# A DISTANCE-BASED METHOD FOR ATTRIBUTE REDUCTION IN INCOMPLETE DECISION SYSTEMS* 

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#### Abstract

There are limitations in recent research undertaken on attribute reduction in incomplete decision systems. In this paper, we propose a distance-based method for attribute reduction in an incomplete decision system. In addition, we prove theoretically that our method is more effective than some other methods.


1. Introduction. Attribute reduction is one of the most important problems in data preprocessing, in knowledge discovery and data mining. Attribute reduction based on rough sets is the process of finding a minimal attribute set, known as reduct, which preserves some necessary information of decision systems. There have been many methods to find reducts of complete decision systems [17], such as positive region methods, discernibility matrix methods, information entropy methods, granular computing methods. In reality, decision systems often contain missing values in the domain values of attributes and these decision systems are called incomplete decision systems. Derived from the idea of rough set

[^0]theory [11], Marzena Kryszkiewicz [5] defines a tolerance relation based on the equivalent relation and proposes tolerance rough set. Recently, much research has been undertaken on measures and methods to find reducts in incomplete decision systems $[1,3,4,7,8,9,12,13,20]$. Though distance has been a popular measure applied to solve some problems in data mining [16, 18, 19], there is limited reseach on attribute reduction in rough set theory. Yuhua Qian et al. $[14,15]$ propose distances between coverings in incomplete decision systems. Long Giang Nguyen [10] proposes a distance-based method to find reduct of a complete decision system.

In this paper, we propose a distance-based method for attribute reduction in incomplete decision systems. We first generalize Liang entropy [6] in incomplete decision systems. Based on generalized Liang entropy, we establish a distance between attributes and study some properties of the distance. As a result, we use the proposed distance to formally define a reduct and the importance of attribute, and later construct a heuristic algorithm to find the best reduct.

This paper consists of six sections. The concept of tolerance rough set in incomplete systems is introduced in Section 2. The generalized Liang entropy and its propeties are proposed in Section 3. Section 4 establishes a distance between two attributes based on the generalized Liang entropy and studies some properties of the distance. Section 5 proposes a distance-based method and example to find the best reduct. Section 6 presents our conclusions.

## 2. Basic concepts.

In this section, we summarize the basic concepts of tolerance rough sets in incomplete decision systems [5].

Let $U$ be a set of objects and Attrbe a set of attributes. Then $I S=$ ( $U, A t t r$ ) is called an information system. A decision system is an information system $D S=(U, \operatorname{Attr} \cup\{d\})$ where $\operatorname{Attr}$ is a conditional attribute and $d$ is a decision attribute. An incomplete decision system is a decision system where there exists an attribute $a \in$ Attr so that acontains a missing value. Further on, a missing value is denoted as ' $*$ '. Table 1 is an example of an incomplete decision system.

Attributes Price, Mileage, Size and Max-speed are called conditional attributes and Decision is the decision attribute. We denote the decision attribute Decision as $d$, and the conditional attributes Price, Mileage, Size and Max-speed as $a_{1}, \ldots, a_{4}$ in order. Consequently, Table 1 is an incomplete decision system $I D S=(U, \operatorname{Attr} \cup\{d\})$ where $U=\left\{x_{1}, x_{2}, x_{3}, x_{4}, x_{5}, x_{6}\right\}$ and Attr $=\left\{a_{1}, a_{2}, a_{3}, a_{4}\right\}$.

Table 1. An example of an incomplete decision system

| Car | Price | Mileage | Size | Max-speed | Decision |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $x_{1}$ | High | High | Full | Low | Good |
| $x_{2}$ | Low | $*$ | Full | Low | Good |
| $x_{3}$ | $*$ | $*$ | Compact | High | Poor |
| $x_{4}$ | High | $*$ | Full | High | Good |
| $x_{5}$ | $*$ | $*$ | Full | High | Excellent |
| $x_{6}$ | Low | High | Full | $*$ | Good |

For any attribute set $A \subseteq A t t r$, a tolerance relation $\operatorname{TLR}(A)$ is defined on $U \times U$ for any $x, y \in U$ as follows:

$$
(x, y) \in T L R(A) \text { rightarrow } \forall_{a \in \operatorname{Attr}}(a(x)=a(y) \vee a(x)=* \vee a(y)=*)
$$

It is clear that $T L R(A)=\cap_{a \in A} T L R(\{a\})$. The tolerance relation $T L R(A)$ determines a covering of $U$ which is denoted by $K(A)$ or $U / T L R(A)$. Then, $K(A)=U / T L R(A)=\left\{T_{A}(x) \mid x \in U\right\}$ where $T_{A}(x)=\{y \in U \mid(x, y) \in$ $T L R(A)\} . T_{A}(x)$ is called a tolerance class. It shows that $T_{A}(x) \neq \emptyset$ for any $x \in U$ and $\cup_{x \in U} T_{A}(x)=U$. The set of all $K(A)$ where $A \subseteq A t t r$ is denoted as $\operatorname{COV}(U)$. For coverings in $\operatorname{COV}(U), \omega=\left\{T_{\text {Attr }}(x)=\{x\} \mid x \in U\right\}$ is called the discrete covering and $\delta=\left\{T_{\text {Attr }}(x)=\{U\} \mid x \in U\right\}$ is called the indiscrete covering. A partial relation is defined on $\operatorname{COV}(U)$ as follows:

Definition 2.1 [9]. Given an incomplete decision system IDS $=$ $(U, \operatorname{Attr} \cup\{d\})$ and two attribute sets $A, B \subseteq$ Attr,

1) $U / \operatorname{TLR}(A)=U / T L R(B)$ if and only if $\forall x \in U, T_{A}(x)=T_{B}(x)$.
2) $U / T L R(A) \preceq U / T L R(B)$ if and only if $\forall x \in U, T_{A}(x) \subseteq T_{B}(x)$.

Property 2.1 [9]. Given an incomplete decision system $I D S=$ $(U, A t t r \cup\{d\})$ and two attribute sets $A, B \subseteq$ Attr, the following properties hold:

1) If $A \subseteq B \subseteq$ Attr then $U / T L R(B) \preceq U / T L R(A)$.
2) If $A, B \subseteq$ Attr then $T_{A \cup B}(x)=T_{A}(x) \cap T_{B}(x)$ for any $x \in U$.

Let $I D S=(U, \operatorname{Attr} \cup\{d\})$ be an incomplete decision system. For any $A \subseteq$ Attr and $x \in U, \partial_{A}(x)=\left\{d(y) \mid y \in T_{A}(x)\right\}$ is called the generalized decision. If $\left|\partial_{\text {Attr }}(x)\right|=1$ for any $x \in U$ then IDS is consistent. Otherwise, it is inconsistent.

One of the most important concepts in tolerance rough sets is reduct. According to Kryszkiewicz [5], a reduct of an incomplete decision system is a minimal subset of a conditional attribute set which keeps the generalized decision unchanged for all objects.

Definition 2.2 [5]. Given an incomplete decision system $I D S=$ $(U$, Attr $\cup\{d\})$, if an attribute set $R \subseteq$ Attr satisfies
(1) $\partial_{R}(x)=\partial_{\text {Attr }}(x)$ for any $x \in U$;
(2) $R-\{r\}$ is not satisfied (1) for any $r \in R$, then $R$ is called a reduct of IDS based on generalized decision.

