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A Functional Estimation Approach to the First-Price Auction Models *

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Abstract

This paper introduces new methods of identification and estimation of the first-price sealed bid auction model and compares them with the previous existing ones.

The first method of estimation allows us to estimate directly (through an iterative algorithm) the cumulative distribution function of the private values without estimating the private values beforehand. In the second method, we use a quantile approach. Although the first-price auction is a complex nonlinear inverse problem, the use of quantile leads to a linearisation of the model. Thus, in contrast with the existing methods we are able to deduce a closed-form solution for the quantile of the private values. This constructive identification allows for a one-stage estimation procedure that can be performed using three regularization methods: the Tikhonov regularization, the Landweber-Friedman regularization and the kernels.

We conduct a Monte Carlo experiment to compare our methods of estimation by c.d.f. and quantiles with the methods of estimation developed by Guerre et al. (2000), Marmer and Shneyerov (2012), and Hickman and Hubbard (2015).

Keywords: first-price auction model, regularization methods, mildly ill-posed inverse problem, quantile methods

JEL Classification: $C73, C40$

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1 Introduction

The first-price sealed bid auction is one of the four standard types of auctions (along with the second-price sealed bid auction, ascending auction and descending auction) and therefore has made the object of an intensive study in the literature for auctions.

A first price auction model is a game of incomplete information. The main features of a game theoretic model of incomplete information are captured by the following model: players receive a signal (or a type) denoted by ξ, generated by a probability characterised by its cumulative distribution function F, or by its quantile function F^{-1}. Players know their type ξ and their distribution F. They play an action X, function of ξ and F:

$$X = \sigma_F(\xi). \tag{1.1}$$

The strategy σ is in general assumed to be strictly increasing with respect to ξ. We assume the statistician observes X and knows σ as a function of ξ and F. The types and their distribution are unknown and the objective of the econometric analysis is to derive an estimator and test on F and on ξ. The strategy σ usually derives from an assumption on the equilibrium of the game, for example σ may be the Nash equilibrium or an approximation.

From a statistical viewpoint, the main characteristic of this model is that it involves simultaneously the unobservable element ξ and its distribution. This is different from most of the econometric literature, where models contain an unobserved residual but the parameters do not depend on its distribution. From an economic perspective, we may make a distinction between the reduced form analysis (the distribution of X) and the structural analysis (estimation of F). The structural analysis allows to do counterfactual analysis. For example, if we change the rule of the game, this would change the strategy σ, but we could still run simulations based on the estimated same F.

In the case of symmetric first-price private values auction models, the Nash equilibrium of the game determines σ of the form [1]:

$$X = \sigma_F(\xi) = \xi - \frac{\int_{\underline{\xi}}^{\xi} F^N(u)du}{F^N(\xi)} \quad \text{with } N \geq 1 \text{ and } \xi \in [\underline{\xi}, \overline{\xi}] \subset \mathbb{R}, \tag{1.2}$$

and $N+1$ is the number of bidders (known by the player and the econometrician). The identification and the estimation of the independent private values first price auction model has already received a lot of attention from the econometricians and the break-

[1] In most applications this specification is too simple and should be extended. The distribution of ξ may depend on conditioning variables Z and the strategy may be function of conditioning variables W and of some unknown element θ. However, this paper will consider the simplest case, in order to focus on the possible resolutions of the functional equations generated by the model.

through in this subfield has been made by the paper of Guerre et al. (2000). By rearranging the terms of the first-order condition associated with the maximization problem of the bidders, the authors prove the nonparametric identification of the distribution of private values, without any kind of restrictions besides those imposed by the economic theory. Moreover the model remains identified even in the case where the econometrician observes only the transaction prices, which in this case correspond to the winning bids. Guerre et al. (2000) pioneered a series of papers based on an inversion of the σ function, which generates a relation $\xi = \lambda(X, G)$, where G is the cumulative distribution function (c.d.f. hereafter) of X (hereafter we will name their approach as the "GPV approach"). From the observations X and the estimation of G, one may derive the ξ's and then estimate F.

The aim of this paper is threefold. Firstly, we propose a new identification and estimation procedure using a functional quantile approach as developed in Enache and Florens (2020), Enache (2015), Enache and Florens (2014) and Enache and Florens (2018) (hereinafter we will name their approach as the "quantile approach"). Loosely speaking this methodology implies rewriting the model in terms of the relation between the quantile of private values and the quantile of bids. At the heart of this relation there is again the concept of Bayesian Nash Equilibrium that maps the types of the players into their actions, i.e. the bids. In the case of a first price auction model, writing the model under a quantile form leads to a linear inverse problem, therefore simplifying the analysis with respect to the cumulative distribution function approach mentioned previously. Moreover, under the quantile approach one does not need to invert the strategy function of the bidders, σ_F. Other papers that use also a quantile-based approach are Gimenes and Guerre (2014), Luo et al. (2015), Marmer and Shneyerov (2012).

Secondly, we develop new estimation procedures using a functional c.d.f. approach, extending the works of Protopopescu (1998), Florens et al. (1998) and Florens and Sbaï (2010). It is based on a functional relation linking F and G in the form of $G = T(F)$. This relation can be non linear and its inversion allows to derive an estimation of F from an estimation of G (hereinafter we will name this approach as the "c.d.f. approach"). As noted above, in the specific case of first-price auction model it is possible to linearize the problem. However, it may not be the case in general. Hence, in order to allow our procedures to be more general, it is important to study the implementation and properties of the direct nonlinear c.d.f. approach.

Thirdly, we aim at comparing the performances of quantile approach, GPV approach and c.d.f. approach.

The paper unfolds as follows: in section 2 we present our new methods of estimation of the first-price auction model, section 3 and 4 describe the c.d.f. approach and, respectively, the quantile approach in more details, section 5 compares our methods with the existing ones in the setting of a Monte Carlo experiment. Section 6 concludes.

2 Two approaches to analyze the model

In an independent private values first-price auction, the strategy function of the Bayesian Nash Equilibrium has been recalled in 1.2. For simplicity we assume that we observe several games with the same N and F and that we get an iid sample of data of size n: $x_1, x_2, \dots, x_n$.

Assumption 1.

1. *The support of ξ is a compact interval $[\underline{\xi}, \overline{\xi}] \subset \mathbb{R}$ and the values $\underline{\xi}$ and $\overline{\xi}$ are assumed to be known.*

2. *The true c.d.f. of ξ is denoted F_0 and is assumed to be to an element of $\mathscr{C}^q[\underline{\xi}, \overline{\xi}]$, $q \geq 1$. The density f_0 is assumed to be bounded from below on $[\underline{\xi}, \overline{\xi}]$. The true c.d.f. of X is denoted $G_0 = T(F_0)$.*

Under the assumption above, the strategy σ defined in 1.2 is strictly increasing.

2.1 The c.d.f. approach

We have the relation:

$$G(x) = F \circ \sigma_F^{-1}(x), \tag{2.1}$$

where $G(x) = \Pr(X \leq x)$ is the cumulative distribution of the data. This implicit relation comes from the observation that: $\Pr(X \leq x) = \Pr(\sigma_F(\xi) \leq x) = \Pr(\xi \leq \sigma_F^{-1}(x))$.

Equation (2.1) may be denoted $G = T(F)$, where T is an operator. It defines a non linear inverse problem: G is estimable and F should be estimated by solving this equation. The structural econometric model corresponding to the first-price auction model can be written in terms of the functional equation:

$$A(F, G) = G - T(F) = 0.$$

We face an inverse problem that is nonlinear and ill-posed, as we will see later.

2.2 The quantile approach

As mentioned in the previous section, the relation between the c.d.f. of the bids and the c.d.f. of the private values described in the equation 2.1 is a nonlinear one. This relation becomes linear (in the quantile functions of the private values and bids) if one inverts the equation (2.1) (as in Enache and Florens (2020), Enache and Florens (2014) and Enache and Florens (2018)):

$$G^{-1}(\alpha) = \sigma_F \circ F^{-1}(\alpha), \tag{2.2}$$

where $\alpha \in [0,1]$. After some manipulations (see appendix A.1)we get that:

$$G^{-1}(\alpha) = \frac{N}{\alpha^N} \int_0^1 \mathbb{1}_{[u \leq \alpha]} u^{N-1} F^{-1}(u) \, \mathrm{d}u$$

Let us denote by $r(\alpha) = \dfrac{\alpha^N}{N} G^{-1}(\alpha)$ and $\varphi(\alpha) = F^{-1}(\alpha)$ the two quantile functions (up to some weighting element). We then have the relation:

$$r = K\varphi, \tag{2.3}$$

where K is the linear operator:

$$(K\varphi)(\alpha) = \int_0^1 \mathbb{1}_{[u \leq \alpha]} u^{N-1} \varphi(u) \, \mathrm{d}u. \tag{2.4}$$

The structural econometric model corresponding to the first-price auction model can be written in terms of the functional equation:

$$A\left(\varphi, G^{-1}\right) = K\varphi - r = 0.$$

This equation generates a linear inverse problem. Let us now discuss the ill-posedness of the two-inverse problems presented in sections 2.1 and 2.2.

2.3 A mildly ill-posed inverse problem

Let $L^2[\underline{\xi}, \overline{\xi}]$ be the set of square real integrable functions on $[\underline{\xi}, \overline{\xi}]$ in $\mathbb{R}$, associated with the uniform measure. Our functional parameter set is a subspace $\mathscr{F}$ of $L^2[\underline{\xi}, \overline{\xi}]$ typically characterised by smoothness conditions. $\mathscr{F}_0$ is the subset of $\mathscr{F}$ of cumulative distribution functions ($F \in \mathscr{F}_0 \subset \mathscr{F}$ if $F(\underline{\xi}) = 0$, $F(\overline{\xi}) = 1$ and if F is non decreasing). Note that $\mathscr{F}_0$ is a closed convex subset of $\mathscr{F}$. We assume that F_0 is an element of $\mathscr{F}_0$.

The true quantile function F_0^{-1} is denoted φ_0 and is an element of the set of square integrable functions (w.r.t. the uniform measure) defined on $[0,1]$. Let us denote this set by $L^2_{[0,1]}$. The two spaces of $L^2_{[\underline{\xi}, \overline{\xi}]}$ and $L^2_{[0,1]}$ are provided with their canonical Hilbert space structure. We consider first the properties of the linear case derived from the analogies of our problem under the quantile approach.

The equation $r = K\varphi$ defines an inverse problem as one cannot directly observe φ, but has instead an indirect measurement of it, which is r. Moreover, even r is measured with some error. This problem is well-posed if the inverse operator K^{-1} exists and it is continuous, such that for a consistent estimation of r, one can recover a consistent estimation of φ. In other words, a problem is well posed in Hadamard sense if a solution exists, it is unique and it is stable, i.e. small errors in the measurement of r do not have a significant impact on the estimation of φ.

