FACTA UNIVERSITATIS (NIŠ)
Ser. Math. Inform. Vol. 36, No 4 (2021), 749–759
https://doi.org/10.22190/FUMI200118055A
Original Scientific Paper

THE STRUCTURE OF UNIT GRUOP OF $\mathbb{F}_{3^n}T_{39}$

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Abstract. Let RG be the group ring of a group G over ring R and let $\mathscr{U}(RG)$ be its unit group. In this paper, we study the structure of the unit group of $\mathbb{F}_{3^n}T_{39}$.
Key words: Group ring, unit group, group modules

1. Introduction

Let FG be the group ring of a group G over a field F and let $\mathscr{U}(FG)$ be its unit group, which is the multiplicative subgroup containing all invertible elements. The study of a unit group is one of the classical topics in ring theory that started in 1940 with a famous paper written by G. Higman [11]. In recent years many new results have been achived; however, only few group rings have been computed. Unit groups are useful, for instance, in the investigation of Lie properties of group rings (for example see [3]) and isomorphism problems (for example see [4]).

Up to now, the structure of unit groups of some group rings has been found. For instance, on an integral group ring [12], on a permutation group ring [18], on a commutative group ring [16], on a linear group ring [13], on a quaternion group ring [6], on a modular group ring [17] and on a pauli group ring [9]. In [7], the authors proved which groups can be unit groups as well as properties of unit elements themselves [2] and also we studied the structure of $\mathscr{U}(\mathbb{F}_{2^n}D_{14})$ in [1].

In this paper we will study the unit group of $\mathbb{F}_{3^n}T_{39}$. So far, some cases, in characteristic 3, have been studied. For instance, in [5], the authors obtained

Received January 18, 2020, accepted: May 5, 2020
Communicated by Alireza Ashrafi and Hassan Daghigh
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2010 *Mathematics Subject Classification.* Primary 16S34; Secondary 20C05; 16U60

the structure of unit group of $\mathbb{F}_{3^k}D_6$, in [8], Gildea determined the structure of unit group of $\mathbb{F}_{3^k}(C_3 \times D_6)$ and in [10] Gildea and Monaghan studied groups of order 12 and recently in [15], Monaghan studied groups of order 24. In this paper we characterize the unit group structure of group T_{39} over any finite field with characteristic 3.

2. Preliminaries and Notations

In this section, we collect some notations and lemma which we need for the proofs of our main results. We denote the order of an element g in the group G by $Ord_G(g)$, the sum of all elements of subset X in ring R by $\widehat{X}$, which is $\sum_{r\in X} r$. Notice there is no need for X to be a subring or subgroup; it defines for any arbitrary subset. In group ring RG, when X is the subset of all different powers of g, an element of group G, we may simply write $\widehat{g}$ instead of $\widehat{X}$. Also when X is the right coset $\langle g\rangle h$, we may write $\widehat{g}h$ for $\widehat{X}$. In group, x^y denotes the conjugate of x by y, that is, $x^y = y^{-1}xy$. Let $f : X \to Y$ be an arbitrary function. Define $Supp_X(f) = \{x \in X \,|\, f(x) \neq 0\}$. Also, we use the following notations: $\mathrm{Ann}_R(a) = \{r \in R \,|\, ra = ar = 0\}$, we denote a finite field of characteristic p with order p^n by $\mathbb{F}_{p^n}$. If E is a vector space over F, then $Dim_F(E)$ is the dimension of E over F. Let $\mathscr{U}(R)$ be the unit group of ring R, which is $\mathscr{U}(R) = \{u \in R \,|\, u^{-1} \in R\}$ and let $J(R)$ be the Jacobson radical of ring R. Now we state a useful definition and recall a lemma.

