Generalization of the Observer Principle for YOULA-Parametrized Regulators

László Keviczky and Csilla Bányász

Computer and Automation Research Institute and
MTA-BME Control Engineering Research Group
Hungarian Academy of Sciences
H-1111 Budapest, Kende u 13-17, HUNGARY
e-mail: keviczky@sztaki.hu ; banyasz@sztaki.hu

Abstract. An equivalent transfer function representation (*TFR*) is introduced to study the state-feedback/observer (*SFO*) topologies of control systems. This approach is used to explain why an observer can radically reduce even large model errors. Then the same principle is combined with YOULA-parametrization (*YP*) introducing a new class of regulators

Keywords: Observer, state-feedback, model error, YOULA-parametrization

1. Introduction, the State Feedback (*SF*)

It is a well known methodology to use the state variable representations (*SVR*) of linear time invariant (*LTI*) single input - single output (*SISO*) systems [1]. The *SVR* proved to be excellent tool to implement both *LQR* (Linear system - Quadratic criterion - Regulator) control and pole placement design. The practical applicability required to introduce the observers, which make this methodology widely applied even for large scale and higher dimension plants [3]. Thousands of theoretical considerations mostly concentrate on the irregularities and special structures in the *SVR* appearing and much less publications deal with the model error properties of these systems.

It is possible to find a proper new way to discuss and investigate the the special properties and limitations of the classical state-feedback (*SF*), state-feedback/observer (*SFO*) topologies if someone replaces the *SVR* by their transfer function representations (*TFR*) [2].

Consider a *SISO* continuous time (t) *LTI* dynamic plant described by the *SVR*

$$P = \frac{\mathcal{B}}{\mathcal{A}} \tag{1}$$

Here P is the *TFR* of the open-loop system with the numerator and denominator polynomials

$$\mathcal{B}(s) = s^n + b_1 s^{n-1} + \ldots + b_{n-1} s + b_n \tag{2}$$

$$\mathcal{A}(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n \tag{3}$$

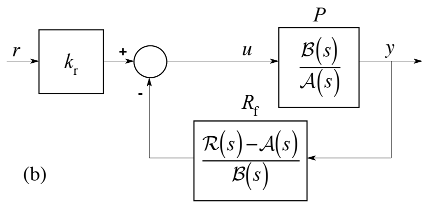


Fig. 1. Equivalent schemes of *SF* using *TFR* forms

If we want to express the operation of the *SF* by equivalent scheme using *TFR* forms, Fig. 1 can be used, where the feedback regulator $R_\mathrm{f} = K_k$ is obtained from the basic equation (complementary sensitivity function, *CSF*) of the closed-loop

$$T_\mathrm{ry}(s) = \frac{k_\mathrm{r}\,\mathcal{B}(s)}{\mathcal{R}(s)} = \frac{k_\mathrm{r}\,\mathcal{B}(s)}{\mathcal{A}(s)+\mathcal{K}(s)} = \frac{k_\mathrm{r}\,P}{1+K_k\,P} \tag{4}$$

where k_r is obtained by requiring that the static gain of T_ry should be equal to one. The calibrating factor k_r is necessary because the closed-loop using *SF* is not an integrating one. Equation (4) clearly shows, that the open-loop zeros remain unchanged and the closed-loop poles will be the required ones. The solution formally makes the characteristic polynomial of the closed-loop equal to the desired polynomial ("placed poles")

$$\mathcal{R}(s) = s^n + r_1 s^{n-1} + \ldots + r_{n-1} s + r_n \tag{5}$$

Here it is obtained that

$$R_\mathrm{f} = K_k(s) = \frac{\mathcal{K}(s)}{\mathcal{B}(s)} = \frac{\mathcal{R}(s)-\mathcal{A}(s)}{\mathcal{B}(s)} \tag{6}$$

which corresponds to the state feedback vector in the classical *SVR*.