To illustrate the ill-posedness characteristic of our problem let us consider the case $N = 1$ (i.e. two bidders). In that case, K reduces to the integral operator:

$$r(\alpha) = \int_0^\alpha \varphi(u)\mathrm{d}u, \alpha \in [0,1].$$

This equation implies that r is differentiable and that K is not invertible without restrictions on the image space of K. The nature of ill-posedness may be illustrated by the following remark:

Remark 1. *In this particular case with two bidders, the singular value decomposition*[2] *implies that the eigenvectors are* $\varphi_j(\alpha) = \cos\left(\pi\left(j+\frac{1}{2}\right)\alpha\right)$ *and the singular values are* $\lambda_j = \dfrac{1}{\pi(j+\frac{1}{2})}$*, j=0,1,... The degree of ill-posedness of this problem with respect to the differential operator L on* $[0,1]$ *is 1 because* $K = L^{-1}$ *(see Engl et al. (2000)). This property is illustrated by the geometrical decline of order 1 of the* λ_j*.*

Below we generalize the remark made above. Let us denote $\psi(\alpha)$ the function G^{-1}. Under regularity conditions, $\psi(\alpha)$ may immediately be derived from equation (2.3). The solution would be

$$\varphi(\alpha) = \psi(\alpha) + \frac{\alpha}{N}\psi'(\alpha). \tag{2.5}$$

The inversion of K also requires that ψ is first order differentiable, which characterizes an order of ill-possedness of 1. A geometric decline of the spectrum of the operator or equivalently a finite order of ill-posedness characterize mildly ill-posed inverse problems.

Consider now the nonlinear equation generated by the model in the cdf form. The known operator T is defined on $\mathscr{F}$ and its image is $\mathscr{G} = T(\mathscr{F})$. The elements of $\mathscr{G}$ are real functions defined on an interval $[\underline{x}, \overline{x}]$, also associated with the uniform measure and we associate $\mathscr{G}$ with the L^2 topology.

As T is nonlinear, it will be approximated locally by a linear operator and we have the following lemma:

[2] Recall that φ_j and λ_j^2 are eigenvalues of K^*K ($K^*K\varphi_j = \lambda_j^2\varphi_j$), where K^* is the adjoint operator of K, studied in Section 4.

Lemma 1. *Under Assumption 1, T is Fréchet differentiable and its derivative is:*

$$dT_{F_0}(\tilde{F})(x) = \frac{F_0 \circ \sigma_{F_0}^{-1}(x)}{N \int_0^{\sigma_{F_0}^{-1}(x)} F_0^N(u)du} \int_0^{\sigma_{F_0}^{-1}(x)} F_0^N(u)\tilde{F}(u)du. \tag{2.6}$$

Proof. See B.1 in Appendix. □

In a nonlinear problem, ill-posedness may be considered globally or locally (see Engl et al. (2000) or Gagliardini and Scaillet (2012)). In this paper we only look to the local ill-posedness. The transform of $\tilde{F}$ by the derivative of T need to be first order differentiable and the resolution of 2.6 wrt $\tilde{F}$ requires a smoothness condition of the left hand side. For the same reason as in the linear case the local degree of ill-posedness is equal to 1 and the problem remains mildly ill-posed.

We are now faced with three possible strategies to estimate F or $\varphi = F^{-1}$. We can estimate G by the usual empirical c.d.f. and solve (2.1), we can estimate G^{-1} by the empirical quantile function and solve (2.2), or we can estimate $G^{-1} = \psi$ by a smooth estimator and estimate φ using formula (2.5). The first approach involves the resolution of a nonlinear ill-posed problem and will be solved by an iterative algorithm, which needs to be stopped at some step in order to regularise the inversion. The second method needs to select a regularized inverse of K depending on some regularisation parameter. In the last method we need to choose a smooth estimator of ψ, which requires for example a bandwidth selection. The ill-posedness of the problem appears in the three equivalent forms of the equation and in each case a regularisation parameter should be selected.

In the simulations we will compare these three methods and the original method presented in Guerre et al. (2000). Remember that, following preliminary results by Jean-Jacques Laffont, these authors construct the estimation of F in the following way. Equation (1.2) implies:

$$\xi = \frac{1}{N-1}\frac{G(x)}{g(x)}, \tag{2.7}$$

where g is the density of G. In a finite step they estimate non parametrically G and g and they reconstruct the $\xi' s$ from the relation above. Then they perform a nonparametric estimation of F and of its density, f.

3 The c.d.f. approach

3.1 A nonlinear inverse problem: iterative algorithm

In this section, we consider the resolution of the equation $G = T(F)$ by an iterative algorithm. Let us first compute the adjoint of the $\mathrm{d}T_{F_0}$:

Lemma 2. *The operator T is a linear bounded operator from $\mathscr{F}$ to $\mathscr{G}$ and its adjoint operator $dT^*_{F_0} : \mathscr{G} \longrightarrow \mathscr{F}$ verifies:*

$$dT^*_{F_0}(\tilde{H})(\xi) = F_0^N(\xi) \int_{\xi}^{\overline{\xi}} \frac{\tilde{H}(\sigma_{F_0}(u))}{F_0^N(u)} du. \tag{3.1}$$

Proof. See B.1 in Appendix. □

Remark 2. *This last adjoint operator is obtained from the L^2 topology on $\mathscr{G}$ even if $dT_{F_0}(\tilde{F})$ is differentiable. If $\tilde{\mathscr{G}}$ is the set of differentiable functions of $\mathscr{G}$, dT_{F_0} is included in $\tilde{\mathscr{G}}$. If $\tilde{\mathscr{G}}$ is associated to a Sobolev topology, the adjoint of dT_{F_0} is different. However, the empirical c.d.f. $\widehat{G}$ is not in $\tilde{\mathscr{G}}$ and this motivates our choice for the computation of $dT^*_{F_0}$.*

We know from Guerre et al. (2000) that the model is globally identified. Local identification, i.e. injectivity of dT_{F_0}, is verified in Florens and Sbaï (2010).
We also know from Protopopescu (1998), Florens et al. (1998) that first-price independent private value model is (mildly) ill-posed of order one.

The equation $G = T(F)$ has a regularised solution computed by a landweber iteration with an Hilbert Scale penalty (see Kaltenbacher et al. (2008)).
Let first consider these two integral operators from $\mathscr{F}$ to $\mathscr{F}$:

$$(M\psi)(t) = \int_{\underline{\xi}}^{t} \psi(u)du \tag{3.2}$$

$$\text{and } (M^*\lambda)(u) = \int_{u}^{\overline{\xi}} \lambda(t)dt. \tag{3.3}$$

Algorithm 1. *The Landweber algorithm is defined by:*

1. *$\widehat{F}_0$ is arbitrarily selected (e.g. $\widehat{F}_0$ is the c.d.f. of a uniform on $[\underline{\xi}, \overline{\xi}]$).*

2. $$\widehat{F}_k = \widehat{F}_{k-1} + wMM^*T'^*_{\widehat{F}_{k-1}} \left(\widehat{G} - T(\widehat{F}_{k-1})\right), \tag{3.4}$$

where w is a fixed number verifying:

$$w \left\| dT_{F_0} MM^* T'^*_{F_0} \right\| < 1. \tag{3.5}$$

The algorithm stops at step k_0. We discuss later on the choice of the stopping rule.

Remark 3. *Algorithm 1 may be motivated by the following argument. As F_0 is differentiable, we may rewrite the equation $G = T(F)$ by $G = T(Mf)$ where $f = F'$. Then the*

Fréchet derivative of TM is $dT_{F_0}M$ and a usual algorithm on f is

$$\widehat{f}_k = \widehat{f}_{k-1} + w(dT_{F_0}M)^* \left(\widehat{G} - T(M\widehat{f}_{k-1})\right). \tag{3.6}$$

Note that $(dT_{F_0}M)^* = M^* dT_{F_0}$. If we transform this relation by M we get formula (3.4).

Let us now discuss the empirical selection of the stopping rule.

3.2 Data driven stopping rule and relaxation parameter

In order for the Landweber iteration to converge, we must find the optimal number of iterations for the algorithm. If we let $k \to \infty$, the algorithm will diverge and we would be likely to observe over-fitting. We are using the stopping rule as given in Fève and Florens (2014), i.e. we select the value of k which minimizes the expression:

$$k||\widehat{G} - T(\widehat{F}_k)||. \tag{3.7}$$

Also, in order to converge, we need $\|dT_F\| \leq 1$ in the algorithm. If it is not the case, a relaxation parameter ω is applied such that:

$$\omega\|dT_F\| \leq 1. \tag{3.8}$$

3.3 Rate of convergence

We can first notice that the c.d.f. G of the observed actions is not available precisely, but the corresponding perturbed function is $G^\delta = \widehat{G}$ with:

$$||\widehat{G} - G||^2 = O(\delta). \tag{3.9}$$

Lemma 3. *We have equation (3.9). If we assume F be β times differentiable ($0 < \beta \leq 2$), then we have:*

$$\|\widehat{F} - F\|^2 = O(\delta^{\frac{\beta}{\beta+1}}). \tag{3.10}$$

Proof. See Appendix B.2. □

Note that with $\widehat{G}$ being the empirical c.d.f., then $\delta = \frac{1}{n}$, and F being twice differentiable w.r.t. ξ, then $\beta = 2$, we have $\|\widehat{F} - F\|^2 = O(n^{-\frac{2}{3}})$.

4 The quantile approach

4.1 A linear inverse problem

We have seen in the Section 2 that the quantile function of the bids, $G^{-1}(\alpha)$ (denoted $\psi(\alpha)$) is related to the quantile function $F^{-1}(\alpha)$ (denoted $\varphi(\alpha)$) by an integral equation:

$$n\alpha^N \psi(\alpha) = \int_0^1 \mathbb{1}_{[u\leq\alpha]} u^{N-1} \varphi(u)\, \mathrm{d}u \quad \text{or} \quad r = K\varphi. \tag{4.1}$$

This equation is a linear Fredholm equation of type I that has been extensively studied in the literature (see Engl et al. (2000) and Carrasco et al. (2007)). This equation may be analyzed from different view points.

Firstly, it may be solved and an easy computation shows that:

$$\varphi(\alpha) = \psi(\alpha) + \frac{\alpha}{N}\psi'(\alpha). \tag{4.2}$$

Indeed equation(4.1) implies the differentiability of ψ and ψ' is its derivative. Then, the estimation of φ by (4.2) requires a differentiable estimation of ψ.

Our quantile approach is similar to the one by Marmer and Shneyerov (2012) (see equation 3 in their paper), but there is an important distinction. Compared with Marmer and Shneyerov (2012) we focalize the "inversion", i.e. the transformation of the distribution of the bids into the distribution of valuations on the transformation between the two quantile functions. Recomputing the distribution F or its density becomes a question independent of the game model. In particular, we never need to estimate the density or the derivative of the density of the bids. If the object of intent is f, then we will simply use the following relation $f = (\varphi^{-1})' = \dfrac{1}{\varphi'(\varphi^{-1})}$.

