Z. Angew. Math. Phys. (2019) 70:92

https://doi.org/10.1007/s00033-019-1136-9

Zeitschrift für angewandte Mathematik und Physik ZAMP

The Galerkin–Fourier method for the study of nonlocal parabolic equations

Fuensanta Andrés and Julio Muñoz

Abstract. The aim of this paper is the study of a type of nonlocal parabolic equation. The formulation includes a convolution kernel k in the diffusion term and a design function h that plays the role of the diffusion coefficient. The main goal is twofold: On the one hand, the existence and uniqueness of nonlocal solution are deduced. Also, a comprehensive and rigorous procedure, which is based on the classical Galerkin–Fourier Method, is performed. As in the classical setting, the appropriate choice of the Gelfand triplet will guarantee the differentiation and therefore the operational technique for the study of the parabolic equation. On the other hand, the convergence of the nonlocal solution as the kernel k converges to a Dirac Delta is studied. The series expansion of the nonlocal solution allows us, in an easy way, to show its convergence to the solution of the corresponding local parabolic equation.

Mathematics Subject Classification. 35J20, 49J22, 45A05, 46N20.

Keywords. Approximation of partial differential equations, Nonlocal parabolic equations, Optimal control, Integral equations.

1. Introduction

There is no doubt about the interest generated by nonlocal models. The modeling of diffusion through integral–differential equations has been studied in multiple works over the last two decades (see [20, 58], and references therein). These integral–differential formulations have appeared in the contexts of Probability (see [12,15] or [57]), and also in Analysis, where it has given rise to seminal results for the study and understanding of nonlocal problems (see for instance [6,18,19,45,46]). Consequently, nonlocal models have been used in many applied fields where the corresponding formulation includes the presence of long-term interactions. These models have turned out to be a fundamental tool in the field of Anomalous Diffusion, where different versions of the fractional Laplacian have been used ([10,14,43,44]). This kind of nonlocality appears also in Finance ([38]), Fluid Dynamics ([23]), Elasticity ([1,54,55]), Biology ([35,36]), Image Processing ([32,41]) or Nonconvex Variational Analysis ([13,48]).

From all these references about the nonlocal formulation, we must highlight the problems of evolution of nonlinear type. A prominent example is the fractional p-Laplacian. The goal of this paper is constrained to the linear case, $p = 2$. This manuscript analyzes the fractional Laplacian from the most basic elements and reproduces the framework of classical parabolic equations in a rigorous way ([37,39,56,59]). Although there are multiple perspectives and many technical procedures in connection with this topic, this work will focus on providing a thorough analysis of the Galerkin–Fourier method for nonlocal parabolic diffusion equations. The general procedure used to obtain nonlocal solutions is nothing else than a faithful version of the classical method for partial differential equations ([30,39]). From a practical point of view, the analysis performed gives rise to a meaningful operative methodology. The obtained results explicitly provide a powerful method to solve the problem because apart from giving a representation for the solutions, it facilitates the comparison between local and nonlocal solutions. Therefore, the practical sense of the approach provided by the article is clear. Besides, the proposed framework is advantageous because the nonlocal nature of the formulation is a feature that favors practical applications. This formulation is

Published online: 04 June 2019

given by means of an integral equation without gradients, and hence, it will be assumed that the involved spaces are not necessarily regular, or at least not so regular as in the classical setting. Consequently, under these circumstances, it is very likely that the set of numerical simulations could be large enough in order to faithfully reproduce the phenomenon we are analyzing.

Despite the fact that there is a lot of work on this type of nonlocal analysis, we have not found any reference dealing in detail with the fundamentals of the Galerkin–Fourier method. The nonlocal functional framework, the right setting for the formulation of the problem and the appropriate compactness results will be the main issues to be addressed in this work. This analysis will serve to attain an explicit representation of the solution and also, to deduce its connection with the classical local model of diffusion. More specifically, the nonlocal procedures will give rise, as a Γ-limit, the classical Galerkin–Fourier method and consequently, to the classical solution of the parabolic evolution equation.

As we have already stated, the literature on the subject is abundant. Without wishing to be exhaustive, we shall review some of the most important works. Concerning the nonlocal evolution equation we shall deal with, it is worth indicating that we analyze the nonlocal linear case with a kernel broadly used in diffusion models. The formulation of the state equation depends on the number δ called *horizon.* This parameter appears in the definition of the kernel, and it plays the role of an internal length scale in the modeling of long range interactions. The kernels we consider are radial functions converging to a Dirac measure, as the horizon tends to zero. These kernels contain a class of Riesz operators, they are singular, but no additional bounds or regularity conditions are assumed. Our hypotheses are not the same as those considered by other authors. For example, the kernel $\gamma(x, y) = \frac{\exp(-|x-y|)}{|x-y|}$ is a case that could be treated in our framework, and yet, it does not satisfy the conditions imposed in other works (see [24,52]).

We consider a smooth bounded domain so that the nonlocal operator is a restricted fractional Laplacian. See [17,42,53] and references therein. Also, a diffusion coefficient depending only on the spacial variable is included. This ingredient endows the problem with great generality since its Γ-limit gives rise to a fairly broad class of diffusion equations. In this sense, we refer [7,24,49,60] or [26]. The boundary value is of homogeneous Dirichlet type; it is a volume constraint because it acts on a neighborhood of the boundary.

A key issue in our analysis is the Spectral Theory. The previous spectral results obtained in the nonlocal context contribute decisively to the representation of the solution as a series. Besides, the asymptotic behavior of the spectra is essential to derive the local limit problem. In relation with this matter, we follow [2,3]. Some of the most interesting works whose context is similar to ours are [52] or [9]. References [11,21] are two interesting papers where the study of the fractional reaction–diffusion equation is approached from the perspective of Fourier's spectral methods. See also [22].

The existence and the asymptotic behavior of solutions for the restricted nonlocal fractional Laplacian have been already analyzed in several contexts. For the linear case, we must highlight the works [6,24, 60,61]. These articles consider nonlocal wave or diffusion equations in which a less singular kernel than ours is used, or some specific behavior of the admissible functions at the nonlocal boundary are assumed. In [31], the existence of solution for parabolic problems is analyzed. The proof is carried out by applying the Lax–Milgram Lemma instead a Galerkin–Fourier approach. In that paper, the authors use nonradial kernels and build the underlying Gelfand triplet. Nevertheless, compared to the present work, they assume other slightly different conditions on the kernel and on the diffusion coefficients, and the asymptotic analysis is not analyzed. [49] or [7] are cases whose models include a kernel similar to the one analyzed here in the sense that they incorporate a diffusion coefficient in the formulation. They also deal with reaction–diffusion equations, even if the reactive term is a nonlinear function depending on a measure.

Complementing the above-mentioned works and within the framework of the numerical analysis, we refer to [27–29,33,34,47,50,51].

As far as we know, there are not too many specific works concerning the study of optimal design governed by nonlocal diffusion equations . Some works, but for the nonlocal elliptic equation, are [3,4,8, 25,26]. See also [16] for the H-convergence study of the nonlocal elliptic operator. [5] is a work that deals with the existence and asymptotic analysis of optimal designs in the context of parabolic equations.

2. Formulation of the problem, tools and results

2.1. The problem

We shall analyze the nonlocal parabolic equation

$$(P_\delta) \doteq \begin{cases} u_t(x,t) + L_\delta(u(x,t)) = f(x,t), \text{ in } \Omega, \\ u(x,t) = 0 \text{ in } \Omega_\delta \setminus \Omega, t \in (0,T), \\ u(x,0) = g(x) \text{ in } \Omega, \end{cases} \tag{1}$$

where δ is a fixed, positive and small parameter, L_δ is an integral operator defined by

$$L_\delta(u(x,t)) = -2 \int\limits_{B(x,\delta)} \frac{k_\delta(|x'-x|)}{|x'-x|^2} H(x',x)(u(x',t) - u(x,t))\, \mathrm{d}x', \tag{2}$$

where

$$H(x',x) = \frac{h(x) + h(x')}{2},$$

$\Omega_\delta = \Omega \cup \{\cup_{y\in\partial\Omega} B(y,\delta)\}$ ($B(x,r)$ is the notation for an open ball centered at $x \in \mathbb{R}^N$ and radius $r > 0$) and $\delta \leq \delta_0$, where δ_0 is a given small number. In addition, the following conditions are assumed:

1. Ω is a smooth bounded domain in $\mathbb{R}^N$.
2. $g \in L^2(\Omega_\delta)$ and $f \in L^2\left([0,T]; L^2(\Omega)\right)$.
3. The design function h satisfies $h \in \mathcal{H}$, with

$$\mathcal{H} \doteq \{h : \Omega_\delta \to [h_{\min}, h_{\max}] \text{ a.e. } x \in \Omega,\ h = 0 \text{ in } \Omega_\delta \setminus \Omega\ \} \tag{3}$$

for given constants $0 < h_{\min} < h_{\max}$.