Definition 2.1. Let RG be group ring of ring R over the group G, let p be a prime number and let S_p be subset of all *p–elements* including identity element of G, which is $S_p = \{g \in G \,|\, \exists n \in \mathbb{Z}^{\geqslant 0};\ \mathrm{Ord}_G(g) = p^n\}$. We define a binary map $T : G \to R$ as follows:

$$T(g) = \begin{cases} 1 & \textbf{\textit{If}} \quad g \in S_p \\ 0 & \textbf{\textit{If}} \quad g \notin S_p \end{cases}$$

As we know that T on G is the base of RG, so we can linearly extend it to whole RG, of course no more remains binary. Also if see elements of RG as functions from G to R, that map every group element (g) to its coefficient (r_g), then their supports will be feasible. Now we can define $\mathrm{Krn}(T) := \{\alpha \in RG \,|\, \forall g \in G;\ \alpha g \in \mathrm{Ker}_{RG}(T)\}$ and $\mathrm{Spr}(\alpha) := \mathrm{Supp}_G(\alpha)$. Also $\mathrm{Anh}(a) := \mathrm{Ann}_{RG}(a)$ and $\mathrm{Dmn}(S) := \mathrm{Dim}_F(S)$.

Lemma 2.1. *Let F be a finite field of characteristic p, let G be a finite group, let T be a function defined as above and $s = \widehat{S}_p$. Then:*
(1) $J(FG) \subseteq \mathrm{Krn}(T)$.
(2) $\mathrm{Krn}(T) = \mathrm{Anh}(s)$.
(3) $J(FG) \subseteq \mathrm{Anh}(s)$.

Proof. [19, Lemma 2.2 on p. 151]. □

In the next section we present our main results.

3. Unit Group of $\mathbb{F}_{3^n}T_{39}$

Let $T_{39} = \langle x, y \mid x^{13} = y^3 = 1,\ x^y = x^3\rangle$, let C_n be the cyclic group of order n and let $GL_n(R)$ be the general linear group of degree n on ring R. Our main result is:

Theorem 3.1. *Let $G = T_{39}$ and $F = \mathbb{F}_{3^n}$. Then the structure of $\mathscr{U}(FG)$ can be obtained as follows:*

$$\mathscr{U}(FG) = C_3^{2n} \times C_{3^n-1} \times GL_3(F)^4.$$

Let $p = 3$, let s be defined as in Lemma 2.1, let $\langle x\rangle$ be the cyclic subgroup generated by x and let $\langle x\rangle y$ be a right coset of $\langle x\rangle$, that is, $\langle x\rangle y = \{x^i y \mid -6 \leqslant i \leqslant +6\}$, or equivalently, $\langle x\rangle y = \{x^{-6}y, x^{-5}y, x^{-4}y, x^{-3}y, x^{-2}y, x^{-1}y, y, xy, x^2y, x^3y, xy^4, x^5y, x^6y\}$. By definition, we have

$$\begin{aligned}
\widehat{x} &= x^{-6} + x^{-5} + x^{-4} + x^{-3} + x^{-2} + x^{-1} + 1 + x + x^2 + x^3 + x^4 + x^5 + x^6\\
\widehat{x}y &= x^{-6}y + x^{-5}y + x^{-4}y + x^{-3}y + x^{-2}y + x^{-1}y + y + xy + x^2y + x^3y\\
&\quad + x^4y + x^5y + x^6y.
\end{aligned}$$

Now we show:

Proposition 3.1. *Let $p = 3$ and $G = T_{39}$. Then the structure of annihilator will be as follows:*

$$\mathrm{Anh}(s) = \{a^- \widehat{x}y^{-1} + a\widehat{x} + a^+ \widehat{x}y \mid a^- + a + a^+ = 0\}.$$

Proof. It is easy to find that the conjugacy classes of G are as below:

$$\begin{aligned}
&\mathscr{C}_0 = \{1\}\\
&\mathscr{C}_{-1} = \{x^{-1},\ x^{-3},\ x^4\}\\
&\mathscr{C}_{+1} = \{x,\ x^3,\ x^{-4}\}\\
&\mathscr{C}_{-2} = \{x^{-2},\ x^{-5},\ x^{-6}\}\\
&\mathscr{C}_{+2} = \{x^2,\ x^5,\ x^6\}\\
&\mathscr{C}_{-3} = \langle x\rangle y^{-1}\\
&\mathscr{C}_{+3} = \langle x\rangle y
\end{aligned} \tag{3.1}$$

It is clear that T_{39} has three types of elements: Identity, elements of the form $x^i y^{\pm 1}$ with order 3 and elements of the form $x^i \neq 1$ with order 13. Therefore, $S_3 = \mathscr{C}_{-3} \cup \mathscr{C}_0 \cup \mathscr{C}_{+3}$, so $\widehat{S_3} = \widehat{\mathscr{C}_{-3}} + \widehat{\mathscr{C}_0} + \widehat{\mathscr{C}_{+3}} = \widehat{x}y^{-1} + 1 + \widehat{x}y$, sum of $3-$*elements* including identity. Let $\alpha = \sum_{i=-3}^{+3} \alpha_i \in \mathrm{Anh}(s)$ where, $\mathrm{Spr}(\alpha_i) \subseteq \mathscr{C}_i$ and $s = \widehat{S_3}$.

