Further Discussions on Induced Bias Matrix Model for the

Pair-wise Comparison Matrix

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Abstract Inconsistency issue of pairwise comparison matrices has been an important subject in the analytical network process. The most inconsistent elements can efficiently be identified by inducing a bias matrix only based on the original matrix. This paper further discusses the induced bias matrix, and integrates all related theorems and corollaries into the induced bias matrix mode. The theorem of inconsistency identification is proved mathematically using the maximum eigenvalue method and the contradiction method. In addition, a fast inconsistency identification method for one pair of inconsistent elements is proposed and proved mathematically. Two examples are used to illustrate the proposed fast identification method. The results show that the proposed new method is easier and faster than the existing method for the special case with only one pair of inconsistent elements in the original comparison matrix.

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1. Introduction

The pair-wise comparison method is a well-established technique, and widely used in multi-criteria decision making (MCDM) methods [1, 2]. The consistency test of pair-wise comparison matrix in the AHP and ANP, two of the widely used MCDM methods, has been studied extensively over the past few decades [3-14]. To improve the consistency ratio of the pair-wise comparison matrix in the AHP and ANP and preserve the original comparison information as much as possible, literature [15] introduced an induced bias matrix (IBM, hereinafter), which can be derived by the original reciprocal pairwise comparison matrix (RPCM hereinafter), to identify and adjust the inconsistent elements.

IBM is based on the theorems of matrix multiplication and vectors dot product as well as the definitions and notations of the pair-wise comparison matrix. The IBM method has been applied in questionnaire design [16], risk analysis [17], and task scheduling [18]. If the comparison matrix A is perfectly consistent, we mathematically proved that the IBM should be a zero matrix in [15]. If the comparison matrix A is approximately consistent, we also mathematically proved that the IBM should be as close as possible to a zero matrix in [19]. This corollary can be used to estimate the uncertain or missing values in an RPCM. If the pair-wise matrix A is inconsistent, there must be some inconsistent elements in the induced bias matrix (IBM) deviating far away from zero (*Corollary 2* in [15]). This corollary shows that

the farthest value should be identified as the most inconsistent element from the induced bias matrix (IBM). However, this critical corollary for identifying the most inconsistent element has not been proved mathematically in [15].

The objective of this paper is to prove the theorem of aforementioned critical corollary mathematically, and integrates all related theorems and corollaries into the induced bias matrix model (IBMM), which simplifies and refines the proposed inconsistency identification method. In addition, this paper proposes a fast inconsistency identification method to extend the proposed IBMM for the special case of one pair of inconsistent elements in the original RPCM.

The remaining parts of this paper are organized as follows. The next section integrates all related theorem and corollaries into one model and provides two mathematic proofs of *Corollary 2.2* (i.e. *Theorem 2.3 in IBMM*) by maximum eigenvalue method and contradiction method. Section 3 analyzes the inconsistency identification method; proposes and proves a fast inconsistency identification method; describes the sign of non-zero of the induced bias matrix; and introduces two numeric examples with one pair of inconsistent elements to demonstrate the proposed fast inconsistency identification method. Section 4 concludes the paper.

2. Theorems and Proofs of the Induced Bias Matrix Model (IBMM)

In order to efficiently identify the inconsistent elements and preserve most of the original pair-wise comparison information, we proposed an induced bias matrix (IBM), which is only based on the original RPCM in [15], and the following theorem and corollaries were derived.

Theorem 2.1: The induced bias matrix (IBM) $C = AA - nA$ should be a zero matrix, if comparison matrix A is perfectly consistent.

Corollary 2.1: The induced bias matrix (IBM) $C = AA - nA$ should be as close as possible to zero matrix, if comparison matrix A is approximately consistent.

Corollary 2.2: There must be some inconsistent elements in induced bias matrix (IBM) C deviating far away from zero, if the pair-wise matrix A is inconsistent.

The correctnesses of *Theorem 2.1* and *Corollary 2.1* have been proved mathematically in Section 3.1 of [15] and in Section 2.1 of [19], respectively. Besides, some special cases, where there are some errors in the original RPCM of order 3, have been addressed as examples in [19]. To determine whether a comparison pairwise matrix is reciprocal, *Corollary 2.3* is proposed and illustrated using 3×3 comparison pair-wise matrix, as an example in [19].

Corollary 2.3: Despite that the comparison matrix A is consistent or not, all entries in the main diagonal of the induced bias matrix (IBM) $C = AA - nA$ should be zeroes, giving that the comparison matrix A is satisfied with the reciprocal condition.

The inconsistent elements are identified by inducing a bias matrix C from the original RPCM, and the critical component of the above mentioned theorem and corollaries is the induced bias matrix (IBM). To simplify and refine the proposed inconsistency identification method, and make the proposed method more comprehensive and systematic, we integrate these theorem and corollaries into one

model, say induced bias matrix model (IBMM), which includes the following three theorems.

