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# Inefficient weights from pairwise comparison matrices with arbitrarily small inconsistency

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**Abstract**

Having a pairwise comparison matrix in a multi-attribute decision problem, two basic problems arise: how to compute the weight vector, and, how to associate an inconsistency index to the matrix. Two key concepts of the Analytic Hierarchy Process, the eigenvector method and inconsistency index $CR$ are discussed. (In)efficiency is a well-known property in multiple objective optimization. We introduce a restriction of it in the paper. Given a pairwise comparison matrix $\mathbf{A} = [a_{ij}]_{i,j=1,\dots,n}$, weight vector $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ is called *internally inefficient* if there exists a weight vector $\mathbf{w}' = (w'_1, w'_2, \dots, w'_n)^T$ such that $a_{ij} \le w'_i/w'_j \le w_i/w_j$ if $a_{ij} \le w_i/w_j$, and $a_{ij} \ge w'_i/w'_j \ge w_i/w_j$ if $a_{ij} \ge w_i/w_j$ for all $i, j$, and there exist $k, \ell$ such that $w'_k/w'_\ell < w_k/w_\ell$ if $a_{k\ell} \le w_k/w_\ell$, and $w'_k/w'_\ell > w_k/w_\ell$ if $a_{k\ell} \ge w_k/w_\ell$. A class of internally inefficient pairwise comparison matrices is provided that includes matrices of arbitrarily small $CR$ inconsistency. The paper is closed by another internally inefficient matrix and an open question of a neccessary and sufficient condition of (internal) inefficiency.

# 1 Introduction

## 1.1 Pairwise comparison matrix

Pairwise comparison matrices are applied in multi-attribute decision making to quantify the importance of the criteria as well as for the evaluation of the actions. It is assumed that decision makers prefer answering questions 'How many times criterion $i$ is more important than criterion $j$?' compared to 'What are the importance of the criteria expressed by numbers?' Pairwise comparison matrix is a key concept of the Analytic Hierarchy Process proposed by Saaty [17].

Let $\mathbb{R}_+^{n\times n}$ denote the set of positive matrices of size $n \times n$ and $\mathbb{R}_+^n$ denote the positive orthant of the $n$-dimensional Euclidean space.

**Definition 1.** $\mathbf{A} = [a_{ij}]_{i,j=1,\ldots,n} \in \mathbb{R}_+^{n\times n}$ *is called a* pairwise comparison matrix *if* $a_{ij} = 1/a_{ji}$ *for all* $i, j = 1, \ldots, n$.

$\mathcal{PCM}_n$ denotes the set of pairwise comparison matrices of size $n \times n$, $\mathcal{PCM}_n \subset \mathbb{R}_+^{n\times n}$.

**Definition 2.** $\mathbf{A}$ *is called* consistent *if* $a_{ij}a_{jk} = a_{ik}$ *holds for all* $i, j, k = 1, \ldots, n$.

Every consistent pairwise comparison matrix can be associated to a weight vector $\mathbf{w} = (w_1, w_2, \ldots, w_n)^T \in \mathbb{R}_+^n$ and be written as $\mathbf{A} = \left[\frac{w_i}{w_j}\right]_{i,j=1,\ldots,n}$ and $\mathbf{w}$ is unique within a positive multiplicative constant.

**Definition 3.** $\mathbf{A}$ *is called* inconsistent *if it is not consistent, that is, there exist* $i, j, k$ *such that* $a_{ij}a_{jk} \neq a_{ik}$.

Pairwise comparison matrices provided by a decision maker are usually inconsistent, therefore, two problems arise. One is how to estimate the weights based on an inconsistent pairwise comparison matrix, in other words, how to approximate $\mathbf{A}$ by a consistent pairwise comparison matrix. A number of weighting methods have been developed during the last 35 years, see Jensen [13], Golany and Kress [11], Choo and Wedley [4], Ishizaka and Lusti [12] for a review and comparative studies. In the paper we deal with the eigenvector method suggested by Saaty [17]. The second question is whether $\mathbf{A}$ can be used at all, i.e., does not it have too many and/or too heavy errors and contradictions. It leads us to the problem of indexing inconsistency. See Golden and Wang [10], Koczkodaj [14], Bozóki and Rapcsák [2], Temesi [19], Brunelli, Canal and Fedrizzi [3] and their references for a detailed overview. In the paper, the $CR$ inconsistency index [17] is discussed.

