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Explicit mathematical models for behavioral science theories

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EXPLICIT MATHEMATICAL MODELS FOR BEHAVIORAL SCIENCE THEORIES

BY

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THESIS

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EXPLICIT MATHEMATICAL MODELS FOR
BEHAVIORAL SCIENCE THEORIES

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Physical science theories are often expressed as precise mathematical relationships from which researchers can derive empirically testable consequences. Mathematical modeling takes on additional tasks in the conduct of behavioral science research because behavioral science theories are typically expressed verbally, permitting a variety of mathematical representations for their conceptual relationships. This paper contains three different applications that (1) specify explicit mathematical models for various behavioral science theories, (2) verify the logical consistency of the formalized set of assumptions, and (3) examine the deductive content of the theories' models.

The first application, "Stability in the Prisoners' Dilemma," corrects some theorems by Robert Axelrod and others asserting the existence of "evolutionarily stable strategies" and extends this work. This is accomplished by: (1) formalizing the concept of strategy for iterated games and showing that the original proofs only establish "pair-distinct" stability, (2) showing that all strategies for playing the Iterated Prisoners' Dilemma (IPD) are dynamically unstable, (3) deriving a measure of the degree of instability of IPD Strategies, and (4) demonstrating that mutual cooperation can reduce instability, even though it does not eliminate it.

The second application, "Models for Long Cycles in War and Production," produces (1) a differential equation model consistent with Joshua Goldstein's long cycle theory that produces simple harmonic motion with fixed cycle times in contrast

to his observation that the duration of individual cycles varies from 30 to 70 years, and (2) a second model that corroborates three primary features of long cycles observed by Goldstein including variable cycle times.

The third application, "Measuring the Rate of War Outbreak," (1) develops a variable intensity Poisson process model, (2) uses this model to explicitly derive statistically precise predictive estimates of the rate of war outbreak, and (3) derives descriptive estimates of the rate of war outbreak that provide strong, unanticipated corroboration of Goldstein's long cycle dating scheme and of the "resource interpretation" of his long cycle theory developed in the second application.

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CHAPTER 1
INTRODUCTION

In this introduction I discuss the role of mathematics in representing and testing non-mathematical scientific theories. The three chapters that follow provide concrete illustrations for this discussion.

In some areas of scientific research, notably the physical sciences, theories are often expressed from the start in the form of precise mathematical relationships. From these explicit mathematical assumptions, the researcher can unambiguously deduce particular testable consequences. If these deductions are falsified by controlled experiments, then the researcher can conclude with certainty that at least one of the theory's assumptions did not hold in the experiment. Iterative scientific progress is facilitated because (1) the theories motivate (potentially) decisive experiments, and (2) the results of these experiments can motivate choices among, or modifications and refinements of, the theories.

In contrast behavioral science theories are often presented non-mathematically, typically employing common language. Even if mathematical constructs are used, the use of precise quantitative laws (e.g., x is a linear function of y) may have no empirical basis; instead, the hypothesized relationships should often be interpreted more generically (e.g., x increases with y). Thus the relationships among concepts in behavioral science theories are typically ambiguous, permitting a variety of mathematical representations for each relationship.[1] Consequently, the

[1]Some researchers have claimed that social and life science theories may (or even must) contain components that cannot, even in principle, be adequately captured mathematically. On the other hand, several methodologists (see, for example, Kaplan 1963, Carley 1981, and Seitz 1983) have espoused the use of explicit mathematical models to make the analysis and testing of social and life science theories more precise and productive. The chapters that follow provide evidence for this second approach in the form of successful explicit mathematical interpretations of three social and life science theories.

testable hypotheses and other conclusions advanced by behavioral science theorists often *cannot* be deduced from the assumptions of their theories: at best, these conclusions are consistent with the theoretical assumptions; failing this, the conclusions are false.

Because behavioral science theories are ambiguous, mathematical modeling acquires additional tasks in the conduct of behavioral science research. The first task is to establish the existence of at least one consistent logical representation of the theory in question, an *interpretive* model *for* the theory.[2] Failing this, mathematical methods can be used to demonstrate that a theory is logically inconsistent, i.e., that there are no models for the theory. In each of the three chapters that follow, I produce interpretive models for various behavioral science theories, verifying the logical consistency of (at least a subset of) their assumptions.

The next task is to examine the content of the theory's interpretive models. The goal is to produce, by introducing auxiliary assumptions, an interpretive model in which the theorist's conclusions can be deduced. Alternatively, mathematical methods can be used to show that a conclusion is false in any model for the theory, thus refuting the theory. In the first chapter to follow, "Stability in the Iterated Prisoners' Dilemma," I correct some published theorems asserting the existence of "evolutionarily stable strategies" (e.g., Axelrod 1984: 217). By formalizing the concept of strategy in the context of iterated games, I show that the original proofs only establish "pair-distinct" stability. I then show that all strategies for playing the Iterated Prisoners' Dilemma (IPD) are dynamically unstable. I use this analysis to derive a measure of the degree of instability of IPD strategies. Finally, I demonstrate

[2]In terms of Kaplan (1963: 267), "Given a formal system, a model is constituted by any interpretation of the system which makes its postulates true. This is the sense in which the term is used by logicians, like Tarski and Suppes – 'a non-linguistic entity in which a theory is satisfied'; ... . We may call them *interpretive models* to make their origin explicit. The system being interpreted is sometimes also called a model, especially when the interpretation is another linguistic system; we may speak here of *formal models*. An interpretive model is thus a model *for* a theory, while a formal model is a model *of* a theory."

that mutual cooperation can reduce instability, even though it does not eliminate it.

Short of such a global refutation, producing any number of interpretive models in which the theorist's conclusions are false or cannot be deduced does not refute the theory. It merely confirms the assertion that behavioral science theories are often ambiguous, that they are deductively inadequate or logically *deficient*. In the second chapter to follow, "Models for Long Cycles in War and Production," I produce a differential equation model consistent with Joshua Goldstein's long cycle theory (Goldstein, 1987: 590, 593) that produces simple harmonic motion with fixed cycle times, in contrast to his observation (and theoretical goal) that the duration of individual long cycles has varied from as low as 30 years to as high as 70 years. Thus Goldstein's theory is empirically deficient.

On the other hand, a single successful interpretive model demonstrates that the theory can be given sufficient explanatory power to achieve the theorist's goals. In "Models for Long Cycles in War and Production," I produce a second model that corroborates three primary features of long cycles observed by Goldstein; in particular cycle times can vary due to perturbation of the variables in this model.

When constructing mathematical models to establish potential theoretical sufficiency, the emphasis is on tractability rather than predictive accuracy: an evocative caricature is preferred to a detailed replication of all empirical observations. Yet this does not necessarily render mathematical models constructed for this purpose empirically impotent. The long cycle model motivates statistics and indicators (empirical data hypothesized to correspond to the model's variables) that may be better able to substantiate the claims of long cycle researchers. Furthermore, in the final chapter to follow, "Measuring the Rate of War Outbreak," I develop a variable intensity Poisson process model that is clearly a caricature, positing that the rate of war outbreak takes on one of only two different levels. Despite this "over-simplification" I use the model to explicitly derive statistically precise predictive estimates of the rate of war outbreak. I also derive descriptive estimates of the rate of war outbreak that provide strong, unanticipated corroboration of Goldstein's long cycle dating scheme (1987: 576-7), and of my "resource model" interpretation of his long cycle theory.

CHAPTER 2

STABILITY IN THE ITERATED PRISONERS' DILEMMA

Introduction

In game theory, a player's *strategy* is a specification of the choice that that player will make in any situation arising in the game. For a given game, we would like to determine the strategy (or set of strategies) that we can expect players to actually use. This has led game theorists to propose various solution concepts. In classical game theory, the solution of a game is determined by the assumption that the players each attempt to maximize their own payoff. In contrast, theoretical biologists studying the population dynamics that arise when the players' survival rates are determined by their game payoffs have introduced the concept of *evolutionary stability* as an alternative solution of a game. A strategy is an *evolutionarily stable strategy* (ESS) if, when used by nearly all of the players in a population, players using any other strategy will receive a lower payoff, on the average, than those players using the ESS. Stability of the ecosystem is achieved because players using other strategies will be less likely to survive.

Axelrod (1984) has adapted this solution concept to the study of the Iterated Prisoners' Dilemma (IPD), introducing the concept of *collective stability*. A strategy A has collective stability if the expected payoff for a player using any other strategy B against A does not exceed the expected payoff for using strategy A against A. This solution concept is easier to satisfy than evolutionary stability: any strategy that has evolutionary stability also has collective stability, but there are strategies that have collective stability but do not have evolutionary stability. Axelrod proved that ALL–D, the strategy of always defecting, always has collective stability. He also showed that, if the value of future payoffs (relative to current payoffs) is sufficiently high, then TIT

FOR TAT, the strategy of beginning with cooperation and reciprocating the other player's previous move, also has collective stability.

Axelrod further asserts that both of these strategies have evolutionary stability (1984: 217). This assertion is incorrect. In fact, I show that *no* strategy for playing IPD can have evolutionary stability.[1] In particular, I show that cooperation is initially viable *without* clustering (compare with Axelrod, 1984: 21, 63). I obtain these results by demonstrating that the behavior (pattern of choices) produced by a given strategy can be mimicked by other strategies. Thus any population saturated with players using one strategy can be invaded by players using a mimicking strategy. However, even after such an invasion, the observable behavior in the population will remain unchanged. Could this behavior then be stable, even though the strategy is not? No: I show that the behavior produced by any strategy is unstable. Based on this analysis, I develop a measure of the *persistence* of IPD strategies; this measure varies inversely with the degree of instability. I conclude by showing that mutual cooperation can enhance the persistence of IPD strategies. Thus, even though we cannot always expect to see IPD players use mutually cooperative strategies, mutually cooperative ecologies can be more persistent than uncooperative ecologies.

Strategies, Payoffs, and Dynamics of the Iterated Prisoners' Dilemma

The Prisoners' Dilemma game is a two-player game in which each player can choose to either cooperate (*C*) or defect (*D*). The matrix of payoffs for the four possible combinations of choices appears in figure 1.1. In the Iterated Prisoners' Dilemma, the two players interact in a sequence of Prisoners' Dilemma games.

Definition 1: Strategy. A strategy for IPD is a decision rule, $d : \mathbb{M} \rightarrow \{C,D\}$, used by a player to choose either *C* or *D* (the two elements in the range of *d*) when this player meets another player. $\mathbb{M}$, the domain of *d*, is the union of all possible

[1] Boyd and Lorberbaum (1987) showed that no pure strategy is an ESS for IPD. This paper presents the stronger result from Pudaite (1985) that no strategy, pure or mixed, is an ESS, in order to

sequences of n moves, $\mathbb{M}_n$; elements of $\mathbb{M}_n$ take the form $\mathbf{M} \Leftarrow \{(x_i,y_i)\}_{i=1}^{n}$,[2] representing the n pairs of choices made by the two players in their first n encounters (the x_i and y_i take on values of C and D). If the players have not previously met, then n=0 and $\mathbb{M}_0 = \{\mathbf{M}_0\}$, where $\mathbf{M}_0$ is an empty sequence (denoted by $\varnothing$). A *mixed* strategy is a strategy for which $d(\mathbf{M})$ is a random variable for at least one sequence of moves $\mathbf{M} \in \mathbb{M}$.

Figure 1.1b displays a list of all of the strategies discussed in this chapter.

Let $M_n(X,Y)$ be the sequence of moves occurring in the first n encounters between players using strategies X and Y. $M_n(X,Y)$ is a random variable with sample space $\mathbb{M}_n$. For a sequence of n moves, $\mathbf{M} \in \mathbb{M}_n$, $\Pr[M_n(X,Y)=\mathbf{M}]$ is the probability of this sequence occurring in the first n encounters between a player using strategy X and a player using strategy Y. $\Pr[M_n(X,Y)=\mathbf{M}]$ can be computed iteratively as follows. Let $\mathbf{M} \Leftarrow \{(x_i,y_i)\}_{i=1}^{n}$, and for each $j \le n$, let $\mathbf{M}_j \Leftarrow \{(x_i,y_i)\}_{i=1}^{j}$ (in particular, $\mathbf{M} = \mathbf{M}_n$) and $\mathbf{M}_j^T \Leftarrow \{(y_i,x_i)\}_{i=1}^{j}$. Then

$$\Pr[M_0(X,Y)=\varnothing] = 1 \qquad [1.0a]$$

$$\Pr[M_j(X,Y)=\mathbf{M}_j] = \Pr[X(\mathbf{M}_{j-1})=x_j]\,\Pr[Y(\mathbf{M}_{j-1}^T)=y_j]\,\Pr[M_{j-1}(X,Y)=\mathbf{M}_{j-1}] \text{ for } j = 1,2,\ldots,n \qquad [1.0b]$$

Figure 1.1c shows the probability of various sequences occurring for $\Sigma_{(\omega)}$, a set comprised of the last two strategies appearing in Figure 1.1b.

In games involving a single move, a player's strategy is identified by the player's choice for that move. One of the critical differences between single move games and iterated games is that a player's sequences of choices in one iterated game does not identify the player's strategy. A strategy for an iterated game can be identified only by determining the player's response to *every* possible sequence of

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introduce a useful ecological framework for analyzing IPD.

[2]Where "$\Leftarrow$" denotes an assignment by definition; "=" denotes an assertion of equality.

moves. But in a given ecology, players may only be confronted by a small subset of the set of all move sequences. This leads to the following criterion for distinguishing among strategies.

Definition 2: Identifying strategies in an ecology. Let Σ be a given set of strategies. Then two strategies $X, X' \in \Sigma$ are *Σ-equivalent* if and only if for all $Y \in \Sigma$, $n \geq 0$, and $\mathbf{M} \in \mathbb{M}_n$ such that $\Pr[M_n(X,Y)=\mathbf{M}] > 0$, $\Pr[M_n(X,Y)=\mathbf{M}] = \Pr[M_n(X',Y)=\mathbf{M}]$. If X and X' are not Σ-equivalent, then they are *Σ-distinct*. In this case, there exists at least one *discriminator* of X and X', $Y \in \Sigma$, for which $\Pr[M_n(X,Y)=\mathbf{M}] \neq \Pr[M_n(X',Y)=\mathbf{M}]$ for some $n \geq 0$ and $\mathbf{M} \in \mathbb{M}_n$ with $\Pr[M_n(X,Y)=\mathbf{M}] > 0$.

Definition 3: Payoffs. Expected payoffs are computed so that the payoff of each move is worth some fraction w of the previous move (w is called the *discount parameter*). The expected payoff to a player using strategy A from encounters with a player using strategy B,

$$E[A \mid B] \Leftarrow (1-w) \sum_{n=1}^{\infty} w^{n-1} \sum_{\mathbf{M} \in \mathbb{M}_n} \Pr[M_n(A,B)=\mathbf{M}]\, V(a_n,b_n) \qquad [1.1]$$

where a_n and b_n are the players' respective choices for move n (i.e., $\mathbf{M} \Leftarrow \{(a_i,b_i)\}_{i=1}^{n}$), and $V(a,b)$ is the payoff to a player choosing a in a Prisoners' Dilemma game when the other player chooses b; see figure 1.1a. The *payoff matrix* for an ecology $\Sigma \Leftarrow \{X_i\}_{i=1}^{N}$ is an $N \times N$ matrix $\mathbf{E}(\Sigma) \Leftarrow \langle E[X_i \mid X_j] \rangle$.

Assume that the players in an IPD ecosystem cannot interact preferentially with players using one particular strategy rather than with players using other strategies. Then the dynamics of this ecology are determined by Σ, the strategies present in the ecology, and the number of players using each strategy. Without loss of generality, the number of players using a given strategy will be measured by their proportion in the entire population. Thus the expected payoff to players using strategy X_i in an ecology $\Sigma \Leftarrow \{X_j\}_{j=1}^{N}$ is

$$E[X_i] \Leftarrow \sum_{j=1}^{N} E[X_i \mid X_j] \; |X_j| = (\mathbf{E}(\Sigma)\,\mathbf{x})_i \qquad [2.1]$$

where $|X|$ is the proportion of players using strategy X, and $\mathbf{x} = \langle |X_1|, \ldots, |X_N| \rangle$. The expected payoff for the entire ecology is

$$E \Leftarrow \sum_{i=1}^{N} E[X_i] \; |X_i| = \mathbf{x}\,(\mathbf{E}(\Sigma)\,\mathbf{x}) \qquad [2.3]$$

Note that $E[X]$ and E are implicit functions of $\mathbf{x}$. Figure 1.1c shows all of the expected payoffs for Σ_ω.

Assume that the expected payoffs and proportions of each strategy in the ecology contribute linearly to changes in the strategy's population. Let $\Delta(\Sigma)$ represent the system of differential equations formed by substituting each $X \in \Sigma$ into the following equation:

$$\frac{d|X|}{dt} = (E[X] - E)\,|X|; \; |X| \in [0,1] \qquad [2.2]$$

Note that because of the dynamics, E, $E[X]$ and $|X|$ are all implicit functions of time. This system of cubic differential equations has "been suggested by Taylor & Jonker (1978) and Zeeman [1980]" (Maynard Smith, 1982: 183). It reflects the assumptions that expected payoffs to and proportions of players using a given strategy in the ecology contribute linearly to changes in the proportion of players using that strategy. Because

$$\sum_{X \in \Sigma} |X| = 1 \qquad [2.4]$$

this system can be reduced to N-1 equations where $N = N(\Sigma)$, the number of Σ-distinct strategies in Σ. The domain of this system of equations is then

$$\sum_{i=1}^{N-1} |X_i| \in [0,1]; \ |X_i| \geq 0 \text{ for } i = 1 \text{ to } N\text{-}1 \quad [2.5]$$

Game Theoretic Stability and Dynamic Stability

Evolutionary stability was defined verbally above. Here is a formal definition.

Definition 4: Evolutionary stability (ES). A strategy X has ES if for any finite set of strategies Σ such that $X \in \Sigma$, then for all $Y \in \Sigma$ with X and Y Σ-distinct, X *repels invasion by* Y, i.e., either (1) $E[X|X] > E[Y|X]$, or (2) if $E[X|X] = E[Y|X]$ then $E[X|Y] > E[Y|Y]$.

The criteria for evolutionary stability in this definition have been relaxed compared to standard versions[3] in two ways: (1) the criteria are checked only for finite sets of strategies (not all possible sets of strategies); (2) the criteria are checked only against Σ-distinct strategies, not all other decision rules. Yet even with these relaxations, there are still no strategies for IPD that have ES. Only by restricting consideration to pairs of strategies can we obtain a non-vacuous version of this solution concept for IPD.

Definition 5: Pair-distinct stability (PS). If Y is $\{X,Y\}$-distinct from X, then Y is *pair-distinct* from X; otherwise Y is *pair-equivalent* to X. A strategy X has PS if for any Y pair-distinct from X, X repels invasion by Y.

Axelrod's assertion (1984: 217) that

> "All propositions in the text remain true if 'evolutionary stability' is substituted for 'collective stability' with the exception of the Characterization Theorem of Appendix B, where the characterization is necessary but no longer sufficient"

is true provided that we replace "evolutionary stability" with "pair-distinct stability." The following theorem shows that an ecology saturated by a strategy is

[3]See, for example, Maynard Smith (1982) p. 14.

asymptotically stable precisely if the strategy can repel "invasion" by strategies from which it is pair-distinct.

Theorem 1: Equivalence of PS and asymptotic stability. If strategy X has PS and $\Sigma = \{X,Y\}$ with Y pair-distinct from X, then $|X| = 1$ is an asymptotically stable equilibrium point of $\Delta(\Sigma)$. Conversely, if for any $\Sigma = \{X,Y\}$ with Y pair-distinct from X, $|X| = 1$ is an asymptotically stable equilibrium point of $\Delta(\Sigma)$, then X has PS.

Proof. From equation 2.2, we obtain the differential equation

$$\frac{dx}{dt} = x\,(1\text{-}x)\left(x\,(E[X|X] - E[Y|X]) + (1\text{-}x)\,(E[X|Y] - E[Y|Y])\right) \qquad [3]$$

where $x = |X|$.

Part 1. PS implies asymptotic stability. If $E[X|X] > E[Y|X]$, then there exists $\delta > 0$ such that for all $x \in (1\text{-}\delta, 1)$, $\frac{dx}{dt} > 0$. If $E[X|X] = E[Y|X]$ and $E[X|Y] > E[Y|Y]$, then for all $x \in (0,1)$, $\frac{dx}{dt} > 0$.

Part 2. Asymptotic stability implies PS. In order for $|X| = 1$ to be an asymptotically stable equilibrium point, we must have $E[X|X] \geq E[Y|X]$. If $E[X|X] = E[Y|X]$ then we must have $E[X|Y] > E[Y|Y]$. #

No Strategies Have Evolutionary Stability

In order to show that no strategies have evolutionary stability, we need the following adaptation of Axelrod's Characterization Theorem.

Theorem 2: Necessity of defection for PS. Let

$$V_n[Y|X] \Leftarrow (1\text{-}w) \sum_{i=1}^{n\text{-}1} w^{i\text{-}1}\, V(y_i, x_i) \qquad [4.1]$$

be Y's discounted cumulative score from the moves before move n (here x_i and y_i are X and Y's choices on move i, respectively). Let X be pair-distinct from Y. Then X has PS only if X defects on the next move whenever

$$V_n[Y|X] > \mathrm{E}[X|X] - w^{n-1}\left(T + \frac{w}{1-w}P\right) \qquad [4.2]$$

Proof. See Axelrod (1981: 313-4). #

Theorem 3: No IPD strategy has ES. Given any strategy X, there exists a set of strategies Σ containing X' Σ-distinct from X, such that X cannot repel invasion by X'.

Sketch of proof. Since PS is a necessary condition for ES, we can ignore all strategies that do not have PS. Then for any strategy X with PS, we produce a set $\Sigma = \{X,X',Y\}$ such that X and X' are Σ-distinct but not pair-distinct.

Remark. The rest of this paper focuses on the dynamic systems produced by sets of strategies of the form just described. Such a set represents a *pair-equivalent invasion ecology of X* (or just *invasion ecology*, for short): X' can invade an ecology saturated by X because they are pair-equivalent. Thus, even though $|X| = 1$ is a stable equilibrium point of $\Delta(\Sigma)$, it is not asymptotically stable. In an invasion ecology, X' is the *pair-equivalent invader* and Y is the discriminator (recall definition 2).

Proof. Let X be an strategy with PS. Let X' be pair-equivalent to X. We seek a strategy Y, and a *discrimination sequence* of n moves $\mathbf{M}^*$ such that $\Pr[M_n(X,X)=\mathbf{M}^*] = 0$ but $\Pr[M_n(X,Y)=\mathbf{M}^*] > 0$. The proof can then be completed by choosing $X'(\mathbf{M}^*) \neq X(\mathbf{M}^*)$.[4]

In the following, for all $j \geq 0$, let $\mathbf{M}_j \Leftarrow \{(a_i,b_i)\}_{i=1}^{j}$ and $\mathbf{M}_j^T \Leftarrow \{(b_i,a_i)\}_{i=1}^{j}$. Now if for any k such that $\Pr[M_k(X,X)=\mathbf{M}_k] > 0$, $\Pr[X(\mathbf{M}_k)=C] \in \{0,1\}$ (i.e., X makes a deterministic choice), then (1) set $Y(\mathbf{M}_j) \Leftarrow X(\mathbf{M}_j)$ and $Y(\mathbf{M}_j^T) \Leftarrow X(\mathbf{M}_j^T)$ for all $j < k$, (2) set $Y(\mathbf{M}_k) \neq X(\mathbf{M}_k)$ by letting $Y(\mathbf{M}_k) \Leftarrow C$ if $\Pr[X(\mathbf{M}_k)=D] > 0$, else $Y(\mathbf{M}_k) \Leftarrow D$, and (3) set $n \Leftarrow k+1$, $\mathbf{M}^* \Leftarrow \mathbf{M}_{k+1}^T$ with $a_n \Leftarrow Y(\mathbf{M}_k)$ and $b_n \Leftarrow X(\mathbf{M}_k^T)$. This choice of Y, n, and $\mathbf{M}^*$ fulfills the conditions required above because $\Pr[M_j(X,Y)=\mathbf{M}_j] = \Pr[M_j(Y;X)=\mathbf{M}_j] = \Pr[M_j(X,X)=\mathbf{M}_j] > 0$ for all $j < n$, but

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[4] For mixed strategies, read $X'(\mathbf{M}^*) \Leftrightarrow X(\mathbf{M}^*)$ as $\Pr[X'(\mathbf{M}^*)=C] \Leftrightarrow \Pr[X(\mathbf{M}^*)=C]$, where "$\Leftrightarrow$" represents either equality ("=") or inequality ("$\neq$").

$$\Pr[M_n(X,X)=\mathbf{M}^*]$$
$$= \Pr[X(\mathbf{M}_k^T)=b_n] \Pr[X(\mathbf{M}_k)=a_n] \Pr[M_k(X;X)=\mathbf{M}_k] = 0 \qquad [4.3]$$

because $\Pr[X(\mathbf{M}_k)=a_n] = 0$, while

$$\Pr[M_n(X,Y)=\mathbf{M}^*]$$
$$= \Pr[X(\mathbf{M}_k^T)=b_n] \Pr[Y(\mathbf{M}_k)=a_n] \Pr[M_k(Y;X)=\mathbf{M}_k] > 0 \qquad [4.4]$$

because $\Pr[Y(\mathbf{M}_k)=a_n] > 0$. Thus the only way we can fail to find Y and $\mathbf{M}^*$ as required is if X is a *completely mixed strategy*, i.e., for all $\mathbf{M} \in \mathbb{M}$, $\Pr[X(\mathbf{M})=C] \in (0,1)$. But by theorem 2, there are sequences for which X *must* defect in order to have PS. Thus X cannot be a completely mixed strategy, and we can always find X' and Y such that X' is pair-equivalent to, but Σ-distinct from X. #

The relationship between strategy and behavior in iterated games resembles the relationship between theory and observation in scientific research. Just as a given sequence of moves can be produced by more than one pair of strategies, a given experiment may be consistent with several theories. Nonetheless, if two theories are not logically equivalent, then there must be a "critical" experiment for which the theories make different predictions. Similarly, if two strategies are not equivalent, then there must be some sequence of moves that can discriminate between them, at least probabilistically. However, in the Iterated Prisoners' Dilemma, this discrimination can only be achieved by giving up strategic stability.

The Initial Viability of Cooperation

Theorem 3 shows that for any strategy X, there is a Σ-distinct strategy X' that has no selective disadvantage in an ecology saturated by X. This theorem plays an important part in establishing that cooperation can be initially viable even without *clustering* (preferential interaction) of the cooperative players. Consider the dynamic system $\Delta(\Sigma_1)$ where $\Sigma_1 = \{X, X', Y\} \Leftarrow$ {ALL-D,TFT-D,TFT-C}, where TFT-C is TIT FOR TAT and TFT-D is the strategy of starting with defection, then reciprocating the other player's previous move. In this system

$$E[X] = P\,(x+x') + \big(T\,(1\text{-}w) + P\,w\big)\,y \qquad [5.1]$$

$$E[X'] = P\,(x+x') + \frac{T + S\,w}{1 + w}\,y \qquad [5.2]$$

$$E[Y] = \big(S\,(1\text{-}w) + P\,w\big)\,x + \frac{S + T\,w}{1 + w}\,x' + R\,y \qquad [5.3]$$

where $x = |X|$, $x' = |X'|$, and $y = |Y| = 1 - x - x'$. Figure 1.2 displays a phase portrait of $\Delta(\Sigma_1)$ using the values $w = 4/5$ and $(S,P,R,T) = (0,1,3,5)$. The corners of the (equilateral) triangle $XX'Y$ represent saturation of the ecology by the respective strategy labelled at that corner. Arrows superimposed on the trajectories show the direction of time evolution of the system. Notice that no arrows appear along the base of the triangle, the line segment XX' representing the region where none of the players use TFT-C ($y = 0$). This is because every point on this line is a critical point, this in turn is because X and X' are pair-equivalent.

