**PAPER • OPEN ACCESS**

# Interactions of Soliton in Weakly Nonlocal Nonlinear Media

View the article online for updates and enhancements.

# Interactions of Soliton in Weakly Nonlocal Nonlinear Media

**N A B Aklan[1], F A Faizar[2] and B A Umarov[3]**

[1,2]Department of Computational and Theoretical Sciences, Kulliyyah of Science, International Islamic University Malaysia, 25200 Kuantan, Pahang, Malaysia
[3]Department of Physics, Kulliyyah of Science, International Islamic University Malaysia, 25200 Kuantan, Pahang, Malaysia


[1]**Email:** noramirah@iium.edu.my

**Abstract**. Solitary waves or solitons is a nonlinear phenomenon which has been studied intensively due to its application in solid-state matter such as Bose-Einstein condensates state, plasma physics, optical fibers and nematic liquid crystal. In particular, the study of nonlinear phenomena occurs in the structure of waves gained interest of scholars since their discovery by John Russell in 1844. The Nonlinear Schrödinger Equation (NLSE) is the theoretical framework for the investigation of nonlinear pulse propagation in optical fibers. Nonlocality can be found in an underlying transport mechanisms or long-range forces like electrostatic interactions in liquid crystals and many-body interactions with matter waves in Bose-Einstein condensate or plasma waves. The length of optical beam width and length of response function are used to classify nonlocality in optical materials. The nonlocality can be categorized as weak nonlocal if the width of the optical beam broader than the length of response function and if the width of the optical beam is narrower than the length of response function, it is considered as highly nonlocal. This work investigates the interactions of solitons in a weakly nonlocal Cubic NLSE with Gaussian external potential. The variational approximation (VA) method was employed to solve non integrable NLSE to ordinary differential equation (ODE). The soliton parameters and the computational program are used to simulate the propagation of the soliton width and its center-of-mass position. In the presence of Gaussian external potential, the soliton may be transmitted, reflected or trapped based on the critical velocity and potential strength. Direct numerical simulation of Cubic NLSE is programmed to verify the results of approximation method. Good agreement is achieved between the direct numerical solution and VA method results.


Keywords: Soliton; nonlinear Schrödinger equation; gaussian potential; nonlinear equation; scattering; variational method; numerical method.

## 1. Introduction
Recently, soliton waves are a nonlinear phenomenon that has been extensively studied due to their universal definition. Specifically, in nonlinear optic, nonlocality has been discovered in photorefractives, atomic vapors, liquid crystals, and thermal nonlinear media in nonlinear optics [1]. Mishra and Hong [2] investigates the propagation properties of a Super-Gaussian beam in extremely nonlocal nonlinear media. While the scattering of solitons of generalized NLSE on a localized external Delta potential in weak nonlocal media using secant trial function is also studied in [3]. The following

is a general representation of the refractive index change $\Delta n$ induced by a beam of intensity $I(x', z)$ in nonlocal nonlinear Kerr-type media:

$$\Delta n(I) = \pm \int_{-\infty}^{\infty} R(x'-x) I(x',z) dx'. \tag{1}$$

The focusing and defocusing nonlinearities are denoted by the positive and negative signs, respectively, while the transverse and propagation coordinates are denoted by $x$ and $z$ [4]. Conversely, optical materials' nonlocality nomenclature is based on the relative length of the optical beam width and the length of the response function. If the length of the response function is shorter than the width of the optical beam, it is called weak nonlocality, but otherwise if the length of the response function is longer than the width of the optical beam, it is designated as highly nonlocality [5].

The study of nonlocality reveals that it is important for very narrow beams, and the effects of nonlocality can vary depending on the degree of nonlocality, which ranges from weak to high. Furthermore, depending on the amplitude of the potential, the external potential or barrier may affect soliton propagation, causing it to be reflected, transmitted or trapped [6]. On the other hand, the study of nonlocality nonlinear with the presence of an external potential is relatively uncommon.

Therefore, the interaction of the soliton in nonlocal nonlinear media, specifically the cubic nonlinear Schrodinger equation in weakly nonlocal nonlinear media with external potential will be investigated in this paper. The scattering process of soliton will be studied by two methods, analytically using the variational approximation method, and the results will be verified numerically by direct numerical simulation of NLSE of the main equation.