Remark 4. *Equation (4.2) exhibits many interesting features: it shows that the deviation between the quantile of bids and the quantile of private values equals $\frac{\alpha}{N}\psi'(\alpha)$ and that this factor is decreasing with the number of participants, N. Moreover, one can see that $\underline{\xi} = \underline{x}$ and $\overline{\xi} = \overline{x} + \frac{1}{N}\psi'(1)$. This last equation may be used to estimate $\overline{\xi}$.*

Remark 5. *If the estimation of ψ is constrained to be in $\mathscr{C}^1[0,1]$, φ may be estimated by replacing ψ by this estimation. The regularization in that case is introduced by the constraint on the estimation of ψ (ψ has to be smooth). This analysis is very standard in nonparametric statistics and is in particular used for the estimation of the density. There is however an over identification question that appears. Even if ψ is a quantile function (or its estimator), φ characterized by (2.5) is not necessarily a quantile function. $\varphi(0) = \psi(0)$, but φ is not necessarily increasing. To obtain this property we need to estimate*

φ such that $\varphi'(\alpha)\left(1+\frac{1}{N}\right)+\frac{\alpha}{N}\psi''(\alpha)$ is nonnegative. In practice we omit generally this constraint and if φ is not increasing, we complete the estimation by constructing an increasing approximation of φ.

A second approach is to estimate φ by a regularized solution of (4.2), which does not need a smooth estimation of ψ. The more common regularizations is the Tikhonov aproach based on the minimisation of least squares penalized by an L^2 norm. Sequential methods may also be applied. The space of functions φ or ψ is $L^2[0,1]$ provided with the uniform measure and the operator K is now an operator from $L^2[0,1]$ into $L^2[0,1]$. This is an Hilbert Schmidt operator because (see Carrasco et al. (2007)):

$$\int_0^1\int_0^1 \left(1(u\leq\alpha)u^{N-1}\right)^2 \mathrm{d}u\,\mathrm{d}\alpha < \infty. \tag{4.3}$$

This operator is then bounded and compact. Its adjoint operator K^* ($L^2[0,1]\to L^2[0,1]$) is characterized by:

$$\int_0^1 (K\varphi)(\alpha)\lambda(\alpha)\,\mathrm{d}\alpha = \int_0^1 \varphi(\beta)(K^*\lambda)(\beta)\,\mathrm{d}\beta, \tag{4.4}$$

that verifies:

$$(K^*\lambda)(\beta) = \frac{\beta^{N-1}}{N}\int_\beta^1 \alpha^N\lambda(\alpha)\,\mathrm{d}\alpha. \tag{4.5}$$

Then:

$$\left(K^*K\varphi\right)(\beta) = \int_0^1 \alpha^{N-1}\beta^{N-1}(1-\max(\alpha,\beta))\varphi(\alpha)\,\mathrm{d}\alpha. \tag{4.6}$$

For more details about the above computations see Appendix C.1.

The Tikhonov regularized solution with an L^2 norm of the densities is defined by:

$$\begin{aligned}\varphi_\mu &= \arg\min\|r-K\varphi\|^2+\mu\left\|M^{-1}\varphi\right\|^2 \\ &= \left(\mu M^{*-1}M^{-1}+K^*K\right)^{-1}K^*r \\ &= M(\mu I+M^*K^*KM)^{-1}M^*K^*r,\end{aligned} \tag{4.7}$$

under the assumption that φ is at least twice differentiable.

This approach may be generalized by replacing M by M^s (with possibly $s=0$), if φ is sufficiently differentiable and defined on a Hilbert scale penalization approach (see Engl et al. (2000) chapter):

$$\varphi_\mu = M^s(\mu I + M^{s*}K^*KM^s)^{-1}M^{s*}K^*r. \tag{4.8}$$

An alternative regularized solution of (4.1) is given by the Landweber algorithm, starting with an arbitrary φ_0:

$$\varphi_{k+1} = \varphi_k - \omega MM^*K^*(r - K\varphi_k),\ k = 0, ..., k_0 \text{ and } \left\|\omega M^*K^*KM\right\| < 1. \tag{4.9}$$

The regularization is given by the stopping rule k_0.

4.2 Estimation of the quantile function φ

4.3 Estimation under smoothness constraint on ψ

If $(x_i)_{i=1,...n}$ is the iid sample of the bids, we denote by $(x_{in})_{i=1,...n}$ the order statistic and the empirical quantile function:

$$\widehat{\psi}(\alpha) = \sum_{i=1}^{n} x_{in}\mathbb{1}\left(\alpha \in \left(\frac{i-1}{n}, \frac{i}{n}\right]\right). \tag{4.10}$$

To implement the method derived from equation (4.1), we need a smooth version of ψ, denoted $\widetilde{\psi}$ equal to:

$$\begin{aligned} \widetilde{\psi}(\alpha) &= \sum_{i=1}^{n} x_{in} \int_{\frac{i-1}{n}}^{\frac{i}{n}} \frac{1}{h} c\left(\frac{\alpha - y}{h}\right) \mathrm{d}y \\ &= \sum_{i=1}^{n} x_{in} \left[\overline{C}\left(\frac{\alpha - \frac{i}{n}}{h}\right) - \overline{C}\left(\frac{\alpha - \frac{i-1}{n}}{h}\right)\right], \end{aligned} \tag{4.11}$$

where $\overline{C}$ is the cumulative distribution of a Nadaraya-Watson kernel c and h is a suitably chosen bandwidth. The estimator is then:

$$\widetilde{\varphi} = \widetilde{\psi}(\alpha) + \frac{\alpha}{N}\widetilde{\psi}'(\alpha). \tag{4.12}$$

We will give in the appendix C.2 the mean and the variance of this estimator where the regularization parameter is the bandwidth of the estimation (4.11). Under usual regularity condition on the kernel and if $\varphi \in \mathscr{C}^3[0,1]$, the rate of convergence of $\mathbb{E}\left(\widetilde{\varphi}-\varphi\right)^2$ is $O\left(\frac{1}{nh}+h^4\right)$ and under a suitable choice of h, the rate is of order $n^{-4/5}$ which is the usual rate of a density of a single variable (equivalent to Guerre et al. (2000)). If $\widetilde{\psi}$ is sufficiently smooth, φ', the quantile density, may be estimated by $\widetilde{\psi}'\left(1+\frac{1}{N}\right)+\frac{\alpha}{N}\widetilde{\psi}''(\alpha)$ which converges at the same rate as the derivative of a density ($n^{-4/5}$ for the square of the norm). From $\widetilde{\varphi}$ and $\widetilde{\varphi}'$ we may recover $F=\varphi^{-1}$ and $f=\frac{1}{\varphi'\circ\varphi}$ which converges at the same rate as φ and φ' (for the transformation of φ and φ' into F and f, see Enache and Florens (2020)).

4.4 Estimation without constraint on ψ

If ψ is not estimated under a differentiability constraint, we need to solve the equation $K\varphi=r$ by one of the regularisation method presented in the section 4.1: the Tikhonov solution ((4.7)) or the Landweber algorithm ((4.8)). In these two approaches, the estimation of ψ is given by (4.10) and the integrals M, M^*, K and K^* are approximated by Riemann sums. For example, the estimator of K^*r is equal to:

$$\left(\widehat{K^*r}\right)(u)=\frac{u^{N-1}}{N}\frac{1}{n}\sum_{i=1}^{n}\mathbb{1}\left(u\leq\frac{i}{n}\right)\left(\frac{i}{n}\right)^{N}x_{in}, \tag{4.13}$$

and $\widehat{\varphi}_\mu$ is estimated by (4.10) where K^*r is replaced by (4.13) and where the integrals M, M^*, K and K^* are approximated by Riemann sums.

Note that both in the Tikhonov and Landweber estimations, the introduction of M implies that $\widehat{\varphi}_\mu$ is differentiable and then φ', F and f may be derived from $\widehat{\varphi}_\mu$ as before.

The rate of convergence of this estimation is discussed in the Appendix C.3.

4.5 Choice of h, μ and k_0

Data driven methods for choosing μ has been developed in several papers. We may select μ which minimises

$$\frac{1}{\mu}\|\widehat{r}-K\widehat{\varphi_\mu^{(2)}}\|, \tag{4.14}$$

where $\widehat{\varphi_\mu^{(2)}}$ is the iterated Tikhonov estimator of order 2 defined by (see (Carrasco et al. (2007)):

$$\widehat{\varphi_\mu^{(2)}}=\widehat{\varphi_\mu}+\mu M(\alpha I+M^*K'KM)^{-1}M^{-1}\widehat{\varphi_\mu}. \tag{4.15}$$

In the iterative Landweber approach, we are using the stopping rule as given in Fève and Florens (2014): we select the value of k_0 which minimizes the expression

$$k\|\widehat{r}-K\widehat{\varphi}_k\|. \tag{4.16}$$

We study in Appendix C.3(see Enache (2015) and Enache and Florens (2014)) the properties of $\widehat{\varphi}_\mu$ (the properties of $\widehat{\varphi}_{k_0}$ would be obtained in a very similar way).

5 Monte Carlo Study

We are going to show simulations results for the First-Price independent private values auction model presented above. The true value for the private information is $F_0(s) = s^2$. We consider two cases for each method: $N = 5$ bidders and 20 rounds, which gives 100 observed bids, or $N = 3$ bidders and 33 rounds, which gives 99 observed bids. Our Monte Carlo experiment consists of 1000 replications. Solutions are evaluated over a grid of values equal to the number of observations. We run the algorithm using Matlab. Some results are shown below. Dashed lines correspond to 90% confidence intervals and the mean of all 1000 estimations. The true function is a plain line.

It should be noted that in our methods based on Tikhonov or Landweber regularisation, we do not impose additional smoothness constraints (by opposition to, for example, smoothness constraints imposed by the use of kernel methods).

We will first present graphical results for existing two-step GPV type methods and then our one-step methods based on inverse problem techniques. Finally, we will make a more precise comparison using Mean Squared Error (MSE) and bias, for the entire range of values or close to the upper boundary (last 10 %).

Figures 1 and 2 are the application of the two steps kernel based method in Guerre et al. (2000), with trimming. We use the Matlab code provided by Hickman and Hubbard (2015). We use trapezoidal numerical integration of the estimated p.d.f. to obtain an estimate of the c.d.f. We do not observe much difference between 3 and 5 players. This method is not supposed be effective, in particular because of sample trimming and boundary issues. Different corrections to GPV approach exist in the literature.

As expected, the results can be improved using the Boundary-Correction GPV estimator (BCGPV) proposed by Hickman and Hubbard (2015), without trimming and with boundary correction. It provides more satisfactory results, as illustrated in Figures 4 and 3. We do not observe much difference between 3 and 5 players. To a lesser extent, the boundary issue remains, but overall the fit appears satisfactory.

Another adaptation of GPV approach is Marmer and Shneyerov (2012), writing the problem in quantile form. They do not need explicit trimming. Also, they present methods to estimate the quantile function and the c.d.f.. Results are in Figures 5 to 8. It seems we gain precision going to 5 players, compared to 3 players (because the distribution of bids is closer to the distribution of private values), but we do not observe dramatic change. Similarly to BCGP estimator, overall their method seems to perform well, but the boundary issue remains. This is true for both quantile and c.d.f. estimators, even if their quantile approach seem to produce larger bias at the boundary than their c.d.f. estimator.