The Theorem of the Induced Bias Matrix Model (IBMM):

Theorem 2.2: The induced bias matrix (IBM) $C = AA - nA$ should be equal (or close) to a zero matrix, if comparison matrix A is perfectly (or approximately) consistent. That is,

$$C = AA - nA \begin{cases} = 0 & \textit{if } a_{ik}a_{kj} = a_{ij}, \\ \approx 0 & \textit{if } a_{ik}a_{kj} \approx a_{ij}, \end{cases} \tag{1}$$

where A is the original RPCM and a_{ij} represents the values of RPCM. The "n" denotes the order of RPCM.

Theorem 2.3: There must be some inconsistent elements in the induced bias matrix (IBM) C deviating far away from zero, if the pair-wise matrix is inconsistent. Especially, any row or column of matrix C contains at least one positive element.

Theorem 2.4: All entries in the main diagonal of the induced bias matrix (IBM) $C = AA - nA$ should be zeroes whether matrix *A* is consistent or not, as long as the comparison matrix *A* satisfies the reciprocal condition.

We provided the proof of *Theorem 2.2* for consistent case in [15]. The principles of *Theorem 2.3 and Theorem 2.4* are demonstrated by introducing some errors into a 3×3 RPCM in [19]. The following subsections mathematically prove *Theorem 2.3* and *Theorem 2.4*.

2.1 The Proof of *Theorem 2.3* by Maximum Eigenvalue Method

Proof: If the RPCM A is inconsistent, the induced bias matrix $C = AA - nA$

cannot be zero. More precisely, any row of C contains at least one positive element.

It is known, e.g. literature [20], that for the maximal eigenvalue λ_{max} of A, $\lambda_{max} \geq n$, and the corresponding unique eigenvector ω_{max} is a positive vector. Furthermore, A is consistent if and only if $\lambda_{max} = n$. By applying

$$A\omega_{max} = \lambda_{max}\omega_{max} \tag{2}$$

at the appropriate places, we get

$$\begin{aligned} C\omega_{max} &= (AA - nA)\omega_{max} = \lambda_{max}A\omega_{max} - n\lambda_{max}\omega_{max} \\ &= \lambda_{max}^2\omega_{max} - n\lambda_{max}\omega_{max} = \lambda_{max}(\lambda_{max} - n)\omega_{max}. \end{aligned} \tag{3}$$

Since $\lambda_{max} > n$, $C\omega_{max}$ is a positive vector. Consequently, C cannot have any row containing only zeros. Moreover, since both $C\omega_{max}$ and ω_{max} are positive vectors, any row of C must contain at least one positive element. $\square$

2.2 The Proof of Theorem 2.3 by Contradiction

Proof: It has been proved in [15] that, if a reciprocal pairwise comparison matrix (RPCM) is perfectly consistent, that is, $a_{ij} = a_{ik}a_{kj}$ for all i, j, k, then

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot a_{kj} - na_{ij} = na_{ij} - na_{ij} = 0. \tag{4}$$

If an RPCM A is inconsistent, $a_{ij} \neq a_{ik}a_{kj}$ holds at least for one of the i, j, k $(i, j, k = 1, 2, \cdots, n)$. Moreover, if A is inconsistent, for any i, there exist j and k such that $a_{ij} \neq a_{ik}a_{kj}$ (*Corollary 2* in [21]). Assume that an RPCM A is inconsistent, but the i-th row of the induced bias matrix C contains only non-positive elements. Then $a_{ij} \neq a_{ik}a_{kj}$ with some j and k, and $c_{i1} \leq 0, c_{i2} \leq 0, \cdots, c_{in} \leq 0$. We get the following inequalities:

$$\begin{cases} c_{i1} = \sum_{k=1}^{n} a_{ik}a_{k1} - na_{i1} \leq 0 \\[2mm] c_{i2} = \sum_{k=1}^{n} a_{ik}a_{k2} - na_{i2} \leq 0 \\[1mm] \quad\vdots \\[1mm] c_{in} = \sum_{k=1}^{n} a_{ik}a_{kn} - na_{in} \leq 0, \end{cases} \qquad (5)$$

$$\Rightarrow \begin{cases} \dfrac{1}{a_{i1}} \sum_{k=1}^{n} a_{ik}a_{k1} \leq n \\[2mm] \dfrac{1}{a_{i2}} \sum_{k=1}^{n} a_{ik}a_{k2} \leq n \\[1mm] \quad\vdots \\[1mm] \dfrac{1}{a_{in}} \sum_{k=1}^{n} a_{ik}a_{kn} \leq n. \end{cases} \qquad (6)$$