Then we have

$$
\begin{aligned}
0 = \alpha.s &= (\sum_{i=-3}^{+3} \alpha_i).(\widehat{S}_3) \\
&= (\alpha_{-3}+\alpha_{-2}+\alpha_{-1}+\alpha_0+\alpha_{+1}+\alpha_{+2}+\alpha_{+3})(\widehat{xy}^{-1}+1+\widehat{xy}) \\
&= (\alpha_{-3}+(\alpha_{-2}+\alpha_{-1}+\alpha_0+\alpha_{+1}+\alpha_{+2})+\alpha_{+3})(\widehat{xy}^{-1}+1+\widehat{xy}) \\
&= (\alpha_{-3}+(\alpha_{-2}+\alpha_{-1}+\alpha_0+\alpha_{+1}+\alpha_{+2})\widehat{xy}^{-1}+\alpha_{+3}\widehat{xy}) \\
&\quad +(\alpha_{-3}\widehat{xy}+(\alpha_{-2}+\alpha_{-1}+\alpha_0+\alpha_{+1}+\alpha_{+2})+\alpha_{+3}\widehat{xy}^{-1}) \\
&\quad +(\alpha_{-3}\widehat{xy}^{-1}+(\alpha_{-2}+\alpha_{-1}+\alpha_0+\alpha_{+1}+\alpha_{+2})\widehat{xy}+\alpha_{+3})
\end{aligned} \tag{3.2}
$$

Notice that for every j, we know:

$$
\begin{aligned}
x^j y \quad .\widehat{xy} \quad &= x^j \ .\widehat{xy}^{-1} = \widehat{xy}^{-1} \\
x^j y^{-1}.\widehat{xy} \quad &= x^j y.\widehat{xy}^{-1} = \widehat{x} \\
x^j y^{-1}.\widehat{xy}^{-1} &= x^j \ .\widehat{xy} \quad = \widehat{xy}
\end{aligned} \tag{3.3}
$$

So the conjugacy classes of three last parentheses of (3.2) are different and since the left hand side is zero, every parentheses should be zero separately. Hence,

$$
\begin{aligned}
&(\alpha_{-3}+(\alpha_{-2}+\alpha_{-1}+\alpha_0+\alpha_{+1}+\alpha_{+2})\widehat{xy}^{-1}+\alpha_{+3}\widehat{xy}) = 0 \\
&(\alpha_{-3}\widehat{xy}+(\alpha_{-2}+\alpha_{-1}+\alpha_0+\alpha_{+1}+\alpha_{+2})+\alpha_{+3}\widehat{xy}^{-1}) = 0 \\
&(\alpha_{-3}\widehat{xy}^{-1}+(\alpha_{-2}+\alpha_{-1}+\alpha_0+\alpha_{+1}+\alpha_{+2})\widehat{xy}+\alpha_{+3}) = 0.
\end{aligned}
$$

Similarly, using (3.3) we can conclude that:

$$
\begin{aligned}
&\alpha_{-3}+\varepsilon((\alpha_{-2}+\cdots+\alpha_{+2})+\alpha_{+3})\widehat{xy}^{-1} = 0 \\
&(\alpha_{-2}+\cdots+\alpha_{+2})+\varepsilon(\alpha_{-3}+\alpha_{+3})\widehat{x} \qquad = 0 \\
&\alpha_{+3}+\varepsilon(\alpha_{-3}+(\alpha_{-2}+\cdots+\alpha_{+2}))\widehat{xy} \quad = 0.
\end{aligned} \tag{3.4}
$$

