New descriptions of the Lovász number, and the weak sandwich theorem

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Abstract. In the seminal work [8] L. Lovász introduced the concept of an orthonormal representation of a graph, and also a related value, now popularly known as the Lovász number of the graph. One of the remarkable properties of the Lovász number is that it lies sandwiched between the stability number and the complementer chromatic number. This fact is called the sandwich theorem.

In this paper, using new descriptions of the Lovász number and linear algebraic lemmas we give three proofs for a weaker version of the sandwich theorem.

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1 Introduction

From the several remarkable properties of the Lovász number of a graph we mention here only the sandwich theorem: the Lovász number lies 'sandwiched' between the stability number, and the chromatic number of the complementer graph. A weaker form of this sandwich theorem will be derived here using new descriptions of the Lovász number. This weak sandwich theorem is the immediate consequence of the sandwich theorem, Brooks' Theorem (concerning an upper bound on the chromatic number), and the counterpart of Brooks' Theorem (concerning a lower bound on the stability number). In this paper our aim is to give more direct proofs.

We begin this paper with stating the above-mentioned results. First we fix some notation. Let $n \in \mathcal{N}$, and let $G = (V(G), E(G))$ be an undirected graph, with vertex set $V(G) = \{1, \ldots, n\}$, and with edge set $E(G) \subseteq \{\{i, j\} : i \neq j\}$. The complementer graph will be denoted by $\overline{G}$. Thus $\overline{G} = (V(\overline{G}), E(\overline{G}))$ where $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{\{i, j\} \subseteq V(G) : i \neq j,\ \{i, j\} \notin E(G)\}$.

In the seminal work [8] L. Lovász introduced the following number, $\vartheta(G)$,

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now popularly known as the *Lovász number* of the graph G ([7]):

$$\vartheta(G) := \inf \left\{ \max_{1\le i\le n} \frac{1}{(a_i a_i^T)_{11}} \;\middle|\; \begin{array}{l} m \in \mathcal{N};\ a_i \in \mathcal{R}^m \ (i=1,\dots,n); \\ a_i^T a_i = 1 \ (i=1,\dots,n); \\ a_i^T a_j = 0 \ (\{i,j\} \in E(\overline{G})) \end{array} \right\}.$$

The feasible solutions (a_i) of the program defining $\vartheta(G)$ are called the *orthonormal representations* of the graph G. (Here $(a_i a_i^T)_{11}$ denotes the upper left corner element of the matrix $a_i a_i^T$, that is the square of the first element of the vector a_i, and though not emphasized in the definition of $\vartheta(G)$, we suppose that $(a_i a_i^T)_{11} \neq 0$ for all $i \in V(G)$.)

By Lemma 3 in [8], the Lovász number $\vartheta(G)$ is an upper bound for the stability number $\alpha(G)$, the maximum cardinality of the (so-called stable) sets $S \subseteq V(G)$ such that $\{i,j\} \subseteq S$ implies $\{i,j\} \notin E(G)$. Moreover, by Theorem 11 in [8] if there exists an orthonormal representation of the graph G with vectors a_i in $\mathcal{R}^m$ then $\vartheta(G) \le m$. Specially, $\vartheta(G)$ is at most the chromatic number of the complementer graph, $\chi(\overline{G})$, where the chromatic number of a graph is the minimal number of stable sets covering the vertex set of the graph. Hence (see [8])

$$\alpha(G) \le \vartheta(G) \le \chi(\overline{G}),$$

a fact known as the sandwich theorem (see [7]).

The Lovász number can also be defined via orthonormal representations of the complementer graph: it is shown in [8] that $\vartheta(G) = \vartheta'(G)$ where the number $\vartheta'(G)$ is defined as

$$\vartheta'(G) := \sup \left\{ \sum_{i=1}^{n} (b_i b_i^T)_{11} \;\middle|\; \begin{array}{l} m \in \mathcal{N};\ b_i \in \mathcal{R}^m \ (i=1,\dots,n); \\ b_i^T b_i = 1 \ (i=1,\dots,n); \\ b_i^T b_j = 0 \ (\{i,j\} \in E(G)) \end{array} \right\}.$$

(We remark that here the values $(b_i b_i^T)_{11}$ are allowed to be zero.) The proof of the equality $\vartheta(G) = \vartheta'(G)$ relies on strong duality between Slater-regular primal-dual semidefinite programs equivalent with the programs defining $\vartheta(G)$ and $\vartheta'(G)$, respectively. (See [8], [10] or [15] for the equivalency results; and, for example, [16], [17] for the duality results.) As a consequence of the sandwich theorem and the equality between the values $\vartheta(G)$ and $\vartheta'(G)$ we have

$$\alpha(G) \le \vartheta'(G) \le \chi(\overline{G}),$$

a fact that can also be derived easily from the definition of $\vartheta'(G)$.