#### 1.1.1 Eigenvector method

The linear algebraic foundation of the eigenvector method is the well known Perron-Frobenius theory [7, 8, 9, 16]. Let $\lambda_{\max}(\mathbf{A})$ denote the Perron eigenvalue of $\mathbf{A}$, also known as the *largest* or *dominant eigenvalue.* $\lambda_{\max}(\mathbf{A}) \geq n$ and equals to $n$ if and only if matrix $\mathbf{A}$ is consistent [17]. Let $\mathbf{w}^{EM(\mathbf{A})} = (w_1^{EM(\mathbf{A})}, w_2^{EM(\mathbf{A})}, \ldots, w_n^{EM(\mathbf{A})})^T$ denote the right eigenvector of

$\mathbf{A}$ corresponding to $\lambda_{\max}(\mathbf{A})$. It follows from the Perron-Frobenius theorem that $\mathbf{w}^{EM(\mathbf{A})}$ is positive and unique up to a scalar multiplication. $\mathbf{w}^{EM(\mathbf{A})}$ is usually normalized to 1, that is, $\sum_{i=1}^{n} w_i^{EM(\mathbf{A})} = 1$. $\mathbf{w}^{EM(\mathbf{A})}$ is also called $EM$ *weight vector*. Let $\mathbf{X}^{EM(\mathbf{A})} \stackrel{\text{def}}{=} \left[ \frac{w_i^{EM(\mathbf{A})}}{w_j^{EM(\mathbf{A})}} \right]_{i,j=1,\ldots,n}$ be the consistent pairwise comparison matrix generated by $\mathbf{w}^{EM(\mathbf{A})}$. It is the approximation of $\mathbf{A}$ by the eigenvector method. $\lambda_{\max}$ is also used for $\lambda_{\max}(\mathbf{A})$ as well as $\mathbf{w}^{EM}$ for $\mathbf{w}^{EM(\mathbf{A})}$ and $\mathbf{X}^{EM}$ for $\mathbf{X}^{EM(\mathbf{A})}$ if it does not cause a misunderstanding.

### 1.1.2 Inconsistency index $CR$

Saaty [17] defined the inconsistency index as

$$CR(\mathbf{A}) \stackrel{\text{def}}{=} \frac{\frac{\lambda_{\max}(\mathbf{A})-n}{n-1}}{\frac{\overline{\lambda_{\max}^{n\times n}}-n}{n-1}} = \frac{\lambda_{\max}(\mathbf{A}) - n}{\overline{\lambda_{\max}^{n\times n}} - n},$$

where $\overline{\lambda_{\max}^{n\times n}}$ denotes the average value of the maximal eigenvalue of randomly generated pairwise comparison matrices of size $n \times n$ such that each element $a_{ij}$ $(i < j)$ is chosen from the ratio scale $1/9, 1/8, \ldots, 1/2, 1, 2, \ldots, 9$ with equal probability. $CR(\mathbf{A})$ is a positive linear transformation of $\lambda_{\max}(\mathbf{A})$. $CR(\mathbf{A}) \geq 0$ and $CR(\mathbf{A}) = 0$ if and only if $\mathbf{A}$ is consistent. Saaty suggested the rule of acceptability $CR < 0.1$. In Section 2 we apply the property that $CR(\mathbf{A})$ can be arbitrarily small if $\lambda_{\max}(\mathbf{A})$ is close enough no $n$.

## 1.2 Inefficiency

Our motivation is the paper of Blanquero, Carrizosa and Conde [1] discussing a general framework of (in)efficiency of a consistent approximation of a pairwise comparison matrix. Their remarkable example on page 282 is as follows.

**Example 1.** *Let $\mathbf{A} \in \mathcal{PCM}_4$, from which one can compute the weight vector $\mathbf{w}^{EM}$. The authors compared $\mathbf{w}^{EM}$ to another weight vector $\mathbf{w}^*$:*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 6 & 2 \\ 1/2 & 1 & 4 & 3 \\ 1/6 & 1/4 & 1 & 1/2 \\ 1/2 & 1/3 & 2 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1 \\ 2.0712416 \end{pmatrix}, \quad \mathbf{w}^* = \begin{pmatrix} 6.01438057 \\ 4.26049429 \\ 1.003 \\ 2.0712416 \end{pmatrix}.$$

