Quantum gravity and space-time foam

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Zusammenfassung

Diese Doktorarbeit besteht aus zwei Teilen. Teil 1 umfasst Kapitel 1 bis 3 und besteht aus einer kurzen Einleitung und einer Einführung in die Quantenfeldtheorie auf gekrümmten Räumen und in die Quantengravitation. Diese Teile haben lediglich einführenden Charakter. Kapitel 4 und 5 enthalten neue Resultate.

Das euklidische Pfadintegral der Quantengravitation $\int \mathcal{D}g_{\mu\nu}e^{-I}$ beinhaltet eine Summation über alle Riemannschen Metriken $g_{\mu\nu}$. Mit den klassischen Einstein'schen Feldgleichungen gilt $R = 4\Lambda$, wobei R der Ricci Skalar und Λ die kosmologische Konstante ist. Die euklidische Wirkung des Gravitationsfeldes wird zu $I = -\frac{1}{16\pi}\int d^4x\,(R - 2\Lambda) = -\frac{\Lambda V}{8\pi}$. Der Krümmungsskalar berechnet sich aus der Metrik. Verschiedene Metriken die die Einstein'schen Feldgleichungen erfüllen, ergeben dann verschiedene numerische Werte für R. Wird die Pfadintegration auf klassische Metriken beschränkt ist sie damit äquivalent zu einer Integration über Λ. Damit wird Λ zu einem variablen Feld über das (mit zu bestimmenden Gewichtungsfaktoren) summiert werden muss.

Aus dem euklidischen Pfadintegral der Gravitation kann man die Entropie des Gravitationsfelds berechnen. Der klassisch beobachtete Wert für Λ im thermodynamischen Gleichgewicht sollte dann durch einen Wert Λ_s gegeben sein, bei dem die Entropie maximal wird.

Das so genannte ßpace-time foamModell von Hawking beinhaltet eine Berechnung des Entropiemaximums aus der Amplitude der euklidischen Quantengravitation. In der vorliegenden Doktorarbeit wird dieses Modell in Kapitel 3 modifiziert und erweitert. Eine Näherung von Hawking, welche nur für $\Lambda < 0$ funktioniert, wird weggelassen und die Renormierungsskala wird auf den in der heutigen Literatur üblichen Wert gesetzt. Zudem werden Materieterme hinzugefügt.

Nach dem Materieterme hinzugefügt wurden erzeugt das modifizierte space-time foam Modell bei niedrigen Ordnungen der Störungstheorie eine kosmologische Konstante $\Lambda \geq 0$ mit einem Wert der nah an den Messergebnissen ist.

Es ist bekannt dass die effektive Wirkung von Materieamplituden in höheren Ordnungen der Störungstheorie topologische Beiträge liefert welche einer $f(R)$

Gravitation mit höheren Ableitungen entsprechen. Teile dieser Beiträge sind proportional zur Euler Characteristik. Diese Beiträge werden teilweise als Resultate der Erzeugung von Wurmlöchern oder schwarzen Löchern interpretiert. Diese Beiträge ändern die kosmologische Konstante im Space-time foam Modell nur leicht.

Der Rest der topologischen Terme der effektiven Materiewirkung ist von einer Form welche die Ostrogradski Instabilität beinhaltet. Das heisst diese Terme beschreiben ohne weitere Modifikation die propagation von negativer Energie. Es wird gezeigt dass die Ostrogradski Instabilität für bestimmte Masseverhältnisse von Bosonen und Fermionen nicht mehr im System vorhanden ist. Die effektive Wirkung entspricht dann dem Starobinski Modell einer $R + R^2$ Gravitation aus der sich die Inflationstheorie ableiten lässt.

Hawking's Amplitude besteht aus einer verschachtelten Pfadintegration. Mittels Störungsentwicklung wird eine one-loop Amplitude ausgehend von einer beliebigen klassischen Raumzeit berechnet. Dann wird diese Amplitude über alle klassischen Raumzeiten nicht-perturbativ integriert. Es wird dargelegt dass der nichtperturbative Teil der Amplitude nur für bestimmte Verhältnisse des Hubbleparameters und Λ konvergiert.

Das Modell ergibt insgesamt eine Raumzeit die aus einem Gas aus Wurmlöchern besteht. Dadurch wird das no-go Theorem von Weinberg umgangen welches eine translationsinvariante Raumzeit als Voraussetzung verlangt.

Ohne Randterme verschwindet der Hamiltonian der Gravitation. Dies erzwingt über die Schrödingergleichung der Quantengravitation dass es keine Dynamik für die Quantenmechanischen Observablen mehr gibt. Es wird angeführt dass das modifizierte space-time foam Modell dieses Problem mittels Randtermen löst.

Die Bewegungsgleichungen von Teilchen welche in ein Gas aus sehr vielen mikroskopisch kleinen schwarzen Löchern oder Wurmlöchern eingebettet sind können sich von Bewegungsgleichungen denen Teilchen in einem klassischen Minkowskiraum unterliegen unterscheiden. Damit beschäftigt sich Kapitel 4 der vorliegenden Doktorarbeit. Teilchen in einem Gas aus zahlreichen mikroskopischen schwarzen Löchern sind Hawkingstrahlung hoher Temperatur ausgesetzt. Dieses Photonengas interagiert mit eingebetteten Teilchen unter anderem durch Streuung und den Comptoneffekt. Ferner hat das Gravitationsfeld der schwarzen Löcher oder der Wurmlöcher auf großen Zeitskalen und bei großen Abständen einen kollektiven Einfluss auf die Teilchen in Form eines Stokes Gesetzes. Setzt man eine Schar klassischer Teilchen niedriger Energie in eine solche Raumzeit und nimmt man an dass die Entropie der Schar durch die Interaktion mit den Wurmlöchern nicht verringert wird, ergibt sich dass dass diese Teilchen durch eine Schrödingergleichung

beschrieben werden.

Für Teilchen hoher Energie wird ein Modell von ’t Hooft herangezogen. In diesem Modell wird aus der Streuung eines Teilchens an einem schwarzen Loch die Wick rotierte Amplitude der Stringtheorie hergeleitet.

Es wird dargelegt dass dieses Modell praktisch ausschließlich für die Streuung an relativ kleinen schwarzen Löchern geeignet ist und zur Beschreibung der Streuung von Teilchen an großen derartigen Objekten nicht nutzbar ist.