Adding all the inequalities together in the system of inequalities (6), we get

$$\frac{1}{a_{i1}} \sum_{k=1}^{n} a_{ik}a_{k1} + \frac{1}{a_{i2}} \sum_{k=1}^{n} a_{ik}a_{k2} + \cdots + \frac{1}{a_{in}} \sum_{k=1}^{n} a_{ik}a_{kn} \leq n^2, \qquad (7)$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{a_{i1}} a_{ik}a_{k1} + \sum_{k=1}^{n} \frac{1}{a_{i2}} a_{ik}a_{k2} + \cdots + \sum_{k=1}^{n} \frac{1}{a_{in}} a_{ik}a_{kn} \leq n^2, \qquad (8)$$

$$\Rightarrow \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{1}{a_{ij}} a_{ik}a_{kj} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{kj} \frac{a_{ik}}{a_{ij}} \leq n^2. \qquad (9)$$

The inequality (8) or (9) can be unfolded to the following matrix form:

$$\begin{aligned}
&\frac{1}{a_{i1}}a_{i1}a_{11} + \boxed{\frac{1}{a_{i1}}a_{i2}a_{21}} + \frac{1}{a_{i1}}a_{i3}a_{31} + \cdots + \frac{1}{a_{i1}}a_{ii}a_{i1} + \cdots + \frac{1}{a_{i1}}a_{ij}a_{j1} + \cdots + \boxed{\frac{1}{a_{i1}}a_{in}a_{n1}} + \\
&\boxed{\frac{1}{a_{i2}}a_{i1}a_{12}} + \frac{1}{a_{i2}}a_{i2}a_{22} + \frac{1}{a_{i2}}a_{i3}a_{32} + \cdots + \frac{1}{a_{i2}}a_{ii}a_{i2} + \cdots + \frac{1}{a_{i2}}a_{ij}a_{j2} + \cdots + \frac{1}{a_{i2}}a_{in}a_{n2} + \\
&\frac{1}{a_{i3}}a_{i1}a_{13} + \frac{1}{a_{i3}}a_{i2}a_{23} + \frac{1}{a_{i3}}a_{i3}a_{33} + \cdots + \frac{1}{a_{i3}}a_{ii}a_{i3} + \cdots + \frac{1}{a_{i3}}a_{ij}a_{j3} + \cdots + \frac{1}{a_{i3}}a_{in}a_{n3} + \\
&\qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\frac{1}{a_{ii}}a_{i1}a_{1i} + \frac{1}{a_{ii}}a_{i2}a_{2i} + \frac{1}{a_{ii}}a_{i3}a_{3i} + \cdots + \frac{1}{a_{ii}}a_{ii}a_{ii} + \cdots + \frac{1}{a_{ii}}a_{ij}a_{ji} + \cdots + \frac{1}{a_{ii}}a_{in}a_{ni} + \\
&\qquad\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\frac{1}{a_{ij}}a_{i1}a_{1j} + \frac{1}{a_{ij}}a_{i2}a_{2j} + \frac{1}{a_{ij}}a_{i3}a_{3j} + \cdots + \frac{1}{a_{ij}}a_{ii}a_{ij} + \cdots + \frac{1}{a_{ij}}a_{ij}a_{jj} + \cdots + \boxed{\frac{1}{a_{ij}}a_{in}a_{nj}} + \\
&\qquad\vdots \qquad\qquad \vdots \\
&\boxed{\frac{1}{a_{in}}a_{i1}a_{1n}} + \frac{1}{a_{in}}a_{i2}a_{2n} + \frac{1}{a_{in}}a_{i3}a_{3n} + \cdots + \frac{1}{a_{in}}a_{ii}a_{in} + \cdots + \boxed{\frac{1}{a_{in}}a_{ij}a_{jn}} + \cdots + \frac{1}{a_{in}}a_{in}a_{nn} \leq n^2.
\end{aligned} \qquad (10)$$

Since matrix A is a reciprocal matrix, that is, $a_{kj} = \dfrac{1}{a_{jk}}$ and $a_{kj} > 0$, $a_{ij} > 0$,

$a_{ik} > 0$, from the expansion inequality (9), any of the above inequalities (7)-(10) can

be simplified as the following inequality:

$$n + \sum_{k>j}\left(a_{kj}\frac{a_{ik}}{a_{ij}} + a_{jk}\frac{a_{ij}}{a_{ik}} \right) \le n^2, \tag{11}$$

$$\Rightarrow \sum_{k>j}\left(a_{kj}\frac{a_{ik}}{a_{ij}} + a_{jk}\frac{a_{ij}}{a_{ik}} \right) \le n^2 - n. \tag{12}$$