*Note that $\mathbf{w}^{EM}$ and $\mathbf{w}^*$ are written unnormalized in order to be compared simpler, on the other hand they differ in the third coordinate only. Computational results in [1] are given with interval arithmetic, however, coordinates* *are now written truncated at 8 correct digits and we emphasize that the origin of the phenomenon in our focus is not a rounding error. The approximations* $\mathbf{X}^{EM}$ *and* $\mathbf{X}^*$ *coincide except for the third row and column, due to reciprocity, the latter is sufficient to be reported:*

| $i$ | $a_{i3}$ | $x_{i3}^{EM}$ | $x_{i3}^*$ | $\lvert a_{i3} - x_{i3}^{EM}\rvert$ | $\lvert a_{i3} - x_{i3}^*\rvert$ |
|---|---|---|---|---|---|
| 1 | 6 | 6.01438057 | 5.99639139 | 0.01438057 | 0.00360859 |
| 2 | 4 | 4.26049429 | 4.24775103 | 0.26049429 | 0.24775103 |
| 3 | 1 | 1 | 1 | 0 | 0 |
| 4 | 2 | 2.07124160 | 2.06504646 | 0.07124160 | 0.06504646 |

*The authors argue that* $\mathbf{X}^*$ *is a* better *approximation of* $\mathbf{A}$ *than* $\mathbf{X}^{EM}$ *because there exist three elements (and their reciprocals), which are closer to the corresponding elements of* $\mathbf{A}$ *while all the other approximations are the same.*

Efficiency, also known as Pareto optimality or non-dominatedness, is a basic concept of multiple objective optimization, see, e.g., the book of Liu, Yang and Whidborne [15, Chapter 4]. However, it is more convenient to use the opposite for our purpose. Let $\mathbf{A} = [a_{ij}]_{i,j=1,\dots,n} \in \mathcal{PCM}_n$ and $\mathbf{w} = (w_1, w_2, \dots, w_n)^T$ be a positive weight vector.

**Definition 4.** $\mathbf{w}$ *is called* inefficient *if there exists a weight vector* $\mathbf{w}' = (w'_1, w'_2, \dots, w'_n)^T$ *such that* $|a_{ij} - w'_i/w'_j| \le |a_{ij} - w_i/w_j|$ *for all* $i, j$*, and there exist* $k, \ell$ *such that* $|a_{ij} - w'_k/w'_\ell| < |a_{ij} - w_k/w_\ell|$.

It follows from the definiton that $\mathbf{w}^{EM}$ in Example 1 is inefficient. A special type of inefficiency is introduced and used in the paper.

**Definition 5.** $\mathbf{w}$ *is called* internally inefficient *if there exists a weight vector* $\mathbf{w}' = (w'_1, w'_2, \dots, w'_n)^T$ *such that* $a_{ij} \le w'_i/w'_j \le w_i/w_j$ *if* $a_{ij} \le w_i/w_j$*, and* $a_{ij} \ge w'_i/w'_j \ge w_i/w_j$ *if* $a_{ij} \ge w_i/w_j$ *for all* $i, j$*, and there exist* $k, \ell$ *such that* $w'_k/w'_\ell < w_k/w_\ell$ *if* $a_{k\ell} \le w_k/w_\ell$*, and* $w'_k/w'_\ell > w_k/w_\ell$ *if* $a_{k\ell} \ge w_k/w_\ell$.

It follows from the definitions that if $\mathbf{w}$ is internally inefficient, then it is inefficient as well.

Blanquero, Carrizosa and Conde [1] investigate the properties of the set of efficient solution and they discuss tests of efficiency, too.

## 1.3 Eigenvalue method as the solution of optimization problems

In this subsection two optimization problems are recalled. They share the property that the optimal solution is the solution of the eigenvector method. As we see through Example 1 and will see in Section 2 that optimality with respect to reasonable and nice objective functions does not exclude inefficiency.

### 1.3.1 $\min\max$ and $\max\min$ problems of Perron and Frobenius

**Theorem 1.** (Perron [16], Frobenius [7, 8, 9]) *Let* $\mathbf{A} \in \mathcal{PCM}_n$*, and the largest eigenvalue of* $\mathbf{A}$ *is denoted by* $\lambda_{\max}$*. Then*

$$\max_{\mathbf{w}\in\mathbb{R}^n_+} \min_{1\le i\le n} \frac{\sum\limits_{j=1}^{n} a_{ij}w_j}{w_i} \le \lambda_{\max} \le \min_{1\le i\le n} \max_{\mathbf{w}\in\mathbb{R}^n_+} \frac{\sum\limits_{j=1}^{n} a_{ij}w_j}{w_i}$$

*where* $\mathbf{w} = (w_1, w_2, \dots, w_n)$*. Furthermore, both inequalities hold with equality if and only if* $\mathbf{w} = \kappa\mathbf{w}^{EM}$*, where* $\kappa$ *is an arbitrary positive number.*

Theorem 1 is discussed and reformulated by Sekitani and Yamaki [18] and it is applied by Fülöp [6] in the development of a fast eigenvalue optimization algorithm.