The existence of these non-isolated equilibria complicates the mathematical analysis (see also Maynard Smith 1982:189), but does not make analysis intractable.[5] The point S_c is a saddle point; all of the points on the line segment S_cX' excluding S_c are unstable equilibria. The points on the line segment XS_c (again, excluding S_c) are stable, but not asymptotically stable, equilibria. Y (saturation by TFT-C) is the only asymptotically stable equilibrium point of $\Delta(\Sigma_1)$. Because of this, the equilibrium

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[5]Zeeman (1980: 488-9) gives a conjectured classification for games with three strategies, according to topological equivalence of their flows. Although the classification is intended to be exhaustive, $\Delta(\Sigma_1)$ does not appear in his list because he limits his analysis to *stable* games, games for which "perturbations of the pay-off matrix do not alter the qualitative nature of the flow," reasoning that "a model can only be an approximation of the reality, and so a perturbation may well be as good a model. Therefore only robust properties are reliable for prediction and testing of the model" (1981: 250). Because single encounter (i.e., non-iterated) games that produce non-isolated equilibria are not stable, he omits the corresponding flows from his list. However, in ecologies of iterated game strategies, non-isolated equilibria may not be eliminated by perturbations of the single game payoff matrix if, as in Σ_1, some of the strategies are pair-equivalent. Thus non-isolated equilibria can be stable features of iterated games, even though they are never stable features of single encounter games.

segment XS_c can be considered *metastable*: given a "small" perturbation[6] from a point on XS_c, the system may return to XS_c (although it is unlikely that the system will return to the original point). However, any sufficiently large perturbation will result in a transition to saturation by TFT-C. Thus, in this example, cooperation is initially viable and perhaps even inevitable.

Why is cooperation initially viable even without clustering? One conceptual explanation is that even if we do not observe cooperation among the players in a particular environment, we cannot conclude that none of the players are willing to cooperate. Some of the players may be waiting for others to take the initial risk of cooperation. If at any time there are enough of these "risk-averse" cooperators, then cooperation will be able to take hold in the environment.

Discriminative Perturbation

The previous section illustrated a method for invading a system saturated by strategies that do not initiate cooperation. But theorem 3 shows that even if saturated by strategies that initiate cooperation, any system can still be invaded. Does this mean that pure defection is also initially viable? In this section, I develop an example which motivates a useful technique for testing initial viability of a strategy. The example suggests that defection may not be initially viable. This hypothesis is confirmed later in the paper.

Consider a second invasion ecology $\Sigma_2 = \{X, X', Y\} \Leftarrow$ {TFT-C,ALL-C,BULLY}, where ALL-C is the strategy of always cooperating and BULLY is the strategy of defecting until the other player defects, and always cooperating after that. In this system

$$E[X] = R\,(x+x') + \left((S + P\,w + T\,w^2)\,(1-w) + R\,w^3\right) y \qquad [6.1]$$

$$E[X'] = R\,(x+x') + S\,y \qquad [6.2]$$

[6]Perturbations might be produced by, for example, mutation (in biological applications) or experimentation (in social applications).

$$E[Y] = \left((T + P\,w + S\,w^2)\,(1\text{-}w) + R\,w^3\right) x + T\,x' + \left(P\,(1\text{-}w) + R\,w\right) y \qquad [6.3]$$

Figure 1.3 displays a phase portrait of $\Delta(\Sigma_2)$ using the values $w = 4/5$ and $(S,P,R,T) = (0,1,3,5)$.

Both $\Delta(\Sigma_1)$ and $\Delta(\Sigma_2)$ display stable equilibria for corresponding line segments XS_c. In particular, if we apply *discriminative perturbation*, raising the proportion of discriminators in the system from y=0 to y=δ>0, then for δ sufficiently small, both systems will return to XS_c. But in $\Delta(\Sigma_1)$ the trajectories near XS_c lead *away* from X; thus discriminative perturbation followed by *selection* (differential survival rates based on payoffs) will tend to reduce the proportion of players using ALL-D. On the other hand, in $\Delta(\Sigma_2)$ the corresponding trajectories lead *toward* X; discriminative perturbation of $\Delta(\Sigma_2)$ followed by selection eventually produces an increase in the proportion of players using TFT-C.

Applying discriminative perturbation continuously over time (instead of one time only) produces an even more dramatic effect. For $\varepsilon > 0$, let $\Delta(\Sigma;\varepsilon)$ be the system of differential equations (perturbed from equation 2.2) produced by introducing discriminators into an invasion ecology at a small but continuous rate:[7]

$$\frac{dx}{dt} = (E[X] - E - \varepsilon)\,x \qquad [7.1]$$

$$\frac{dx'}{dt} = (E[X'] - E - \varepsilon)\,x' \qquad [7.2]$$

$$\frac{dy}{dt} = (E[Y] - E - \varepsilon)\,y + \varepsilon \qquad [7.3]$$

Figures 1.4 and 1.5 display phase portraits of $\Delta(\Sigma_1;\varepsilon)$ and $\Delta(\Sigma_2;\varepsilon)$, respectively, with $\varepsilon \Leftarrow .001$. Because of discriminative perturbation, there are no longer any equilibria on the line XX' in either $\Delta(\Sigma_1;\varepsilon)$ or $\Delta(\Sigma_2;\varepsilon)$; furthermore, the remaining equilibria are all isolated.

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[7]For symmetry and generality, we could add perturbation terms corresponding to introduction of the two other strategies. These effects would become important for small levels of x or x'. However, since we are primarily interested in the behavior of the system for small y, this would just introduce unnecessary complication.

In $\Delta(\Sigma_1;\varepsilon)$, for ε sufficiently small, S_c is a saddle point while S_e is an unstable equilibrium (numerical values appear in figure 1.4). The only stable point in $\Delta(\Sigma_1;\varepsilon)$ corresponds to saturation by TFT-C. On the other hand, in $\Delta(\Sigma_2;\varepsilon)$ S_c is a stable equilibrium, while S_e is an unstable equilibrium (numerical values appear in figure 1.5).

Stability and Persistence of Behavior

If we let ε approach 0, then in both $\Delta(\Sigma_1;\varepsilon)$ and $\Delta(\Sigma_2;\varepsilon)$, S_c approaches the point X (corresponding to saturation of the ecology by strategy X). However, in $\Delta(\Sigma_1;\varepsilon)$, S_c remains an unstable (saddle) point, while in $\Delta(\Sigma_2;\varepsilon)$, S_c remains a stable equilibrium point. Is S_c a stable equilibrium approaching X for any invasion ecology with X = TFT-C? The following example shows this conjecture is false. The example motivates a new solution concept that is formally stronger than PS but weaker than ES.

Consider a third invasion ecology $\Sigma_3 = \{X,X',Y\}$ = {TFT-C,PR,1D}, where PR is the strategy of permanent retaliation (cooperate until the other player defects, then always defect after that; see Axelrod 1984, 15-6), and 1D is the strategy of defecting on the first move, then always cooperating after that. In this system

$$E[X] = R\,(x+x') + \big((S + T\,w)\,(1\text{-}w) + R\,w^2\big)\,y \qquad [8.1]$$

$$E[X'] = R\,(x+x') + \big(S\,(1\text{-}w) + T\,w\big)\,y \qquad [8.2]$$

$$E[Y] = \big((T + S\,w)\,(1\text{-}w) + R\,w^2\big)\,x + \big(T\,(1\text{-}w) + S\,w\big)\,x' + \big(P\,(1\text{-}w) + R\,w\big)\,y \qquad [8.3]$$

Figure 1.6 displays a phase portrait of $\Delta(\Sigma_3;\varepsilon)$ for the same parameter values used previously. For this invasion ecology, the equilibrium point S_c (representing near saturation of the ecology by TFT-C) is not asymptotically stable. The phase portrait of $\Delta(\Sigma_3;\varepsilon)$ differs from those of both $\Delta(\Sigma_1;\varepsilon)$ and $\Delta(\Sigma_2;\varepsilon)$ in two ways. First of all, the equilibrium point S_e does not appear. Second, there is a new, asymptotically stable equilibrium point S_C' on the line $X'Y$. If we let ε approach 0, then S_C' approaches the point X' (corresponding to saturation of the ecology by PR).

But note that if all of the players use either PR or TFT-C, then their choices produce mutual cooperation. Thus, even though TFT-C is not asymptotically stable under discriminative perturbation in this invasion ecology, as ε approaches 0, the *behavior* of the players at the only stable equilibrium point of this system becomes indistinguishable from an ecology saturated by TFT-C. This motivates the following solution concept for iterated games.

Definition 6: *Evolutionary stability of behavior* (ESB). A strategy produces ESB if it can produce an asymptotically stable equilibrium for which the proportion of discriminators is arbitrarily small. Formally, a strategy X produces ESB if and only if (1) X has PS, and (2) for any invasion ecology $\Sigma = \{X, X', Y\}$ and some $\delta > 0$, there exists a sequence of sets of equilibrium points $C(\varepsilon)$ such that (a) for all $\varepsilon \in (0,\delta)$, the entire set $C(\varepsilon)$ is asymptotically stable in $\Delta(\Sigma;\varepsilon)$, and (b) $\lim_{\varepsilon \to 0^+} \sup_{(x,x',y) \in C(\varepsilon)} y = 0$.[8]

The conditions in this definition guarantee an asymptotically stable equilibrium for which the predominant behavior is identical to that observed in an ecology saturated by players using X, even though the ecology may contain a substantial portion of players using X'. I characterize this solution concept in game-theoretic terms below. Unfortunately, this solution concept is vacuous for IPD (although it may not be vacuous for other games). In the long run, all IPD strategies, and even the behavior they produce, are transient. However, proving the following characterization of ESB motivates a way to measure how long we can expect a behavioral equilibrium to survive.

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[8]The definition incorporates sets of equilibrium points because, as we will see, when $E[X \mid Y] = E[X' \mid Y]$, none of the equilibria with small y are isolated. However, taking all of these equilibria together, the set may be asymptotically stable. Since the proportion of discriminators at any point in this entire set can be made arbitrarily small, the system still displays stability of the behavior produced by X.

Characterization of ESB in Terms of Expected Payoffs

Theorem 4: Characterization of ESB. A strategy X produces ESB if and only if (1) X has PS, and (2) for any invasion ecology $\Sigma = \{X, X', Y\}$, either (a) $E[X \mid Y] > E[X' \mid Y]$, or (b) X' repels invasion by Y.

To prove theorem 4, we first need to determine the location of all equilibria with small y. The theorem can then be proved by evaluating eigenvalues of $\Delta(\Sigma;\varepsilon)$ for these equilibria. The cases in which the largest eigenvalue is 0 require some additional analysis. These components of the proof are provided by the following lemmas.

Lemma 4a: Location of equilibria. Given $\varepsilon > 0$ sufficiently small, critical points $\{(x(\varepsilon), x'(\varepsilon), y(\varepsilon)) : \lim_{\varepsilon \to 0^+} y(\varepsilon) = 0\}$ occur if and only if the following conditions are satisfied.

1. When $E[X \mid X] > E[Y \mid X']$, there is a critical point at

$$x = 0, x' = 1\text{-}y, y = \frac{\varepsilon}{E[X \mid X] - E[Y \mid X']} + O(\varepsilon^2) \qquad [11.1]$$

2. When $E[X \mid X] = E[Y \mid X']$ and $E[X' \mid Y] > E[Y \mid Y]$, there is a critical point at

$$x = 0, x' = 1\text{-}y, y = \sqrt{\frac{\varepsilon}{E[X' \mid Y] - E[Y \mid Y]}} \qquad [11.2]$$

3. When $E[X \mid X] > E[Y \mid X]$, there is a critical point at

$$x = 1\text{-}y, x' = 0, y = \frac{\varepsilon}{E[X \mid X] - E[Y \mid X]} + O(\varepsilon^2) \qquad [11.3]$$

4. When $E[X \mid X] = E[Y \mid X]$ and $E[X \mid Y] > E[Y \mid Y]$, there is a critical point at

$$x = 1\text{-}y, x' = 0, y = \sqrt{\frac{\varepsilon}{E[X \mid Y] - E[Y \mid Y]}} \qquad [11.4]$$

5. When $E[X \mid Y] = E[X' \mid Y]$, every point of the conic section

$$f(x,y) \Leftarrow \alpha y^2 + (\beta - \gamma x) y + \varepsilon = 0 \qquad [11.5]$$

(where $\alpha \Leftarrow E[X \mid X] - E[X \mid Y] - E[Y \mid X'] + E[Y \mid Y]$, $\beta \Leftarrow -E[X \mid X] + E[Y \mid X']$, and γ

$\Leftarrow - E[Y|X] + E[Y|X'])$ intersecting the domain of $\Delta(\Sigma;\varepsilon)$, $\mathbb{D} \Leftarrow \{(x,y){:}x\in[0,1\text{-}y], y\in[0,1]\}$, is a critical point. Critical points such that $\lim_{\varepsilon\to 0^+} y(\varepsilon) = 0$ occur on this "conic segment" only if at least one of conditions 1 through 4 occurs. Let $C(\varepsilon)$ be the maximal connected segment of critical points containing this point. Then $\lim_{\varepsilon\to 0^+} \sup_{(x,x',y)\in C(\varepsilon)} y = 0$ if and only if both (a) condition 1 or 2 holds and (b) condition 3 or 4 holds.

Proof. See Appendix. #

Lemma 4b: Eigenvalues at equilibria. The equilibria listed in lemma 4a have, respectively, the following eigenvalues.

1. When $E[X|X] > E[Y|X']$, the critical point given in equation 11.1 has eigenvalues:

$$\lambda_1 = (E[X|Y] - E[X'|Y])\, y \qquad [12.1.1]$$

$$\lambda_2 = -(E[X|X] - E[Y|X']) + O(\varepsilon) < 0 \qquad [12.1.2]$$

2. When $E[X|X] = E[Y|X']$ and $E[X'|Y] > E[Y|Y]$, the critical point given in equation 11.2 has eigenvalues

$$\lambda_1 = (E[X|Y] - E[X'|Y])\, y \qquad [12.2.1]$$

$$\lambda_2 = -2\,(E[X'|Y] - E[Y|Y])\, y + O(\varepsilon) < 0 \qquad [12.2.2]$$

3. When $E[X|X] > E[Y|X]$, the critical point given in equation 11.3 has eigenvalues:

$$\lambda_1 = (E[X'|Y] - E[X|Y])\, y \qquad [12.3.1]$$

$$\lambda_2 = -(E[X|X] - E[Y|X]) + O(\varepsilon) < 0 \qquad [12.3.2]$$

4. When $E[X|X] = E[Y|X]$ and $E[X|Y] > E[Y|Y]$, the critical point given in equation 11.4 has eigenvalues

$$\lambda_1 = (E[X' \mid Y] - E[X \mid Y])\, y \qquad [12.4.1]$$

$$\lambda_2 = -\,2\,(E[X \mid Y] - E[Y \mid Y])\, y + O(\varepsilon) < 0 \qquad [12.4.2]$$

Proof. See Appendix. #

Lemma 4c: Stability of the conic segment of critical points. Let $E[X' \mid Y] = E[X \mid Y]$. Let $C(\varepsilon)$ be as defined in item 5 of lemma 4a. Then (1) any proper subset of $C(\varepsilon)$ is not asymptotically stable, and (2) $C(\varepsilon)$ is asymptotically stable with $\lim_{\varepsilon \to 0^+} \sup_{(x,x',y) \in C(\varepsilon)} y = 0$ if and only if both (a) condition 1 or 2 of lemma 4a holds and (b) condition 3 or 4 of lemma 4a holds.

Proof. See Appendix. #

Proof of theorem 4 (characterization of ESB). We first show that to satisfy condition 2 of ESB (in definition 6), one of the three following conditions must hold: (1) $E[X \mid Y] > E[X' \mid Y]$ and X repels invasion by Y, (2) $E[X \mid Y] < E[X' \mid Y]$ and X' repels invasion by Y, or (3) $E[X \mid Y] = E[X' \mid Y]$ and both X and X' repel invasion by Y.

If $E[X \mid Y] > E[X' \mid Y]$, lemma 4b shows that asymptotic stability occurs if and only if either condition 1 or 2 of lemma 4a holds, i.e., if X repels invasion by Y. Similarly, if $E[X \mid Y] < E[X' \mid Y]$, lemma 4b shows that asymptotic stability occurs if and only if either condition 3 or 4 of lemma 4a holds, i.e., if X' repels invasion by Y. Finally if $E[X \mid Y] < E[X' \mid Y]$, lemma 4c shows that asymptotic stability occurs if and only if both (a) condition 1 or 2 of lemma 4a holds and (b) condition 3 or 4 of lemma 4a holds, i.e., if both X and X' repel invasion by Y.

The proof is concluded by applying condition 1 of ESB: X has PS, hence repels invasion by Y. If $E[X \mid Y] > E[X' \mid Y]$, then X produces ESB. Otherwise, X produces ESB if and only if X' repels invasion by Y. #

No IPD Strategy Produces ESB

Theorem 5. For w sufficiently large, no IPD strategy produces ESB. In particular, for any strategy X that has PS there exists a pair-equivalent strategy X' and a discriminator Y such that Y invades X' and $E[X \mid Y] \leq E[X' \mid Y]$.

Sketch of proof. Theorem 3 showed how X' and Y can reveal to each other that neither is X. After these revelations, X' always cooperates with Y; Y defects against X' often enough to invade X', and always defects against X.

Proof. Define the relationship $\mathbf{M} \subset \mathbf{L}$ if, given $\mathbf{L} = \{(a_i,b_i)\}_{i=1}^{l}$, $\mathbf{M} = \{(a_i,b_i)\}_{i=1}^{m}$ for some $m < l$. As in the proof of theorem 3, select a sequences of n moves following which X makes a deterministic choice, i.e., $\mathbf{M}^0$ such that $\Pr[M_n(X,X)=\mathbf{M}^0] > 0$ and $\Pr[X(\mathbf{M}^0)=C] \in \{0,1\}$. Let $\mathbb{M}^0 \Leftarrow \{\mathbf{M} \in \mathbb{M} : \mathbf{M}^0 \subset \mathbf{M} \text{ or } \mathbf{M}^{0T} \subset \mathbf{M}\}$; this is the set of sequences of moves starting with $\mathbf{M}^0$. Let

$$E_0 \Leftarrow (1-w) \sum_{i=1}^{\infty} w^{i-1} \sum_{\mathbf{M}_i \in \mathbb{M}_i \setminus \mathbb{M}^0} \Pr[M_i(A,B)=\mathbf{M}_i]\, V(a_n,b_n) \qquad [13.1]$$

This is the payoff to X from sequences other than those starting with $\mathbf{M}^0$.

Again, as in the proof of theorem 3, choose X' and Y so that they are pair-equivalent to X, except that $Y(\mathbf{M}^0) \neq X(\mathbf{M}^0)$. Since X, X' and Y make the same choices for $\mathbf{M} \in \mathbb{M} \setminus \mathbb{M}^0$,

$$E[A \mid B] = E_0 + (1-w) \sum_{i=1}^{\infty} w^{i-1} \sum_{\mathbf{M}_i \in \mathbb{M}^0} \Pr[M_i(A,B)=\mathbf{M}_i]\, V(a_n,b_n) \qquad [13.2]$$

for any $A,B \in \Sigma$. Since we only need to make relative comparisons between expected payoffs, we can assume without loss of generality that $\mathbf{M}^0 = \varnothing$ (i.e., that the first choice by X is deterministic), whence $\mathbb{M}^0 = \mathbb{M}$ and $E_0 = 0$.

Because X's first choice is deterministic and $Y(\varnothing) \neq X(\varnothing)$, Y's first choice informs the other player that it is not X or X'. Let $X'(M_1(X,Y)) \Leftarrow D$ if $\Pr[X(M_1(X,Y)) = D] \leq 1/2$; otherwise, let $X'(M_1(X,Y)) \Leftarrow C$. Choose $\gamma > 0$ so that $E_1 \Leftarrow \gamma R + (1-\gamma) S > P$. Let $X'(\mathbf{M}) \Leftarrow C$ and $\Pr[Y(\mathbf{M}^T) = C] \Leftarrow \gamma$ for all $\mathbf{M} \in \mathbb{M}$ such that $M_1(X,Y) \subset \mathbf{M}$. Then, for constants $a,a',b,c,c' \in [S,T]$, and w sufficiently large (i.e., sufficiently close to 1)

$$E[X \mid X] = E[X' \mid X'] \leq R \qquad [13.3a]$$

$$E[X \mid Y] \leq (a + bw)\,(1\text{-}w) + \tfrac{1}{2}(E_1 + P)\,w^2 \qquad [13.3b]$$

$$E[X' \mid Y] = (a + cw)\,(1\text{-}w) + E_1\,w^2 > E[X \mid Y] \qquad [13.3c]$$

$$E[Y \mid X'] = (a' + c'w)\,(1\text{-}w) + (\gamma R + (1\text{-}\gamma)\,T)\,w^2 > R > E[X' \mid X'] \qquad [13.3d]$$

Equation 13.3c holds because $E_1 > \frac{1}{2}(E_1 + P)$; equation 13.3d holds because $\gamma R + (1\text{-}\gamma)\,T > R$ for $\gamma > 0$. Thus Y can invade X' (equation 13.3d), while X' receives a higher payoff from Y than X does (equation 13.3c). This eliminates the potential for any stable equilibria with a small proportion of discriminators. #

To illustrate theorem 5, consider the invasion ecology $\Sigma_4 = \{X, X', Y\} \Leftarrow$ {TFT-C,ALL-C,CFD2C }, where CFD2C is the strategy of starting with defection, then cooperating forever after if the other player starts by defecting once or cooperating twice and defecting forever otherwise. In this system

$$E[X] = R\,(x+x') + \big(S\,(1\text{-}w) + P\,w\big)\,y \qquad [14.1]$$

$$E[X'] = R\,(x+x') + \big(S\,(1\text{-}w^2) + R\,w^2\big)\,y \qquad [14.2]$$

$$E[Y] = \big(T\,(1\text{-}w) + P\,w\big)\,x + \big(T\,(1\text{-}w^2) + R\,w^2\big)\,x' + \big(P\,(1\text{-}w) + R\,w\big)\,y \qquad [14.3]$$

Figure 1.7 displays a phase portrait of $\Delta(\Sigma_4;\varepsilon)$ for the same parameter values used previously. For this invasion ecology, the equilibrium point S_c (representing near saturation of the ecology by TFT-C) is an unstable saddle point. Y (saturation of the ecology by CFD2C) is the only stable equilibrium point for this ecology.

Persistence of Strategies and Their Behavior

The previous section shows that any IPD ecology cannot remain saturated by one strategy or even one mode of behavior forever. The lemmas used to characterize the ESB solution concept can be used to determine how long a saturated ecology can persist, relative to the rate of discriminative perturbation.

Definition 7: Persistence of strategic behavior. Given a strategy X and an invasion ecology of X, $\Sigma = \{X, X', Y\}$, if S, one of the critical points listed in lemma 4a, exists, then let $\lambda(S)$ be the largest eigenvalue of $\Delta(\Sigma;\varepsilon)$ at S. Let $\lambda(\Sigma;\varepsilon) \Leftarrow \min(\lambda(S))$, and let $r(\Sigma) \Leftarrow \lim_{\varepsilon \to 0^+} \frac{\lambda(\Sigma;\varepsilon)}{\varepsilon}$. $r(\Sigma)$ represents the rate of departure from the "most persistent" equilibrium saturated with the behavior produced by X, relative to the rate of discriminative perturbation. Define the persistence of X:

$$\mathrm{Per}(X) \Leftarrow \inf_{\Sigma} \frac{1}{r(\Sigma)} \qquad [15]$$

Per(X) varies directly with the shortest possible "doubling" time of discriminators in an invasion ecology of X, relative to the rate of discriminative perturbation. The larger Per(X), the longer we can expect the behavior produced by X to persist.

From the equilibria and eigenvalue calculations in lemmas 4a and 4b, we find that

$$\mathrm{Per}(X) = \inf_{\Sigma \in \mathbb{S}} \frac{E[X\,|\,X] - E[Y\,|\,X]}{E[X'\,|\,Y] - E[X\,|\,Y]} \qquad [16]$$

where $\mathbb{S}$ includes only those invasion ecologies in which Y invades X' and $E[X'\,|\,Y] > E[X\,|\,Y]$.

Because $E[X\,|\,X]$ varies directly with the amount of cooperation when players using X play each other, the more a strategy cooperates with itself, the greater its persistence. Theorem 6 makes this relationship explicit.

Theorem 6. Let X be a strategy with PS. Then

$$\mathrm{Per}(X) \le \frac{E[X\,|\,X] - P}{R - P} + O(1 - w) \qquad [17.1]$$

Proof. Using the invasion ecology of theorem 5, we find that

$$\mathrm{Per}(X) \le \frac{E[X\,|\,X] - P}{\frac{1}{2}(R-P)} + O(1-w) \qquad [17.2]$$

The factor of $\frac{1}{2}$ appears in equation 17.2 because $\Pr[X(M_1(X,Y))=X'(M_1(X,Y))] \le \frac{1}{2}$. Thus X has at most one chance in two of imitating X', avoiding permanent retaliation from Y. If Y observes its partner's choices for n turns, it can distinguish X from X' with correspondingly greater certainty: the probability that X avoids permanent retaliation by Y is then at most 2^{-n}. Thus

$$\mathrm{Per}(X) \le \frac{E[X\,|\,X] - P}{(1-2^{-n})\,(R-P)} + O(1-w) \qquad [17.3]$$

Since we can vary n independently of w, the theorem is proved. #

For a particular strategy, Per(X) can be explicitly computed. For example, for w sufficiently large

$$\mathrm{Per}(\text{ALL-D}) = \frac{(P-S)\,(1-w)}{(\gamma_0 R + (1-\gamma_0)T)\,w^2 - Pw + Sw(1-w)} \qquad [18.1]$$

where γ_0 satisfies

$$(R-S)\,w^2\,\gamma + (S(1-w) + T\,(w-w^2) + Sw^2) = P \qquad [18.2]$$

As guaranteed by theorem 6, Per(ALL-D) tends to 0 as w increases to 1. The infimum is approached by using the following set of pairs of strategies $X'(\gamma)$ and Y for $\gamma \in (\gamma_0, 1]$. Y cooperates on the first move and reciprocates the other player's first choice on the second move. Y then reciprocates the other player's second choice forever after. $X'(\gamma)$ defects on the first move (in order to be pair-equivalent to ALL-D), then reciprocates the other player's first choice on the second move. If the other player defects on the first move, then $X'(\gamma)$ defects forever after (again, in order to be pair-equivalent to ALL-D). Otherwise, starting with the third move, $X'(\gamma)$ cooperates with probability γ.

The infimum is not achieved because Y cannot invade $X'(\gamma_0)$, although Y can invade $X'(\gamma)$ for any $\gamma > \gamma_0$.

As another example, for w sufficiently large

$$\mathrm{Per(PR)} = \frac{R - Pw - T(1-w)}{Rw^2 - Pw + Sw(1-w)} \qquad [16.2]$$

which tends to 1 as w increases to 1. This shows that the upper bound guaranteed by theorem 6 can be achieved when $E[X \mid X] = R$. The infimum is achieved using $\{X', Y\}$ = {ALL-C,CFD2C }.

Summary

There are no asymptotically stable IPD strategies. All ecologically stable strategies allow unstable strategies to coexist with them. Thus, under discriminative perturbation, all strategies, and even the behavior they produce, are unstable. However, the rate of invasion under discriminative perturbation varies. Mutual cooperation can reduce this rate of invasion. Thus the persistence of an IPD ecology is constrained by the level of mutual cooperation.

		Player *Y*'s Choice	
	$V(X,Y)$	Cooperate (C)	Defect (D)
Player *X*'s Choice	Cooperate (C)	R	S
	Defect (D)	T	P

$S < P < R < T$ $\qquad$ $R + R > T + S$

Figure 1.1a

Prisoners' Dilemma Payoff Matrix

ALWAYS DEFECT

ALL-D(M) $\Leftarrow D$ for all $M \in \mathbb{M}$ [F1.1]

TIT FOR TAT

TFT(∅) $\Leftarrow X$ [F1.2a]

TFT(M) $\Leftarrow y_n$ for all $M \in \mathbb{M}_n$ with n = 1,2,... [F1.2b]

where X is a random variable with sample space $\{C,D\}$. If Pr[X=C] = 1, this is the traditional TIT FOR TAT strategy denoted by TFT-C; if Pr[X=C] = 0, this is TIT FOR TAT starting with defection denoted by TFT-D.