Referring to Table 1, $T_{\text {Attr }}\left(x_{1}\right)=\left\{x_{1}\right\}, T_{\text {Attr }}\left(x_{2}\right)=\left\{x_{2}, x_{6}\right\}, T_{\text {Attr }}\left(x_{3}\right)=$ $\left\{x_{3}\right\}, T_{\text {Attr }}\left(x_{4}\right)=\left\{x_{4}, x_{5}\right\}, T_{\text {Attr }}\left(x_{5}\right)=\left\{x_{5}, x_{4}, x_{6}\right\}, T_{\text {Attr }}\left(x_{6}\right)=\left\{x_{6}, x_{2}, x_{5}\right\}$, we have the covering $K($ Attr $)=\left\{\left\{x_{1}\right\},\left\{x_{2}, x_{6}\right\},\left\{x_{3}\right\},\left\{x_{4}, x_{5}\right\},\left\{x_{4}, x_{5}, x_{6}\right\}\right.$, $\left.\left\{x_{2}, x_{5}, x_{6}\right\}\right\}$.

For $R=\left\{a_{3}, a_{4}\right\}$, we obtain the covering

$$
\begin{aligned}
& K(R)=U / T L R(R)=\left\{T_{R}(x) \mid x \in U\right\} \\
& =\left\{\left\{x_{1}, x_{2}, x_{6}\right\},\left\{x_{1}, x_{2}, x_{6}\right\},\left\{x_{3}\right\},\left\{x_{4}, x_{5}, x_{6}\right\},\left\{x_{4}, x_{5}, x_{6}\right\},\left\{x_{1}, x_{2}, x_{4}, x_{5}, x_{6}\right\}\right\}
\end{aligned}
$$

For the attribute set Attr, we have $\partial_{A t t r}\left(x_{1}\right)=\partial_{A t t r}\left(x_{2}\right)=\{\operatorname{good}\}$, $\partial_{\text {Attr }}\left(x_{3}\right)=\{$ poor $\}, \partial_{\text {Attr }}\left(x_{4}\right)=\partial_{\text {Attr }}\left(x_{5}\right)=\partial_{\text {Attr }}\left(x_{6}\right)=\{$ good, excellent $\}$. For the attribute set $R$, we have $\partial_{R}\left(x_{1}\right)=\partial_{R}\left(x_{2}\right)=\{$ good $\}, \partial_{R}\left(x_{3}\right)=\{$ poor $\}$, $\partial_{R}\left(x_{4}\right)=\partial_{R}\left(x_{5}\right)=\partial_{R}\left(x_{6}\right)=\{$ good, excellent $\}$. As a result, we obtain $\partial_{R}(x)=$ $\partial_{\text {Attr }}(x)$ for any $x \in U$. In addition, $\partial_{\left\{a_{3}\right\}}(x)=\partial_{A t t r}(x)$ and $\partial_{\left\{a_{4}\right\}}(x)=\partial_{\text {Attr }}(x)$ is incorrect for any $x \in U$. According to Definition $2.2, R$ is a reduct based on generalized decision.

## 3. Generalized Liang Entropy and Properties. <br> 3.1. Generalized Liang Entropy.

Definition 3.1. Given an incomplete decision system $I D S=(U, \operatorname{Attr} \cup$ $\{d\})$ where $U=\left\{x_{1}, \ldots, x_{|U|}\right\}, A \subseteq$ Attr and $U / T L R(A)=\left\{T_{A}\left(x_{1}\right), T_{A}\left(x_{2}\right), \ldots\right.$, $\left.T_{A}\left(x_{|U|}\right)\right\}$. We define generalized Liang entropy of $P$ as

$$
I E(A)=\sum_{i=1}^{|U|} \frac{1}{|U|}\left(1-\frac{\left|T_{A}\left(x_{i}\right)\right|}{|U|}\right)
$$

where $\left|T_{A}(x)\right|$ is the cardinality of $T_{A}(x)$. If $U / T L R(A)=\omega$ then $\operatorname{IE}(A)$ has the maximum value $I E(A)=1-\frac{1}{|U|}$. If $U / T L R(A)=\delta$ then $I E(A)$ has the minimum value $I E(A)=0$. Obviously, $0 \leq I E(A) \leq 1-\frac{1}{|U|}$.

The following Proposition 3.1 proves that Liang entropy $E(A)$ in [6] is a particular case of our generalized Liang entropy.

Proposition 3.1. Given a complete decision system $D S=(U$, Attr $\cup$ $\{d\}), A \subseteq$ Attr, $U=\left\{x_{1}, \ldots, x_{|U|}\right\}$ and $U / A=\left\{A_{1}, A_{2}, \ldots, A_{m}\right\}$, then

$$
I E(A)=\sum_{i=1}^{|U|} \frac{1}{|U|}\left(1-\frac{\left|T_{A}\left(x_{i}\right)\right|}{|U|}\right)=\sum_{i=1}^{m} \frac{\left|A_{i}\right|}{|U|}\left(1-\frac{\left|A_{i}\right|}{|U|}\right)=E(A)
$$

where $E(A)$ is Liang entropy in $[6]$.
Proof. Suppose that $A_{i}=\left\{x_{i 1}, x_{i 2}, \ldots, x_{i p_{i}}\right\}$ where $\left|A_{i}\right|=p_{i}$ and $\sum_{i=1}^{m} p_{i}=|U|$.

$$
\begin{aligned}
A_{i} & =T_{A}\left(x_{i 1}\right)=T_{A}\left(x_{i 2}\right)=\cdots=T_{A}\left(x_{i p_{i}}\right), \\
\left|A_{i}\right| & =\left|T_{A}\left(x_{i 1}\right)\right|=\left|T_{A}\left(x_{i 2}\right)\right|=\cdots=\left|T_{A}\left(x_{i p_{i}}\right)\right|=p_{i} \\
\frac{\left|A_{i}\right|}{|U|}\left(1-\frac{\left|A_{i}\right|}{|U|}\right) & =\frac{1}{|U|}\left(\left|A_{i}\right|-\frac{\left|A_{i}\right|\left|A_{i}\right|}{|U|}\right) \\
& =\frac{1}{|U|}\left(1-\frac{\left|T_{A}\left(x_{i 1}\right)\right|}{|U|}+1-\frac{\left|T_{A}\left(x_{i 2}\right)\right|}{|U|}+\cdots+1-\frac{\left|T_{A}\left(x_{i p_{i}}\right)\right|}{|U|}\right) \\
E(A) & =\sum_{i=1}^{m} \frac{\left|A_{i}\right|}{|U|}\left(1-\frac{\left|A_{i}\right|}{|U|}\right)=\sum_{i=1}^{m} \sum_{k=1}^{p_{i}} \frac{1}{|U|}\left(1-\frac{\left|T_{A}\left(x_{i k}\right)\right|}{|U|}\right) \\
& =\sum_{i=1}^{|U|} \frac{1}{|U|}\left(1-\frac{\left|T_{A}\left(x_{i}\right)\right|}{|U|}\right)=I E(A) .
\end{aligned}
$$

Consequently, we have $E(A)=I E(A)$. The proposition is proved.
Definotion 3.2. Given an incomplete decision system $\operatorname{IDS}=(U$, Attr $\cup$ $\{d\})$, where $U=\left\{x_{1}, \ldots, x_{|U|}\right\}$ and $A, B \subseteq$ Attr. We define generalized Liang entropy of $A \cup B$ as

$$
I E(A \cup B)=\sum_{i=1}^{|U|} \frac{1}{|U|}\left(1-\frac{\left|T_{A \cup B}\left(x_{i}\right)\right|}{|U|}\right)=\sum_{i=1}^{|U|} \frac{1}{|U|}\left(1-\frac{\left|T_{A}\left(x_{i}\right) \cap T_{B}\left(x_{i}\right)\right|}{|U|}\right) .
$$

### 3.2. Conditional Generalized Liang Entropy.

Definition 3.3. Given an incomplete decision system $I D S=(U$, Attr $\cup$ $\{d\}$ ), where $U=\left\{x_{1}, \ldots, x_{|U|}\right\}$, two attribute sets $A, B \subseteq$ Attr and two coverings $U / T L R(A)=\left\{T_{A}\left(x_{1}\right), \ldots, T_{A}\left(x_{|U|}\right)\right\}$ and $U / T L R(B)=\left\{T_{B}\left(x_{1}\right), \ldots, T_{B}\left(x_{|U|}\right)\right\}$.