2. Observer-Based State-Feedback with Equivalent *TFR* Forms

The practical applicability of the *SF* theory was introduced by the development of the observers capable to calculate the unmeasured state variables. The most general *SF*/Observer (*SFO*) topology discussed above can also be given using equivalent *TFR* forms of *SF* and is shown in Fig. 2.

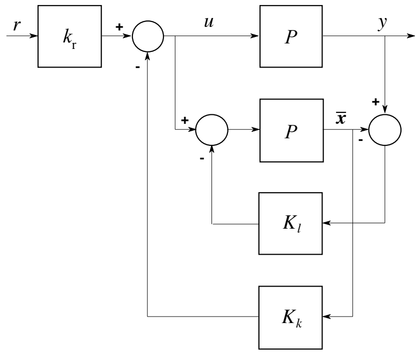


Fig. 2. Equivalent topology of the general basic *SFO* scheme using *TFR* forms

The usual classical design goal for the observer is to determine the observer feedback so that its feedback closed-loop system has the characteristic polynomial

$$\mathcal{Q}(s) = s^n + q_1 s^{n-1} + \ldots + q_{n-1} s + q_n \tag{7}$$

The *TFR* $K_l(s) = \mathcal{L}(s)/\mathcal{B}(s)$ in Fig. 2 corresponds to the observer feedback vector in the classical *SVR*.

The pole-placement design goals for the *SF* and observer dynamics require

$$\mathcal{K}(s) = \mathcal{R}(s) - \mathcal{A}(s) \quad \text{and} \quad \mathcal{L}(s) = \mathcal{Q}(s) - \mathcal{A}(s) \tag{8}$$

After some long, but straightforward block manipulations the equivalent *SFO* scheme can be transformed into another unity feedback closed-loop form given in Fig. 3.

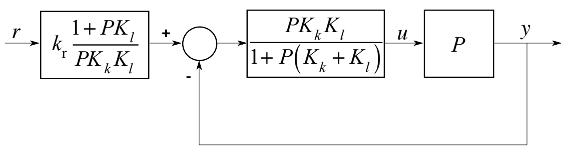


Fig. 3. Reduced equivalent topology of the general basic *SFO* scheme

It is interesting to observe that the transfer function of the closed-loop in Fig. 3 has a very special structure

$$\frac{P^2 K_k K_l}{1 + P(K_k + K_l) + P^2 K_k K_l} = \frac{PK_k}{1 + PK_k} \frac{PK_l}{1 + PK_l} = \frac{\mathcal{K}}{\mathcal{R}} \frac{\mathcal{L}}{\mathcal{Q}} \tag{9}$$

It is formally two simpler closed-loops cascaded, which dynamically completely corresponds to the characteristic equation: $\mathcal{R}(s)=0$ and $\mathcal{Q}(s)=0$. The overall transfer function of the *SFO* system is

$$T_{\mathrm{ry}}(s)=k_{\mathrm{r}}\frac{1+PK_l}{PK_kK_l}\frac{PK_k}{1+PK_k}\frac{PK_l}{1+PK_l}=\frac{k_{\mathrm{r}}P}{1+PK_k}=\frac{k_{\mathrm{r}}\mathcal{B}}{\mathcal{R}} \tag{10}$$

3. Model Error Properties

The above widely applied methodology has a common problem, that in all regulator and observer equations the true process P is used instead of the estimated model $\hat{P}$ of the process. The equivalent *TFR* form of the *SF* using the model of the process is shown in Fig. 4.

