MTA SZTAKI

Computer and Automation Institute,
Hungarian Academy of Sciences

Research Division

H-1518 Budapest, P.O.Box 63.

PRINCIPLE OF VIRTUAL WORK SUBJECT TO NONHOLONOMIC AND RHEONOMOUS CONSTRAINTS

Rapcsák, T.

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Principle of virtual work subject to nonholonomic and rheonomous constraints[1]

Rapcsák, T.[2]

Abstract

In the paper, it is shown that the principle of virtual work considered an axiom of mechanics by Lagrange (1788) and Farkas (1906) results in a new general equilibrium system, assuming nonholonomic and rheonomous constraints. Then, the dual forms of the principle of virtual work are formulated in these cases, and some examples are discussed.

1. Introduction

In 1687, Newton laid down the solid foundation of mechanics by describing the motion of a mass point in the *3*-dimensional Euclidean space, based on Newton's second law: "*mass times acceleration equals moving force*". It was deduced from the motion of a mass point in the field of gravity on the earth, and was then applied to the motion of planets under the action of the sun.

The analytical form of mechanics was introduced by Euler and Lagrange considering a system of mass points and also taking constraints for the motion into account. A nice synthesis of these efforts is Lagrangian mechanics the purpose of which is the description of the motion and equilibrium of a system of mass points subject to ideal holonomic constraints, i.e., a differentiable manifold, and in which the motion is governed by the Lagrangian function (see, e.g., Arnold, 1989). In the Newtonian case, the differentiable manifold is equal to the *3*-dimensional Euclidean space and the Lagrangian function to the difference of the kinetic energy depending on the velocities and the potential energy depending on the position only. An important property of Lagrangian mechanics is, by using the D'Alembert principle, that the description of the motion of a system of mass points is equivalent to that of equilibrium positions.

The original Newton's idea, to govern a motion by the force, is more general than that of the Lagrangian mechanics, to govern a motion by a function. This was the motivation to follow the original idea of Newton by characterizing mechanical equilibrium through the principle of virtual work, assuming force fields and holonomic-scleronomic constraints (Rapcsák, 2003).

In the paper, it is shown that the principle of virtual work considered an axiom of mechanics by Lagrange (1788) and Farkas (1906) results in a new general equilibrium system, assuming nonholonomic and rheonomous constraints. Then, the dual forms of the principle of virtual work are formulated in these cases, and some examples are discussed.

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[2]Adviser in Science, Computer and Automation Research Institute, Hungarian Academy of Sciences, H-1518 Budapest, P.O. Box 63, Hungary, e-mail: rapcsak@sztaki.hu, Internet: http://www.oplab.sztaki.hu/rapcsak

2. Statement of the mechanical force equilibrium problem

One of the most general settings of the problem in Newtonian mechanics to be discussed is as follows. Let n mass points, on which active forces $P_1, \dots, P_n$ have some effect, be given in R^3 (in the *3*-dimensional Euclidean space). Besides, let the constraints which put the only restriction on the motion of system of mass points be given. The position and motion of the system of n mass points can be given by time functions of the space coordinates in R^3. In the paper, the case will be discussed when a possible motion of the point system in an interval of time is given by a twice continuously differentiable vector function $\mathbf{x} : [t_0, t_1] \subseteq R \to R^{3n}$ satisfying the constraints where $t_0, t_1 \in R$ are given values. The vector functions $\dot{\mathbf{x}}(t), \ddot{\mathbf{x}}(t)$, $t \in [t_0, t_1] \subseteq R$, denote the velocity and acceleration vector at time t, respectively.

We assume that the active forces affecting the system are continuous functions of position, velocity and time. The mechanical system is considered to be in force equilibrium if the sum of active forces and reaction forces is zero on every mass point. (Reaction forces cause the points to maintain constraints.) The main problem is to determine the mechanical states of the system of mass points or, in a special case, the characterization of the equilibrium position of the system.

The principle of virtual work is one of the oldest laws of classical mechanics stated for the case of equality type constraints, first, by Bernoulli in 1717. In the famous work of Lagrange (1788), this principle appeared as an axiom of mechanics (see details in Mach, 1960 and in Bussotti, 2003). For the case of inequality constraints, the principle was stated by Fourier in 1798, thereupon by Gauss in 1829. But as early as in 1838, Ostrogradsky, putting the principle of virtual displacements into use, met the difficulty that virtual displacements were given implicitly. Therefore, the virtual displacements were eliminated in the case of equality and inequality type constraints, i.e., the statement of Farkas theorem was used but not proved. It was proved by Farkas (1901). A survey on the principle of virtual work can be found in Banach (1951), more historical details in Mach (1960) and Farkas (1895).

By the principle of virtual work, if a system of mass points is in an equilibrium position, then the work of the active forces in the directions of the virtual displacements are less than or equal to zero, i.e., if the active forces are denoted by the vector $\mathbf{P} \in R^{3n}$, then by the principle of virtual work, the inequality

$$\mathbf{P}^T \mathbf{v} \leq 0 \tag{2.1}$$

fulfils for every virtual displacement $\mathbf{v} \in R^{3n}$.

An important special case is the conservative force field where there exists a potential energy function $V : R^{3n} \to R$ depending on the position only, for which the equation

$$\mathbf{P}(\mathbf{x})^T = -\nabla V(\mathbf{x}), \qquad \mathbf{x} \in R^{3n}, \tag{2.2}$$

holds. (By agreement, the gradient of a function is a row vector.) Furthermore, again in a conservative force field, the principle of virtual work is replaced by the Courtivron principle which says that a mechanical equilibrium position is a Karush-Kuhn-Tucker point of the potential function V subject to the given constraints, i.e., the mechanical equilibrium problem leads to a nonlinear optimization one (see, e.g, [2]).

3. Principle of virtual work in a force field, subject to scleronomic and holonomic constraints

In the case of force fields where the forces depend on the position only, subject to scleronomic and holonomic constraints, the following statements were proved. Let us introduce the notations

$$A = \{\mathbf{x} \in R^{3n} \mid h_j(\mathbf{x}) = 0, \quad j = 1, \dots, p; \qquad g_i(\mathbf{x}) \geq 0, \quad i = 1, \dots, m\} \tag{3.1}$$

where the functions h_j, $j = 1, \dots, p$; and g_i, $j = 1, \dots, m$, are twice continuously differentiable; p and m are positive integers,

$$I(\mathbf{x}) = \{i \mid g_i(\mathbf{x}) = 0, \qquad i \in \{1, \dots, m\}\}, \qquad \mathbf{x} \in A,$$

(the active indices of the inequality constraints),

$$D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}) = \{\mathbf{v} \in R^{3n} \mid \nabla h_j(\mathbf{x})\mathbf{v} = 0, \quad j = 1, \dots, p; \quad \nabla g_i(\mathbf{x})\mathbf{v} \geq 0, \quad i \in I(\mathbf{x})\} \\ \mathbf{x} \in A, \tag{3.2}$$

where the gradients are row vectors and

$$I(\mathbf{x}, \mathbf{v}) = \{i \in I(\mathbf{x}) \mid \nabla g_i(\mathbf{x})\mathbf{v} = 0\}, \qquad \mathbf{x} \in A, \qquad \mathbf{v} \in D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}).$$

A set $K \subseteq R^n$ is a cone with apex at $\bar{\mathbf{x}} \in cl\, K$ if

$$\mathbf{x} \in K, \quad \alpha \in (0, +\infty) \;\Rightarrow\; \bar{\mathbf{x}} + \alpha(\mathbf{x} - \bar{\mathbf{x}}) \in K. \tag{3.3}$$

If $\bar{\mathbf{x}} = \mathbf{0}$, then K is a cone with apex at the origin or simply a cone.