We define conditional generalized Liang entropy of $B$ about $A$ as

$$
I E(B \mid A)=\frac{1}{|U|} \sum_{i=1}^{|U|}\left(\frac{\left|T_{A}\left(x_{i}\right)\right|-\left|T_{B}\left(x_{i}\right) \cap T_{A}\left(x_{i}\right)\right|}{|U|}\right)
$$

The following Proposition 3.2 proves that conditional Liang entropy $E(B \mid A)$ in [6] is a particular case of our conditional generalized Liang entropy $I E(B \mid A)$.

Proposition 3.2. Given a complete decision system $D S=(U$, Attr $\cup$ $\{d\})$, where $U=\left\{x_{1}, \ldots, x_{|U|}\right\}$, two attribute sets $A, B \subseteq$ Attr and two partitions $U / A=\left\{A_{1}, A_{2}, \ldots, A_{m}\right\}$ and $U / B=\left\{B_{1}, B_{2}, \ldots, B_{n}\right\}$, then

$$
\begin{aligned}
I E(B \mid A) & =\frac{1}{|U|} \sum_{i=1}^{|U|}\left(\frac{\left|T_{A}\left(x_{i}\right)\right|-\left|T_{B}\left(x_{i}\right) \cap T_{A}\left(x_{i}\right)\right|}{|U|}\right) \\
& =\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\left|B_{i} \cap A_{j}\right|}{|U|} \frac{\left|B_{i}^{c}-A_{j}^{c}\right|}{|U|}=E(B \mid A)
\end{aligned}
$$

where $B_{i}^{c}=U-B_{i}, A_{j}^{c}=U-A_{j}$ and $E(B \mid A)$ is the conditional Liang entropy in [6].

Proof. Suppose that $B_{i} \cap A_{j}=\left\{x_{i 1}, x_{i 2}, \ldots, x_{i s_{j}}\right\}$, here $\left|B_{i} \cap A_{j}\right|=p_{j}$ and $\left|B_{i}\right|=q_{i}$. We have $\sum_{j=1}^{m} p_{j}=q_{i}$ and $\sum_{i=1}^{n} q_{i}=|U|$. Then

$$
\begin{aligned}
& B_{i} \cap A_{j}=T_{B}\left(x_{i 1}\right) \cap T_{A}\left(x_{i 1}\right)=T_{B}\left(x_{i 2}\right) \cap T_{A}\left(x_{i 2}\right)=\cdots=T_{B}\left(x_{i p_{j}}\right) \cap T_{A}\left(x_{i p_{j}}\right), \\
& \left|B_{i} \cap A_{j}\right|=\left|T_{B}\left(x_{i 1}\right) \cap T_{A}\left(x_{i 1}\right)\right|=\left|T_{B}\left(x_{i 2}\right) \cap T_{A}\left(x_{i 2}\right)\right|=\cdots \\
& \quad=\left|T_{B}\left(x_{i p_{j}}\right) \cap T_{A}\left(x_{i p_{j}}\right)\right|=p_{j}, \\
& \left|B_{i} \cap A_{j}\right|\left|B_{i}^{c}-A_{j}^{c}\right|=\left|B_{i} \cap A_{j}\right|\left|B_{i}^{c} \cap A_{j}\right|=\left|B_{i} \cap A_{j}\right|\left|A_{j}-\left(B_{i} \cap A_{j}\right)\right| \\
& \quad=\left|T_{A}\left(x_{i 1}\right)-\left(T_{B}\left(x_{i 1}\right) \cap T_{A}\left(x_{i 1}\right)\right)\right|+\cdots+\left|T_{A}\left(x_{i s_{i}}\right)-\left(T_{B}\left(x_{i p_{j}}\right) \cap T_{A}\left(x_{i p_{j}}\right)\right)\right| \\
& \quad=\sum_{k=1}^{p_{j}}\left|T_{A}\left(x_{i k}\right)-\left(T_{B}\left(x_{i k}\right) \cap T_{A}\left(x_{i k}\right)\right)\right|=\sum_{k=1}^{p_{j}}\left|T_{A}\left(x_{i k}\right)\right|-\left|T_{B}\left(x_{i k}\right) \cap T_{A}\left(x_{i k}\right)\right| .
\end{aligned}
$$

Hence

$$
\begin{aligned}
& \begin{aligned}
\sum_{j=1}^{m}\left|B_{i} \cap A_{j}\right|\left|B_{i}^{c}-A_{j}^{c}\right|= & \sum_{j=1}^{m} \sum_{k=1}^{p_{j}}\left|T_{A}\left(x_{i k}\right)\right|-\left|T_{B}\left(x_{i k}\right) \cap T_{A}\left(x_{i k}\right)\right| \\
& =\sum_{k=1}^{q_{i}}\left|T_{A}\left(x_{i k}\right)\right|-\left|T_{B}\left(x_{i k}\right) \cap T_{A}\left(x_{i k}\right)\right|,
\end{aligned} \\
& \begin{aligned}
\sum_{i=1}^{n} \sum_{j=1}^{m}\left|B_{i} \cap A_{j}\right|\left|B_{i}^{c}-A_{j}^{c}\right| & =\sum_{i=1}^{n} \sum_{k=1}^{q_{i}}\left|T_{A}\left(x_{i k}\right)\right|-\left|T_{B}\left(x_{i k}\right) \cap T_{A}\left(x_{i k}\right)\right| \\
& =\sum_{i=1}^{|U|}\left|T_{A}\left(x_{i}\right)\right|-\left|T_{B}\left(x_{i}\right) \cap T_{A}\left(x_{i}\right)\right|,
\end{aligned} \\
& I E(B \mid A)=\frac{1}{|U|} \sum_{i=1}^{|U|}\left(\frac{\left|T_{A}\left(x_{i}\right)\right|-\left|T_{B}\left(x_{i}\right) \cap T_{A}\left(x_{i}\right)\right|}{|U|}\right) \\
& =\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{\left|B_{i} \cap A_{j}\right|}{|U|} \frac{\left|B_{i}^{c}-A_{j}^{c}\right|}{|U|}=E(B \mid A) .
\end{aligned}
$$

Consequently, $I E(B \mid A)=E(B \mid A)$. The proposition is proved.

### 3.3 Some Properties of Generalized Liang Entropy.

Proposition 3.3. Given an incomplete decision system IDS $=(U$, Attr $\cup$ $\{d\}$ ), where $U=\left\{x_{1}, \ldots, x_{|U|}\right\}$ and $A, B, C \subseteq$ Attr, the following properties hold:
a) If $U / T L R(A) \preceq ; U / T L R(B)$ then $\operatorname{IE}(A) \geq I E(B)$.
$I E(A)=I E(B)$ if and only if $U / T L R(A)=U / T L R(B)$.
b) If $U / T L R(A) \preceq U / T L R(B)$ then $\operatorname{IE}(A \cup B)=I E(A)$.
c) $I E(A \cup B) \geq I E(A), I E(A \cup B) \geq I E(B)$.
d) $I E(A \cup B)=I E(A)+I E(B \mid A)=I E(A)+I E(A \mid B)$.
e) $0 \leq I E(B \mid A) \leq 1-1 /|U| \cdot I E(B \mid A)=0$ if and only if $U / T L R(A) \preceq U / T L R(B)$. $\operatorname{IE}(B \mid A)=1-1 /|U|$ if and only if $U / T L R(A)=\delta$ and $U / T L R(B)=\omega$.
f) If $U / T L R(A) \preceq U / T L R(B)$ then $\operatorname{IE}(C \mid B) \geq I E(C \mid A)$.
g) If $U / T L R(A) \preceq U / T L R(B)$ then $I E(A \mid C) \geq I E(B \mid C)$.