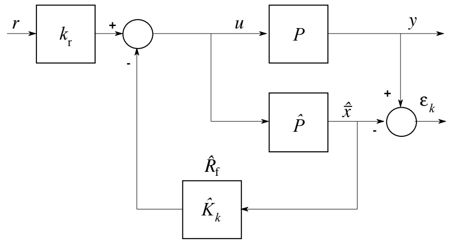


Fig. 4. The model based *SF* scheme and error

The parallel scheme in Fig. 4 is used to compute the model error. Using (4) the $\hat{T}_{\mathrm{ry}}$ model-based version of T_{ry} is

$$\hat{T}_{\mathrm{ry}}=\frac{k_{\mathrm{r}}P}{1+K_k\hat{P}}=\frac{k_{\mathrm{r}}\mathcal{B}}{\mathcal{R}}\frac{\hat{\mathcal{A}}}{\mathcal{A}}=T_{\mathrm{ry}}\frac{\hat{\mathcal{A}}}{\mathcal{A}}\hat{\mathcal{A}} \tag{11}$$

and its relative uncertainty

$$\ell_{\mathrm{T}}=\frac{\hat{T}_{\mathrm{ry}}-T_{\mathrm{ry}}}{\hat{T}_{\mathrm{ry}}}=\frac{\hat{\mathcal{A}}-\mathcal{A}}{\mathcal{A}}=\ell_{\mathcal{A}} \tag{12}$$

which shows that $\ell_{\mathrm{T}}=0$ for $\ell_{\mathcal{A}}=0$. Introducing the additive $\Delta=P-\hat{P}$ and relative plant model error

$$\ell=\frac{\Delta}{\hat{P}}=\frac{P-\hat{P}}{\hat{P}} \tag{13}$$

the modeling error ε_k in Fig. 4 can be expressed as

$$\varepsilon_k = \frac{k_{\mathrm{r}}\hat{\mathcal{B}}}{\mathcal{B}}\ell r = T_{\mathrm{ry}}\frac{\hat{\mathcal{B}}}{\mathcal{B}}\ell r = \hat{P}\ell u \tag{14}$$

The *SFO* scheme is widely applied in the practice with model-based *SVR*, so it is interesting how the model-based scheme in Fig. 5 influences the original modeling error ε_k.

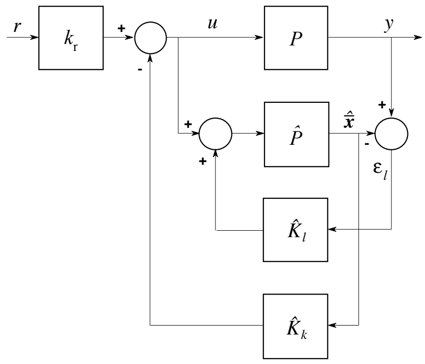


Fig. 5. Model based *SFO* scheme with *TFR* forms

After some long but straightforward computations

$$\varepsilon_l = \frac{\hat{P}}{1+K_l\hat{P}}\ell u = \frac{\hat{\mathcal{B}}}{\mathcal{Q}}\ell u = \frac{1}{1+K_l\hat{P}}\varepsilon_k \tag{15}$$

is obtained. Equation (15) clearly shows the influence of the *SFO* scheme, because it decreases the modeling error ε_k by $(1+K_l\hat{P})$. Selecting fast observer poles, one can reach quite small "virtual" modeling error ε_l in the major frequency domains of the tracking task.

Besides the radical model error attenuating behavior of the model-based *SFO* scheme, unfortunately it has a very important drawback, the nice cascade (9) structure changes to

$$\left.\frac{\hat{P}^2 K_k K_l\left(1+\ell\right)}{1+\hat{P}\left(K_k+K_l\right)+\hat{P}^2 K_k K_l\left(1+\ell\right)}\right|_{\ell\to 0} = \frac{PK_k}{1+PK_k}\frac{PK_l}{1+PK_l} = \frac{\mathcal{K}}{\mathcal{R}}\frac{\mathcal{L}}{\mathcal{Q}} \tag{16}$$

which form is not factorable except for the exact model matching case, when $\ell \to 0$. On the basis of Fig. 5 and (16) it is easy to see that the poles of the observer feedback loop remain unchanged using the placement design equation forms model-based *SFO* (8), thus the only solution is to use the available model of the process, in this case $\hat{\mathcal{A}}$, i.e.,

$$\mathcal{K}(s)=\mathcal{R}(s)-\hat{\mathcal{A}}(s) \quad \text{and} \quad L(s)=\mathcal{Q}(s)-\hat{\mathcal{A}}(s) \tag{17}$$

for the pole placing equations.