The following is a breakdown of the paper's structure. The model and governing equations are introduced in Sec. 2. In Sec. 3, we present the variational approximation method and results obtained. The numerical simulation of variational approximation is compared to the exact solution in Sec. 4. Finally, in Sec. 5 we summarize our discoveries.

## 2. The Model of Main Equation

With external potential based generalized NLSE, the main equation for soliton in weakly nonlocal nonlinear media is as follows [4]:

$$i\psi_t + \frac{1}{2}\psi_{xx} + \Delta n(I)\psi + V(x)\psi = 0, \tag{2}$$

$V(x)$ is the external potential, and $\psi(x,t)$ is a slowly varying envelop and also a complex function by definition. Moreover, we consider the following potential to be a Gaussian potential that can be represented by a function below [7]

$$V(x) = U_0 \exp(-x^2 / c^2) \tag{3}$$

with a potential strength coefficient of $U_0$ and a width of c. On interesting note, the sign of $U_0$ determine the potential shape either potential wall or potential well, as shown in Figure 1.
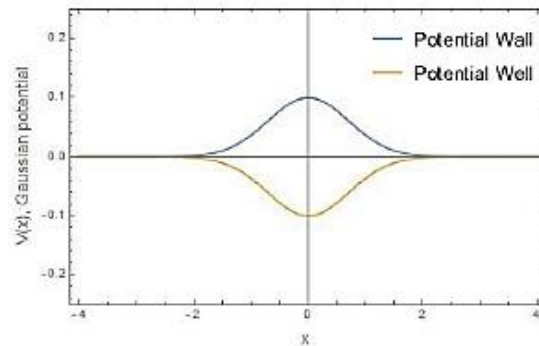
**Figure 1.** The profile of Gaussian potential at $U_0 > 0$ (potential wall)

and $U_0 < 0$ (potential well).

We defined $\Delta n(|\psi(x,t)|^2)$ as the local Kerr nonlinearity in the previous section, where

$$\Delta n(I) = \pm\left(I + \gamma\partial_x^2 I\right),\tag{4}$$

as well as the nonlocality parameter $\gamma > 0$, which is given by

$$\gamma = \frac{1}{2}\int_{-\infty}^{\infty} R(x)x^2 dx,\tag{5}$$

will results to revised nonlinear Schrodinger equation [4] as in equation (6) where $\gamma < 1$ is a positive coefficient.

$$i\psi_t + \frac{1}{2}\psi_{xx} + (|\psi|^2 + \gamma\partial_x^2(|\psi|^2))\psi + V(x)\psi = 0.\tag{6}$$

## 3. Variational Approximation Analysis

Pursuant to Anderson [8], the VA method can provide an explicit approximation of the analytical expression for the pulse compression/decompression factor, as well as the maximum pulse amplitude and induced frequency chirp. Since these parameters are the most critical for characterising pulse propagation, this will enable successful investigation of soliton scattering. This method not only provides a useful approximate expression for the evolution of the characteristics pulse parameter by converting partial differential equations (PDE) to ordinary differential equations (ODE), but also a suggestive explanation of the relationship between disperse and nonlinear effects. In this work, this method is employed to find the approximated results of coupled ordinary differential equation of soliton's width and center-of-mass position.

Considering Lagrangian density derived by Bezuhanov, 2008 in [9], we acquired:

$$\mathcal{L} = \frac{i}{2}(\psi\psi_t^* - \psi^*\psi_t) + \frac{1}{2}|\psi_x|^2 - V(x)|\psi|^2 - \frac{g}{2}|\psi|^4 - \frac{\gamma}{2}(\partial_x|\psi|^2)^2\tag{7}$$

where $g$ and $\gamma$ are positive nonlinearity coefficients that lead to attractive interactions between atoms in a condensate of focusing nonlinearity in optics applications, enabling the device to accommodate bright matter-wave solitons [10]. Using the Lagrangian density (7) and the Euler-Lagrange equation [11], the equation (6) can be easily confirmed. Then, as our trial function, we choose a Gaussian function with time-dependent parameters as follows;

$$\psi(x,t) = Ae^{-\frac{(x-\xi)^2}{2a^2}+ib(x-\xi)^2+iv(x-\xi)+i\varphi} . \tag{8}$$

The amplitude, distance, centre of mass location, chirp parameter, velocity, and initial step of the soliton are represented by $A$, $a$, $\xi$, $b$, $v$, and $\varphi$ respectively.