Definition 3.1 *The gradient-type Karush-Kuhn-Tucker (GKKT) constraint qualification holds at a point* $\mathbf{x} \in A$ *if for every* $\mathbf{v} \in D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x})$*, the gradients*

$$\nabla g_i(\mathbf{x}), \quad i \in I(\mathbf{x}, \mathbf{v}); \qquad \nabla h_j(\mathbf{x}), \quad j = 1, \dots, p,$$

are linearly independent. The GKKT constraint qualification fulfils on the set A *if it fulfils at every point of* A*.*

Definition 3.2 *Let* H *be a real Hilbert space,* $A \subseteq H$ *a subset,* $F : A \to H$ *a mapping and* $K : A \rightrightarrows H$ *a set-valued mapping. Quasi-complementarity systems (in short, QCS) are equilibrium systems where the aim is to find a point* $x \in A$ *such that* $x \in K(x)$ *and*

$$< F(x), y - x > \geq 0, \qquad \forall y \in K(x),$$

where $K(x)$*,* $x \in A$*, are convex cones with apex at the origin or different points (shifted convex cones).*

Theorem 3.1 [20] *Consider the mechanical equilibrium problems with constraints* (3.1) *in the force field defined by a continuous force function* $\mathbf{P} : A \subseteq R^{3n} \to R^{3n}$*. If the GKKT constraint qualification fulfils on the feasible set* A*, then, the determination of an equilibrium position is equivalent to the solution of a quasi-complementarity system (QCS) where a point* $\mathbf{x} \in A$ *has to be found for which*

$$\mathbf{P}(\mathbf{x})^T \mathbf{v} \leq 0, \qquad \mathbf{v} \in D_{[\mathbf{h},g]}(\mathbf{x}). \tag{3.4}$$

Now, the dual problem of *QSC* (3.4) is formulated based on *the Farkas theorem*. Let the symbols *span*{ } and *cone conv*{ } denote the subspace and convex cone spanned by the vectors between brackets, respectively, furthermore, let the sum be the Minkowski sum.

Theorem 3.2 [20] *Consider the mechanical equilibrium problems with constraints* (3.1) *in the force field defined by a continuous force function* $\mathbf{P} : A \subseteq R^{3n} \to R^{3n}$. *If the GKKT constraint qualification fulfils on the feasible set A, then, the determination of an equilibrium position is equivalent to the solution of the dual form of QCS* (3.4) *where a point* $\mathbf{x} \in A$ *has to be found for which*

$$-\mathbf{P}(\mathbf{x}) \in span\{\nabla h_j(\mathbf{x})^T,\, j = 1,\dots,p\} + cone\ conv\{\nabla g_i(\mathbf{x})^T,\, i \in I(\mathbf{x})\}. \tag{3.5}$$

It follows from *Theorem 3.2* that the determination of a mechanical equilibrium position, by using the dual form, is a solution of the feasibility problem

$$-\mathbf{P}(\mathbf{x})^T = \sum_{j=1}^{p} \lambda_j \nabla h_j(\mathbf{x}) + \sum_{i \in I(\mathbf{x})} \mu_i \nabla g_i(\mathbf{x}), \quad \boldsymbol{\lambda} \in R^p, \quad \mu_i \geq 0, \quad i \in I(\mathbf{x}), \quad \mathbf{x} \in A. \tag{3.6}$$

Theorem 3.3 *Consider the mechanical equilibrium problems with constraints* (3.1) *in the force field defined by a continuous force function* $\mathbf{P} : A \subseteq R^{3n} \to R^{3n}$. *If the configuration space A is convex and compact, the GKKT constraint qualification holds on it, both* $\mathbf{P}$ *and* $D_{[\mathbf{h},\mathbf{g}]}$ *are continuous, then problem* (3.6) *has at least one solution.*

Proof. If the GKKT constraint qualification fulfils on the set A, then by *Theorem 3.1*, the determination of an equilibrium position of the mechanical equilibrium problem is equivalent to the solution of (3.4). By *Theorem 3.2*, the solution of (3.4) is equivalent to the solution of (3.6). Since the set A is convex and compact, both $\mathbf{P}$ and $D_{[\mathbf{h},\mathbf{g}]}$ are continuous, all the sets $D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x})$, $\mathbf{x} \in A$, are nonempty, closed and convex, thus by *Theorem 6.1* in [4], problem (3.6) has at least one solution. ■

Example 3.1 [2] *A heavy point of mass m subject to the action of the force* $\mathbf{P} \in R^3$ *is constrained to remain on the surface of the sphere*

$$\frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2 - r^2 = 0. \tag{3.7}$$

We assume that the x_3*-axis has a vertical direction and an upward sense. Determine the position of equilibrium, assuming that friction does not take place.*

The virtual displacements $\mathbf{v}^T = (v_1, v_2, v_3) \in R^3$ *satisfy the equation which we get by differentiating* (3.7)*:*

$$x_1v_1 + x_2v_2 + x_3v_3 = 0. \tag{3.8}$$

The virtual work of the force $\mathbf{P}$ *is*

$$P_1v_1 + P_2v_2 + P_3v_3,$$

and that of the force of gravity $-mgv_3$*. In the position of equilibrium, consequently, we have*

$$P_1v_1 + P_2v_2 + (P_3 - mg)v_3 = 0. \tag{3.9}$$

Applying the dual form of the principle of virtual work (Theorem 3.2), we have that

$$P_1 + \lambda x_1 = 0, \quad P_2 + \lambda x_2 = 0, \quad P_3 - mg + \lambda x_3 = 0, \tag{3.10}$$

where λ *is the Lagrange multiplier.*

From equations (3.7) *and* (3.10)*, we can determine the coordinates* x_1, x_2, x_3 *of the position of equilibrium and* λ*:*

$$\lambda = \pm\sqrt{P_1^2 + P_2^2 + (P_3 - mg)^2}/\sqrt{2}\,r, \tag{3.11}$$

$$x_1 = -P_1/\lambda, \quad x_2 = -P_2/\lambda, \quad x_3 = -(P_3 - mg)/\lambda. \tag{3.12}$$