Proof. a) This result obtains directly from Definition 3.1 and Definition 2.1.
b) This result obtains directly from Definition 3.1, Definition 3.2, Definition 2.1 and Property 2.1.
c) This result obtains directly from $a$ ).
d) From Definition 3.1, Definition 3.2 and Definition 3.3, we have

$$
\begin{aligned}
& I E(B \mid A) \quad=\frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{A}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right) \cap T_{B}\left(x_{i}\right)\right|}{|U|} \\
& =1-\frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{A}\left(x_{i}\right) \cap T_{B}\left(x_{i}\right)\right|}{|U|}-1+\frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{A}\left(x_{i}\right)\right|}{|U|} \\
& =\frac{1}{|U|} \sum_{i=1}^{|U|} 1-\frac{\left|T_{A}\left(x_{i}\right) \cap T_{B}\left(x_{i}\right)\right|}{|U|}-\frac{1}{|U|} \sum_{i=1}^{|U|} 1-\frac{\left|T_{A}\left(x_{i}\right)\right|}{|U|}=\operatorname{IE}(A \cup B)-\operatorname{IE}(A) .
\end{aligned}
$$

Consequently, we have $I E(A \cup B)=I E(A)+I E(A \mid B)$. By symmetric property of $I E(A \cup B)$ we have $I E(A \cup B)=I E(B)+I E(A \mid B)$.
e) It is clear that $I E(B \mid A) \geq 0$. From $d)$ we have $I E(B \mid A)=I E(A \cup$ $B)-I E(A) . I E(B \mid A)=0 \Leftrightarrow I E(A \cup B)=I E(A)$. Property 2.1 shows that $U / T L R(A \cup B) \preceq U / T L R(A)$. From $a)$ we obtain $I E(A \cup B)=I E(A) \Leftrightarrow$ $U / T L R(A \cup B)=U / T L R(A) \Leftrightarrow U / T L R(A) \preceq U / T L R(B)$. In addition, it follows from $d$ ) and Definition 3.1 that $I E(B \mid A)=I E(A \cup B)-I E(A), I E(A \cup$ $B) \leq 1-1 /|U|, I E(A) \geq 0$. So we obtain $\operatorname{IE}(B \mid A) \leq 1-1 /|U|$. The conditional equality is $I E(A)=0 \wedge I E(A \cup B)=1-1 /|U|$, that is $U / T L R(A)=\delta$ and $U / T L R(A \cup B)=\omega$. This is equivalent to $U / T L R(A)=\delta$ and $U / T L R(B)=\omega$.
$f$ ) Suppose that $U / T L R(C)=\left\{T_{C}\left(x_{1}\right), T_{C}\left(x_{2}\right), \ldots, T_{C}\left(x_{|U|}\right)\right\}$. Since $U / T L R(A) \preceq U / T L R(B)$, we have $T_{A}\left(x_{i}\right) \subseteq T_{B}\left(x_{i}\right)$ for $\forall x_{i} \in U, i=1 \ldots|U|$ and

$$
\begin{align*}
& \left(T_{B}\left(x_{i}\right)-T_{A}\left(x_{i}\right)\right) \cap T_{C}\left(x_{i}\right) \subseteq T_{B}\left(x_{i}\right)-T_{A}\left(x_{i}\right) \\
& \Leftrightarrow\left(T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right)-\left(T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right) \subseteq T_{B}\left(x_{i}\right)-T_{A}\left(x_{i}\right) \\
& \Leftrightarrow\left|\left(T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right)-\left(T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right)\right| \leq\left|T_{B}\left(x_{i}\right)-T_{A}\left(x_{i}\right)\right| \tag{3.1}
\end{align*}
$$

Since $T_{A}\left(x_{i}\right) \subseteq T_{B}\left(x_{i}\right)$ we have $T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right) \subseteq T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)$ and Equation 3.1 is equivalent to

$$
\begin{aligned}
& \left|T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right| \leq\left|T_{B}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right)\right| \\
& \Leftrightarrow\left|T_{B}\left(x_{i}\right)\right|-\left|T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right| \geq\left|T_{A}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right| \\
& \Leftrightarrow \frac{1}{|U|} \sum_{i=1}^{n} \frac{\left|T_{B}\left(x_{i}\right)\right|-\left|T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right|}{|U|} \geq \frac{1}{|U|} \sum_{i=1}^{n} \frac{\left|T_{A}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right|}{|U|} \\
& \Leftrightarrow I E(C \mid B) \geq I E(C \mid A) .
\end{aligned}
$$

g) Since $U / T L R(A) \preceq U / T L R(B)$, we have $T_{A}\left(x_{i}\right) \subseteq T_{B}\left(x_{i}\right)$ for $\forall x_{i} \in U$, $i=1 \ldots|U|$. Suppose that $U / \operatorname{TLR}(C)=\left\{T_{C}\left(x_{1}\right), T_{C}\left(x_{2}\right), \ldots, T_{C}\left(x_{|U|}\right)\right\}$, we obtain

$$
\begin{aligned}
& T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right) \subseteq T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right) \\
& \Leftrightarrow\left|T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right| \leq\left|T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right| \\
& \Leftrightarrow\left|T_{C}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right| \geq\left|T_{C}\left(x_{i}\right)\right|-\left|T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right| \\
& \Leftrightarrow \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{C}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right|}{|U|} \geq \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{C}\left(x_{i}\right)\right|-\left|T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right|}{|U|} \\
& \Leftrightarrow I E(A \mid C) \geq I E(B \mid C) .
\end{aligned}
$$

4. Distance between Coverings and Properties. Let $X$ be the set of objects. A distance between two objects $x, y \in X$, denoted as $d(x, y)$, is a measure which satisfies three conditions [2]:

$$
\begin{align*}
d(x, y) & \geq 0, \quad d(x, y)=0 \Leftrightarrow x=y  \tag{C1}\\
d(x, y) & =d(y, x) ;  \tag{C2}\\
d(x, y)+d(y, z) & \geq d(x, z) \text { for any } z \in X . \tag{C3}
\end{align*}
$$

In this section, a distance is established between two coverings generated by two attributes based on the generalized Liang entropy. Some properties of the distance are also investigated.

Lemma 4.1. Given an incomplete decision system $\operatorname{IDS}=(U, \operatorname{Attr} \cup\{d\})$ where $U=\left\{x_{1}, \ldots, x_{|U|}\right\}$ and $A, B, C \subseteq$ Attr, the following properties hold:
a) $I E(A \mid C)+I E(B \mid A \cup C)=I E(A \cup B \mid C)$;
b) $I E(B \mid A)+I E(A \mid C) \geq I E(B \mid C)$.

Proof. Suppose that

$$
\begin{aligned}
U / \operatorname{TLR}(A) & =\left\{T_{A}\left(x_{1}\right), T_{A}\left(x_{2}\right), \ldots, T_{A}\left(x_{|U|}\right)\right\}, \\
U / \operatorname{TLR}(B) & =\left\{T_{B}\left(x_{1}\right), T_{B}\left(x_{2}\right), \ldots, T_{B}\left(x_{|U|}\right)\right\}, \\
U / T L R(C) & =\left\{T_{C}\left(x_{1}\right), T_{C}\left(x_{2}\right), \ldots, T_{C}\left(x_{|U|}\right)\right\} .
\end{aligned}
$$

$$
\begin{aligned}
& \text { a) } I E(A \mid C)+I E(B \mid A \cup C)= \\
= & \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{C}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right|+\left|T_{A \cup C}\left(x_{i}\right)\right|-\left|T_{A \cup C}\left(x_{i}\right) \cap S_{B}\left(x_{i}\right)\right|}{|U|} \\
= & \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{C}\left(x_{i}\right)\right|-\left|T_{A \cup C}\left(x_{i}\right)\right|+\left|T_{A \cup C}\left(x_{i}\right)\right|-\left|T_{A \cup C}\left(x_{i}\right) \cap T_{B}\left(x_{i}\right)\right|}{|U|} \\
= & \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{C}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right) \cap T_{B}\left(x_{i}\right) \cap T_{C}\left(x_{i}\right)\right|}{|U|} \\
= & \frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{C}\left(x_{i}\right)\right|-\left|T_{C}\left(x_{i}\right) \cap T_{A \cup B}\left(x_{i}\right)\right|}{|U|}=I E(A \cup B \mid C) .
\end{aligned}
$$