Since there are $\dfrac{n(n-1)}{2}$ sum term at the left side of the inequality (12), and

$a_{kj}\dfrac{a_{ik}}{a_{ij}} + a_{jk}\dfrac{a_{ij}}{a_{ik}} = a_{kj}\dfrac{a_{ik}}{a_{ij}} + \dfrac{1}{a_{kj}\dfrac{a_{ik}}{a_{ij}}} \ge 2$, the inequality (12) holds if and only if

$a_{kj}\dfrac{a_{ik}}{a_{ij}} = 1$, namely, $a_{ij} = a_{ik}a_{jk}$ for all j and k. However, this result contradicts the

previous assumption that $a_{ij} \ne a_{ik}a_{kj}$ for some j and k. Therefore, one of the

inequalities, at least, does not hold. Thus, inequality (12) holds with $>$ sign. This

entails that at least one of elements in the i-th row the induced bias matrix C is

positive.   □

Based on the above two proofs for rows, the same proofs for columns can also be

induced. If A is a pairwise comparison matrix with the reciprocal property, the

transpose of A is also a pairwise comparison matrix with the reciprocal property. In

addition, A is consistent if and only if the transpose of A is consistent.

The transpose of the IBM C generated by A is the IBM generated by the

transpose of A. Consequently, if C is inconsistent, any column of C contains at least

one positive element. The same statement for the rows was stated earlier

2.3 The Proof of Theorem 2.4

Proof: According to the principle of matrix multiplication, all values in the main diagonal of the induced bias matrix C can be calculated by the formula (13):

$$c_{ii} = \sum_{k=1}^{n} a_{ik} \cdot a_{ki} - na_{ii}. \tag{13}$$

If $a_{ik} = \dfrac{1}{a_{ik}}$, and $a_{ii} = 1$, then

$$c_{ii} = \sum_{k=1}^{n} a_{ik} \cdot a_{ki} - na_{ii} = \sum_{k=1}^{n} a_{ik} \cdot \frac{1}{a_{ik}} - na_{ii} = n - n = 0, \quad i = 1,2,\cdots,n. \tag{14}$$

$\square$

3. The Inconsistency Identification Method

In this section, the mathematic principles of the proposed "*Method of Maximum*", "*Method of Minimum*" and "*Method for identifying a_{ij}*" in [15], are firstly discussed. Next, a fast inconsistency identification method for special case is proposed and proved mathematically. Two numerical examples are introduced to illustrate the proposed method in Section 3.3. Details are given next.

3.1 Clarification of the Proposed Inconsistency Identification Method

If the RPCM A is inconsistent, the above proof shows that any row of the IBM C contains at least one non-zero element, that is, $a_{ij} \neq a_{ik}a_{kj}$ holds at least for one group of i, j, k, which means that there is at least one pair of inconsistent elements existing in the original RPCM A. Suppose c_{ij}, the element with largest absolute value in the IBM C, is identified. The second step is to analyze that which element makes c_{ij} to be far away from zero. According to the rule of matrix multiplication, the value

of c_{ij} is calculated by all values on the ith row and j^{th} column of matrix A and a_{ij}, that is,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot a_{kj} - na_{ij} \tag{15}$$

$$= a_{i1}a_{1j} - a_{ij} + a_{i2}a_{2j} - a_{ij} + \cdots + a_{ik}a_{kj} - a_{ij} + \cdots + a_{in}a_{nj} - a_{ij}.$$

Clearly, the farthest value of c_{ij} can be impacted by any term of $a_{ik}a_{kj} - a_{ij}$ on the right side of the sum equality (15). In order to identify the inconsistent elements that caused the value of c_{ij} to be far away from zero, the scalar product of vectors in n dimension technique is introduced. The impact of each term can easily be observed by the scalar product of vectors in n dimension technique, that is,

$$b = r_i \cdot c_j^T = (a_{i1}, a_{i2}, \cdots, a_{in}) \cdot (a_{1j}, a_{2j}, \cdots, a_{nj}) = (a_{i1}a_{1j}, a_{i2}a_{2j}, \cdots, a_{in}a_{nj}), \tag{16}$$

and

$$f = b - a_{ij} = (a_{i1}a_{1j} - a_{ij}, a_{i2}a_{2j} - a_{ij}, \cdots a_{ik}a_{kj} - a_{ij}, \cdots, a_{in}a_{nj} - a_{ij}). \tag{17}$$

If $a_{ij} \neq a_{ik}a_{kj}$, then $a_{ik}a_{kj} - a_{ij} \neq 0$. Therefore, the non-zero element(s), which caused the value of c_{ij} to be far away from zero, can be identified through observing all elements in the bias identifying vector f. In addition, the inequality $a_{ij} \neq a_{ik}a_{kj}$ can be caused by a_{ij} or any $a_{ik}a_{kj}$ $(k = 1, 2, \cdots, n)$, or both.