Since λ *has two values, two equilibrium positions exist.*

4. Principle of virtual work subject to scleronomic and nonholonomic constraints

In this part, the relation between the principle of virtual work and equilibrium systems are studied under holonomic and nonholonomic constraints. The active forces $\mathbf{P} : A \subseteq R^{6n} \to R^{3n}$ affecting the system are continuous functions of position and velocity defined on a subset A of R^{6n}. The constraints of the mechanical systems are as follows:

$$\begin{aligned} h_j(\mathbf{x}, \dot{\mathbf{x}}) = 0, &\quad j = 1, \ldots, p, \\ g_i(\mathbf{x}, \dot{\mathbf{x}}) \geq 0, &\quad i = 1, \ldots, m, \\ \mathbf{A}_1(\mathbf{x})\dot{\mathbf{x}} = \mathbf{0}&, \\ \mathbf{A}_2(\mathbf{x})\dot{\mathbf{x}} \geq \mathbf{0}&, \\ (\mathbf{x}^T, \dot{\mathbf{x}}^T)^T \subseteq R^{3n} &\times R^{3n}, \end{aligned} \tag{4.1}$$

where $g_i \in C^2$, $i = 1, \ldots, m; h_j \in C^2$, $j = 1, \ldots, p$; $\mathbf{A}_1(\mathbf{x})$ a $k_1(\mathbf{x})\times$ $3n$ matrix and $\mathbf{A}_2(\mathbf{x})$ a $k_2(\mathbf{x})\times$ $3n$ matrix at every $\mathbf{x}$, respectively.

Let us introduce the notation $A \subseteq R^{3n} \times R^{3n}$ of the feasible set of constraints (4.1). In other words, the states of the investigated mechanical systems are represented by the vectors of the set A. If the constraints do not explicitly depend on the time, the system is called scleronomic. In the tangent space at each point of the configuration manifold of a nonholonomic system, there exists a fixed subspace or a cone to which the velocity vector must belong. An interesting fact is that the nonholonomic linear equality and inequality constraints, appearing in the paper of Farkas published in 1906, seem to be the most general ones even recently [10].

A simple example of a nonholonomic physical system is a ball rolling on a table without slipping.

Example 4.1 *Certain kinematical conditions do not always show up as equations between the coordinates of the mass points. A characteristic example is the rolling of a ball on a table. The ball, moving freely in the space, has six degrees of freedom. Since the ball rests on the surface of the table, the height of the centre is a given constant, which reduces the number of degrees of freedom to five. The position of the ball can be characterized by two coordinates of* *the point of the centre, plus three angles which fix the position of the ball relative to its centre. If the ball can slide along the surface, it can make use of all of its five degrees of freedom. However, if it is confined to rolling, the point of the contact has to be momentarily at rest, and the instantaneous axis of rotation has to go through the point of contact. If the instantaneous axis is confined to some line which has to stay within the surface of the table, we have "pure rolling", otherwise, "rolling and pivoting".*

Pure rolling cuts down the degrees of freedom to two. If the path of the point of the centre is determined by two coordinates, as functions of time, the condition of rolling determines the position of the ball at any time uniquely. This would suggest that perhaps the angles can be given as functions of the two coordinates of the position. This, however, is not possible.

In order to extend the principle of virtual work to nonholonomic constraints, some definitions and lemmas are necessary.

Definition 4.1 *The displacements*

$$\mathbf{x}(t), \qquad t \in [t_0, t_1], \qquad \mathbf{x}(t_0) = \mathbf{x}_0, \qquad \mathbf{x}(t_1) = \mathbf{x}_1,$$

of the given system of mass points from the point $\mathbf{x}_0$ *to the point* $\mathbf{x}_1$ *and the velocities* $\dot{\mathbf{x}}(t), t \in [t_0, t_1]$, *belonging to them are possible if the functions* $(\mathbf{x}(t)^T, \dot{\mathbf{x}}(t)^T)^T$, $t \in [t_0, t_1]$, *satisfy the constraints. Virtual displacements in the case of scleronomic constraints* (4.1) *are the ones which point in the direction of possible velocities.*

In nonlinear optimization, the sets

$$I(\mathbf{x}, \dot{\mathbf{x}}) = \Big\{ i \mid g_i(\mathbf{x}, \dot{\mathbf{x}}) = 0,\ i \in \{1, \ldots, m\} \Big\},$$

$$\left(\mathbf{x}^T, \dot{\mathbf{x}}^T\right)^T \in A = \Big\{ \left(\mathbf{x}^T, \dot{\mathbf{x}}^T\right)^T \in R^{6n} \mid h_j(\mathbf{x}, \dot{\mathbf{x}}) = 0,\ j = 1, \ldots, p;$$
$$g_i(\mathbf{x}, \dot{\mathbf{x}}) \geq 0,\ i = 1, \ldots, m \Big\}$$

denote the active indices of the inequality constraints.

Definition 4.2 *Let*

$$\begin{aligned} C_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}}) = \Big\{ (\mathbf{v}_1^T, \mathbf{v}_2^T)^T \in R^{6n} \mid \exists \left(\mathbf{x}^T,\ \dot{\mathbf{x}}^T\right)^T : [t, t+\varepsilon] \subseteq R \to R^{6n}, \quad \varepsilon > 0, \\ h_j(\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau)) = 0, \qquad j = 1, \ldots, p; \\ g_i(\mathbf{x}(\tau), \dot{\mathbf{x}}(\tau)) \geq 0, \qquad i \in I(\mathbf{x}, \dot{\mathbf{x}}); \\ \mathbf{x}(t) = \mathbf{x}, \quad \frac{d\mathbf{x}(\tau)}{d\tau} \mid_{\tau=t} = \mathbf{v}_1 = \dot{\mathbf{x}}, \quad \frac{d^2\mathbf{x}(\tau)}{d\tau^2} \mid_{\tau=t} = \mathbf{v}_2 \Big\}, \\ x_l \in C^2, \qquad l = 1, \ldots, 3n, \qquad (\mathbf{x}^T, \dot{\mathbf{x}}^T)^T \in A. \end{aligned} \tag{4.2}$$

The set $C_{[\mathbf{h},\mathbf{g}]}$ at every feasible point is the cone of attainable directions (see recent characterizations in [14]) related to the equality and inequality constraints. It is emphasized that the values ε appearing in *Definition 4.2* depend on the position and velocity. Curves satisfying the constraints are called feasible.