As mentioned above $\alpha = \sum_{i=-3}^{+3} \alpha_i = \alpha_{-3}+\alpha_{-2}+\alpha_{-1}+\alpha_0+\alpha_{+1}+\alpha_{+2}+\alpha_{+3}$ where $\mathrm{Spr}(\alpha_i) \subseteq \mathscr{C}_i$ and by definition of $\mathscr{C}_i$'s from (3.1), we can write:

$$
\begin{aligned}
\alpha_0 &= a_0 \\
\alpha_{-1} &= a_{-1}x^{-1}+a_{-3}x^{-3}+a_4x^4 \\
\alpha_{+1} &= a_1x+a_3x^3+a_{-4}x^{-4} \\
\alpha_{-2} &= a_{-2}x^{-2}+a_{-5}x^{-5}+a_{-6}x^{-6} \\
\alpha_{+2} &= a_2x^2+a_5x^5+a_6x^6 \\
\alpha_{-3} &= \sum_{i=-6}^{6} a_i^- x^i y^{-1} \\
\alpha_{+3} &= \sum_{i=-6}^{6} a_i^+ x^i y
\end{aligned}
$$

By substitution of each α_i's in (3.4), we can calculate the coefficients of each element of the group in the left hand sides of equations and since the right hand sides are zero, so each coefficient must be zero too. Thus for every h, i and j we have

$$\begin{array}{lll}
a_h^- = -\varepsilon(\sum_{r=-2}^{+2} \alpha_r + \alpha_{+3}) & a_i = -\varepsilon(\alpha_{+3} + \alpha_{-3}) & a_j^+ = -\varepsilon(\alpha_{-3} + \sum_{r=-2}^{+2} \alpha_r) \\
a_h^- = -\sum_{r=-6}^{6} (a_r + a_r^+) & a_i = -\sum_{r=-6}^{6} (a_r^+ + a_r^-) & a_j^+ = -\sum_{r=-6}^{6} (a_r^- + a_r) \\
a_{-6}^- = \cdots = a_6^- & a_{-6} = \cdots = a_6 & a_{-6}^+ = \cdots = a_6^+
\end{array}$$

So by knowing a_0^-, a_0 and a_0^+, all coefficients can be computed. Also since we deal with a field of characteristic 3, so $13 = 1$, therefore, we have $a_0^- + a_0 + a_0^+ = 0$, thus:

$$\mathrm{Anh}(s) = \{a_0^- \widehat{x}y^{-1} + a_0\widehat{x} + a_0^+\widehat{x}y \mid a_0^- + a_0 + a_0^+ = 0\}.$$

□

Let s be as in Proposition 3.1, that is $s = \widehat{S}_3$, then we have

Proposition 3.2. Anh(s) *is a nilpotent ideal.*

Proof. Let $\alpha, \beta, \gamma \in \mathrm{Anh}(s)$. According to Proposition 3.1, we have

$$\begin{aligned}
\alpha &= a^-\widehat{x}y^{-1} + a\widehat{x} + a^+\widehat{x}y \\
\beta &= b^-\widehat{x}y^{-1} + b\widehat{x} + b^+\widehat{x}y \\
\gamma &= c^-\widehat{x}y^{-1} + c\widehat{x} + c^+\widehat{x}y.
\end{aligned} \tag{3.5}$$

So their production is:

$$\begin{aligned}
\alpha.\beta.\gamma &= (a^-\widehat{x}y^{-1} + a\widehat{x} + a^+\widehat{x}y).(b^-\widehat{x}y^{-1} + b\widehat{x} + b^+\widehat{x}y).(c^-\widehat{x}y^{-1} + c\widehat{x} + c^+\widehat{x}y) \\
&= (a^+ - a^-)(b^+ - b^-)\widehat{G}.(c^-\widehat{x}y^{-1} + c\widehat{x} + c^+\widehat{x}y) \\
&= (a^+ - a^-)(b^+ - b^-)(c^- + c + c^+)\widehat{G}|\langle x\rangle|
\end{aligned} \tag{3.6}$$

By Proposition 3.1, $\alpha.\beta.\gamma = 0$, thus $\mathrm{Anh}^3(s) = 0$, therefore, Anh(s) is a nilpotent ideal. □

Let s be as in Proposition 3.2, that is $s = \widehat{S}_3$, then we have

Proposition 3.3. $\mathrm{Anh}(s) \subseteq J(FG)$.