   The following effective or averaged Lagrangian density (9) is obtained by spatial integration of the Lagrangian density $L = \int_{-\infty}^{\infty} \mathcal{L}\, dx$ using the trial function. The number of atoms in the condensate region is the wave function norm, $N = \int_{-\infty}^{\infty} |\psi|^2\, dr = \sqrt{\pi} A^2 a$, which is a conserved quantity.

$$L = N\left[ \frac{a^2 b_t}{2} + \phi_t + \frac{1}{4a^2} + a^2 b^2 - \frac{\xi_t^2}{2} - \frac{U_0 c e^{\frac{-\xi^2}{a^2+c^2}}}{\sqrt{a^2+c^2}} - \frac{N}{2\sqrt{2\pi}a} + \frac{\gamma N}{2\sqrt{2\pi}a^3} \right] \tag{9}$$

   The reduced variational principal is represented by equation (9) which leads to a system of Euler-Lagrange equations (10) that specify the time-dependent Gaussian parameters $a$, $\xi$, and $b$, also known as the modulation equation [12].

$$\frac{d}{dt}\cdot\frac{\partial L}{\partial a_t} - \frac{\partial L}{\partial a} = 0$$

$$\frac{d}{dt}\cdot\frac{\partial L}{\partial b_t} - \frac{\partial L}{\partial b} = 0 \tag{10}$$

$$\frac{d}{dt}\cdot\frac{\partial L}{\partial \xi_t} - \frac{\partial L}{\partial \xi} = 0$$

From equation (9) calculated through equation (10), we have

$$b_t = -2b^2 + \frac{1}{2a^4} - U_0 c \frac{\left(2\xi^2 - a^2 - c^2\right)}{\left(a^2+c^2\right)^{5/2}} e^{\frac{-\xi^2}{a^2+c^2}} - \frac{N}{2\sqrt{2\pi}a^3} + \frac{3\gamma N}{2\sqrt{2\pi}a^5} \tag{11}$$

$$a_t = 2ab \tag{12}$$

$$-\xi_{tt} - \frac{2\xi U_0 c}{\left(a^2+c^2\right)^{\frac{3}{2}}} e^{\frac{-\xi^2}{a^2+c^2}} = 0 \tag{13}$$

Below are the two corresponding coupled equations for width and center-of-mass location retrieved

$$a_{tt} = \frac{1}{a^3} - \frac{2ac U_0\left(2\xi^2 - a^2 - c^2\right)}{\left(a^2+1\right)^{\frac{5}{2}}} e^{\frac{-\xi^2}{a^2+c^2}} - \frac{N}{\sqrt{2\pi}a^2} + \frac{3\gamma N}{\sqrt{2\pi}a^4} \tag{14}$$

$$\xi_{tt} = \frac{2\xi U_0 c}{\left(a^2+c^2\right)^{\frac{3}{2}}} e^{\frac{-\xi^2}{a^2+c^2}} \tag{15}$$

When there is no external potential, i.e. $U_0 = 0$, equations (14) and (15) become decouple. The estimated width of a stationary $a_s$ soliton solution can be calculated using equation (14) when $\left(a_{tt} = 0\right)$

$$a_s = \sqrt{\frac{\sqrt{2}}{A^2} + 3\gamma} \qquad (16)$$

Perturbation can cause the width to oscillate around this fixed point. The constant free parameter in this case is velocity. In the presence of Gaussian potential, the evolution of soliton width and its centre of position were coupled. Since the soliton is located far from inhomogeneity, we conclude that its parameters are unchanged. We can apparently obtain some qualitative results regarding soliton's evolution if we assume that the width of soliton will not be influenced by the potential. On the localized barrier, equation (15) describes the scattering of effective classical particles.