Figures 9 and 10 are the application of our kernel quantile estimation method, with boundary correction, which reduces the bias at the boundary of the kernel estimation of the quantile of bids. It behaves similarly to the quantile approach suggested in Marmer and Shneyerov (2012), also with some clear boundary issues. Even though this method is not based on GPV two-step approach, boundary issues remain as it is still kernel based.

Figures 11 and 11 correspond to the application of our Landweber c.d.f. estimation method presented in Algorithm 1. We do not impose any constraints. In particular, we do not use a kernel based approach and there are no smoothness or monotonicity constraints. As we do not have a Sobolev penalty, following our notation, it means that M is the identity matrix. In the case of 3 players, the method performs remarkably badly, in particular in terms of bias. When we have 5 players, the accuracy is somehow remarkable, given the total lack of constraints. In particular, we observe a clear improvement at the boundary compared to all previous kernel based approaches.

Figures 15 and 16 correspond to the application of the Tikhonov quantile estimation method presented in 4.13. We impose a Sobolev penalty. Contrary to the Landweber c.d.f. approach, the procedure is stable also with 3 players, even though we clearly gain precision with 5 players. This inverse problem also improves results at the boundary. However, we should note that we get aberrant numerical values when we arrive very close to the upper bound and, therefore, we dropped them.

Figures 13 and 14 are the application of the Landweber quantile estimation method, where we also impose a Sobolev penalty. It can be considered as the iterative counterpart of 4.13 and hence could hopefully provide smoother results. Indeed, we observe less variation in the results, even if the average bias does not seem to improve. In addition, we still observe an improvement at the boundary. We can notice that this approach does not produce much aberrant numerical values very close to the upper bound.

It is useful to complement our graphical interpretations with a report of Mean Integrated Squarred Errors (MISE) and average bias values. We report results for the full range of observations and for the last 10 % (i.e., close to the boundary). For each case, the best result is in red.

A general comment is that quantile based methods have good small sample properties and our method based on the equation (4.2) is certainly the easiest to be applied by the practitioners. The method does not require a preliminary estimation of the private values and the choice of the regularization parameter (the bandwidth) is well known and easily implemented.

Our estimation confirms that the original GPV method, which has been a seminal contribution to theoretical researchers can not be applied without some corrections as in the BCGPV extension.

The quantile Landweber estimation gives, by Monte Carlo simulations, small confidence intervals and has good performance at the upper boundary of the distribution of the private values. Due to the usual few number of observations, this part of the distribution is the most difficult to be estimated. We should also note that alternative approaches, (such as the quantile or c.d.f. estimation by Marmer and Shneyerov (2012)), have also good properties.

Even if the use of quantiles seems the more powerful tool for the estimation of the first-price auction models, the functional approach based on c.d.f. and using an iterative algorithm for the inversion has good properties. Moreover, it can be generalized to other games of incomplete information up to the difficult computation of the adjoint operator.

Finally, we want to recall that functional approaches do not require observation of bids but only estimation of the quantile (or the c.d.f.) of the bids. Functional estimation may be extended for example in the case where the bids are observed with error or, more generally when the estimation of the distribution of the bids comes from another model.

6 Conclusions

The first-price auction model has been intensively studied in the econometric literature. Probably one of the best well-known papers in this field is the one by Guerre et al. (2000) where the authors show the nonparametric identification of the distribution of the private values within the independent private values paradigm and provide a two-steps estimation method. In our paper, we treat the same problems of identification and estimation but while using a functional approach. The first-price auction gives rise to a complex nonlinear inverse problem. Although the information contained in the quantile functions is the same as the information contained in the cumulative distribution functions, the use of quantiles leads to a linearization of the problem. By contrast with previous approaches used in the literature of first-price auctions, within this framework, we are able to find a closed-form solution for the quantile of the private values. Therefore we obtain a constructive identification (see Matzkin (2013)) that allow for a one stage estimation procedure. The estimation is performed using three regularization methods: Tikhonov regularization, Landweber-Friedman regularization and estimation by kernels. We also show that it is also possible to implement a one stage estimation of the c.d.f. directly, for example using an iterative regularization. The Monte Carlo simulations show the performances of the functional approach, in particular a better performance of the quantile-based estimators. The functional estimation derived from the c.d.f. approach may be extended to other game theoretic models.

A Appendix section 2

A.1 Solution of the quantile equation

While the equation $G(x) = F \circ \sigma_F^{-1}(x)$ has no closed-form solution in F, the quantile version of it, i.e. $G^{-1}(\alpha) = \sigma_F \circ F^{-1}(\alpha)$ can be fully solved. After performing the composition one obtains:

$$G^{-1}(\alpha) = F^{-1}(\alpha) - \frac{\int_{\underline{\xi}}^{F^{-1}(\alpha)} F^N(u)\,\mathrm{d}u}{F^N(F^{-1}(\alpha))},$$

or, equivalently:

$$G^{-1}(\alpha) = F^{-1}(\alpha) - \frac{1}{\alpha^N} \int_{\underline{\xi}}^{F^{-1}(\alpha)} F^N(u)\,\mathrm{d}u.$$

After the change of variable $F(u) = v$ we obtain that:

$$G^{-1}(\alpha) = F^{-1}(\alpha) - \frac{1}{\alpha^N} \int_0^{\alpha} F^N(F^{-1}(v)) \frac{1}{f(F^{-1}(v))},\mathrm{d}v$$

or, equivalently:

$$G^{-1}(\alpha) = F^{-1}(\alpha) - \frac{1}{\alpha^N} \int_0^{\alpha} v^N F^{-1\prime}(v)\mathrm{d}v.$$

After integration by parts we get that:

$$G^{-1}(\alpha) = \frac{N}{\alpha^N} \int_0^{\alpha} v^{N-1} F^{-1}(v)\,\mathrm{d}v.$$

B Appendix section 3

B.1 Computation of derivatives

Let us first derive a general result for game theoretic inverse problems. This computation partly reproduces some results of Florens and Sbaï (2010). The computation of the adjoint is new.

Let us consider the operator $T(F) = F \circ \sigma_F^{-1} = F \circ \varphi_F$ with

$$\varphi_F = \sigma_F^{-1}.$$

We first compute the Gâteaux derivative of T in F_0 in the direction $\tilde{F}$ (i.e. $\frac{\delta}{\delta\alpha}T(F_0 + \alpha\tilde{F})|_{\alpha=0}$), which gives:

$$dT_{F_0}(\tilde{F}) = (f_0 \circ \varphi_{F_0}).d\varphi_{F_0}(\tilde{F}) + \tilde{F} \circ \varphi_{F_0}, \tag{B.1}$$

where $d\varphi_{F_0}(\tilde{F})$ is the derivative of φ as a function of F.
Moreover from the identity $\sigma_F(\varphi_(x)) = x$ we derive:

$$d\varphi_{F_0}(\tilde{F}) = -\frac{1}{\sigma'_F \circ \varphi_{F_0}} d\sigma_{F_0}(\tilde{F}), \tag{B.2}$$

where $\sigma'_F = \frac{\delta}{\delta\xi}\sigma_F(\xi, F_0)$.

The:

$$dT_{F_o}(\tilde{F}) = \tilde{F} \circ \varphi_{F_0} - \frac{f_0 \circ \varphi_{F_0}}{\sigma'_{F_0} \circ \varphi_{F_0}}.d\sigma_{F_0}(\tilde{F}). \tag{B.3}$$

This derivative computed using Gâteaux derivative is actually a Fréchet derivative, as shown by Florens and Sbaï (2010).

In the case of private values first price models, this formula becomes:

$$dT_{F_0}(\tilde{F})(x) = \frac{F_0 \circ \varphi_{F_0}(x)}{N \int_{\underline{\xi}}^{\varphi_{F_0}(x)} {F_0}^N(u)\,du} \int_{\underline{\xi}}^{\varphi_{F_0}(x)} {F_0}^N(u)\tilde{F}(u)\,du.$$

Under our assumptions, this linear operator from $\mathscr{F}$ to $\mathscr{G}$ is compact and then bounded.

At a more general level, we may compute the adjoint of $dT_{F_0}(\tilde{F})$ for $T(F) = F \circ \sigma_F$, where F is an element of L^2 on an interval provided with a density ρ. If $\tilde{H} \in \mathscr{G}$, we have by definition

$$\left\langle dT_{F_0}(\tilde{F}), \tilde{H} \right\rangle = \left\langle \tilde{F}, dT^*_{F_0}(\tilde{H}) \right\rangle.$$

Then:

$$\begin{aligned} &\int \left[\tilde{F} \circ \varphi_{F_0} - \frac{f_0 \circ \varphi_{F_0}}{\sigma'_{F_0} \circ \varphi_{F_0}}.d\sigma_{F_0}(\tilde{F}) \right](x).\tilde{H}(x)\rho(x)dx \\ = &\int \tilde{F}(\xi)\{\tilde{H}(\sigma_{F_0}(\xi))\frac{\rho(\sigma_{F_0}(\xi))\sigma'_{F_0}(\xi)}{\pi(\xi)}\}\pi(\xi)d\xi - \int d\sigma^*_{F_0}\{\frac{f_0 \circ \varphi_{F_0}}{\sigma'_{F_0} \circ \varphi_{F_0}}\tilde{H}\}\pi(\xi d\xi), \end{aligned}$$

where $d\sigma^*_{F_0} : \mathscr{G} \to \mathscr{F}$ is the adjoint of $d\sigma_{F_0}$. Then:

$$dT^*_{F_0}(\tilde{H})(\xi) = \tilde{H}(\sigma_{F_0}(\xi))\frac{\rho(\sigma_{F_0}(\xi))\sigma'_{F_0}(\xi)}{\pi(\xi)} - d\sigma^*_{F_0}\{\frac{f_0 \circ \varphi_{F_0}}{\sigma'_{F_0} \circ \varphi_{F_0}}\tilde{H}\}(\xi). \tag{B.4}$$

B.2 Proof of Lemma 3

Proof. Under the source condition $F \in \mathfrak{R}[\left(T_F^{'*}T_F^{'}\right)^{\frac{\beta}{2}}]$, our result is obtained by generalising Theorem 2.13 in Kaltenbacher et al. (2008) with $\beta = 2$. ($\beta = 2\mu$ in their Theorem 2.13, where they have at most $\mu = \frac{1}{2}$) We need now to check the condition. In the case of a game of incomplete information, we have seen that the Fréchet derivative of T in the direction $\tilde{F}$ has the form

$$dT_F(\tilde{F}) = a\int_{\xi}^{c} b\tilde{F}.$$

This implies that $F \in \mathfrak{R}[\left(T_F^{'*}T_F^{'}\right)^{\frac{\beta}{2}}]$ is equivalent to F being β times differentiable w.r.t. ξ. Hence our result. $\square$