### 1.3.2 Fichtner's metric

Fichtner proved that the eigenvector method can also be written as a distance minimizing method.

**Theorem 2.** (Fichtner, [5, pp. 37–38]) *Let* $\delta : \mathcal{PCM}_n \times \mathcal{PCM}_n \to \mathbb{R}$ *be as follows:*

$$\delta(\mathbf{A},\mathbf{B}) \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^{n} \left(w_i^{EM(\mathbf{A})} - w_i^{EM(\mathbf{B})}\right)^2} + \frac{|\lambda_{\max}(\mathbf{A}) - \lambda_{\max}(\mathbf{B})|}{2(n-1)} + \\ + \chi(\mathbf{A},\mathbf{B}) \frac{|\lambda_{\max}(\mathbf{A}) + \lambda_{\max}(\mathbf{B}) - 2n|}{2(n-1)},$$

*where*

$$\chi(\mathbf{A},\mathbf{B}) = \begin{cases} 0 & \text{if } \mathbf{A} = \mathbf{B}, \\ 1 & \text{if } \mathbf{A} \neq \mathbf{B}. \end{cases}$$

*Then,* $\delta$ *is a metric in* $\mathcal{PCM}_n$ *with the following properties:*

*(a) for every* $\mathbf{A} \in \mathcal{PCM}_n$, $\mathbf{X}^{EM}$ *is the optimal solution of the problem* $\min\{\delta(\mathbf{A},\mathbf{X})|\mathbf{X}$ *is consistent*$\}$*;*
*(b)* $\min\{\delta(\mathbf{A},\mathbf{X})|\mathbf{X}$ *is consistent*$\} = \delta(\mathbf{A},\mathbf{X}^{EM}) = \frac{\lambda_{\max}(\mathbf{A})-n}{n-1}$.

It is emphasized that the distance function above is not continuous.

# 2 Inefficient weights from matrices with arbitrarily small $CR$ inconsistency

There are examples of extremely high inconsistency as in the paper of Jensen [13, Section 6] that are particularly interesting from mathematical point of view but their relevance in real decision problems seems to be low. In this section a class of pairwise comparison matrices is constructed with arbitrarily small $CR$ inconsistency such that the $EM$ weight vector is inefficient. Although we apply a specific structure, the phenomenon of inefficiency is present in an essentially wider subset of pairwise comparison matrices as Example 1 witnesses.

Let $n \geq 4$ and $\mathbf{A}(p,q) \in \mathcal{PCM}_n$ as follows:

$$\mathbf{A}(p,q) = \begin{pmatrix} 1 & p & p & p & \dots & p & p \\ 1/p & 1 & q & 1 & \dots & 1 & 1/q \\ 1/p & 1/q & 1 & q & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots & \vdots \\ 1/p & 1 & 1 & 1 & \dots & 1 & q \\ 1/p & q & 1 & 1 & \dots & 1/q & 1 \end{pmatrix}, \tag{1}$$

where $p,q$ are arbitrary positive numbers. Formally, $a_{ii} = 1$ $(i = 1,2,\dots,n)$; $a_{1i} = p$ $(i = 2,3,\dots,n)$; $a_{i,i+1} = q$ $(i = 2,3,\dots,n)$; $a_{2,n} = q$ and all other elements above the main diagonal are equal to 1. Apply reciprocity rule $a_{ji} = 1/a_{ij}$ to get the elements below the main diagonal. $\mathbf{A}(p,q)$ is consistent if and only if $q = 1$. Hereafter, $q \neq 1$ is assumed.