Definition 4.3 *Let*

$$\begin{aligned}
D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}}) = \big\{ (\mathbf{v}_1^T, \mathbf{v}_2^T)^T \in R^{6n} \mid \\
\nabla h_j(\mathbf{x},\dot{\mathbf{x}})(\mathbf{v}_1^T, \mathbf{v}_2^T)^T = 0, \qquad j = 1,\ldots,p; \\
\nabla g_i(\mathbf{x},\dot{\mathbf{x}})(\mathbf{v}_1^T, \mathbf{v}_2^T)^T \geq 0, \qquad i \in I\left(\mathbf{x},\dot{\mathbf{x}}\right) \big\}, \\
(\mathbf{x}^T, \dot{\mathbf{x}}^T) \in A.
\end{aligned} \tag{4.3}$$

Lemma 4.1 *If $h_j, g_i \in C^1$, $j = 1,\ldots,p$; $i = 1,\ldots,m$, then,*

$$C_{[\mathbf{h},\mathbf{g}]} \subseteq D_{[\mathbf{h},\mathbf{g}]} \tag{4.4}$$

at every feasible point.

Proof. Let us consider an arbitrary vector $(\mathbf{v}_1^T, \mathbf{v}_2^T)^T$ of $C_{[\mathbf{h},\mathbf{g}]}\left(\mathbf{x},\dot{\mathbf{x}}\right)$, where $(\mathbf{x}^T, \dot{\mathbf{x}}^T)^T$ is an arbitrary point of A. By differentiating the functions $h_j\left(\mathbf{x}(\tau),\dot{\mathbf{x}}(\tau)\right)$, $j = 1,\ldots,p$; $g_i\left(\mathbf{x}(\tau),\dot{\mathbf{x}}(\tau)\right)$, $i \in I(\mathbf{x},\dot{\mathbf{x}})$, $\tau \in [t, t+\varepsilon]$ by τ and tending to t by τ, we have that

$$\begin{aligned}
\frac{d}{d\tau} h_j\left(\mathbf{x}\left(\tau\right), \dot{\mathbf{x}}\left(\tau\right)\right)_{|\tau=t} = \nabla h_j(\mathbf{x}\left(\tau\right), \dot{\mathbf{x}}\left(\tau\right)) \left(\dot{\mathbf{x}}\left(\tau\right)^T, \ddot{\mathbf{x}}\left(\tau\right)^T\right)^T_{|\tau=t} = \\
\nabla h_j(\mathbf{x},\dot{\mathbf{x}})(\mathbf{v}_1^T, \mathbf{v}_2^T)^T = 0, \qquad j = 1,\ldots,p, \qquad (\mathbf{v}_1^T, \mathbf{v}_2^T)^T \in C_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}}); \\
\frac{d}{d\tau} g_i\left(\mathbf{x}\left(\tau\right), \dot{\mathbf{x}}\left(\tau\right)\right)_{|\tau=t} = \nabla g_i(\mathbf{x}\left(\tau\right), \dot{\mathbf{x}}\left(\tau\right)) \left(\dot{\mathbf{x}}\left(\tau\right)^T, \ddot{\mathbf{x}}\left(\tau\right)^T\right)^T_{|\tau=t} = \\
\nabla g_i(\mathbf{x},\dot{\mathbf{x}})(\mathbf{v}_1^T, \mathbf{v}_2^T)^T \geq 0, \qquad i \in I\left(\mathbf{x},\dot{\mathbf{x}}\right), \qquad (\mathbf{v}_1^T, \mathbf{v}_2^T)^T \in C_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}}),
\end{aligned}$$

which is the statement. ■

Let

$$\begin{aligned}
I\left(\left(\mathbf{x}^T, \dot{\mathbf{x}}^T\right)^T, (\mathbf{v}_1^T, \mathbf{v}_2^T)^T\right) = \left\{ i \in I\left(\mathbf{x},\dot{\mathbf{x}}\right) \mid \nabla g_i(\mathbf{x},\dot{\mathbf{x}})(\mathbf{v}_1^T, \mathbf{v}_2^T)^T = 0 \right\}, \\
\left(\mathbf{x}^T, \dot{\mathbf{x}}^T\right)^T \in A, \qquad (\mathbf{v}_1^T, \mathbf{v}_2^T)^T \in D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}}).
\end{aligned} \tag{4.5}$$

Definition 4.4 *The gradient-type Karush-Kuhn-Tucker constraint qualification holds at a point $(\mathbf{x}^T, \dot{\mathbf{x}}^T)^T \in A$ if for every $(\mathbf{v}_1^T, \mathbf{v}_2^T)^T \in D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}})$, the gradients*

$$\begin{aligned}
\nabla g_i(\mathbf{x},\dot{\mathbf{x}}), \quad i \in I\left(\left(\mathbf{x}^T, \dot{\mathbf{x}}^T\right)^T, (\mathbf{v}_1^T, \mathbf{v}_2^T)^T\right); \\
\nabla h_j(\mathbf{x},\dot{\mathbf{x}}),\ i = 1,\ldots,p,
\end{aligned} \tag{4.6}$$

are linearly independent.

Lemma 4.2 *If $h_j, g_i \in C^1$, $j = 1,\ldots,p$; $i = 1,\ldots,m$, and the GKKT constraint qualification holds at a point $(\mathbf{x}^T, \dot{\mathbf{x}}^T)^T \in A$, then,*

$$D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}}) = C_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}}). \tag{4.7}$$

Proof. By *Lemma 4.1*,

$$C_{[\mathbf{h},\mathbf{g}]} \subseteq D_{[\mathbf{h},\mathbf{g}]}$$

at every feasible point, consequently, it is sufficient to prove that

$$D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}}) \subseteq C_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}}), \qquad \left(\mathbf{x}^T,\dot{\mathbf{x}}^T\right)^T \in A.$$

Let us consider an arbitrary $(\mathbf{v}_1^T,\mathbf{v}_2^T)^T \in D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}})$. By fulfilling the GKKT constraint qualification, $(\mathbf{v}_1^T,\mathbf{v}_2^T)^T$ belongs to the intersection of a finite number of hyperspaces with the linearly independent normal vectors ∇h_j, $j = 1,\ldots,p$; ∇g_i, $i \in I\left(\left(\mathbf{x}^T,\dot{\mathbf{x}}^T\right)^T, (\mathbf{v}_1^T,\mathbf{v}_2^T)^T\right)$. The corresponding constraints are active, thus, the intersection of these constraints determines a differentiable manifold in a neighbourhood of the given point where it can be endowed with the induced Euclidean metric (see, e.g., [19]). Since the vector $(\mathbf{v}_1^T,\mathbf{v}_2^T)^T$ belongs to the tangent space of this Riemannian manifold at the investigated point, it follows that a uniquely determined geodesic of the Riemannian manifold starts from the point $\left(\mathbf{x}^T,\dot{\mathbf{x}}^T\right)^T$ in the direction $(\mathbf{v}_1^T,\mathbf{v}_2^T)^T$ in a neighbourhood of the investigated point.