Proof. Since every nilpotent ideal is a nil ideal, so Proposition 3.2 shows $\mathrm{Anh}(s)$ is a nil ideal. On the other hand, by [14, Lemma 2.7.13 on p. 109], Jacobson radical contains all of the nil ideals, so,

$$\mathrm{Anh}(s) \subseteq J(FG).$$

□

In the next corollary, we will show that the equality hold:

Corollary 3.1. $J(FG) = \mathrm{Anh}(s)$.

Proof. By Proposition 3.3, $\mathrm{Anh}(s) \subseteq J(FG)$ and we know from Lemma 2.1 part (3) that $J(FG) \subseteq \mathrm{Anh}(s)$, so the equality is hold:

$$J(FG) = \mathrm{Anh}(s).$$

□

We will need the following proposition in the next steps:

Proposition 3.4. $\mathrm{Dmn}(J(FG)) = \mathrm{Dmn}(\mathrm{Anh}(s)) = 2$.

Proof. By Proposition 3.1 and Corollary 3.1 we have

$$J(FG) = \mathrm{Anh}(s) = \{a_0^- \widehat{x}y^{-1} + a_0\widehat{x} + a_0^+\widehat{x}y \mid a_0^- + a_0 + a_0^+ = 0\}. \tag{3.7}$$

That means, $J(FG)$ and $\mathrm{Anh}(s)$ are generated by three elements, with one restriction. Hence,

$$\mathrm{Dmn}(J(FG)) = \mathrm{Dmn}(\mathrm{Anh}(s)) = 3 - 1 = 2.$$

□

Let $H := \langle x \rangle = \{x^{-6}, x^{-5}, x^{-4}, x^{-3}, x^{-2}, x^{-1}, 1, x, x^2, x^3, x^4, x^5, x^6\} \trianglelefteq G$, a normal subgroup of G. Also we recall augmentation ideals $\Delta(G,H) := \langle h-1 | \, h \in H\rangle$, that in special case $H = G$, we denote $\Delta(G) := \Delta(G,G)$. Now it is obvious that, by using [14, Proposition 3.3.3 on p. 135], we have

$$\mathrm{Dmn}(\Delta(G,H)) = |G| - [G:H] = 39 - 3 = 36$$
$$\mathrm{Dmn}(\Delta(G,G)) = |G| - [G:G] = 39 - 1 = 38.$$

Therefore we obtain the following remark:

Remark 3.1. Dimensions of $\Delta(G,H)$ and $\Delta(G)$ can be computed as follows:

$$\mathrm{Dmn}(\Delta(G,H)) = 36$$
$$\mathrm{Dmn}(\Delta(G,G)) = 38.$$

We want to represent a decomposition for $\Delta(G)$ over $J(FG)$ and $\Delta(G,H)$. As both of them are included in $\Delta(G)$, first we show that they are disjoint:

Proposition 3.5. $J(FG)\cap\Delta(G,H)=0$.

Proof. Let $\alpha\in J(FG)\cap\Delta(G,H)$. By (3.7), $J(FG)=\langle\widehat{G}\rangle$. Now we compute $\alpha.\widehat{x}$ in two different ways, in order to see α as an element of $J(FG)$ or $\Delta(G,H)$ separately:

$$\begin{array}{cc} \alpha\in J(FG)=\langle\widehat{G}\rangle & \alpha\in\Delta(G,H)=\langle x-1\rangle \\ \alpha=a.\widehat{G} & \alpha=\beta(x-1) \\ \alpha\widehat{x}=a\widehat{G}\widehat{x}=a\widehat{G}|\langle x\rangle| & \alpha\widehat{x}=\beta(x-1)\widehat{x}=\beta(x\widehat{x}-1\widehat{x}) \\ =a.\widehat{G}.n=a.\widehat{G}=\alpha & =\beta.(\widehat{x}-\widehat{x})=\beta.0=0 \end{array}$$

So we conclude that:

$$\alpha=\alpha.\widehat{x}=0. \tag{3.8}$$

And therefore we have

$$J(FG)\cap\Delta(G,H)=0.$$

□

Now the decomposition can be achieved:

Proposition 3.6. $\Delta(G)=J(FG)\oplus\Delta(G,H)$.