Es wird hergeleitet, dass im space-time foam Modell durch die Expansion des Universums Änderungen der Topologie erzeugt werden sollten welche die Euler-Charakteristik ändern. Ein Theorem von Geroch impliziert dass daraus die Entwicklung von Singularitäten in der Raumzeit folgt.

DeWitt zeigte dass die Quantenfeldtheorie in gekrümmten Räumen und die Quantengravitation bei singulären Raumzeiten inkonsistent werden. In dieser Doktorarbeit werden die mathematischen Details dieser Probleme dargestellt und weiter entwickelt. Diese Diskussion zeigt dass die Inkonsistenzen bei Topologieänderungen generell für Quantenfeldtheorien gelten (dazu zählt auch die Stringtheorie bei der man annehmen muss dass eine singuläre Hintergrundraumzeit zuerst desingularisiert wird bevor diese Theorie überall auf einer Raumzeit definiert werden kann).

Daraus wird der Schluss gezogen dass die Quantenmechanik durch eine Theorie ersetzt werden sollte die eine mathematische Modellierung beinhaltet welche mit Singularitäten verträglich ist.

Die Arbeit bemerkt dass das stochastische Modell aus dem die Schrödingergleichung hergeleitet wurde nur auf grossen Längenskalen Differentialgleichungen mit glatten Lösungen beschreibt. Auf kleinen Längenskalen nutzt das Modell nicht differenzierbare Trajektorien.

Die Arbeit schliesst mit dem Vorschlag, dass ein ähnliches Modell für die Quantengravitation entwickelt werden sollte. Damit könnte man in der Nähe der Singularität während eines Topologieübergangs die nicht-differenzierbare Beschreibung nutzen. Man hätte dann eine Theorie mit der sich Topologieänderungen konsistent beschreiben lassen.

Abstract

This thesis consists of two parts. Part one includes chapters 1-3 and consists of a short introduction into Quantum field theory and Quantum gravity. Chapters 4 and 5 contain new results.

The euclidean path integral of gravity $\int \mathcal{D}g_{\mu\nu}e^{-I}$ can be regarded as a weighed sum over all Riemannian metrics $g_{\mu\nu}$. Einstein's classical field equations are given by $R = 4\Lambda$, where R is the curvature scalar which can be computed from the metric and Λ is the cosmological constant. With the Euclidean action $I = \frac{-1}{16\pi}\int d^4x(R - 2\Lambda)$, different metrics lead to different numerical values of R and thus of Λ. If path integral is restricted to a sum over classical metrics, it becomes an integral over Λ with some weighting factors.

The entropy of the gravitational field can be computed from the path integral of Euclidean quantum gravity. The observed value of Λ at thermodynamical equilibrium should then be given by a state of maximum entropy.

The so-called space-time foam model of Hawking contains a computation of the gravitational entropy. In this thesis, this model is modified and extended.

An approximation of Hawking which only works for $\Lambda < 0$ is removed, and for the renormalization scale, a value is assumed which is nowadays used in the modern literature. Finally, matter terms are added to the model.

We show that the modified space-time foam model predicts a cosmological constant $\Lambda \geq 0$ whose value is close to the observed magnitude.

For higher orders of perturbation theory of the matter amplitudes, it is known that one gets an effective matter action with topological terms in the form of an $f(R)$ gravity with derivative terms of higher orders. Some of these terms are proportional to the Euler characteristic. We argue why these terms should be interpreted as resulting from black- or worm hole creation.

The rest of the higher order terms of the effective matter amplitude are such that they give rise to the so-called Ostrogradski instability. Without further modification, the effective matter amplitude therefore describes the propagation of negative energies. We show that for a certain relationship between the masses of fermions and bosons, the Ostrogradski instability can be removed and the Starobinski model $R + R^2$ is obtained, from which the theory of inflation can be derived.

Hawking's article describes an amplitude that is given by a nested path integration. Using perturbative methods, a one loop amplitude is computed with respect to an arbitrary classical background. Then one integrates this amplitude non-perturbatively over all classical backgrounds. We argue that the non-perturbative integral does only converge for a certain relationship between the Hubble constant and Λ which is consistent with observations.

In general, the model yields a space-time which can be described as a gas of worm holes. Thereby it circumvents Weinberg's theorem on the cosmological constant, which assumes a translationally invariant space-time.

Without boundary terms, the Hamiltonian of gravity is zero, which leads to stationary observables. It is argued that the modified space-time foam model solves this problem of time. One has to add boundary terms of the wormholes to the Hamiltonian, which then becomes non-vanishing.

Particles inserted in a space-time that consists of a gas of microscopic black- or wormholes may exhibit a different behaviour than those embedded in vacuum. The implications of this are discussed in chapter 4. Particles in a gas of gravitational instantons with an event horizon are under the influence of a heath bath with high temperature from Hawking radiation and they are under the influence of the collective gravitational field of the worm holes. The former interacts with the particles, for example trough Compton scattering. The latter has, on long time scales and large distances, a collective effect on the particles in form of a Stokes law. It is argued that if one inserts a set of non relativistic and classical particles into such a space-time, and if one assumes that the particles do not loose entropy to the black-hole gas on the average, then these particles are governed by Schrödinger's equation.

In order to describe particles of high energy, a model of 't Hooft is used where a particle scatters with a Schwarzschild black-hole in a Kruskal space-time. 't Hooft shows that this yields an S-matrix that is equal to a Wick rotated amplitude from string theory. It is argued in this thesis that this model does not to hold for large astronomical black-holes but that it works for small black-holes.

We point out that the space-time foam model implies topological changes during the expansion of the universe in which the Euler characteristic should change.

According to a theorem of Geroch, this leads to the emergence of singularities. DeWitt has argued that quantum field theory in curved space-times and quantum gravity become inconsistent when singularities appear. In this thesis, we give additional details why these inconsistencies emerge. The arguments show that these problems exist for quantum field theories in general and also hold for string theory, where a desingularization of the background space-time has to be assumed if the theory should be defined consistently on the entire background. Unfortunately, by Geroch's theorem, such a desingularization is incompatible with the space-time undergoing a topological change. We therefore conclude that we have to replace quantum mechanics by different mathematical tools.

We mention that the stochastic model from which we derived the single particle Schroedinger equation in the thesis is smooth only on the average and on a large scale. The model uses non-differentiable trajectories on small scales. We close the thesis with the open suggestion that such an underlying model for quantum mechanics may be developed for quantum gravity. The theory could then be used in the vicinity of a singularity during a topology change.

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Part I

Introduction

The first part of this thesis begins in chapter 1 with an introduction to quantum field theory in curved space-times. Then chapter 2 follows with an introduction to quantum gravity. This part of the thesis is just of introductory value.