ALWAYS COOPERATE

ALL-C(M) $\Leftarrow C$ for all $M \in \mathbb{M}$ [F1.3]

BULLY

BULLY(M) $\Leftarrow C$ for all $M \in \mathbb{M}$ such that at least one $y_i = D$. [F1.4a]

$\Leftarrow D$ else [F1.4b]

PERMANENT RETALIATION

PR(M) $\Leftarrow D$ for all $M \in \mathbb{M}$ such that at least one $y_i = D$. [F1.5a]

$\Leftarrow C$ else [F1.5b]

DEFECT ONCE

1D(∅) $\Leftarrow D$ [F1.6a]

1D(M) $\Leftarrow C$ for all $M \in \mathbb{M}_n$ with n = 1,2,... [F1.6b]

COOPERATE FOR DEFECTION OR TWO COOPERATIONS

CFD2C(M) $\Leftarrow C$ for all $M \in \mathbb{M}$ such that $y_1 = D$

or both $y_1 = C$ and $y_2 = C$ [F1.7a]

$\Leftarrow D$ else [F1.7b]

TIT FOR SECOND TAT

TF2ndT(∅) $\Leftarrow C$ [F1.8a]

TF2ndT(M) $\Leftarrow y_1$ for all $M \in \mathbb{M}_1$ [F1.8b]

$\Leftarrow y_2$ for all $M \in \mathbb{M}_n$ with n = 2,3,... [F1.8c]

$X'(\gamma)$

$X'(\gamma)$(∅) $\Leftarrow D$ [F1.9a]

$X'(\gamma)(M)$ $\Leftarrow y_1$ for all $M \in \mathbb{M}_1$ [F1.9b]

$\Leftarrow D$ for all $M \in \mathbb{M}_n$ with n = 2,3,... such that $y_1 = D$ [F1.9c]

$\Leftarrow Q$ for all $M \in \mathbb{M}_n$ with n = 2,3,... such that $y_1 = C$ [F1.9d]

where Q is a random variable with sample space $\{C,D\}$ and Pr[Q=C] $\Leftarrow \gamma$.

Figure 1.1b

Cast of Characters

When a player using $\text{TF2}^{\text{nd}}\text{T}$ or $X'(\gamma)$ meets a player using the same strategy, the sequence of moves proceeds deterministically.

$$M_n(\text{TF2}^{\text{nd}}\text{T},\text{TF2}^{\text{nd}}\text{T}) = CC_n \qquad \text{[F1.10]}$$

where CC_n is a sequence of n mutual cooperations, $\{(C,C),(C,C),\ldots,(C,C)\}$.

$$M_n(X'(\gamma),X'(\gamma)) = DD_n \qquad \text{[F1.11]}$$

where DD_n is a sequence of n mutual defections, $\{(D,D),(D,D),\ldots,(D,D)\}$. When a player using $\text{TF2}^{\text{nd}}\text{T}$ meets a player using $X'(\gamma)$, the sequence of moves becomes random.

$$M_1(\text{TF2}^{\text{nd}}\text{T},X'(\gamma)) = \{(C,D)\} \qquad \text{[F1.12a]}$$

$$M_2(\text{TF2}^{\text{nd}}\text{T},X'(\gamma)) = \{(C,D),(D,C)\} \qquad \text{[F1.12b]}$$

$$\Pr[M_n(\text{TF2}^{\text{nd}}\text{T},X'(\gamma)) = \{(C,D),(D,C),(C,b_3),\ldots(C,b_n)\}]$$

$$= \prod_{i=3}^{n} p_i \text{ for } n = 3,4,\ldots \qquad \text{[F1.12c]}$$

where $p_i \Leftarrow \gamma$ if $b_i = C$, and $p_i \Leftarrow 1-\gamma$ if $b_i = D$. The expected payoffs for this set of strategies is

$$E[\text{TF2}^{\text{nd}}\text{T} \mid \text{TF2}^{\text{nd}}\text{T}] = R \qquad \text{[F1.13a]}$$

$$E[\text{TF2}^{\text{nd}}\text{T} \mid X'(\gamma)] = (1-w)\,(S+wT) + w^2\,(\gamma R+(1-\gamma)S) \qquad \text{[F1.13b]}$$

$$E[X'(\gamma) \mid \text{TF2}^{\text{nd}}\text{T}] = (1-w)\,(T+wS) + w^2\,(\gamma R+(1-\gamma)T) \qquad \text{[F1.13c]}$$

$$E[X'(\gamma) \mid X'(\gamma)] = P \qquad \text{[F1.13d]}$$

$$E[\text{TF2}^{\text{nd}}\text{T}] = R\,x + ((1-w)\,(S+wT) + w^2\,(\gamma R+(1-\gamma)S))\,y \qquad \text{[F1.13e]}$$

$$E[X'(\gamma)] = ((1-w)\,(T+wS) + w^2\,(\gamma R+(1-\gamma)T))\,x + P\,y \qquad \text{[F1.13f]}$$

$$E = R\,x^2 + \alpha\,xy + P\,y^2 \qquad \text{[F1.13g]}$$

where $x \Leftarrow |\text{TF2}^{\text{nd}}\text{T}|$, $y \Leftarrow |X'(\gamma)|$, and $\alpha \Leftarrow (1-w^2\gamma)\,(S+T) + 2w^2\gamma R$. The system of differential equations

$$\frac{dx}{dt} = (E[\text{TF2}^{\text{nd}}\text{T}] - E)\,x \qquad \text{[F1.14a]}$$

$$\frac{dy}{dt} = (E[X'(\gamma)] - E)\,y \qquad \text{[F1.14b]}$$

can be reduced to

$$\frac{dx}{dt} = (E[\text{TF2}^{\text{nd}}\text{T} \mid X'(\gamma)] - P)\,x + (R+2P-\alpha)\,x^2 + (\alpha-R-P)\,x^3 \qquad \text{[F1.14c]}$$

Figure 1.1c

Sequence Probabilities, Expected Payoffs, and Differential Equations for $\Sigma_\omega = \{\text{TF2}^{\text{nd}}\text{T},X'(\gamma)\}$

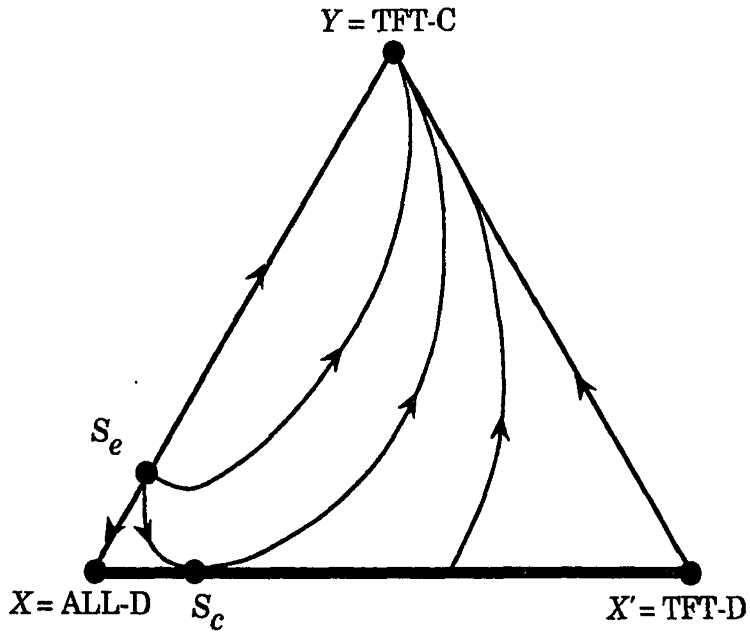

Figure 1.2

Phase portrait of $\Delta(\Sigma_1)$

$T = 5, R = 3, P = 1, S = 0, w = 4/5$. Solid circles denote critical points of $\Delta(\Sigma_1)$. $S_c = (x = 55/64, x' = 9/64, y = 0)$; $S_e = (x = 6/7, x' = 0, y = 1/7)$. All of the points on the line XX' are critical points of $\Delta(\Sigma_1)$.

For a given ecology of three strategies $\Sigma = \{X, X', Y\}$, let $\Delta'(\Sigma)$ represent the system of differential equations formed by substituting each $A \in \Sigma$ into the following equation.

$$\frac{d|A|}{dt} = (E[A] - E)\ |A|\ /\ |Y|;\ |A| \in [0,1] \qquad \text{[F2.1]}$$

The points X, X' and S_c are critical points of $\Delta'(\Sigma_1)$.

Note: In this and all of the following phase portraits, all critical points lie on the domain's boundary. Thus there are no periodic orbits in any of these phase portraits (see, for example, Coddington and Levinson, 1955: 400).

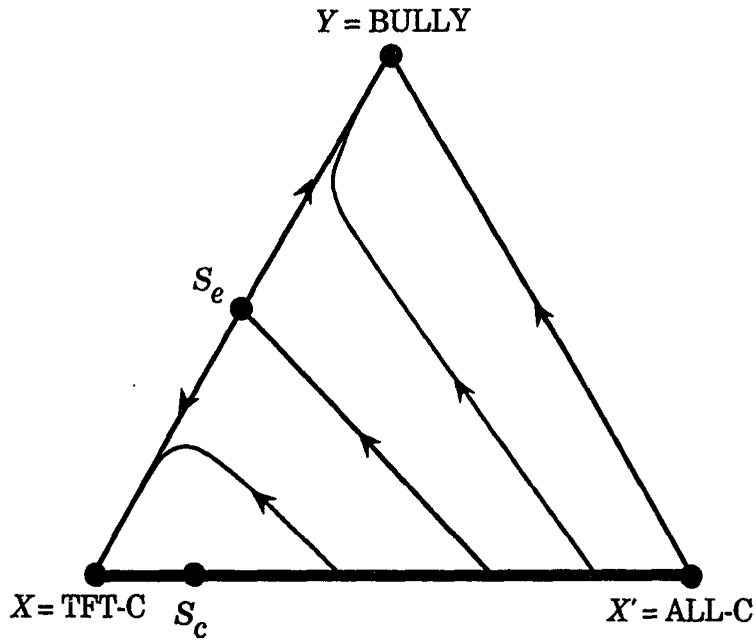

Figure 1.3

Phase portrait of $\Delta(\Sigma_2)$

$T = 5, R = 3, P = 1, S = 0, w = 4/5$. Solid circles denote critical points of $\Delta(\Sigma_2)$. $S_c = (x = 125/144, x' = 19/144, y = 0)$; $S_e = (x = 33/71, x' = 0, y = 38/71)$. All of the points on the line $\overline{XX'}$ are critical points of $\Delta(\Sigma_2)$. The points X, X' and S_c are critical points of $\Delta'(\Sigma_2)$ (see figure 1.2).

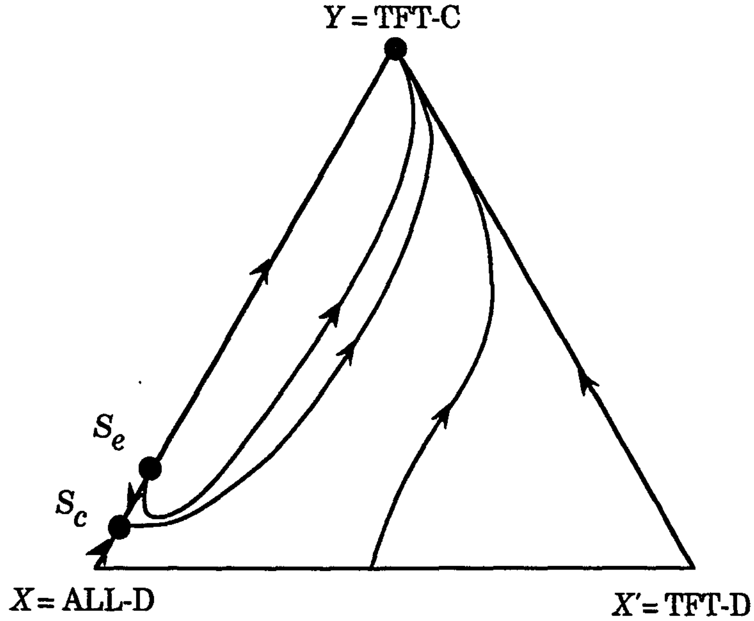

Figure 1.4

Phase portrait of $\Delta(\Sigma_1;\varepsilon)$

$T = 5, R = 3, P = 1, S = 0, w = 4/5, \varepsilon = .001$. Solid circles denote critical points of $\Delta(\Sigma_1;\varepsilon)$. The eigenvalues and eigenvectors of S_c ($x = .9948, x' = 0, y = .0052$)[1] are $\lambda_1 = .0051$, $\mathbf{v}_1 = \langle 1,-.963,-.037\rangle$ and $\lambda_2 = -.18$, $\mathbf{v}_2 = \langle 1,0,-1\rangle$. The eigenvalues and eigenvectors of S_e ($x = .86, x' = 0, y = .14$) are $\lambda_1 = .16$, $\mathbf{v}_1 = \langle 1,0,-1\rangle$ and $\lambda_2 = .13$, $\mathbf{v}_2 = \langle -.83,-.17,1\rangle$.

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[1]In order to show details in the graph, S_c has been displayed at an exaggerated distance from the point X.

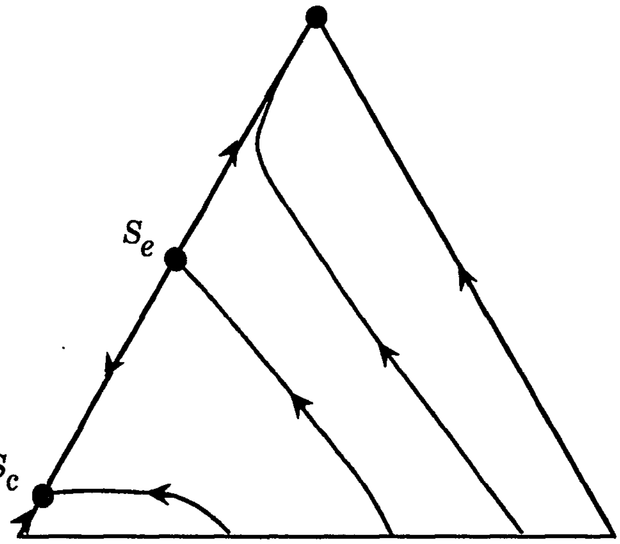

Figure 1.5

Phase portrait of $\Delta(\Sigma_2;\varepsilon)$

$T = 5, R = 3, P = 1, S = 0, w = 4/5, \varepsilon = .001$. Solid circles denote critical points of $\Delta(\Sigma_2;\varepsilon)$. The eigenvalues and eigenvectors of S_c ($x = .9967, x' = 0, y = .0033$)[2] are $\lambda_1 = -.0077$, $\mathbf{v}_1 = \langle 1,-.975,-.025\rangle$ and $\lambda_2 = -.30$, $\mathbf{v}_2 = \langle 1,0,-1\rangle$. The eigenvalues and eigenvectors of S_e ($x = .47, x' = 0, y = .53$) are $\lambda_1 = .14$, $\mathbf{v}_1 = \langle 1,0,-1\rangle$ and $\lambda_2 = -1.24$, $\mathbf{v}_2 = \langle .12,-1.12,1\rangle$.

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[2]In order to show details in the graph, S_c has been displayed at an exaggerated distance from the point X.

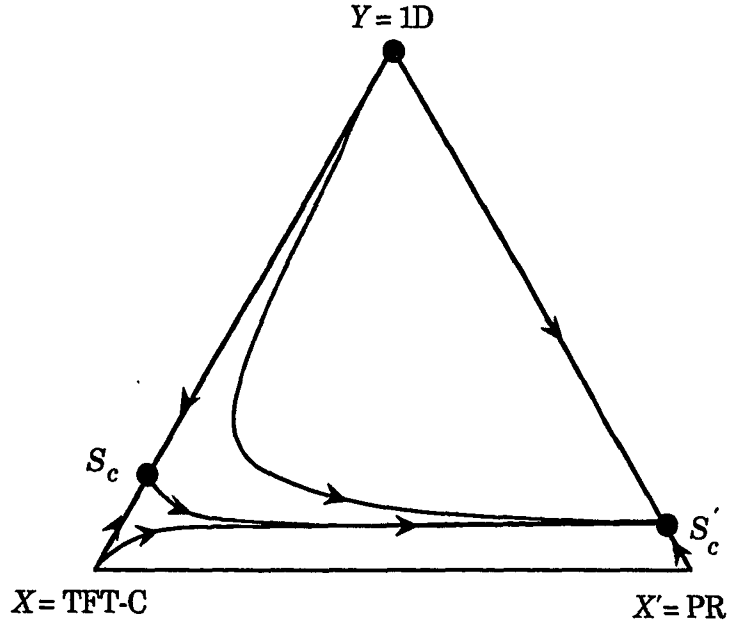

Figure 1.6

Phase portrait of $\Delta(\Sigma_3;\varepsilon)$

$T = 5, R = 3, P = 1, S = 0, w = 4/5, \varepsilon = .001$. Solid circles denote critical points of $\Delta(\Sigma_3;\varepsilon)$. The eigenvalues and eigenvectors of S_c ($x = .988, x' = 0, y = .012$)[3] are $\lambda_1 = .016$, $\mathbf{v}_1 = \langle 1,-1.33,.33\rangle$ and $\lambda_2 = -.080$, $\mathbf{v}_2 = \langle 1,0,-1\rangle$. The eigenvalues and eigenvectors of S_c' ($x = 0, x' = .99950, y = .00050$) are $\lambda_1 = -.00064$, $\mathbf{v}_1 = \langle 1,-1.00048,.00048\rangle$ and $\lambda_2 = -2.00$, $\mathbf{v}_2 = \langle 0,-1,1\rangle$.

---

[3]In order to show details in the graph, S_c and S_c' have been displayed at exaggerated distances from the points X and X', respectively. Their relative positions have been preserved.

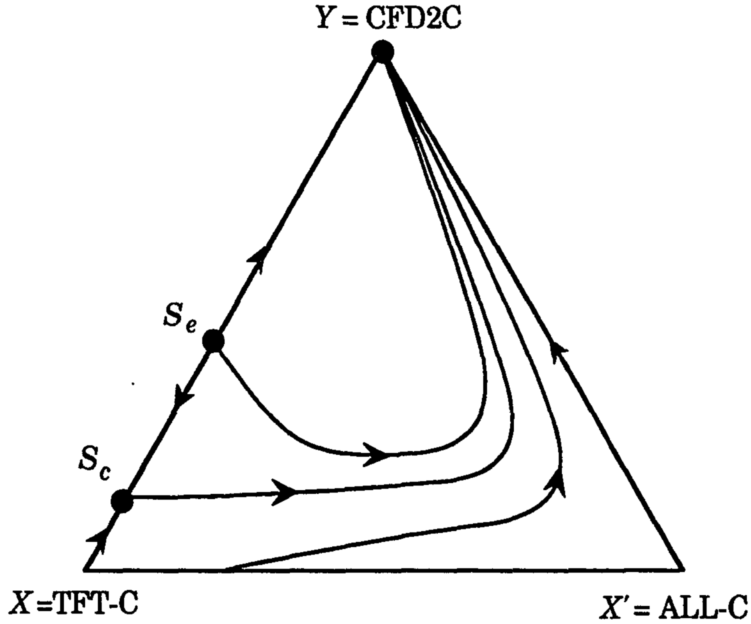

Figure 1.7

Phase portrait of $\Delta(\Sigma_4;\varepsilon)$

$T = 5, R = 3, P = 1, S = 0, w = 4/5, \varepsilon = .001$. Solid circles denote critical points of $\Delta(\Sigma_4;\varepsilon)$. The eigenvalues and eigenvectors of S_c ($x = .99916, x' = 0, y = .00084$)[4] are $\lambda_1 = .00094$, $\mathbf{v}_1 = \langle 1,-.9987,-.0013\rangle$ and $\lambda_2 = -1.19$, $\mathbf{v}_2 = \langle 1,0,-1\rangle$. The eigenvalues and eigenvectors of S_e ($x = .60, x' = 0, y = .40$) are $\lambda_1 = .45$, $\mathbf{v}_1 = \langle -.039,-.961,1\rangle$ and $\lambda_2 = .72$, $\mathbf{v}_2 = \langle 1,0,-1\rangle$.

[4]In order to show details in the graph, S_c has been displayed at an exaggerated distance from the point X.

CHAPTER 3

MODELS FOR LONG CYCLES IN WAR AND PRODUCTION

Long Cycles: Observation and Theory

In his 1987 article "Long Waves in War, Production, Prices, and Wages", Joshua Goldstein claims to have developed a "theoretical model consistent with [his empirical findings of long cycles in various historical time series]" (p. 573). I examine this claim by developing and analyzing some mathematical models based on hypotheses from his theory of the long wave. Specifically, I investigate whether or not some simple mathematical representations of his theory are capable of replicating three key features (enumerated below) of his empirical findings.

Goldstein establishes a "base dating scheme" for 10 observed long waves determined by successive peaks and troughs in prices (p. 576-7). Goldstein summarizes the sequence and timing of the long waves identified in his research in his figure 9 (p. 592). In particular, "... war severity *leads* the nominal long wave dates, by something like 1-5 years on the average" (p. 582; emphasis in original), while production waves lead "prices by 10 to 15 years and hence war severity by about a decade" (p. 589).

The cycles Goldstein identifies are not periodic in calendar time. Instead he defines "social cycles in terms of 'cycle time'" (p. 575) so that he defines "the long wave as a pattern of alternating historical phase periods – upswings and downswings – that are only roughly equal in length" (p. 576). This definition is somewhat problematic: any non-monotonic time series is going to have ups followed by downs. As an example, in examining his figure 2 (p. 581), the reader may wonder if Goldstein has skipped over some peaks in order to synchronize war peaks with his base dating scheme. The duration of cycles from Goldstein's base dating scheme varies from as short 30 years (1529 to 1559) to as long as 70 years (1650 to 1720). The mean duration over all cycles (measured from one peak to the next) is 48.5 years with a standard deviation of 12.6 years.

In order to explain these long cycle features he has identified, Goldstein proposes

> a new theory of the long wave, based on a two-way causal relationship between economic and political variables. Sustained economic growth both promotes (enables) war and is disrupted by war. Figure 10 illustrates the cyclical sequence of production and war in this theory. Faster growth gives rise to increased great power war severity. Higher war severity in turn dampens long-term economic growth. Lower growth leads to less severe war, which in turn allows faster economic growth. This sequence takes roughly 50 years – one long wave – to complete. While war and economic growth are the driving variables, prices react to war, and real wages react to war and prices. (p. 590)

These arguments are illustrated in Goldstein's figure 10 (p. 593).

Is Goldstein's theoretical account capable of replicating his empirical observations? In particular:

1. Does the two-way causal relationship hypothesized by Goldstein generate a cyclical sequence of production and war?

2. If Goldstein's theory is capable of generating a cyclical sequence of production and war, does it generate the lag structure he observes?

3. Can Goldstein's theory account for the rhythmic, but non-periodic pattern of the long waves he observes?

In raising these three questions, I have attempted to identify the most important qualitative aspects of Goldstein's empirical findings. In order to evaluate Goldstein's theory relative to the criteria represented by these questions, I develop two differential equation models that are consistent with his hypotheses. I then determine whether or not these models satisfy the three empirical criteria I have extracted from his empirical findings.

Before developing and analyzing these models, here are some nuances to consider when interpreting the results of this exercise. Because Goldstein's theory is expressed verbally, there is some ambiguity in representing his argument mathematically, i.e., there are many mathematical models consistent with Goldstein's theory. Ideally, all of these models would replicate his empirical findings, including the three criteria I have selected. Unfortunately, the first model developed below cannot account for the third criterion (the rhythmic, but non-periodic pattern of long waves). This demonstrates that Goldstein's theory is *deficient*, requiring additional hypotheses to eliminate such inadequate models from models capable of accounting for his empirical findings. However, the existence of a deficient model for Goldstein's theory does not refute it because there may be another mathematical representation of his argument that *can* account for his findings. And indeed, the second model I develop does satisfy all of the empirical criteria identified above.

A "Growth" Model for Goldstein's Theory

As specified by Goldstein's theory, this model has two "driving" variables: (1) W, war severity, presumably takes on non-negative values; and (2) G, production growth, takes on real values (positive values indicate actual growth in production, and negative values indicate decline in production). Goldstein's theory posits the following relationships among these variables:

1. War severity reduces production growth. In the absence of war ($W = 0$), production growth rises. The greater W, the less G rises, and for sufficiently large W, G will decrease.

2. Production growth augments war severity. When G is low (or negative), war severity is reduced. The greater G, the less W decreases, and for sufficiently large G, W will increase.

The following "growth" model, an autonomous system of two differential equations, is consistent with the above relationships:

$$\frac{dW}{dt} = \alpha G + \beta \qquad [2.1a]$$

$$\frac{dG}{dt} = -\gamma W + \delta \qquad [2.1b]$$

α, β, γ, and δ are constants, with α, γ, and δ positive. This linear system of differential equations can be solved explicitly:

$$G = \eta \sin(\omega t + \theta) - \frac{\beta}{\alpha} \qquad [2.2a]$$

$$W = \frac{-\eta\omega}{\gamma} \cos(\omega t + \theta) + \frac{\delta}{\gamma} \qquad [2.2b]$$

where $\omega \Leftarrow \sqrt{\alpha\gamma}$, while η and θ are determined from the initial conditions.

This growth model for Goldstein's theory displays pure harmonic oscillation of both production growth and war severity, satisfying question 1. In these harmonic cycles, production growth leads war severity by exactly 1/4 cycle, which is close to, but does not exactly match Goldstein's estimate of a 10 year lead within the 48.5 year cycle (of course, Goldstein's estimate of the lead time is quite rough). Thus the growth model provides a satisfactory account of the second criterion.

However, the periodicity of the oscillation in the growth model is fixed: perturbing the model's variables (*G* and *W*) does not change ω. The only way the periodicity can change is if the structure changes, i.e., through the parameters α and δ. Allowing these parameters to vary over time represents an additional level of long cycle dynamics that Goldstein does not discuss. Because the model cannot account for the non-periodic pattern of long waves in war and production, both the model and (consequently) Goldstein's theory are deficient. Yet this does not refute his theory because, even though it is consistent with his theory, the growth model cannot be deduced from it. By interpreting Goldstein's theory in a different way, we may still be able to demonstrate that it has the potential to be adequate empirically.

<u>A Resource Model for Goldstein's Elaborated Theory</u>

In this section I develop an alternate model motivated by Goldstein's discussion of his long wave theory. In elaborating on his figure 9, Goldstein provides much additional information relevant to the relationship between war and production. In particular, it becomes apparent that "economic growth" has two separate components. We find that

> "When treasuries are full, countries will be able to wage big wars; when they are empty, countries will not wage such wars. Thus, when the growth of production accelerates, the war-supporting capacity of the system increases, and bigger wars ensue" [page 591].

> "Resources allocated to war are not available for productive economic purposes (including both consumption and investment), and economic assets destroyed by war (houses, factories, farms, etc.) are no longer available for productive purposes either" [page 593].

Thus it seems that economic growth can be partitioned into resources (treasuries or assets) and production (productive economic purposes). It appears that war severity is directly driven (or enabled) by resources, and is indirectly driven by production (which replenishes the resources spent on or destroyed by war).

This leads me to propose the following model for Goldstein's elaborated theory. This model has three "driving" variables: (1) W, war severity, takes on non-negative values; (2) P, production, also takes on non-negative values; and (3) R, resources, which takes on real values (positive values indicate surplus or abundance, and negative values indicate deficit or shortage). A fourth variable, production growth (G) is now determined by observing the change in P. This model will be referred to as the "resource" model. Here are four hypotheses about the relationships among these variables:

1. War depletes resources, both by consumption and destruction. The greater W, the greater the reduction of R.

2. Production augments resources. The greater P, the greater the augmentation of R.

3. A surplus of resources ($R > 0$) augments both war severity and production. The greater R, the greater the augmentation of both W and P. Similarly, a shortage

of resources ($R < 0$) reduces both war severity and production. The greater $-R$, the greater the reduction of both W and P.

4. All other things held constant, changes in war severity and production are proportional to their current levels.

The first three hypotheses are suggested by Goldstein's elaborated theory. I appended the fourth hypothesis in order to dynamically maintain the non-negativity constraints on W and P. These relationships can be captured in the following autonomous system of three differential equations:

$$\frac{dW}{dt} = \alpha W R \qquad [2.3a]$$

$$\frac{dP}{dt} = \beta P R \qquad [2.3b]$$

$$\frac{dR}{dt} = \gamma P - \delta W \qquad [2.3c]$$

α, β, γ, and δ are (non-negative) constants of proportionality. By choosing appropriate units of measurement for P, W, R, and t (algebraically, replacing P by γP, W by δW, R with $\beta^{1/2} R$ and t with $\beta^{1/2} t$), we obtain

$$\frac{dW}{dt} = \rho W R \qquad [2.4a]$$

$$\frac{dP}{dt} = P R \qquad [2.4b]$$

$$\frac{dR}{dt} = P - W \qquad [2.4c]$$

where $\rho \Leftarrow \frac{\alpha}{\beta}$. ρ can be thought of as the "war vs. production allocation ratio" of the international system.. For example, if $\rho > 1$, then war severity will respond more strongly than production to surpluses or shortages of resources.

Analysis of the Resource Model

Initial conditions: $(t,W,P,R) = (0,W_0,P_0,R_0)$.

First look at how the system behaves in the absence of war ($W_0 = 0$). Without war, resources will always increase; but they may or may not increase without limit.

