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Vector Soliton in Coupled Nonlinear Schrödinger Equation

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Abstract. Researchers are currently interested in studying the dynamics of the wave field in a nonlinear and dispersive medium. The Nonlinear Schrödinger Equation (NLSE), which is the fundamental equation that explains the phenomenon, has paved the way for research in a variety of fields, including soliton scattering. However, if the fields have a large number of components, the Coupled NLSE should be considered. We used orthogonally polarised and equal-amplitude vector solitons with two polarization directions to model the interactions. The effect of vector soliton scattering by external Delta potential in Coupled NLSE was studied in this paper. The scattering process is primarily determined by the initial velocity, amplitude of the soliton and potential strength. The variational approximation and direct numerical methods of Coupled NLSE were used to investigate the scattering process. The variational approximation (VA) method was used to analyse the dynamics of soliton's width and center of mass position. The soliton may thus be reflected, transmitted or trapped within the potential. Uncoupled solitons may initially create a coupled state if their kinetic energy is less than the attractive interaction potential between solitons, but once their velocity surpasses the critical velocity, the soliton will easily pass through each other. To validate the approximation, a direct numerical simulation of CNLSE was performed. The results of the VA method and direct numerical simulation of Coupled NLSE are in good agreement when the parameters for both solutions are set to the same value. The initial velocity, potential strength and soliton amplitude play a role in the scattering of the vector soliton with Delta potential.

Keywords: Vector soliton; coupled nonlinear Schrödinger equation; delta potential; nonlinear equation; scattering; variational method; numerical method.

1. Introduction

Solitary wave collisions in science and engineering are a common phenomenon in the physical field. In the mathematical field, solitary-wave collision is a significant branch of nonlinear waves [1], for example, wavelength-division multiplexing (WDM). The NLSE is the fundamental equation which describes the dynamics of a wave field, also known as a soliton, in a nonlinear and dispersive medium. Coupled NLSE, on the other hand, is a solitary wave with multiple components coupled together while maintaining its shape during the propagation.

We investigated the interactions of equal-amplitude vector soliton and orthogonally polarized with two polarization directions [5]. Based on the previous study in [1-3], the Coupled NLSE is the best method for governing solitary wave interaction. Gromov et al. [4] discovered vector soliton scattering in the absence of potential, whereas Din et al. [5] demonstrated vector soliton interaction in Coupled NLSE with Gaussian potential. The collisions are generally inelastic, according to Ueda et al. [3], and may lead to a phase shift, but it remains unchanged in terms of shape and velocity, as the velocity is constant when a vector soliton collides [2].

A significant generalisation in fiber optics is to consider two polarizations of light (or two separate wavelengths in one fiber), by using the polarized light propagation model in fiber optics, resulting to Coupled NLS equations system.

$$\begin{aligned} iJ_t + J_{xx} + (|J|^2 + \beta |K|^2)J = 0 \\ iK_t + K_{xx} + (|K|^2 + \beta |J|^2)K = 0 \end{aligned} \tag{1}$$

In the two polarization directions, the variables J and K define the wave packets of vector soliton, respectively. β is the coefficient of cross-phase modulation (XPM), which is the nonlinear phase shift of an optical field caused by another field with a different wavelength, direction, or polarisation state. The system is completely integrable in the cases of $\beta = 0$ and $\beta = 1$. The systems reduce to a pair of uncoupled NLSE when $\beta = 0$, and when $\beta = 1$, Manakov mechanism occurs [7].

The purpose of this study is to investigate the dynamics of a vector soliton with an external Delta potential. Analytical and numerical studies are used to construct the two components of soliton scattering with potential within the Coupled NLSE. The inverse scattering method is well known for solving the integrable PDE, particularly the NLSE. However, this equation is non-integrable and has no analytical solution with the existence of external potential and the extended form of NLSE. As a result, we must rely on approximate methods to explain the dynamic properties of the soliton scattering, such as the VA method proposed by Anderson [6].