Proof. By Proposition 3.4 and Remark 3.1, we have

$$\mathrm{Dmn}(J(FG))+\mathrm{Dmn}(\Delta(G,H))=2+36=38=\mathrm{Dmn}(\Delta(G))$$

Now Proposition 3.5 together with above equality shows that:

$$\Delta(G)=J(FG)\oplus\Delta(G,H).$$

□

In the next Proposition, we prove that $\Delta(G,H)$ is a semisimple ring:

Proposition 3.7. $\Delta(G,H)$ *is a semisimple ring.*

Proof. By Proposition 3.6, we have $\Delta(G,H)=\Delta(G)/J(FG)\subseteq FG/J(FG)$. From [14, Theorem 6.6.1 on p. 214], the group ring of a field over a finite group is Artinian, so FG is an Artinian ring, and [14, Lemma 2.4.9 on p. 87], implies its quotient ring, $FG/J(FG)$, is an Artinian ring too. Also from [14, Lemma 2.7.5 on p. 107] we know that $J(FG/J(FG))=0$. Now by using [14, Theorem 2.7.16 on p. 111] we can conclude that $FG/J(FG)$ is semisimple, and by [14, Proposition 2.5.2 on p. 91], all of its subrings are semisimple too. So $\Delta(G,H)$ is semisimple. □

By the Artin-Wedderburn Theorem, semisimple ring $\Delta(G,H)$, decomposes to its simple components that are division rings of matrices over extensions of F. Now we need to know their numbers and dimensions. First we show that the center of $\Delta(G,H)$ is included in the center of FG:

Proposition 3.8. $Z(\Delta(G,H)) \subseteq Z(FG)$.

Proof. For the proof of this proposition, we need show that each element of $Z(FG)$ must commute with all of elements of FG. Since F is commutative and G is generated by x and y, so it suffices to show they commute with x and y. Let $\alpha \in Z(\Delta(G,H))$, so it commutes with $x-1$ as it is in $\Delta(G,H)$:

$$\begin{gathered}\alpha.(x-1) = (x-1).\alpha\\ \alpha.x - \alpha = x.\alpha - \alpha\\ \alpha.x = x.\alpha\end{gathered}$$

So α commutes with x. Now we show that α also commutes with y. First we show that $\alpha y - y\alpha$ is in $\mathrm{Anh}(x-1)$. Notice we know that $(x-1)y = y(x^{-1}-1) \in \Delta(G,H)$, so,

$$\begin{array}{cc}(x-1)y \in \Delta(G,H) & y(x-1) \in \Delta(G,H)\\ \alpha.(x-1).y = (x-1).y.\alpha & \alpha.y.(x-1) = y.(x-1).\alpha\\ (x-1).\alpha y = (x-1).y\alpha & \alpha y.(x-1) = y\alpha.(x-1)\\ (x-1)(\alpha y - y\alpha) = 0 & (\alpha y - y\alpha)(x-1) = 0\end{array}$$

So $(\alpha y - y\alpha) \in \mathrm{Anh}(x-1)$ and by [14, Lemma 3.4.3 on p. 139] we know that $\mathrm{Anh}(x-1) = \mathrm{Anh}(\Delta(G,H)) = FG\widehat{x}$. Now we compute $(\alpha y - y\alpha).\widehat{x}$ in two different ways, directly itself or consider $(\alpha y - y\alpha)$ as an element of $FG.\widehat{x}$ separately. Note that $\alpha \in Z(\Delta(G,H)) \subseteq \Delta(G,H)$, so by (3.8), $\alpha.\widehat{x} = 0$, and although x does not commute with y, but $\widehat{x}$ does, also $|\langle x\rangle| = Ord_G(x) = 7 = 1$. So we have

$$\begin{array}{cc}(\alpha y - y\alpha).\widehat{x} = \alpha.y.\widehat{x} - y.\alpha.\widehat{x} = & (\alpha y - y\alpha).\widehat{x} = \beta.\widehat{x}.\widehat{x} =\\ \alpha\widehat{x}.y - y.\alpha\widehat{x} = 0.y - y.0 = 0 & \beta.\widehat{x}.|\langle x\rangle| = \beta\widehat{x} = (\alpha y - y\alpha)\end{array}$$

Hence $\alpha y - y\alpha = (\alpha y - y\alpha).\widehat{x} = 0$. Thus $\alpha y = y\alpha$, which means α also commutes with y and therefore

$$Z(\Delta(G,H)) \subseteq Z(FG).$$

□

In the next proposition, we obtain the exact structure of $Z(\Delta(G,H))$:

Proposition 3.9. $Z(\Delta(G,H)) = \langle \widehat{\mathscr{C}_1}, \widehat{\mathscr{C}_2}, \widehat{\mathscr{C}_3}\rangle$.