Consequently, we have $I E(A \mid C)+I E(B \mid A \cup C)=I E(A \cup B \mid C)$.
b) Using Proposition 3.3, item $a)$, it follows from $U / T L R(A \cup C) \preceq$ $U / T L R(A), U / T L R(A \cup B) \preceq U / T L R(B)$ that $I E(B \mid A) \geq I E(B \mid A \cup C)$ and $I E(A \cup B \mid C) \geq I E(B \mid C)$. Using Lemma 4.1 item $a)$ we have

$$
I E(B \mid A)+I E(A \mid C) \geq I E(B \mid A \cup C)+I E(A \mid C)=I E(A \cup B \mid C) \geq I E(B \mid C) .
$$

Consequently, we have $I E(B \mid A)+I E(A \mid C) \geq I E(B \mid C)$.
Theorem 4.1. Given an incomplete decision system $\operatorname{IDS}=(U, A t t r \cup$ $\{d\})$ and two attributes $A, B \subseteq$ Attr, for any $K(A), K(B) \in \operatorname{COV}(U)$, the mapping $d_{E}: \operatorname{COV}(U) \times \operatorname{COV}(U) \rightarrow[0, \infty)$ determined by

$$
d_{E}(K(A), K(B))=I E(A \mid B)+I E(B \mid A)
$$

is a distance between $K(A)$ and $K(B)$.
Proof. (C1) According to Proposition 3.3 item $e$ ) we have $d_{E}(K(A)$, $K(B)) \geq 0$ for any $K(A), K(B) \in \operatorname{COV}(U), d_{E}(K(A), K(B))=0$
$\Leftrightarrow(I E(B \mid A)=0.) \wedge(I E(A \mid B)=0$.
$\Leftrightarrow(U / T L R(A) \preceq U / T L R(B)) \wedge(U / T L R(B) \preceq U / T L R(A)) \Leftrightarrow K(A)=K(B)$.
(C2) According to the definition of the distance $d_{E}$, we have $d_{E}(K(A)$, $K(B))=d_{E}(K(B), K(A))$ for any $K(A), K(B) \in \operatorname{COV}(U)$.
(C3) For any $K(A), K(B), K(C) \in \operatorname{COV}(U)$, from Lemma 4.1 item b) we have

$$
\begin{equation*}
I E(B \mid A)+I E(A \mid C) \geq I E(B \mid C) \tag{4.1}
\end{equation*}
$$

$$
\begin{equation*}
I E(C \mid A)+I E(A \mid B) \geq I E(C \mid B) \tag{4.2}
\end{equation*}
$$

From Equation (4.1) and Equation (4.2), we obtain

$$
d_{E}(K(B), K(A))+d_{E}(K(A), K(C)) \geq d_{E}(K(B), K(C))
$$

From (C1), (C2), (C3) we conclude that $d_{E}(K(A), K(B))$ is a distance on $\operatorname{COV}(U)$. The theorem is proved.

Proposition 4.1. Given an incomplete decision system IDS $=(U$, Attr $\cup$ $\{d\}$ ), where $U=\left\{x_{1}, \ldots, x_{|U|}\right\}$ and $A \subseteq$ Attr, then

$$
d_{E}(K(A), K(A t t r))=\frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{A}\left(x_{i}\right)\right|-\left|T_{A t t r}\left(x_{i}\right)\right|}{|U|} .
$$

Proof. Since $A \subseteq$ Attr we have $U / T L R(A t t r) \preceq U / T L R(A)$ (Property 2.1). From Proposition 3.3 item $e$ ) we obtain $I E(A \mid A t t r)=0$. In addition, it follows from $A \subseteq \operatorname{Attr}$ that $T_{A t t r}\left(x_{i}\right) \subseteq T_{A}\left(x_{i}\right)$ or $T_{A}\left(x_{i}\right) \cap T_{\text {Attr }}\left(x_{i}\right)=T_{A t t r}\left(x_{i}\right)$ for $\forall x_{i} \in U, i=1 \ldots|U|$. Consequently,

$$
\begin{aligned}
& d_{E}(K(A), K(\operatorname{Attr}))=\operatorname{IE}(A \mid \operatorname{Attr})+\operatorname{IE}(\operatorname{Attr} \mid A)=I E(\operatorname{Attr} \mid A) \\
& \quad=\frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{A}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right) \cap T_{A t t r}\left(x_{i}\right)\right|}{|U|}=\frac{1}{|U|} \sum_{i=1}^{|U|} \frac{\left|T_{A}\left(x_{i}\right)\right|-\left|T_{A t t r}\left(x_{i}\right)\right|}{|U|} .
\end{aligned}
$$

The proposition is proved.
Proposition 4.2. Given an incomplete decision system $\operatorname{IDS}=(U, A t t r \cup$ $\{d\})$, if $A \subseteq$ Attr, then $d_{E}(K(A), K(A \cup\{d\})) \geq d_{E}(K(A t t r), K(A t t r \cup\{d\}))$.

Proof. Suppose that $U=\left\{x_{1}, x_{2}, \ldots, x_{|U|}\right\}$ and $A \subseteq$ Attr. For $\forall x_{i} \in U$, $i=1 \ldots|U|$, it is clear that $T_{\text {Attr }}\left(x_{i}\right) \subseteq T_{A}\left(x_{i}\right)$. So we have

$$
\begin{align*}
& \left(T_{A}\left(x_{i}\right)-T_{A t t r}\left(x_{i}\right)\right) \cap T_{\{d\}}\left(x_{i}\right) \subseteq T_{A}\left(x_{i}\right)-T_{A t t r}\left(x_{i}\right) \\
& \Leftrightarrow\left(T_{A}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)-\left(T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right) \subseteq T_{A}\left(x_{i}\right)-T_{A t t r}\left(x_{i}\right) \\
& \Leftrightarrow\left|\left(T_{A}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)-\left(T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right| \leq\left|T_{A}\left(x_{i}\right)-T_{A t t r}\left(x_{i}\right)\right| . \tag{4.3}
\end{align*}
$$

It follows from $T_{\text {Attr }}\left(x_{i}\right) \subseteq T_{A}\left(x_{i}\right)$ that $T_{\text {Attr }}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right) \subseteq T_{A}\left(x_{i}\right) \cap$ $T_{\{d\}}\left(x_{i}\right)$. So Equation 4.3 is equivalent to

$$
\begin{align*}
& \left|T_{A}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right|-\left|T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right| \leq\left|T_{A}\left(x_{i}\right)\right|-\left|T_{A t t r}\left(x_{i}\right)\right|  \tag{4.4}\\
& \quad \Leftrightarrow\left|T_{A}\left(x_{i}\right)\right|-\left|T_{A}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right| \geq\left|T_{A t t r}\left(x_{i}\right)\right|-\left|T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right| .
\end{align*}
$$

Since $T_{A}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right) \subseteq T_{A}\left(x_{i}\right), T_{\text {Attr }}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right) \subseteq T_{A t t r}\left(x_{i}\right)$, Equation 4.4 is equivalent to

$$
\begin{align*}
& \left|T_{A}\left(x_{i}\right) \cup\left(T_{A}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right|-\left|T_{A}\left(x_{i}\right) \cap\left(T_{A}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right| \geq  \tag{4.5}\\
& \left|T_{\text {Attr }}\left(x_{i}\right) \cup\left(T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right|-\left|T_{A t t r}\left(x_{i}\right) \cap\left(T_{\text {Attr }}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right| .
\end{align*}
$$

$$
\text { Since } T_{A \cup\{d\}}\left(x_{i}\right)=T_{A}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right), T_{A t t r \cup\{d\}}\left(x_{i}\right)=T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right) \text {, }
$$