Chapter 1 begins with a review of the definition of the functional integral in mathematically rigorous terms, which is based on [1]. This discussion is extended to a definition of the path integral and quantum field theory in curved space-times. Amplitudes in curved space-times are computed with zeta function renormalization. It is observed that in curved space-times, quantum field theory implies that all amplitudes carry an undetermined renormalization scale parameter that must be fixed by measurement. One finds that with a reasonable guess for this scale, one gets a large cosmological constant from matter amplitudes.

The effective action of matter fields in curved space-times contain terms with higher derivatives. We review the theorem of Ostrogradski[2] in section 1.5, which says that these theories are usually unphysical because they describe the propagation of negative energy forward in time. We note that one can avoid this instability by adding an R^2 term only.

Weinberg's theorem is reviewed in section 1.6, which shows that one can not have a small and non-vanishing cosmological constant together with a translationally invariant metric.

In chapter 2, the covariant and canonical quantisation methods for gravity are reviewed. Amplitudes for quantum gravity in the covariant formalism are regularized with zeta functions. The canonical approach that is based on the Wheeler-DeWitt [3] equation is also reviewed. Quantum gravity is usually assumed to have a vanishing Hamiltonian constraint. Since the Wheeler-DeWitt equation is essentially similar to a Schrödinger equation, one can not have time dependent observables in quantum gravity. This so-called the problem of time in quantum gravity is reviewed shortly.

Arguments from Hartle and Hawking are discussed that show the gravitational path integral to be an approximate solution of the Wheeler-DeWitt equation [4]. These results imply that the problem of time is also there in the covariant formalism of quantum gravity.

We then review various aspects of the Euclidean version of quantum gravity that can be used to define partition functions. We mention that the Euclidean action of gravity is usually assumed not to be bounded from below, which leads to diverging amplitudes. We note that Dasgupta has shown recently in [5, 6] that if one adds ghost and gauge fixing terms, the effective Euclidean action of gravity becomes bounded from below.

As an example of how to use the quantum theory of gravity, the derivation of the black-hole entropy with the path integral from [7] is reviewed.

In subsection 3.3, we review Hawking's proof from [7] that space-time curvature can not cause the Euclideanized gravitational action to contribute to the entropy unless the curvature is strong enough to imply the presence of a boundary in the space-time.

The second part of this thesis consists of chapters 4 and 5 where new results are obtained. They have been partly published by the author as original paper in [8] during his work for this phd thesis.

In section 4.2, a discussion of the space-time foam model of Hawking [9] begins. Several problems of the model are pointed out and it is discussed how the theory has to be modified.

Hawking assumes that one integrates the Euclidean path integral $\int \mathcal{D}g_{\mu\nu}e^{-I}$ non-perturbatively over all classical space-times. These space-times fulfil Einstein's equation $R = 4\Lambda$. Since for classical gravitational fields, the Einstein-Hilbert action is given by $I = -\frac{\Lambda V}{8\pi}$, integrating the path integral over all metrics which fulfil $R = 4\Lambda$ is equivalent to an integral over Λ with some weighting factors.

Hawking's work from [9] describes a method to determine these weighting factors: Before the integration over all classical backgrounds, a perturbative expansion with zeta function renormalization is done for each background, which adds a contribution to the weighting factor. From the resulting amplitude, a number of microstates can be computed. This leads to the definition of an entropy which can be maximized.

It is noted that Hawking's calculation contains an approximation of the gravitational action which implies that the model, at least in its original form, seems to work only for universes with a negative cosmological constant $\Lambda < 0$. Furthermore, the renormalization scale that Hawking used differs from the one that one finds in the recent literature.

We remove Hawking's approximation for the action and set the renormalization scale to a value that is currently used in most articles. We then repeat some of Hawking's calculations with the modified model.

We find that the non-perturbative path integration in the modified space-time foam model only converges if the renormalization scale has a certain relation to Λ. We argue that this is consistent with observations. We also note that the model predicts a cosmological constant $\Lambda \geq 0$ and that $\Lambda \to 0$ in the limit of the four volume $V \to \infty$. Furthermore, the theory appears to solve the so-called coincidence problem of cosmology. This is the problem how to use a quantum mechanical amplitude in order to explain the coincidence that $\Lambda \approx H_0^2$, where H_0 is Hubble's constant.

Then, a matter amplitude is added to the theory. For low orders in perturbation theory, it is argued that the space-time foam calculation provides a solution to the cosmological constant problem. This is the problem that the matter terms by themselves would naively lead to a large cosmological constant.

In higher orders of perturbation theory, parts of the matter terms are proportional to the Euler characteristic. These terms do contribute to the classical equations of motion or to in the one loop expansion. Nevertheless, in the non-perturbative amplitude, they appear to make a small contribution to the cosmological constant.

We argue that these terms may be interpreted as results from black or worm hole creation that modify the thermodynamic properties of the partition function.

Without further modification the other terms of the effective matter action are such that they give rise to the Ostrogradski instability. We show that if a specific relation between the sums of all fermion and boson masses holds, the Ostrogradski instability can be removed and the effective matter action reduces to a Starobinski model of an $R + R^2$ gravity.

A general result of the space-time foam model is that space-time should be filled with "holes" of negative Euler characteristic. This implies that space-time can not be simply connected. Such a topology usually signals the presence of worm holes. Noting that intra-universe worm holes are space-times which contain a boundary, we associate a boundary term to each hole.

We add this boundary term to the Euclideanized gravitational action and bring it into the form of the cosmological constant term. Since the worm holes of space-time foam can vary in number and size, the boundary term, and thus the cosmological constant Λ should be a variable field. The gravitational entropy of the space-time depends on Λ. Thus, the value of Λ should be found my maximizing the entropy, as it was done before. It is argued that the boundary terms of the space-time foam model also can be used to solve the problem of time in quantum gravity.

It has been pointed out by Hawking that a space-time which can be described as a gas of microscopic black- or worm holes might modify the trajectories and scattering amplitudes of particles in a way that they disagree with recent experiments in collider facilities.

In subsection 5.2 of this thesis, we try to give arguments why the modified space-time foam model of this thesis appears not to imply changes of the observed physics at low energies.

Because the micro black- or worm holes of space-time do have a small lifetime, an observer who can not measure short timescales does not observe the process of the particle falling into the hole and getting replaced by Hawking radiation.