For $i \in V(G)$ let $N(i)$ denote the set of vertices $j \in V(G)$ such that $\{i,j\} \in E(G)$. Let us denote by d_i the cardinality of the set $N(i)$, and let $d_{\max}$ denote the maximum of the values d_i $(i \in V(G))$. We define similarly $\overline{N}(i)$, $\overline{d}_i$ and $\overline{d}_{\max}$ for the complementer graph $\overline{G}$ instead of G.

The following theorem is well-known (see for example [9]):

THEOREM 1.1. (Brooks) *The chromatic number* $\chi(G)$ *is at most* $d_{\max}+1$, *with equality for a connected graph* G *if and only if the graph is a clique or an odd cycle.*

As a corollary of Theorem 1.1 and the sandwich theorem we obtain

COROLLARY 1.1. *The value* $\vartheta(G)$ *(*$\vartheta'(G)$ *also) is at most* $\overline{d}_{\max}+1$.

The counterpart of the Brooks' Theorem can be found in [1]. For further lower bounds on the stability number, see [3], [18].

THEOREM 1.2. (Alon-Spencer) *The stability number* $\alpha(G)$ *is at least* $\sum_{i\in V(G)} 1/(d_i+1)$, *with equality if and only if the graph* G *is the disjoint union of cliques.*

Similarly as in the case of Theorem 1.1 we have the following corollary:

COROLLARY 1.2. *The value* $\vartheta'(G)$ *(*$\vartheta(G)$ *also) is at least* $\sum_{i\in V(G)} 1/(d_i+1)$.

We will call the results described in Corollaries 1.1 and 1.2 together the weak sandwich theorem. In Sections 2 and 3 we give two proofs for this theorem using linear algebraic lemmas and new descriptions of the Lovász number. In Section 4 we present a new proof for Theorem 1.2 thus obtaining a third proof for the weak sandwich theorem.

2 First proof for the weak sandwich theorem

In the first proof of the weak sandwich theorem we will need the following lemma, implicit in the proof of Theorem 3 in [8]:

LEMMA 2.1. *Let* PSD *denote the set of* n *by* n *real symmetric positive semidefinite matrices. Let* S *denote the following set of matrices:*

$$S := \left\{ \left(\frac{a_i^T a_j}{e_1^T a_i \cdot e_1^T a_j} - 1 \right) \middle| \begin{array}{l} m \in \mathcal{N};\ a_i \in \mathcal{R}^m\ (1 \le i \le n); \\ a_i^T a_i = 1\ (1 \le i \le n) \end{array} \right\}.$$

Then $\mathrm{PSD} = S$. *(Here* e_1 *denotes the first column vector of the identity matrix* E*. Though not emphasized in the definition of the set* S*, we suppose that the vectors* a_i *have nonzero first coordinates, that is* $e_1^T a_i \neq 0$ *for* $i = 1, \ldots, n$*.)*

Proof. First we will prove the inclusion $S \subseteq \mathrm{PSD}$. Let $a_1, \ldots, a_n$ be unit vectors. Then the vectors $a_i \cdot (e_1^T a_i)^{-1}$ can be written as $(1, x_i^T)^T$ with appropriate vectors x_i. We have

$$\left(\frac{a_i^T a_j}{e_1^T a_i \cdot e_1^T a_j} - 1 \right) = (x_i^T x_j) \in \mathrm{PSD}.$$

Thus the elements of the set S are positive semidefinite.