Equation 4.5 is equivalent to

$$
\begin{equation*}
\sum_{i=1}^{n} \frac{\left|T_{A}\left(x_{i}\right)\right|-\left|T_{A \cup\{d\}}\left(x_{i}\right)\right|}{|U|^{2}} \geq \sum_{i=1}^{n} \frac{\left|T_{A t t r}\left(x_{i}\right)\right|-\left|T_{A t t r \cup\{d\}}\left(x_{i}\right)\right|}{|U|^{2}} \tag{4.6}
\end{equation*}
$$

From Proposition 4.1 and $A \subset A \cup\{d\}$, Attr $\subset A t t r \cup\{d\}$, Equation 4.6 is equivalent to $d_{E}(K(A), K(A \cup\{d\})) \geq d_{E}(K(\operatorname{Attr}), K(\operatorname{Attr} \cup\{d\}))$. The proposition is proved.
5. Distance-based Attribute Reduction Method. Deriving from results in Section 3 and 4, we propose a distance-based method for attribute reduction in incomplete decision systems. First, we define a reduct based on the distance. Second, we define the importance of an attribute based on the distance as the classification ability of the attribute. As a result, we propose a heuristic algorithm to find the best reduct by using the importance of an attribute as an attribute selection criterion.

Definition 5.1. Given an incomplete decision system $I D S=(U, A t t r \cup$ $\{d\}$ ), if an attribute set $R \subseteq$ Attr satisfies
(1) $\quad d_{E}(K(R), K(R \cup\{d\}))=d_{E}(K(A t t r), K(A t t r \cup\{d\}))$;
(2) $\forall r \in R, d_{E}(K(R-\{r\}), K((R-\{r\}) \cup\{d\})) \neq d_{E}(K(A t t r), K(\operatorname{Attr} \cup\{d\}))$, then $R$ is called a reduct of IDS based on distance.

The following Proposition 5.1 shows the relationship between the reduct based on generalized decision and the reduct based on distance.

Proposition 5.1. Given an incomplete decision system IDS $=(U$, Attr $\cup$ $\{d\})$ and $R \subseteq \operatorname{Attr}$, if $d_{E}(K(R), K(R \cup\{d\}))=d_{E}(K(\operatorname{Attr}), K(\operatorname{Attr} \cup\{d\}))$, then $\forall x_{i} \in U, \partial_{R}\left(x_{i}\right)=\partial_{A t t r}\left(x_{i}\right)$.

Proof. Suppose that $U=\left\{x_{1}, x_{2}, \ldots, x_{|U|}\right\}$.
Since $d_{E}(K(R), K(R \cup\{d\}))=d_{E}(K(\operatorname{Attr}), K(\operatorname{Attr} \cup\{d\}))$, according to Propo-
sition 4.1 we have:

$$
\begin{equation*}
\frac{1}{|U|} \sum_{i=1}^{|U|}\left(\frac{\left|T_{R}\left(x_{i}\right)\right|-\left|T_{R \cup\{d\}}\left(x_{i}\right)\right|}{|U|}\right)=\frac{1}{|U|} \sum_{i=1}^{|U|}\left(\frac{\left|T_{\text {Attr }}\left(x_{i}\right)\right|-\left|T_{\text {Attr } \cup d\}}\left(x_{i}\right)\right|}{|U|}\right) \tag{5.1}
\end{equation*}
$$

$\Leftrightarrow\left|T_{R}\left(x_{i}\right)\right|-\left|T_{R \cup\{d\}}\left(x_{i}\right)\right|=\left|T_{A t t r}\left(x_{i}\right)\right|-\left|T_{A t t r \cup\{d\}}\left(x_{i}\right)\right|$ for any $x_{i} \in U$.
It is clear that $T_{R \cup\{d\}}\left(x_{i}\right) \subseteq T_{R}\left(x_{i}\right), T_{A t t r \cup\{d\}}\left(x_{i}\right) \subseteq T_{A t t r}\left(x_{i}\right)$, so Equation 5.1 is equivalent to

$$
\begin{equation*}
\left|T_{R}\left(x_{i}\right)-T_{R \cup\{d\}}\left(x_{i}\right)\right|=\left|T_{A t t r}\left(x_{i}\right)-T_{A t t r \cup\{d\}}\left(x_{i}\right)\right| \text { for any } x_{i} \in U . \tag{5.2}
\end{equation*}
$$

Since $T_{\text {Attr }}\left(x_{i}\right) \subseteq T_{R}\left(x_{i}\right)$ we have $T_{\text {Attr }}\left(x_{i}\right)-T_{\{d\}}\left(x_{i}\right) \subseteq T_{R}\left(x_{i}\right)-T_{\{d\}}\left(x_{i}\right)$
$\Leftrightarrow T_{A t t r}\left(x_{i}\right)-T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right) \subseteq T_{R}\left(x_{i}\right)-T_{R}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)$
$\Leftrightarrow T_{A t t r}\left(x_{i}\right)-T_{A t t r \cup\{d\}}\left(x_{i}\right) \subseteq T_{R}\left(x_{i}\right)-T_{R \cup\{d\}}\left(x_{i}\right)$.
So Equation 5.2 is equivalent to

$$
\begin{equation*}
T_{R}\left(x_{i}\right)-T_{R \cup\{d\}}\left(x_{i}\right)=T_{A t t r}\left(x_{i}\right)-T_{A t t r \cup\{d\}}\left(x_{i}\right) \text { for any } x_{i} \in U . \tag{5.3}
\end{equation*}
$$

In addition, we have

$$
\begin{aligned}
& T_{R}\left(x_{i}\right)=\left(T_{R}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right) \cup\left(T_{R}\left(x_{i}\right)-\left(T_{R}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right), \\
& T_{A t t r}\left(x_{i}\right)=\left(T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right) \cup\left(T_{A t t r}\left(x_{i}\right)-\left(T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right) .
\end{aligned}
$$

Suppose that $d_{i}=d\left(x_{i}\right), R_{i}=\left\{d\left(y_{i}\right) \mid y_{i} \in T_{R}\left(x_{i}\right)-\left(T_{R}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right\}$, $A_{i}=\left\{d\left(y_{i}\right) \mid y_{i} \in T_{A t t r}\left(x_{i}\right)-\left(T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right\}$. Then we have

$$
\begin{aligned}
\partial_{R}\left(x_{i}\right)= & \left\{d\left(y_{i}\right) \mid y_{i} . \in\left(T_{R}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right) \cup\left(T_{R}\left(x_{i}\right)-\left(T_{R}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right)\right\} \\
= & \left\{d_{i}\right\} \cup R_{i} \\
\partial_{\text {Attr }}\left(x_{i}\right)= & \left\{d\left(y_{i}\right) \mid y_{i} \in\left(T_{\text {Attr }}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right. \\
& \left.\cup\left(T_{A t t r}\left(x_{i}\right)-\left(T_{A t t r}\left(x_{i}\right) \cap T_{\{d\}}\left(x_{i}\right)\right)\right)\right\}=\left\{d_{i}\right\} \cup A_{i} .
\end{aligned}
$$

According to Equation 5.3, we obtain $R_{i}=A_{i}$, thus $\partial_{R}\left(x_{i}\right)=\partial_{A t t r}\left(x_{i}\right)$ for any $x_{i} \in U$. The proposition is proved.

Proposition 5.1 shows that if $R_{D}$ is a reduct based on metric then there exists a reduct based on generalized decision $R_{\partial}$ so that $R_{\partial} \subseteq R_{D}$.