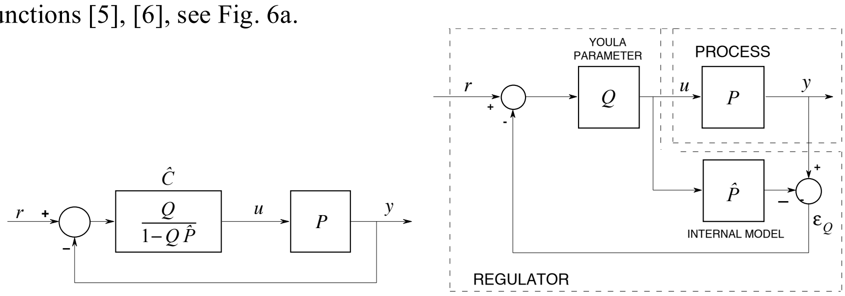
Because this design ensures the required poles only for small ℓ (see (16)), a serious robust stability investigation is required first. Next it is important to investigate where the actual pole is located for non zero ℓ, so how big the performance loss is coming from the model based *SFR*. These steps are usually neglected in most of the published papers, books and applications.

4. Introducing the Observer Based YOULA-Regulator

For open-loop stable processes the all realizable stabilizing (*ARS*) model based regulator $\hat{C}$ is the *YOULA-parametrized* one:

$$\hat{C}\left(\hat{P}\right)=\left.\frac{Q}{1-Q\hat{P}}\right|_{\hat{P}\to P}=\frac{Q}{1-QP}=C\left(P\right) \tag{18}$$

where the "parameter" Q ranges over all proper ($Q\left(\omega=\infty\right)$ is finite), stable transfer functions [5], [6], see Fig. 6a.



(a) (b)

Fig. 6. The equivalent *IMC* structure of an *ARS* regulator

It is important to know that the *Y-parametrized* closed-loop with the *ARS* regulator is equivalent to the well-known form of the so-called *Internal Model Control* (*IMC*) principle [6] based structure shown in Fig. 6b.

Q is anyway the transfer function from r to u and the *CSF* of the whole closed-loop for $\hat{P}=P$, when $\ell\to 0$

$$\hat{T}_{\text{ry}}=\frac{\hat{C}P}{1+\hat{C}P}=QP\left.\frac{1+\ell}{1+\left(1-QP\right)\ell}\right|_{\ell\to 0}=QP=T_{\text{ry}} \tag{19}$$

is linear (and hence convex) in Q.

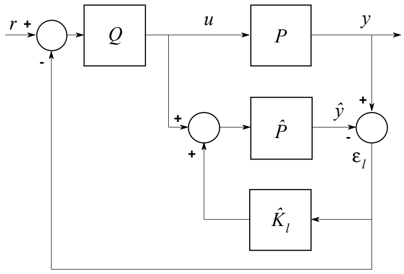


Fig. 7. The observer-based *IMC* structure

It is interesting to compute the relative error ℓ_{T} of $\hat{T}_{\mathrm{ry}}$

$$\ell_{\mathrm{T}} = \frac{T_{\mathrm{ry}} - \hat{T}_{\mathrm{ry}}}{\hat{T}_{\mathrm{ry}}} = \frac{T_{\mathrm{ry}}}{\hat{T}_{\mathrm{ry}}} - 1 = Q\left(P - \hat{P}\right) = QP\frac{\ell}{1+\ell} = T_{\mathrm{ry}}\frac{\ell}{1+\ell} \tag{20}$$

The equivalent *IMC* structure performs the feedback from the model error ε_Q. Similarly to the *SFO* scheme it is possible to construct an internal closed-loop, which virtually reduces the model error to

$$\varepsilon_l = \frac{1}{1+\hat{K}_l\hat{P}}\left(y - \hat{P}u\right) = \frac{1}{1+\hat{K}_l\hat{P}}\varepsilon_Q = \frac{1}{1+\hat{L}_l}\varepsilon_Q = \hat{H}\,\varepsilon_Q \quad ; \quad \hat{L}_l = \hat{K}_l\hat{P} \tag{21}$$

and performs the feedback from ε_l (see Fig. 7), where $\hat{L}_l$ is the internal loop transfer function. In this case the resulting closed-loop will change to the scheme shown in Fig. 8.