However, since we have associated a boundary to the black- or worm holes of space-time foam and since these cavities are small, they should produce Hawking radiation of a very high temperature. For low a low energy particle, flying through space-time foam would mean to fly through a thermal heath bath of high temperature. Additionally, a calculation of Chandrasekhar [10] shows that there would be a collective effect of the gravitational field of the micro black- or worm holes that affects the particle over long timescales. We argue in subsection 5.2.1 that together, these two effects would imply that a classical particle in a space-time foam of micro black- or worm holes undergoes a Brownian motion. If we put a family of classical particles into such a space-time and demand that on average, they do not loose entropy to the worm hole gas, the probability amplitude of the particles is then given by the Schrödinger equation.

The discussion in subsection 5.2.1 is somewhat preliminary. For example, the behaviour of relativistic particles with spin is not discussed. Furthermore, we do not derive entangled states. Nevertheless, we review Bell's theorem in a mathematically rigorous form in subsection 5.2.2 and we make a short and preliminary discussion what this theorem implies for the stochastic model from subsection 5.2.1. Specifically, we argue that the model is not a hidden variable theory in the sense of Bell because its outcomes are not predetermined. We argue that if it is possible to extend the model and describe entangled states, one would have to assume that the worm holes of space-time foam are entangled in the sense that they produce a correlated heath bath. This may be interesting in view of the so-called ER=EPR correspondence conjecture of Suesskind.

For the description of the scattering process of particles of high energy with a black-hole, a model of 't Hooft [11] is reviewed in section 5.3.1. The amplitude of this process turns out to be the usual string theory amplitude over the Polyakov action in Wick rotated form.

In 't Hooft's formalism, the mass of the black-hole with which the scattering is described is arbitrary. We give some arguments in section 5.3.2 which indicate that the model appears mostly to be valid for small black-holes and does not hold for large astronomical black-holes.

In section 5.4, we point out that the space-time foam model outlined in this article implies topological changes during the expansion of the universe in which the Euler characteristic should change.

A theorem of Geroch [12] implies that a topology change will lead to the developments of singularities. DeWitt has given two arguments that show inconsistencies in quantum field theory and quantum gravity when singularities appear. DeWitt's second argument was just a short statement without proof in a non-technical paper. In this thesis, we provide this proof and give additional explanations why these inconsistencies emerge and can not be avoided.

In [13], DeWitt has proposed the use of string theory to investigate topology changes because it is sometimes written that this theory could be exactly defined on singular spaces like orbifolds. Mathematically, an orbifold is a so-called a stratified space consisting of a manifold and singularities as strata.

The famous article of Dixon et al. [14] that tries to define string theory on orbifolds introduces boundary conditions for so-called twisted sectors of the string theory on an orbifold. The authors note that their amplitude for strings on orbifolds would yield results that were equal to that of strings on a manifold with boundary that one gets from the orbifold after one has made a blow-up in the limit of an infinitesimally small ϵ neighbourhood around the singularity.

Using a mathematically rigorous construction of the path integral and arguments from constructive quantum field theory, we argue that without a previous blow up of the orbifold singularity, string theory suffers from similar problems and inconsistencies as conventional quantum gravity whenever a singularity appears in the target space.

Noting recent progress by mathematicians [15, 16], we find in section 5.4 that Dixon et al. can use their boundary conditions because the exponential map is well behaved in the ϵ neighbourhood of the orbifold singularity if the latter was blown up. If one takes the limit $\epsilon \rightarrow 0$, one just removes a point from the orbifold. For the string theory, this means one removes an end point from each curve over which one writes a path integral. Since this point has no measure, the implicit assumption of a blow-up is often not noticed in discussions of these amplitudes for singular target spaces.

For the description of topology changes in Lorentzian 4D manifolds, this result causes problems. Because the theorem of Geroch implies singularities whenever a topological change takes place, one can not, e.g. desingularize a target space into two Lorentzian manifolds A and B of separate topology and then have a non-singular path from A to B on which the string theory could be defined exactly.

The incompatibilities of quantum field theory with singular space-times seem to be quite general, since the field operators are always assumed to be tempered distributions which are defined by using smooth test functions.

We therefore conclude that we may have to consider theories that yield quantum field theory only in some approximation if we want to describe topological changes.

Finally, we mention that the stochastic model from which we derived the single particle Schroedinger equation in the thesis is smooth only on average and on a large scale. It uses non-differentiable trajectories on small scales. We close the thesis with the open suggestion that such an underlying model for quantum mechanics may be developed for the Wheeler-DeWitt equation of quantum gravity. Such a theory could then be used in the vicinity of a singularity during a topology change.

As it was pointed out, the material of chapters 1 and 2 is just review material. Parts of subsection 1.4, and subsections 2.2, 2.3, and 3.2 are adapted with some changes (shortened derivations, added references, some additional explanations and text) from my diploma thesis [17].

In my diploma thesis [17], one could also find a description of Hawking's space-time foam model. However, what I wrote at that time about this model was just review material that enabled me to understand the model. My diploma thesis does not contain the extensive modifications of Hawking's model that are described in this thesis.

The material of chapters 4 and 5 contain original and new results. They were published to a large part by the author in [8] in the course of his work on this phd thesis. The publication [8] contains the results of subsections 4.2, 4.3, 4.4, 4.5 and 5.2.1.

Subsection 5.2.2.1 is closely adapted from an old unpublished preprint [18] that the author of this thesis wrote some time ago in his spare time. It contains only review material.

The contents of subsections 5.2.2.2, 5.3.2, 5.4 and 5.5 have not yet been submitted. The author thinks he may want to extend this material and add further calculations before considering a publication in scientific journals.

Part II

A short review of quantum field theory in curved space-times and quantum gravity

Chapter 1

Quantum field theory in curved space-times

1.1 Introduction

Since much of the later calculations use functional integrals, we begin this section by reviewing the mathematical definition of this integral in section 1.2. In section 1.3, we then give a rudimentary review of some properties of quantum field theory in curved space-times. In section 1.4, we describe how to regularize amplitudes in curved space-times with zeta function renormalization. As an example, we review the computation of the cosmological constant in a curved background. We end this section with a discussion about Weinberg's no-go theorem for the cosmological constant.

Section 1.4 is adapted from my diploma thesis with some added material and modifications. Other sources from which material was taken as a basis for this review section are indicated in the sections.

1.2 The functional integral

This section is concerned with the mathematical definition of the functional integral that we need later on for most calculations. It is review material based on the work of DeWitt-Morette [1, 19–21].