The system can be solved explicitly:

$$R^2 - R_0^2 = 2(P - P_0) \text{ (for } P \geq 0) \qquad [2.5]$$

See figure 1. Note that the system cannot proceed to the left of the P=0 axis. So either resources and production will increase without bound, or if there is a sufficiently severe shortage of resources, producution will tend to 0, and the shortage of resources will persist indefinitely.

To help analyze the system when there is war, we can solve explicitly for W in terms of P. We obtain (for $\rho \geq 0$):

$$W = W_0 \left(\frac{P}{P_0}\right)^{\rho} \qquad [2.6]$$

Thus W and P display a direct, monotonic relationship. This reduces the analysis to the following system of two autonomous differential equations.

$$\frac{dP}{dt} = PR \qquad [2.7a]$$

$$\frac{dR}{dt} = P - W_0 \left(\frac{P}{P_0}\right)^{\rho} \qquad [2.7b]$$

We can now explicitly solve the relationship between P and R. We obtain $R^2 = f(P)$ where

$$f(P) = 2\left(P - W_0 \log P\right) + \left(R_0^2 - 2P_0 + 2W_0 \log P_0\right) \text{ (for } \rho = 0) \qquad [2.8a]$$

$$f(P) = 2\left(P - \frac{W_0}{\rho}\left(\frac{P}{P_0}\right)^{\rho}\right) + \left(R_0^2 - 2P_0 + 2\frac{W_0}{\rho}\right) \text{ (for } \rho > 0) \qquad [2.8b]$$

It is now relatively easy to produce phase portraits for the (P,R) system. (For convenience, I'll just display phase portraits of this system. It's easy to "read in" the missing variable W because it varies monotonically with P.) Figure 2 shows the phase portrait when $\rho < 1$. Figures 3a and 3b show the phase portrait when $\rho = 1$.

Figure 4 shows the phase portrait when $\rho > 1$. It's important to note that changing the initial value of W or P does not simply produce a different trajectory on the same two variable (P,R) phase portrait; it also results in a different phase portrait for P and R. This is because the (P,R) phase portrait is actually just one "slice" from the complete (W,P,R) phase portrait. Changing either W_0 or P_0 moves us to a different slice. However, if W_0 and P_0 are held fixed, changing R_0 leaves the phase portrait unchanged. More on this in the following section on perturbations.

When $\rho > 1$, we finally obtain phase portraits with cyclical orbits, satisfying the first criterion. So for long waves to occur, the waging of war must be more sensitive than production to the level of resources. What we observe is a "boom and bust" cycle. But if the level of resources is too large (either positive or negative), the cycle eventually "busts": production goes to zero, and although war severity does also, a shortage of resources persists indefinitely.

The lead time between phases of production growth $(\frac{dP}{dt})$ and war severity varies depending on ρ, the initial values of W, P, and R, and even on the position within the phase (production peaks may lead war peaks by a different length of time than production troughs lead war troughs). This model not only may account for the observed lead time, it also promotes a more careful evaluation of the concept and measurement of lead time in long waves.

But can this model account for non-periodicity in long waves without introducing any additional dynamics? Without perturbation (displacement of the values of W, P, and R), the model produces strictly periodic oscillation. The effects of perturbation are described in the next section.

Perturbation of the Resource Model

The effect of perturbations on the orbits can be determined by evaluating partial derivatives of $f(P)$ with respect to the initial values of W, P, and R (for the rest of this paper assume $\rho > 1$). Here are the results:

Bumping R_0, as noted above, just changes trajectories on the (P,R) phase portrait. These changes either produce larger or smaller cycles (i.e., higher peaks and lower troughs, or lower peaks and higher troughs, respectively, in both war and production).

Bumping W_0 up raises $f(P)$ for $P < P_0$, and reduces $f(P)$ for $P > P_0$. This in turn reduces both the peak and the trough in levels of production. Reducing W_0 raises both the production peak and trough.

Compared with perturbation of W_0, perturbations of P_0 have exactly the opposite effect on production cycles. Raising P_0 raises both the production peak and trough. Reducing P_0 reduces both the production peak and trough. If we continually boost production, this will produce a spiral in which each trough is higher than the past. Figure 5 displays a numerical simulation of the following dynamic modification of the resource model:

$$\frac{dW}{dt} = 2WR \qquad [2.9a]$$

$$\frac{dP}{dt} = PR + 1 \qquad [2.9b]$$

$$\frac{dR}{dt} = P - W \qquad [2.9c]$$

To estimate the effect of perturbations on cycle times, I linearized the reduced system of differential equations and determined the cycle time at the critical point $(W,P,R) = (W_{eq},P_{eq},0)$ for which production is positive (see figure 4). I then determined the cycle time T for this linearized system:

$$T = \frac{2\pi}{\sqrt{(\rho-1)\,P_{eq}}} \qquad [2.10]$$

where

$$P_{eq} = \left(\frac{P^{\rho}}{W}\right)^{\frac{1}{\rho-1}} \qquad [2.11]$$

This shows that the cycle time may vary from one orbit to another, and does not just depend on the parameter ρ. The cycle time is thus sensitive to perturbations of both W and P. The cycle time is particularly sensitive to such perturbations when P is low or W is high.

The cycle time can be explicitly solved when $\rho \Leftarrow 2$:

$$T = 2\pi \sqrt{\frac{W}{P^2 - (P - W)^2 - W R^2}} \qquad [2.12]$$

Since the cycle time is the same for all points of a given orbit, equation 2.12 is valid for any point in the resource model's phase space, provided that the denominator in the radical is positive, i.e., provided the trajectory produces a complete cycle. Conversely, if the denominator is not positive, then we know that the trajectory does not cycle. For example, cycling occurs if and only if $W < 2P$ when $R = 0$. Even though the allocation ratio is fixed, equation 2.12 shows that the cycle time can take on any (non-negative) value. This variation in cycle time does not require the introduction of any additional dynamic hypotheses.

These estimates of the cycle time[1] show that the resource model can produce cyclic behavior with the potential for highly variable cycle times, satisfying the third criterion for evaluating long wave models. This also makes it possible that, despite the apparent variation in cycle time, a constant allocation ratio may provide a good empirical fit across the entire historical time span.

Implications for Research

Statistical inference. The resource model motivates more precise detection schemes for long wave researchers to employ. First of all, long wave researchers have primarily relied on linear statistics. But the resource model suggests that W and P display a power relationship, not a linear one. Also, researchers have

[1]For any parameter value, equation 2.10 is a local estimate of the cycle time in the phase space; conversely, equation 2.12 provides the cycle time for the entire phase space for a single point in the parameter space.

attempted to discover whether one variable leads or lags another. But if lead times vary depending on the position within the cycle, such tests will be confounded. Many of the tests which have been employed are also weakened if the cycles are not uniform in duration. The tests should be adapted for relative positions within cycles rather than measurements based on years.

Indicators. Goldstein has selected battle deaths in great power wars as his indicator for war severity, and tracked the production of several different commodities in order to get production figures for as long a time span as possible. But now that I have selected certain portions of his theoretical argument in order to develop a model which appears to capture many of the characteristic of long cycles, we should focus on these specific arguments in order to refine the indicators. These indicators can then be used, for example, (1) to test the hypothesized power relationship between war severity and production; and (2) to detect perturbations; we can see if the perturbations have the predicted effect on cycle times and peak levels of production and war severity.

Scaling. The resource model, being a mathematical abstraction, need not apply only to the major sequence of long waves. By fitting the model to indices from various nations over different time periods, we can more accurately segregate the core countries from the periphery. We may then be able to detect similar war and production cycles outside of the core. We may also be able to detect sub-cycles within the core. The interplay between war (and other destructive human enterprises) and production may exist at many geographic and temporal scales.

War vs. production allocation ratio. As noted above, a constant allocation ratio may provide a good empirical fit across the entire historical time series. If, on the other hand, the allocation ratio is observed to change over time, it will be very useful to provide dynamic and historical explanations of this change. In particular, what conditions tend to reduce the allocation ratio?

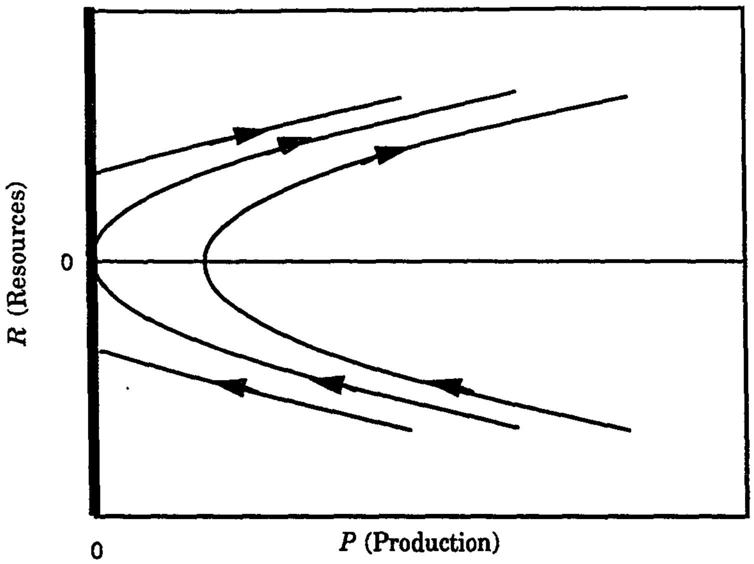

Figure 1

Resource Model with No War

All of the points on the line $P = 0$ are critical points.

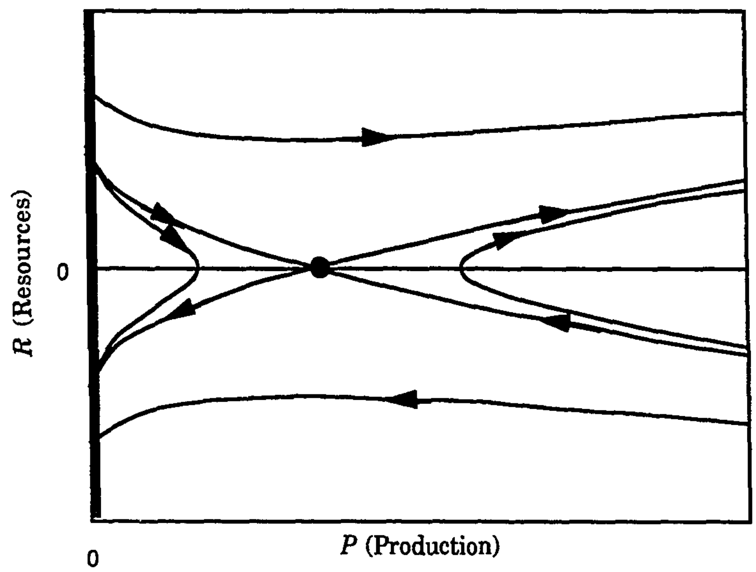

Figure 2

Resource Model with $\rho < 1$

All of the points on the line $P = 0$ are critical points. The solid circle is an isolated critical point.

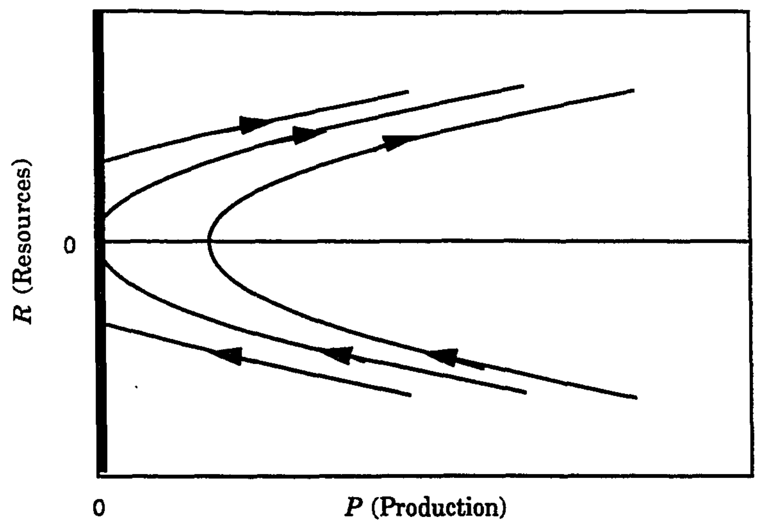

Figure 3a

Resource Model with $\rho = 1$, $W < P$

All of the points on the line $P = 0$ are critical points.

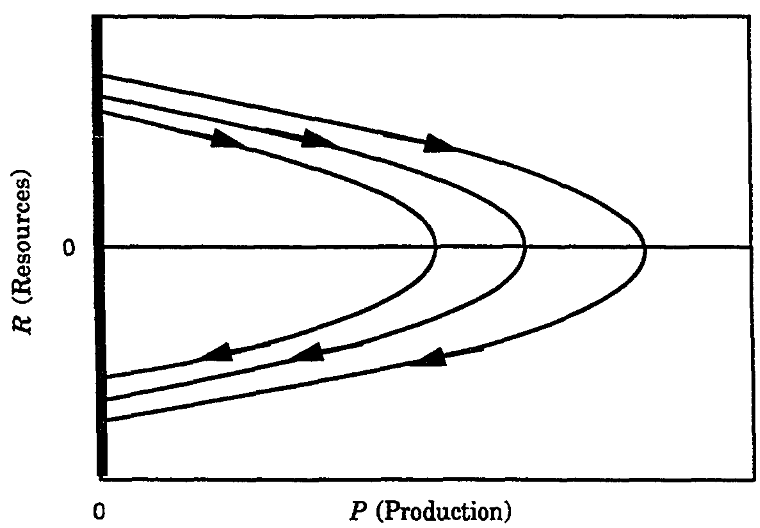

Figure 3b

Resource Model with $\rho = 1$, $W > P$

All of the points on the line $P = 0$ are critical points.

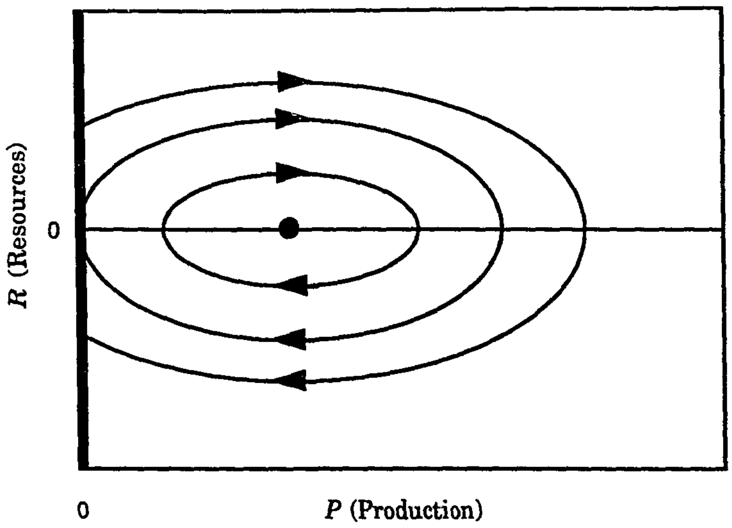

Figure 4

Resource Model with ρ > 1

All of the points on the line $P = 0$ are critical points. The solid circle is an isolated critical point.

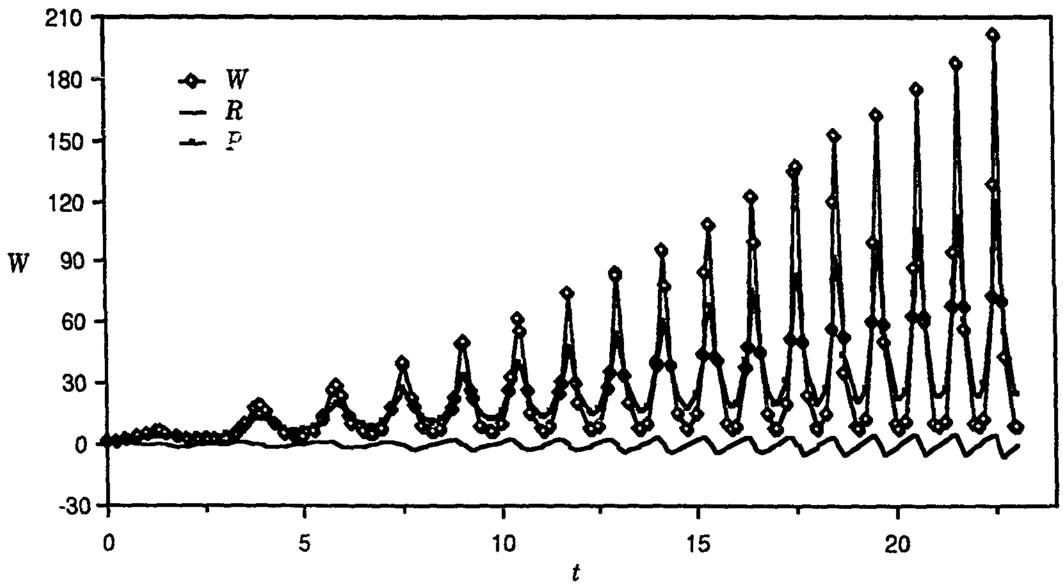

Figure 5

Resource Model (Additional Production)

CHAPTER 4

MEASURING THE RATE OF WAR OUTBREAK

Introduction

We observe N outbreaks of war at a sequence of times $\mathbf{T} = \{t_i\}_{i=1}^{N}$ occurring during an interval $[t_0,t_{N+1})$. For example, Small and Singer (1982) list 224 outbreaks of wars during the interval t_0 = Jan. 1, 1816 to t_{225} = Jan. 1, 1981; $t_{225} - t_0$ = 60266 days.

We want to measure $\lambda(t)$, the rate at which war outbreaks occur in the observed interval $[t_0,t_{N+1}]$ as a function of time. In order to produce this measurement, I first discuss a general event process model. I then select and analyze one of the simplest cases of this general model, the random hazard doubly stochastic Poisson process. I estimate the parameters of the random hazard model in order to produce both predictive and descriptive estimates of $\lambda(t)$. I show that the predictive estimates are statistically precise: not only are the expected values accurate, the amount of error is also accurately predicted. In particular, we can reject two alternative hypotheses: that war outbreaks are generated by a simple Poisson process, or some more general time-invariant event process. The descriptive estimates provide strong, unanticipated corroboration of Goldstein's long cycle dating scheme (1987: 576-7), and for my "resource model" interpretation of his long cycle theory.

The General Intensity-Transition Model of War Outbreak

The following model of war outbreak will be used to produce a daily estimate of the rate of war outbreak, together with the statistical precision of the estimate. This will enable statistically thorough evaluations of theories explaining variations in the rate of war outbreak. For example, by estimating the stochastic component of variation, R^2, the coefficient of determination, can be more accurately interpreted.

Assumption 1. War outbreak events follow a Poisson process with *intensity* (the expected rate at which events occur) $\lambda(t)$.

Assumption 2. $\lambda(t)$ is constant except at events called *intensity transitions*, or just "transitions" for short.

Assumption 3. Intensity transition events follow a Poisson process with intensity $\mu(t)$.

To distinguish between these two types of intensity, $\lambda(t)$ will be referred to as the *war intensity*, and $\mu(t)$ will be referred to as the *transition intensity*.

Assumption 4. The war intensity following a transition is given by the transition p.d.f. $f(x,y,t)$, where x and y are, respectively, the war intensity before and after the transition.

Even though $\lambda(t)$ is assumed to be constant except at intensity transitions, this general form of the model can approximate any given continuously variable war intensity to any desired level of precision. For example, given $\lambda_v(t)$, a continuous function of time, let $f(x,y,t) \Leftarrow \delta(\lambda_v(t))$, i.e., whenever an intensity transition occurs, the war intensity jumps to the desired value $\lambda_v(t)$. As $\mu(t)$ grows arbitrarily large, the expected value of $\lambda(t)$ converges pointwise to $\lambda_v(t)$.[1]

---

[1]If $\mu(t)$ is large, then $\lambda(t)$ does not remain constant for very long periods of time. Thus the applicability of $\hat{\lambda}(t)$, our estimated intensity, as a predictor of $\lambda(t')$ $(t' > t)$ varies inversely with $\mu(t)$: the larger $\mu(t)$ is, the shorter the duration through which we can expect $\hat{\lambda}(t)$ to remain valid as an estimate. In addition, we will see that $\hat{\mu}(t)$, our estimate of $\mu(t)$, governs, roughly speaking, the total weight we can apply to our observations of the sequence of war outbreaks preceding t. The larger $\hat{\mu}(t)$, the faster we must discount our observations further back in time. Thus $\hat{\lambda}(t)$ will display larger variance than it would have if $\hat{\mu}(t)$ were smaller. For these reasons, we hope to observe "small" $\mu(t)$.

Despite this apparent motivation to underestimate $\mu(t)$, it is still best to estimate $\mu(t)$ as accurately as possible. $\hat{\mu}(t)$ is a measure of the volatility of the information gained through the estimation process, indicating how fast our uncertainty grows as we attempt to predict further into the future. If we underestimate $\mu(t)$, then we will give too much weight to observations further back in time, biasing $\hat{\lambda}(t)$ towards $\lambda(t')$ $(t' < t)$ and away from $\lambda(t)$. Biasing the estimate does more damage than raising the estimate's variance because we can estimate the variance, but we cannot estimate the bias. Underestimating $\mu(t)$ will also cause us to overestimate the accuracy of the model's predictions.

The Random Hazard Model of War Outbreak

The following simplifying assumptions are used to make estimation and application of the general model tractable.

Simplifying assumption 1. The transition intensity is constant, i.e., $\mu(t) \equiv \mu^*$.

Simplifying assumption 2. The war intensity following a transition is independent of the war intensity before the transition, and its probability distribution does not change over time, i.e., $f(x,y,t) \equiv f(y)$.

Simplifying assumption 3. $f(y)$ is a member of $\mathfrak{F} \Leftarrow \{f(\lambda,\sigma^2;y)\}_{\lambda,\sigma^2 \geq 0}$, a set of discrete p.d.f.'s for which $f(\lambda,\sigma^2;y)$ has a sample space $\{\lambda_L,\lambda_H\}$ such that

$$f(\lambda,\sigma^2;\lambda_L) \Leftarrow \frac{1}{2} \quad [1.4a]$$

$$f(\lambda,\sigma^2;\lambda_H) \Leftarrow \frac{1}{2} \quad [1.4b]$$

$$\lambda_L \Leftarrow \lambda - \sigma \quad [1.4c]$$

$$\lambda_H \Leftarrow \lambda + \sigma \quad [1.4d]$$

for positive mean value and standard deviation, $\lambda \geq \sigma > 0$. If the standard deviation is 0, then the sample space of $f(\lambda,0;y)$ is $\{\lambda\}$, and $f(\lambda,0;\lambda) = 1$.

These assumptions form the *random hazard doubly stochastic Poisson process model* (Lawrance, 1972: 228), or "random hazard model" for short. This model is one of the simplest realizations of the general model that still incorporates time variation of intensity. Gaver (1963: 223-5) introduced a somewhat more general version of this model and derived expected values and survival probabilities. Grandell discusses inference for doubly stochastic Poisson processes but concludes that "In many cases it is not natural, or at least not practical to [compute the likelihood of the observed sequence of war outbreaks] in order to derive estimates of [the model's parameters]" (1972: 91). He suggests three alternatives: (1) linear estimation, (2) a regression model, and (3) estimates of the covariance sequence. Another approach is to *define* an estimator and assess its statistical efficiency; see, for example, Jacobsen's presentation of the Aalen estimator (1982: 129). Basawa and Rao (1980: 109-111) show that the expectation of the intensity conditioned by the observed event sequence produces a minimum mean-square estimate of the intensity. However, because this expectation "is, in general, difficult to evaluate" (p. 110) they develop a linear approximation of this estimate. Much behavioral research uses the regression model approach; see, for example, Allison (1984) and King (1989). I complete the procedure described by Basawa and Rao by producing a computationally feasible method of computing the likelihood of an arbitrary sequence of war outbreaks assuming the random hazard model.

If the rate of war outbreak does change over time, the random hazard model might appear to be an over-simplified model for this process because it posits only two distinct rates of war outbreak. However, the rest of this chapter demonstrates that despite its simplicity, the random hazard model is nonetheless capable of producing statistically precise estimates of the rate of war outbreak over the time period observed by Small and Singer.

Distribution of War Outbreaks

Using the random hazard model, what is the distribution of the total number of war outbreaks during the time interval $[0,T]$, given the initial war intensity, $\lambda(0) \in \{\lambda_L, \lambda_H\}$? Let $X_n(t)$ be the event that exactly n wars occur during the time interval $[0,t]$. Let $P_n(t) \Leftarrow \Pr[X_n(t)]$, the probability of observing n wars during an interval of duration t. Let $Y_n(t)$ and $Z_n(t)$ be, respectively, the events that exactly n wars occur during $[0,t]$ and $\lambda(t)$ is λ_L or λ_H, respectively. Let $Q_n(t) \Leftarrow \Pr[Y_n(t)]$ and $R_n(t) \Leftarrow \Pr[Z_n(t)]$. Then

$$P_0(0) = 1 \qquad [1.5a]$$

$$P_0(t) = Q_0(t) + R_0(t) \qquad [1.5b]$$

$$Q_0'(t) = -(\lambda_L + \mu)\, Q_0(t) + \mu\, R_0(t) \qquad [1.5c]$$

$$R_0'(t) \quad = \quad \mu\, Q_0(t) - (\lambda_H + \mu)\, R_0(t) \qquad [1.5d]$$

where $\mu \Leftarrow \frac{\mu^*}{2}$, and $Q_0(0)$ is the probability that the initial intensity is λ_L; by equation 1.5b, $R_0(0) = 1 - Q_0(0)$. Equation 1.5c governs the probability of $Y_0(t)$, that $\lambda(t) = \lambda_L$ and no war occurs during $[0,t]$. In this equation, the first term is an "exit" term, representing (1) the probability of exit from $Y_0(t)$ into $Y_1(t)$ produced by the occurrence of war and (2) the probability of transition *out of* $Y_0(t)$ into $Z_0(t)$; the second term is an "entry" term, representing the probability of transition *into* $Y_0(t)$ from $Z_0(t)$. Equation 1.6d governs the probability of $Z_0(t)$, that $\lambda(t) = \lambda_H$ and no war has occurs during $[0,t]$. In this equation, the first term is produced by the entry transition from $\lambda(t) = \lambda_L$ to λ_H; the second term combines the exit transition from $\lambda(t) = \lambda_H$ to λ_L and the probability of exit from $Z_0(t)$ produced by the occurrence of war.

For $n = 1,2,\ldots$

$$P_n(0) \quad = \quad 0 \qquad [1.5e]$$

$$P_n(t) \quad = \quad Q_n(t) + R_n(t) \qquad [1.5f]$$

$$Q_n(0) \quad = \quad 0 \qquad [1.5g]$$

$$R_n(0) \quad = \quad 0 \qquad [1.5h]$$

$$Q_n'(t) \quad = \quad -(\lambda_L + \mu)\, Q_n(t) + \mu\, R_n(t) + \lambda_L\, Q_{n\text{-}1}(t) \qquad [1.5i]$$

$$R_n'(t) \quad = \quad \mu\, Q_n(t) - (\lambda_H + \mu)\, R_n(t) + \lambda_H\, R_{n\text{-}1}(t) \qquad [1.5j]$$

In equations 1.5i and 1.5j, the third term represents the entry into $Y_n(t)$ and $Z_n(t)$ from $Y_{n\text{-}1}(t)$ and $Z_{n\text{-}1}(t)$, respectively, produced by the occurrence of war.