This means that the introduction of the observer feedback changes the *YOULA-parametrized* regulator to

$$\hat{C}'\left(\hat{P}'\right) = \frac{Q}{1 - Q\,\hat{P}\big/\left(1+\hat{K}_l\hat{P}\right)} = \frac{Q\left(1+\hat{K}_l\hat{P}\right)}{1+\hat{K}_l\hat{P} - Q\hat{P}} \tag{22}$$

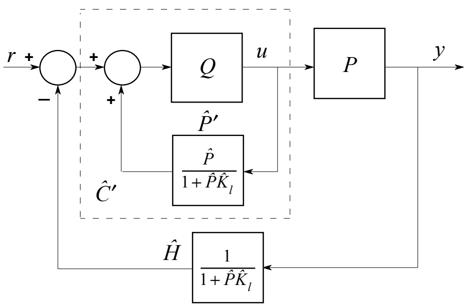


Fig. 8. Equivalent closed-loop for the observer-based *IMC* structure

The form of $\hat{C}'$ shows that the regulator virtually controls a fictitious plant $\hat{P}'$ which is also demonstrated in Fig. 8. Here the fictitious plant is

$$\hat{P}' = \frac{\hat{P}}{1+\hat{K}_l \hat{P}} = \frac{\hat{P}}{1+\hat{L}_l} \tag{23}$$

The closed-loop transfer function is now

$$\hat{T}'_{\mathrm{ry}} = \frac{\hat{C}'P}{1+\hat{C}'P} = \frac{QP\left(1+\hat{K}_l \hat{P}\right)}{1+\hat{K}_l \hat{P} - Q\hat{P} + QP} = QP \left. \frac{1}{1+QP\dfrac{1}{1+\hat{K}_l \hat{P}}\dfrac{\ell}{1+\ell}} \right|_{\ell \to 0} = QP = T_{\mathrm{ry}} \tag{24}$$

The relative error ℓ'_{T} of $\hat{T}'_{\mathrm{ry}}$ becomes

$$\ell'_{\mathrm{T}} = \frac{T_{\mathrm{ry}} - \hat{T}'_{\mathrm{ry}}}{\hat{T}'_{\mathrm{ry}}} = \frac{T_{\mathrm{ry}}}{\hat{T}'_{\mathrm{ry}}} - 1 = QP \frac{\ell}{1+\ell} \frac{1}{\left(1+\hat{K}_l \hat{P}\right)} = \ell_{\mathrm{T}} \frac{1}{1+\hat{L}_l} \tag{25}$$

which is smaller than ℓ_{T}. The reduction is by $\hat{H} = 1 \big/ \left(1+\hat{L}_l\right)$.

5. An Observer Based *PID*-Regulator

The ideal form of a YOULA-regulator based on reference model design [4], [5] is

$$C_{\mathrm{id}} = \frac{\left(R_{\mathrm{n}} P^{-1}\right)}{1-\left(R_{\mathrm{n}} P^{-1}\right)P} = \frac{Q}{1-QP} = \frac{R_{\mathrm{n}}}{1-R_{\mathrm{n}}} P^{-1} \tag{26}$$

when the inverse of the process is realizable and stable. Here the operation of R_{n} can be considered a reference model (desired system dynamics). It is generally required that the reference model has to be strictly proper with unit static gain, i.e., $R_{\mathrm{n}}\left(\omega = 0\right) = 1$.