To prove the reverse inclusion PSD $\subseteq S$, let X be a positive semidefinite matrix. Then there exist vectors x_i such that $X = (x_i^T x_j)$. Let $a_i := \lambda_i(1, x_i^T)^T$ where the constants λ_i are chosen appropriately so that $a_i^T a_i = 1$ holds. With these definitions we have

$$X = (x_i^T x_j) = ((1, x_i^T)(1, x_j^T)^T - 1) = \left(\frac{a_i^T a_j}{e_1^T a_i \cdot e_1^T a_j} - 1 \right).$$

Thus $X \in S$, which was to be shown. $\square$

From Lemma 2.1 follows immediately that the program defining $\vartheta(G)$ and the following program are equivalent:

$$\inf \max_{1 \le i \le n} x_{ii} + 1,\ x_{ij} = -1\ (\{i,j\} \in E(\overline{G})),\ X \in \text{PSD}. \tag{1}$$

(We remark that program (1) in an equivalent form was studied previously by Meurdesoif, see program $(\mathcal{P}_{\mathcal{L}})$ in [11].) We can see

THEOREM 2.1. *The optimal value of program (1) is equal to $\vartheta(G)$, and it is attained.* $\square$

Now let X be the following matrix:

$$X := (x_{ij}),\ \text{where } x_{ij} := \begin{cases} \overline{d}_i, & \text{if } i = j, \\ 0, & \text{if } \{i,j\} \in E(G), \\ -1, & \text{if } \{i,j\} \in E(\overline{G}). \end{cases}$$

Then $x_{ii} \ge \sum_{i \ne j} |x_{ij}|$ holds for $1 \le i \le n$, so the matrix X is positive semidefinite, see [14]. (We can also use the fact that X is the Laplacian matrix corresponding to the adjacency matrix of $\overline{G}$, see [16].) Moreover, the matrix X is a feasible solution of program (1), with corresponding value $\overline{d}_{\max} + 1$. Thus we have $\vartheta(G) \le \overline{d}_{\max} + 1$, and Corollary 1.1 is proved. $\square$

Similarly on the dual side we can apply the variable transformation described in Lemma 2.1 to the program defining $\vartheta'(G)$. This way we obtain the following program:

$$\sup \sum_{i=1}^{n} \frac{1}{y_{ii} + 1},\ y_{ij} = -1\ (\{i,j\} \in E(G)),\ Y \in \text{PSD}. \tag{2}$$

The optimal value of program (2) is a lower bound of $\vartheta'(G)$, as when writing program (2) we considered only the representations (b_i) where the vectors b_i had nonzero first coordinates. From these considerations Corollary 1.2 follows similarly as in the case of Corollary 1.1 above. $\square$

We remark that the program defining $\vartheta'(G)$, and the program (2) are not equivalent generally. Really, let G_0 be the cherry graph:

$$G_0 := (\{1,2,3\}, \{\{1,2\},\{1,3\}\}).$$

Then $\vartheta'(G_0) = 2$ by the sandwich theorem, but the program (2) has no feasible solution with corresponding value 2. Otherwise there would exist

$$Y = \begin{pmatrix} x & -1 & -1 \\ -1 & y & a \\ -1 & a & z \end{pmatrix} \in \mathrm{PSD}$$

such that

$$\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} = 2.$$

But then $xy \geq 1$, $y, z > 0$, and

$$x = \frac{1 - yz}{2yz + y + z}$$

would hold. From these relations $(1 - yz)y \geq 2yz + y + z$, that is $0 \geq z(y+1)^2$ would follow, which is a contradiction. This contradiction shows that there exist graphs such that in every optimal orthonormal representation (b_i) there exist at least one vector b_i with zero first coordinate.

In the next two propositions we describe two lower bounds for the optimal value of program (2).

PROPOSITION 2.1. *The optimal value of program (2) is at least* $n/\vartheta(\overline{G})$.

Proof. Let $\varepsilon > 0$, and let $X = X(\varepsilon) \in \mathcal{R}^{n \times n}$ be a feasible solution of program (1) with $\overline{G}$ instead of G, such that

$$\max_{1 \leq i \leq n} x_{ii} + 1 \leq \vartheta(\overline{G}) + \varepsilon.$$

Then the matrix X is a feasible solution of program (2), and

$$n = \sum_{i=1}^{n} \frac{1}{x_{ii}+1} \cdot (x_{ii}+1) \leq \max_{1 \leq i \leq n} (x_{ii}+1) \cdot \sum_{i=1}^{n} \frac{1}{x_{ii}+1}.$$

We can see that $n/(\vartheta(\overline{G}) + \varepsilon)$ is a lower bound for the optimal value of program (2), and $\varepsilon \to 0$ gives the statement. $\square$

PROPOSITION 2.2. *The optimal value of program (2) is at least* $\alpha(G)$.