Proof. Let $\alpha \in Z(\Delta(G,H))$, from [14, Theorem 3.6.2 on p. 151] we know that $Z(FG) = \langle \widehat{\mathscr{C}_{-3}}, \widehat{\mathscr{C}_{-2}}, \widehat{\mathscr{C}_{-1}}, \widehat{\mathscr{C}_0}, \widehat{\mathscr{C}_{+1}}, \widehat{\mathscr{C}_{+2}}, \widehat{\mathscr{C}_{+3}}\rangle$, so for center of augmentation ideal we have $Z(\Delta(G,H)) \subseteq \langle \widehat{\mathscr{C}_{-3}}, \widehat{\mathscr{C}_{-2}}, \widehat{\mathscr{C}_{-1}\mathscr{C}_0}, \widehat{\mathscr{C}_{+1}}, \widehat{\mathscr{C}_{+2}}, \widehat{\mathscr{C}_{+3}}\rangle$, by using Proposition 3.8. So

$\alpha = \sum_{i=-3}^{+3} r_i\widehat{\mathscr{C}_i} = r_{-3}\widehat{\mathscr{C}}_{-3} + r_{-2}\widehat{\mathscr{C}}_{-2} + r_{-1}\widehat{\mathscr{C}}_{-1} + r_0\widehat{\mathscr{C}_0} + r_{+1}\widehat{\mathscr{C}}_{+1} + r_{+2}\widehat{\mathscr{C}}_{+2} + r_{+3}\widehat{\mathscr{C}}_{+3}$. By (3.8), $\alpha.\widehat{x} = 0$ and notice that $x^i\widehat{x} = \widehat{x}$, so for $i \in \{-2,-1,+1,+2\}$ we have $\widehat{\mathscr{C}_i}\widehat{x} = 3\widehat{x} = 0$. Hence,

(3.9)

$$\begin{aligned} 0 = \alpha\widehat{x} = \sum_{i=-3}^{+3} r_i\widehat{\mathscr{C}_i}\widehat{x} &= r_{-3}\widehat{\mathscr{C}}_{-3}\widehat{x} + (\sum_{i=-2}^{-1} r_i\widehat{\mathscr{C}_i}\widehat{x}) + r_0\widehat{\mathscr{C}_0}\widehat{x} + (\sum_{i=+1}^{+2} r_i\widehat{\mathscr{C}_i}\widehat{x}) + r_{+3}\widehat{\mathscr{C}}_{+3}\widehat{x} \\ &= r_{-3}y^{-1}\widehat{x} + 0 + r_0.1.\widehat{x} + 0 + r_{+3}y\widehat{x} = r_{-3}\widehat{x}y^{-1} + r_0.1.\widehat{x} + r_{+3}\widehat{x}y \end{aligned}$$

Since the left hand side of (3.9) is zero, so the right hand side coefficients must be zero too, hence we have $r_{-3} = r_0 = r_{+3} = 0$, terefore we conclude that $\alpha = r_{-2}\widehat{\mathscr{C}}_{-2} + r_{-1}\widehat{\mathscr{C}}_{-1} + r_{+1}\widehat{\mathscr{C}}_{+1} + r_{+2}\widehat{\mathscr{C}}_{+2}$. As α was an arbitrary element in center of $\Delta(G,H)$, thus $Z(\Delta(G,H)) \subseteq \langle \widehat{\mathscr{C}}_{-2}, \widehat{\mathscr{C}}_{-1}, \widehat{\mathscr{C}}_{+1}, \widehat{\mathscr{C}}_{+2}\rangle$. Now it suffices to show that all of these types of elements are included in $\Delta(G,H)$. We must show that there is a β such that $\alpha = \beta(x-1)$. It is straightforward to find β's coefficients by solving a system of linear equations. So $\alpha \in \Delta(G,H)$, and therefore

$$Z(\Delta(G,H)) = \langle \widehat{\mathscr{C}}_{-2}, \widehat{\mathscr{C}}_{-1}, \widehat{\mathscr{C}}_{+1}, \widehat{\mathscr{C}}_{+2}\rangle.$$

□

Now the dimension of the center of $\Delta(G,H)$ can be computed:

Corollary 3.2. $\mathrm{Dmn}(Z(\Delta(G,H))) = 4$.