4. The kernel k_δ satisfies:
   (a) supp $k_\delta \subset B(0,\delta)$.
   (b) $k_\delta \geq 0$.
   (c) $(k_\delta)_\delta$, with $\delta > 0$, is a sequence of radial functions such that:

$$\frac{1}{N} \int\limits_{B(0,\delta)} k_\delta(|s|)\, \mathrm{d}s = 1. \tag{4}$$

   (d) The function $K_r : B(0,\delta) \setminus \{0\} \to]0,+\infty[$ defined by $K_r(z) = \frac{k_\delta(|z|)}{|z|}$, is integrable,

$$K_r \in L^1(B(0,\delta)). \tag{5}$$

   (e) The function $K_s : B(0,\delta) \setminus \{0\} \to]0,+\infty[$ defined by $K_s(z) = \frac{k_\delta(|z|)}{|z|^2}$ is singular near the origin in the sense that

$$\lim_{\theta\to 0^+} \int\limits_{B(0,\delta)-B(0,\theta)} K_s(z)\, \mathrm{d}z = +\infty. \tag{6}$$

In order to understand the statement of the problem, we consider the spaces involved. First of all, we define $L_0^2(\Omega_\delta)$ and X as

$$L_0^2(\Omega_\delta) \doteq \left\{v : \Omega_\delta \to \mathbb{R} : v \in L^2(\Omega;\mathbb{R}) \text{ and } v = 0 \text{ in } \Omega_\delta \setminus \Omega\right\},$$

and

$$X := \left\{ v \in L_0^2(\Omega_\delta) : B_h(v,v) < \infty \right\},$$

where

$$B_h(u,v) \doteq \int\limits_{\Omega_\delta}\int\limits_{\Omega_\delta} H(x',x) \frac{k_\delta(|x'-x|)}{|x'-x|^2} (u(x',t)-u(x,t))(v(x',t)-v(x,t))\, \mathrm{d}x'\mathrm{d}x. \tag{7}$$

We also define the following subspace of X,

$$X_0 = \overline{C_{co}^\infty(\Omega_\delta)},$$

where

$$C_{co}^\infty(\Omega_\delta) \doteq \{v : \Omega_\delta \to \mathbb{R} : v \in C_c^\infty(\Omega) \text{ and } v = 0 \text{ in } \Omega_\delta \setminus \Omega\} \subset X.$$

The closure is defined with respect to the norm $\|\cdot\|$ given in X through the quadratic form $B_h(\cdot,\cdot)$, that is:

$$X_0 = \left\{ v \in X : \text{there is } (v_j) \subset C_{co}^\infty(\Omega_\delta) \text{ such that } \lim_j B_h(v_j - v, v_j - v) = 0 \right\}.$$

We notice that, for each h and δ fixed, both X and X_0 are Hilbert spaces with the inner product $B_h(\cdot,\cdot)$. They depend on δ but not on h because the underlying norms are equivalent (see [2]).

Another important issue, for each δ, is the chain of embeddings

$$X_0 \subset L_0^2(\Omega_\delta) \subset X_0', \tag{8}$$

where X_0' denotes the dual of X_0. We shall be able to identify:

1. X_0 with a dense subspace of $L_0^2(\Omega_\delta)$ (as in the classical setting $H_0^1(\Omega) \subset L^2(\Omega)$ is dense with respect to the $L^2(\Omega)$-norm).
2. $L_0^2(\Omega_\delta)$ with itself, by means of its own inner product (as usual for a Hilbert space).
3. $L_0^2(\Omega_\delta)$ with a dense subspace of X_0', so that any function of $L_0^2(\Omega_\delta)$ acts on X_0 via the inner product of $L_0^2(\Omega_\delta)$.

In order to perform a rigorous formulation of the parabolic equation and to reproduce the classical procedures, the embeddings (8) must be dense and continuous (see Sect. 3). Recall that in such a case, the above three separable Hilbert spaces constitute what is called a *Gelfand triplet* (see [59] or [56]).

We finally define

$$Y_0 \doteq L^2([0,T]; X_0) = \left\{ u(\cdot,t) \in X_0 : \int\limits_0^T \|u\|_{X_0}^2 \,\mathrm{d}t < \infty \right\}, \tag{9}$$

which is also a Hilbert space with the inner product

$$(u,v)_{Y_0} = \int\limits_0^T (u(t), v(t))_{X_0} \,\mathrm{d}t.$$

Again, we notice that this space depends upon the parameter $\delta, Y_0 = Y_0(\delta)$.

The solution of the problem (P_δ) defined in (1) must be understood in a weak sense:

Definition 1. *([30])* It is said that $u \in Y_0$ is a weak solution of the problem (1), if

1. $u_t \in L^2([0,T]; X_0')$,
2. u satisfies the evolution equation of (P_δ), namely, for any $v \in X_0$ and a.e. $t \in [0,T]$,
$$\langle u_t, v\rangle + B_h(u,v) = (f(x,t), v) \tag{10}$$
3. $u(x,0) = g(x),\ x \in \Omega.$

In the above definition u_t must be understood as the weak derivative of u with respect to t. Also, in (10), $\langle\cdot,\cdot\rangle$ stands for the pairing of X_0' and X_0 since u_t belongs to X_0'.

Once the embeddings (8) are a Gelfand Triplet, we will be able to apply a classical result (see for instance [37,40]) ensuring that any solution of (1) can be redefined in an appropriate null set, in order to belong to the space $C\left([0,T];L_0^2(\Omega_\delta)\right)$. This fact gives sense to the initial condition used in Definition 1 (3).

2.2. Organization and results

The manuscript is organized as follows: Sect. 3 contains the proof of the Gelfand Triplet and its consequences in terms of differentiation (Theorems 6 and 7). Section 4 is devoted to obtaining the solution of the problem for a class of parabolic equations parametrized through the *horizon parameter* δ. For each parameter δ, we achieve existence and uniqueness of solution for the problem (1) (Theorem 8). The obtained solution is expressed as a series, which simplifies the asymptotic analysis when δ tends to zero. The underlying limit problem we derive is the classical local parabolic equation. This is analyzed in Sect. 5 (Theorem 9).

Since in Sect. 5 the asymptotic behavior of the sequence of solutions $(u_\delta)_\delta$ when $\delta\to 0$ is analyzed, some notes related to the limit problem are in order. The limit problem is the classical one, which is formulated as follows: Find $u\in W(0,T)$ such that

$$(P)=\begin{cases} w_t(x,t)-\operatorname{div}(h(x)\nabla w(x,t))=f(x,t)\,,\text{in }\Omega\\ w(x,t)=0\text{ in }\partial\Omega\\ w(x,0)=g(x)\end{cases}\tag{11}$$

where $W(0,T)$ is the Hilbert space described as

$$W(0,T)=\left\{v\in L^2\left([0,T];H_0^1(\Omega)\right):v_t\in L^2\left([0,T];H^{-1}(\Omega)\right)\right\}$$

and whose inner product is given by the formula

$$(u,v)=\int_0^T(\nabla_x u(t),\nabla_x v(t))_{L^2(\Omega)}\,\mathrm{d}t+\int_0^T\langle u_t(t),v_t(t)\rangle_{H^{-1}(\Omega)}\,\mathrm{d}t.$$

As in the nonlocal case, u_t must be understood in a weak sense. Since we are looking for a solution u such that $u_t\in L^2\left(0,T;H^{-1}(\Omega)\right)$, the action of u_t, as an element of $H^{-1}(\Omega)$, upon a function $v\in L^2\left(0,T;H_0^1(\Omega)\right)$, will be denoted by $\langle u_t,v\rangle$.

We recall that $u\in W(0,T)$ is a weak solution of (11) if

$$\langle u_t,v\rangle+\int_\Omega h(x)\nabla u\nabla v\mathrm{d}x=\int_\Omega fv\mathrm{d}x$$

for each $v\in H_0^1(\Omega)$ a.e. $t\in[0,T]$ and $u(x,0)=g(x)$ (the reader can look at [30,39,56] for the details). Here, the product $\langle\cdot,\cdot\rangle$ must be understood as follows: $\langle U,V\rangle_{H^{-1}(\Omega)}=\langle RU,RV\rangle_{H_0^1(\Omega)}$ where $R:H^{-1}(\Omega)\to H_0^1(\Omega)$ is a duality map derived from the Riesz Representation Theorem.

2.3. Preliminaries: the nonlocal steady case

We show the basic tool we shall employ to solve (1): the existence of a basis of eigenfunctions $\left\{w_\delta^{(k)}\right\}_k\subset X_0$, with eigenvalues $\gamma_k^{nl}(\delta)$ for the nonlocal operator B_{h_δ}, which can be expressed by writing

$$B_{h_\delta}\left(w_\delta^{(k)},v\right)=\gamma_k^{nl}(\delta)\left(w_\delta^{(k)},v\right)_{L^2(\Omega)\times L^2(\Omega)}$$

for any $v \in X_0$. The sequence of eigenvalues $\gamma_k^{nl}(\delta)$ is non-decreasing and $\gamma_k^{nl}(\delta) \to +\infty$ if $k \to +\infty$. Concerning the eigenfunctions, we know $\left\{ w_\delta^{(k)} \right\}_k$ is an orthonormal basis in $L_0^2(\Omega_\delta)$ and orthogonal in X_0. In practice, it is essential to determine the eigenfunctions. The basis they constitute and the associated eigenvalues are calculated by means of the following iterative process:

$$\gamma_k^{nl}(\delta) = B_h\left(w_\delta^{(k)}, w_\delta^{(k)}\right) = \min_{u \in X^{(k)}} B_h(v, v)$$

where

$$X^{(k)} \doteq \left\{ w \in X_0 : B_h\left(v, w_\delta^{(k)}\right) = 0, \ j = 1, 2, \ldots, k-1 \right\}$$

for $k \geq 2$ and $X^{(1)} = X_0$ (see [2,57,60,61]).