$$\xi_{tt} = \frac{-2\xi U_0 c e^{\frac{-\xi^2}{a^2+1}}}{\left(a^2 + c^2\right)^{\frac{3}{2}}} = \frac{dV_p\left(\xi\right)}{d\xi} \qquad (17)$$

$$\xi_t = \sqrt{2V_p\left(\xi\right)} \qquad (18)$$

Integration is used to minimise equation (17) to equation (18) in which

$$V_p\left(\xi\right) = \frac{U_0 c e^{\frac{-\xi^2}{a^2+1}}}{\left(a^2 + c^2\right)^{\frac{1}{2}}} . \qquad (19)$$

$V_p$ stands for the effective potential, which represents the initial localised Gaussian potential's impact on the solitons velocity. Equation (18) indicates that depending on whether the velocity is above or below the critical value $\left(\xi = 0\right)$, the effective particle can be transmitted or reflected from the potential.

$$v_c = \sqrt{\frac{2U_0}{\left(a^2 + c^2\right)^{\frac{1}{2}}}} \qquad (20)$$

The scattering of soliton in weakly nonlocal nonlinear media, in accordance to the results obtained in equations (14) and (15) are then interpreted by numerical simulations in Section 4.1. Direct numerical simulations of PDE of equation (6) support these findings in Section 4.2.

## 4. Numerical Simulations of Soliton Scattering

*4.1. A Numerical Simulation of Variational Approximation Results of ODE.*
Firstly, NDsolve in MATHEMATICA is used to solve the coupled second ODE of equations (14) and (15), which describe the evolution of width and centre of mass position respectively. This command solves a variety of ODEs and PDEs and it returns results in the form of

`InterpolatingFunction`. The approximated results were then compared to the exact solution of the governing equation (6) using direct numerical simulation.

For $A = 1.0, \gamma = 0.2, v = 0.4, \xi_0 = -6$, the usual parameters are set with constant coefficients and initial conditions of $a(0) = a_s, a'(0) = 0, \xi(0) = \xi_0, \xi'(0) = v$. As the soliton approaches the potential, it experiences a perturbation effect in its width and velocity. The examples of numerical solution results of the variational coupled equations (14) and (15) for different potential strengths, $U_0$ are shown in the Figures (2) – (5).
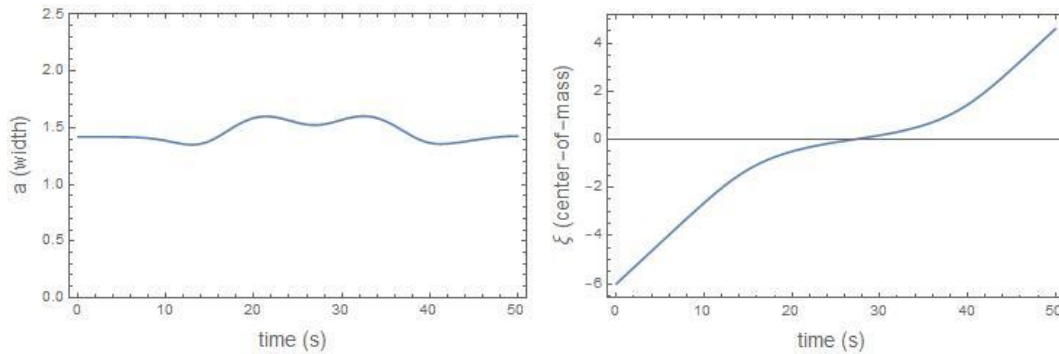