This system of differential equations can be solved using generating functions. Let

$$Q(t,x) \Leftarrow \sum_{n=0}^{\infty} Q_n(t)\,(x+1)^n \qquad [1.6a]$$

$$R(t,x) \Leftarrow \sum_{n=0}^{\infty} R_n(t)\,(x+1)^n \qquad [1.6b]$$

$$P(t,x) \Leftarrow \sum_{n=0}^{\infty} P_n(t)\,(x+1)^n \qquad [1.6c]$$

Obviously, $P(t,x) = Q(t,x) + R(t,x)$. Because $0 \le P_n(t) \le 1$ for all n, these power series are all uniformly convergent for (at least) $-2 < x < 0$. Differentiating with respect to t:

$$\begin{bmatrix} Q_t(t,x) \\ R_t(t,x) \end{bmatrix} = \big(A + (x+1)\,B\big) \begin{bmatrix} Q(t,x) \\ R(t,x) \end{bmatrix} \qquad [1.7a]$$

where

$$A \Leftarrow \begin{bmatrix} -(\lambda_L+\mu) & \mu \\ \mu & -(\lambda_H+\mu) \end{bmatrix} \qquad [1.7b]$$

$$B \Leftarrow \begin{bmatrix} \lambda_L & 0 \\ 0 & \lambda_H \end{bmatrix} \qquad [1.7c]$$

Holding x fixed, equation 1.7a becomes a pair of ordinary differential equations. Thus

$$Q(t,x) = a(x)\,e^{k_1(x)\,t} + b(x)\,e^{k_2(x)\,t} \qquad [1.8a]$$

$$R(t,x) = c(x)\,e^{k_1(x)\,t} + d(x)\,e^{k_2(x)\,t} \qquad [1.8b]$$

where

$$\begin{bmatrix} a(x) & c(x) \\ b(x) & d(x) \end{bmatrix} \Leftarrow \begin{bmatrix} 1 & 1 \\ k_1(x) & k_2(x) \end{bmatrix}^{-1} \begin{bmatrix} Q_0(0) & (\lambda_L\,x - \mu)\,Q_0(0) + \mu\,R_0(0) \\ R_0(0) & \mu\,Q_0(0) + (\lambda_H\,x - \mu)\,R_0(0) \end{bmatrix}^T \qquad [1.8c]$$

$$k_1(x) \Leftarrow \lambda\,x - \mu + \Delta(x) \qquad [1.8d]$$

$$k_2(x) \Leftarrow \lambda\,x - \mu - \Delta(x) \qquad [1.8e]$$

$$\Delta(x) \Leftarrow \sqrt{\sigma^2\,x^2 + \mu^2} \qquad [1.8f]$$

Thus, in particular,

$$P(t,x) = \frac{\Delta(x) + \mu - \kappa\,\sigma\,x}{2\,\Delta(x)}\,e^{k_1(x)\,t} + \frac{\Delta(x) - \mu + \kappa\,\sigma\,x}{2\,\Delta(x)}\,e^{k_2(x)\,t} \qquad [1.9]$$

where $\kappa \Leftarrow (2\,Q_0(0) - 1)$. Equation 1.9 shows that for any t the Maclaurin series of $P(t,x)$ in x converges for all x, and converges uniformly on any bounded set of x.

Because $Q_n(t)$ and $R_n(t)$ are positive and bounded by $P_n(t)$, the same must hold true for $Q(t,x)$ and $R(t,x)$. The following results can now be obtained by setting $x = -1$ in equations 1.6a, 1.6b and 1.6c.

$$Q_0(t) = a(-1)\, e^{k_1(-1)\, t} + b(-1)\, e^{k_2(-1)\, t} \quad [1.10a]$$

$$R_0(t) = b(-1)\, e^{k_1(-1)\, t} + d(-1)\, e^{k_2(-1)\, t} \quad [1.10b]$$

$$P_0(t) = \big(a(-1) + c(-1)\big)\, e^{k_1(-1)\, t} + \big(b(-1) + d(-1)\big)\, e^{k_2(-1)\, t} \quad [1.10c]$$

Solutions for $n > 0$ can be determined in a similar fashion:

$$Q_n(t) \Leftarrow \frac{1}{n!} \frac{\partial^n}{\partial x^n} Q(t,x)\Big|_{x=-1} \quad [1.10d]$$

$$R_n(t) \Leftarrow \frac{1}{n!} \frac{\partial^n}{\partial x^n} R(t,x)\Big|_{x=-1} \quad [1.10e]$$

$$P_n(t) \Leftarrow \frac{1}{n!} \frac{\partial^n}{\partial x^n} P(t,x)\Big|_{x=-1} \quad [1.10f]$$

Setting $x = 0$ also produces useful information.

$$P(t,0) = \sum_{n=0}^{\infty} P_n(t) \equiv 1 \quad [1.11a]$$

Equation 1.11a verifies that the total probability that a non-negative number of wars has occurred during an interval of duration t is 1. Setting, in addition, $Q_0(0) = \frac{1}{2}$:

$$P_x(t,0) = \sum_{n=0}^{\infty} n\, P_n(t) = \lambda\, t \quad [1.11b]$$

$$P_{xx}(t,0) = \sum_{n=0}^{\infty} n\,(n-1)\, P_n(t) = \lambda^2\, t^2 + f_1(\sigma,\mu,t) \quad [1.11c]$$

$$P_{xxx}(t,0) = \sum_{n=0}^{\infty} \frac{n!}{(n-3)!} P_n(t) = \lambda^3\, t^3 + 3\, f_1(\sigma,\mu,t)\, \lambda\, t \quad [1.11d]$$

$$P_{xxxx}(t,0) = \sum_{n=0}^{\infty} \frac{n!}{(n-4)!} P_n(t) = \lambda^4\, t^4 + 6\, f_1(\sigma,\mu,t)\, \lambda^2\, t^2 + f_2(\sigma,\mu,t) \quad [1.11e]$$

where

$$f_1(\sigma,\mu,t) \Leftarrow \frac{\sigma^2}{2\mu^2}\left(e^{-2\mu t} - (1 - 2\mu t)\right) \quad [1.11f]$$

$$f_2(\sigma,\mu,t) \Leftarrow \frac{3\,\sigma^4}{2\mu^4}\left((3 - 4\mu t + 2\mu^2 t^2) - (3 + 2\mu t)\, e^{-2\mu t}\right) \quad [1.11g]$$

Let $N(\Delta t)$ be the number of war outbreaks during an interval of duration Δt when the initial intensity has equal probability of being either λ_L or λ_H. Then

$$E[N(\Delta t)] = \lambda\, \Delta t \quad [1.12a]$$

$$\mathrm{Var}[N(\Delta t)] = \lambda\, \Delta t + f_1(\sigma,\mu,\Delta t) \quad [1.12b]$$

Estimating the Parameters

Based on equation 1.12a, λ was estimated as the mean rate of war outbreak over all observations, $\hat{\lambda} = 1.36$ wars/year. To estimate σ^2 and μ, consider the statistic

$$\theta(\Delta t) \Leftarrow \frac{n \sum N^2(\Delta t) - \sum^2 N(\Delta t)}{n\,(n\text{-}1)} - \frac{\sum N(\Delta t)}{n} \quad [1.13a]$$

where the summations are taken over n observed intervals of duration Δt. $\theta(\Delta t)$ is an estimate of the difference between the variance in the number of war outbreaks from the simple Poisson process to the random hazard model. By equations 1.12a and 1.12b:

$$E[\theta(\Delta t)] = f_1(\sigma,\mu,\Delta t) \quad [1.13b]$$

Consequently, σ^2 and μ were estimated by minimizing the following weighted sum of squared errors between observed and predicted values of $\theta(\Delta t)$:

$$S(\sigma^2,\mu) \Leftarrow \sum \rho(\Delta \mathrm{t})\, (E[\theta(\Delta t)] - \theta(\Delta t))^2 \quad [1.13c]$$

$\theta(\Delta t)$ was observed for intervals of duration $\Delta t \in \{F_{13}, [\sqrt{F_{13}\, F_{14}}], F_{14}, [\sqrt{F_{14}\, F_{15}}], \ldots, F_{20}\}$ (in days);[2] here F_n is the nth Fibonacci number $\{F_1{=}1, F_2{=}1, F_3{=}2, \ldots, F_n = F_{n\text{-}1} +$

[2]This set of durations was chosen in an attempt to obtain as many potentially statistically independent observations of $\theta(\Delta t)$ as possible. Three heuristics were considered in selecting this set of durations. 1. Choose durations that are relatively prime (no common divisors) or nearly so, so

$F_{n-2}, \ldots\}$. For durations that were not exact divisors of $t_{225} - t_0$ (60266 days), intervals of equal duration at each end of the time series were excluded from the sample. For example, if $\Delta t \Leftarrow 21$ days, there would be 2869 intervals starting with [1816-1-10,1816-1-31) and ending with [1980-12-2,1980-12-23). The weights were chosen to standardize the squared errors:

$$\rho(\Delta t) \Leftarrow \frac{1}{\mathrm{Var}[\theta(\Delta t)]} \qquad [1.13d]$$

where

$$\mathrm{Var}[\theta(\Delta t)] = \frac{2\,(\lambda\,\Delta t + f_1(\sigma,\mu,\Delta t))^2}{n-1} + \frac{2\,f_1(\sigma,\mu,\Delta t) - 3\,f_1^2(\sigma,\mu,\Delta t) + f_2(\sigma,\mu,\Delta t)}{n} \qquad [1.13e]$$

The best fit was obtained using $\hat{\sigma} = .406$ wars/year, and $\hat{\mu} = .089$ transitions/year.

Table 1 lists the interval durations selected, together with observed and predicted values of $\theta(\Delta t)$, and the standard error of the observations from the expectations. Figure 3.1 displays the same information graphically. H_λ is the hypothesis that the war outbreaks were generated by a simple Poisson process ($\sigma^2 = 0$, $\mu = 0$).

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that sets of consecutive intervals using a given duration do not systematically coincide with intervals when using a longer duration. From the set of durations chosen, 84 of 105 pairs of durations are relatively prime. Only two of the pairs have common divisors larger than 10: (296 = 37 * 8, 4181 = 37 * 113) and (775 = 31 * 25, 3286 = 31 * 106); neither of these pairs produced similar standard errors from the model to the observations of $\theta(\Delta t)$ (which would be indicative of covariation). 2. Avoid durations for which the majority of the intervals contain no war outbreaks. If $N(\Delta t) = 0$ for an interval starting on a given day, then $N(\Delta t') = 0$ for all intervals starting on that day with $\Delta t' < \Delta t$, i.e., $N(\Delta t')$ is completely determined by $N(\Delta t)$. Conversely, the larger $N(\Delta t)$, the greater the freedom of $N(\Delta t')$, i.e., the greater the number of values $N(\Delta t')$ might take. Thus I eliminated all but one observation for which $\Delta t < 267$ days (the average duration between wars). This heuristic was empirically validated in that the standard error from the model to the observations of $\theta(\Delta t)$ was positive for each of the seven durations $\{F_8, [\sqrt{F_8 F_9}], \ldots, F_{11}\}$. For these short durations, we thus conclude that this was not a statistically independent set of observations of $\theta(\Delta t)$. 3. Avoid durations that are so long that $\theta(\Delta t)$ is computed from too small a sample of $N(\Delta t)$. This is pretty much a computational convenience because $\mathrm{Var}[\theta(\Delta t)]$ is correspondingly large for such durations, hence the parameter estimates would be relatively unaffected by these observations of $\theta(\Delta t)$ anyway. I arbitrarily eliminated durations longer than $\Delta t \Leftarrow F_{20}$ days because a sample size of 8 seemed quite small to me.

The selection of the set of durations used was successful in that the standard errors from fitting the model display negligible serial correlation. However, I did not attempt to determine whether this set was the largest possible. Using hindsight, if I were going to select another set, I would choose a set of the form $\{p_{nm}\}$, where p_{nm} is the prime number closest to $e^{n/m}$ and start exploring how large m could be without producing noticeable serial correlation.

Predicting War Outbreaks

Given the current level of intensity, we can use $P(t,x)$ (see equation 1.9) to predict the distribution of the total number war outbreak at any time in the future. To make this dependence explicit, let $P(q,t,x)$, $P_n(q,t)$, $Q_n(q,t)$ and $R_n(q,t)$ be $P(t,x)$, $P_n(t)$, $Q_n(t)$ and $R_n(t)$, respectively, where $q \Leftarrow \Pr[\lambda(0)=\lambda_L] = Q_0(0)$. Thus, to make predictions of future war outbreaks, we need to estimate q based on the past observations of war outbreaks.

In order to apply any statistical method to compute $\hat{q}$ (our estimate of q), we will need to evaluate the probability of a sequence of war outbreaks as a function of q. The sequence of war outbreaks in the data provided by Small and Singer is recorded in units of days. Note that because the random hazard model is time-symmetric (neither the transition intensity nor the transition p.d.f. change over time), the probability of past sequences can be computed by reversing the direction of time in the model, particularly in the system of differential equations (1.5a-j) and its solution(s). Thus it suffices to demonstrate how to compute the probability of sequences of the form $\mathbf{W}_{m,n} \Leftarrow \{W_i\}_{i=m}^{n}$, given $q_m \Leftarrow \Pr[\lambda(\text{day } m)=\lambda_L]$, where W_i is the number of war outbreaks on day i, i.e., during the interval [day i,day i+1); note that λ(day m) is the intensity at the *beginning* of day m. $\Pr[\mathbf{W}_{m,n} \mid q_m]$ can be broken down iteratively using the following identities. First, by partitioning the event $\mathbf{W}_{i,j}$ into independent observations of q_i (viz., $q_i = 1$ and $q_i = 0$).

$$\Pr[\mathbf{W}_{i,j} \mid q_i] = q_i \Pr[\mathbf{W}_{i,j} \mid 1] + (1 - q_i) \Pr[\mathbf{W}_{i,j} \mid 0] \qquad [1.14c]$$

Second, for any $i \leq j \leq k$:

$$\Pr[\mathbf{W}_{i,k} \mid q_i] = \Pr[\mathbf{W}_{i,j\text{-}1} \mid q_i] \Pr[\mathbf{W}_{j,k} \mid q(q_i, \mathbf{W}_{i,j\text{-}1})] \qquad [1.14a]$$

where

$$q(q_i,\mathbf{W}_{i,j\text{-}1}) \Leftarrow \Pr[\lambda(\text{day } j)=\lambda_L \mid q_i,\mathbf{W}_{i,j\text{-}1}] \quad [1.14b]$$

because the events $\mathbf{W}_{i,j\text{-}1}$ and $\mathbf{W}_{j,k}$ are independent aside from $q(q_i,\mathbf{W}_{i,j\text{-}1})$. Finally, by definition

$$\Pr[\mathbf{W}_{i,i} \mid q_i] \equiv P_{W_i}(q_i,1 \text{ day}) \quad [1.14d]$$

and

$$q(q_i,\mathbf{W}_{i,i}) \equiv \frac{Q_{W_i}(q_i,1 \text{ day})}{P_{W_i}(q_i,1 \text{ day})} \quad [1.14e]$$

Applying these identities:

$$\Pr[\mathbf{W}_{i,j} \mid q_i] = Q_{W_i}(q_i,1 \text{ day}) \Pr[\mathbf{W}_{i+1,j} \mid 1] + R_{W_i}(q_i,1 \text{ day}) \Pr[\mathbf{W}_{i+1,j} \mid 0] \quad [1.14f]$$

$\Pr[\mathbf{W}_{i,j} \mid q_i]$ can now be efficiently calculated for all $i \leq j$ by first iteratively computing $\Pr[\mathbf{W}_{i,j} \mid 1]$ and $\Pr[\mathbf{W}_{i,j} \mid 0]$ using equations 1.14f with $q_i \Leftarrow 0$ and $q_i \Leftarrow 1$.

In the random hazard model, the probability of the intensity at a given time is generated by a regular Markov process for which the equilibrium state has p.d.f. $f(\lambda,\sigma^2;y)$. This motivates the use of Bayes' rule with $g(q)$, the discrete uniform distribution on the sample space {0,1} (i.e., $g(0) \Leftarrow g(1) \Leftarrow \frac{1}{2}$), as the prior distribution to compute $\hat{q}$. It should be noted that other statistical methods, such as maximum likelihood estimation, could be used to estimate q now that $\Pr[\mathbf{W}_{i,j} \mid q_i]$ can be computed. Also, Bayes' estimation could be used with other prior distributions, such as the continuous uniform distribution on the sample space [0,1]. A second motivation for Bayes' estimation with $g(q)$ as prior is that, as shown below, this produces a *summary estimate* of $\Pr[\mathbf{W}_{i,j}]$, encapsulating all of the information contained in $\mathbf{W}_{i,j}$ that is required to produce estimates and test hypotheses (by any statistical method) under the random hazard model (see King, 1989: 12). Beyond this, I will not compare statistical methods in this paper. Instead, I will show that Bayes' estimation is sufficient for the empirical task at hand.

Using $g(q)$ as a prior, the Bayes' estimate of $\Pr[\lambda(\text{day } i)=\lambda_L \mid \mathbf{W}_{i,j}]$ is given by

$$\hat{q}_i(\mathbf{W}_{i,j}) = \frac{\Pr[\mathbf{W}_{i,j} \mid 1]}{\Pr[\mathbf{W}_{i,j} \mid 1] + \Pr[\mathbf{W}_{i,j} \mid 0]} \quad [1.15a]$$

Note that this relationship is invertible in the sense that, given $\hat{q}_i(\mathbf{W}_{i,j})$, we can determine the likelihood ratio between $\Pr[\mathbf{W}_{i,j} \mid 1]$ and $\Pr[\mathbf{W}_{i,j} \mid 0]$:

$$\frac{\Pr[\mathbf{W}_{i,j} \mid 1]}{\Pr[\mathbf{W}_{i,j} \mid 0]} = \frac{\hat{q}_i(\mathbf{W}_{i,j})}{1-\hat{q}_i(\mathbf{W}_{i,j})} \quad [1.15b]$$

Thus $\hat{q}_i(\mathbf{W}_{i,j})$ is a *summary estimate* of $\Pr[\lambda(\text{day } i)=\lambda_L \mid \mathbf{W}_{i,j}]$. In particular, any other statistical estimate can be expressed as a function of q_i. For example, let $\hat{q}^{\mathrm{ML}}{}_i(\mathbf{W}_{i,j})$ be the maximum likelihood estimate of $\Pr[\lambda(\text{day } i)=\lambda_L \mid \mathbf{W}_{i,j}]$. Then

$$\hat{q}^{\mathrm{ML}}{}_i(\mathbf{W}_{i,j}) \Leftarrow 1 \text{ if } \hat{q}_i(\mathbf{W}_{i,j}) > \tfrac{1}{2} \quad [1.15c]$$

$$\Leftarrow \tfrac{1}{2} \text{ if } \hat{q}_i(\mathbf{W}_{i,j}) = \tfrac{1}{2} \quad [1.15d]$$

$$\Leftarrow 0 \text{ if } \hat{q}_i(\mathbf{W}_{i,j}) < \tfrac{1}{2} \quad [1.15e]$$

Note, however, that $\hat{q}_i(\mathbf{W}_{i,j})$ cannot be recovered from $\hat{q}^{\mathrm{ML}}{}_i(\mathbf{W}_{i,j})$.

Applying equations 1.14c, 1.14f and 1.15b to equation 1.15a yields an iterative method for computing these estimates:

$$\hat{q}_i(\mathbf{W}_{i,j}) = f(W_i, \hat{q}_{i+1}(\mathbf{W}_{i+1,j})) \qquad \text{for } i \leq j \quad [1.16a]$$

$$\hat{q}_{j+1}(\mathbf{W}_{j+1,j}) = \tfrac{1}{2} \quad [1.16b]$$

where

$$f(W,q) \Leftarrow \frac{Q_W(1,1\text{ day})\, q + R_W(1,1\text{ day})\,(1-q)}{2\, Q_W(1/2,1\text{ day})\, q + 2\, R_W(1/2,1\text{ day})\,(1-q)} \quad [1.16c]$$

$$\mathbf{W}_{j+1,j} \Leftarrow \varnothing \quad [1.16d]$$

Applying the time symmetry of the random hazard model, the Bayes' estimate of $\Pr[\lambda(\text{day } j+1)=\lambda_L \mid \mathbf{W}_{i,j}]$,

$$\hat{q}_{j+1}(\mathbf{W}_{i,j}) = f(W_j, \hat{q}(\mathbf{W}_{i,j-1})) \quad \text{for } j \geq i \tag{1.17a}$$

$$\hat{q}_i(\mathbf{W}_{i,i-1}) = \frac{1}{2} \tag{1.17b}$$

Note that $\lambda(\text{day } j+1)$ is the intensity just after the end of day j. Just as $\hat{q}_i(\mathbf{W}_{i,j})$ is a summary estimate of $\Pr[\lambda(\text{day } i)=\lambda_L \mid \mathbf{W}_{i,j}]$, $\hat{q}_{j+1}(\mathbf{W}_{i,j})$ is also a summary estimate of $\Pr[\lambda(\text{day } j+1)=\lambda_L \mid \mathbf{W}_{i,j}]$. Figure 3.2a displays $\hat{q}_i(\mathbf{W}_{i0,i-1})$, where day $0 \Leftarrow$ Jan 1, 1904, day $i_0 = t_0$ (whence i_0 = -32141) and i ranges from i_0 to i_1, where day $i_1 \Leftarrow t_{225}$ (whence i_1 = 28124). The minimum mean-square estimate of the expected rate of war outbreak is thus given by (Basawa and Rao, 1980: 110)

$$E[\lambda \mid q] = q\ \lambda_L + (1\text{-q})\ \lambda_H \tag{1.17c}$$

Figure 3.2b displays the expected rate of war outbreak over historical time.

Because $\hat{q}_i(\mathbf{W}_{i0,i-1})$ is a summary estimate of $\Pr[\lambda(\text{day } i)=\lambda_L \mid \mathbf{W}_{i0,i-1}]$, any prediction about the sequence of war outbreaks from day i onwards can be derived from it. One of the most basic predictions is the distribution of future daily cumulations of war outbreaks. Let $\hat{p}(\mathbf{W}_{i,j})$ be the estimate of $\Pr[\mathbf{W}_{i,j} \mid \mathbf{W}_{i0,i-1}]$. Then

$$\hat{p}(\mathbf{W}_{i,j}) = \Pr[\mathbf{W}_{i,j} \mid \hat{q}_i(\mathbf{W}_{i0,i-1})] \tag{1.18a}$$

As another example, let $N_i(\Delta t)$ be the number of war outbreaks during the interval [day i, day $i + \Delta t$). Let $\hat{N}_i(\Delta t) \Leftarrow E[N_i(\Delta t) \mid q = \hat{q}_i(\mathbf{W}_{i0,i-1})]$ be the predicted number of war outbreaks during this interval. Then

$$\hat{N}_i(\Delta t) = P_x(\hat{q}_i(\mathbf{W}_{i0,i-1}), \Delta t, 0) \tag{1.18b}$$

$$= \hat{\lambda}\ \Delta t + \frac{\hat{\sigma}}{2\hat{\mu}}(1 - 2\,\hat{q}_i(\mathbf{W}_{i0,i-1}))\,(1 - e^{-2\hat{\mu}\Delta t}) \tag{1.18c}$$

Figure 3.3 shows $N_i(\Delta t)$ vs. $\hat{N}_i(\Delta t)$ sampled at yearly intervals (more precisely, $\Delta t \Leftarrow 365$ days). Even though the stochastic component of the variance is far larger than the systematic component, the linear regression estimate of the slope ($m = .81$) is tolerably close to the expected value ($m = 1$). The likelihood of observing this relationship between $N_i(\Delta t)$ and $\hat{N}_i(\Delta t)$ is three times greater assuming the random hazard model than assuming that $N_i(\Delta t)$ and $\hat{N}_i(\Delta t)$ are independent.[3]

A third example. Let w_i be the waiting time to the next war outbreak from the beginning of day i. Let $\hat{\Phi}_i(w)$ be the predicted c.d.f. of w_i. Then

$$\hat{\Phi}_i(w) = \Pr[w_i \le w \mid \hat{q}_i(\mathbf{W}_{i0,i-1})] = 1 - P_0(\hat{q}_i(\mathbf{W}_{i0,i-1}),w) \qquad [1.18d]$$

For a set of observations $\{w_i\}_{i=1}^{n}$, let

$$\Xi(\{\Phi_i\}_{i=1}^{n},x) \Leftarrow \frac{\mid \{w_i : \Phi_i(w_i) \le x\} \mid}{n} \qquad [1.18e]$$

Figure 3.4a compares $\Xi(\{\hat{\Phi}_i\}_{i=1}^{n},x)$ with $\Xi(\{\hat{\Phi}_\lambda\}_{i=1}^{n},x)$, where $\hat{\Phi}_\lambda(w)$ is the c.d.f. of w assuming the events were generated by a simple Poisson process, i.e., $\hat{\Phi}_\lambda(w) \Leftarrow 1 - e^{-\hat{\lambda}w}$, and i is sampled at intervals of duration $\Delta t = 365$ days. If $\{\Phi_i\}_{i=1}^{n}$ is a set of accurate c.d.f.'s of the corrresponding $\{w_i\}_{i=1}^{n}$, then $E[\Xi(\{\Phi_i\}_{i=1}^{n},x)] \equiv x$ and $\mathrm{Var}[\Xi(\{\Phi_i\}_{i=1}^{n},x)] \equiv \frac{x(1-x)}{n}$. If $\Xi(\{\Phi_i\}_{i=1}^{n},x) > x$, this indicates a tendency to underestimate $\Phi_i(w_i)$. This in turn indicates a tendency to overestimate the probability of waiting times longer than $\Phi_i^{-1}(x)$. Figure 3.4b shows the standard errors produced by using $\{\hat{\Phi}_i\}_{i=1}^{n}$ and $\{\hat{\Phi}_\lambda\}_{i=1}^{n}$ as estimates of the c.d.f. of waiting times. Quadratic regression curves are used to emphasize the overall trends. Figure 3.4b shows that $\{\hat{\Phi}_i\}_{i=1}^{n}$ has a slight (statistically tolerable) tendency to underestimate the proportion of waiting times longer than $\Phi_i^{-1}(x)$ for most x. $\{\hat{\Phi}_\lambda\}_{i=1}^{n}$ displays the same tendency to underestimate the proportion of longer waiting times; however, as x increases, the

[3]For $m = 1$, the t statistic $t_1 = -.372$; for $m = 0$, the t statistic $t_0 = 1.575$. Using the p.d.f. of the t distribution with r degrees of freedom, $p(r;t) = c(r)\,(1 + t^2/r)^{(r+1)/2}$, we find that the ratio of likelihoods between the two hypotheses, $p(163;t_1)/p(163;t_0) = 3.22$.

underestimation becomes more pronounced. Compared to the hypothesis that war outbreaks are generated by a simple Poisson process, the random hazard model produces a more accurate estimate of Φ_i by attributing greater likelihood to the hypothesis that the war outbreak intensity is low as we observe longer periods without war.

Testing the Random Hazard Model

In this section, I test the random hazard model in two ways. First, I show that the random hazard model is stronger than two competing hypotheses. One competing hypothesis, first proposed by Lewis Fry Richardson (1945: 242-250), is that war outbreaks are generated by a simple Poisson process. A second competing hypothesis, the null hypothesis, is that war outbreaks are generated by an event process that does not change over time. The random hazard model motivates statistical tests which force us to reject these competing hypotheses.

The second test is to see how well the random hazard model accounts for the systematic and stochastic components of the observed sequence of war outbreaks.[4] These tests show that the random hazard model accurately predicts both the expected value (the systematic component) and the variance (a key feature of the stochastic component) of $N_i(\Delta t)$, w_i, and $\hat{q}_i(\mathbf{W}_{i,j})$. Thus, from a statistical perspective, the random hazard model produces a rather complete account of a wide variety of features of the observed sequence of war outbreaks.

Test 1: Rejecting Alternative Hypotheses

In "The distribution of wars in time" Richardson (1945: 242-250) proposed and was not able to reject the hypothesis that the war outbreaks are generated by a simple Poisson process. Richardson contrasted this hypothesis with "the popular belief that a war can usually be blamed on one or two named persons." This contrast was not in principle irreconcilable, as "... there are similar contrasts in other social affairs; the

[4] King (1989: 7-12) provides a more thorough description of this inferential scheme.

statistics of marriage for example are in contrast with any biographer's account of the incidents that led two named persons to marry each other." Richardson used the statistical regularity of war outbreaks as evidence for the role of chance in history, proposing

> *as a working hypothesis, that every finite set of historical events is only a sample of what might have happened. Any quantitative theory of history is therefore not required to agree precisely with actual historical events but to agree only within the range of uncertainty ascribable to sampling.* (1960: 132; original emphasis)

This hypothesis was not merely intended to replace a deterministically causal framework with a statistical perspective permitting both systematic and stochastic components in scientific inference. Richardson argued that if the dependent variable's variation can be accounted for stochastically, then we need not seek systematic explanations.

> Various explanations of the increase from 143 [the number of wars between A.D. 1500 to 1715] to 156 [the number of wars between A.D. 1716 to 1931] are conceivable such as the growth of the world-population, or fuller information about the more recent years, or increase of capitalism. But it is idle to discuss any of them if the difference between 143 and 156 can be explained by chance. (1960: 137)

Houweling and Kuné (1984) state that "uncritical acceptance of Richardson's model of the war generating process has discouraged researchers from focusing on the number of outbreaks and their causes" because of "the impression ... that the Richardson model rules out meaningful causal questions for research on war outbreaks" (1984: 53-54). This assertion appears to be true: it is hard to find research in which the dependent variable is the number of war outbreaks in a given time period. Instead, researchers typically substitute other indicators in place of war outbreaks when war is their dependent variable. For example, in a section entitled '*The dependent variable – international war*', Wallace (1971: 25) writes

> Since the frequency of the outbreak of war is hardly a sensitive indicator either of conflict or the *amount* of war occurring in the international system, two more sophisticated indices compiled by Singer and Small were employed here. One is ... the number of nation-months of war experienced by each nation. The second is ... the number of battle deaths suffered ... . (original emphasis)

Houweling and Kuné argue that even if war outbreaks "occur randomly in time" (1984: 54), causal questions for research on war outbreaks can still be meaningful if we find systematic variation in war outbreaks among certain subintervals of time or among different actors. But they reach a conclusion much like Richardson's: if these efforts were unsuccessful, then "meaningful causal research ... would come to a standstill because of the lack of variation in substantive variables" (1984: 55).