Obviously, if the inconsistent element is a_{ij}, other elements are consistent. Assume $a_{ik}a_{kj} = a'_{ij} > a_{ij}$, namely, a_{ij} is too small. We can get that all values in the bias identifying vector f are positive except $k = i, j$, as $a_{ii}a_{ij} - a_{ij} = 0$ and $a_{ij}a_{jj} - a_{ij} = 0$. Vice versa, if a_{ij} is too large, all the values in the bias identifying vector f will be negative except two values $k = i, j$. Therefore, the "*Method for identifying a_{ij}*" inconsistency identification method is proposed [15].

Besides, the farthest value of c_{ij} must be caused by some outliers either too large or too small located at the bias identifying vector f; therefore, "*Method for Maximum*" and "*Method for Minimum*" inconsistency identification methods are proposed [15].

In order to further identify the inconsistent element for those elements whose values are close to the largest or smallest simultaneously, therefore the "*Method of matrix order reduction*" inconsistency identification method is proposed [15].

3.2 Fast Identification Method for Special Case and Its Proof

In this section, one fast inconsistency identification method is proposed to quickly identify the inconsistent elements when there is only one pair of inconsistent elements in the original RPCM.

Assume that RPCM A is inconsistent, and there is one pair of inconsistent elements a_{ip} and its corresponding reciprocal element $a_{pi} = \dfrac{1}{a_{ip}}$, while other elements are consistent, namely, $a_{ik}a_{kj} = a_{ij}$ for all k except $k = p$ ($a_{ip}a_{pj} \neq a_{ij}$). Therefore, the two inconsistent elements are elements at the ith and p^{th} rows, and the pth and i^{th} columns. According to the rule of matrix multiplication, all elements, which are located at the ith, p^{th} rows, and the ith, p^{th} columns in the induced bias matrix $C = AA - nA$, will be impacted by a_{ip} and a_{pi}. Since it is assumed that $a_{ik}a_{kj} = a_{ij}$ ($k \neq p; j \neq p,$) and $a_{ik}a_{kp} \neq a_{ip}$, suppose $a_{ik}a_{kp} = a'_{ip}$, all the values in the ith row of the IBM C can be computed by formula (18), that is,

$$c_{ij} = \sum_{k=1}^{n} a_{ik} \cdot a_{kj} - na_{ij} = \sum_{k=1,\neq p}^{n} a_{ik} \cdot a_{kj} + a_{ip}a_{pj} - na_{ij}, \quad j = 1,2,\cdots,n$$

$$= \begin{cases} (n-1)a_{ij} + a_{ip}a_{pj} - na_{ij} = a_{ip}a_{pj} - a_{ij}; & j \neq i, p \\ (n-1)+1-n = 0; & j = i \\ (n-2)a'_{ip} + a_{ii}a_{ip} + a_{ip}a_{pp} - na_{ip} = (n-2)(a'_{ip} - a_{ip}); & j = p. \end{cases} \tag{18}$$

In order to analyze the sign change of each element on the ith row, the equalities in (18) are further unfolded, as shown below.

$$\begin{cases} c_{i1} = a_{ip}a_{p1} - a_{i1} \\ c_{i2} = a_{ip}a_{p2} - a_{i2} \\ \quad \vdots \\ c_{ii} = 0 \\ \quad \vdots \\ c_{ip} = (n-1)a'_{ip} + a_{ii}a_{ip} + a_{ip}a_{pp} - na_{ip} = (n-2)(a'_{ip} - a_{ip}) \\ \quad \vdots \\ c_{in} = a_{ip}a_{pn} - a_{in}. \end{cases} \tag{19}$$

If $a_{ip} \uparrow$, then

$$\begin{cases} c_{i1} = a_{ip}a_{p1} - a_{i1} > 0 \\ c_{i2} = a_{ip}a_{p2} - a_{i2} > 0 \\ \quad \vdots \\ c_{ii} = 0 \\ \quad \vdots \\ c_{ip} = (n-1)a'_{ip} + a_{ii}a_{ip} + a_{ip}a_{pp} - na_{ip} = (n-2)(a'_{ip} - a_{ip}) < 0 \\ \quad \vdots \\ c_{in} = a_{ip}a_{pn} - a_{in} > 0. \end{cases} \tag{20}$$

If $a_{ip} \downarrow$, then

$$\begin{cases} c_{i1} = a_{ip}a_{p1} - a_{i1} < 0 \\ c_{i2} = a_{ip}a_{p2} - a_{i2} < 0 \\ \quad \vdots \\ c_{ii} = 0 \\ \quad \vdots \\ c_{ip} = (n-1)a'_{ip} + a_{ii}a_{ip} + a_{ip}a_{pp} - na_{ip} = (n-2)(a'_{ip} - a_{ip}) > 0 \\ \quad \vdots \\ c_{in} = a_{ip}a_{pn} - a_{in} < 0, \end{cases} \tag{21}$$

where the symbols "$\uparrow$" and "$\downarrow$" denote "increase" and "decrease", respectively

(hereinafter).