These facts provide the appropriate framework to solve the following nonlocal elliptic problem: fixed $\delta > 0$ and $G \in L^2(\Omega)$, there is a unique solution in X_0 of the stationary elliptic problem

$$L_\delta(v(x)) = G(x), \ v = 0 \text{ in } \Omega_\delta - \Omega. \tag{12}$$

That is, there exists $u_\delta \in X_0$ such that

$$B_h(u_\delta, w) = (G, w)_{L^2(\Omega) \times L^2(\Omega)} \text{ for any } w \in X_0.$$

More specifically, it has been proved that the only solution of this problem, u_δ, can be written as a series:

$$u_\delta(x) = \sum_k \frac{G_{\delta k}}{\gamma_k^{nl}} w_\delta^{(k)}(x) \tag{13}$$

where $(G_{\delta k})_k$ is the sequence of Fourier coefficients of the given function $G \in L^2(\Omega)$ with respect to the basis $\left\{ w_\delta^{(k)} \right\}_k$. The proof of this existence theorem is, basically, a consequence of the Nonlocal Poincaré inequality (see (14) below) and the compact embedding $X_0 \subset L_0^2(\Omega_\delta)$ (see [2] or [4]).

Remark 2. Even though these results are proved under the hypotheses (4)–(6), these assumptions on the kernels could be replaced by other classical inequalities without any consequence and therefore, without any change in the remain of the paper (see [2, p. 501 and Remark 3.2]).

Next result requires a little bit of precise notation: We know that for each $\delta > 0$, the operator B_h has a unitary basis of eigenfunctions. If $\gamma_k^{nl}(\delta)$ denotes the k-th eigenvalue, then the set $\left\{ w_{i\delta}^{(k)} \right\}_{i=1}^{n_k}$ is made of the corresponding eigenfunctions.

Theorem 3. ([2]) *There is subsequence of δ's for which the following limits hold:*

$$\lim_{\delta \to 0^+} \gamma_k^{nl}(\delta) = \gamma_k$$

and

$$\lim_{\delta \to 0^+} w_{i\delta}^{(k)} = w_i^{(k)} \text{ strongly in } L^2, \text{ for } i = 1, \ldots, n_k,$$

where $w_i^{(k)}, i = 1, \ldots, n_k$, are eigenfunctions of the $-\operatorname{div}(h\nabla\cdot)$ operator in $H_0^1(\Omega)$ and γ_k, are the corresponding eigenvalues. Moreover, $\lim_{\delta \to 0^+} w_{i\delta}^{(k)} = w_i^{(k)}$ strongly in X_0, in the sense that

$$\lim_{\delta \to 0} B_h\left(w_{i\delta}^{(k)} - w_i^{(k)}, w_{i\delta}^{(k)} - w_i^{(k)}\right) = 0.$$

The proof of this theorem is based on the results by Brezis, Ponce et al., and in particular, it makes use of this essential compactness statement:

Theorem 4. ([2,18,19,45]) *If $(\psi_\delta)_\delta$ is a bounded sequence in $L_0^2(\Omega_\delta)$ and there is a positive constant C such that $B_h(\psi_\delta, \psi_\delta) \leq C$, for any $\delta \leq \delta_0$, then $(\psi_\delta)_\delta$ is relatively compact in $L_0^2(\Omega_\delta)$. Moreover, there if $\left(\psi_{\delta_j}\right)_j$ is a subsequence such that $\psi_{\delta_j} \to \psi$ strong in $L^2(\Omega)$, if $j \to +\infty$, then $\psi \in H^1(\Omega)$.*

Thanks to the above results, a convergence result toward the local problem can be easily obtained:

Theorem 5. ([2]) *If $(u_\delta)_\delta$ is the sequence of solutions of the nonlocal elliptic problem (12), then:*

1. *There is a subsequence of it (still denoted by $(u_\delta)_\delta$) such that $u_\delta \to u$ strongly in L^2 if $\delta \to 0$ and $u \in H^1(\Omega)$.*
2. *The above convergence is also strongly in X_0, in the sense that $\lim_{\delta\to 0} B_h(u_\delta - u, u_\delta - u) = 0$.*
3. *The function u is the solution of the local elliptic problem $-\operatorname{div}(h\nabla u) = G$ and it is written like*

$$u(x) = \sum_k \frac{G_k}{\gamma_k} w^{(k)}(x)$$

where $(G_k)_k$ is the sequence of Fourier coefficients of the given function $G \in L^2(\Omega)$ with respect to the basis $\left\{w^{(k)}\right\}$.

Although the procedures may change slightly, the steps to follow in the parabolic case are very similar to the elliptic one.

3. The Gelfand triplet

The result we give is fundamental since it provides the rules of differentiation we need to look into the parabolic equations.

Theorem 6. *The chain of embeddings $X_0 \subset L_0^2(\Omega_\delta) \subset X_0'$ is a Gelfand triplet.*

Proof. We must prove the above embeddings are dense and continuous. On the basis of the definition of X_0, the denseness for the first embedding is obvious. The continuity is straightforwardly derived by using the nonlocal version of Poincaré inequality (see [2,6,45]): For any $v \in X_0$, there is a positive constant C such that

$$C\|v\|_{L_0^2}^2 \le B_h(v,v). \tag{14}$$

Concerning the second embedding, it is clear that any function $f \in L_0^2$ can be injected onto X_0'. This is due to the fact that any function f defines the linear functional T_f whose action on any $v \in X_0$ is

$$T_f(v) = (f,v)_{L^2(\Omega_\delta)\times L^2(\Omega_\delta)}.$$

To verify that T_f is continuous, we note

$$\begin{aligned}\|T_f\|_{X_0'} &= \sup_{\|w\|_{X_0}=1} \left|(f,w)_{L^2(\Omega_\delta)\times L^2(\Omega_\delta)}\right| \\ &\le \sup_{\|w\|_{X_0}=1} \|f\|_{L^2(\Omega_\delta)} \|w\|_{L^2(\Omega_\delta)}.\end{aligned}$$

If we use (14), we conclude that there is a positive constant $c > 0$ such that

$$\|T_f\|_{X_0'} \le \|f\|_{L^2(\Omega_\delta)} \sup_{\|w\|_{X_0}=1} c\|w\|_{X_0}.$$

and hence

$$\|T_f\|_{X_0'} \le c\|f\|_{L_0^2},$$

which amounts to state the continuity of the embedding $L_0^2 \subset X_0'$.

We prove now the denseness. By using the Riesz Theorem, we can identify each $F \in X_0'$ with one element $f \in X_0$, in such a way that the action of F on any $v \in X_0$ can be expressed by the formula

$$F(v) = T_f(v) = B_h(f,v).$$

Besides, we know that there exists a sequence $(f_n)_n \subset C_{co}^{\infty}(\Omega_\delta)$ such that $f_n \to f$ strongly in X_0. This means

$$\lim_{n\to\infty} \left(B_h(f_n, v) - B_h(f, v)\right) = 0$$

and this convergence is uniformly on the set $\left\{v \in X_0 : \|v\|_{X_0} = 1\right\}$.