A Banach space X is a space with a norm $||\cdot||_X$ where for every Cauchy sequence

$$\{x_n\} \in X \exists x \in X : \lim_{n\to\infty} \{x_n\} = x. \tag{1.1}$$

One can define a Banach space of fields $\phi : M^D \to \mathbb{C}$, where M^D is the D dimensional physical space-time with covariant vectors $x^\mu, \mu \in 1, \ldots, D$. One also can define a dual Banach space X' to X which consists of the covectors, or linear forms $x' : M_D \to \mathbb{C}$: and we define a scalar product $\langle x', x\rangle$ that yields the value of the linear form $x' \in X'$ at point $x \in X$. Furthermore, one defines two continuous linear maps $D : X \to X', G : X' \to X$ that fulfil

$$DG = \mathbf{1},\ GD = \mathbf{1},\ \langle Dx, y\rangle = \langle Dy, x\rangle,\ \langle x', Gy'\rangle = \langle y', Gx'\rangle. \tag{1.2}$$

Additionally, one defines two quadratic forms

$$Q(x) \equiv \langle Dx, x\rangle \tag{1.3}$$

and

$$W(x') \equiv \langle x', Gx'\rangle, \tag{1.4}$$

where one sets

$$W(x', y') = \langle x', Gx'\rangle. \tag{1.5}$$

One can define a Gaussian volume element on X by the Fourier transform

$$(\mathcal{F}\Gamma_{a,Q})(x') \equiv \int_X d\Gamma_{a,Q}(x) e^{-2\pi i \langle x', x\rangle} = e^{-a\pi W(x')} \tag{1.6}$$

For arbitrary $x' \in X'$. The parameter a can be either i for Lorentzian path integrals, or 1 for Euclidean ones. Finally, one defines a formal volume element $D_{aQ}(x)$ by

$$\mathcal{F}(\Gamma_{a,Q})(x') = \int_X d\Gamma_{a,Q}(x) e^{-2\pi i \langle x', x\rangle} = \int_X \mathcal{D}_{a,Q}(x) e^{-\frac{\pi}{a} Q(x)} e^{-2\pi i \langle x', x\rangle}. \tag{1.7}$$

In both Euclidean and Lorentzian cases it can be shown that the integrals are rigorously defined [1, 21].

For the quadratic form Q one can make various choices. For example, Q can be the kinetic energy of a field plus any existing quadratic terms in the potential energy. In quantum field theory, Q is usually the second term in a Taylor expansion of the of the classical action around some background. In physicists notation, the above path integral is given by

$$Z_0(J) \equiv \int \mathcal{D}\phi e^{iS(\phi) - i\langle J, \phi\rangle} \equiv e^{iW(J)}. \tag{1.8}$$

Here, S is the quadratic part of the action, J is the so-called source and W is called energy functional. It is given by

$$W(J) \equiv \langle J, GJ \rangle \equiv \int\int d^4x d^4y G(x,y)J(x)J(y) \tag{1.9}$$

G is usually called two-point function or propagator since it can also be defined by

$$\frac{a}{2\pi}G(x,y) = \int \mathcal{D}_{a,Q}\phi e^{-\frac{\pi}{a}Q(\phi)}\phi(x)\phi(y), \tag{1.10}$$

where Q is given by

$$Q(\phi) = \langle D\phi, \phi \rangle. \tag{1.11}$$

One can expand $Z_0(J)$ in powers of J and get n point functions

$$G_n(x_1, \ldots x_n) = 0 \tag{1.12}$$

for odd n and

$$G_{2m}(x_1, \ldots x_{2m}) = \sum G(x_{i_1}, x_{j_1})G(x_{i_2}, x_{j_2}), \ldots G(x_{i_m}, x_{j_m}), \tag{1.13}$$

where one sums over the permutations $i_1, \ldots, i_m$ and $j_1, \ldots j_m$ of $1, 2, \ldots, 2m$ such that $i_1 < i_2 < \ldots i_m$ and $i_1 < j_1, \ldots i_m < j_m$.

Most field theories have an interaction part S_i in their action in addition to the free field description S_{free}

$$S = S_{free} + \lambda S_i. \tag{1.14}$$

In that case one can define an amplitude

$$Z_{int}(J) = \int \mathcal{D}\phi e^{iS_i} e^{-2\pi i \langle J, \phi \rangle} \tag{1.15}$$

and one can define W_{int} by

$$e^{iW_{int}(J)} = \frac{Z_{int}(J)}{Z_{int}(0)}. \tag{1.16}$$

Finally, one can expand W_{int} as a perturbation series.

Path integrals often diverge and this also holds for the perturbation series. Therefore, one may insert a convergence factor with a physical dimension into the functional integral. In order to restore the proper units of the physical expression, one has then to insert dimensional parameters (e.g a mass or energy scale). These parameters then must be determined by measurement before the renormalized theory becomes predictive.

If one does a perturbation expansion with the functional integral and sorts the terms according to higher and higher energy, it may be that additional convergence factors, so-called counter-terms are needed for the higher order terms. If the number of these terms gets infinite at some order of the perturbation expansion, the theory would depend on infinitely many free renormalization scale parameters. Thus the theory would become unpredictive. Such models are called non-renormalizable.

1.3 Quantum field theory and the functional integral in curved space-times

This section describes some of the basic problems that arise if one does computations with quantum fields in curved space-times. It is review material based on the famous article [22] of DeWitt and the book of Birrell and Davies [23], which is basically an extension of DeWitt's review article in book form.

For a free field ϕ that propagates in a curved space-time, the action is given by

$$S = \frac{1}{2}\int \phi F \phi d^4x \tag{1.17}$$

and the field equations have the form $F\phi = 0$, where F is a self adjoint differential operator that fulfils

$$\int \psi_1^* F \psi_2 d^4x = \int (F\psi_1)^* \psi_2 d^4x. \tag{1.18}$$

where $\psi_{1/2}$ are any two smooth complex functions with compact support. With two solutions of the field equations u_1 and u_2 defined on a Cauchy hyper-surface Σ, one can define an inner product

$$\langle u_1, u_2 \rangle = -i \int_\Sigma u_1^*(Fu_2) - (Fu_1)^* u_2 d^4x. \tag{1.19}$$