Proof. Let $S \subseteq V(G)$ be a stable set with cardinality $\alpha(G)$, and let $\varepsilon > 0$. Let us define the matrix $Y = Y(\varepsilon) \in \mathcal{R}^{n \times n}$ the following way:

$$Y := (y_{ij}), \text{ where } y_{ij} := \begin{cases} \varepsilon, & \text{if } i = j \in S, \\ 0, & \text{if } i, j \in S,\ i \neq j, \\ \lambda, & \text{if } i = j \notin S, \\ -1, & \text{otherwise.} \end{cases}$$

Here let $\lambda = \lambda(\varepsilon) \in \mathcal{R}$ be the minimum number such that Y is positive semidefinite, that is

$$\lambda := \left(1 + \frac{1}{\varepsilon}\right) \cdot (n - \alpha(G)) - 1.$$

(Schur complements [12] can be used to determine λ.) Then Y is a feasible solution of program (2). It can be easily seen that the corresponding value increases to $\alpha(G)$ while $\varepsilon > 0$ decreases to 0. $\square$

From Propositions 2.1 and 2.2 equality between the optimal value of program (2) and $\vartheta'(G)$ follows:

- for vertex-transitive graphs where the lower bound $n/\vartheta(\overline{G})$ (Proposition 2.1) and the upper bound $\vartheta(G)$ are equal, see Theorem 8 in [8];
- for perfect graphs where the lower bound $\alpha(G)$ (Proposition 2.2) and the upper bound $\vartheta(G)$ are equal by the sandwich theorem and the perfect graph theorem [9].

Note that in the case of vertex-transitive graphs the optimal value of program (2) is attained, while in the case of perfect graphs non-attainment is possible.

Equality holds in the general case as well:

THEOREM 2.2. *The optimal value of program (2) and $\vartheta'(G)$ are equal.*

Proof. Let us denote by $\mathrm{TH}\,(G)$ the set of vectors $x = (x_1, \ldots, x_n) \in \mathcal{R}^n$ satisfying $x \geq 0$ and the so-called orthogonality constraints,

$$\sum_{i=1}^{n} (e_1^T a_i)^2 x_i \leq 1,$$

where (a_i) is an orthonormal representation of the graph G.

It can be shown (see [7]) that $\mathrm{TH}\,(G)$ can be described alternatively as the set of vectors $x = (x_1, \ldots, x_n) \in \mathcal{R}^n$ such that

$$x_i = (e_1^T b_i)^2 \ (i = 1, \ldots, n)$$

for some (b_i) orthonormal representation of the complementer graph $\overline{G}$.

Let $\mathrm{TH}_+(G)$ denote the set of positive vectors of $\mathrm{TH}(G)$. Then $\mathrm{TH}_+(G)$ is a convex set (as $\mathrm{TH}(G)$ is a convex set), and it is nonempty (as every graph can be represented by vectors with nonzero first elements). From this observation easily follows that

$$\mathrm{TH}_+(G) \subseteq \mathrm{TH}(G) \subseteq \mathrm{cl}\,\mathrm{TH}_+(G),$$

where cl denotes closure. Consequently we obtain the same value optimizing any linear function over $\mathrm{TH}(G)$ and $\mathrm{TH}_+(G)$; which, for the linear function $(x_1,\dots,x_n) \mapsto \sum_i x_i$, is exactly the statement. □

3 Second proof for the weak sandwich theorem

In this section we give an alternative proof for the weak sandwich theorem using a completely different technique than the one used in the previous section.

Let $\sigma(n)$ denote the number of integers s in the range $0 < s < n$ such that $s \equiv 0, 1, 2$ or $4 \pmod 8$. For small values of n, the value $\sigma(n)$ can be read out from the following table:

n	1	2	3,4	5,6,7,8
$\sigma(n)$	0	1	2	3
n	9	10	11,12	13,14,15,16
$\sigma(n)$	4	5	6	7

The table can be continued in a similar manner for larger values of n. With this notation the following combinatorial lemma holds:

LEMMA 3.1. *If $n \geq 2$ then there exist n of $\sigma(n)$-letter words made up from the letters a, b, c, d such that the number of letter-pairs (a, b) and (c, d) on the same position in any two of the words is altogether odd. (For example in the words "aa" and "cb" there is only one such letter-pair: (a, b), on the second position.)*

Proof. For the values $2 \leq n \leq 9$ the following word-sets have the desired property:

$n = 2,\ \sigma(n) = 1:$	a, b
$n = 3$ or $4,\ \sigma(n) = 2:$	any n words from the word-set aa, cb, ba, db
$n = 5, 6, 7$ or $8,\ \sigma(n) = 3:$	any n words from the word-set
	$aaa, ccb, cba, cdb, baa, dab, dbc, dbd$
$n = 9,\ \sigma(n) = 4:$	$aaaa, accb, acba, acdb, abaa$
	$adab, adbc, cdbd, ddbd.$