Likewise, for the pth row:

$$c_{pj} = \sum_{k=1}^{n} a_{pk} \cdot a_{kj} - na_{pj} = \sum_{k=1,\neq i}^{n} a_{pk} \cdot a_{kj} + a_{pi}a_{ij} - na_{pj}, \quad j = 1,2,\cdots,n$$

$$= \begin{cases} (n-1)a_{pj} + a_{pi}a_{ij} - na_{pj} = a_{pi}a_{ij} - a_{pj}; & j \neq i, p \\ 0; & j = p \\ (n-1)a'_{pj} + a_{pi}a_{ii} + a_{pp}a_{pi} - na_{pi} = (n-2)(a'_{pi} - a_{pi}); & j = i. \end{cases} \quad (22)$$

Therefore, if $a_{ip} \uparrow$, then $a_{pi} = \dfrac{1}{a_{ip}} \downarrow$,

$$c_{pj} = \begin{cases} (n-1)a_{pj} + a_{pi}a_{ij} - na_{pj} = a_{pi}a_{ij} - a_{pj} < 0; & j \neq i, p \\ 0; & j = p \\ (n-1)a'_{pj} + a_{pi}a_{ii} + a_{pp}a_{pi} - na_{pi} = (n-2)(a'_{pi} - a_{pi}) > 0; & j = i. \end{cases} \quad (23)$$

Likewise, if $a_{ip} \downarrow$, then $a_{pi} = \dfrac{1}{a_{ip}} \uparrow$,

$$c_{pj} = \begin{cases} (n-1)a_{pj} + a_{pi}a_{ij} - na_{pj} = a_{pi}a_{ij} - a_{pj} > 0; & j \neq i, p \\ 0; & j = p \\ (n-1)a'_{pj} + a_{pi}a_{ii} + a_{pp}a_{pi} - na_{pi} = (n-2)(a'_{pi} - a_{pi}) < 0; & j = i. \end{cases} \quad (24)$$

If $a_{ip} \uparrow$, all values on the ith row of IBM C will be more than zeroes ($c_{ij} > 0, j = 1,2,\cdots,n$ and $j \neq p$) except $c_{ip} < 0$, and all values on the pth row of IBM C will be less than zeroes ($c_{ij} < 0, j = 1,2,\cdots,n$ and $j \neq p$) except $c_{ip} > 0$.

Therefore, only the elements on the ith row and p^{th} row are non-zeroes, and the sign form of the values on the ith row and p^{th} row of the IBM C can be derived, as shown in the following matrix,

$$
\begin{array}{c}
\quad\quad i^{th} \quad\quad\quad\quad p^{th} \\
\left(
\begin{array}{ccccccccc}
& & & & & & & & \\
+ & \cdots & + & 0 & +\cdots & - & + & \cdots & + \\
& & & & & & & & \\
- & \cdots & - & + & -\cdots & 0 & - & \cdots & - \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\right)
\begin{array}{c}
\\
\\
i^{th} \\
\\
p^{th}
\end{array}
\end{array}
\tag{25}
$$

Likewise, the signs of each element on the ith column and p^{th} column can be derived similarly.

$$
c_{ji} = \sum_{k=1}^{n} a_{jk} \cdot a_{ki} - na_{ji} = \sum_{k=1,\neq p}^{n} a_{jk} \cdot a_{ki} + a_{jp}a_{pi} - na_{ji}, \quad j=1,2,\cdots,n
$$

$$
= \begin{cases}
(n-1)a_{ji} + a_{jp}a_{pi} - na_{ji} = a_{jp}a_{pi} - a_{ji}; & j \neq p,i \\
0; & j = i \\
(n-1)a'_{pi} + a_{pi}a_{ii} + a_{pp}a_{pi} - na_{pi} = (n-2)\left(a'_{pi} - a_{pi}\right); & j = p.
\end{cases}
\tag{26}
$$

If $a_{ip} \uparrow$, then $a_{pi} = \dfrac{1}{a_{ip}} \downarrow$,

$$
c_{ji} = \begin{cases}
(n-1)a_{ji} + a_{jp}a_{pi} - na_{ji} = a_{jp}a_{pi} - a_{ji} < 0; & j \neq p,i \\
0; & j = i \\
(n-1)a'_{pi} + a_{pi}a_{ii} + a_{pp}a_{pi} - na_{pi} = (n-2)\left(a'_{pi} - a_{pi}\right) > 0; & j = p.
\end{cases}
\tag{27}
$$