On the other hand, all the functions g_i, $i \in I(\mathbf{x},\dot{\mathbf{x}})\backslash I\left(\left(\mathbf{x}^T,\dot{\mathbf{x}}^T\right)^T, (\mathbf{v}_1^T,\mathbf{v}_2^T)^T\right)$ must increase along this curve, because

$$\nabla g_i\left(\mathbf{x}^T,\dot{\mathbf{x}}^T\right)(\mathbf{v}_1^T,\mathbf{v}_2^T)^T > 0, \qquad i \in I\left(\mathbf{x},\dot{\mathbf{x}}\right)\backslash I\left(\left(\mathbf{x}^T,\dot{\mathbf{x}}^T\right)^T, (\mathbf{v}_1^T,\mathbf{v}_2^T)^T\right).$$

Thus, this geodesic is a feasible curve, it meets the requirements of a possible displacement, which is the statement. ■

Definition 4.5 *Let us consider a polyhedron $\mathcal{P}$ in the (n_1+n_2)-dimensional Euclidean space of the n_1-dimensional variables $\mathbf{x}$ and the n_2-dimensional variables $\mathbf{y}$, respectively. Then, the projection of the polyhedron $\mathcal{P}$ into the n_1-dimensional subspace of the variables $\mathbf{x}$ is*

$$\mathrm{Pr}_{\mathbf{x}}\left(\mathcal{P}\right) = \left\{\mathbf{x} \in R^{n_1} \mid \exists\, \mathbf{y} \in R^{n_2} \text{ such that } \left(\mathbf{x}^T,\mathbf{y}^T\right)^T \in \mathcal{P}\right\}. \tag{4.8}$$

Lemma 4.3 *If $h_j,\ g_i \in C^1$, $j = 1,\ldots,p$; $i = 1,\ldots,m$, and the GKKT constraint qualification holds at a point $\left(\mathbf{x}^T,\dot{\mathbf{x}}^T\right)^T \in A$, then, the cone of the virtual displacements for the equality and inequality constraints in* (4.1) *at $\left(\mathbf{x}^T,\dot{\mathbf{x}}^T\right)^T$ is* $\mathrm{Pr}_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}})$.

Proof.

I. Let $\mathbf{v}_1 \in R^{3n}$ be a virtual displacement for the equality and inequality constraints in (4.1) at $(\mathbf{x}^T,\dot{\mathbf{x}}^T)^T$. By *Definition 4.1*, there exists a possible displacement with a possible velocity function, i.e., there are feasible functions $\left(\mathbf{x}\left(\tau\right)^T,\dot{\mathbf{x}}\left(\tau\right)^T\right)^T$, $\tau \in [t+\varepsilon\left(\mathbf{x},\mathbf{v}_1\right)]$, $\varepsilon(\mathbf{x},\mathbf{v}_1 > 0$, such that

$$\mathbf{x}\left(t\right) = \mathbf{x}, \quad \frac{d\mathbf{x}\left(\tau\right)}{d\tau}\mid_{\tau=t} = \mathbf{v}_1 = \dot{\mathbf{x}}, \quad \frac{d^2\mathbf{x}\left(\tau\right)}{d\tau^2}\mid_{\tau=t} = \mathbf{v}_2.$$

By *Lemma 4.2*, $C_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}}) = D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}})$, thus, $(\mathbf{v}_1^T,\mathbf{v}_2^T)^T \in D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}})$ and $\mathbf{v}_1 \in \mathrm{Pr}_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x},\dot{\mathbf{x}})$.

II. If $\mathbf{v}_1 \in \Pr_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}})$, then, there exists $\mathbf{v}_2 \in R^{3n}$ such that

$$(\mathbf{v}_1^T, \mathbf{v}_2^T)^T \in D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}}).$$

By *Lemma 4.2*, $D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}}) = C_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}})$, thus, there exist feasible functions

$$\left(\mathbf{x}(\tau)^T, \dot{\mathbf{x}}(\tau)^T\right)^T, \qquad \tau \in [t, t+\varepsilon(\mathbf{x}, \mathbf{v}_1)], \qquad \varepsilon(\mathbf{x}, \mathbf{v}_1) > 0,$$

such that

$$\mathbf{x}(t) = \mathbf{x}, \quad \frac{d\mathbf{x}(\tau)}{d\tau}\,|_{\tau=t} = \mathbf{v}_1 = \dot{\mathbf{x}}, \quad \frac{d^2\mathbf{x}(\tau)}{d\tau}\,|_{\tau=t} = \mathbf{v}_2,$$

consequently $\mathbf{v}_1$ is a virtual displacement. ■

Sufficient conditions are given for the explicit formulation of the cone of virtual displacements $\Pr_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}})$ in [18].

Definition 4.6 *Let H and H_1 be two real Hilbert spaces, $A \subseteq H_1$ a subset, $F : A \to H$ a mapping and $K : A \rightrightarrows H$ a set-valued mapping. Generalized quasi-complementarity systems (in short, GQCS) are equilibrium systems where the aim is to find a point $x \in A$ such that*

$$< F(z),\, v > \geq 0, \qquad \forall v \in K(z),$$

where $K(z)$, $z \in A$, are convex cones.

GQCS are a generalization of QCS. If $H = H_1$, $\mathcal{K}(z) = z + K(z)$, $y = z + v$, then GQCS reduces to QCS.

The main result of this section is as follows:

Theorem 4.1 *Consider the mechanical equilibrium problems with constraints* (4.1) *and a continuous force function $\mathbf{P} : A \subseteq R^{6n} \to R^{3n}$. If the GKKT constraint qualification holds on the feasible set A, then, the determination of a mechanical equilibrium position is equivalent to the solution of an equilibrium problem GQCS where a point $(\mathbf{x}^T,\, \dot{\mathbf{x}}^T)^T \in A$ has to be found for which*

$$\begin{gathered}\mathbf{P}(\mathbf{x}, \dot{\mathbf{x}})^T\mathbf{v} \leq 0, \quad \mathbf{v} \in \mathcal{K}(\mathbf{x}, \dot{\mathbf{x}}) = \\ \{\mathbf{v} \in R^{3n} |\, \mathbf{v} \in \Pr_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}}), \quad \mathbf{A}_1(\mathbf{x})\,\mathbf{v} = 0, \quad \mathbf{A}_2(\mathbf{x})\,\mathbf{v} \geq 0\}.\end{gathered} \tag{4.9}$$

Proof. By the principle of virtual work, the inequalities

$$\mathbf{P}(\mathbf{x}^*, \dot{\mathbf{x}}^*)^T\mathbf{v} \leq 0 \tag{4.10}$$

hold for all the virtual displacements in an equilibrium state $(\mathbf{x}^*, \dot{\mathbf{x}}^*)^T \in A$. By *Lemma 4.3*, the set $\mathcal{K}(\mathbf{x}, \dot{\mathbf{x}})$, $(\mathbf{x}^T, \dot{\mathbf{x}}^T)^T \in A$ consists of all the virtual displacements of constraints (4.1).