We analyze the way of writing $B_h(f_n, v)$: since $B_h(f_n, v) < C$ and f_n is assumed to be smooth then

$$\begin{aligned}
B_h(f_n, v) &= \int_{\Omega_\delta}\int_{\Omega_\delta} k_\delta(|x'-x|)\, H \frac{(f_n(x') - f_n(x))(v(x') - v(x))}{|x'-x|^2} \mathrm{d}x' \mathrm{d}x \\
&= -2 \int_{\Omega_\delta} \left(\int_{B(x,\delta)} k_\delta(|x'-x|)\, H \frac{(f_n(x') - f_n(x))}{|x'-x|^2} \mathrm{d}x' \right) v(x)\, \mathrm{d}x \\
&= (L_\delta(f_n), v)_{L^2(\Omega_\delta)\times L^2(\Omega_\delta)},
\end{aligned}$$

where $L_\delta(f_n)$ is given by

$$L_\delta(f_n) = -2 \int_{B(x,\delta)} k_\delta(|x'-x|)\, H \frac{(f_n(x') - f_n(x))}{|x'-x|^2} \mathrm{d}x'.$$

The function $L_\delta(f_n)$ defined in this way belongs to $L^2(\Omega_\delta)$ for any n, and gives rise to the linear operator S_n on X_0 defined by:

$$S_n(v) = (L_\delta(f_n), v)_{L^2(\Omega_\delta)\times L^2(\Omega_\delta)}, \ v \in X_0.$$

To conclude, it is enough to check that

$$\lim_{n\to\infty} \|S_n - T_f\|_{X_0'} = 0.$$

But this is straightforward because

$$\begin{aligned}
\lim_{n\to\infty} \|S_n - T_f\|_{X_0'} &= \lim_{n\to\infty} \sup_{v\in X_0:\|v\|_{X_0}=1} |S_n(v) - T_f(v)| \\
&= \lim_{n\to\infty} \sup_{v\in X_0:\|v\|_{X_0}=1} |B_h(f_n, v) - B_h(f, v)| \\
&= 0,
\end{aligned}$$

where the last identity is true thanks to the uniform convergence of $B_h(f_n, \cdot)$ toward $B_h(f, \cdot)$. □

By using the above Gelfand triplet, we are in the position to reformulate a classical result on differentiation:

Theorem 7. ([37,56,59]) *Let $X_0 \subset L_0^2(\Omega_\delta) \subset X_0'$ be a Gelfand triplet. If $u \in L^2(0, T, X_0)$ and $u_t \in L^2(0, T, X_0')$ then $u \in C\left([0, T]; L_0^2(\Omega_\delta)\right)$. Moreover:*

1. *For any $v \in X_0$, the real-valued function $t \to (u, v)_{L_0^2(\Omega_\delta)}$ is weakly differentiable in $(0, T)$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}(u, v)_{L_0^2(\Omega_\delta)} = \langle u_t, v\rangle$$

2. *The real-valued function $t \to \|u(t)\|^2_{L^2(\Omega_\delta)}$ is absolutely continuous with*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u(t)\|^2_{L_0^2(\Omega_\delta)}\right) = \langle u_t(t), u(t)\rangle$$

for a.e. $t \in [0, T]$.

3. *There is a constant* $C = C(T)$ *such that*

$$\|u(t)\|_{L^\infty([0,T];L^2_0(\Omega_\delta))} \leq C\left(\|u(t)\|_{L^2([0,T];X_0)} + \|u_t\|_{L^2([0,T];X_0')}\right)$$

4. Existence and uniqueness of solutions

As we have commented in the introduction, the construction of solutions completely hinges on the classical methods of partial differential equations. In order to accomplish the existence in (1), we adapt and combine the separation variables and the Galerkin–Fourier methods. As in the local cases, we shall obtain an expression under the format of a series for the solution of the nonlocal problem.

Theorem 8. (Existence)*For each* $h \in \mathcal{H}$ *and each* $\delta > 0$ *fixed, there exists a unique solution* $u_\delta \in Y_0$ *of Eq.* (1) *in the sense of Definition* 1*. Moreover, if* $f(x,t) = \sum_{k=1}^{\infty} f_{\delta k}(t)\, w_\delta^{(k)}(x)$*, where* $(f_{\delta k})_k$ *is the sequence of Fourier coefficients given by* $f_{\delta k}(t) = \int_\Omega f(x,t)\, w_\delta^{(k)}(x)\, \mathrm{d}x$*, and* $(g_{\delta k})_k$ *is the sequence of Fourier coefficients of* g*, then*

$$u_\delta(x,t) = \sum_{k=1}^{\infty} d_\delta^{(k)}(t) w_\delta^{(k)}(x), \tag{15}$$

where $d_\delta^{(k)}$ *is the solution to the initial value problem*

$$\begin{cases} \left(d_\delta^{(k)}(t)\right)' + \gamma_k^{nl}(\delta)\, d_\delta^{(k)}(t) - f_{\delta k}(t) = 0, \\ d_\delta^{(k)}(0) = g_{\delta k}. \end{cases} \tag{16}$$

We split the proof into several steps:

4.1. Step 0: uniqueness

We first address the proof of the uniqueness. The procedure is standard and is entirely based on the Principle of Conservation of the Energy: If both f and g are identically zero, and z is a solution of (10), then the energy vanishes, that is

$$\langle z_t, z\rangle + B_h(z,z) = 0.$$

Since

$$\int_0^T \langle z_t, z\rangle\, \mathrm{d}t = \frac{1}{2}\|z\|^2_{L^2(\Omega)}(T),$$

then

$$\int_0^T B_h(z,z)\, \mathrm{d}t + \frac{1}{2}\|z\|^2_{L^2(\Omega)}(T) = 0$$

and thereby $z = 0$.

4.2. Step 1: Galerkin approximation

For each $\delta \in \mathbb{R}^+$, we shall seek u_δ, a solutions of Eq. (10). Firstly, we shall deal with a finite-dimensional version of (10). We set $u_\delta^{(k)}(x,t) = d_\delta^{(k)}(t) w_\delta^{(k)}(x)$, where $w_\delta^{(k)}(x)$ are the eigenfunctions with eigenvalues $\gamma_k^{nl}(\delta)$ (see Sect. 2.3) and $d_\delta^{(k)}(t)$ are functions that will be defined later. Clearly

$$B_h\left(u_\delta^{(k)}(x,t), v(x)\right) = \gamma_k^{nl}(\delta)\left(u_\delta^{(k)}(x,t), v(x)\right). \tag{17}$$

Also, if $g_{\delta k}$ are the Fourier coefficients of g, $g_\delta^M(x) = \sum_{k=1}^{M} g_{\delta k} w_\delta^{(k)}(x)$, and the functions $d_\delta^{(k)}$are assumed to satisfy the initial value problem (16), then the function

$$u_\delta^M(x,t) \doteq \sum_{k=1}^{M} d_\delta^{(k)}(t) w_\delta^{(k)}(x) \tag{18}$$

is a weak solution in Y_0 of the nonlocal equation

$$\begin{cases} u_t(x,t) + L_\delta(u(x,t)) = f(x,t) \\ u(x,t) = 0 \text{ in } \Omega_\delta - \Omega \\ u(x,0) = g_\delta^M(x) \end{cases} \tag{19}$$

onto the finite-dimensional subspace $\mathcal{L}\left\{\cup_{k=1}^{M} w_\delta^{(k)}\right\}$.

4.3. Step 2: convergence

By using (19), we have

$$\left(\left(u_\delta^M\right)_t, u_\delta^M\right)_{L^2(\Omega)\times L^2(\Omega)} + B_h\left(u_\delta^M, u_\delta^M\right) = \left(f, u_\delta^M\right)_{L^2(\Omega)\times L^2(\Omega)} \tag{20}$$

and therefore, thanks to Theorem 7 and the orthogonality of the functions $w_\delta^{(k)}$, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|u_\delta^M\right\|_{L^2(\Omega_\delta)}^2\right) + \sum_{k=1}^{M}\left(d_\delta^{(k)}\right)^2 \gamma_k^{nl} \leq \frac{1}{2}\|f\|_{L^2(\Omega)}^2 + \frac{1}{2}\left\|u_\delta^M\right\|_{L^2(\Omega_\delta)}^2. \tag{21}$$

Thus, Gronwall's inequality (see [30, Appendix B, p. 624] or [37, Th. 6.41, p. 208-209]) ensures the existence of a positive constant C such that

$$\left\|u_\delta^M\right\|_{L^2(\Omega_\delta)}^2(t) \leq e^{Ct}\left(\left\|u_\delta^M\right\|_{L^2(\Omega_\delta)}^2(0) + \int_0^t \|f\|_{L^2(\Omega)}^2\,\mathrm{d}t\right), \tag{22}$$

and hence

$$\max_{t\in[0,T]}\left\|u_\delta^M\right\|_{L^2(\Omega_\delta)}^2 \leq C\left(\|g\|_{L^2(\Omega)}^2 + \|f\|_{L^2([0,T];L^2(\Omega))}^2\right). \tag{23}$$

By performing integration in t in (21) and using (23), we derive

$$\int_0^T \left\|u_\delta^M\right\|_{X_0}^2 \mathrm{d}t \leq C\left(\|g\|_{L^2(\Omega)}^2 + \|f\|_{L^2([0,T];L^2(\Omega))}^2\right), \tag{24}$$

where a C is a positive constant independent of M. In particular $\int_0^T \left\|u_\delta^M\right\|_{L^2(\Omega_\delta)}^2 \mathrm{d}t \leq C$ for every M. The bounds we have just found allow us to state $u_\delta^M \rightharpoonup z$ both weakly in $L^2(\Omega_\delta)$ and weakly in Y_0 for any t if $M \to +\infty$.