For larger values of n we can use the following induction argument. Let us denote by $S_1, \dots, S_9$ the words defined above in the case $n = 9$. Suppose that for some n we have appropriate $\sigma(n)$-letter words $T_1, \dots, T_n$. Then the word-set

$$S_1 \& T_1, \dots, S_9 \& T_1,\ bdbd \& T_2, \dots, bdbd \& T_n,$$

where & denotes concatenation, is made up of $n+8$ of $(\sigma(n)+4)$-letter words, and also has the desired property. Thus the statement in the lemma is dealt with for all the values of n. $\square$

Now let

$$A := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, B := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, C := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, D := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These matrices are orthogonal, furthermore from the matrix set

$$A^T B, A^T C, A^T D, B^T C, B^T D, C^T D$$

the matrices $A^T B$ and $C^T D$ are skew-symmetric, the others are symmetric. Given a word made up of the letters a, b, c and d we can define a matrix by Kronecker-multiplying the corresponding matrices: for example the word "dbc" is transformed into the 8 by 8 matrix $D \otimes B \otimes C$ where $\otimes$ denotes Kronecker product. (For the definition of the Kronecker product see for example [12].) The matrices obtained this way are orthogonal, as they are the Kronecker products of orthogonal matrices.

Using this construction, from Lemma 3.1 immediately follows

LEMMA 3.2. *If $m = 2^{\sigma(n)}$ then there exist m by m orthogonal matrices $C_1, \dots, C_n$ such that for each $i \neq j$, the matrix $C_i^T C_j$ is skew-symmetric.*

Proof. Transform a word-set with the properties described in Lemma 3.1 into a matrix-set using the construction described before Lemma 3.2. We claim that this matrix-set meets the requirements. For example consider the matrix-set

$$A \otimes A, C \otimes B, B \otimes A, D \otimes B.$$

As we have noted already, these m by m matrices are orthogonal. On the other hand,

$$\begin{aligned}(A \otimes A)^T \cdot (C \otimes B) &= (A^T \otimes A^T) \cdot (C \otimes B) = (A^T C) \otimes (A^T B) = \\ &= (C^T A) \otimes (-B^T A) = -(C^T \otimes B^T) \cdot (A \otimes A) = -(C \otimes B)^T \cdot (A \otimes A),\end{aligned}$$

and similarly for the other matrix-pairs:

$$(A \otimes A)^T \cdot (B \otimes A) = -(B \otimes A)^T \cdot (A \otimes A) \dots \text{ etc.}$$

In the general case similar argument can be applied, so the lemma is proved. $\square$

We remark that in [13] Radon proved that there exist m by m orthogonal matrices $\tilde{C}_1, \ldots, \tilde{C}_n$ such that for each $i \neq j$ the matrix $\tilde{C}_i^T \tilde{C}_j$ is skew-symmetric if and only if $m \equiv 0 \pmod{2^{\sigma(n)}}$ (see also [6], [12]). The "if" part is an easy consequence of Lemma 3.2: just Kronecker-premultiply the C_i matrices with an identity matrix of appropriate dimension. For a similar proof of this part of Radon's Theorem, see [4].

We will need one further lemma, concerning new descriptions of the Lovász number. The idea is to represent the graph G with matrices instead of vectors. Let us define

$$\hat{\vartheta}(G) := \inf \left\{ \max_{1 \le i \le n} \frac{1}{(A_i A_i^T)_{11}} \;\middle|\; \begin{array}{l} m, k \in \mathcal{N};\ A_i \in \mathcal{R}^{m \times k}\ (i = 1, \ldots, n); \\ A_i^T A_i = E\ (i = 1, \ldots, n); \\ A_i^T A_j = 0\ (\{i, j\} \in E(\overline{G})) \end{array} \right\}$$

and

$$\check{\vartheta}(G) := \sup \left\{ \sum_{i=1}^{n} (B_i B_i^T)_{11} \;\middle|\; \begin{array}{l} m, k \in \mathcal{N};\ B_i \in \mathcal{R}^{m \times k}\ (i = 1, \ldots, n); \\ B_i^T B_i = E\ (i = 1, \ldots, n); \\ B_i^T B_j = 0\ (\{i, j\} \in E(G)) \end{array} \right\}.$$

It is obvious that $\hat{\vartheta}(G) \le \vartheta(G)$ and $\vartheta'(G) \le \check{\vartheta}(G)$. Equalities here follow from Lemma 3.3.