Now, it will be shown that system (4.9) is a GQCS. The cone $D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}^*, \dot{\mathbf{x}}^*)$ is given by a finite number of linear equalities and inequalities, thus, it is a polyhedral convex cone. Since a projection in *Definition 4.5* is a linear transformation, by *Theorem 19.3* (Rockafellar, 1970), $Pr_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}^*, \dot{\mathbf{x}}^*)$ is a polyhedral convex set, more precisely, a polyhedral convex cone as well. If $H = R^{3n}$ and $H_1 = R^{6n}$, $F(z) = -\mathbf{P}(\mathbf{x}, \dot{\mathbf{x}})$,

$$K(z) = \mathcal{K}(\mathbf{x}, \dot{\mathbf{x}}), \qquad z = (\mathbf{x}, \dot{\mathbf{x}}^T)^T \in A, \qquad v = \mathbf{v},$$

then equilibrium system (4.9) gives a GQCS, which is the statement. ■

In the next example, a mechanical problem is considered subject to scleronomic and nonholonomic constraints.

Example 4.2 A problem of Galilei's
Two bodies $\mathcal{A}_1$ and $\mathcal{A}_2$ of equal masses and total weight $2p_1$ are connected with an inextensible thread through two fixed points. The stretched thread is horizontal. A third mass $\mathcal{A}_3$ of weight p_2 is attached to the center of the thread (see Figure 1).

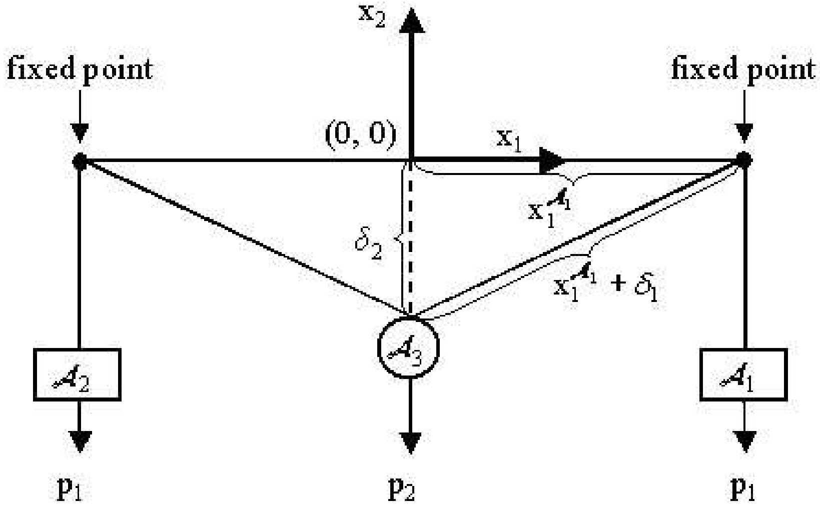

Figure 1

The coordinates of the bodies are

$$\mathcal{A}_1 : \left(x_1^{\mathcal{A}_1}, x_2^{\mathcal{A}_1}\right), \qquad \mathcal{A}_2 : \left(-x_1^{\mathcal{A}_1}, x_2^{\mathcal{A}_1}\right), \qquad \mathcal{A}_3 : \left(x_1^{\mathcal{A}_3}, x_2^{\mathcal{A}_3}\right), \quad \textit{and}$$
$$x_2^{\mathcal{A}_1} = C + \delta_1, \quad x_1^{\mathcal{A}_3} = 0, \quad x_2^{\mathcal{A}_3} = -\delta_2,$$

where $C < 0$ is a constant, δ_1, $\delta_2 \geq 0$ are the vertical displacements of $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_3$, respectively.

The vertical displacements fulfil the constraint

$$\left(x_1^{\mathcal{A}_1}\right)^2 + \left(x_2^{\mathcal{A}_3}\right)^2 = \left(x_1^{\mathcal{A}_1} + x_2^{\mathcal{A}_1} - C\right)^2,$$

from which we have that

$$h\left(x_1^{\mathcal{A}_1}, x_2^{\mathcal{A}_1}, x_1^{\mathcal{A}_3}, x_2^{\mathcal{A}_3}\right) = \left(x_2^{\mathcal{A}_3}\right)^2 - \left(x_2^{\mathcal{A}_1}\right)^2 - 2x_1^{\mathcal{A}_1}x_2^{\mathcal{A}_1} + 2\left(x_1^{\mathcal{A}_1} + x_2^{\mathcal{A}_1}\right)C - C^2 = 0,$$
$$\left(x_1^{\mathcal{A}_1}, x_2^{\mathcal{A}_1}, x_1^{\mathcal{A}_3}, x_2^{\mathcal{A}_3}\right) \in R^4.$$

The virtual displacements at the given point $\left(x_1^{\mathcal{A}_1}, x_2^{\mathcal{A}_1}, x_1^{\mathcal{A}_3}, x_2^{\mathcal{A}_3}\right)$ are as follows:

$$\nabla h\left(x_1^{\mathcal{A}_1}, x_2^{\mathcal{A}_1}, x_1^{\mathcal{A}_3}, x_2^{\mathcal{A}_3}\right) \begin{pmatrix} v_{11} \\ v_{12} \\ v_{31} \\ v_{32} \end{pmatrix} = 0, \tag{4.11}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{12} \\ v_{31} \\ v_{32} \end{pmatrix} = 0. \tag{4.12}$$

This system results in the relation

$$\left(x_1^{\mathcal{A}_1} + \delta_1\right) v_{12} + \delta_2 v_{32} = 0. \tag{4.13}$$

The coordinates of the active forces are

$$\mathbf{P}_1 \left(x_1^{\mathcal{A}_1}, x_2^{\mathcal{A}_1}\right)^T = (0, -p_1), \quad \mathbf{P}_2 \left(-x_1^{\mathcal{A}_1}, x_2^{\mathcal{A}_1}\right)^T = (0, -p_1), \quad \mathbf{P}_3 \left(0, x_2^{\mathcal{A}_3}\right)^T = (0, -p_2).$$

By the principle of virtual work, the condition of the equilibrium is as follows:

$$-2p_1 v_{12} - p_2 v_{32} \leq 0.$$

By (4.13), $-v_{32} = \dfrac{\left(x_1^{\mathcal{A}_1} + \delta_1\right)}{\delta_2} v_{12}$*, thus, we have that*

$$p_2 \frac{\left(x_1^{\mathcal{A}_1} + \delta_1\right)}{\delta_2} v_{12} \leq 2p_1 v_{12}.$$

Because v_{12} *can be positive and negative at an equilibrium state, the necessary condition of the equilibrium is the equality*

$$\frac{2p_1}{p_2} = \frac{x_1^{\mathcal{A}_1} + \delta_1}{\delta_2}.$$

This condition was stated by Farkas in 1893, [5], [22], Galilei's condition (Dialoghi delle nuove Scienze, 1638) given by

$$\frac{2p_1}{p_2} = \frac{\delta_2}{\delta_1}.$$

5. Dual form of the principle of virtual work subject to scleronomic and nonholonomic constraints

In this part, the dual problem of (4.9) is formulated based on the *Farkas* theorem. If K is a cone with apex at the origin, then its polar is a closed and convex cone, again with apex at the origin, given by

$$K^* = \{\mathbf{y} \in R^n \mid \mathbf{y}^T \mathbf{x} \leq 0, \quad \forall \mathbf{x} \in K\}. \tag{5.1}$$