In addition, we are going to prove convergence for the sequence $\left(\left(u_\delta^M\right)_t\right)_M$: We fix $v \in X_0$ such that $\|v\|_{X_0} = 1$ and we write $v = v_1 + v_2$, where $v_1 \in \mathcal{L}\left\{\cup_{j=1}^M w_\delta^{(j)}\right\}$ and $\left(v_2, w_\delta^{(k)}\right) = 0$, for $k = 1, \ldots, M$. By using (20), we get

$$\left(\left(u_\delta^M\right)_t, v_1\right)_{L^2(\Omega)\times L^2(\Omega)} = (f, v)_{L^2(\Omega)\times L^2(\Omega)} - B_h\left(u_\delta^M, v_1\right)$$

so that

$$\left|\left(\left(u_\delta^M\right)_t, v_1\right)_{L^2(\Omega)\times L^2(\Omega)}\right| \le C\left(\|f\|_{L^2(\Omega)} + \left\|u_\delta^M\right\|_{X_0}\right).$$

This estimation serves to define $\left(u_\delta^M\right)_t$ as a bounded operator in X_0', namely

$$\left\|\left(u_\delta^M\right)_t\right\|_{X_0'} \le C\left(\|f\|_{L^2(\Omega)} + \left\|u_\delta^M\right\|_{X_0}\right).$$

Thus, by (24) we have

$$\int_0^T \left\|\left(u_\delta^M\right)_t\right\|_{X_0'}^2 \mathrm{d}t \le C\left(\int_0^T \|f\|_{L^2(\Omega)}^2 \mathrm{d}t + \int_0^T \left\|u_\delta^M\right\|_{X_0}^2 \mathrm{d}t\right)$$
$$\le C\left(\|f\|_{L^2([0,T];L^2(\Omega))}^2 + \|g\|_{L^2(\Omega)}^2\right).$$

Then, $\left(u_\delta^M\right)_t$ is bounded in $L^2\left([0,T];X_0'\right)$, and for a subsequence, $\left(u_\delta^M\right)_t \rightharpoonup z'$ weakly in $L^2\left([0,T];X_0'\right)$. In particular we derive $z \in C\left([0,T];X_0\right)$.

4.4. Step 3: existence

Let us check that the function z is the solution to our problem. We fix $t \in (0,T)$ and $k \in \mathbb{N}$, and we take $M > k$. By multiplying in the equation

$$\left(\left(u_\delta^M\right)_t, w_\delta^{(k)}(x)\right) + B_h\left(u_\delta^M, w_\delta^{(k)}(x)\right) = \left(f, w_\delta^{(k)}(x)\right)_{L^2(\Omega)\times L^2(\Omega)}$$

by any function $\varphi(t) \in C^\infty([0,T])$, with $\varphi(T) = 0$, and denoting $\varphi_k(x,t) = \varphi(t)\, w_\delta^{(k)}(x)$, we perform integration by parts to get

$$\int_0^T \left(-u_\delta^M, (\varphi_k)_t\right) \mathrm{d}t + \int_0^T B_h\left(u_\delta^M, \varphi_k\right) \mathrm{d}t$$
$$= \int_0^T (f, \varphi_k)_{L^2(\Omega)\times L^2(\Omega)} \mathrm{d}t + \left(u_\delta^M(x,0), w_\delta^{(k)}(x)\right)\varphi(0).$$

If we pass to the limit when $M \to +\infty$, then

$$\int_0^T \left(-z, w_\delta^{(k)}\right)\varphi'(t)\,\mathrm{d}t + \int_0^T B_h\left(z, w_\delta^{(k)}(x)\right)\varphi(t)\,\mathrm{d}t$$
$$= \int_0^T \left(f, w_\delta^{(k)}\right)_{L^2(\Omega)\times L^2(\Omega)} \varphi(t)\,\mathrm{d}t + \left(g, w_\delta^{(k)}(x)\right)\varphi(0). \tag{25}$$

Since we can use any $\varphi \in C_c^\infty([0,T])$ then, in particular, the above identity implies

$$\left\langle z', w_\delta^{(k)}\right\rangle + B_h\left(z, w_\delta^{(k)}(x)\right) = \left(f, w_\delta^{(k)}\right)_{L^2(\Omega)\times L^2(\Omega)},$$

where the derivative must be understood in the sense of the distributions.

Since the set of functions $w_\delta^{(k)}$ is dense in X_0, then the formula from the above can be generalized, namely:

$$\langle z', v\rangle + B_h(z,v) = (f,v)_{L^2(\Omega)\times L^2(\Omega)} \tag{26}$$

for any $v \in X_0$ and for a.e. $t \in (0,T)$.
In order to check $z(0) = g$, we integrate by parts in (25) to obtain

$$\left(z(0), w_\delta^{(k)}\right)\varphi(0) = \left(g, w_\delta^{(k)}(x)\right)\varphi(0)$$

for any $w_\delta^{(k)}$ and any φ. Therefore, $z(x,0) = g(x)$.

4.5. Step 4: strong convergence

Throughout the remain of the paper, we shall denote by u_δ the solution of the problem (1). To verify that u_δ can be written as a series, we shall prove the sequence $\left(u_\delta^M\right)_M$ converges strongly in Y_0 to the function

$$z = u_\delta(x,t) = \sum_{k=1}^{\infty} d_\delta^{(k)}(t) w_\delta^{(k)}(x),$$

where $d_\delta^{(k)}$ is the solution of (16), that is

$$d_\delta^{(k)}(t) = \int_0^t f_{\delta k}(s)\exp\left(\gamma_k^{nl}(s-t)\right)ds + g_\delta^{(k)}\exp\left(-\gamma_k^{nl}t\right),\ d_\delta^{(k)}(0) = g_{\delta k}.$$

In other words, we shall see

$$\lim_{M\to\infty}\int_0^T B_h\left(u_\delta^M - z, u_\delta^M - z\right)\mathrm{d}t = 0. \tag{27}$$

We note this convergence serves to state that, for each t, the sequence $\left(u_\delta^M\right)_M$ strongly converges to z in the norm X_0 (and consequently strongly in L^2), that is

$$\lim_{M\to\infty} B_h\left(u_\delta^M - z, u_\delta^M - z\right) = 0. \tag{28}$$

Indeed, if we assume $\lim_{M\to\infty} B_h\left(u_\delta^M - z, u_\delta^M - z\right) > 0$ in a set of $t's$ of positive measure, for instance, for all $t \in S \subset [0,T]$, then, thanks to the Fatou's Lemma

$$0 < \int_S \lim_{M\to\infty} B_h\left(u_\delta^M - z, u_\delta^M - z\right)\mathrm{d}t \le \lim_{M\to\infty}\int_0^T B_h\left(u_\delta^M - z, u_\delta^M - z\right)\mathrm{d}t = 0.$$

Whence we get a contradiction and thereby we prove (28).

Now, it is enough to follow the lines given in [39] to prove (27): by using the variational equality

$$\left(\left(u_\delta^M\right)_t, u_\delta^M\right) + B_h\left(u_\delta^M, u_\delta^M\right) = \left(f, u_\delta^M\right),$$

we derive the following identity:

$$\begin{aligned} I &\doteq \int_0^T B_h\left(u_\delta^M - z, u_\delta^M - z\right) \mathrm{d}t + \frac{1}{2}\left\|u_\delta^M(T) - z(T)\right\|^2_{L_0^2(\Omega_\delta)} \\ &= \int_0^T \left(f, u_\delta^M\right) \mathrm{d}t - \frac{1}{2}\left\|u_\delta^M(T)\right\|^2_{L_0^2(\Omega_\delta)} + \frac{1}{2}\left\|u_\delta^M(0)\right\|^2_{L_0^2(\Omega_\delta)} \\ &\quad + \frac{1}{2}\left\|u_\delta^M(T)\right\|^2_{L_0^2(\Omega_\delta)} + \frac{1}{2}\left\|z(T)\right\|^2_{L_0^2(\Omega_\delta)} - \left(u_\delta^M(T), z(T)\right) \\ &\quad - \int_0^T B_h\left(u_\delta^M, z\right) \mathrm{d}t - \int_0^T B_h\left(z, u_\delta^M - z\right) \mathrm{d}t. \end{aligned}$$

If we take into account the limits

$$\int_0^T \left(f, u_\delta^M\right) \mathrm{d}t \to \int_0^T (f, z)\, \mathrm{d}t, \quad \int_0^T B_h\left(u_\delta^M, z\right) \mathrm{d}t \to \int_0^T B_h(z, z)\, \mathrm{d}t,$$
$$\left(u_\delta^M(T), z(T)\right) \to \|z(T)\|^2_{L_0^2(\Omega_\delta)}, \quad \left\|u_\delta^M(0)\right\|^2_{L_0^2(\Omega_\delta)} \to \|z(0)\|^2_{L_0^2(\Omega_\delta)}, \text{ if } M \to +\infty,$$

(where the last limit is true since u_δ^M is the solution of (19) and g_δ^M converges to g strongly in L^2), then we deduce

$$\begin{aligned} \lim_{M\to\infty} I &= \int_0^T (f, z)\, \mathrm{d}t + \frac{1}{2}\|z(T)\|^2_{L_0^2(\Omega_\delta)} \\ &\quad + \frac{1}{2}\|z(0)\|^2_{L_0^2(\Omega_\delta)} - \|z(T)\|^2_{L_0^2(\Omega_\delta)} - \int_0^T B_h(z, z)\, \mathrm{d}t \\ &= 0, \end{aligned}$$

where the last equality is due to the fact that z is the solution of the nonlocal problem.