LEMMA 3.3. *With the above definitions the inequality $\check{\vartheta}(G) \le \hat{\vartheta}(G)$ holds.*

Proof. We adapt the proof of Lemma 4 in [8].

Let $A_1, \ldots, A_n$ and $B_1, \ldots, B_n$ be matrices with the properties described in the definition of $\hat{\vartheta}(G)$ and $\check{\vartheta}(G)$, respectively. Then

$$(A_i \otimes B_i)^T \cdot (A_j \otimes B_j) = (A_i^T A_j) \otimes (B_i^T B_j) = 0\ (1 \le i, j \le n;\ i \neq j).$$

Thus the column vectors of the matrices $A_i \otimes B_i$ $(1 \le i \le n)$ altogether form an orthonormal system. Hence

$$\sum_{i=1}^{n} ((A_i \otimes B_i)(A_i \otimes B_i)^T)_{11} \le 1,$$

which can be written as

$$\sum_{i=1}^{n} (A_i A_i^T)_{11} \cdot (B_i B_i^T)_{11} \le 1.$$

From this inequality

$$\min_{1 \le i \le n} (A_i A_i^T)_{11} \cdot \sum_{i=1}^{n} (B_i B_i^T)_{11} \le 1$$

follows, and so

$$\sum_{i=1}^{n}(B_iB_i^T)_{11} \leq \max_{1\leq i\leq n} \frac{1}{(A_iA_i^T)_{11}}$$

holds. We can see that $\check{\vartheta}(G) \leq \hat{\vartheta}(G)$, and the proof of the lemma is finished. □

Lemma 3.3, together with the equality $\vartheta(G) = \vartheta'(G)$, gives

THEOREM 3.1. *The values $\hat{\vartheta}(G)$ and $\check{\vartheta}(G)$ are equal to $\vartheta(G)$ ($\vartheta'(G)$ also), and are attained.* □

The weak sandwich theorem is an easy consequence of Lemmas 3.2 and 3.3. Let $C_1, \dots, C_n$ be m by m orthogonal matrices with the property described in Lemma 3.2. Let us define the matrices $A_1, \dots, A_n$ the following way: the matrix will be $(1+\overline{e})m$ by m where $\overline{e}$ denotes the cardinality of $E(\overline{G})$. The first m by m block in A_i is $\alpha_i C_i$ where

$$\alpha_i := \frac{1}{\sqrt{\overline{d}_i + 1}}.$$

The further m by m blocks correspond to the edges of the complementer graph $\overline{G}$: let the block corresponding to the edge $\{i,j\}$ be $\alpha_i C_j$ in A_i, $\alpha_j C_i$ in A_j, and the zero matrix otherwise. The matrix set $A_1, \dots, A_n$ defined this way has the properties described in the definition of $\hat{\vartheta}(G)$, so

$$\max_{1\leq i\leq n} \frac{1}{(A_iA_i^T)_{11}} \geq \hat{\vartheta}(G).$$

On the other hand,

$$\max_{1\leq i\leq n} \frac{1}{(A_iA_i^T)_{11}} = \max_{1\leq i\leq n} \frac{\overline{d}_i + 1}{(C_iC_i^T)_{11}} = \overline{d}_{\max} + 1$$

(note that the matrices C_i are orthogonal so the matrix $C_iC_i^T$ is the identity matrix). We obtained $\overline{d}_{\max} + 1 \geq \hat{\vartheta}(G)$. Similar construction on the dual side shows that $\sum_{i\in V(G)} 1/(d_i+1) \leq \check{\vartheta}(G)$. The weak sandwich theorem now follows from Lemma 3.3. and the obvious inequalities $\check{\vartheta}(G) \geq \vartheta'(G)$, $\hat{\vartheta}(G) \leq \vartheta(G)$. □

Note that instead of the matrices C_i in the above construction we can also use matrices D_i with the following properties: the matrices D_i are orthogonal; the matrices $D_i^T D_j$ are symmetric and have zero trace ($i \neq j$). (The only change is that the block corresponding to the edge $\{i,j\}$ is $\alpha_i D_j$ in A_i and $-\alpha_j D_i$ in A_j.)

