s+2) & -1 \\ 1 & -(s+1) \end{bmatrix} \quad (25)$$

Let us choose a reference model for achieving the decoupling and the speed up of the response as follows

$$\mathbb{R}_n(s) = \frac{1}{(1+0.5s)} I = \frac{1}{(1+0.5s)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (26)$$

Now we can perform the design of the controller by substituting the above calculated parameters into (14) to obtain

$$\mathbb{C}(s) = \frac{2(s^2+3s+3)}{s(s^2+3s+1)} \begin{bmatrix} -(s+1) & 1 \\ -1 & -(s+2) \end{bmatrix} \quad (27)$$

6. Conclusion

A linear output feedback structure has been presented in this paper for a special subset of Lotka-Volterra systems, where the input terms are assumed to be input-affine and bilinear. It is also assumed that the open-loop LV system has a locally stable equilibrium point.

The design is based on the underlying dynamically similar linear MIMO LTI model, that can be associated to a Lotka-Volterra system model with a positive equilibrium point, while the centred versions of the physical input and output variables are used. It is shown that the zero eigenvalues associated to the new variables during the embedding of a QP model do not appear in the resulting input-output model.

The developed model was used to design a decoupling controller based on Youla-parametrization when the centred version of the original QP-variables were used as output variables.

The method is illustrated on a simple nonlinear fermentation model, where the structure of the designed feedback can be related to the structure of the original open-loop system.

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7. References

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