Proof. By Proposition 3.9,we know that $Z(\Delta(G,H)) = \langle \widehat{\mathscr{C}}_{-2}, \widehat{\mathscr{C}}_{-1}, \widehat{\mathscr{C}}_{+1}, \widehat{\mathscr{C}}_{+2}\rangle$. So,

$$\mathrm{Dmn}(Z(\Delta(G,H))) = 4.$$

□

Let $M_n(R)$ be the ring of the square matrices of order n on the ring R and let $GL_n(R)$ be its unit group. Also R^n be the direct sum of n copy of the ring R, which is $R^n = \oplus_{i=1}^n R$ and let F_n be the extension of the finite field F of the order n that is $[F_n : F] = n$. Now we are ready to prove Theorem 3.1:

Proof. [Proof of Theorem 3.1] Let $\alpha \in Z(\Delta(G,H))$. From Proposition 3.9, we know that α can be written as $\alpha = r_{-2}\widehat{\mathscr{C}}_{-2} + r_{-1}\widehat{\mathscr{C}}_{-1} + r_{+1}\widehat{\mathscr{C}}_{+1} + r_{+2}\widehat{\mathscr{C}}_{+2}$. Since $char(F) = 3$, we have

$$\begin{aligned} \alpha &= r_{-2}\widehat{\mathscr{C}}_{-2} + r_{-1}\widehat{\mathscr{C}}_{-1} + r_{+1}\widehat{\mathscr{C}}_{+1} + r_{+2}\widehat{\mathscr{C}}_{+2} \\ \alpha^3 &= r_{-2}^3\widehat{\mathscr{C}}_{-2}^3 + r_{-1}^3\widehat{\mathscr{C}}_{-1}^3 + r_{+1}^3\widehat{\mathscr{C}}_{+1}^3 + r_{+2}^3\widehat{\mathscr{C}}_{+2}^3 \\ \alpha^3 &= r_{-2}^3\widehat{\mathscr{C}}_{-2} + r_{-1}^3\widehat{\mathscr{C}}_{-1} + r_{+1}^3\widehat{\mathscr{C}}_{+1} + r_{+2}^3\widehat{\mathscr{C}}_{+2} \\ \alpha^{3n} &= r_{-2}^{3n}\widehat{\mathscr{C}}_{-2} + r_{-1}^{3n}\widehat{\mathscr{C}}_{-1} + r_{+1}^{3n}\widehat{\mathscr{C}}_{+1} + r_{+2}^{3n}\widehat{\mathscr{C}}_{+2} \\ \alpha^{3n} &= r_{-2}\widehat{\mathscr{C}}_{-2} + r_{-1}\widehat{\mathscr{C}}_{-1} + r_{+1}\widehat{\mathscr{C}}_{+1} + r_{+2}\widehat{\mathscr{C}}_{+2}. \end{aligned}$$

Since $|F| = 3^n$, we know $r_i^{3^n} = r_i$, so $\alpha^{3^n} = \alpha$. Therefore we have

$$\Delta(G,H) \cong M_3(F)^4.$$

By [14, Proposition 3.6.7 on p. 153], $FG \cong F(G/H) \oplus \Delta(G,H)$, therefore, $\mathscr{U}(FG) \cong \mathscr{U}(F(C_3)) \times \mathscr{U}(\Delta(G,H))$. So we have

$$\mathscr{U}(FG) = C_3^n \times C_{3^n-1} \times GL_3(F)^4.$$

□

Acknowledgment

The authors would like to thank the anonymous referee for careful reading and the helpful comments for improving this paper. Also we are grateful to prof. Ali Rajaei, for his helpful guidance.

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