For a simple, but robust *PID* regulator design method assume that the process can be well approximated by its two major time constants, i.e.,

$$P \cong \frac{A}{\mathcal{A}_2} \quad \text{where} \quad \mathcal{A}_2 = \left(1+sT_1\right)\left(1+sT_2\right) \tag{27}$$

According to (26) the ideal YOULA-regulator is

$$C_{\text{id}} = \frac{R_{\text{n}} P^{-1}}{1 - R_{\text{n}}} = \frac{R_{\text{n}} \left(1 + sT_1\right)\left(1 + sT_2\right)}{A\left(1 - R_{\text{n}}\right)} \quad ; \quad T_1 > T_2 \tag{28}$$

Let the reference model R_{n} be of first order

$$R_{\text{n}} = \frac{1}{1 + sT_{\text{n}}} \tag{29}$$

which means that the first term of the regulator is an integrator

$$\frac{R_{\text{n}}}{1 - R_{\text{n}}} = \frac{1/\left(1 + sT_{\text{n}}\right)}{1 - 1/\left(1 + sT_{\text{n}}\right)} = \frac{1}{1 + sT_{\text{n}} - 1} = \frac{1}{sT_{\text{n}}} \tag{30}$$

whose integrating time is equal to the time constant of the reference model. Thus the resulting regulator corresponds to the design principle, i.e., it is an ideal *PID* regulator

$$C_{\text{PID}} = A_{\text{PID}} \frac{\left(1 + sT_{\text{I}}\right)\left(1 + sT_{\text{D}}\right)}{sT_{\text{I}}} = A_{\text{PID}} \frac{\left(1 + sT_1\right)\left(1 + sT_2\right)}{sT_1} \tag{31}$$

with

$$A_{\text{PID}} = T_1 / A T_{\text{n}} \quad ; \quad T_{\text{I}} = T_1 \quad ; \quad T_{\text{D}} = T_2 \tag{32}$$

The YOULA-parameter Q in the ideal regulator is

$$Q = R_{\text{n}} P^{-1} = \frac{1}{A} \frac{\left(1 + sT_1\right)\left(1 + sT_2\right)}{1 + sT_{\text{n}}} \tag{33}$$

It is not necessary, but desirable to ensure the realizability, i.e., to use

$$Q = R_{\text{n}} P^{-1} = \frac{1}{A} \frac{\left(1 + sT_1\right)\left(1 + sT_2\right)}{\left(1 + sT_{\text{n}}\right)\left(1 + sT\right)} \tag{34}$$

where T can be considered the time constant of the derivative action ($0.1T_{\text{D}} \le T \le 0.5T_{\text{D}}$). The regulator $\hat{C}'$ and the feedback term $\hat{H}$ must be always realizable. In the practice the *PID* regulator and the YOULA-parameter is always model-based, so

$$\hat{C}_{\text{PID}}\left(\hat{P}\right) = \hat{A}_{\text{PID}} \frac{\left(1 + s\hat{T}_1\right)\left(1 + s\hat{T}_2\right)}{s\hat{T}_1} \quad ; \quad \hat{A}_{\text{PID}} = \frac{\hat{T}_1}{\hat{A} T_{\text{n}}} \tag{35}$$

$$\hat{Q} = R_n \hat{P}^{-1} = \frac{1}{\hat{A}} \frac{\left(1+s\hat{T}_1\right)\left(1+s\hat{T}_2\right)}{1+sT_n} \tag{36}$$

The sheme of the observer based *PID* regulator is shown in Fig. 9, where a simple *PI* regulator

$$\hat{K}_l = A_l \frac{1+sT_l}{sT_l} \tag{37}$$

is applied in the observer-loop. Here T_l must be in the range of T, i.e., considerably smaller than T_1 and T_2.

Note that the frequency characteristic of $\hat{H}$ cannot be easily designed to reach a proper error suppression. For example, it is almost impossible to design a good realizable high cut filter in this architecture. The high frequency domain is always more interesting to speed up a control loop, so the target of the future research is how to select $\hat{K}_l$ for the desired shape of $\hat{H}$.