5. Convergence to the local problem

The aim of this section is to prove

Theorem 9. (Convergence to the local problem) *If $(u_\delta)_\delta$ is the sequence of solutions of the nonlocal parabolic problem (1), then:*

1. *There is a subsequence of it (still denoted by $(u_\delta)_\delta$) such that $u_\delta \to u$ strongly in L^2 if $\delta \to 0$ and $u \in H_0^1(\Omega)$, for any $t \in [0, T]$.*
2. *$u \in W(0, T)$ and it is the weak solution of the local problem (11).*
3. *The above convergence is also strongly in Y_0, in the sense that*

$$\lim_{\delta\to 0} \int_0^T B_h(u_\delta - u, u_\delta - u) = 0. \tag{29}$$

In particular, u_δ strongly converges to u in X_0, in the sense that

$$\lim_{\delta\to 0} B_h(u_\delta - u, u_\delta - u) = 0, \;\; a.e.\ t \in [0, T]. \tag{30}$$

The proof consists of three steps.

5.1. Step 1: classical estimates and strong convergence in $L^2(\Omega)$

Under the present framework, we have $V = X_0, H = L_0^2(\Omega_\delta)$ and the underlying X_0'. The classical procedure of calculus in abstract spaces (see Theorem 7) enables us to state that the function $t \to \|u_\delta(t)\|^2_{L^2(\Omega_\delta)}$ is absolutely continuous and

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_\delta(t)\|^2_{L^2(\Omega_\delta)}\right) = \langle (u_\delta)_t(t), u_\delta(t)\rangle \text{ and a.e. } t \in [0,T].$$

Therefore, by repeating the arguments from Sect. 4.3, we can write

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_\delta\|^2_{L^2(\Omega_\delta)}\right) + B_h(u_\delta,u_\delta) \le \frac{1}{2}\|f\|^2_{L^2(\Omega_\delta)} + \frac{1}{2}\|u_\delta\|^2_{L^2(\Omega)}, \tag{31}$$

which, by using Gronwall's inequality, yields

$$\|u_\delta\|^2_{L^2(\Omega_\delta)} \le e^{Ct}\left[\|u_\delta\|^2_{L^2(\Omega_\delta)}(0) + \int_0^t \|f\|^2_{L^2(\Omega)}\,\mathrm{d}t\right].$$

Since $\|u_\delta\|^2_{L^2(\Omega_\delta)}(0) = \|g\|^2_{L^2(\Omega)}$, we can write

$$\max_{t\in[0,T]} \|u_\delta\|^2_{L^2(\Omega_\delta)} \le C\left(\|g\|^2_{L^2(\Omega)} + \|f\|^2_{L^2([0,T];L^2(\Omega))}\right) \tag{32}$$

uniformly in δ. Combining Theorem 8 with (31) and (32), we ensure there is a positive constant C such that

$$\|u_\delta\|^2_{Y_0} = \int_0^T B_h(u_\delta,u_\delta)\mathrm{d}t = \int_0^T \sum_{k=1}^\infty \left(d_\delta^{(k)}\right)^2 \gamma_k^{nl}\mathrm{d}t \le C \tag{33}$$

for every δ. From the above estimations, we deduce the existence of a function $u \in L^2(\Omega)$ such that for a subsequence of $\delta' s$, $u_\delta \rightharpoonup u$ weakly in $L^2(\Omega)$. Moreover, by Theorem 8 we know $u_\delta(x,t) = \sum_{k=1}^\infty d_\delta^{(k)}(t) w_\delta^{(k)}(x)$ where

$$d_\delta^{(k)}(t) = \int_0^t f_\delta^{(k)}(s)\exp\left(\gamma_k^{nl}(s-t)\right)ds + g_\delta^{(k)}\exp\left(-\gamma_k^{nl}t\right).$$

By using Hölder and other basic inequalities, we deduce this chain of inequalities:

$$\begin{aligned} &B_h(u_\delta,u_\delta)\\ &\le \sum_{k=1}^\infty \left\{2\int_0^t \left(f_\delta^{(k)}(s)\right)^2 \mathrm{d}s \int_0^t \gamma_k^{nl}\exp\left(2\gamma_k^{nl}(s-t)\right)\mathrm{d}s + 2\left(g_\delta^{(k)}\right)^2 \gamma_k^{nl}\exp\left(-2\gamma_k^{nl}t\right)\right\}\\ &\le \sum_{k=1}^\infty \int_0^t \left(f_\delta^{(k)}(s)\right)^2 \mathrm{d}s + 2\sum_{k=1}^\infty \left(g_\delta^{(k)}\right)^2 \gamma_k^{nl}\exp\left(-2\gamma_k^{nl}t\right)\\ &\le \sum_{k=1}^\infty \int_0^T \left(f_\delta^{(k)}(s)\right)^2 \mathrm{d}s + \frac{1}{t}\sum_{k=1}^\infty \left(g_\delta^{(k)}\right)^2 \end{aligned}$$

Consequently, for each $t \in (0,T]$, there is a constant $C = C(t)$ (not depending on δ) such that for any δ

$$B_h(u_\delta,u_\delta) = \sum_{k=1}^\infty \left(d_\delta^{(k)}\right)^2 \gamma_k^{nl} \le C.$$

Then, thanks to Theorem 4 we ensure that the sequence $(u_\delta)_\delta$ or a subsequence of it strongly converges in $L^2(\Omega)$ to a function $u \in H^1(\Omega)$ if $\delta \to 0$. It remains to prove that $u \in H_0^1(\Omega)$: By extending the functions u_δ by zero to an arbitrary domain O containing Ω, we deduce likewise the sequence of extensions $(\overline{u}_\delta)_\delta$ converges in $L^2(O)$, to the extension by zero of u, $\overline{u}$, and $\overline{u} \in H^1(O)$. The arbitrariness of the set O implies that $\overline{u} \in H^1(\mathbb{R}^N)$, and since $u \in H^1(\Omega)$, then $u \in H_0^1(\Omega)$.

5.2. Step 2: the convergence of nonlocal Galerkin–Fourier expansion toward the solution

What we prove in this step is the convergence of nonlocal Galerkin–Fourier expansion toward the local one.

We recall again that the sequence of solutions of (1) admits the specific writing:

$$\begin{aligned} u_\delta(x,t) &= \sum_{k=1}^{\infty} d_\delta^{(k)}(t) w_\delta^{(k)}(x) \\ &= \sum_{k=1}^{\infty} \left(\int_0^t f_\delta^{(k)}(s) \exp\left(\gamma_k^{nl}(s-t)\right) \mathrm{d}s + g_\delta^{(k)} \exp\left(-\gamma_k^{nl} t\right) \right) w_\delta^{(k)}(x). \end{aligned}$$

We know from Theorem 3 that

$$\begin{aligned} \lim_{\delta\to 0} \gamma_k^{nl}(\delta) &= \gamma_k, \\ \lim_{\delta\to 0} w_\delta^{(k)}(x) &= w^{(k)}(x) \text{ strongly in } L^2(\Omega), \end{aligned}$$

where γ_k is the corresponding eigenvalue of the eigenfunction $w^{(k)}(x)$ for the local operator b_h. We recall that b_h is defined by means of the formula

$$b_h(\psi,\varphi) = \int_\Omega h(x) \left(\nabla\psi(x) . \nabla\varphi(x)\right) \mathrm{d}x, \text{ for any } (\psi,\varphi) \in H^1(\Omega) \times H^1(\Omega).$$

Then, as result, we deduce

$$\begin{aligned} \lim_{\delta\to 0} f_\delta^{(k)}(t) &= \lim_{\delta\to 0} (f, w_\delta^{(k)}) = (f, w^{(k)}) = f_k, \\ \lim_{\delta\to 0} g_\delta^{(k)} &= \lim_{\delta\to 0} \left(g, w_\delta^{(k)}\right) = (g, w^{(k)}) = g_k. \end{aligned}$$

Due to the weak convergence $u_\delta \rightharpoonup u$ in L^2 and the strong convergence $w_\delta^{(k)} \to w^{(k)}$ (see Theorem 4), we also deduce

$$\lim_{\delta\to 0} (u_\delta)_k = \lim_{\delta\to 0} \left(u_\delta, w_\delta^{(k)}(x)\right) = \left(u, w^{(k)}\right) = u_k,$$

which is the same to write

$$\begin{aligned} \lim_{\delta\to 0} (u_\delta)_k = \lim_{\delta\to 0} d_\delta^{(k)}(t) &= \lim_{\delta\to 0} \int_0^t f_\delta^{(k)}(s) \exp\left(\gamma_k^{nl}(s-t)\right) \mathrm{d}s + g_\delta^{(k)} \exp\left(-\gamma_k^{nl} t\right) \\ &= \int_0^t f_k(s) \exp\left(\gamma_k(s-t)\right) \mathrm{d}s + g_k \exp\left(-\gamma_k t\right). \end{aligned}$$

Since the function u is the weak limit in L^2 of u_δ, u can expressed as a series in the Fourier basis $\left\{w^{(k)}(x)\right\}_k$ and therefore, necessarily,

$$u(x,t)=\sum_{k=1}^{\infty} d_k(t) w^{(k)}(x) \tag{34}$$

where

$$d_k(t)=\int_0^t f_k(s)\exp\left(\gamma_k(s-t)\right)\mathrm{d}s+g_k\exp\left(-\gamma_k t\right) \tag{35}$$

Now, it is automatic to be convinced that this limit u, defined by (34)–(35), is the only solution of the local problem (11). It would remain to check that this function u belongs to the space $W(0,T)$, but this is precisely what we know from the classical theory ([39]).