It is an open problem to characterize the numbers m such that there exist m by m matrices $D_1, \dots, D_n$ with the properties described above; but any power of 2, greater than or equal to n meets the requirements: n words of the same length $\log_2 m$, and made up from the letters a and d (or a and c) translate into appropriate matrices (see the proof of Lemma 3.2).

Using simultaneous diagonalization, the open problem described above can be cast also in the following form: characterize the numbers (m, n) such that there exists a matrix $M \in \{\pm 1\}^{m\times n}$ such that $M^T M = E$. This is the Hadamard determinantal problem (see [5]); the conjecture is that the (m, n) pairs satisfying the requirements are:

- $(m, 1)$ such that $m \geq 1$;
- $(m, 2)$ such that $m \geq 2$ is even;
- (m, n) such that $m \geq n$ and $m \equiv 0 \pmod 4$.

4 New proof for the Alon-Spencer Theorem

In this section we will prove the Alon-Spencer Theorem (Theorem 1.2), the counterpart of Brooks' Theorem (Theorem 1.1). We also describe the counterpart of Turán's Theorem.

First we will show that

$$\alpha(G) \geq \sum_{i=1}^{n} \frac{1}{d_i + 1} \tag{3}$$

holds. We apply induction on the cardinality n of $V(G)$. In the case when $n = 1$, the statement is trivial; in what follows we will suppose that the number of vertices is $n > 1$ and that for graphs with smaller number of vertices the inequality (3) already holds. Note that we can suppose also that the graph G is α-critical (that is leaving out any edge the stability number becomes larger). Really, otherwise delete edges from the graph until this operation does not change the stability number. In the end we get an α-critical graph, and the value on the right hand side of (3) became larger, while the value on the left hand side of (3) stayed the same. We can suppose also that the graph G is connected: if it has more than one components, then by induction the inequality (3) holds true for its components, and this implies the validity of (3) for the whole graph. Hence it is enough to consider the case when the graph G is α-critical and connected.

Let v be a vertex of G such that $d_v = d_{\max}$. It is easy to prove that there exists a stable set of the size $\alpha(G)$ such that it does not contain the vertex v (see Exercise 8.12 in [9]). Let us denote by $G - v$ the graph with vertex-set $\{1,\ldots,n\} \setminus \{v\}$, and with edge-set $\{\{i,j\} \in E(G) : i,j \neq v\}$. Then $\alpha(G-v) = \alpha(G)$. By induction, for the graph $G-v$ (3) holds, that is

$$\alpha(G-v) \geq \sum_{i \in N(v)} \frac{1}{d_i} + \sum_{i \notin N(v),\, i \neq v} \frac{1}{d_i+1}. \tag{4}$$

As $d_v \geq d_i$ for all $i \in V(G)$, we have

$$\frac{1}{d_i} \geq \frac{1}{d_i+1} + \frac{1}{d_v(d_v+1)} \ (i \in N(v)) \tag{5}$$

Writing this bound into (4) we obtain the following inequality:

$$\alpha(G-v) \geq \sum_{i \in N(v)} \frac{1}{d_i+1} + \sum_{i \notin N(v),\, i \neq v} \frac{1}{d_i+1} + \frac{1}{d_v+1}.$$

As $\alpha(G-v) = \alpha(G)$, this inequality is in fact (3), and the first half of Theorem 1.2 is proved.

To prove the second half of the theorem we will show that if

$$\alpha(G) = \sum_{i=1}^{n} \frac{1}{d_i+1} \tag{6}$$

holds then the graph G is the disjoint union of cliques (the other direction is obvious). Again we apply induction on n. Note that if (6) holds then the graph G is α-critical (otherwise G would have an edge such that after deleting this edge the stability number stays unchanged, while the value on the right hand side of (6) becomes larger, contradicting (3)). We can suppose also that G is connected (if (6) holds then it holds for the components also). Thus it suffices to prove that if the graph G is α-critical and connected, furthermore (6) holds then G is a clique.