**Figure 2.** The scattering of soliton's width (left) and the center-of-mass position (right) in the presence of a Gaussian potential, according to the ODE (14) and (15) with parameters $U_0 = 0.1, \gamma = 0.2, A = 1.0, v = 0.4, \xi_0 = -6$.
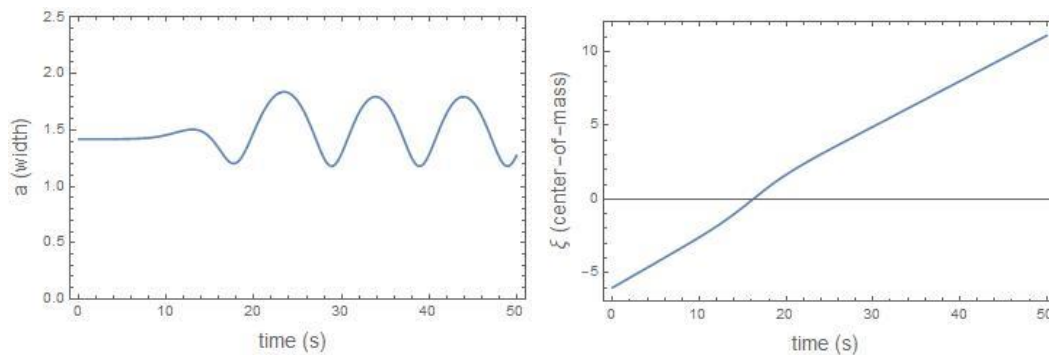


**Figure 3.** The scattering of soliton's width (left) and the center-of-mass position (right) in the presence of a Gaussian potential, according to the ODE (14) and (15) with parameters $U_0 = -0.1, \gamma = 0.2, A = 1.0, v = 0.4, \xi_0 = -6$.

When reaching the potential wall, the soliton behaves like a particle, as shown in Figure 2, and is transmitted through the potential wall. The energy of the soliton is preserved and managed to transmit through the potential strength of $U_0 = 0.1$ with a very small perturbation of the soliton width. While having a different form and strength of potential, the soliton in Figure 3 is also transmitted through the potential well of $U_0 = -0.1$. The soliton with initial velocity of $v = 0.4$ has enough energy to propagate passing the potential well.
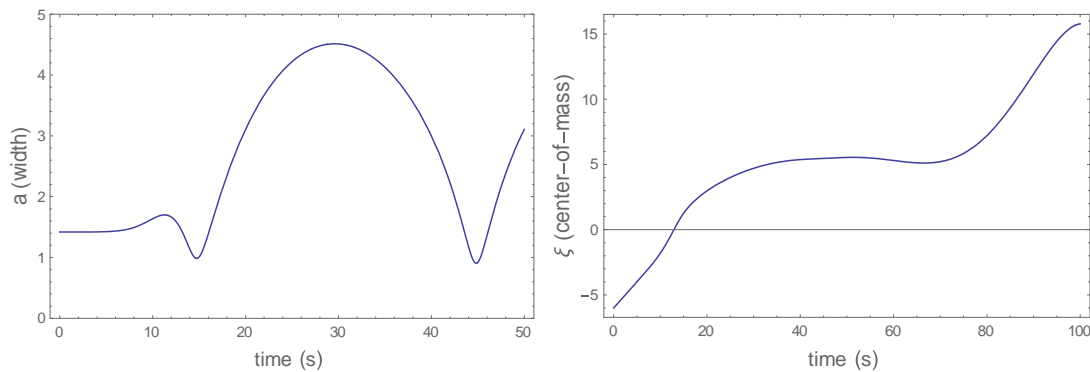
**Figure 4.** The scattering of soliton's width (left) and the center-of-mass position (right) in the presence of a Gaussian potential, according to the ODE (14) and (15) with parameters
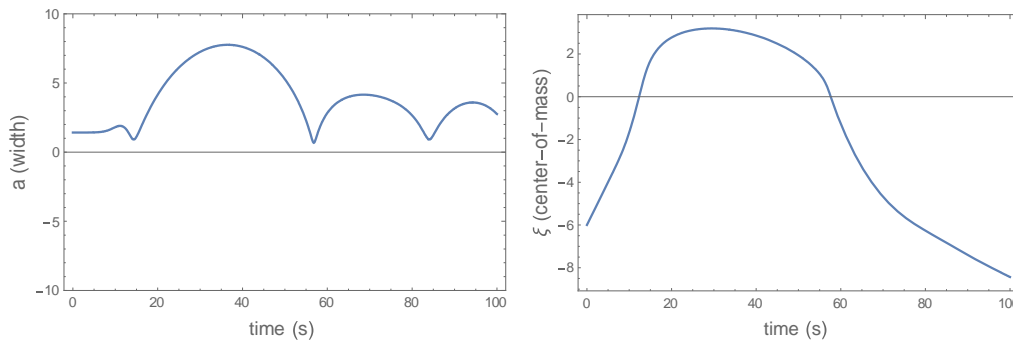$$U_0 = -0.3, \gamma = 0.2, A = 1.0, v = 0.4, \xi_0 = -6.$$