**Lemma 1.** *The maximal eigenvalue of* $\mathbf{A}(p,q)$ *and the right eigenvector are* *as follow:*

$$\lambda_{\max} = \frac{\sqrt{q^4 + (2n-8)(q^3+q) + (n^2-4n+14)q^2 + 1} + q^2 + (n-2)q + 1}{2q},$$

$$w_1^{EM} = p\frac{\sqrt{q^4 + (2n-8)(q^3+q) + (n^2-4n+14)q^2 + 1} - [q^2 + (n-4)q + 1]}{2q},$$

$$w_i^{EM} = 1, \qquad i = 2, 3, \ldots, n.$$

*Proof.* The verification of the eigenvalue-eigenvector equation $\mathbf{A}(w_1^{EM}, w_2^{EM}, \ldots, w_n^{EM})^T = \lambda_{\max}(w_1^{EM}, w_2^{EM}, \ldots, w_n^{EM})^T$ with the formulas of Lemma 1 is elementary but requires a lot of space, therefore it is omitted. We also need to confirm that the maximal eigenvalue and the associated eigenvector are found. It follows from the assumptions $n \geq 4$ and $p, q > 0$ that $w_1^{EM} > 0$ and certainly $w_i^{EM} > 0$ $(i = 2, 3, \ldots, n)$, meaning that the eigenvector is positive. Sekitani and Yamaki proved that any positive eigenvector belongs to $\lambda_{\max}$ [18, Lemma 5], which completes the proof. □

In order to have shorter formulas,

$$Q \stackrel{\text{def}}{=} q + \frac{1}{q},$$

$$f(Q) \stackrel{\text{def}}{=} \frac{\sqrt{(Q+n-4)^2 + 4n - 4} - (Q+n-4)}{2}, \qquad Q \in [2, \infty)$$

are introduced.

**Lemma 2.** *The consistent approximation of* $\mathbf{A}$ *denoted by* $\mathbf{X}^{EM} = [x_{ij}^{EM}]_{i,j=1,\ldots,n}$ *and computed from the EM weight vector by* $x_{ij}^{EM} = \frac{w_i^{EM}}{w_j^{EM}}$ $(i, j = 1, \ldots, n)$ *is as follows:*

$$x_{1j}^{EM} = pf(Q), \qquad j = 2, 3, \ldots, n,$$

$$x_{j1}^{EM} = \frac{1}{x_{1j}^{EM}}, \qquad j = 2, 3, \ldots, n,$$

$$x_{ij}^{EM} = 1, \text{ everywhere else.}$$

*Furthermore,* $x_{1j}^{EM} \leq p$ $(j = 2, 3, \ldots, n)$ *and* $x_{1j}^{EM} = p$ $(j = 2, 3, \ldots, n)$ *if and only if* $Q = 2$*, being equivalent to* $q = 1$.

*Proof.* Lemma 1 can be rewritten as

$$\lambda_{\max} = \frac{\sqrt{(Q+n-4)^2 + 4n - 4} + Q + n - 2}{2} \tag{2}$$

$$w_1^{EM} = pf(Q).$$

It can be seen that $f(Q)$ is continuous and differentiable on its domain. One can show with elementary calculus that $\lim_{Q\to\infty} f(Q) = 0$; $f'(Q) < 0$ and $f''(Q) > 0$ for all $Q \in [2,\infty)$; $0 < f(Q) \leq 1$ for all $Q \in [2,\infty)$; $f(Q) = 1 \Leftrightarrow Q = 2$, which completes the proof. □

**Corollary 1.** $\lim_{Q\to 2+} \lambda_{\max} = n$, *that is, $CR$ inconsistency can be arbitrarily small if $q$ is close enough to* 1.

*Proof.* Note that $\lambda_{\max}$ does not depend on $p$. Apply (2) to verify $\lim_{Q\to 2+} \lambda_{\max} = n$. □

**Proposition 1.** *Let $q$ be positive and $q \neq 1$. Then $\mathbf{w}^{EM}$ is internally inefficient, therefore inefficient.*

*Proof.* We show that weight vector $\mathbf{w}^* = (w_1^*, w_2^*, \ldots, w_n^*)^T$ defined as $w_1^* = p$, $w_j^* = 1$ $(j = 2,3,\ldots,n)$ provides a better approximation, because $\mathbf{X}^* = [x_{ij}^*]_{i,j=1,\ldots,n}$ with $x_{ij}^* = w_i^*/w_j^*$ $(i,j = 1,\ldots,n)$ is at least as good as $\mathbf{X}^{EM}$ in every positions and there exist $n-1$ positions (and their reciprocals) in which the approximation is strictly better. $\mathbf{X}^*$ can be written as $x_{1j}^* = p$ $(j = 2,3,\ldots,n)$, $x_{j1}^* = 1/p$ $(j = 2,3,\ldots,n)$, $x_{ij}^* = 1$ everywhere else. The bottom-right $(n-1)\times(n-1)$ submatrices of $\mathbf{X}^*$ and $\mathbf{X}^{EM}$ are equal. $\mathbf{X}^*$ approximates $\mathbf{A}$ perfectly in all entries of the first row and column. However, by Lemma 2, $\mathbf{X}^{EM}$ does not provide perfect approximation in the first row and column (except for the diagonal element). We have proven that $\mathbf{w}^{EM}$ is internally inefficient, consequently inefficient. □