One can now find solutions u_i, u_i^* which fulfil orthogonality relations

$$\langle u_i, u_j \rangle = \delta_{ij}, \quad \langle u_i^*, u_j \rangle = 0 \tag{1.20}$$

and one can expand φ as

$$\varphi = \sum_i (a_i u_i + a_i^\dagger u_i) \tag{1.21}$$

where the a_i fulfil

$$[a_i, a_j^\dagger]_\pm = \delta_{ij},\ [a_i, a_j]_\pm = 0. \tag{1.22}$$

In general, there exist infinitely many sets u_i, u_i* and the field can be expanded in form of

$$\varphi = \sum_j (\overline{a}_j \overline{u}_j + \overline{a}_j^\dagger \overline{u}_j^*). \tag{1.23}$$

Different sets u_i, u_i^* and $\overline{u}_i, \overline{u}_i^*$ are related by

$$\overline{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*),\quad u_i = \sum_j (\alpha_{ji}^* \overline{u}_j - \beta_{ji} \overline{u}_j^*) \tag{1.24}$$

where

$$\alpha_{ij} = (\overline{u}_i, u_j),\quad \beta_{ij} = -(\overline{u}_i, u_j^*) \tag{1.25}$$

and

$$a_i = \sum_j (\alpha_{ji} \overline{a}_j + \beta_{ji}^* \overline{a}_j^\dagger) \tag{1.26}$$

and

$$\overline{a}_j = \sum_i (\alpha_{ji} * \overline{a}_i - \beta_{ji}^* a_i^\dagger). \tag{1.27}$$

Furthermore, α_{ij},β_{ij} fulfil

$$\sum_k (\alpha_{ik} \alpha_{jk}^* - \beta_{ik} \beta_{jk}^*) = \delta_{ij} \tag{1.28}$$

and

$$\sum_k (\alpha_{ik} \beta_{jk} - \beta_{ik} \alpha_{jk}) = 0. \tag{1.29}$$

If we have chosen a set such that $\overline{a}_j|0\rangle = 0$ then

$$a_i|0\rangle = \beta_{ji}^* \overline{a}_j^\dagger |0\rangle \neq 0. \tag{1.30}$$

Considering that the particle number operator is given by $N_i = a_i^\dagger a_i$, this means that the particle number for different vacua is different.

Quantized matter fields have an effect on the curvature of space-time. One is seeking the expectation value of the energy stress tensor $T_{\mu\nu}$ which gets into the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi \langle T_{\mu\nu} \rangle. \tag{1.31}$$

In order to compute that, one considers the generating functional

$$Z(J) = \int \mathcal{D}\varphi e^{iS_m(\phi)+i\int J\varphi d^4x} \tag{1.32}$$

with a matter action S_m. One can now compute the vacuum expectation value by the variation

$$\delta Z(0) = i\int \mathcal{D}\varphi \delta S_m e^{iS_m} = i\langle out, 0|\delta S_m|0, in\rangle \tag{1.33}$$

and since in a classical field theory

$$\frac{2\delta S_m}{\sqrt{-g}\delta g^{\mu\nu}} = T_{\mu\nu} \tag{1.34}$$

one has

$$\frac{2\delta Z(0)}{\sqrt{-g}\delta g^{\mu\nu}} = i\langle out, 0|\delta T_{\mu\nu}|0, in\rangle. \tag{1.35}$$

One problem is that this path integral is usually divergent. And since each vacuum has a different particle number it is difficult to regularize it "mode by mode".

1.4 Review on zeta function renormalization

The following short introduction is a review of [24] and [25] p. 45, where zeta function renormalization is described in detail to regularize the divergent amplitude in curved space-times. This section is an adaption from my earlier diploma thesis [17] with some shortened derivations, added material and references.

For a scalar field with an Euclidean action

$$S = \frac{1}{2}\int d^4x\sqrt{g}(g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi + V\varphi^2) \tag{1.36}$$

zeta function renormalization of a divergent path integral like $Z = \int \mathcal{D}\varphi e^{-S}$ proceeds according to the following algorithm:

One first integrates the action by parts:

$$\begin{aligned} S &= \frac{1}{2}\int d^4x(-\varphi\nabla_\nu(\sqrt{g}g^{\mu\nu}\nabla_\mu\varphi) + \sqrt{g}V\varphi^2) \\ &= \frac{1}{2}\int d^4x(\varphi F\varphi) \end{aligned} \tag{1.37}$$

where

$$F = -\Box + V \tag{1.38}$$

is a self adjoint operator with a well defined eigenvalue problem:

$$F\varphi_n = \lambda_n \varphi_n \tag{1.39}$$

The eigenfunctions φ_n of F form an orthonormal base

$$\int d^4x \sqrt{g} \varphi_n \varphi_m = \delta_{nm} \tag{1.40}$$

because F is self adjoint. The existence of the orthonormal base implies that

$$\varphi = \sum_{n=0}^{\infty} c_n \varphi_n, \tag{1.41}$$

with coefficients

$$c_n = \int d^4x \sqrt{g} \varphi \varphi_n. \tag{1.42}$$

The action can therefore be written as weighed sum of eigenvalues:

$$S = \frac{1}{2} \int d^4x \sqrt{g} \sum_{m,n} c_m c_n \lambda_m \varphi_m \varphi_n = \frac{1}{2} \sum_n c_n^2 \lambda_n \tag{1.43}$$

Given that the Feynman path "measure" $\mathcal{D}\varphi$ must be covariant and that c_n are coordinate independent, one makes the guess

$$\mathcal{D}\varphi = \prod_n \mu dc_n \tag{1.44}$$

where μ is a renormalization scale which has to be determined from experiment. The Euclidean path integral $Z = \int \mathcal{D}\varphi e^{-S}$ is then given by

$$Z = \int \prod_{n=0}^{\infty} dc_n \mu e^{-\lambda_n c_n^2} = \prod_{n=0}^{\infty} \frac{1}{2} \mu \pi^{\frac{1}{2}} \lambda_n^{-1/2} = \left(det \left(4\mu^{-2} \pi^{-1} F \right) \right)^{-\frac{1}{2}}. \tag{1.45}$$

The determinant of F is equal to the product of its eigenvalues and since F is a differential operator, this product is infinite. One must find some way to regularize it. We will do this via analytic continuation.