If $I D S$ is consistent, it follows from the condition $\forall x_{i} \in U,\left|\partial_{R}\left(x_{i}\right)\right|=$ $\left|\partial_{A t t r}\left(x_{i}\right)\right|=1$ that $T_{R}\left(x_{i}\right)=T_{R \cup\{d\}}\left(x_{i}\right)$ and $T_{A t t r}\left(x_{i}\right)=T_{A t t r \cup\{d\}}\left(x_{i}\right)$ for any $x_{i} \in U$. Acoording to Proposition 4.1 we have

$$
d_{E}(K(R), K(R \cup\{d\}))=d_{E}(K(A t t r), K(A t t r \cup\{d\}))=0
$$

Consequently, $d_{E}(K(R), K(R \cup\{d\}))=d_{E}(K($ Attr $), K(\operatorname{Attr} \cup\{d\}))$ if and only if $\forall x_{i} \in U, \partial_{R}\left(x_{i}\right)=\partial_{\text {Attr }}\left(x_{i}\right)$. This means that reduct based on metric is equivalent to reduct based on generalized decision.

Definition 5.2. Given an incomplete decision system $I D S=(U, \operatorname{Attr} \cup$ $\{d\})$ and $A \subset$ Attr, the importance of attribute $a \in \operatorname{Attr}-A$ is defined as

$$
I M P_{A}(a)=d_{E}(K(A), K(A \cup\{d\}))-d_{E}(K(A \cup\{a\}), K(A \cup\{a\} \cup\{d\}))
$$

According to Proposition 4.2 we have $I M P_{A}(a) \geq 0$. When $a$ is added into $A$, the distance $d_{E}(K(A), K(A \cup\{d\}))$ changes, which impacts on the importance of the attribute $a$ in the way that the larger the value of $I M P_{A}(a)$ is, the more important is the attribute $a$. Using the importance of an attribute as an attribute selection criterion, we design a heuristic algorithm to find the best reduct.

Algorithm 5.1. The algorithm to find the best reduct of an incomplete decision system.

Input: An incomplete decision system $I D S=(U, \operatorname{Attr} \cup\{d\})$.
Output: The best reduct $R$.

1. $R=\emptyset$;
2. Calculate $d_{E}(K(R), K(R \cup\{d\})), d_{E}(K($ Attr $), K(\operatorname{Attr} \cup\{d\}))$;
3. While $d_{E}(K(R), K(R \cup\{d\})) \neq d_{E}(K(\operatorname{Attr}), K(\operatorname{Attr} \cup\{d\}))$ do
4. Begin
5. For each $a \in \operatorname{Attr}-R$
6. Begin
7. Calculate $d_{E}(K(R \cup\{a\}), K(R \cup\{a\} \cup\{d\}))$;
8. Calculate $I M P_{R}(a)=d_{E}(K(R), K(R \cup\{d\}))-d_{E}(K(R \cup\{a\})$, $K(R \cup\{a\} \cup\{d\})) ;$
9. End;
10. Select $a_{m} \in \operatorname{Attr}-R$ so that $I M P_{R}\left(a_{m}\right)=\underset{a \in A t t r-R}{\operatorname{Max}}\left\{I M P_{R}(a)\right\}$;
11. $R=R \cup\left\{a_{m}\right\}$;
12. Calculate $d_{E}(K(R), K(R \cup\{d\}))$;
13. End;
14. For each $a \in R$
15. Begin
16. Calculate $d_{E}(K(R-\{a\}), K(R-\{a\} \cup\{d\}))$;
17. if $d_{E}(K(R-\{a\}), K(R-\{a\} \cup\{d\}))=d_{E}(K(\operatorname{Attr}), K(\operatorname{Attr} \cup\{d\}))$
then $R=R-\{a\}$;
18. End;
19. Return $R$;

Let us consider the command lines of Algorithm 5.1. From 3 to 13, the obtained attribute set $R$ satisfies $d_{E}(K(R), K(R \cup\{d\}))=d_{E}(K(A t t r), K(A t t r \cup$ $\{d\})$ ). From 14 to $18, R$ is minimal, that is

$$
\forall r \in R, d_{E}(K(R-\{r\}), K((R-\{r\}) \cup\{d\})) \neq d_{E}(K(\operatorname{Attr}), K(\operatorname{Attr} \cup\{d\})) .
$$

According to Definition 5.1, $R$ is a reduct. Consequently, Algorithm 5.1 is complete.

Complexity of Algorithm 5.1. First we analyse the complexity of While Loop from 3 to 13 . Since $T_{R}\left(u_{i}\right)$ and $T_{R \cup\{d\}}\left(u_{i}\right)$ are calculated in the previous step, we calculate $T_{R \cup\{a\}}\left(u_{i}\right), T_{R \cup\{a\} \cup\{d\}}\left(u_{i}\right)$ only. The complexity of calculating $T_{R \cup\{a\}}\left(u_{i}\right)$ for $\forall u_{i} \in U$ when $T_{R}\left(u_{i}\right)$ calculated is $O\left(|U|^{2}\right)$. So the complexity of calculating all $I M P_{R}(a)$ is:

$$
\begin{aligned}
(|\operatorname{Attr}|+(|\operatorname{Attr}|-1)+\cdots & +1) *|U|^{2} \\
& =(|\operatorname{Attr}| *(|\operatorname{Attr}|-1) / 2) *|U|^{2}=O\left(|\operatorname{Attr}|^{2}|U|^{2}\right) .
\end{aligned}
$$

where the cardinality $|A t t r|$ is the number of conditional attributes and $|U|$ is the number of objects. The complexity of obtaining the attribute with maximum
importance is $|\operatorname{Attr}|+(|\operatorname{Attr}|-1)+\cdots+1=|\operatorname{Attr}| *(|\operatorname{Attr}|-1) / 2=O\left(|A t t r|^{2}\right)$. Hence, the complexity of While Loop is $O\left(|A t t r|^{2}|U|^{2}\right)$. Second, in a similar way, the complexity of For Loop from 14 to 18 is $O\left(|A t t r|^{2}|U|^{2}\right)$. Finally, the complexity of Algorithm 5.1 is $O\left(|A t t r|^{2}|U|^{2}\right)$. Consequently, this complexity is better than the complexity of algorithms in [1, 3, 4, 20].

For example, let us consider the incomplete decision system in Table 1. We have the following coverings:

$$
\begin{aligned}
& U / T L R(A t t r)=\left\{\left\{x_{1}\right\},\left\{x_{2}, x_{6}\right\},\left\{x_{3}\right\},\left\{x_{4}, x_{5}\right\},\left\{x_{4}, x_{5}, x_{6}\right\},\left\{x_{2}, x_{5}, x_{6}\right\}\right\}, \\
& U / T L R\left(\left\{a_{1}\right\}\right)=\left\{\left\{x_{1}, x_{3}, x_{4}, x_{5}\right\},\left\{x_{2}, x_{3}, x_{5}, x_{6}\right\}, U,\left\{x_{1}, x_{3}, x_{4}, x_{5}\right\}, U,\right. \\
& U / T L R\left(\left\{a_{2}\right\}\right)=\{U, U, U, U, U, U\}, \\
& U / T L R\left(\left\{a_{3}\right\}\right)=\left\{\left\{x_{1}, x_{2}, x_{4}, x_{5}, x_{6}\right\},\left\{x_{1}, x_{2}, x_{4}, x_{5}, x_{6}\right\},\left\{x_{3}\right\},\left\{x_{1}, x_{2}, x_{4}, x_{5}, x_{6}\right\},\right. \\
&\left.\left\{x_{1}, x_{2}, x_{4}, x_{5}, x_{6}\right\},\left\{x_{1}, x_{2}, x_{4}, x_{5}, x_{6}\right\}\right\}, \\
& U / T L R\left(\left\{a_{4}\right\}\right)=\left\{\left\{x_{1}, x_{2}, x_{6}\right\},\left\{x_{1}, x_{2}, x_{6}\right\},\left\{x_{3}, x_{4}, x_{5}, x_{6}\right\},\left\{x_{3}, x_{4}, x_{5}, x_{6}\right\},\right. \\
&\left.\left\{x_{3}, x_{4}, x_{5}, x_{6}\right\}, U\right\}, \\
& U / T L R(\{d\})=\left\{\left\{x_{1}, x_{2}, x_{4}, x_{6}\right\},\left\{x_{1}, x_{2}, x_{4}, x_{6}\right\},\left\{x_{3}\right\},\left\{x_{1}, x_{2}, x_{4}, x_{6}\right\},\left\{x_{5}\right\},\right. \\
&\left.\left\{x_{1}, x_{2}, x_{4}, x_{6}\right\}\right\} .
\end{aligned}
$$