C Section 4

C.1 Computation of the adjoint operator

The adjoint operator of K, denoted by K^*, is characterized by $\langle K\varphi, \lambda\rangle = \langle \varphi, K^*\lambda\rangle$ or equivalently:

$$\int_0^1 \lambda(\alpha)\,\mathrm{d}\alpha \int_0^1 \mathbb{1}_{[u\leq\alpha]} u^{N-1}\varphi(u)\,\mathrm{d}u = \int_0^1 \varphi(u)\,\mathrm{d}u \int_0^1 u^{N-1}\mathbb{1}_{[u\leq\alpha]}\lambda(\alpha)\,\mathrm{d}\alpha,$$

which amounts to:

$$(K^*\lambda)(u) = u^{N-1}\int_0^1 \mathbb{1}_{[u\leq\alpha]}\lambda(\alpha)\,\mathrm{d}\alpha.$$

This computation implies in particular that:

$$K^*K\varphi = v^{N-1}\int_0^1 u^{N-1}\left[\int_0^1 \mathbb{1}_{[v\leq\alpha]}\mathbb{1}_{[u\leq\alpha]}\mathrm{d}\alpha\right]\varphi(u)\mathrm{d}u = v^{N-1}\int_0^1 u^{N-1}(1-max(u,v))\varphi(u)\mathrm{d}u.$$

We also have that:

$$(K^*r)(u) = \frac{u^{N-1}}{N}\int_0^1 \mathbb{1}_{[u\leq\alpha]} r(\alpha)\alpha^N\,\mathrm{d}\alpha.$$

C.2 Properties of the estimator derived from a smooth estimation of the quantile function of the bids

For the estimation of ψ we follow the approach considered in several papers (see Jones (1992) for a survey and Enache and Florens (2014)). Other possible methods may be used, in particular the inversion of a smooth version of the empirical c.d.f.. We assume that ψ is a $\mathscr{C}^3$ function and then the quantile density, ψ' is twice differentiable. The bias of our estimation of φ verifies:

$$\mathbb{E}\left(\widehat{\varphi}(\alpha)-\varphi(\alpha)\right)=\mathbb{E}\left(\widehat{\psi}(\alpha)-\psi(\alpha)\right)+\frac{\alpha}{N}\mathbb{E}\left(\widehat{\psi}'(\alpha)-\psi'(\alpha)\right). \tag{C.1}$$

Following Jones (1992) this expression is approximated by:

$$\mathbb{E}\left(\widehat{\varphi}(\alpha)-\varphi(\alpha)\right)\simeq\frac{1}{2}h^2\sigma_c^2\left(\frac{\varphi''(\alpha)}{\varphi'^2}+\frac{\alpha}{N}\left(\frac{\varphi'(\alpha)\varphi'''(\alpha)-3\left(\varphi''(\alpha)\right)^2}{\varphi'^3(\alpha)}\right)\right), \tag{C.2}$$

where c is the kernel function and σ_c its variance. The variance term of the estimation is only driven by the estimation of of ψ' because the variance of ψ has a rate $\frac{1}{n}$ faster than the variance of ψ'. Then:

$$V\left(\widehat{\varphi}(\alpha)\right)\simeq\frac{1}{nh}\varphi'^2(\alpha)\frac{\alpha^2}{N^2}\int c^2. \tag{C.3}$$

As usual for nonparametric estimation a optimality rate for h is $n^{-\frac{1}{5}}$ (for a second order kernel) leading to an optimal rate of convergence of $n^{-\frac{4}{5}}$. Under integrability conditions this rate is also the rate of $\mathbb{E}\left\|\widehat{\varphi}_\mu-\varphi\right\|^2$.

C.3 Properties of the Tikhonov solution of the quantile function

We consider:

$$\widehat{\varphi}_\mu-\varphi=M\left(\mu I+M^*K^*KM\right)^{-1}M^*K^*(\widehat{r}-K\varphi)+M\left(\mu I+M^*K^*KM\right)^{-1}M^*K^*K\varphi-\varphi=A+B, \tag{C.4}$$

and we look at the behavior of these two terms.

The first term, A, converges to a Gaussian Process for a fixed μ. Indeed note first that:

$$\sqrt{n}\,(\widehat{r}-K\varphi)\Rightarrow\frac{\alpha^N}{g\circ G^{-1}(\alpha)}\text{ (Brownian Bridge)}. \tag{C.5}$$

The Brownian Bridge is the Gaussian process on $[0,1]$ with mean 0 and covariance function $\min(\alpha,\beta)-\alpha\beta$ (see Van der Vaart (1998)). This convergence is actually true on $[\varepsilon,1]$ for any $\varepsilon>0$, but the transformation by K^* regularizes the origin (see Enache and Florens (2020)). We denote by Σ the asymptotic variance of $\sqrt{n}(\widehat{r}-K\varphi)$ characterized by the covariance function:

$$\frac{\alpha^N\beta^N}{g\circ G^{-1}(\alpha)g\circ G^{-1}(\beta)}. \tag{C.6}$$

Then $\sqrt{n}A$ coverges to a Gaussian process on $[0,1]$ with 0 mean and variance Ω where:

$$\Omega = M\left(\mu I+M^*K^*KM\right)^{-1}M^*K^*\Sigma KM\left(\mu I+M^*K^*K\mu\right)^{-1}M^*. \tag{C.7}$$

The first term of $\mathbb{E}\left\|\widehat{\varphi}_\mu-\varphi\right\|^2$ is $\frac{1}{n}\operatorname{Tr}\Omega$. Let us introduce the following assumption:

Assumption 2. *Let $\left(\lambda_j^2,\varphi_j\right)_{j\geq 0}$ be the spectral decomposition of M^*K^*KM (where $M^*K^*KM\varphi_j=\lambda_j^2\varphi_j$, $\forall j$). We assume that $\exists \ \in[0,1]$ such that $\sum_j \frac{<\Sigma\varphi_j,\varphi_j>}{\lambda_j^{2\rho}}<\infty$.*

Then:

Lemma 4. $\operatorname{Tr}\Omega = O\left(\frac{1}{\mu^{1-\rho}}\right)$. *Proof:*

$$\begin{aligned}\operatorname{Tr}\Omega &= O\left(\sum\frac{\lambda_j^2<\Sigma\varphi_j,\varphi_j>}{\left(\mu+\lambda_j^2\right)^2}\right) && \text{(C.8)}\\ &= O\left(\sum\frac{\lambda_j^{2(1+\rho)}}{\left(\mu+\lambda_j^2\right)^2}\frac{<\Sigma\varphi_j,\varphi_j>}{\lambda_j^{2\rho}}\right). && \text{(C.9)}\end{aligned}$$

As $\frac{\lambda_j^{2(1+\rho)}}{\left(\mu+\lambda_j^2\right)^2}$ is of order $\frac{1}{\mu^{1-\rho}}$ we have the result.

Let us look now at the bias term B in (C.4). We assume:

Assumption 3. $\varphi\in\operatorname{Range}$ *of M and $\varphi=Mp$, with $p\in\operatorname{Range}\left(M^*K^*KM\right)^{-\frac{\beta}{2}}$.*

Then:

Lemma 5. $\|B\|^2=O\left(\mu^\beta\right)$, *if $\beta\leq 2$ and $\|B\|^2=O\left(\mu^2\right)$ if $\beta>2$.*

The proof is given in Carrasco et al. (2007) and it is very similar to the proof of the previous lemma after remarking that our assumption is equivalent to:

$$\sum_{\rho \geq 0} \frac{<\varphi_j, \varphi_j>^2}{\lambda_j^{2\varphi}} < \infty. \tag{C.10}$$

Then this lemma implies that

$$\mathbb{E}\left\|\widehat{\varphi}_\mu - \varphi\right\|^2 = O\left(\frac{1}{n\mu^{1-\rho}} + \mu^\beta\right). \tag{C.11}$$

The optimal rate for μ is $n^{-\frac{1}{1+\beta-\rho}}$ given a rate for $\mathbb{E}\left\|\widehat{\varphi}_\mu - \varphi\right\|^2$ proportional to $n^{-\frac{1}{1+\beta-\rho}}$. We will comment the previous assumption but note that if $\beta = 2$ and $\rho = \frac{1}{2}$, this rate is exactly $n^{-\frac{4}{5}}$, the rate obtained in the previous results.

Let us comment the assumptions. The assumption 3 means that φ is in the range of M which is equivalent to $\varphi(0) = 0$ and to φ differentiable. Moreover $p \in \text{Range}\,(M^*K^*KM)^{\frac{\beta}{2}}$ when $\beta = 2$ is equivalent to $p = M^*K^*\nu$ ($\nu \in L^2[0,1]$) which requires p twice differentiable (plus some boundary conditions). In particular, Assumption 2 requires $\varphi \in \mathscr{C}^3$.

Assumption 2 is difficult to verify. Consider a particular case where M is not introduced in the estimation (s=0 in the notation of (4.8)) and if $N = 2$ with ξ uniform on $[0,1]$. In this case, $x = \frac{1}{2}\xi$ and the bids are uniform $[0, \frac{1}{2}]$. Under these assumptions K is the integral operator of M and the spectral decomposition of K^*K is $\left(\lambda_j^2, \varphi_j\right)$ with $\lambda_j^2 = c\frac{1}{j^2}$ and $\varphi_j\left(\xi\right) = \cos(\frac{c\xi}{j})$. The operator Σ reduces to the Brownian Bridge and $< \sum \varphi_j, \varphi_j >$ is also proportional to $\frac{1}{j^2}$. Then if $\rho = \frac{1}{2} - \varepsilon$ for any small positive ε, Assumption 2 is verified and the rate of convergence of $\widehat{\varphi}_\mu$ is $n^{-\frac{4}{5-2\varepsilon}}$.