Let v be the same point as in the first half of the proof, and again consider the graph $G-v$. Let us denote by $S(G)$ the sum on the right hand side of (6). As we have seen in the first half of the proof,

$$\alpha(G) = \alpha(G-v) \geq S(G-v) \geq S(G).$$

As now $\alpha(G) = S(G)$, we have equalities instead of inequalities, that is

$$\alpha(G) = \alpha(G-v) = S(G-v) = S(G).$$

It follows from the $S(G-v) = S(G)$ equality that $d_i = d_v$ $(i \in N(v))$ (as otherwise (5) would hold with strict inequality). Moreover by the $\alpha(G-v) = S(G-v)$ equality and by induction the graph $G-v$ is the disjoint union of cliques. As the graph G is connected, the set $N(v)$ intersects with all of these cliques. Let us chose one of the cliques, and a vertex $i \in N(v)$ from this clique. Then d_i equals d_v as well as the cardinality of the clique. Hence the components of $G - v$ all have the same cardinality d_v. Then $\alpha(G - v) = (n - 1)/d_v$. If the graph $G - v$ would have more than one components then we could chose from each component a vertex from $\overline{N}(v)$. These vertices together with the vertex v would constitute a stable set in G with cardinality larger than $\alpha(G - v)$. This would contradict the fact that $\alpha(G - v) = \alpha(G)$, so $G - v$ is a clique with cardinality d_v with vertices in $N(v)$. Thus the graph G is a clique, and the proof of the second half of Theorem 1.2 is finished also. □

We remark that Turán's Theorem can be derived as a consequence of the Alon-Spencer Theorem, see [1]. Here the graph $T_{n,m}$ is defined as follows: Divide the vertex set $V(T_{n,m}) := \{1, \dots, n\}$ into m disjoint subsets $S_1, \dots, S_m$ such that the cardinality of S_i and S_j differ by at most one for each $i \neq j$. Then the edge set of the graph $T_{n,m}$ is

$$E(T_{n,m}) := \cup_{\ell=1}^{m} \{\{i, j\} \subseteq S_\ell : i \neq j\}.$$

COROLLARY 4.1. (Turán) *Let G be a simple graph on n vertices with stability number $\alpha(G) \leq m$. Minimizing the number of the edges of G under these assumptions, the unique extremal graph is $T_{n,m}$.*

The following corollary describes the counterpart of Turán's Theorem which in turn is a simple consequence of Brooks' Theorem.

COROLLARY 4.2. *Let G be a simple graph on n vertices with chromatic number $\chi(G) \geq m$. Then the number of the edges of G is at least $m(m-1)/2$. Equality holds if and only if G is the disjoint union of a clique and a stable set on m and $n - m$ vertices, respectively.*

Proof. It is well-known that the number of the edges of any simple graph G on n points is at least $\chi(G)(\chi(G) - 1)/2$, so it is enough to prove that if the number of the edges is $m(m-1)/2$ and the chromatic number is m then G is isomorphic with the graph described in the statement.

Hence we can suppose that the vertex set is the disjoint union of m stable sets with exactly one edge going between each two of them. Let us choose a connected component G' of the graph G such that its chromatic number is $\chi(G') = m$. By Brooks' Theorem, then

$$m = \chi(G') \leq d_{\max}(G') + 1 \leq d_{\max}(G) + 1 \leq m,$$

and we can see that G' is a clique on m vertices; the statement is proved. □

Finally we mention an open problem. Wilf proved the following result (see [2]): the chromatic number $\chi(G)$ is at most $\alpha_{\max} + 1$ (where $\alpha_{\max}$ denotes the maximum eigenvalue of the adjacency matrix of G), with equality for a connected graph G if and only if the graph is a clique or an odd cycle. As $\alpha_{\max} \leq d_{\max}$ always holds (with equality for a connected graph if and only if the graph is regular), Wilf's Theorem is stronger than Brooks' Theorem. It would be interesting to see how Theorem 1.2 could be strengthened using spectral information. (The bound $n/(\alpha_{\max} + 1)$ [18] is not a strengthening of the Alon-Spencer bound, as — using the convexity of the function

$$x \mapsto \frac{1}{d^T x + 1} \ (0 \leq x \in \mathcal{R}^n),\ d = (d_1, \ldots, d_n)^T,$$

and Rayleigh's Theorem [14] — it can be easily shown that

$$\frac{n}{\alpha_{\max} + 1} \leq \sum_{i=1}^{n} \frac{1}{d_i + 1}$$

holds, with equality if and only if the graph is regular.)

Conclusion. In this paper we presented a new proof for the counterpart of Brooks' Theorem (the Alon-Spencer Theorem) concerning a simple lower bound on the stability number. As a consequence of the sandwich theorem, Brooks' Theorem and the Alon-Spencer Theorem we derived a weaker version of the sandwich theorem. For this weak sandwich theorem we gave another two, more direct proofs also, which are based on linear algebraic lemmas and new descriptions of the Lovász number.

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