We define a generalized zeta function

$$\zeta(s) = \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_n}\right)^s \tag{1.46}$$

which will converge for $Re(s) > 2$ and can be analytically extended to a meromorphic function of s with poles only at $s = 1$ and $s = 2$. The derivative $\zeta'(s)$ is formally given by

$$\zeta'(s) = \frac{d}{ds} \sum_{n=0}^{\infty} e^{-sln(\lambda_n)} = -\sum_{n=0}^{\infty} e^{-sln(\lambda_n)} ln(\lambda_n) = -\sum_{n=0}^{\infty} \lambda_n^{-s} \ln \lambda_n. \tag{1.47}$$

The logarithm of the partition function becomes

$$ln(Z) = \ln \prod_{n=0}^{\infty} \frac{1}{2} \mu \pi^{\frac{1}{2}} \lambda_n^{-1/2} \tag{1.48}$$

$$= \lim_{s \to 0} \left(\sum_{n=0}^{\infty} \lambda_n^{-s} \ln \left(\frac{1}{2} \mu \pi^{\frac{1}{2}} \lambda_n^{-\frac{1}{2}} \right) \right) \tag{1.49}$$

$$= \lim_{s \to 0} \left(-\frac{1}{2} \sum_{n=0}^{\infty} \lambda_n^{-s} \ln \lambda_n + \sum_{n=0}^{\infty} \lambda_n^{-s} \ln \left(\frac{1}{2} \pi^{\frac{1}{2}} \mu \right) \right) \tag{1.50}$$

$$= \lim_{s \to 0} \left(\frac{1}{2} \zeta'(s) + \frac{1}{2} \zeta(s) \ln \left(\frac{1}{4} \pi \mu^2 \right) \right). \tag{1.51}$$

These formulas are formal because the sum $-\sum_n ln(\lambda_n)$ is not finite. The results only hold in analytic continuation which should remove the divergences of the path integral.

Notably is the undetermined normalization scale parameter μ. In flat space, one generally has $\zeta(0) = 0$, so amplitudes in flat space do not depend on this parameter. This is different in curved space-times. There, any amplitude should depend on μ, which must be determined by making a measurement.

In order to compute the zeta function, one employs the heath kernel $F(x, y, \tau)$ which is a solution of the generalized heath equation

$$\frac{d}{d\tau} K(x, y, \tau) + F K(x, y, \tau) = 0, \tag{1.52}$$

where x, y are points in space-time, τ is an additional parameter, and F is an operator acting on the last argument of $K(x, y, \tau)$.

The heath kernel can be expressed in terms of eigenvalues of F as

$$K(x,y,\tau) = \sum_{n=0}^{\infty} e^{-\lambda_n \tau} \varphi_n(x)\varphi_n(y). \tag{1.53}$$

Since

$$\frac{d}{d\tau}\sum_n e^{-\lambda_n \tau}\varphi_n(x)\varphi_n(y) = \sum_{n=0}^{\infty}(-\lambda_n)e^{-\lambda_n\tau}\varphi_n(x)\varphi_n(y) = -FK(x,y,\tau), \tag{1.54}$$

one defines the "trace" of the heath kernel as

$$tr(K(\tau)) = \int d^4x\sqrt{g}K(x,x,\tau) = \int d^4x\sqrt{g}\sum_{n=0}^{\infty} e^{-\lambda_n\tau}\varphi_n(x)\varphi_n(x) = \sum_{n=0}^{\infty} e^{-\lambda_n\tau}. \tag{1.55}$$

The generalized zeta function is related to the trace of the heath kernel by a Mellin transformation

$$\zeta(s) = \sum_{n=0}^{\infty}\lambda_n^{-s} = \frac{1}{\Gamma(s)}\int_0^{\infty} dtt^{s-1}tr(K(\tau)). \tag{1.56}$$

One determines $K(x,y,\tau)$ by solving the heath equation (1.52) for the operator F and then compute $tr(K(\tau))$ from which one gets the generalized zeta function.

One application of zeta function renormalization can be the computation of the vacuum expectation value for the cosmological constant for the various matter fields. For a massive scalar field with mass m, the zeta function becomes in Euclidean space

$$\zeta(0) = \frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left(\frac{1}{2}m^4a_0 - m^2a_1 + a_2\right), \tag{1.57}$$

where a_i are the so called Seley-DeWitt coefficients which were computed by DeWitt in [26]. For a scalar field, they become

$$a_0 = 1, \;\; a_1 = \left(\frac{1}{6} - \zeta\right)R \tag{1.58}$$

and

$$a_2 = \frac{1}{80}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} - \frac{1}{80}R_{\mu\nu}R^{\mu\nu} - \frac{1}{6}\left(\frac{1}{5} - \zeta\right)\Box R + \frac{1}{2}\left(\frac{1}{6} - \zeta\right)^2 R^2. \tag{1.59}$$

The $\zeta'(0)$ term becomes [23]

$$\zeta'(0) = \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\sqrt{g}\left(\left(\gamma - \frac{3}{2}\right)\left(\frac{1}{2}m^4 - m^2a_1 + a_2\right)\right.$$
$$\left. - \int_0^\infty ln(is)\frac{\partial^3}{\partial(is)^3}(K(x,x,is)e^{-ism^2})ids\right). \tag{1.60}$$

From this one can compute the amplitude for a scalar matter field with mass m. The first Seley-DeWitt coefficient does not depend on the geometry. Its integral over the volume gives an additional term $\Lambda_m V$ that corrects the gravitational action. The term a_1 depends on R. it will yield a renormalization of the gravitational constant.

The renormalized effective action up to the first Seley-DeWitt coefficient a_0 leads to a contribution to the cosmological constant of

$$\Lambda_{eff} = \Lambda_g + \Lambda_m, \tag{1.61}$$

where Λ_g is the unobserved, bare gravitational constant that should be created without the matter fields, and

$$\Lambda_m = \frac{m^4}{(64\pi)^2}\ln\frac{m^2}{\mu^2} \tag{1.62}$$

is the matter contribution to Λ_{eff} [23, 27].

One observes that this value depends on the renormalization scale μ^2 which must be determined from measurement.

According to the more recent literature on this topic, one can make an estimate. Because of its units, the square of the scale parameter μ should be of the order of the energy of the gravitons involved in the amplitude [27–29]. A simple method for estimating μ was given by Shapiro and Sola in [28].