Theorem 5.1 *Consider the mechanical equilibrium problems with constraints* (4.1) *and a continuous force function* $\mathbf{P} : A \subseteq R^{6n} \to R^{3n}$. *If the GKKT constraint qualification fulfils on the feasible set A, then, the determination of a mechanical equilibrium position is equivalent to the solution of the dual form of* (4.9) *where a point* $\left(\mathbf{x}^T, \dot{\mathbf{x}}^T\right)^T \in A$ *has to be found for which*

$$\begin{gathered}\mathbf{P}(\mathbf{x}, \dot{\mathbf{x}}) \in \textit{cone conv}\{\text{a finite generator of the polar of } \mathrm{Pr}_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}})\} + \\ \textit{cone conv}\{\text{the rows of } -\mathbf{A}_2(\mathbf{x})\} + \textit{span}\{\text{the rows of } \mathbf{A}_1(\mathbf{x})\}, \\ (\mathbf{x}^T, \dot{\mathbf{x}}^T)^T \in A.\end{gathered} \tag{5.2}$$

Proof. By *Theorem 4.1*, if $(\mathbf{x}^{*T}, \dot{\mathbf{x}}^{*T}) \in A$ is an equilibrium position, then,

$$\mathbf{P}(\mathbf{x}^*, \dot{\mathbf{x}}^*)^T \mathbf{v} \le 0, \qquad \forall \mathbf{v} \in \mathcal{K}(\mathbf{x}^*, \dot{\mathbf{x}}^*). \tag{5.3}$$

The cone $D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}^*, \dot{\mathbf{x}}^*)$ is given by a finite number of linear equalities and inequalities, thus, it is a polyhedral convex cone. Since a projection in *Definition 4.5* is a linear transformation, by *Theorem 19.3* (Rockafellar, 1970), $\mathrm{Pr}_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}})$ is a polyhedral convex set, more precisely, a polyhedral convex cone as well. By *Theorem 19.1* (Rockafellar, 1970), $\mathrm{Pr}_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}})$ is finitely generated, by *Corollary 19.2.2* (Rockafellar, 1970), its polar is a polyhedral convex cone, that is, a finite generator of the polar exists. By *Corollary 19.3.2* (Rockafellar, 1970), in case of two polyhedral convex sets in R^n, their Minkowski sum is polyhedral. It follows that the set $\mathcal{K}(\mathbf{x}^*, \dot{\mathbf{x}}^*) \subseteq R^{3n}$ and the set in (5.2) are polyhedral convex cones.

By (5.3), the inequality $\mathbf{P}(\mathbf{x}^*, \dot{\mathbf{x}}^*)^T \mathbf{v} \le 0$ is a consequence of the system defined by $\mathcal{K}(\mathbf{x}^*, \dot{\mathbf{x}}^*)$. *Farkas* theorem states that relation (5.3) holds if and only if relation (5.2) is true. ■

We remark that relation (5.2), containing elements from the dual space only, can be considered the dual form of (4.9). Since the dual form of the principle of virtual work does not contain the virtual displacements, thus, in concrete problems, this form seems to be more advantageous than the original one. Duality question in QVI were considered by Giannessi (1995).

Example 5.1 ***The dual form of a problem of Galilei's***

By Theorem 5.1 and the equality

$$\left(x_1^{\mathcal{A}_1} + \delta_1\right) v_{12} + \delta_2 v_{32} = 0,$$

we obtain that

$$\begin{pmatrix} 0 \\ 2p_1 \\ 0 \\ p_2 \end{pmatrix} = \lambda \begin{pmatrix} 0 \\ x_1^{\mathcal{A}_1} + \delta_1 \\ 0 \\ \delta_2 \end{pmatrix},$$

which is equivalent to

$$\begin{gathered} 2p_1 = \lambda(x_1^{\mathcal{A}_1} + \delta_1), \\ p_2 = \lambda \delta_2, \end{gathered}$$

from which

$$2p_1 = \frac{p_2}{\delta_2}(x_1^{\mathcal{A}_1} + \delta_1),$$

i.e.,

$$\frac{2p_1}{p_2} = \frac{x_1^{A_1} + \delta_1}{\delta_2},$$

which is the necessary condition of the equilibrium.

6. Principle of virtual work subject to rheonomous and nonholonomic constraints

We assume that the active forces

$$\mathbf{P} : A \subseteq R^{6n+1} \to R^{3n}$$

affecting the system are continuous functions of position, velocity and time defined on a subset A of R^{6n+1}. The constraints of the mechanical systems are as follows:

$$\begin{aligned} h_j(\mathbf{x}, \dot{\mathbf{x}}, t) = 0, \quad & j = 1, \dots, p, \\ g_i(\mathbf{x}, \dot{\mathbf{x}}, t) \geq 0, \quad & i = 1, \dots, m, \\ \mathbf{A}_1(\mathbf{x}, t)\dot{\mathbf{x}} = \mathbf{0}&, \\ \mathbf{A}_2(\mathbf{x}, t)\dot{\mathbf{x}} \geq \mathbf{0}&, \\ (\mathbf{x}^T, \dot{\mathbf{x}}^T, t)^T \subseteq R^{3n} \times & R^{3n} \times R_{\geq}, \end{aligned} \tag{6.1}$$

where $R_{\geq}$ denotes the nonnegative real numbers, $g_i \in C^1, i = 1, \dots, m; h_j \in C^1$, $j = 1, \dots, p$; $\mathbf{A}_1(\mathbf{x}, t)$ a $k_1(\mathbf{x}, t) \times 3n$ matrix and $\mathbf{A}_2(\mathbf{x}, t)$ a $k_2(\mathbf{x}, t) \times 3n$ matrix at every point $(\mathbf{x}^T, \dot{\mathbf{x}}^T, t)^T$, respectively. Let us introduce the notation $A \subseteq R^{3n} \times R^{3n} \times R_{\geq}$ of the feasible set of constraints (6.1). In other words, the states of the investigated mechanical systems are represented by the vectors of the set A.

If constraints (6.1) do not explicitly depend on the time, the system is called scleronomic. If at least one of the constraints depends on the time, the system is called rheonomous. It follows that a scleronomic system is a special case of a rheonomous one. For example, a system of n mass points, constrained by time-dependent holonomic constraints, can be defined with the help of a time-dependent submanifold of the configuration space of a free system. The next definition follows Banach's book (1951).