In a more general setting, King argues against these conclusions:

> The stochastic component is not a technical annoyance, as it is sometimes treated, but is instead a critical part of the theoretical model: "The fundamental intellectual breakthrough that has accompanied the development of the modern science of statistical inference is the recognition that the random component has its own tenuous regularities that may be regarded as part of the underlying structure of the phenomenon" (Pollock, 1979: 1). (1989: 9).

Both systematic and stochastic regularities deserve scientific attention. If exactly one war occurred each year, would this lack of variation render causal research on the pattern of war outbreaks futile? On the contrary, wouldn't such precise regularity *demand* causal explanation? A simple Poisson process, despite its additional stochastic component, is just as precisely regular because both its systematic component, the intensity of the process, and its stochastic component are constant. Furthermore, the stochastic component of a simple Poisson process is completely determined by its systematic component. If we could account for the distribution of war outbreaks over the past several hundred years using a simple Poisson process, this would be a fascinating result. What mechanisms could guarantee the same rate of war outbreak, despite the vast changes in population, mobility, weapons, governments, etc., that have taken place throughout this time period? Unfortunately, we cannot investigate any of these causal hypotheses because we are forced to *reject* the simple Poisson process hypothesis!

If we hypothesize that the rate of war outbreak varies over time, then a more appropriate and more general null hypothesis is that war outbreaks follow some time-invariant event process. The simple Poisson process and a process that generates events at constant intervals are two examples of time-invariant event processes. The distribution of dependent variables, such as w_i and $N_i(\Delta t)$, of the particular time-invariant event process that might generate observed war outbreaks can be inferred by aggregating observations of these variables independent of their location in time. Thus I will compare three competing hypotheses: H_λ, the hypothesis that war outbreaks are generated by a simple Poisson process, H_0, the hypothesis that war outbreaks are generated by a time-invariant event process, and H_μ, the hypothesis that war outbreaks are generated according to the random hazard model.

Researchers employing a χ^2 test have found that the Poisson distribution fits the observed distribution of war outbreaks fairly well. Houweling and Kuné argue that this test is inappropriate because "it is impossible to make inferences on the sequence of war outbreaks in time from the distribution of outbreaks across years" (1984: 56). But the χ^2 test on the distribution of events is *not* logically inappropriate for assessing whether the events were generated by a simple Poisson process. If the "true" event process did not produce a Poisson distribution of events across years, then, given a large enough sample, the χ^2 test would allow us to reject H_λ. In particular, the random hazard model's parameters were estimated from information derived only from the distribution of outbreaks across years (viz., the mean and the variance), yet, as the examples at the end of the previous section and the statistical tests below demonstrate, the random hazard model *can* produce accurate inferences on the sequence of war outbreaks in time.

The correct reason that the χ^2 test does not allow us to reject H_λ is because it is a statistically weak test: β, the power of the test (i.e., the probability of rejecting the null hypothesis if the alternate hypothesis is true), is not much larger than the significance level α (the probability of false rejection, i.e., rejecting the null hypothesis if the null hypothesis is true). I ran a Monte Carlo experiment to estimate β

assuming that H_μ was actually true. Out of 1000 experimental trials, only 69 of the trials produced distributions of war outbreak for which the χ^2 test allowed us to reject H_λ when $\alpha = .05$. The probability of false retention of H_λ using the χ^2 test is about 93%.

The predictions of the random hazard model can be used to design stronger statistical tests than the χ^2 test. The Neyman-Pearson Theorem shows how to construct the most powerful test of the null hypothesis for a given sample (the test for which β is maximized), assuming a specific alternative hypothesis (see, for example, Hogg and Craig, 1978: 244). Unfortunately, it can be quite difficult to compute the n-dimensional rejection region for the test (n is the sample size). However, we can approximate this rejection region by computing the best rejection region among those that can be mapped linearly to a connected subset of $\Re$ (the real numbers). This procedure yields the following linear test statistic for testing one hypothesis H against another hypothesis H': let

$$S(H,H',\mathbf{X}) \Leftarrow \sum \rho_i(H',H)\, z_i(H) \qquad [3.1a]$$

where $\mathbf{X} = \{X_i\}_{i=1}^{n}$ is the sample, $z_i(H)$ is the standard error of the expectation of X_i assuming hypothesis H,

$$z_i(H) \Leftarrow \frac{X_i - E[X_i \mid H]}{\sqrt{\mathrm{Var}[X_i \mid H]}} \qquad [3.1b]$$

and $\rho_i(H',H)$ is the "weight" applied by H' to observation i,

$$\rho_i(H',H) \Leftarrow \frac{E[z_i(H) \mid H']}{\sqrt{\sum E^2[z_j(H) \mid H']}} \qquad [3.1c]$$

(see footnote[5] for simplifying assumptions and derivation). Defined as above,

[5]Assume that the $\{z_i\}$ are independent random variables. Assume that $\mathrm{Var}[z_i(H) \mid H'] = \sigma^2$ for all i. Without loss of generality, assume that $\sum \rho_i^2 = 1$, where $\rho_i \Leftarrow \rho_i(H',H)$. For brevity, let $\mu_i \Leftarrow \mathrm{E}[z_i(H) \mid H']$. Then $E[S(H,H',\mathbf{X}) \mid H] = \sum \rho_i\, \mu_i$ and $\mathrm{Var}[S(H,H',\mathbf{X}) \mid H] = \sum \rho_i^2\, \sigma^2 = \sigma^2$. Thus for any s we can maximize $\Pr[S(H,H',\mathbf{X}) > s \mid H]$ by maximizing $\sum \rho_i\, \mu_i$, subject to $\sum \rho_i^2 = 1$. Using the method of Lagrange multipliers, we find that $\rho_i = \lambda\, \mu_i$. Solving for λ yields equation 3.1c.

$E[S(H,H',\mathbf{X}) \mid H] = 0$ and $\mathrm{Var}[S(H,H',\mathbf{X}) \mid H] = 1$; thus $S(H,H',\mathbf{X})$ is approximately a standard normal random variable assuming H.[6] For a given significance level α, the power of the test $\beta = \Pr[S(H,H',\mathbf{X}) > S_\alpha \mid H']$, where $S_\alpha \Leftarrow F^{-1}(1 - \alpha)$ and $F(x)$ is the cumulative distribution of a standard normal random variable. Thus $\beta \approx 1 - F(\zeta_\alpha)$, where

$$\zeta \Leftarrow \frac{S(H,H',\mathbf{X}) - E[S(H,H',\mathbf{X}) \mid H']}{\sqrt{\mathrm{Var}[S(H,H',\mathbf{X}) \mid H']}} \qquad [3.1d]$$

$$\zeta_\alpha \Leftarrow \frac{S_\alpha - E[S(H,H',\mathbf{X}) \mid H']}{\sqrt{\mathrm{Var}[S(H,H',\mathbf{X}) \mid H']}} \qquad [3.1e]$$

$$E[S(H,H',\mathbf{X}) \mid H'] = \sqrt{\sum E^2[z_i(H) \mid H']} \qquad [3.1f]$$

$$\mathrm{Var}[S(H,H',\mathbf{X}) \mid H'] = \frac{\sum E^2[z_i(H) \mid H'] \,\mathrm{Var}[z_i(H) \mid H']}{\sum E^2[z_i(H) \mid H']} \qquad [3.1g]$$

because ζ is approximately a standard normal random variable under hypothesis H'.

Table 2 shows the results of several tests of H_μ vs. H_0 and H_λ, using $X_i \Leftarrow N_i(\Delta t)$ with durations $\Delta t \in \{p_{n,5}\}_{n=14}^{46}$ (in days) where $p_{n,m}$ is the prime number closest to $e^{n/m}$. Figure 3.5a displays the results of the tests of H_μ vs. H_0. Because $\mathrm{Var}[S(H_0,H_\mu,\mathbf{X}) \mid H_\mu] \approx 1$ over the entire range of sample durations Δt, the power of these tests pretty much varies directly with $E[S(H_0,H_\mu,\mathbf{X}) \mid H_\mu]$. In particular, when $E[S(H_0,H_\mu,\mathbf{X}) \mid H_\mu] > \zeta_\alpha$, the power of the test, $\beta > .5$ at significance level α, indicating that the test is more likely than not to succeed. Figure 3.5a shows how $\beta(\Delta t,\alpha)$ (the power of this test using duration Δt at significance level α) decreases as Δt increases; this is due primarily to the smaller sample size for longer durations. Figure 3.5a includes three reference lines: $\zeta = \zeta_{.05}$ (= 1.64, the critical value for significance level $\alpha = .05$), $\zeta = \zeta_{.01}$ (= 2.33, the critical value for significance level $\alpha = .01$), and $\zeta = 0$ (= $E[S(H_0,H_\mu,\mathbf{X}) \mid H_0]$, the expected value if H_0 were true).

For small Δt (the left side of the figure), the $S(H_0,H_\mu,\mathbf{X})$ are all very close to $E[S(H_0,H_\mu,\mathbf{X}) \mid H_\mu]$. This is because, as discussed in heuristic 2 of footnote 2 above, the

[6]To be precise, under the assumptions in the previous footnote, the distribution approaches a standard normal distribution as the sample size increases.

$S(H_0,H_\mu,\mathbf{X})$ will covary strongly for $\Delta t < 267$. Indeed, the mean square standard error of $S(H_0,H_\mu,\mathbf{X})$ from $E[S(H_0,H_\mu,\mathbf{X}) \mid H_\mu]$ for $\Delta t \le p_{27,5} = 223$ days, $\overline{z^2} = .04$. Even though this low error value shows that the random hazard model is quite accurate, $E[\overline{z^2}] = 1$ for statistically independent observations of $S(H_0,H_\mu,\mathbf{X})$. To reduce this source of bias in interpreting the test results, consider the tests only for the set of durations $\Delta t \in \{p_{n,5}\}_{n=27}^{46}$ (in days; $p_{28,5} = 271$). For this set of durations, the mean square standard error increases to $\overline{z^2} = .16$. Unfortunately, this value still indicates covariation among the $S(H_0,H_\mu,\mathbf{X})$, so that there is no simple way to combine the results of tests of H_μ vs. H_0 or H_λ to obtain greater confidence in the rejection of H_0 or H_λ, respectively.

Figures 3.5b and 3.5c display the results of tests of H_μ vs. H_0 and H_λ, respectively, for the reduced set of durations $\{p_{n,5}\}_{n=27}^{46}$ (in days). These tests show that we can confidently reject both H_0 and H_λ in favor of H_μ. At $\alpha = .05$, we reject H_0 in 8 of 20 tests, and H_λ in 12 of 20 tests; at $\alpha = .01$, we reject H_0 and H_λ once each out of 20 tests. These rates of rejection agree fairly well with the estimated powers of the tests.

Test 2: Predictive Precision of the Random Hazard Model

The previous series of tests lead to rejecting the hypothesis that war outbreaks are generated by any process in which the rate of war outbreak does not change over time. These tests, as well as the examples at the end of the previous section, also showed that the random hazard model accurately predicted the expected value of diverse features of the sequence of war outbreaks. In this section, I present a similar series of tests of the random hazard model's systematic *and* stochastic predictions.

Table 3 shows the results of several tests of H_0 and H_λ vs. H_μ, using $X_i \Leftarrow N_i(\Delta t)$ with durations $\Delta t \in \{p_{n,5}\}_{n=27}^{46}$ (in days). In these tests, the test statistic $S(H_\mu,H,\mathbf{X})$ gives greater weight to the more "extreme" predictions of the random hazard model when compared to the predictions of the other two models. Figures 3.6a and 3.6b display the results of tests of H_μ vs. H_0 and H_λ, respectively, for the reduced set of durations $\{p_{n,5}\}_{n=27}^{46}$ (in days). These tests show that we cannot reject H_μ in favor of either H_0 or H_λ.[7]

Perhaps the most direct test of the random hazard model is to see how well the predictive and postdictive summary estimates agree. We test whether $\hat{q}_i(\mathbf{W}_{t0,i-1})$, the probability that the war intensity is low at day i estimated from preceding observations of the sequence of war outbreaks, is consistent with, i.e., accurately predicts, $\hat{q}_i(\mathbf{W}_{i,t225})$, the estimated probability for day i given succeeding observations of war outbreaks. To conduct this test, we need to determine the distribution of $\hat{q}_i(\mathbf{W}_{i,t225})$ as a function of q_i.

Let $F_i(q,x)$ be the c.d.f. of $\hat{q}_i(\mathbf{W}_{i,t225})$ given that $q_i = q$, i.e.,

$$F_i(q,x) \Leftarrow \Pr\left[\hat{q}_i(\mathbf{W}_{i,t225}) \leq x \,\middle|\, \Pr[\lambda(\text{day } i) = \lambda_L] = q\right] \qquad [3.2a]$$

Note that $F_i(q,x) = q\, F_i(1,x) + (1-q)\, F_i(0,x)$. Then applying equation 1.16a

$$F_i(q,x) = \sum_{W=0}^{+\infty} \left(Q_W(q,1\,\text{day})\, F_{i+1}(1,h_W(x)) + R_W(q,1\,\text{day})\, F_{i+1}(0,h_W(x)) \right) \qquad [3.2b]$$

where $f(W,h_W(x)) \equiv x$ (with $f(W,q)$ as defined by equation 1.16c). The first term in the summation represents the contribution to $\hat{q}_i(\mathbf{W}_{i,t225})$'s c.d.f. when (1) W wars occur, (2) $q_i = q$, and (3) $q_{i+1} = 1$; similarly, the second term represents the contribution when W wars occur, (2) $q_i = q$, and (3) $q_{i+1} = 0$. $F_i(q,x)$ can now be computed for each day by first iteratively computing $F_i(1,x)$ and $F_i(0,x)$. $F_i(1,x)$ and $F_i(0,x)$ converge slowly to the limiting distributions $F(1,x)$ and $F(0,x)$; p.d.f.'s corresponding to these distributions appear in figure 3.7. The mean values and standard deviations of these limiting distributions are, respectively, $\bar{q}_1 = .612$, $\sigma_1 = .201$, $\bar{q}_0 = .388$, and $\sigma_0 = .216$. Thus

[7]Just as occurred in the tests of H_μ vs. H_0 and H_λ (table 2), the observed values of the test statistic actually lie somewhat closer to 0 than might be expected for statistically independent standardized normal random variables.

$$\lim_{(i-t_{225})\to-\infty}\left(E[\hat{q}_i(\mathbf{W}_{i,t225}) \mid \hat{q}_i(\mathbf{W}_{t0,i-1}) = q]\right)$$
$$= \overline{q}_1\, q + \overline{q}_0\,(1-q)$$
$$= .388 + .224\, q \qquad [3.2c]$$

$$\lim_{(i-t_{225})\to-\infty}\left(\mathrm{Var}[\hat{q}_i(\mathbf{W}_{i,t225}) \mid \hat{q}_i(\mathbf{W}_{t0,i-1}) = q]\right)$$
$$= q\,\sigma_1^2 + (1-q)\,\sigma_0^2 + q\,(1-q)\,(\overline{q}_1-\overline{q}_0)^2$$
$$= .0466 + .0437\, q - .0499\, q^2 \qquad [3.2d]$$

Thus the standard deviation has a range of [.201,.237] (the maximum is achieved at $q = .438$). The rate of convergence of the expected value to its limiting value is about .6 per year. Thus the distance from the limit shrinks by a factor of 10 in less than five years. Since the observations take place over a much longer time span than five years, and the limiting variance is larger than the initial error, the limiting estimate should be tolerably accurate for all observations.

Figure 3.8 shows $\hat{q}_i(\mathbf{W}_{i,t225})$ plotted against $\hat{q}_i(\mathbf{W}_{t0,i-1})$, sampled at intervals of 271 days. The figure also displays the linear regression estimate of this relationship, $\hat{q}_i(\mathbf{W}_{i,t225}) = .343 + .324\,\hat{q}_i(\mathbf{W}_{t0,i-1})$. This estimate is consistent with the predicted limiting relationship (equation 3.2c) The likelihood of observing this relationship between $\hat{q}_i(\mathbf{W}_{i,t225})$ and $\hat{q}_i(\mathbf{W}_{t0,i-1})$ is 200,000 times greater assuming the random hazard model than assuming that the two estimates are independent.[8] The estimated slope is actually slightly larger (not statistically significant), indicating that the predictions are more accurate than expected! This in turn suggests that the intensity transition rate may actually be lower than estimated. Additional evidence for this hypothesis appears below.

Measuring the Rate of War Outbreak

To obtain our best possible estimate of the rate of war outbreak at each point in

[8]For equation 3.2c, the t statistic $t_\mu = 1.605$; assuming independence, the t statistic $t_0 = 5.191$. The ratio of likelihoods between the two hypotheses, $p(221;t_\mu)/p(221;t_0) = 2.31\times10^5$.

time, we can combine both predictive and "postdictive" estimates to form a descriptive esimate of the rate of war outbreak. In order to combine these estimates, define the *odds* of an event whose probability is p,

$$\rho(p) \Leftarrow \frac{p}{1-p} \qquad [3.3a]$$

The odds that the intensity is low at a particular time given the entire observed sequence of war outbreaks is a product of the odds that the intensity is low given the sequence of war outbreaks preceding and succeeding that time:

$$\rho(\hat{q}_i(\mathbf{W}_{t0,t225})) = \rho(\hat{q}_i(\mathbf{W}_{t0,i\text{-}1}))\ \rho(\hat{q}_i(\mathbf{W}_{i,t225})) \qquad [3.3b]$$

To compute $\hat{q}_i(\mathbf{W}_{t0,t225})$, note that

$$p = \frac{\rho(p)}{1+\rho(p)} \qquad [3.3c]$$

$\hat{q}^{\mathrm{ML}}{}_i(\mathbf{W}_{i,j})$ can be computed from equations 1.15cde; $\hat{\lambda}_i(\mathbf{W}_{t0,t225})$ and $\hat{\lambda}^{\mathrm{ML}}{}_i(\mathbf{W}_{t0,t225})$ can be computed using equation 1.17c.

Figure 3.9a displays $\hat{\lambda}_i(\mathbf{W}_{t0,t225})$ and $\hat{\lambda}^{\mathrm{ML}}{}_i(\mathbf{W}_{t0,t225})$, respectively, the expected value and maximum likelihood descriptive estimates of the rate of war outbreak. There are three prominent "cycles" in the figure: (1) a period of low intensity, starting in 1816 and ending around 1846, followed by a period of high intensity until 1870; (2) a second period of low intensity lasting until 1893, followed by a period of moderately high intensity lasting until 1914; (3) a third period of low intensity lasting until 1944, followed by high intensity until 1981, the end of the observations.

There is an unanticipated and surprisingly close correspondence between these dates and Joshua Goldstein's base dating scheme of long cycles in prices: 1814, 1872, 1917, and 1980 are peak years for prices; while 1848, 1893 and 1940 are trough years for prices (1987: 577). The probability that two sets of five numbers out of 165 being as close as {1846,1870,1893,1914,1944} and {1848,1872,1893,1917,1940} is less than

1.7×10^{-6}.[9]

Goldstein used war severity (specifically, battle fatalities from great power war; 1988: 248) as an indicator of the war variable because he could not detect variation in the rate of war outbreak between the long cycle phases. Because the random hazard model allows us to estimate the rate of war outbreak, we can easily observe the relationship between prices and war. Figure 3.9b displays the observed mean rate of war outbreak for Goldstein's rising and falling price long cycle phases superimposed on the estimates of figure 3.9a. Table 4 shows the actual number of war outbreaks during each long cycle phase.

Thus we observe that when the rate of war outbreak is high, prices rise; when the rate of war outbreak is low, prices fall (or do not rise as fast; see Goldstein 1988: 185-9). By inferring resource levels from prices, this observation provides empirical motivation for the effect of war on resources in my "resource model" interpretation for Goldstein's long cycle theory (see chapter 2). Resource levels can be inferred from prices because prices are low when resources are abundant, and high when resources are scarce. In both the resource model and during the time period 1816 to 1980, resources are depleted when the rate of war outbreak is high, and resource levels increase when the rate of war outbreak is low.

Figure 3.9b also suggests that the rate of change in the rate of war outbreak varies inversely with major power price levels. When prices are at their lowest, the rate at which wars are initiated rises the fastest; when prices are at their highest, the rate at which wars are initiated falls the fastest. In my interpretation of Goldstein's verbal theory, the war derivative varies directly with the relative level of resources. This provides empirical motivation for the war derivative equation of the resource

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[9] There are $\binom{165}{5} = 958683033$ different sets of five distinct integers that can be drawn from {1816,...,1980}. To be as close as these two sets, one number must hit exactly, two other numbers must differ by no more than 2, the 4th number must differ at most 3, and the 5th number must differ by at most 4. Thus $p = \frac{1\times5\times5\times7\times9}{958683033} = 1.64\times10^{-6}$.

model.[10]

On another note, it appears highly likely that only five or seven transitions took place during the observed time span; thus it appears that $\mu \leq 7/165 = .042$ transitions/year (compared to the least squares estimate, $\hat{\mu} = .089$ transitions/year, whence we would expect 14 or 15 transitions between 1816 and 1981). As noted above, this may account for the stronger than predicted correspondence between intensity estimates based on the preceding and succeeding sequences of war outbreaks.

Conclusion

The random hazard model produces accurate predictive estimates of several features of the observed sequence of war outbreaks, including (1) the number of war outbreaks in a given duration, (2) the waiting time to the next war outbreak, and (3) the "postdictive" estimate of the rate of war outbreak. It is important to note that the model's parameters were *not* estimated by fitting the predicted values of any of these features to observed values. Instead, the parameters were fit only to cross-sectional summaries of the data[11] that individually eliminated information about the time sequence of the events. The accuracy of the model's estimates allow us to produce statistical tests powerful enough to reject the hypothesis that the rate of war outbreak does not vary over time.

Once variation in the rate of war outbreak has been established, we can more easily pursue empirical analysis into the determinants of war initiation. In order to use all available information to estimate the rate of war outbreak, I combined predictive and postdictive estimates to produce a descriptive estimate of the rate of war oubreak. The accuracy of these descriptive estimates cannot be assessed directly;

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[10]However, the theoretical account of why the rate of war outbreak should rise or fall when price levels are low or high, respectively, remains incomplete. In particular, Goldstein argues only that war *severity* rises and falls because of the availability of resources (1987: 591).
[11]Viz., the "excess variance" beyond that expected from a simple Poisson process.

however, the accuracy of the predictive estimates gives us confidence in forming the descriptive estimates. The descriptive estimates produce unanticipated corroboration of Goldstein's base long cycle dating scheme, and of my interpretive "resource model" for his long cycle theory.

Descriptive estimates produced using the simplified transition model, or refinements of this model, should be useful in testing other theories that hypothesize variation in the rate of war outbreak or other event processes. Because the simplified transition model produces estimates of precision as well as expectation, tests of these theories can be conducted with greater statistical thoroughness: for example, we can compute the power of statistical tests used.

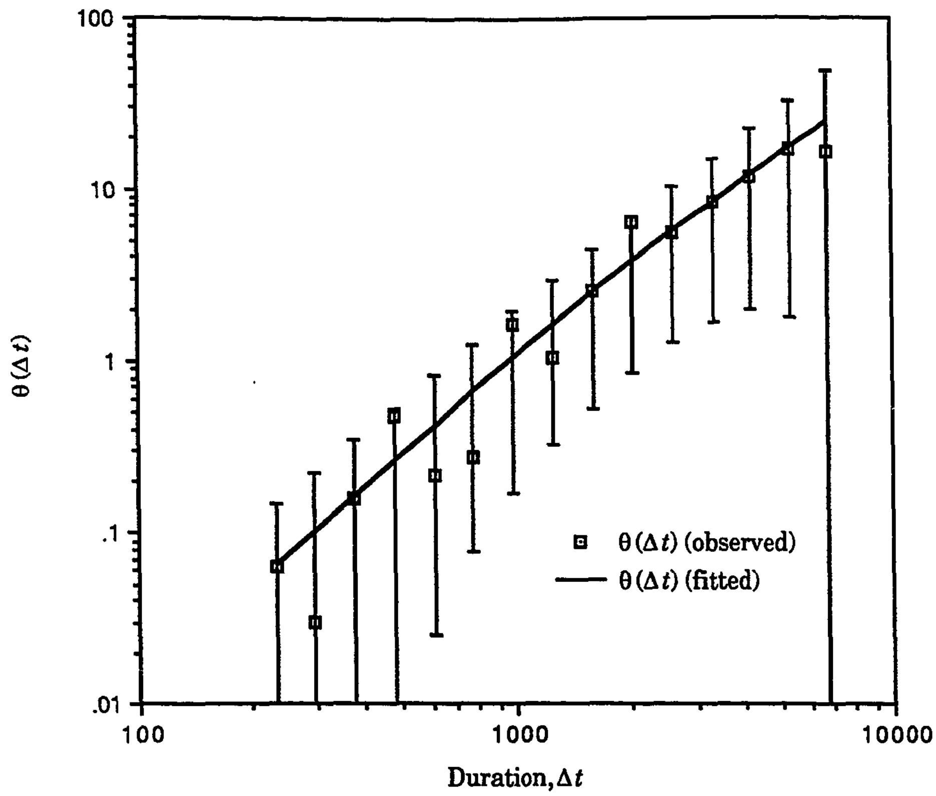

Figure 3.1

Observed and Fitted Values of θ(Δt) ("Excess Variance")

Error bars extend one standard error in each direction from fitted values.