5.3. Step 3: proof of the strong convergence

We prove strong convergence in Y_0, in the sense given at (29). The strategy to follow is as in the proof of Theorem 8. From the equality

$$\langle (u_\delta)_t, u_\delta\rangle + B_h(u_\delta,u_\delta)=(f,u_\delta)$$

we perform integration to get

$$\begin{aligned}\int_0^T B_h(u_\delta,u_\delta)\,\mathrm{d}t &= \int_0^T (f,u_\delta)\,\mathrm{d}t-\int_0^T \langle (u_\delta)_t,u_\delta\rangle\,\mathrm{d}t\\ &=\int_0^T (f,u_\delta)\,\mathrm{d}t-\frac{1}{2}\int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\|u_\delta\|^2_{L^2(\Omega_\delta)}\right)\mathrm{d}t\\ &=\int_0^T (f,u_\delta)\,\mathrm{d}t-\frac{1}{2}\|u_\delta(T)\|^2_{L_0^2(\Omega_\delta)}+\frac{1}{2}\|u_\delta(0)\|^2_{L_0^2(\Omega_\delta)}.\end{aligned}$$

If we let $\delta\to 0$, the strong convergence of $(u_\delta)_\delta$ to u in $L_0^2(\Omega_\delta)$ ensures the limits

$$\lim_{\delta\to 0}\int_0^T (f,u_\delta)\,\mathrm{d}t=\int_0^T (f,u)\,\mathrm{d}t.$$

and

$$\lim_{\delta\to 0}\|u_\delta(T)\|^2_{L_0^2(\Omega_\delta)}=\|u(T)\|^2_{L_0^2(\Omega_\delta)}.$$

Besides, it is obvious that

$$\lim_{\delta\to 0}\|u_\delta(0)\|^2_{L_0^2(\Omega_\delta)}=\|u(0)\|^2_{L_0^2(\Omega_\delta)}=\|g\|^2_{L_0^2(\Omega_\delta)}.$$

Thus, all the previous convergences, together with the fact that u is the solution of the local problem, lead us to

$$\begin{aligned}
\lim_{\delta\to 0}\int_0^T B_h\left(u_\delta,u_\delta\right)\mathrm{d}t &= \lim_{\delta\to 0}\int_0^T \left(f,u_\delta\right)\mathrm{d}t - \lim_{\delta\to 0}\int_0^T \left\langle \left(u_\delta\right)_t, u_\delta\right\rangle \mathrm{d}t \\
&= \int_0^T (f,u)\,\mathrm{d}t - \lim_{\delta\to 0}\left(\frac{1}{2}\left\|u_\delta\left(T\right)\right\|^2_{L^2_0(\Omega_\delta)} - \frac{1}{2}\left\|u_\delta\left(0\right)\right\|^2_{L^2_0(\Omega_\delta)}\right) \\
&= \int_0^T (f,u)\,\mathrm{d}t - \left(\frac{1}{2}\left\|u\left(T\right)\right\|^2_{L^2_0(\Omega_\delta)} - \frac{1}{2}\left\|u\left(0\right)\right\|^2_{L^2_0(\Omega_\delta)}\right) \\
&= \int_0^T (f,u)\,\mathrm{d}t - \int_0^T \langle u,u_t\rangle\,\mathrm{d}t \\
&= \int_0^T b_h(u,u)\mathrm{d}t.
\end{aligned}$$

Now, if we consider the identity $\left\langle \left(u_\delta\right)_t, u\right\rangle + B_h\left(u_\delta,u\right) = (f,u)$ and we carry out the same procedure, then we arrive at

$$\begin{aligned}
\lim_{\delta\to 0}\int_0^T B_h\left(u_\delta,u\right)\mathrm{d}t &= \int_0^T (f,u)\,\mathrm{d}t - \frac{1}{2}\left(\left\|u\left(T\right)\right\|^2_{L^2_0(\Omega_\delta)} - \left\|g\right\|^2_{L^2_0(\Omega_\delta)}\right) \\
&= \int_0^T (f,u)\,\mathrm{d}t - \int_0^T \langle u,u_t\rangle\,\mathrm{d}t = \int_0^T b_h(u,u)\mathrm{d}t.
\end{aligned}$$

By using the above results, we obtain

$$\begin{aligned}
\lim_{\delta\to 0}\int_0^T B_h\left(u_\delta-u,u_\delta-u\right)\mathrm{d}t &= \lim_{\delta\to 0}\int_0^T B_h\left(u_\delta,u_\delta\right)\mathrm{d}t - 2\lim_{\delta\to 0}\int_0^T B_h\left(u_\delta,u\right)\mathrm{d}t \\
&\quad + \lim_{\delta\to 0}\int_0^T B_h\left(u,u\right)\mathrm{d}t \\
&= -\int_0^T b_h(u,u)\mathrm{d}t + \lim_{\delta\to 0}\int_0^T B_h\left(u,u\right)\mathrm{d}t.
\end{aligned}$$

From Corollary 1 of [18], we derive the existence of a positive constant C such that

$$B_h\left(u,u\right) \le C\left\|\nabla u\right\|^2_{L^2}$$

for any δ, and

$$\lim_{\delta\to 0} B_h\left(u,u\right) = b_h(u,u)$$

(see also [3]). These asserts joined with the Dominated Convergence Theorem guarantee the limit

$$\lim_{\delta\to 0}\int_0^T B_h\left(u,u\right)\mathrm{d}t = \int_0^T b_h(u,u)\mathrm{d}t,$$

and therefore, the limit (29) holds. As we have seen in Sect. 4, the limit (30) is a straightforward consequence of (29).

Acknowledgements

This work was supported by Spanish Project MTM2017-87912-P. The authors want to acknowledge the referees for their criticism and valuable suggestions.

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References

[1] Alali, B., Lipton, R.: Multiscale dynamics of heterogeneous media in the peridynamic formulation. J. Elast. **106**, 71–103 (2012)

[2] Andrés, F., Muñoz, J.: A type of nonlocal elliptic problem: existence and approximation through a Galerkin–Fourier Method. SIAM J. Math. Anal. **47**(1), 498–525 (2015)

[3] Andrés, F., Muñoz, J.: Nonlocal optimal design: a new perspective about the approximation of solutions in optimal design. J. Math. Anal. Appl. **429**, 288–310 (2015)

[4] Andrés, F., Muñoz, J.: On the convergence of a class of nonlocal elliptic equations and related optimal design problems. J. Optim. Theory Appl. **172**, 33–55 (2017). https://doi.org/10.1007/s10957-016-1021-z

[5] Andrés, F., Muñoz, J., Rosado, J.: Existence and approximation of nonlocal design problems driven by parabolic equations. Math. Methods Appl. Sci. (2019)

[6] Andreu-Vaillo, F., Mazón, J.M., Rossi, J.D., Toledo-Melero, J.J.: Nonlocal Diffusion Problems. Mathematical Surveys and Monographs, vol. 165. American Mathematical Society, Providence (2010)

[7] Andreu, F., Rossi, J.D., Toledo-Melero, J.J.: Local and nonlocal weighted p-Laplacian evolution equations with Neumann boundary conditions. Publ. Mat. **55**, 27–66 (2011)

[8] Antil, H., Otárola, E., Salgado, A.J.: Optimization with respect to order in a fractional diffusion model: analysis, approximation and algorithmic aspects. J. Sci. Comput. **77**, 204–224 (2018). https://doi.org/10.1007/s10915-018-0703-0

[9] Bai, C.: Existence results for non-local operators of elliptic type. Nonlinear Anal. **83**, 82–90 (2013)

[10] Bakunin, O.: Turbulence and Diffusion: Scaling Versus Equations. Springer, Berlin (2008)

[11] Barrios, B.: Nonlocal Problems in Partial Differential Equations. Memoria Tesis Doctoral. Facultad Ciencias. Departamento de Matemáticas. Universidad Autónoma de Madrid (2013)

[12] Bass, R.F., Kassmann, M., Kumagai, T.: Symmetric jump processes: localization, heat kernels and convergence. Ann. Inst. Henri Poincaré -Porbabilités et Statistiques **46**, 59–71 (2010)

[13] Bazant, Z.P., Jirasek, M.: Nonlocal integral formulation of plasticity and damage: survey of progress. J. Eng. Mech. **128**, 1119–1149 (2002)

[14] Benson, D.A., Wheatcraft, S., Meerchaert, M.M.: Application of a fractional advection-dispersion equation. Water Resour. Res. **36**, 1403–1412 (2000)

[15] Bogdan, K., Burdzy, K., Cehn, Z.Q.: Censored stable processes. Probab. Theory Relat. Fields **127**, 89–152 (2003)