The paper is structured as follows; section 2 introduces the model and its governing equations. Section 3 presents an analysis of the VA method and section 4 discusses numerical simulations of vector soliton scattering in the presence of external potential. Section 5 will bring the paper to a close.

2. The Model of Main Equation

Based on the generalized NLSE in the presence of potential term as discussed in [8],

$$i\psi_t + \frac{1}{2}\psi_{xx} + F(|\psi|^2)\psi + V(x)\psi = 0 \tag{2}$$

where $V(x)$ stands for the external potential, $\psi(x,t)$ is a solitary wave-function, the real variables t and x are frequently time and space coordinates, respectively, and the subscripts indicate partial derivatives. The main equation of our model in equation (1) is recreated as Coupled NLSE below [1],

$$\begin{aligned} iJ_t + J_{xx} + \left(|J|^2 + \beta |K|^2 - V(x)\right)J = 0 \\ iK_t + K_{xx} + \left(|K|^2 + \beta |J|^2 - V(x)\right)K = 0 \end{aligned} \tag{3}$$

where soliton J and K is the wave function of vector soliton. The interaction of two identical, orthogonally polarized, equal-amplitude vector solitons with the presence of external delta potential in equation (4) below is under consideration.

$$V(x)=U_0\,\delta(x) \tag{4}$$

If U_0 is positive, $V(x)$ represents a potential wall, whereas if U_0 is negative, $V(x)$ represents a potential well [10]. Figure 1 depicts the two vector solitons of equation (3) propagating in opposite directions and interacting with the external potential, $V(x)$[5].

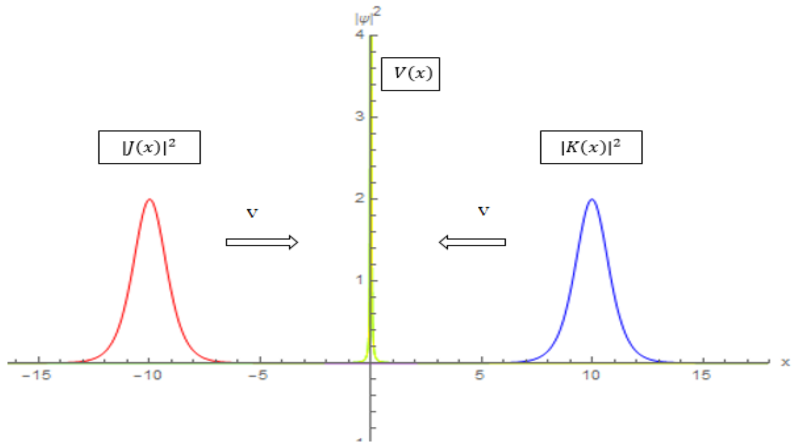

Figure 1. Two vector solitons propagated in opposite directions.

The scattering mechanism was studied using the VA method and direct numerical simulation of Coupled NLSE, as given in equation (3). This VA method was used to investigate the dynamic of a soliton's width and center of mass position during the scattering process. Meanwhile, using the same set of parameters and initial conditions, a direct numerical simulation of Coupled NLSE was performed to verify the accuracy of the approximation. The development of a nonlinear surface wave at the interface, as well as the soliton reflection and transmission across the interface are investigated.

The main equation (3) is built into the VA method, which yields an approximate system of ordinary differential equations for soliton parameters, while the direct numerical experiments yield an exact solution. Although the direct solution of Coupled NLSE is sufficient, the VA method is much faster in terms of time, in which we reduce approximately partial differential equation (PDE) to a system of ordinary differential equation (ODE).

3. Variational Approximation Analysis

One of the most important computational methods for analysing non-integrable soliton bearing equations is the VA method [9]. The VA approach simplifies the process of evaluating the evolution of important pulse parameters by transforming an initial infinite-dimensional PDE system for pulse waveforms into a finite-dimensional ODE system for pulse parameters [3]. The variational approach is employed with trial functions (ansatz) determined by NLS to explain the main characteristics of the pulse evolution. In this case, the main benefit of the variational approach is that it offers an explicit solution.