**Figure 5.** The scattering of soliton's width (left) and the center-of-mass position (right) in the presence of a Gaussian potential, according to the ODE (14) and (15) with parameters
$$U_0 = -0.5, \gamma = 0.2, A = 1.0, v = 0.4, \xi_0 = -6.$$

Figure 4 depicts the soliton being trapped for a short time before being transmitted via the potential well at $U_0 = -0.3$. From 20s to 70s, some solitons are stuck in the well due to small less energy to perform, while the majority is transmitted through the potential. On the other hand, Figure 5 indicates that the soliton is also trapped for a short time during the interaction with the external potential well of strength $U_0 = -0.5$, but substantially reflected after the meeting as the solitons do not have energy to pass through the external potential. Several cases are reported [13] – [15] for soliton trapping which lead to interesting results in soliton scattering investigation history.

These findings are confirmed by the same results occurred in Figures (6)-(9) with direct numerical simulation of PDE in equation (6).

### 4.2. A Direct Numerical Simulation of NLSE.

While the objectives are predominantly analytical, verification with direct numerical simulation is necessary to confirm the physical behavior of the soliton of the approximation result. To find the exact solution of a nonlinear equation (6), the numerical method of Split-Step Fourier Transform (SSFT) method is applied in order to model the pulse propagation of the Cubic NLSE in weakly nonlocal with external potential since its analytic solutions are generally not available. This approach uses split-step schemes to perform spatial sub-steps by discrete Fourier transformation that only consider the

linearities of the focusing effect term in Cubic NLSE, and sub-steps by discrete Fourier transformation that only estimate the influence of the nonlinear terms in an alternating manner [16].

The numerical simulation is based on the propagation of a single soliton $\psi(x)$ moving at a certain distance. The time evolution of Cubic NLSE in weakly nonlocal with external potential at $x = 0$ is depicted in Figures (6) – (9). The results show that the soliton behaves like classical particles, and that the soliton is either reflected, transmitted or both, when interacting with Gaussian external potential.
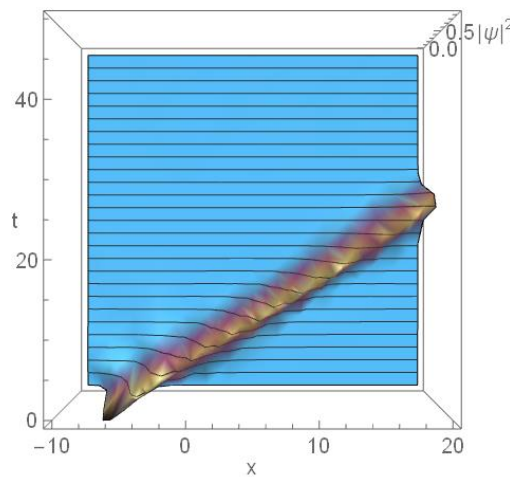


**Figure 6.** Scattering of soliton on potential wall, $U_0 = 0.1$ according to Cubic NLSE (6) with parameters $\gamma = 0.2, A = 1.0, v = 0.4, \xi_0 = -6$.
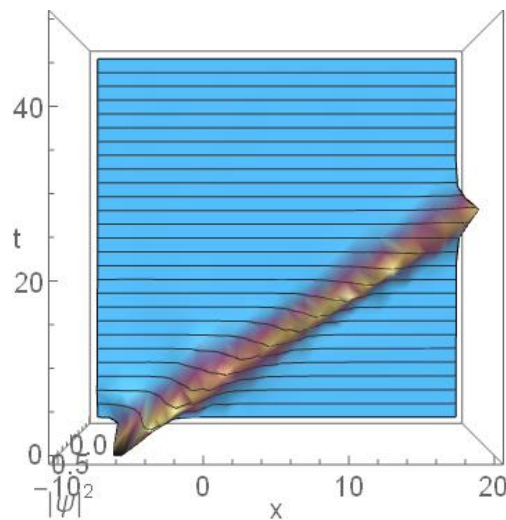


**Figure 7.** Scattering of soliton on potential wall, $U_0 = -0.1$ according to Cubic NLSE (6) with parameters $\gamma = 0.2, A = 1.0, v = 0.4, \xi_0 = -6$.