6. Simulation Examples

The simulation experiments were performed in using the observer based *PID* scheme shown in Fig. 9.

Example 1

The process parameters are: $T_1 = 20$, $T_2 = 10$ and $A = 1$. The model parameters are: $\hat{T}_1 = 25$, $\hat{T}_2 = 12$ and $\hat{A} = 1.2$. The purpose of the regulation is to speed up the basic step response by 4, i.e., $T_n = 5$ is selected in the first order R_n. In the observer loop a simple proportional regulator $\hat{K}_l = 0.01$ is applied. The ideal form of Q (33) was used. Figure 10 shows some step responses in the operation of the observer based *PID* regulator.

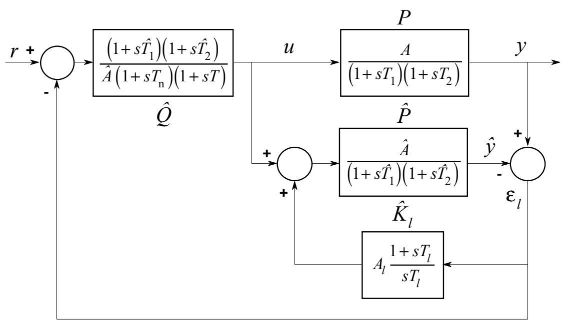
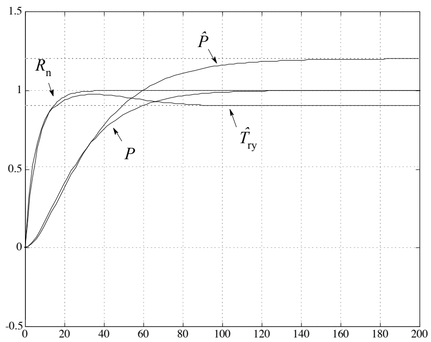
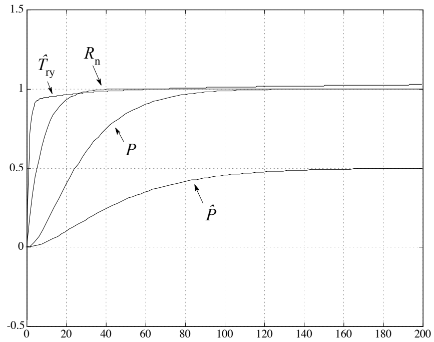


Fig. 9. An observer based *PID* regulator





Step responses using the observer based *PID* regulator

Fig. 10. **Fig. 11.**

It is easy to see that the $\hat{T}'_{ry}$ very well approximates R_n in the high frequencies (for small time values) in spite of the very bad model $\hat{P}$.

Example 2

The process parameters and the selected first order R_n are the same as in the previous example. The model parameters are: $\hat{T}_1 = 30$, $\hat{T}_2 = 20$ and $\hat{A} = 0.5$. In the observer loop a *PI* regulator (37) is applied with $A_I = 0.001$ and $T_I = 2$. The ideal form of Q (33) was used. Figure 11 shows some step responses in the operation of the observer based *PID* regulator.

It is easy to see that the $\hat{T}'_{ry}$ well approximates R_n in the high frequencies (for small time values) in spite of the very bad model $\hat{P}$.

7. Conclusions

The *TFR* of the classical methods are introduced to get a simple and useful tool to analyze and explain further behaviors, which are difficult to obtain using *SVR*. Using *TFR* it was shown, if the *SVR* used in the *SFO* scheme is model-based then the original (without observer) model error decreases by the sensitivity function of the observer feedback loop. This model error reducing capability gives the theoretical background of the success of practical model-based *SFO* applications.

Finally the *SFO* method was applied for the classical *IMC* structure, opening a new class of methods for open-loop stable processes. This new method combines the classical YOULA-parametrization based regulators with the *SFO* scheme. Using this new approach an observer based *PID* regulator was also introduced. This regulator works well even in case of large model errors as some simulations showed.

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