As shown by Din et al. in [5], a Lagrangian density in equation (5) was first obtained from the Coupled NLSE before substituting the trial function.

$$\begin{aligned}\mathcal{L} &= i\left(JJ_t^* - J_tJ^*\right) + i\left(KK_t^* - K_tK^*\right) + \left(2\left|J_x\right|^2 - |J|^4\right) + \left(2\left|K_x\right|^2 - |K|^4\right) \\ &\quad - 2\beta|J|^2|K|^2 + 2V(x)|J|^2 + 2V(x)|K|^2\end{aligned} \tag{5}$$

and taking into account two appropriate choice of trial functions, the Gaussian ansatz as below,

$$J(x,t) = A \cdot \exp\left[-\frac{1}{2}\left(\frac{x+\xi/2}{a}\right)^2 + i\left(b(x+\xi/2)^2 + v(x+\xi/2)+\varphi\right)\right] \tag{6}$$

$$K(x,t) = A \cdot \exp\left[-\frac{1}{2}\left(\frac{x-\xi/2}{a}\right)^2 + i\left(b(x-\xi/2)^2 - v(x-\xi/2)+\varphi\right)\right] \tag{7}$$

where the center-of-mass position, amplitude, chirp parameter, width, velocity and phase of the soliton are presented by ξ, A, b, a, v, φ respectively. The effective Lagrangian with spatial integration of the Lagrangian density, $L = \int_{-\infty}^{\infty} \mathcal{L}\, dx$ is calculated by substituting the chosen ansatz into the Lagrangian density.

The wave function norm, $N = \int_{-\infty}^{\infty} |\psi|^2\, dr = \sqrt{\pi} A^2 a$, is a conserved quantity equals the number of atoms in the condensate area. By calculating the effective Lagrangian, it arrives at the total averaged Lagrangian below,

$$L = N\left[2a^2 b_t + 2v\xi_t + 4\varphi_t + \frac{2}{a^2} + 8a^2b^2 + 4v^2 - \frac{2N}{a\sqrt{2\pi}} - \frac{2N}{a\sqrt{2\pi}}\beta e^{-\left(\frac{\xi^2}{2a^2}\right)} + \frac{4V_0}{a\sqrt{\pi}} e^{-\left(\frac{\xi}{2a}\right)^2}\right]. \tag{8}$$

By using Euler-Lagrange equations of $d/dt\left(\partial L/\partial \dot{q}_i\right) - \partial L/\partial q_i = 0$, equation (8) can be employed to generate collective coordinate equations for variational parameters, and it can also be used to calculate the time-dependent Gaussian parameters *a, ξ,* and *b*. The two coupled equations for width in equation (9) and center-of-mass position in equation (10) are obtained as a result of this. Both equations define the dispersion of vector soliton on Delta potential in equation (4).

$$a_{tt} = \frac{4}{a^3} - \frac{2N}{a^2\sqrt{2\pi}} + \frac{8N\beta}{\sqrt{2\pi}}\left(\frac{\xi^2}{a^4} - \frac{1}{a^2}\right) e^{-\frac{1}{2}\left(\frac{\xi}{a}\right)^2} - \frac{4V_0}{\sqrt{\pi}}\left(\frac{\xi^2}{2a^4} - \frac{1}{a^2}\right) e^{-\left(\frac{\xi}{2a}\right)^2} \tag{9}$$

$$\xi_{tt} = -\frac{4N\beta\xi}{a^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\xi}{a}\right)^2} + \frac{4V_0\xi}{a^3\sqrt{\pi}} e^{-\left(\frac{\xi}{2a}\right)^2} \tag{10}$$

In this case, the velocity ξ_t is the constant free parameter. A slight vibrations in the width can be caused by perturbations around this stationary point. With the existence of Delta potential, the development of its width is connected to the time evolution of the solitons's center-of-mass-position. When the soliton is not inhomogeneous, the parameters of the soliton remain constant. If we disregard the influence of potential on the soliton's width, we can obtain some qualitative conclusions on the evolution of soliton.