Remark 6. *Under usual regularity conditions, the rate of the estimation of φ by solving equation (4.2) is $n^{-\frac{4}{5}}$ and it is a minimax rate under the assumption that ψ is nonparametrically estimated as a smooth version of the empirical distribution. This rate is not verified if $G = \psi^{-1}$ is estimated differently. For example, the bids may be observed with error and G may be estimated by deconvolution of the distribution of the observed bids. The rate obtained in Lemma 3 is more general in the sense that it can be applied for any estimation of G. In the case where G is estimated by the empirical c.d.f., our rate $n^{-\frac{2}{3}}$ is only an upper bound and may be improved using stronger assumptions. The estimation by equation (4.8) (or (4.9)), equivalently leads to identical rate as the nonlinear case if $\rho = 0$.*

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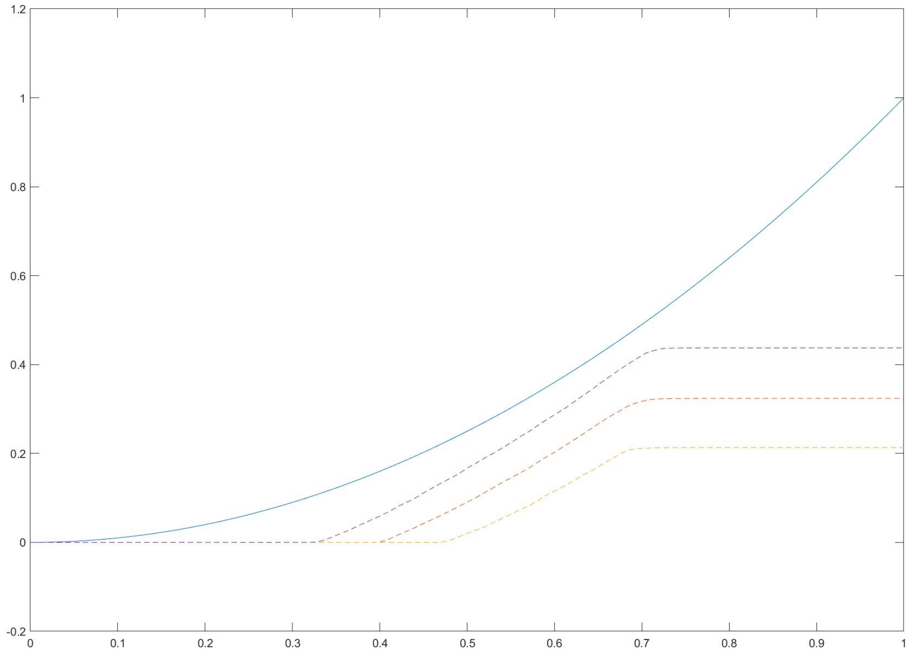