As seen in Figure 6, the soliton tends to be transmitted through the potential wall of strength $U_0 = 0.1$, which is the same as in Figure 1. Figure 7 illustrates the same outcome as compared to Figure 2, with the soliton appearing to be transmitted via the potential well of $U_0 = -0.1$.
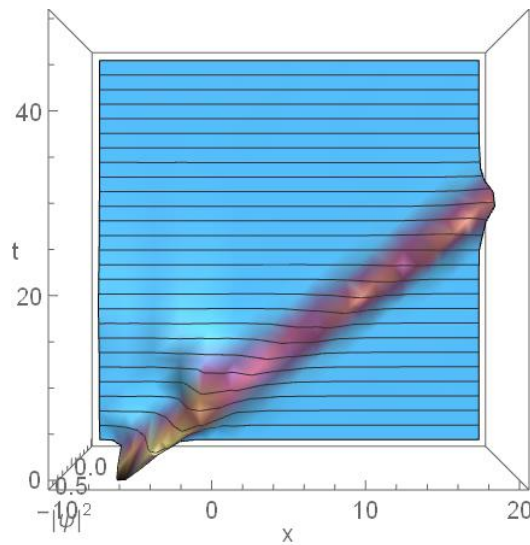
**Figure 8.** Scattering of soliton on potential wall, $U_0 = -0.3$ according to
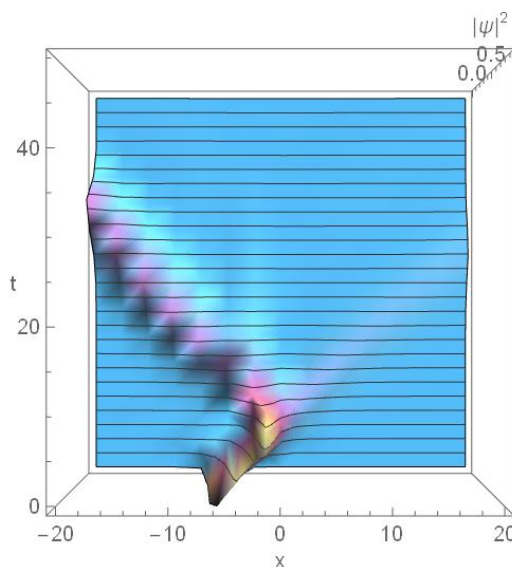Cubic NLSE (6) with parameters $\gamma = 0.2, A = 1.0, v = 0.4, \xi_0 = -6.$



**Figure 9.** Scattering of soliton on potential wall, $U_0 = -0.5$ according to
Cubic NLSE (6) with parameters $\gamma = 0.2, A = 1.0, v = 0.4, \xi_0 = -6.$

The direct numerical simulations results exhibit the same behaviour as the results of the variational approach in section 3. Figures (8) and (9) may depict a trapped soliton with a very small difference in time scale compared to Figures (4) and (5).

The interaction of soliton in nonlinear media with weakly nonlocality has been investigated. From the results, it is showed that the soliton behaves like a particle when impacted by the Gaussian external potential (3), where the soliton is transmitted when interacting with weak potentials such as potential strength of $U_0 = 0.1$ and $U_0 = -0.1$. Instead, the soliton is reflected for strong potentials such as when $U_0 = -0.5$. In general, the results of direct numerical simulation of SSFT show good agreement with the analytical results of VA method.