# 3 Inefficient weights from matrices with high $CR$ inconsistency

We have seen in Section 2 that internal inefficiency can be observed in case of arbitrarily small inconsistency.

Now we do not assume any special structure as in the previous section. An additional example of internal inefficiency has been found. Even if the following matrix has high inconsistency ($CR = 0.78$) it may help us to understand why $EM$ weight vector can be (internally) inefficient.

**Example 2.** *Let $\mathbf{A} \in \mathcal{PCM}_6$, the $EM$ weight vector and a competing weight* *vector* $\mathbf{w}^*$ *(which differs from* $\mathbf{w}^{EM}$ *in three coordinates) be as follow:*

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 1/9 & 9 & 1/9 & 1/8 \\ 1/4 & 1 & 1/8 & 1/4 & 1/7 & 1/5 \\ 9 & 8 & 1 & 8 & 4 & 1/2 \\ 1/9 & 4 & 1/8 & 1 & 7 & 1/3 \\ 9 & 7 & 1/4 & 1/7 & 1 & 1/5 \\ 8 & 5 & 2 & 3 & 5 & 1 \end{pmatrix}, \quad \mathbf{w}^{EM} = \begin{pmatrix} 0.1281 \\ 0.0180 \\ 0.3028 \\ 0.1237 \\ 0.1440 \\ 0.2835 \end{pmatrix}, \quad \mathbf{w}^* = \begin{pmatrix} 0.1281 \\ 0.0206 \\ 0.3471 \\ 0.1237 \\ 0.1440 \\ 0.3249 \end{pmatrix}.$$

*Approximations are*

$$\mathbf{X}^{EM} = \begin{pmatrix} 1 & 7.1326 & 0.4229 & 1.0354 & 0.8892 & 0.4518 \\ 0.1402 & 1 & 0.0593 & 0.1452 & 0.1247 & 0.0633 \\ 2.3649 & 16.8678 & 1 & 2.4487 & 2.1028 & 1.0684 \\ 0.9658 & 6.8885 & 0.4084 & 1 & 0.8587 & 0.4363 \\ 1.1246 & 8.0216 & 0.4756 & 1.1645 & 1 & 0.5081 \\ 2.2135 & 15.7877 & 0.9360 & 2.2919 & 1.9681 & 1 \end{pmatrix},$$

$$\mathbf{X}^* = \begin{pmatrix} 1 & \boldsymbol{6.2242} & \boldsymbol{0.3690} & 1.0354 & 0.8892 & \boldsymbol{0.3942} \\ \boldsymbol{0.1607} & 1 & 0.0593 & \boldsymbol{0.1664} & \boldsymbol{0.1429} & 0.0633 \\ \boldsymbol{2.7100} & 16.8678 & 1 & \boldsymbol{2.8061} & \boldsymbol{2.4097} & 1.0684 \\ 0.9658 & \boldsymbol{6.0112} & \boldsymbol{0.3564} & 1 & 0.8587 & \boldsymbol{0.3808} \\ 1.1246 & \boldsymbol{7.0000} & \boldsymbol{0.4150} & 1.1645 & 1 & \boldsymbol{0.4434} \\ \boldsymbol{2.5365} & 15.7877 & 0.9360 & \boldsymbol{2.6264} & \boldsymbol{2.2554} & 1 \end{pmatrix}.$$

*It can be observed that* $\mathbf{X}^*$ *yields better approximations in nine positions (and their reciprocal) marked by bold.*

Note that all off-diagonal entries of the sixth row and the second column of $\mathbf{A}$ are greater than 1. This property is probably related to inefficiency, however, it is certainly not a neccessary condition in general, because the class of matrices discussed in Section 2 contains the case $p = 1$, when the matrices have no row or column having off-diagonal elements that are all greater than one. Research is continued to find a necessary and sufficient condition of (internal) inefficiency.


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