Figure 1: CDF after Guerre-Perrigne-Vuong, Silvermann rule. N=3, L=33

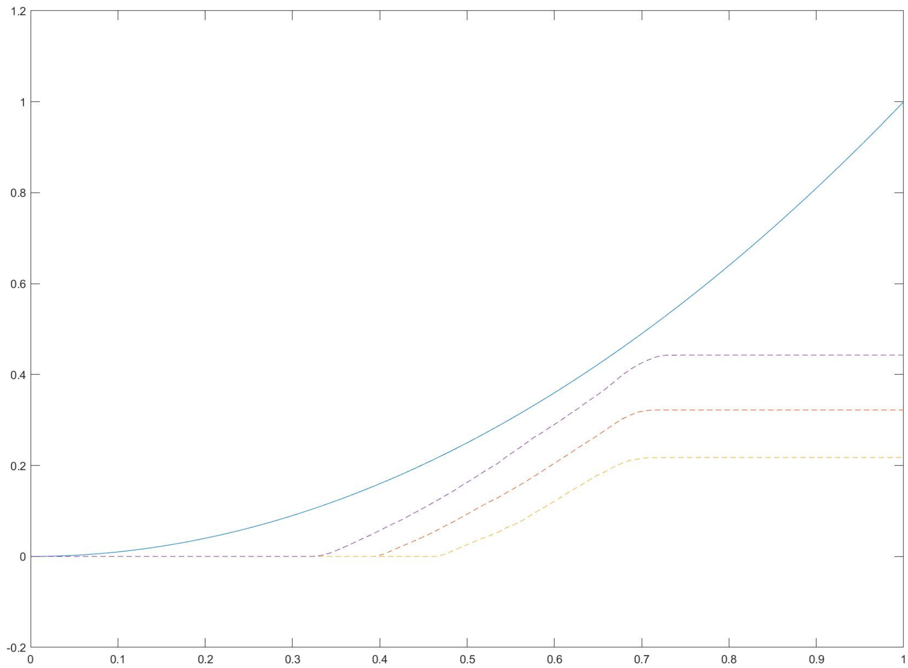

Figure 2: CDF after Guerre-Perrigne-Vuong, Silvermann rule. N=5, L=20

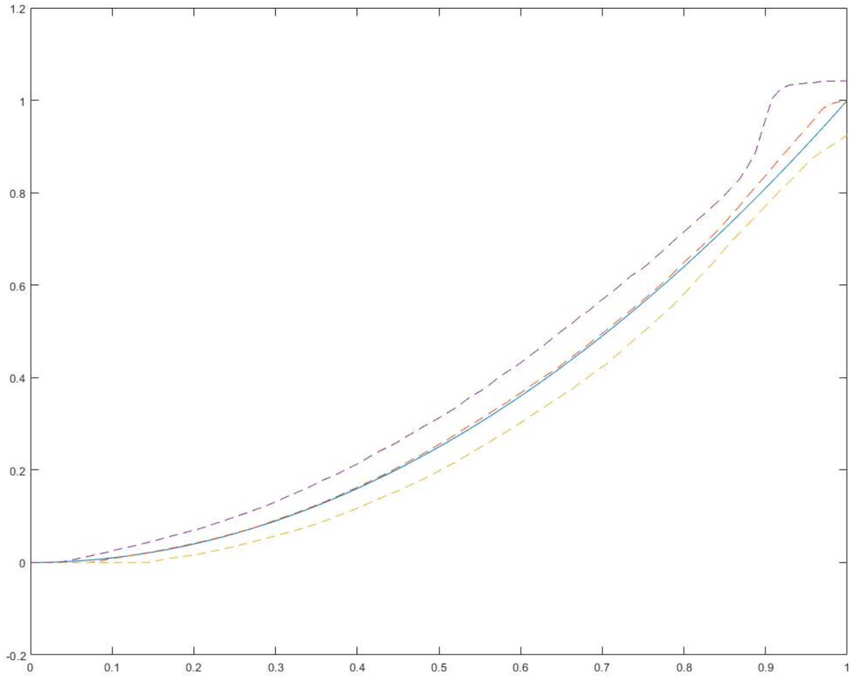

Figure 3: CDF after Hickman-Hubbard with boundary correction. N=5, L=20

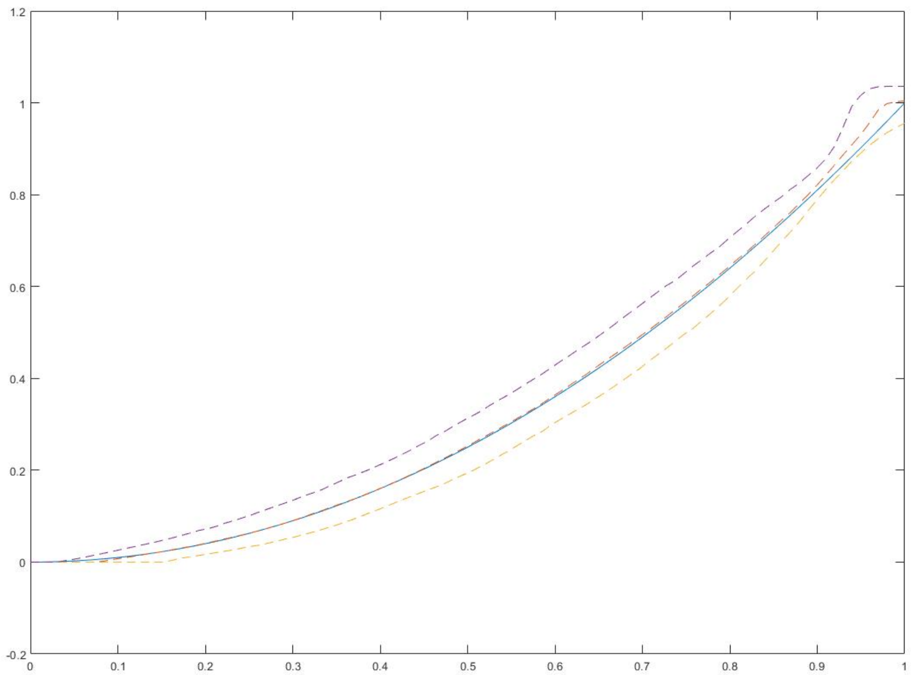

Figure 4: CDF after Hickman-Hubbard with boundary correction. N=3, L=33

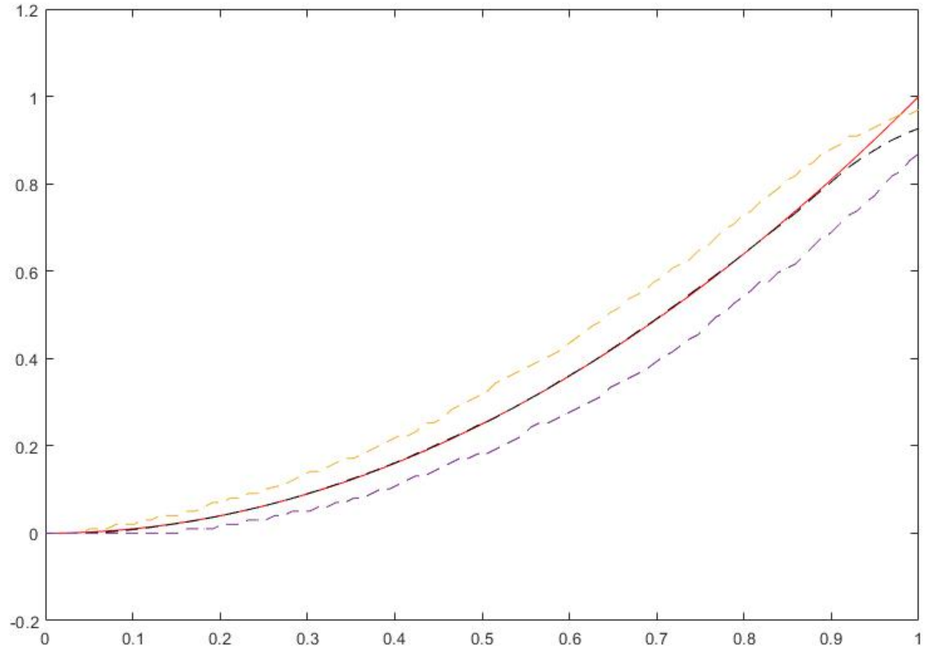

Figure 5: CDF Marmer-Shneyerov. N=3, L=33

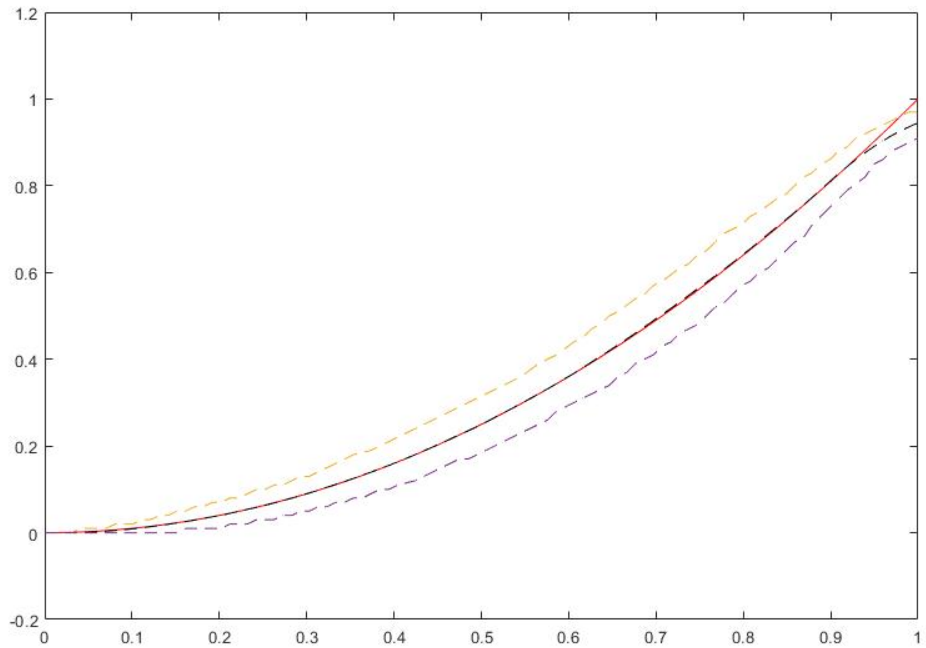

Figure 6: CDF Marmer-Shneyerov. N=5, L=20

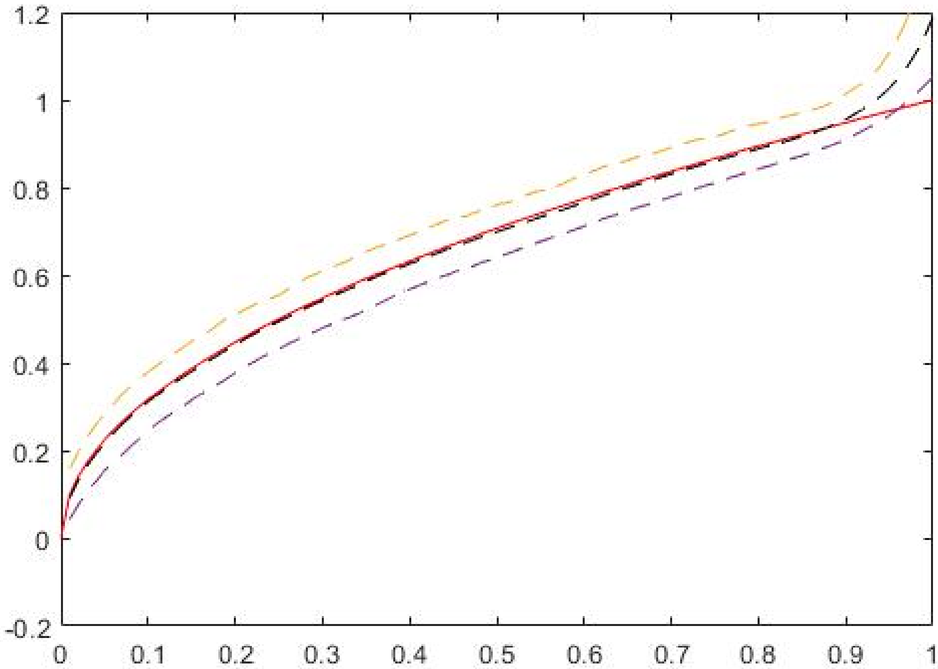

Figure 7: Quantile Marmer-Shneyerov. N=3, L=33

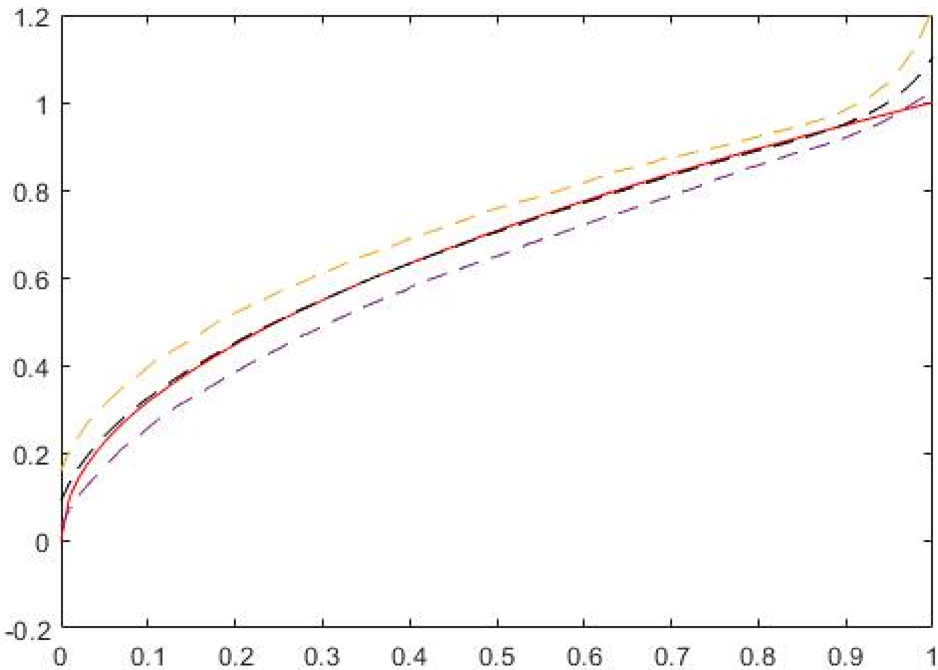

Figure 8: Quantile Marmer-Shneyerov. N=5, L=20

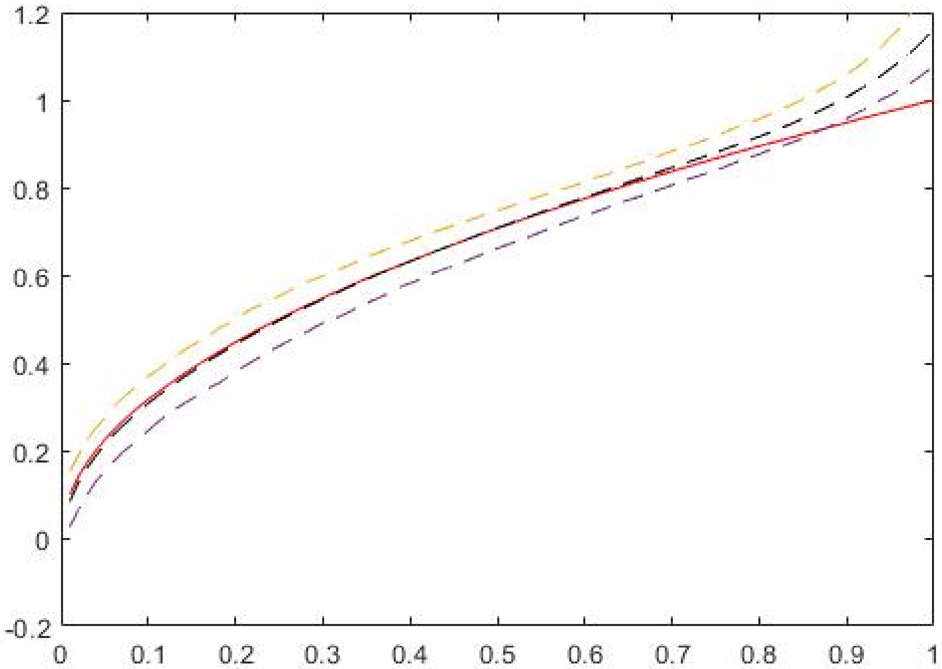

Figure 9: Quantile kernel with boundary correction. N=3, L=33

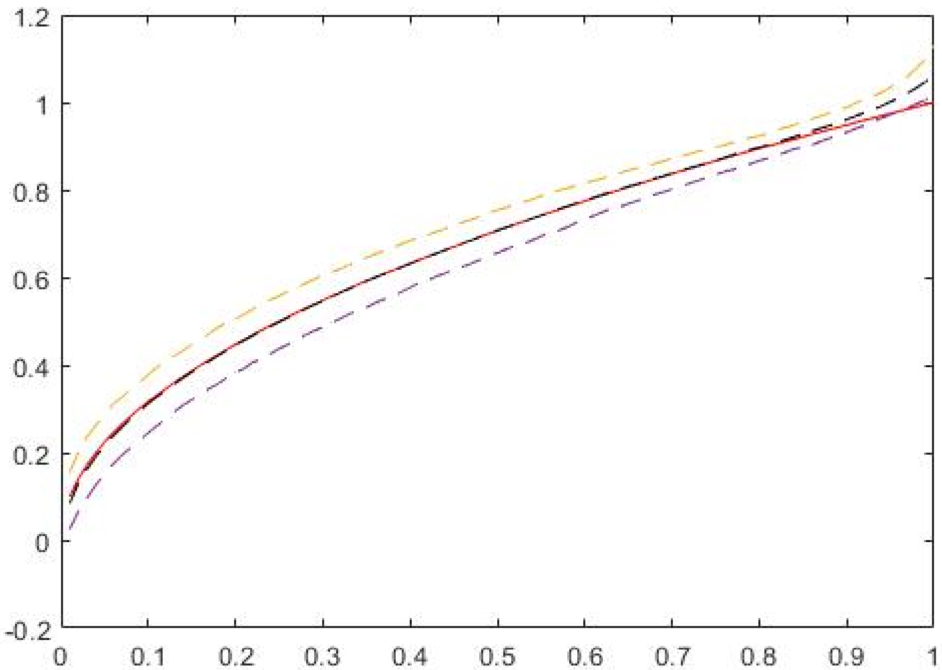

Figure 10: Quantile kernel with boundary correction. N=5, L=20

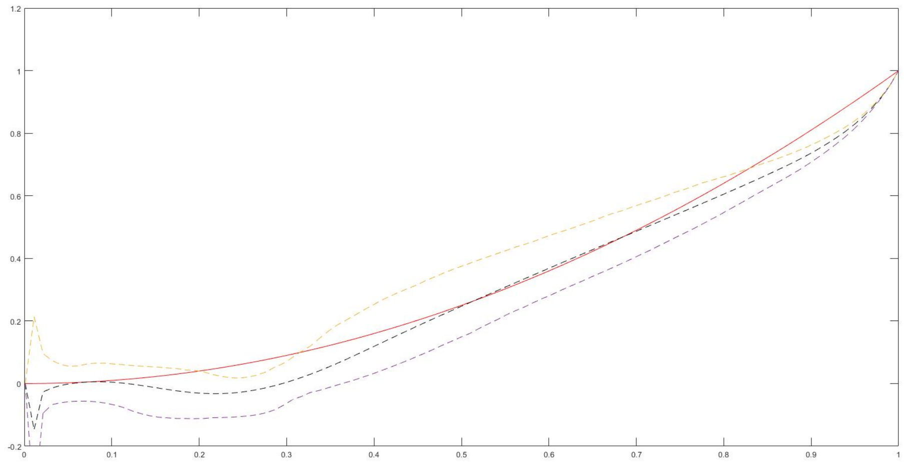

Figure 11: CDF estimation using Landweber. N=3, L=33

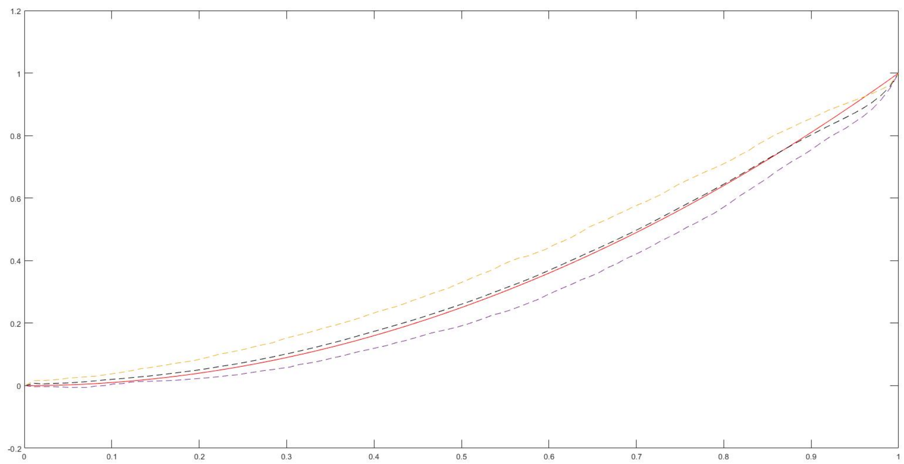