For the pth column,

$$
c_{jp} = \sum_{k=1}^{n} a_{jk} \cdot a_{kp} - na_{jp} = \sum_{k=1,\neq i}^{n} a_{jk} \cdot a_{kp} + a_{ji}a_{ip} - na_{jp}, \quad j=1,2,\cdots,n
$$

$$
= \begin{cases}
(n-1)a_{jp} + a_{ji}a_{ip} - na_{jp} = a_{ji}a_{ip} - a_{jp}; & j \neq p,i \\
0; & j = p \\
(n-1)a'_{ip} + a_{ii}a_{ip} + a_{ip}a_{pp} - na_{ip} = (n-2)\left(a'_{ip} - a_{ip}\right); & j = i,
\end{cases}
\tag{28}
$$

If $a_{ip} \uparrow$,

$$
c_{jp} = \begin{cases}
(n-1)a_{jp} + a_{ji}a_{ip} - na_{jp} = a_{ji}a_{ip} - a_{jp} > 0; & j \neq p,i \\
0; & j = p \\
(n-1)a'_{ip} + a_{ii}a_{ip} + a_{ip}a_{pp} - na_{ip} = (n-2)\left(a'_{ip} - a_{ip}\right) < 0; & j = i.
\end{cases}
\tag{29}
$$

Therefore, the sign forms of the elements on the ith and p^{th} columns of the

IBM C can be obtained, as shown in (30)

$$
\begin{array}{cc}
i^{th} & p^{th}
\end{array}
$$

$$
\left(
\begin{array}{ccc}
- & & + \\
\vdots & & \vdots \\
- & \cdots & + \\
0 & & - \\
- & & + \\
+ & \cdots & 0 \\
- & & + \\
\vdots & & \vdots \\
- & & +
\end{array}
\right)
\begin{array}{l}
\\
\\
\\
i^{th} \\
\\
p^{th} \\
\\
\\
\end{array}
\qquad (30)
$$

To sum up, if $a_{ip}\uparrow$, the signs of all the values, which are located at the ith row

and the ith column, the pth row and p^{th} column, become:

$$
\begin{array}{cc}
i^{th} & p^{th}
\end{array}
$$

$$
\left(
\begin{array}{ccccccccc}
& & & - & & + & & & \\
& & & \vdots & & \vdots & & & \\
& & & - & & + & & & \\
+ & \cdots & + & 0 & + & - & + & \cdots & + \\
& & & - & & + & & & \\
- & \cdots & - & + & - & 0 & - & \cdots & - \\
& & & - & & + & & & \\
& & & \vdots & & \vdots & & & \\
& & & - & & + & & &
\end{array}
\right)
\begin{array}{l}
\\
\\
\\
i^{th} \\
\\
p^{th} \\
\\
\\
\end{array}
\qquad (31)
$$

Therefore, we can obtain the following fast inconsistency identification method:

Method of non-zero rows (columns) and signs identification:

If there are two rows (the ith row and p^{th} row) and two columns (the ith column and p^{th} column) with non-zeroes, and other elements are zero, the inconsistent elements must be a_{ip} and a_{pi}.

If the $c_{ip}<0$ and other elements located at the ith row or p^{th} column are more than zeroes, a_{ip} is too large and should be decreased. Vice versa, a_{ip} is too small and should be increased.

If the $c_{pi} > 0$ and other elements located at the pth row or i^{th} column are less than zeroes, a_{ip} is too large and should be decreased. Vice versus, a_{ip} is too small and should be increased.

In addition,

$$c_{ip} = \sum_{k=1}^{n} a_{ik} \cdot a_{kp} - na_{ip} = \sum_{k=1, \neq i, p}^{n} a_{ik} \cdot a_{kp} + a_{ii}a_{ip} + a_{pp}a_{ip} - na_{ip}. \tag{32}$$

To make the RPCM A be consistent, the value of c_{ip} should be equal to zero. Therefore, inconsistent element a_{ip} can be adjusted by formula (33).

$$a_{ip} = \frac{1}{n-2} \sum_{k=1, \neq i, p}^{n} a_{ik} \cdot a_{kp}. \tag{33}$$

In this case, since there is only one pair of inconsistent elements, a_{ip} and a_{pi} in the RPCM, any $a_{ik}a_{kp} = a'_{ip}$ $(k \neq i, p,)$ can be used as the revised value of a_{ip}.

3.3 Illustrative Examples for Fast Inconsistency Identification Method

The *Example 2* and *Example 3* in [15] are used in this study as *Example 3.1* and *Example 3.2*, respectively, to demonstrate the proposed fast inconsistency identification method.