4. Numerical Simulations of Vector Soliton Scattering

The numerical simulation of the approximation calculation of ODE in equation (9) and equation (10) is compared to the direct numerical solution of the Coupled NLSE to confirm the result. In order to obtain numerical solutions to the ODEs, we use *NDsolve* function in Mathematica software with the independent variable time to illustrate the evolution of centre-of-mass position, ξ and width of vector soliton, a over time. Simultaneously, we run the split-step FFT approach to examine the behaviour of vector soliton in Coupled NLSE with external delta potential. The scattering of the vector solitons with the potential is caused by different potential strength of the Delta potential.

The initial conditions for the ODEs of equation (9) and equation (10) have been defined as below,

$$x(0)=1.189207113,\ x'(0)=0,\ \xi(0)=-10,\ \xi'(0)=0.2,\ \beta=0.2. \tag{11}$$

When the soliton reaches the potential, the perturbation affects both the centre-of-mass position, ξ and the width of vector soliton, a. The initial position of the vector soliton which is $x_0=10$ and the initial velocity of $v=2.0$ are used in the simulation. Second order split step approach is run for the numerical scheme.

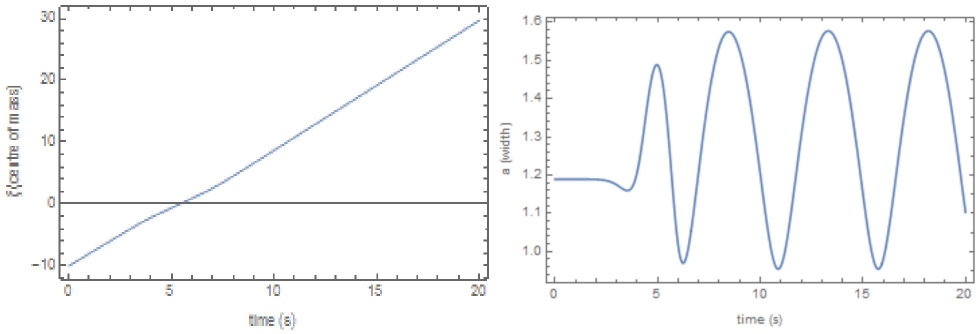

Figure 2. Center of mass position (left) and vector soliton's width (right) for ODE in equation (9) and equation (10) with the existence of Delta potential wall at $V(x)=0.30$.

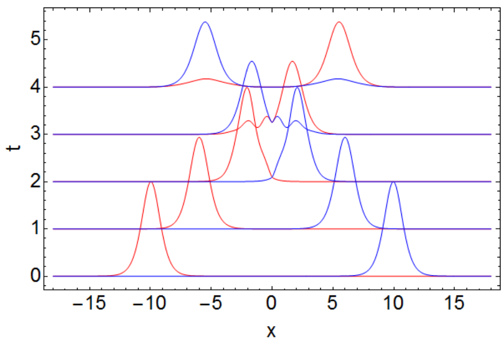

Figure 3. The scattering of vector soliton of PDE in equation (3) with the existence of Delta potential wall at $V(x)=0.30$.

Figure 2 showed the center-of-mass position and the vector soliton's width for ODE in equation (9) and equation (10) with Delta potential wall at strength of $V(x)=0.30$. The vector solitons are transmitted to each other at $t=4s$ in Figure 2 (left). Meanwhile, Figure 2 (right) displays that a minor perturbation occured at $t=4s$ and both solitons maintained their amplitude after the collision. Figure 3 supported and validated the findings in Figure 2 by the direct numerical simulation of PDE in equation (3) where the vector soliton is totally transmitted after the collision.