Figure 12: CDF estimation using Landweber. N=5, L=20

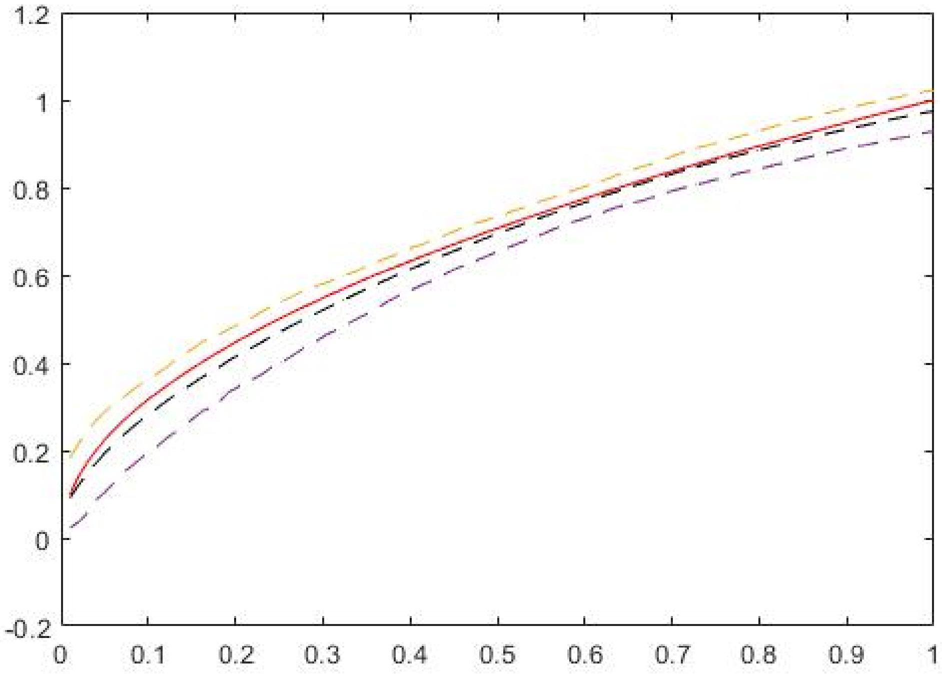

Figure 13: Quantile Landweber. N=3, L=33

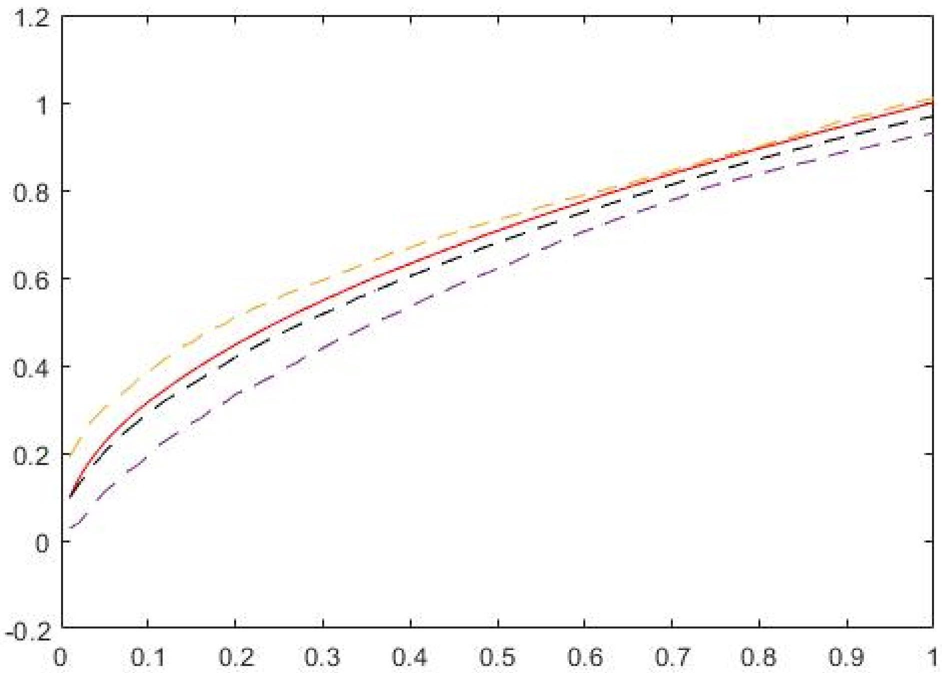

Figure 14: Quantile Landweber. N=5, L=20

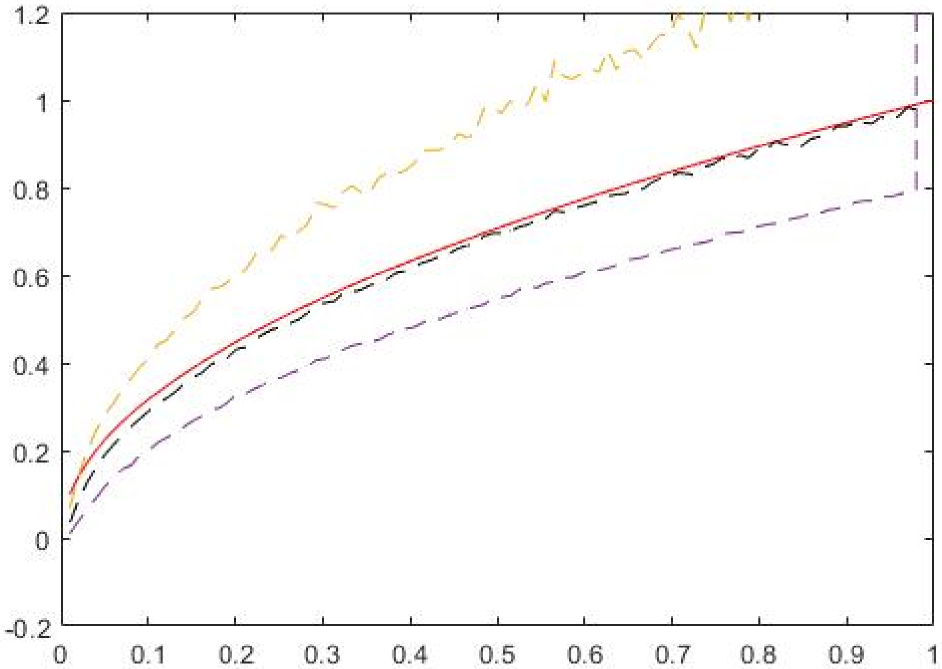

Figure 15: Quantile Tikhonov. N=3, L=33

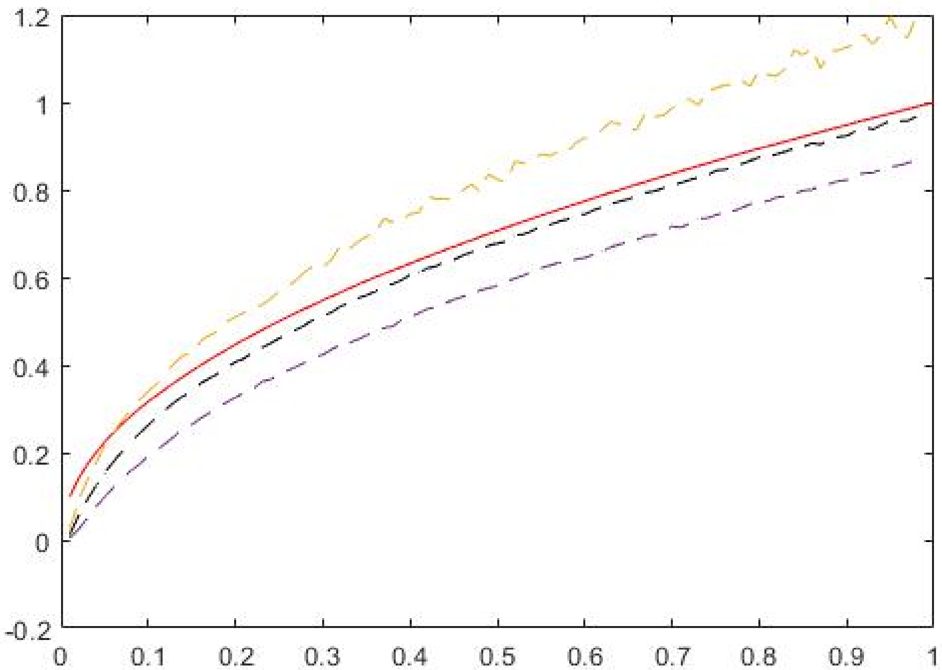

Figure 16: Quantile Tikhonov. N=5, L=20

Table 1: MISE and Bias Comparison

	MISE values last 10%		mean bias last 10%		MISE full		mean bias full	
	NP = 3, L =33	NP=5,L=20	NP = 3, L =33	NP=5,L=20	NP = 3, L =33	NP=5,L=20	NP = 3, L =33	NP=5,L=20
GPV_SH	0.3429	0.3416	-0.5784	-0.5774	0.0693	0.0686	-0.1918	-0.1909
BCGPV	0.0420	0.0018	0.0324	0.0252	0.0014	0.0010	0.0078	0.0052
CDF MS	0.0380	0.0015	-0.0279	-0.0175	0.0019	0.0012	-0.0032	-0.0014
Quantile MS	0.0116	0.0035	0.0695	0.0366	0.0026	0.0014	0.0022	-0.0009
CDF Landweber	0.0042	0.0010	-0.0593	-0.0210	0.0054	0.0013	-0.0350	0.0056
Quantile	0.0131	0.0017	0.1009	0.0305	0.0024	0.0011	0.0145	0.0030
Quantile Landweber	0.0012	0.0008	0.0050	-0.0036	0.1477	0.0265	0.0280	0.1400
Quantile Tikhonov	0.0362	0.0110	0.0241	0.0126	0.0476	0.0295	0.1521	0.1360