We calculate the distance

$$
d_{E}(K(A t t r), K(\operatorname{Attr} \cup\{d\}))=\frac{1}{|U|^{2}} \sum_{i=1}^{|U|}\left(\left|T_{A t t r}\left(u_{i}\right)-\left(T_{A t t r}\left(u_{i}\right) \cap T_{\{d\}}\left(u_{i}\right)\right)\right|\right)=\frac{4}{36} .
$$

Set $R=\emptyset$ and suppose that $T_{\emptyset}(x)=U$ for any $x \in U$. We calculate

$$
\begin{aligned}
T_{\emptyset}\left(x_{i}\right) & =U \text { for } \forall x_{i} \in U, \quad i=1 \ldots|U| . \\
S I G_{\emptyset}\left(a_{1}\right) & =\frac{1}{|U|^{2}} \sum_{i=1}^{|U|}\left(\left|T_{\emptyset}\left(u_{i}\right)-T_{\{d\}}\left(u_{i}\right)\right|-\left|T_{\left\{a_{1}\right\}}\left(u_{i}\right)-T_{\left\{a_{1}, d\right\}}\left(u_{i}\right)\right|\right)=0, \\
S I G_{\emptyset}\left(a_{2}\right) & =\frac{1}{|U|^{2}} \sum_{i=1}^{|U|}\left(\left|T_{\emptyset}\left(u_{i}\right)-T_{\{d\}}\left(u_{i}\right)\right|-\left|T_{\left\{a_{2}\right\}}\left(u_{i}\right)-T_{\left\{a_{2}, d\right\}}\left(u_{i}\right)\right|\right)=0,
\end{aligned}
$$

$S I G_{\emptyset}\left(a_{3}\right)=\frac{1}{|U|^{2}} \sum_{i=1}^{|U|}\left(\left|T_{\emptyset}\left(u_{i}\right)-T_{\{d\}}\left(u_{i}\right)\right|-\left|T_{\left\{a_{3}\right\}}\left(u_{i}\right)-T_{\left\{a_{3}, d\right\}}\left(u_{i}\right)\right|\right)=\frac{10}{36}$,
$S I G_{\emptyset}\left(a_{4}\right)=\frac{1}{|U|^{2}} \sum_{i=1}^{|U|}\left(\left|T_{\emptyset}\left(u_{i}\right)-T_{\{d\}}\left(u_{i}\right)\right|-\left|T_{\left\{a_{4}\right\}}\left(u_{i}\right)-T_{\left\{a_{4}, d\right\}}\left(u_{i}\right)\right|\right)=\frac{8}{36}$.
We choose $a_{3}$ which has the most importance and $R=\left\{a_{3}\right\}$, and calculate the distance

$$
d_{E}\left(K\left(\left\{a_{3}\right\}\right), K\left(\left\{a_{3}, d\right\}\right)\right)=\frac{1}{|U|^{2}} \sum_{i=1}^{|U|}\left(\left|T_{\left\{a_{3}\right\}}\left(u_{i}\right)-\left(T_{\left\{a_{3}\right\}}\left(u_{i}\right) \cap T_{\{d\}}\left(u_{i}\right)\right)\right|\right)=\frac{8}{36} .
$$

So we have $d_{E}\left(K\left(\left\{a_{3}\right\}\right), K\left(\left\{a_{3}, d\right\}\right)\right) \neq d_{E}(K(\operatorname{Attr}), K(\operatorname{Attr} \cup\{d\}))$.
We perform the second loop.

$$
\begin{aligned}
\operatorname{SIG}_{\left\{a_{3}\right\}}\left(a_{1}\right) & =\frac{1}{|U|^{2}} \sum_{i=1}^{|U|}\left(\left|T_{\left\{a_{3}\right\}}\left(u_{i}\right)-T_{\left\{a_{3}, d\right\}}\left(u_{i}\right)\right|-\left|T_{\left\{a_{1}, a_{3}\right\}}\left(u_{i}\right)-T_{\left\{a_{1}, a_{3}, d\right\}}\left(u_{i}\right)\right|\right) \\
& =\frac{2}{36}, \\
\operatorname{SIG}_{\left\{a_{3}\right\}}\left(a_{2}\right) & =\frac{1}{|U|^{2}} \sum_{i=1}^{|U|}\left(\left|T_{\left\{a_{3}\right\}}\left(u_{i}\right)-T_{\left\{a_{3}, d\right\}}\left(u_{i}\right)\right|-\left|T_{\left\{a_{2}, a_{3}\right\}}\left(u_{i}\right)-T_{\left\{a_{2}, a_{3}, d\right\}}\left(u_{i}\right)\right|\right) \\
& =0, \\
S_{I G_{\left\{a_{3}\right\}}\left(a_{4}\right)} & =\frac{1}{|U|^{2}} \sum_{i=1}^{|U|}\left(\left|T_{\left\{a_{3}\right\}}\left(u_{i}\right)-T_{\left\{a_{3}, d\right\}}\left(u_{i}\right)\right|-\left|T_{\left\{a_{3}, a_{4}\right\}}\left(u_{i}\right)-T_{\left\{a_{3}, a_{4}, d\right\}}\left(u_{i}\right)\right|\right) \\
& =\frac{4}{36} .
\end{aligned}
$$

We choose $a_{4}$ which has the most importance and we set $R=\left\{a_{3}, a_{4}\right\}$, and calculate

$$
d_{E}\left(K\left(\left\{a_{3}, a_{4}\right\}\right), K\left(\left\{a_{3}, a_{4}, d\right\}\right)\right)=\frac{4}{36}=d_{E}(K(\operatorname{Attr}), K(\operatorname{Attr} \cup\{d\})) .
$$

Hence, we go to For Loop. According to the above calculation, we obtain

$$
d_{E}\left(K\left(\left\{a_{3}\right\}\right), K\left(\left\{a_{3}, d\right\}\right)\right) \neq d_{E}(K(A t t r), K(\operatorname{Attr} \cup\{d\})) .
$$

In addition,

$$
d_{E}\left(K\left(\left\{a_{4}\right\}\right), K\left(\left\{a_{4}, d\right\}\right)\right)=\frac{10}{36} \neq d_{E}(K(\operatorname{Attr}), K(\operatorname{Attr} \cup\{d\})) .
$$

Consequently, the algorithm finishes and $R=\left\{a_{3}, a_{4}\right\}$ is the best reduct of $\operatorname{Attr}$.
6. Conclusions. Attribute reduction is the most important problem in both classical rough sets and tolerance rough sets. In this paper, a generalized Liang entropy is proposed based on Liang entropy [6] and some properties of the generalized Liang entropy are considered. Based on the generalized Liang entropy, a distance is established between attributes and a distance-based method to find the best reduct is proposed. To construct this method, we define a reduct based on the distance, the importance of an attribute based on the distance. We use the importance of an attribute as heuristic information to design an effective heuristic algorithm to find the best reduct. We prove theoretically that the complexity of our algorithm is less than that of the algorithms in [1, 3, 4, 20].

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