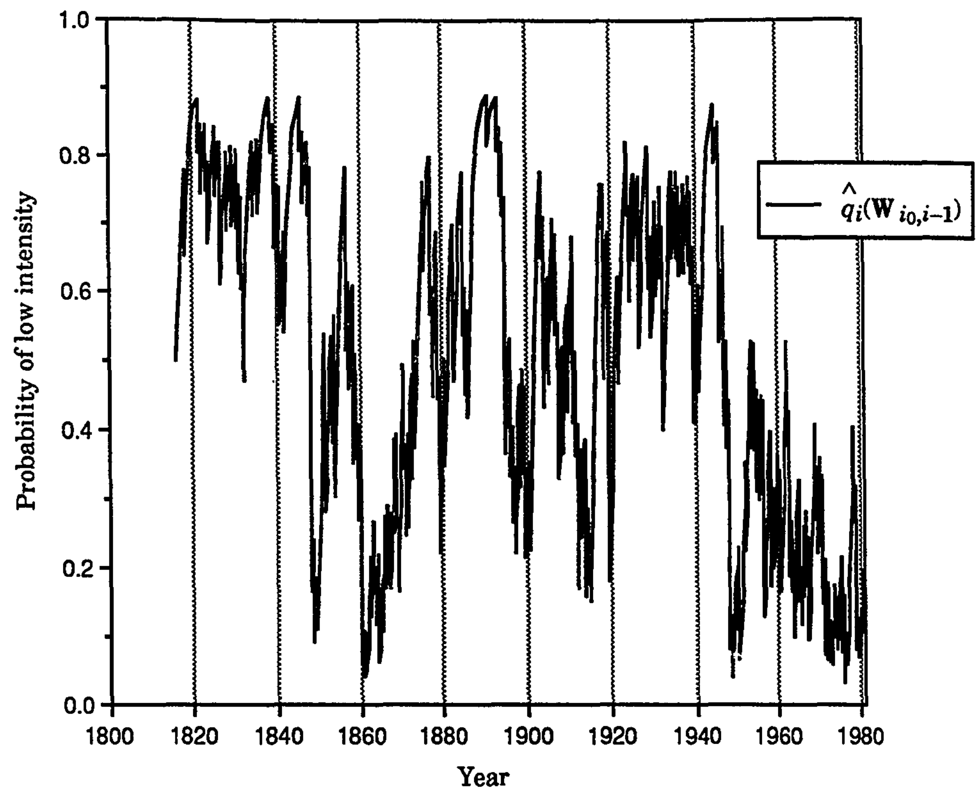

Figure 3.2a

Estimated Probability of Low Intensity

(observed war outbreaks precede date of estimate)

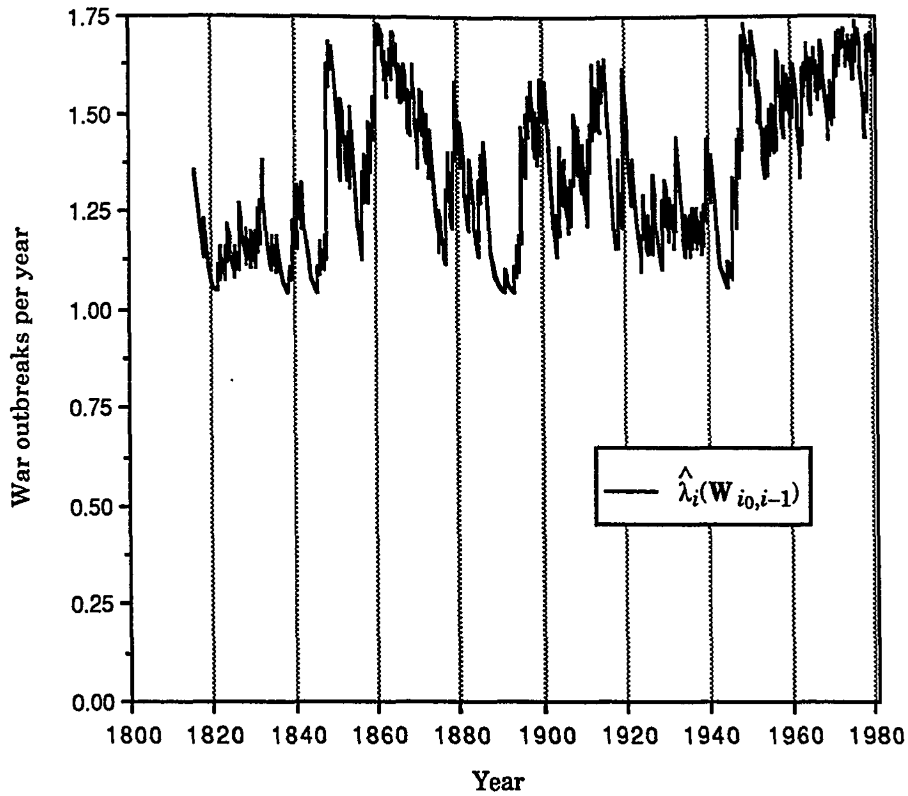

Figure 3.2b

Predicted Rate of War Outbreak

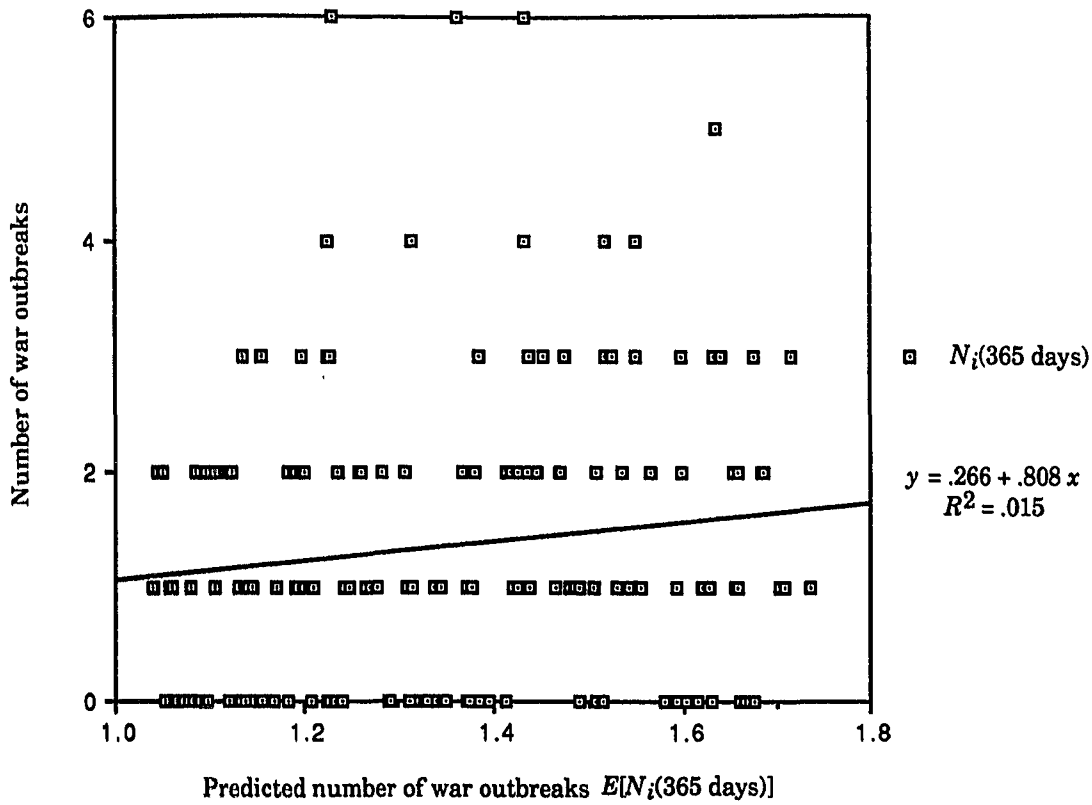

Figure 3.3

Actual vs. Predicted Number of War Outbreaks

(sampled at yearly intervals)

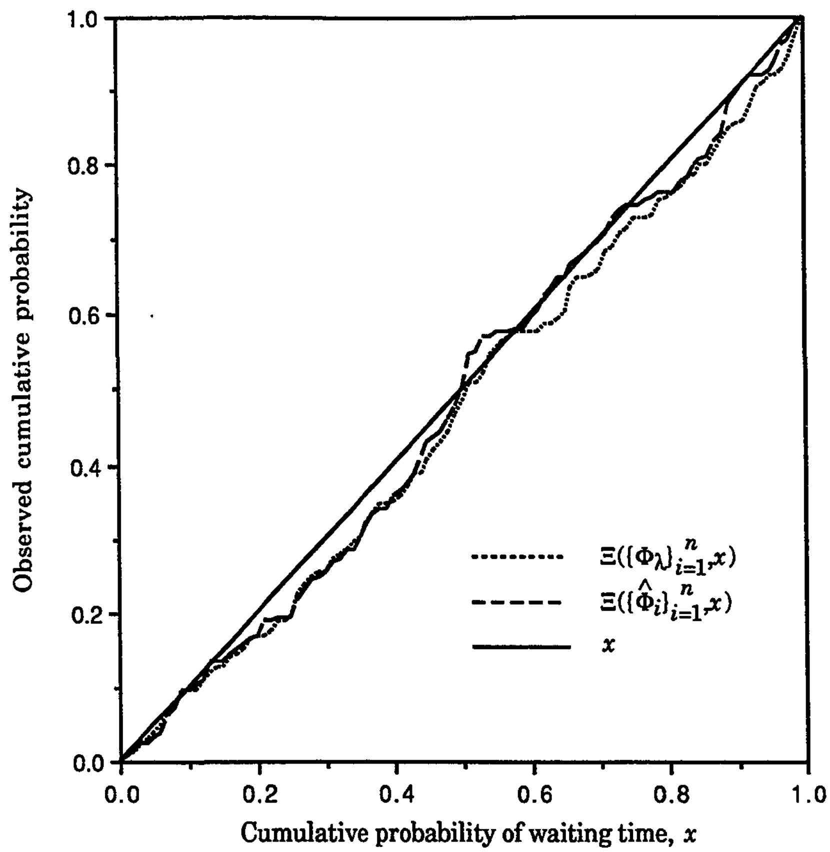

Figure 3.4a

Waiting time comparison

(sampled at yearly intervals)

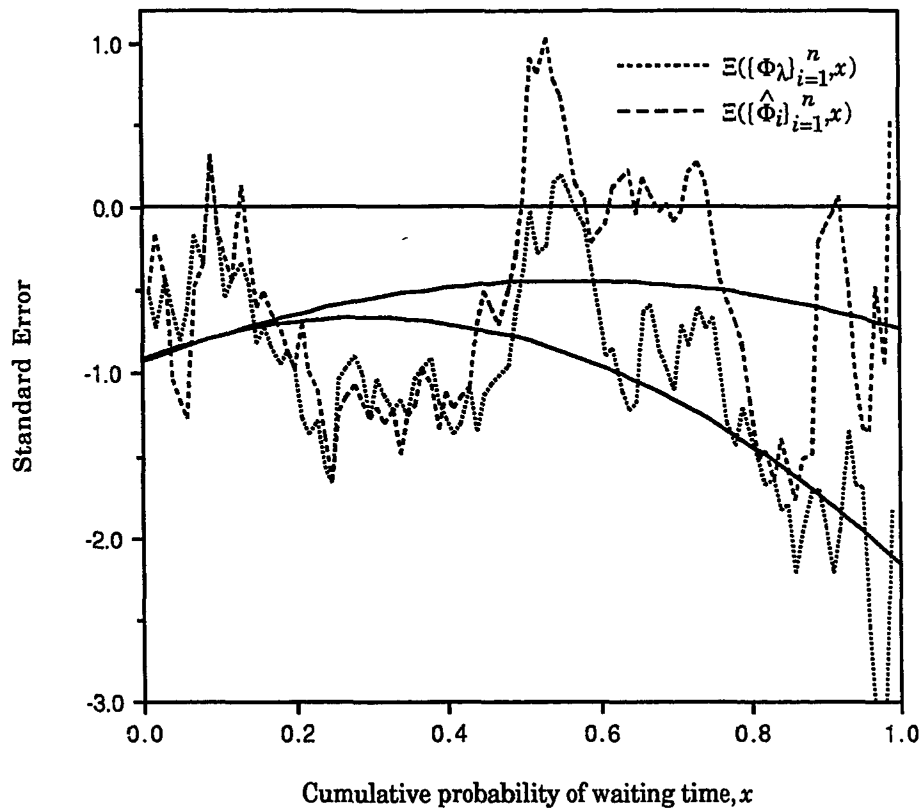

Figure 3.4b

Standard Errors of Waiting Time Predictions

(sampled at yearly intervals)

Positive (negative) standard error indicates under- (over-) estimated x, which in turn indicates over- (under-) estimated probability of longer waiting times. Quadratic regression curves are used to emphasize the overall trends: for $\Xi(\{\hat{\Phi}_\lambda\}_{i=1}^{n}, x)$, $y = -.92 + 1.74\,x - 2.98\,x^2$ ($R^2 = .422$); for $\Xi(\{\hat{\Phi}_i\}_{i=1}^{n}, x)$, $y = -.94 + 1.74\,x - 1.54\,x^2$ ($R^2 = .036$).

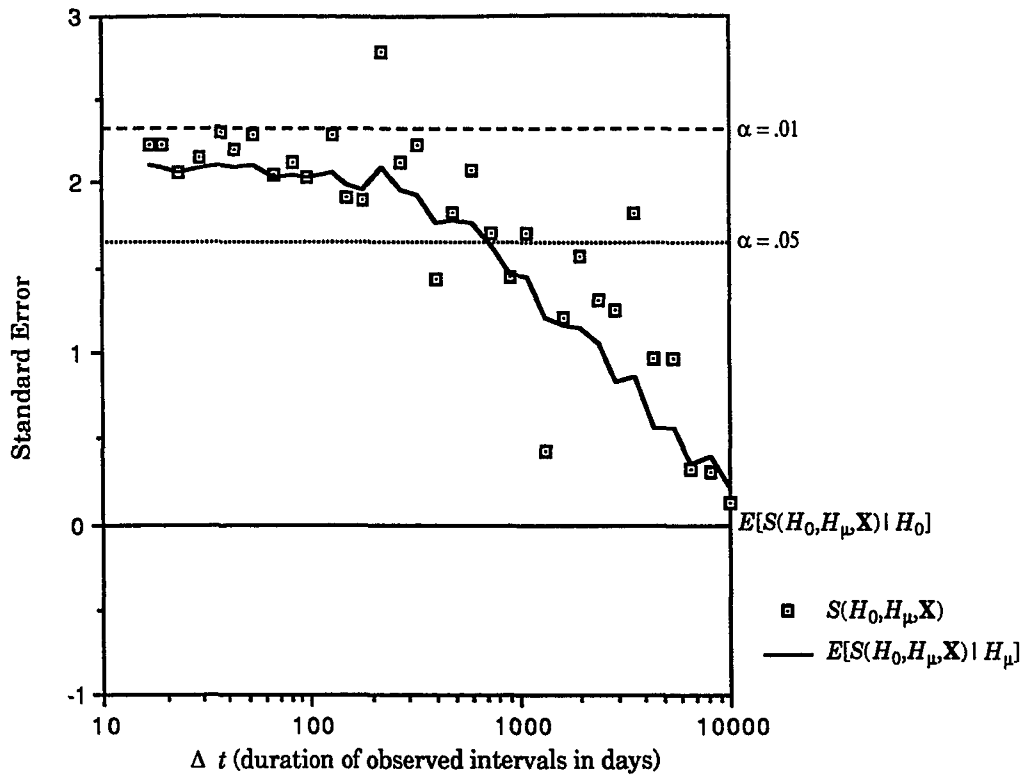

Figure 3.5a

Test of H_μ vs. H_0 Using $\mathbf{X} = \{N_i(\Delta t)\}$

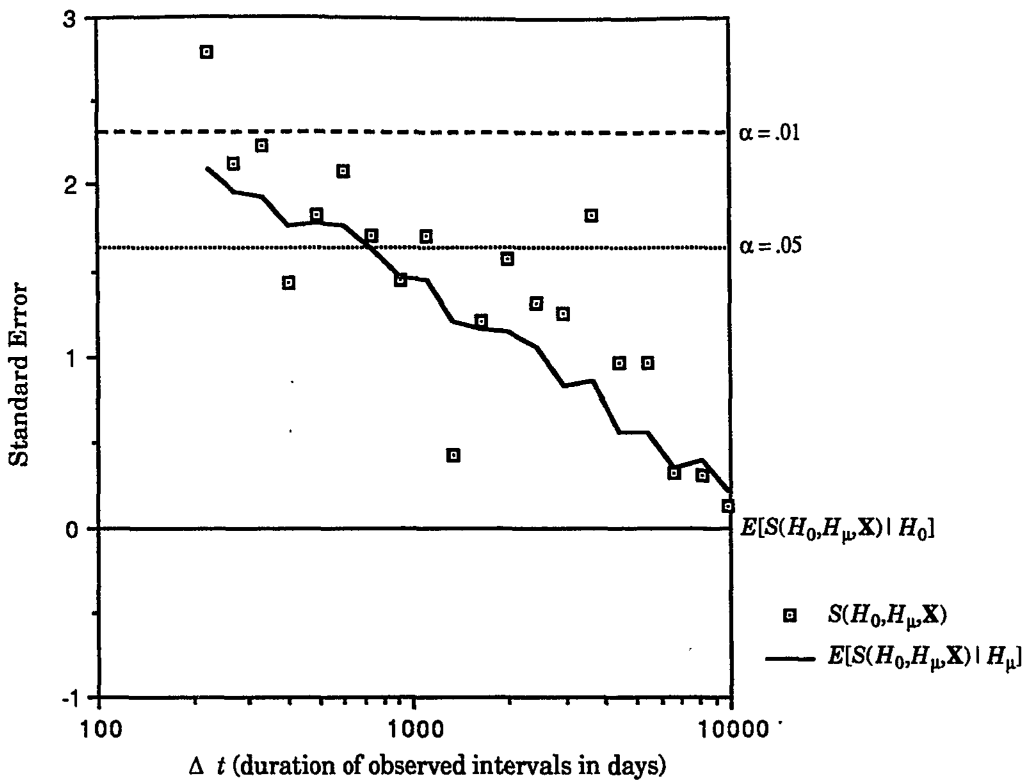

Figure 3.5b

Test of H_μ vs. H_0 Using $\mathbf{X} = \{N_i(\Delta t)\}$

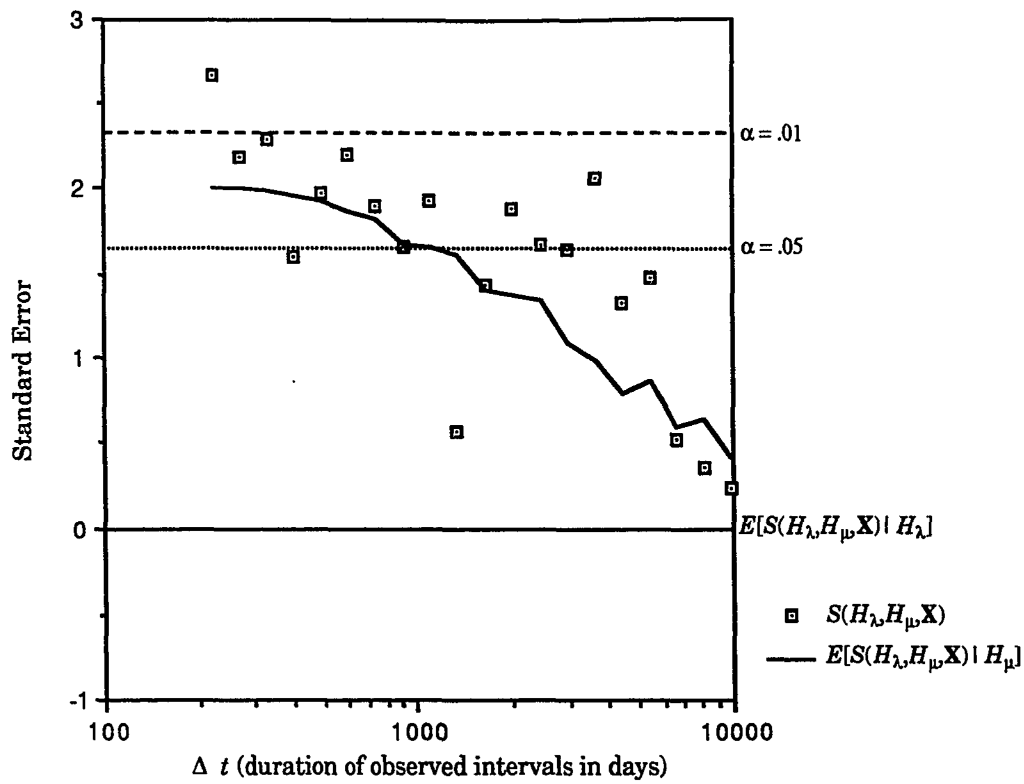

Figure 3.5c

Test of H_{μ} vs. H_{λ} Using $\mathbf{X} = \{N_i(\Delta t)\}$

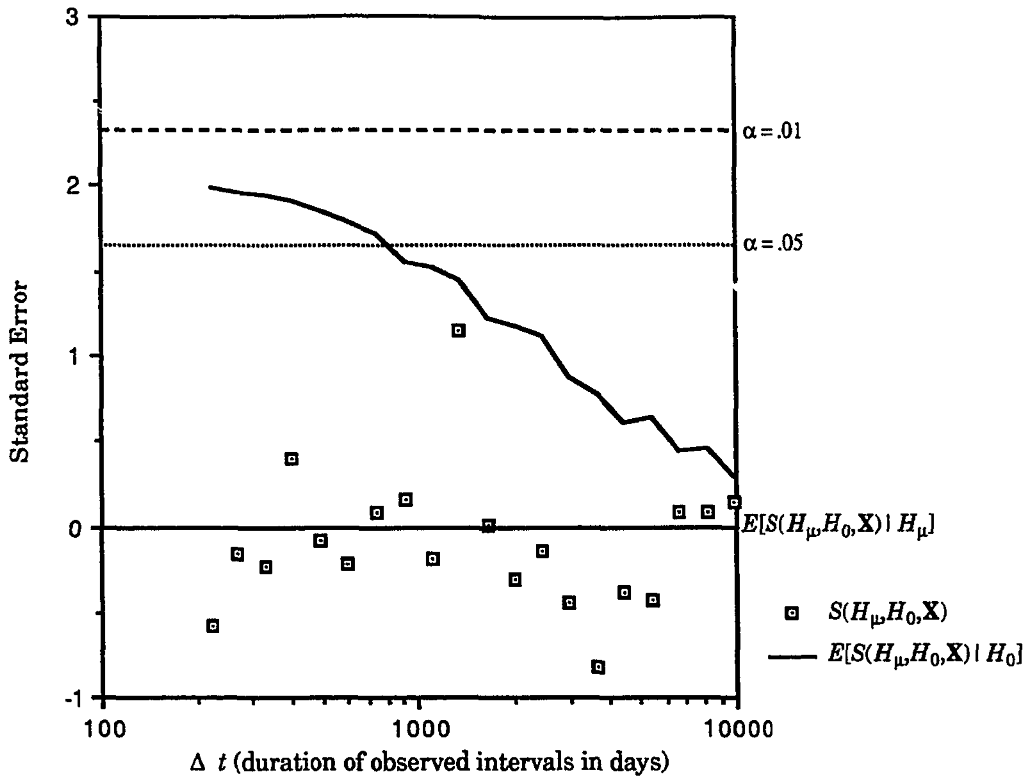

Figure 3.6a

Test of H_0 vs. H_μ Using $\mathbf{X} = \{N_i(\Delta t)\}$

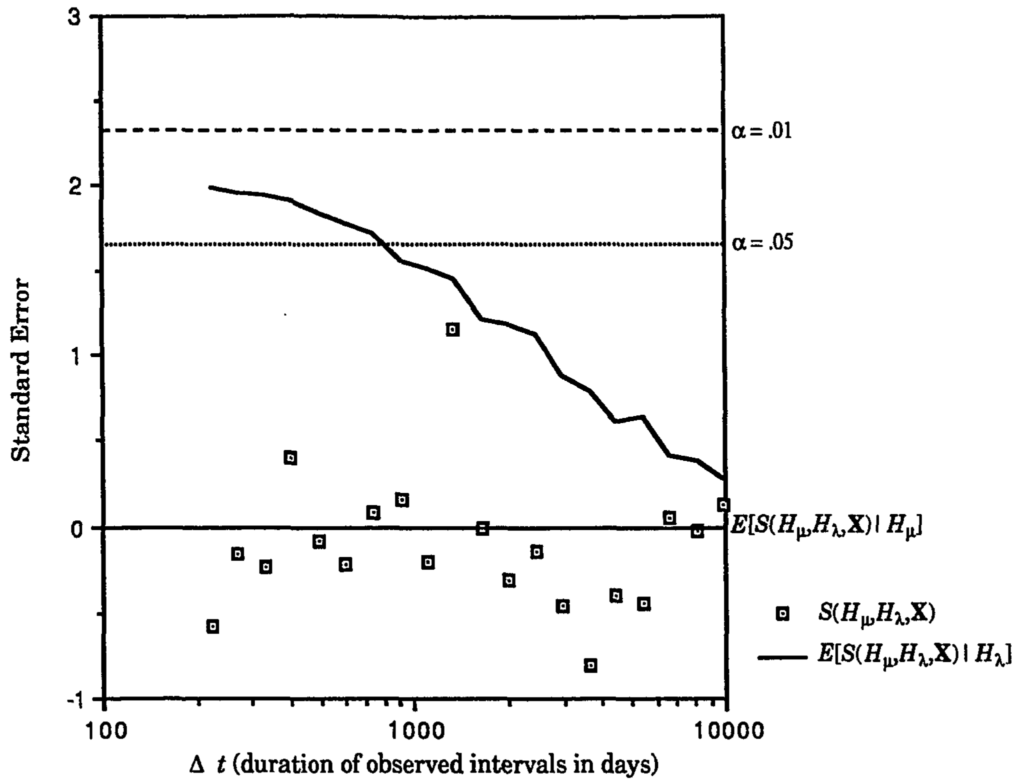

Figure 3.6b

Test of H_λ vs. H_μ Using $\mathbf{X} = \{N_i(\Delta t)\}$

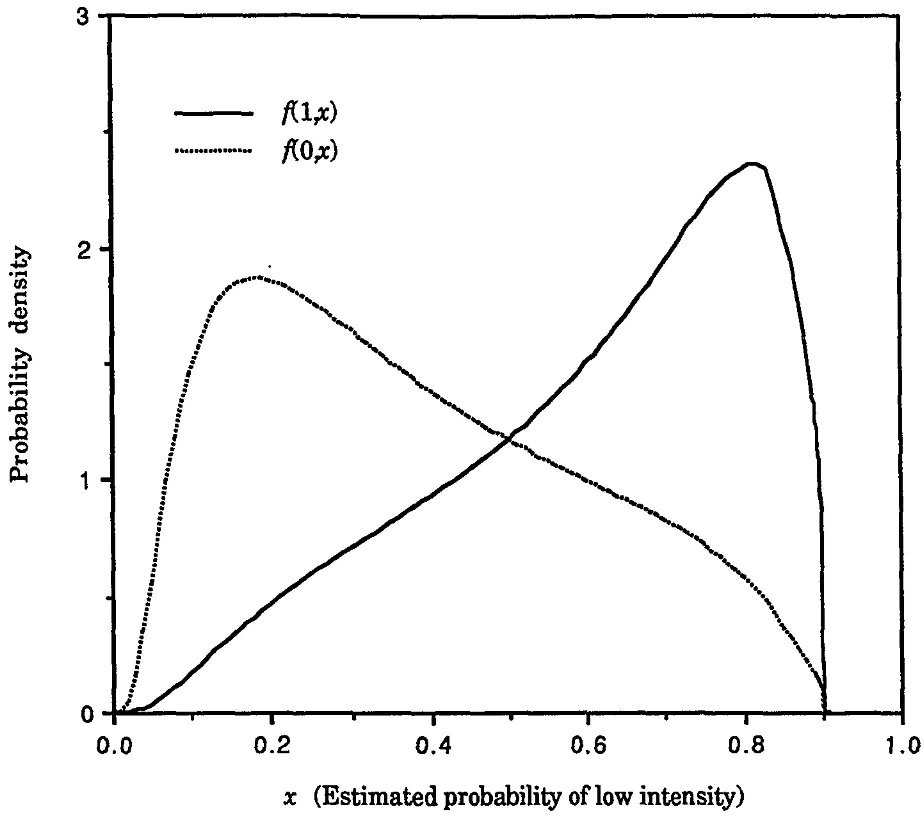

Figure 3.7

Density of Estimated Probability of Low Intensity

$f(q,x)$ is the limiting probability density function of x, the probability that the intensity is low, given q, the actual probability of low intensity.