## 5. Conclusion

The aim of the research is to figure out how soliton interacts with external potential in weakly nonlocal systems. The results show that the soliton behaves like classical particles, and that the soliton is reflected and transmitted during the interaction with Gaussian external potential. Analytical and computational methods are used throughout the work to achieve the study's goals. Finally, more information about the transmission and trapping of critical energies will help us understand the soliton's interaction with external potential.

The result from VA method is compared with direct numerical simulation to verify the findings and the identification of both solutions has consistent results. The VA method has proven to be faster in terms of time for solving the Cubic NLSE compared to direct numerical simulation because the time consumed by the VA method to reduce the PDE into system of ODE is faster than SSFT schemes that require considering several time steps to reduce PDE until the result does not depend on time step.

As a result, future research into soliton in weakly nonlocal nonlinear media with various trial functions and potential functions, especially with higher structure potentials or other shape of potentials (square shape or periodic), would be very interesting.

## References
[1]  Rotschild, C., Cohen, O., Manela, O., Segev, M., & Carmon, T. (2005). Solitons in nonlinear media with an infinite range of nonlocality: first observation of coherent elliptic solitons and of vortex-ring solitons. *Physical review letters*, 95(21), 213904.
[2]  Mishra, M., & Hong, W. P. (2011). Investigation on propagation characteristics of super-Gaussian beam in highly nonlocal medium. *Progress In Electromagnetics Research*, 31, 175-188.
[3]  Aklan, N. A. B., & Umarov, B. A. (2017). The Soliton Interaction in Weakly Nonlocal Nonlinear Media on the External Potentials. In *Journal of Physics: Conference Series* (Vol. 819, No. 1, p. 012024). IOP Publishing.
[4]  Królikowski, W., & Bang, O. (2000). Solitons in nonlocal nonlinear media: Exact solutions. *Physical Review E*, 63(1), 016610.
[5]  Mishra, M., & Hong, W. P. (2011). Investigation on propagation characteristics of super-Gaussian beam in highly nonlocal medium. *Progress In Electromagnetics Research*, 31, 175-188.
[6]  Sakaguchi, H., & Tamura, M. (2004). Scattering and trapping of nonlinear Schrödinger solitons in external potentials. *Journal of the Physical Society of Japan*, 73(3), 503-506.
[7]  Umarov, B. A., Messikh, A., Regaa, N., & Baizakov, B. B. (2013, April). Variational analysis of soliton scattering by external potentials. In *Journal of Physics: Conference Series* (Vol. 435, No. 1, p. 012024). IOP Publishing.
[8]  Anderson, D. (1983). Variational approach to nonlinear pulse propagation in optical fibers. *Physical review A*, 27(6), 3135.
[9]  Bezuhanov, K. S., Dreischuh, A. A., & Krolikowski, W. (2008). Bright optical beams in weakly nonlocal media: Variational analysis. *Physical Review A*, 77(3), 033825.
[10]  Biswas, A., & Konar, S. (2006). Introduction to non-Kerr law optical solitons. CRC Press.
[11]  Hazewinkel, M. (2001). Lagrange equations (in mechanics)", Encyclopedia of Mathematics. EMS Press.
[12]  MacNeil, J. M. (2016, May 19). Solitary Waves in Focussing and Defocussing Nonlinear, Nonlocal Optical Media. The University of Edinburgh.
[13]  Islam, M. N., Poole, C. D., & Gordon, J. P. (1989). Soliton trapping in birefringent optical fibers. *Optics lette*rs, 14(18), 1011-1013.

[14]  Al Khawaja, U., Stoof, H. T. C., Hulet, R. G., Strecker, K. E., & Partridge, G. B. (2002). Bright soliton trains of trapped Bose-Einstein condensates. *Physical review letters*, 89(20), 200404.

[15]  Davidson, A., Dueholm, B., Kryger, B., & Pedersen, N. F. (1985). Experimental investigation of trapped sine-Gordon solitons. *Physical review letters*, 55(19), 2059.

[16]  Deiterding, R., Glowinski, R., Oliver, H., & Poole, S. (2013). A reliable split-step Fourier method for the propagation equation of ultra-fast pulses in single-mode optical fibers. *Journal of Lightwave Technology*, 31(12), 2008-2017.