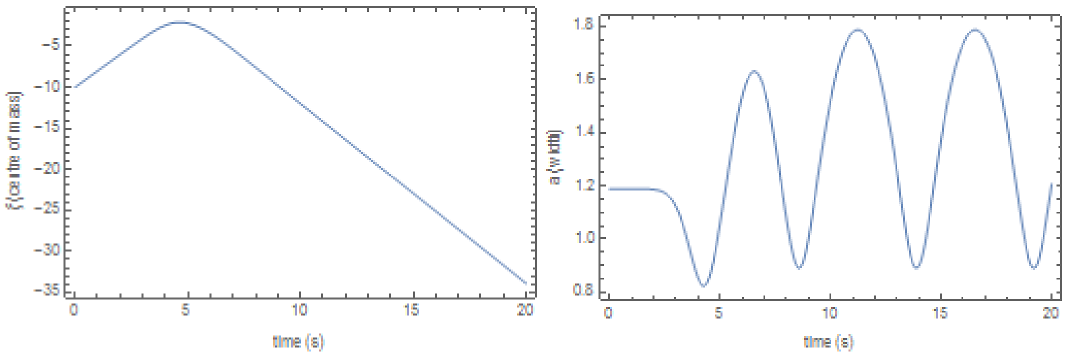

Figure 4. Center of mass position (left) and width of vector soliton (right) for ODE in equation (9) and equation (10) with the existence of Delta potential wall at $V(x)=1.30$.

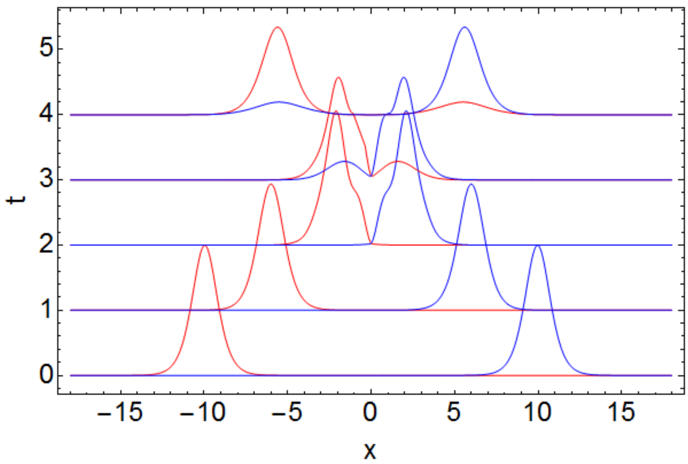

Figure 5. The scattering of vector soliton of PDE equation (3) with the existence of Delta potential wall at $V(x)=1.30$.

The center-of-mass position and the vector soliton's width for ODE in equation (9) and equation (10) with the existence of Delta potential wall at $V(x)=1.30$ are shown in Figure 4. The vector solitons in Figure 4 (left) are reflected to each other after colliding at $t=4s$. Meanwhile, the width in Figure 4 (right) began to display a very slight perturbation when $t=4s$ and above. The direct numerical simulation of PDE in equation (3) as presented in Figure 5, confirms these findings where the vector soliton is fully reflected after the collision with the potential.

The partial transmission and reflection of vector soliton are depicted in Figure 6 (left). A very small perturbation on the width of vector soliton is illustrated in Figure 6 (right). It can be proven by direct numerical simulation of equation (3) and shown in Figure 7 that at $t=3s$, the collision occurred and vector solitons are partially split into reflection and transmission.