Example 3.1: The induced bias matrix C, computed by the proposed IBM method in the *Example 2* in [15], is

$$C = \begin{pmatrix} 0 & -1.9688 & -3.9375 & 15.7500 \\ 31.5000 & 0 & 0 & -3.9375 \\ 15.7500 & 0 & 0 & -1.9688 \\ -15.7500 & 15.7500 & 31.5000 & 0 \end{pmatrix}.$$

It can easily be observed that there are only two non-zero rows (the 1st and 4th), and two non-zero columns (the 1st and 4th) in the IBM C, which is identical to the

formula (31). In addition, there is only one sign different from those elements whether located at the 1st and 4th rows or columns. Therefore, according to the principle of above proposed fast inconsistency identification method, the inconsistent elements are a_{14} and a_{41}. Specifically, a_{14} is too small and should be increased since $c_{14} = 15.75 > 0$. According to the revising formula (33) of a_{ip}, we get

$$a_{14} = \frac{1}{4-2} \sum_{k=2}^{3} a_{1k} a_{k4} = \frac{1}{2} (2 \times 4 + 4 \times 2) = 8, \ a_{41} = \frac{1}{a_{14}} = \frac{1}{8}.$$

The identified inconsistent element and its revised result are the same as shown in *Example 2* of [15].

Example 3.2： The induced bias matrix C, computed by the proposed IBMM in the *Example 3* of [15], is showed below.

$$C = \begin{pmatrix} 0 & -1.8750 & 7.5000 & 0 & 0 & 0 & 0 & 0 \\ 7.5000 & 0 & -22.5000 & 15.0000 & 3.7500 & 15.0000 & 3.7500 & 15.0000 \\ -1.8750 & 22.5000 & 0 & -3.7500 & -0.9375 & -3.7500 & -0.9375 & -3.7500 \\ 0 & -0.9375 & 3.7500 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.7500 & 15.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.9375 & 3.7500 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3.7500 & 15.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.9375 & 3.7500 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It can easily be observed that there are only two non-zero rows (the 2nd and 3rd), and two non-zero columns (the 2nd and 3rd) in above IBM C. Likewise, there is only one sign different from those elements whether located at the 2nd and 3rd rows or columns. Both are identical to the formula (31). Therefore, according to the above fast identification method, the inconsistent elements are a_{23} and a_{32}. Besides, since $c_{23} = -22.5 < 0$ and $c_{32} = 22.5 > 0$, we can get that a_{23} is too large, and a_{32} is too small.

Since $c_{23} = \sum_{k=1}^{8} a_{2k} a_{k3} - 8a_{23} = \sum_{k=1,\neq 2,3}^{8} a_{2k} a_{k3} + a_{22} a_{23} + a_{23} a_{33} - 8a_{23}$, assume $c_{23} = 0$,

$a_{23} = \frac{1}{6} \sum_{k=1,\neq 2,3}^{8} a_{2k} a_{k3} = \frac{1}{6} \left(\frac{1}{2} \times \frac{1}{2} + 1 \times \frac{1}{4} + \frac{1}{4} \times 1 + 1 \times \frac{1}{4} + \frac{1}{4} \times 1 + 1 \times \frac{1}{4} \right) = \frac{1}{4}$, and $a_{32} = 4$. For

simplicity, since a_{23} is inconsistent, assume $a_{2k} a_{k3} = a'_{23}$, $(k = 1,4,5,6,7,8)$; we can

use a'_{23} to be the value of a_{23} in order to let it be consistent. Clearly, any value of

pair of $a_{2k} a_{k3} = a'_{23}$ $(k = 1,4,5,6,7,8)$ is $\frac{1}{4}$, which is the same as shown in *Example 3*

of [15].

The above examples show that the inconsistent elements can be determined by

observing and analyzing the non-zero row (column), and sign identification of the bias

elements in the IBM C instead of following the seven steps of inconsistency

identification developed in [15]. Therefore, the proposed method is simpler and faster

than the previous one for the special case with only one pair of inconsistent elements

in the original RPCM.

4. Conclusion

This paper discussed the principle of the proposed inconsistency identification

method. A fast inconsistency identification method for the special case with only one

pair of inconsistent elements in the original RPCM was also proposed and proved

mathematically. Two examples indicate that the inconsistent elements can be easily

and quickly identified by the proposed fast inconsistency identification method if

there are only two rows and corresponding two columns with non-zero elements.

Although the special case with only one pair of inconsistent elements in the

original RPCM can easily and quickly be identified by the proposed fast inconsistency

identification method, such cases where there are more than one pair of inconsistent elements in RPCM will be relatively complicated by the proposed method, and it remains to be studied in future.

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