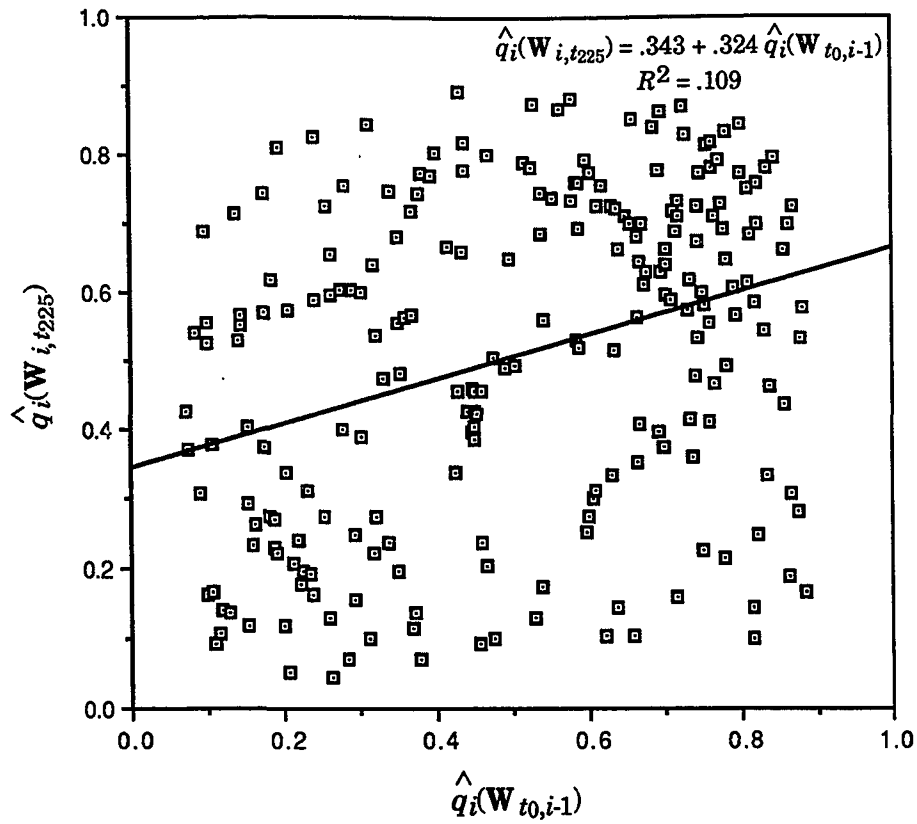

Figure 3.8

Postdicted vs. Predicted Estimates of Probability of Low Intensity

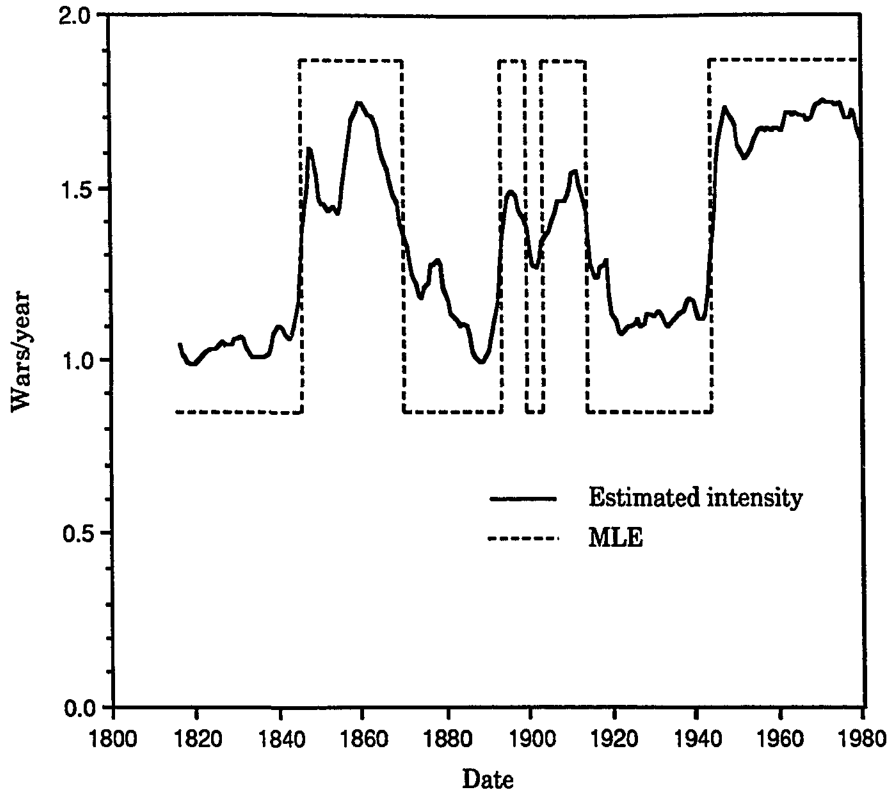

Figure 3.9a

Expected Value and Maximum Likelihood Estimates of the Rate of War Outbreak

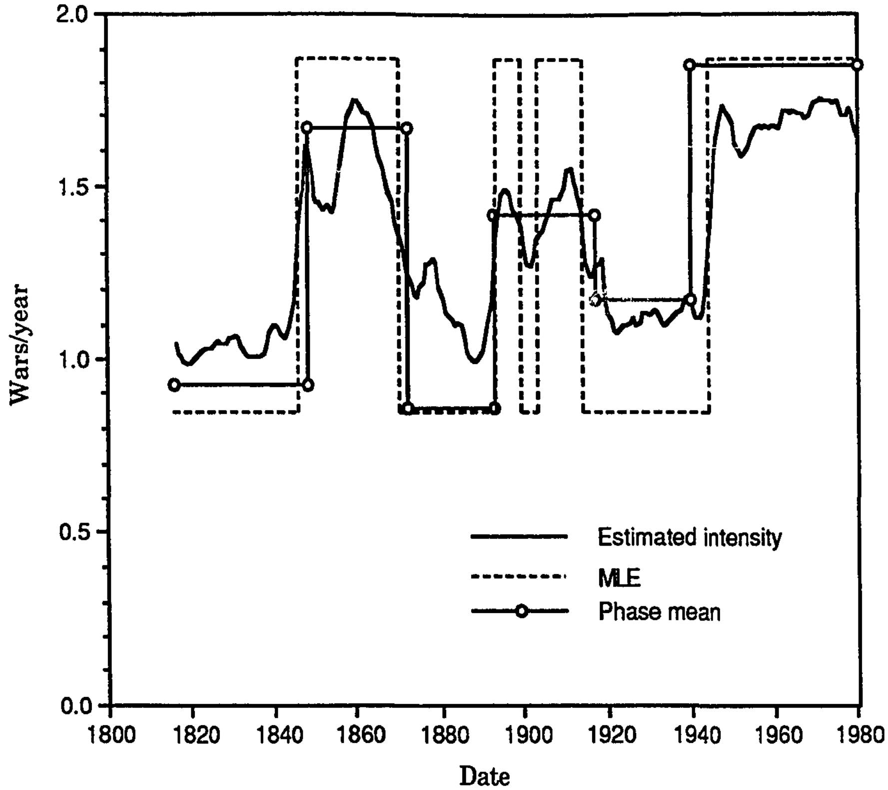

Figure 3.9b

Estimated and Observed Rate of War Outbreak

The phase mean is the observed mean rate of war outbreak during a long cycle phase (during which prices are rising or falling).

Table 1

Observed and Predicted Values of "Excess Variance"

Duration (in days) Δt	Intervals n	$\theta(\Delta t)$ Observed	Predicted by Best fit	$H\lambda$	Standard error Bestfit	$H\lambda$
233	258	6.38e–2	6.45e–2	0	-.01	.83
296	203	2.94e–2	1.03e–1	0	-.60	.27
377	159	1.56e–1	1.65e–1	0	-.05	.99
479	125	4.67e–1	2.62e–1	0	.77	2.07
610	98	2.15e–1	4.17e–1	0	-.52	.66
775	77	2.77e–1	6.56e–1	0	-.65	.59
987	61	1.62e+0	1.03e+0	0	.68	2.42
1255	48	1.05e+0	1.60e+0	0	-.43	1.09
1597	37	2.55e+0	2.47e+0	0	.04	1.83
2031	29	6.35e+0	3.76e+0	0	.89	3.15
2584	23	5.58e+0	5.65e+0	0	-.01	1.93
3286	18	8.41e+0	8.35e+0	0	.01	2.01
4181	14	1.18e+1	1.22e+1	0	-.04	1.93
5318	11	1.70e+1	1.73e+1	0	-.02	1.93
6765	8	1.61e+1	2.43e+1	0	-.33	1.20

Table 2

Test of H_μ vs. H_0 and H_λ using $X_i \Leftarrow N_i(\Delta t)$

Duration (in days)	Intervals	$S \Leftarrow S(H_0,H_\mu,\mathbf{X})$					$S \Leftarrow S(H_\lambda,H_\mu,\mathbf{X})$				
Δt	n	S	$E[S]$	$Var[S]$	$\beta(.05)$	$\beta(.01)$	S	$E[S]$	$Var[S]$	$\beta(.05)$	$\beta(.01)$
17	3545	2.233*	2.109	.990	.679	.413	2.263*	2.136	1.016	.687	.425
19	3171	2.227*	2.105	.988	.678	.412	2.258*	2.134	1.016	.686	.424
23	2620	2.071*	2.074	.961	.669	.398	2.130*	2.133	1.017	.686	.424
29	2078	2.164*	2.095	.983	.675	.408	2.202*	2.132	1.018	.685	.424
37	1628	2.307*	2.119	1.013	.681	.418	2.314*	2.125	1.019	.683	.421
43	1401	2.209*	2.090	.992	.673	.406	2.240*	2.120	1.021	.681	.419
53	1137	2.286*	2.105	1.013	.676	.413	2.296*	2.115	1.023	.679	.417
67	899	2.057*	2.043	.961	.658	.386	2.123*	2.108	1.024	.677	.415
83	726	2.125*	2.056	.989	.660	.393	2.169*	2.098	1.029	.672	.411
97	621	2.030*	2.039	.972	.655	.385	2.091*	2.100	1.031	.673	.412
127	474	2.293*	2.064	1.027	.660	.398	2.297*	2.068	1.031	.661	.399
149	404	1.913*	1.991	.951	.639	.365	1.998*	2.081	1.038	.666	.405
181	332	1.901*	1.956	.949	.625	.352	1.991*	2.051	1.044	.655	.394
223	270	2.788†	2.103	1.149	.665	.417	2.667†	2.012	1.052	.640	.380
271	222	2.132*	1.955	1.013	.621	.356	2.184*	2.004	1.065	.636	.378
331	182	2.226*	1.928	1.012	.611	.346	2.293*	1.986	1.074	.629	.371
401	150	1.439	1.767	.872	.552	.275	1.600	1.968	1.082	.622	.365
491	122	1.832*	1.786	.951	.558	.290	1.977*	1.929	1.108	.606	.353
601	100	2.076*	1.770	.999	.550	.289	2.197*	1.875	1.122	.586	.335
733	82	1.706*	1.634	.917	.496	.235	1.906*	1.828	1.147	.568	.321
907	66	1.452	1.464	.892	.424	.181	1.664*	1.683	1.179	.514	.277
1097	54	1.710*	1.453	.919	.420	.181	1.933*	1.657	1.196	.505	.270
1327	45	.430	1.210	.701	.302	.091	.568	1.618	1.253	.490	.263
1637	36	1.207	1.167	.911	.308	.112	1.430	1.405	1.318	.417	.211
1999	30	1.575	1.154	.973	.309	.117	1.880*	1.378	1.388	.411	.211
2441	24	1.309	1.056	.894	.267	.089	1.671*	1.345	1.452	.402	.208
2971	20	1.253	.827	.885	.192	.056	1.643*	1.092	1.542	.328	.160
3643	16	1.824*	.866	1.268	.245	.097	2.068*	.980	1.623	.301	.145
4447	13	.963	.570	.885	.127	.031	1.325	.786	1.683	.254	.118
5431	11	.969	.563	.763	.108	.022	1.477	.865	1.801	.280	.138
6637	9	.319	.350	.629	.051	.006	.512	.598	1.840	.220	.101
8101	7	.309	.404	.779	.080	.015	.347	.636	1.932	.234	.112
9901	6	.131	.213	.541	.026	.002	.228	.411	2.022	.193	.089

*Reject H_0 or H_λ with significance $\alpha = .05$.

†Reject H_0 or H_λ with $\alpha = .01$.

Table 3

Test of H_0 and H_λ vs. H_μ using $X_i \Leftarrow N_i(\Delta t)$

Duration (in days) Δt	Intervals n	$S \Leftarrow S(H_\mu, H_0, \mathbf{X})$ S	$E[S]$	Var[S]	β(.05)	β(.01)	$S \Leftarrow S(H_\mu, H_\lambda, \mathbf{X})$ S	$E[S]$	Var[S]	β(.05)	β(.01)
223	270	-.584	1.990	.921	.640	.363	-.585	1.987	1.004	.634	.368
271	222	-.158	1.968	1.040	.624	.363	-.160	1.964	.987	.626	.358
331	182	-.235	1.942	1.041	.615	.353	-.236	1.941	.980	.618	.349
401	150	.395	1.916	1.205	.598	.354	.393	1.911	.968	.606	.336
491	122	-.086	1.852	1.100	.578	.326	-.087	1.849	.941	.583	.311
601	100	-.214	1.790	1.044	.556	.300	-.218	1.783	.926	.557	.286
733	82	.084	1.723	1.132	.529	.285	.081	1.716	.901	.530	.260
907	66	.163	1.560	1.150	.469	.238	.159	1.554	.868	.461	.203
1097	54	-.181	1.526	1.120	.455	.225	-.200	1.505	.850	.440	.187
1327	45	1.158	1.456	1.467	.438	.236	1.152	1.446	.816	.413	.165
1637	36	.007	1.228	1.116	.347	.149	-.013	1.207	.762	.308	.100
1999	30	-.309	1.175	1.046	.323	.130	-.310	1.174	.733	.291	.089
2441	24	-.146	1.120	1.134	.311	.129	-.141	1.125	.701	.267	.076
2971	20	-.442	.882	1.141	.237	.088	-.453	.873	.651	.169	.036
3643	16	-.813	.771	.794	.163	.040	-.802	.782	.626	.138	.025
4447	13	-.388	.607	1.136	.165	.053	-.392	.604	.597	.089	.013
5431	11	-.431	.644	1.311	.191	.071	-.446	.637	.555	.088	.012
6637	9	.078	.441	1.593	.170	.068	.056	.417	.543	.048	.005
8101	7	.076	.458	1.284	.147	.050	-.016	.377	.515	.039	.003
9901	6	.140	.289	1.851	.160	.067	.126	.276	.492	.026	.002

*Reject H_μ with significance $\alpha = .05$.

†Reject H_μ with $\alpha = .01$.

Table 4

War Outbreaks During Long Cycle Phases

Price Phase	Year Started	Year Ended	Duration (in Years)	War Outbreaks	War Outbreaks per Year
Falling	1816	1848	32.5	30	.92
Rising	1848	1872	24.0	40	1.67
Falling	1872	1893	21.0	18	.86
Rising	1893	1917	24.0	34	1.42
Falling	1917	1940	23.0	27	1.17
Rising	1940	1980	40.5	75	1.85
Total over falling price phases			76.5	75	.98
Total over rising price phases			88.5	149	1.68

Let H_0 be the hypothesis that the rate of war outbreak is the same during both falling and rising price phases. Then the chi-square statistic, $\chi^2(1 \text{ d.f.}) = 14.95$. Since $\Pr[\chi^2(1 \text{ d.f.}) \geq 14.95 \mid H_0] < .0002$, we can reject H_0.

APPENDIX

PROOFS OF LEMMAS IN CHAPTER 2, "STABILITY IN THE ITERATED PRISONERS' DILEMMA"

Proof of Lemma 4a

Factoring equation 7.1 shows that a critical point must satisfy either $x = 0$ or:

$$(-E[X \mid Y] + E[X' \mid Y] - E[Y \mid X] + E[Y \mid X'])\, x\, y = (E[X \mid X] - E[X' \mid Y] - E[Y \mid X'] + E[Y \mid Y])\, y^2 - (E[X \mid X] + E[X \mid Y] - E[X' \mid Y] - E[Y \mid X'])\, y + \varepsilon \qquad \text{[A1.1]}$$

Equation 7.2 can be factored in the same way. Thus a critical point must satisfy either $x' = 0$ or:

$$(-E[X' \mid Y] + E[X \mid Y] - E[Y \mid X'] + E[Y \mid X])\, x'\, y = (E[X \mid X] - E[X \mid Y] - E[Y \mid X] + E[Y \mid Y])\, y^2 - (E[X \mid X] + E[X' \mid Y] - E[X \mid Y] - E[Y \mid X])\, y + \varepsilon \qquad \text{[A1.1']}$$

Because finding zeros of any two of equations 7.1, 7.2, and 7.3 determines a critical point of $\Delta(\Sigma;\varepsilon)$, all critical points of $\Delta(\Sigma;\varepsilon)$ can be found by (I) setting $x = 0$ and $x' = 0$, (II) substituting $x = 0$ into equation 7.3, (III) substituting $x' = 0$ into equation 7.3, or (IV) solving equations A1.1 and A1.1' simultaneously.

Part I: $x = 0$ and $x' = 0$. This yields $y = 1$, which does not tend to 0 with ε.

Part II: $x = 0$. Equation 7.3 can be factored after setting $x = 0$:

$$\left.\frac{dy}{dt}\right|_{x=0} = (1-y)\left((E[X \mid X] - E[X' \mid Y] - E[Y \mid X'] + E[Y \mid Y])\, y^2 - (E[X \mid X] - E[Y \mid X'])\, y + \varepsilon\right) \qquad \text{[A1.2]}$$

The critical point $y = 1$ was handled in part I. Let $a \Leftarrow E[X|X] - E[X'|Y] - E[Y|X'] + E[Y|Y]$ and $b \Leftarrow E[X|X] - E[Y|X']$. Then we seek "small" positive values of y satisfying the equation:

$$a\,y^2 - b\,y + \varepsilon = 0 \qquad [A1.3]$$

Part IIA: $b < 0$. For ε sufficiently small, there is no "small" critical point with $x = 0$. Part IIA1: If $a > 0$, then $\sqrt{b^2 - 4\,a\,\varepsilon} < -b$, whence the two roots of equation A1.3 are both negative. Part IIA2: If $a = 0$, then equation A1.3 has one negative root. Part IIA3. If $a < 0$, then the positive root of equation A1.3, $y = \frac{b}{a} + O(\varepsilon)$, which does not tend to 0 with ε.

Part IIB: $b = 0$. Then $a = -E[X'|Y] + E[Y|Y]$. For ε sufficiently small ($\varepsilon < -a$ to be precise), we obtain a critical point if and only if $a < 0$. This critical point occurs at

$$x = 0, x' = 1\text{-}y, y = \sqrt{\frac{\varepsilon}{-a}} \qquad [11.1]$$

Part IIC: $b > 0$. For ε sufficiently small, we obtain a critical point with

$$x = 0, x' = 1\text{-}y, y = \frac{\varepsilon}{b} + O(\varepsilon^2) \qquad [11.2]$$

Equation A1.3 has another root only if $a \neq 0$. This root occurs at $y = \frac{b}{a} + O(\varepsilon)$ which does not tend to 0 with ε.

Part III. $x' = 0$. Because of the symmetry between X and X', any results for $x = 0$ can be converted into results for x'=0 by exchanging X with X' and x with x'. Let $a' \Leftarrow E[X|X] - E[X|Y] - E[Y|X] + E[Y|Y]$ and $b' \Leftarrow E[X|X] - E[Y|X]$. Then we obtain the desired critical points if and only if either one of the two following conditions is satisfied.

Part IIIA: $b' = 0$. Then $a' = -E[X|Y] + E[Y|Y]$. For ε sufficiently small, we obtain a critical point if and only if $a' < 0$. This critical point occurs at

$$x = 1\text{-}y, x' = 0, y = \sqrt{\frac{\varepsilon}{-a'}} \qquad [11.3]$$

Part IIIB: $b' > 0$. For ε sufficiently small, we obtain a critical point with

$$x = 1\text{-}y, x' = 0, y = \frac{\varepsilon}{b'} + O(\varepsilon^2) \quad [11.4]$$

Part IV: Simultaneous solution of equations A1.1 and A1.1'. Equation A1.1 can be expressed as

$$\gamma x y = \alpha y^2 + \beta y + \varepsilon \quad [A1.4]$$

where $\alpha \Leftarrow E[X|X] - E[X'|Y] - E[Y|X'] + E[Y|Y]$, $\beta \Leftarrow -E[X|X] - E[X|Y] + E[X'|Y] + E[Y|X']$, and $\gamma \Leftarrow -E[X|Y] + E[X'|Y] - E[Y|X] + E[Y|X']$. Substituting equation A1.4 into equation A1.1' produces

$$\delta y = 0 \quad [A1.5]$$

where $\delta \Leftarrow E[X|Y] - E[X'|Y]$. We can assume that $y \neq 0$. This is because $E\big|_{y=0} = E[X]$, hence $\frac{dy}{dt}\Big|_{y=0} = \varepsilon$, hence no point for which $y = 0$ can be a critical point of $\Delta(\Sigma;\varepsilon)$ (for $\varepsilon > 0$). Thus critical points are produced only if $\delta = 0$. Substituting $E[X'|Y] \Leftarrow E[X|Y]$ into equation A1.1 shows that critical points appear on the conic section:

$$\alpha y^2 + (\beta - \gamma x) y + \varepsilon = 0 \quad [A1.6]$$

with $\beta = -E[X|X] + E[Y|X']$ and $\gamma = -E[Y|X] + E[Y|X']$.

Part IVA: $\gamma = 0$. The analysis of equation A1.3 can be adapted by substituting α for a and -β for b. We obtain a line of critical points if and only if either one of the two following conditions is satisfied.

Part IVA1: $\beta = 0$. Then $\alpha = -E[X|Y] + E[Y|Y]$. For ε sufficiently small, we obtain a critical point if and only if $\alpha < 0$. Critical points occur on the line

$$x \in [0, 1 - y], x' = 1 - x - y, y = \sqrt{\frac{\varepsilon}{-\alpha}} \quad [A1.7]$$

Part IVA2: $\beta < 0$. For ε sufficiently small, critical points occur on the line

$$x \in [0,1-y], x' = 1-x-y, y = \frac{\varepsilon}{-\beta} + O(\varepsilon^2) \qquad \text{[A1.8]}$$

Part IVB: $\gamma \neq 0$. In this case, we can solve for x explicitly as a function of y:

$$x = f(y) \Leftarrow \frac{\alpha y^2 + \beta y + \varepsilon}{\gamma y} \qquad \text{[A1.9]}$$

Critical points occur where the graph of (x,y) satisfying equation A1.9 intersects $\mathbb{D} = \{(x,y): x \in [0,1-y], y \in [0,1]\}$, the domain of $\Delta(\Sigma;\varepsilon)$. Thus all critical points can be determined by solving

$$0 \leq f(y) \leq 1-y \qquad \text{[A1.10]}$$

Because $\lim_{y \to 0^+} f(y) = \pm\infty \notin [0,1]$, if there are any critical points with "small" positive y, then there must be one such critical point satisfying either $f(y) = 0$ or $f(y) = 1-y$. This occurs if and only if (at least) one of the following four conditions holds:

Part IVB1: $\beta = 0$. Then $\alpha = -E[X \mid Y] + E[Y \mid Y]$. For ε sufficiently small, we obtain a critical point if and only if $\alpha < 0$. This critical point occurs at

$$x = 0, x' = 1-y, y = \sqrt{\frac{\varepsilon}{-\alpha}} \qquad \text{[A1.11]}$$

provided that $\varepsilon \leq E[X' \mid Y] - E[Y \mid Y]$.

Part IVB2: $\beta < 0$. For ε sufficiently small, we obtain a critical point at

$$x = 0, x' = 1-y, y = \frac{\varepsilon}{-\beta} + O(\varepsilon^2) \qquad \text{[A1.12]}$$

Part IVB3: $\beta = \gamma$. Then $E[Y \mid X] = E[X \mid X]$. For ε sufficiently small, we obtain a critical point if and only if $\alpha + \gamma = -E[X \mid Y] + E[Y \mid Y] < 0$. This critical point occurs at

$$x = 1-y, x' = 0, y = \sqrt{\frac{\varepsilon}{-(\alpha + \gamma)}} \qquad \text{[A1.13]}$$

Part IVB4: $\beta < \gamma$. For ε sufficiently small, we obtain a critical point with

$$x = 1\text{-}y,\ x' = 0,\ y = \frac{\varepsilon}{\gamma - \beta} + O(\varepsilon^2) \qquad \text{[A1.14]}$$

Part IVB (continued). If any of the above cases are satisfied, then the critical point obtained lies on a connected segment of the hyperbola given by equation A1.9. Let $C(\varepsilon)$ be the maximal connected segment of critical points containing this point. Because x is a continuous function of y for all $y > 0$, $C(\varepsilon)$ must intersect $\partial\mathbb{D}$, the boundary of the domain of the system, at least twice. Thus $\lim_{\varepsilon\to 0^+} \sup_{(x,x',y)\in C(\varepsilon)} y = 0$ only if two of the above cases are satisfied (specifically, one of cases IVB1 or IVB2 and one of cases IVB3 or IVB4). On the other hand, if two of the above cases are satisfied, then $\lim_{\varepsilon\to 0^+} \sup_{(x,x',y)\in C(\varepsilon)} y > 0$ can only occur if $f(y)$ intersects $\partial\mathbb{D}$ tangentially in one of these cases. For $f(y) = 0$ and $f'(y) = 0$ to hold, we must have $\alpha > 0$ and $\varepsilon = \frac{\beta^2}{4\,\alpha}$, which cannot occur for ε sufficiently small, but positive. For $f(y) = 1 - y$ and $f'(y) = -1$ to hold, we must have $\alpha + \gamma > 0$ and $\varepsilon = \frac{(\beta-\gamma)^2}{4\,(\alpha+\gamma)}$, which also cannot occur for ε sufficiently small, but positive. #

Proof of Lemma 4b

In cases 1 and 2, $x = 0$. The first partial derivatives of $\Delta(\Sigma;\varepsilon)$ at these points can be computed by first evaluating at $x = 0$:

$$\left.\frac{\partial\dot{x}}{\partial x}\right|_{x=0} = (E[X\,|\,Y] - E[X'\,|\,Y])\,y \qquad \text{[A2.1]}$$

$$\left.\frac{\partial\dot{x}}{\partial y}\right|_{x=0} = 0 \qquad \text{[A2.2]}$$

$$\left.\frac{\partial\dot{y}}{\partial x}\right|_{x=0} = (E[Y\,|\,X] - E[Y\,|\,X'])\,y - (E[X\,|\,Y] - E[X'\,|\,Y] + E[Y\,|\,X] - E[Y\,|\,X'])\,y^2 \qquad \text{[A2.3]}$$

$$\left.\frac{\partial\dot{y}}{\partial y}\right|_{x=0} = -\,(E[X\,|\,X] - E[Y\,|\,X']) + (3\,E[X\,|\,X] - 3\,E[Y\,|\,X'] + 2\,E[Y\,|\,Y] - 2\,E[X'\,|\,Y])\;y - 2\,(E[X\,|\,X] - E[X'\,|\,Y] - E[Y\,|\,X'] + E[Y\,|\,Y])\,y^2 \qquad \text{[A2.4]}$$

Because $\left.\frac{\partial \dot{x}}{\partial y}\right|_{x=0} = 0$, the eigenvalues are $\lambda_1 = \left.\frac{\partial \dot{x}}{\partial x}\right|_{(x,y)=(0,y_0)}$ and $\lambda_2 = \left.\frac{\partial \dot{y}}{\partial y}\right|_{(x,y)=(0,y_0)}$, where y_0 satisfies equation A1.3. Eigenvalues for case 2 are obtained by setting $E[X|X] = E[Y|X']$, also. Eigenvalues for cases 3 and 4 can be obtained by exchanging X with X' and x with x' in cases 1 and 2, respectively. #

Proof of Lemma 4c

Part 1. Let S be a proper subset of $C(\varepsilon)$, ($S \neq \varnothing$ and $S^c \Leftarrow C(\varepsilon) - S \neq \varnothing$). Then because $C(\varepsilon)$ is connected, either S or S^c has an accumulation point belonging to the other set. Thus for any $\delta > 0$, there exist $s_1 \in S$ and $s_2 \in S^c$ such that the distance between these two points, $d(s_1,s_2) < \delta$. Thus there are arbitrarily small perturbations from S that do not return to S.

Part 2. Lemma 4a showed that $\lim_{\varepsilon \to 0^+} \sup_{(x,x',y) \in C(\varepsilon)} y = 0$ if and only if the conditions required hold. The proof is completed by showing that these conditions also guarantee asymptotic stability. Let

$$V(x,y) \Leftarrow f^2(x,y) \qquad \text{[A3.1]}$$

where $f(x,y)$ is given in equation 11.5. Then

$$V(x,y)\big|_{(x,x',y) \in C(\varepsilon)} = 0 \qquad \text{[A3.2]}$$

Because $C(\varepsilon)$ is maximal, there exists $\delta > 0$ such that

$$V(x,y)\big|_{(x,x',y) \in P(\delta)} > 0 \qquad \text{[A3.3]}$$

where $P(\delta) \Leftarrow \{s : s \in D\text{-}C(\varepsilon);\ d(s,C(\varepsilon)) < \delta\}$ and D is the domain of $\Delta(\Sigma;\varepsilon)$. The proof is completed by showing that there exists $\delta > 0$ such that for all $(x(0),x'(0),y(0)) \in P(\delta)$, $\lim_{t \to \infty} V(x(t),y(t)) = 0$. We have

$$\dot{V}(x,y) = 2\, V(x,y)\, g(x,y) \qquad \text{[A3.4]}$$

where

$$g(x,y) = -x\, f_x(x,y) + (1-y)\, f_y(x,y) \qquad \text{[A3.5]}$$

$$g(x,y)\Big|_{(x,x',y)\in C(\varepsilon)} = -\frac{\varepsilon - 2\,\varepsilon\, y + (E[X\,|\,Y] - E[Y\,|\,Y])\, y^2}{y} \qquad \text{[A3.6]}$$

Thus if $g(x,y) < 0$, then $\lim_{t\to\infty} V(x(t),y(t)) = 0$. Applying either condition 1 or 3 of lemma 4a also produces $g(x,y) < 0$ because

$$g(x,y) = -\frac{\varepsilon + O(\varepsilon^2)}{y} < 0 \qquad \text{[A3.7]}$$

Applying either condition 2 or 4 of lemma 4a:

$$g(x,y) = -\frac{2\,\varepsilon\,(1-y)}{y} < 0 \qquad \text{[A3.8]}$$

Thus $C(\varepsilon)$ is asymptotically stable with $\lim_{\varepsilon\to 0^+} \sup_{(x,x',y)\in C(\varepsilon)} y = 0$ if and only if the required conditions hold. #

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