Definition 6.1 *The displacements*

$$\mathbf{x}(t), \quad t \in [t_0, t_1], \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad \mathbf{x}(t_1) = \mathbf{x}_1,$$

of the given system of mass points from the point $\mathbf{x}_0$ *to the point* $\mathbf{x}_1$ *and the velocities* $\dot{\mathbf{x}}(t), t \in [t_0, t_1]$, *belonging to them are possible if the functions* $(\mathbf{x}(t)^T, \dot{\mathbf{x}}(t)^T, t)^T$, $t \in [t_0, t_1]$, *satisfy the constraints. Virtual displacements at* t_0 *are the ones which point in the direction of possible velocities at fixing* $t = t_0$ *in constraints* (6.1).

Consequently, in rheonomous systems, the virtual displacements, in general, are not proportional to possible velocities as in scleronomic systems. The virtual displacements are as if the feasible set was fixed having the position which it occupies at time t.

In the scleronomic case, a mechanical system is considered to be in force equilibrium if the sum of active forces and reaction forces is zero on every mass point. (Reaction forces cause the points to maintain constraints.) By [2], this definition of equilibrium, however, is not suitable for rheonomous systems. That is to say, if a system of material points is constrained to remain constantly in a horizontal plane moving vertically upwards with a uniform motion, then the system can, at no time, remain at rest. According to the preceding definition, we could not say that any system of forces is in equilibrium.

For rheonomous systems without friction, the principle of virtual work can provide with the definition of the equilibrium of active forces.

Definition 6.2 [2] *The forces acting on a rheonomous system without friction are in equilibrium at a certain time t if the principle of virtual work holds.*

The next statements are direct consequences of *Theorems 4.1* and *5.1.*, furthermore, *Definitions 6.1* and *6.2.*

Theorem 6.1 *Consider the mechanical equilibrium problems with constraints* (6.1) *and a continuous force function $\mathbf{P} : A \subseteq R^{6n+1} \to R^{3n}$. Let us fix a moment $t = t^*$. If the GKKT constraint qualification holds on the feasible set A at t^*, then, the determination of a mechanical equilibrium position at t^* is equivalent to the solution of an equilibrium problem GQCS where a point $(\mathbf{x},^T \dot{\mathbf{x}}^T, t^*)^T \in A$ has to be found for which*

$$\begin{gathered} \mathbf{P}(\mathbf{x}, \dot{\mathbf{x}}, t^*)^T \mathbf{v} \leq 0, \quad \mathbf{v} \in \mathcal{K}(\mathbf{x}, \dot{\mathbf{x}}, t^*) = \\ \{\mathbf{v} \in R^{3n} | \, \mathbf{v} \in \mathrm{Pr}_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}}, t^*), \quad \mathbf{A}_1 (\mathbf{x}, t^*) \, \mathbf{v} = 0, \quad \mathbf{A}_2 (\mathbf{x}, t^*) \, \mathbf{v} \geq 0\}, \\ (\mathbf{x}^T, \dot{\mathbf{x}}^T, t^*)^T \in A. \end{gathered} \tag{6.2}$$

The dual problem of (6.2) can be formulated as follows:

Theorem 6.2 *Consider the mechanical equilibrium problems with constraints* (6.1) *and a continuous force function $\mathbf{P} : A \subseteq R^{6n+1} \to R^{3n}$. Let us fix a moment $t = t^*$. If the GKKT constraint qualification holds on the feasible set A at t^*, then, the determination of a mechanical equilibrium position at t^* is equivalent to the solution of the dual form of* (6.2) *where a point $\left(\mathbf{x}^T, \dot{\mathbf{x}}^T, t^*\right) \in A$ has to be found for which*

$$\begin{gathered} \mathbf{P}(\mathbf{x}) \in \mathit{cone\ conv}\{\text{a finite generator of the polar of } \mathrm{Pr}_{\mathbf{v}_1} D_{[\mathbf{h},\mathbf{g}]}(\mathbf{x}, \dot{\mathbf{x}}, t^*)\} + \\ \mathit{cone\ conv}\{\text{the rows of } -\mathbf{A}_2(\mathbf{x}, t^*)\} + \mathit{span}\{\text{the rows of } \mathbf{A}_1(\mathbf{x}, t^*)\}. \end{gathered} \tag{6.3}$$

Example 6.1 [2] *Let a material point $\mathcal{A}(x_1, x_2, x_3)$ be constrained to remain on the surface of a certain sphere which moves with a uniform advancing motion.*

Let r denote the radius of the sphere, $\xi_1^0, \xi_2^0, \xi_3^0$ the coordinates of the centre of the sphere at time $t_0 = 0$, ξ_1, ξ_2, ξ_3 the coordinates at time t, and a_1, a_2, a_3 the coordinates of velocity. At time t, we have $\xi_1 = \xi_1^0 + a_1 t$, $\xi_2 = \xi_2^0 + a_2 t, \xi_3 = \xi_3^0 + a_3 t$. The sphere, therefore, has the equation

$$\frac{1}{2}\left(x_1 - \xi_1^0 - a_1 t\right)^2 + \frac{1}{2}\left(x_2 - \xi_2^0 - a_2 t\right)^2 + \frac{1}{2}\left(x_3 - \xi_3^0 - a_3 t\right)^2 - r^2 = 0. \tag{6.4}$$

Hence, the coordinates of the point $\mathcal{A}$ must satisfy equation (6.4) *at each moment.*

The virtual displacements satisfy the equation

$$\left(x_1 - \xi_1^0 - a_1 t\right) v_1 + \left(x_2 - \xi_2^0 - a_2 t\right) v_2 + \left(x_3 - \xi_3^0 - a_3 t\right) v_3 = 0,$$
$$\left(v_1, v_2, v_3\right)^T \in R^3.$$

Based on the dual form of the principle of virtual work and Example 3.1, the coordinates x_1, x_2, x_3 *of the position of equilibrium and* λ *at a fixed* t *are as follows:*

$$\lambda = \pm\sqrt{P_1^2 + P_2^2 + (P_3 - mg)^2}/\sqrt{2}\, r,$$

$$x_1 = -P_1/\lambda + \xi_1^0 + \alpha_1 t, \qquad x_2 = -P_2/\lambda + \xi_2^0 + a_2 t, \qquad x_3 = (P_3 - mg)/\lambda + \xi_3^0 + a_3 t.$$

7. Concluding remarks

A novelty in the resulted equilibrium systems is that the variables in *Definition 4.6* belong to different spaces. Thus, the existence of solution(s) seems to be an open question. Since the mechanical equilibrium problems subject to scleronomic and holonomic constraints often lead to nonconvex configuration spaces (e.g., a subset of a differentiable manifold), the question of existence should be answered in these cases as well.

In order to solve the resulted equilibrium systems, efficient methodologies taking the specialities of the concrete problems into consideration should be developed, e.g., for feasibility problems (3.6).

Based on the results, physical phenomenons should be modelled in a wider circle, e.g., nonholonomic or/and time-dependent phenomenons with or without friction. The aim should be to describe and solve physical problems which cannot be solved by Lagrangian mechanics.

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