[16] Bonder, J.F., Ritorto, A., Martín, A.: H-convergence result for nonlocal Elliptic-type problems via Tartar's method. SIAM J. Math. Anal. **49**(4), 2387–2408 (2017)

[17] Bonforte, M., Sire, Y., Vázquez, J.L.: Existence, Uniqueness and Asymptotic behavior for fractional porous medium equations on bounded domains. Am. Inst. Math. Sci. **35**(12), 5725–5767 (2015)

[18] Bourgain, J., Brezis, H., Mironescu, P.: Another look at Sobolev spaces. In: Menaldi, J.L., et al. (eds.) Optimal Control and Partial Differential Equations, pp. 439–455. IOS Press, Amsterdam (2001). (A volume in honour of A. Benssoussan's 60th birthday)

[19] Brezis, H.: How to recognize constant functions. Connections with Sobolev spaces. Uspekhi Mat. Nauk. **57**(4(346)), 59–74 (2002)

[20] Bucur, C., Valdinoci, E.: Nonlocal Diffusion and Applications. Lecture Notes of the Unione Matematica Italiana, vol. 20. Springer, Berlin (2016). (Unione Matematica Italiana, Bologna)

[21] Bueno-Orobio, A., Kay, D., Burrage, K.: Fourier spectral methods for fractional-in-space reaction-diffusion equations. BIT Numer. Math. **54**, 937–954 (2014). https://doi.org/10.1007/s10543-014-0484-2

[22] Chen, Z.Q., Song, R.: Two-sided estimates for subordinate processes in domains. J. Funct. Anal. **226**(1), 90–113 (2005)

[23] Constantin, P.: Euler Equations, Navier–Stokes Equations and Turbulence, In Mathematical foundation of Turbulent Viscous Flows. Lecture Notes in Mathematics, pp. 1–43. Springer, Berlin (2006)
[24] Du, Q., Gunzburger, M.D., Lehoucq, R.B., Zhou, K.: Analysis and approximation of nonlocal diffusion problems with volume constraints. SIAM Rev. **54**(4), 667–696 (2012)
[25] D'Elia, M., Gunzburger, M.: Optimal distributed control of nonlocal steady diffusion problems. SIAM. J. Control Optim. **52**(1), 243–273 (2014)
[26] D'Elia, M., Gunzburger, M.: Identification of the diffusion parameter in nonlocal steady diffusion problems. Appl. Math. Optim. **73**(2), 227–249 (2016)
[27] D'Elia, M., Du, Q., Gunzburger, M.: Recent progress in mathematical and computational aspects of peridynamics. In: Voyiadjis, G. (ed.) Handbook of Nonlocal Continuum Mechanics for Materials and Structures. Springer, Cham (2018)
[28] D'Elia, M., Perego, M., Bochev, P., Littlewood, D.: A Coupling Strategy for Nonlocal and Local di Usion Models with Mixed Volume Constraints and Boundary Conditions. Center for Computing Research Sandia National Laboratories. Report (2015)
[29] D'Elia, M., Gunzburger, M.: The fractional Laplacian operator on bounded domains as a special case of the nonlocal diffusion operator. Comput. Math. Appl. **66**, 1245–1260 (2013)
[30] Evans, L.C.: Partial Differential Equations. American Mathematical Society, Providence (1999)
[31] Felsinger, M., Kassmann, M., Vooigt, P.: The Dirichlet problem for nonlocal operators. Math. Z **279**, 779–809 (2015). https://doi.org/10.1007/s00209-014-1394-3
[32] Gilboa, G., Osher, S.: Nonlocal operators with applications to image processing. Multiscale Model. Simul. **7**, 1005–1028 (2008)
[33] Guan, Q., Gunzburger, M.: Stability and convergence of time-stepping methods for a nonlocal model for diffusion. Discrete Cont. Dyn. Syst. B. **20**(5), 1315–1335 (2015)
[34] Guan, Q., Gunzburger, M.: Stability and accuracy of time-stepping schemes and dispersion relations for a nonlocal wave equation. Numer. Methods Partial Differ. Equ. **31**, 500–516 (2015)
[35] Hutson, V., Martínez, S., Michaidow, K., Vickers, G.T.: The evolution of dispersal. J. Math. Biol. **47**(6), 483–517 (2003)
[36] Humphries, N.E., et al.: Environmental context explains Levy and Brownian movement patterns of marine predators. Nature **465**, 1066–1069 (2010)
[37] Hunter, J.K.: Notes on Partial Differential Equations. Manuscript. https://www.math.ucdavis.edu/~hunter/pdes/pde_notes.pdf
[38] Levendorski, S.Z.: Pricing of the American put under Levy processes. Int. J. Theor. Appl. Finance **7**, 303–335 (2014)
[39] Lions, J.L.: Contrôle optimal de systèmes gouvernes par des équations aux dérivées partielles. Ed. Dunod-Guthier-Villars, Paris (1968)
[40] Lions, J.L., Magenes, E.: Problèmes aux limites non homogénes et applications, 1,2,3. Ed. Dunod-Guthier-Villars, Paris (1968)
[41] Lou, Y., Zhang, X., Osher, S., Bertozzi, A.: Image recovery via nonlocal operators. J. Sci. Comput. **42**, 185–197 (2010)
[42] Mazón, J.M., Rossi, J.D., Toledo-Melero, J.J.: Fractional p-Laplacian evolution equations. J. Math. Pures Appl. **105**, 810–844 (2016)
[43] Metzler, R., Klafter, J.: The random walk's guide to anomalous diffusion: a fractional dynamics approach. Phys. Rep. **339**, 1–77 (2000)
[44] Neuman, S.P., Tartakosky, D.M.: Perspective on theories of non-Fickian transport in heterogeneous media. Adv. Water Resour. **32**, 670–560 (2009)
[45] Ponce, A.C.: An estimate in the spirit of Poincaré's inequality. J. Eur. Math. Soc. (JEMS) **6**, 1–15 (2004)
[46] Ponce, A.C.: A new approach to Sobolev Spaces and connections to Γ-convergence. Calc. Var. **19**, 229–255 (2004)
[47] Puhst, D.: On the evolutionary fractional p-Laplacian. Appl. Math. Res. Express. **2**, 253–273 (2015)
[48] Rogers, R.C.: A nonlocal model for the exchange energy in ferromagnetic materials. J. Integral Equ. Appl. **3**, 85–127 (1991)
[49] Sastre, S.: Nonlocal Diffusion Problems. Memoria Tesis Doctoral, Facultad CC. Matemáticas. Departamento de Matemática Aplicada. Universidad Complutense de Madrid (2014)
[50] Seleson, P., Gunzburger, M., Parks, M.: Interface problems in nonlocal diffusion and sharp transitions between local and nonlocal domains. Comput. Methods Appl. Mech. Eng. **266**, 185–204 (2013)
[51] Seleson, P., Parks, M., Gunzburger, M.: Peridynamic solid mechanics and the embedded-atom model. Commun. Comput. Phys. **15**, 179–205 (2014)
[52] Servadei, R., Valdinoci, E.: Variational methods for non-local operators of elliptic type. Discrete Cont. Dyn. Syst. **33**, 2105–2137 (2013)
[53] Servadei, R., Valdinoci, E.: On the spectrum of two different fractional operators. Proc. R. Soc. Edinb. Sect. A Math. **144**(4), 831–855 (2014)
[54] Silling, S.A.: Reformulation of elasticity theory for discontinuities and long-range forces. J. Mech. Phys. Solids **48**, 175–209 (2000)

[55] Silling, S.A., Epton, M., Weckner, O., Xu, J., Askari, E.: Peridynamics states and constitutive modelling. J. Elast. **88**, 151–184 (2007)
[56] Tröltzsch, F.: Optimal Control of Partial Differential Equations, Theory, Methods and Applications. American Mathematical Society, Providence (2010)
[57] Valdinoci, E.: From the long jump random walk to the fractional Laplacian. Bol. Soc. Esp. Mat. Apl. SeMA **49**, 33–44 (2009)
[58] Vázquez, J.L.: The mathematical theories of diffusion: nonlinear and fractional diffusion. In: Bonforte, M., Grillo, G. (eds.) Nonlocal and Nonlinear Diffusions and Interactions: New Methods and Directions. Lecture Notes in Mathematics, vol. 2186, pp. 205–278. Springer, Cham (2017)
[59] Wloka, J.: Partial Differential Equations. Cambridge University Press, Cambridge (1987)
[60] Zhou, K., Du, Q.: Mathematical and numerical analysis of linear peridynamic models with nonlocal boundary conditions. SIAM J. Numer. Anal. **48**(5), 1759–1780 (2010)
[61] Zhou, K., Du, Q.: Mathematical analysis for the peridynamic nonlocal continuum theory. ESAIM: Math. Model. Numer. Anal. **45**, 217–234 (2011)

Fuensanta Andrés and Julio Muñoz
Departamento de Matemáticas
Universidad de Castilla-La Mancha
Avda. Carlos III s/n
Toledo
Spain
e-mail: Julio.Munoz@uclm.es

(Received: July 5, 2018; revised: May 19, 2019)