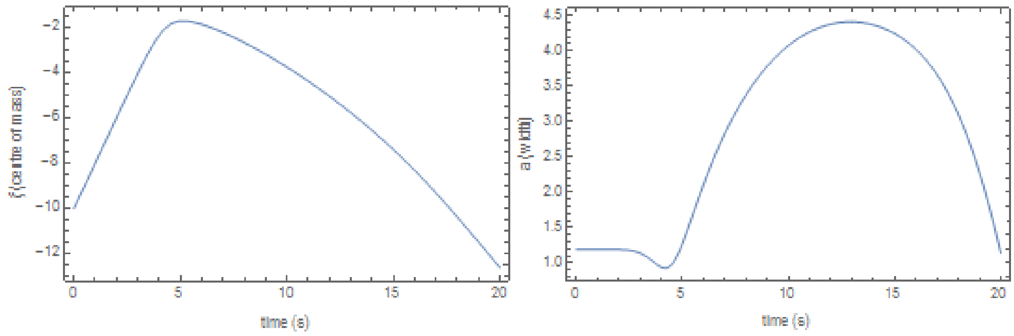

Figure 6. Center of mass position (left) and width of vector soliton (right) for ODE equation (9) and (10) with the existence of Delta potential wall at $V(x)=0.98$.

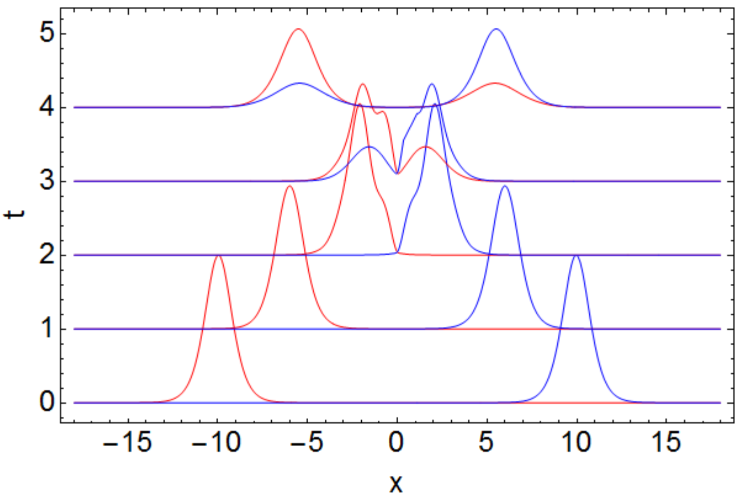

Figure 7. The scattering of vector soliton of PDE equation (3) with the existence of Delta potential wall at $V(x)=0.98$.

The numerical simulation of the VA method results of equation (9) and equation (10) were displayed in Figure 2, Figure 4 and Figure 6. In addition, the direct numerical simulation results of Coupled NLSE of equation (3) were shown in Figure 3, Figure 5 and Figure 7. The figures have confirmed the result obtained from the VA method where the vector soliton moves with constant velocity when it is far from the potential using the same parameters. The analysis of an approximation method with trial function has proven the vector soliton scattering on the potential, according to the findings. The vector solitons are either fully transmitted, fully reflected or partially transmitted and reflected. When potential strength is less than 0.35, $V(x)<0.35$ the vector solitons collide with the Delta potential and are transmitted through each other. Meanwhile, at the potential strength of $0.86<V(x)<1.1$, the vector solitons are partially split into reflection and transmission to each other upon the collision. The vector solitons are reflected from each other at the potential strength of $V(x)>1.2$.

5. Conclusion and Future Work

The scattering of vector solitons of Coupled NLSE with an external Delta potential is investigated. The ultimate goal of this study is to develop an analytical VA method and numerical analysis for explaining the interaction of vector soliton with the existence of Delta potential. The presented results of numerical calculations with full Coupled NLSE show that the VA method was able to define the vector soliton scattering mechanism on the external potential, taking into account the choice of trial function.

Further research can be proposed for vector solitons interactions with potential well, the negative value of $V(x)$, for a variety of shapes and different forms of soliton wave. Also, the external factors, for example, different initial velocity of vector solitons, different types of external potential, and other forms of generalized Coupled NLSE.

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