Currently, we live in a Friedmann universe with Hubble constant

$$H_0^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}\left(\rho + \Lambda - \frac{k}{a^2}\right), \tag{1.63}$$

where a is a time dependent scale factor and

$$\rho = \rho_m + \rho_R \tag{1.64}$$

is the energy density of matter and radiation. We set $k = 0$ because the universe is very flat at present. From experiment, one knows that Λ and ρ are of the same order as the critical density

$$\rho_c = \frac{3H_0^2}{8\pi}, \tag{1.65}$$

therefore

$$H_0 \approx \sqrt{\rho_c} \tag{1.66}$$

and

$$T^\mu_\mu \approx \rho_c. \tag{1.67}$$

The scale parameter μ^2 has the units of an energy. Therefore, a reasonable guess is that

$$\mu \approx \sqrt{T^\mu_\mu}. \tag{1.68}$$

In a curved background, one has to expect quantum corrections to $T_{\mu\nu}$. Unfortunately, $T_{\mu\nu}$ diverges and has to be regulated. However, we may assume that quantum corrections to the observable energy of the universe are small. Therefore, we neglect the quantum corrections and use Eq. (1.68) but this implies with Eq. (1.67) that

$$\mu \approx \sqrt{T^\mu_\mu} \approx \sqrt{R} \approx H_0. \tag{1.69}$$

From Friedmann's equations, one has $\Lambda_{eff} \approx H_0^2$ and this would imply that $\mu^2 \approx \Lambda_{eff}$.

The entire matter contribution to the cosmological constant includes phase transitions of the electroweak theory $\rho^{EW}_{vac} = -1.2 \cdot 10^8 GeV^4$ and qcd $\rho^{QCD}_{vac} = 10^{-2} GeV^4$.

For fermions, the Seley-DeWitt coefficients become a trace over a matrix. After computing that trace, one finds that each fermionic degree of freedom contributes to Λ_m with a different sign as a bosonic degree.

The matter contribution to the cosmological constant then becomes

$$\Lambda_m = \rho^{EW}_{vac} + \rho^{QCD}_{vac} + \sum_i \frac{n_i m_i^4}{(64\pi)^2} \ln \frac{m_i^2}{H_0^2} \tag{1.70}$$

where $n_{Higgs} = 1,\ n_{quarks} = -4,\ n_{leptons} = 4,\ n_{Z/W^\pm} = 3,\ n_{neutrinos} = 4$ and m_i is the mass of the particle species i.

The result of $\Lambda_m \approx -3.2 \cdot 10^8 GeV^4$ differs largely from the observed value of $\Lambda_{eff} \approx 10^{72} GeV^4$. This is due to the dependence of Λ_m by a factor $\propto m^4$ on the particle masses. One therefore wants a mechanism which adjusts Λ_g automatically, such that Λ_{eff} becomes the observed value. This is called the cosmological constant problem. The problem to explain why $\Lambda_{eff} \approx H_0^2$ from a quantum mechanical model is called coincidence problem.

1.5 The Ostrogradski instability

The Seley-DeWitt coefficient a_2 leads to R^2, $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ terms in the effective action. Stelle [30, 31] has shown that gravity with higher derivative terms of the form $R_{\mu\nu}R^{\mu\nu} + R^2$ or $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + R^2$ added is a renormalizable theory. The theory becomes renormalizable because at high energy, one finds propagators $\propto \frac{1}{q^4}$. However, these terms lead to problems with negative energies from the famous Ostrogradski instability [2, 32].

To demonstrate this instability, one starts with a Lagrangian $L(q, \dot{q}, \ddot{q})$ that depends on higher derivatives and one makes a non-degeneracy assumption. This means one assumes that $\frac{\partial L}{\partial \ddot{q}}$ depends on $\ddot{q}$. Thereby, one can state an initial value problem where $q(t)$ is a function

$$q(t) = Q\left(t, q_0, \dot{q}_0, \ddot{q}_0, \frac{d^3 q}{dt^3}\right) \tag{1.71}$$

One can express the phase space which depends on 4 different initial data values with 4 canonical coordinates

$$Q_1 = q,\ Q_2 = \dot{q} \tag{1.72}$$

$$P_1 = \frac{\partial L}{\partial \dot{q}} - \frac{d \partial L}{dt \partial \ddot{q}},\ P_2 = \frac{\partial L}{\partial \ddot{q}}. \tag{1.73}$$

The non-degeneracy assumption implies that there exists a function $a(Q_1, Q_2, P_2)$ such that

$$\left.\frac{\partial L}{\partial \ddot{q}}\right|_{q=Q_1, \dot{q}=Q_2, \ddot{q}=a} = P_2 \tag{1.74}$$

and after a Legendre transformation, one finds a Hamiltonian

$$H = P_1 Q_2 + P_2 a(Q_1, Q_2, P_2) - L(Q_1, Q_2, a(Q_1, Q_2, P_2)). \tag{1.75}$$

Since P_1 can go through the entire phase space, H can have arbitrarily negative energies.

A central assumption of the Ostrogradski instability is the assumption of non-degeneracy. If one violates this assumption, one can (but this is not a must, see [33]) avoid the Ostrogradski instability.

It is known that $\kappa R + aR^2$ gravity (where κ, a are arbitrary real coefficients) violates the non-degeneracy assumption [32] and one can show a positive energy theorem [34] for this theory. Recently, Alvarez-Gaume et al. [35] argued that pure aR^2 gravity is ghost free and that $\kappa R + aR^2$ gravity has a physical spectrum.

More precisely, from the $\kappa^2 R + aR^2$ theory, one gets a propagator

$$\Delta^{(0)}_{\mu\nu,\rho\sigma} = \frac{-1}{(6q^2 a + \kappa^2) q^2} \tag{1.76}$$

where

$$P^{(0)}_{\mu\nu,\rho\sigma} = \left(\eta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\left(\eta_{\rho\sigma} - \frac{q_\rho q_\sigma}{q^2}\right) \tag{1.77}$$

and $\eta_{\mu\nu}$ is the flat space metric.

This propagator looks at first like it would be $\propto 1/q^4$. However, one can expand it according to

$$\Delta^{(0)}_{\mu\nu,\rho\sigma} = \frac{1}{\kappa}\left(\frac{1}{q^2 + \kappa/6\hat{a} - i\epsilon} - \frac{1}{q^2 + i\epsilon}\right) P^{(0)}_{\mu\nu,\rho\sigma}. \tag{1.78}$$

Hence, this propagator can be described by the sum of a ghost propagator with negative energy and the propagator of a massive particle. The apparent ghost of the $\kappa^2 R + aR^2$ gravity is, fortunately, not a physical field because one can use residual gauge freedom to remove it [35]. The remaining field content of the theory is then the massive spin 0 mode with a propagator $\propto 1/q^2$. From this one can conclude that the $\kappa^2 R + aR^2$ gravity is ghost free.

From the results of Starobinski [36], one knows that the pure R^2 gravity also leads to an inflationary potential, therefore, the addition of an R^2 term appears to be useful. Unfortunately, the pure $\kappa R + aR^2$ model is non-renormalizable [37].

The effective action of quantized matter fields unfortunately contains not only the R^2 term. Therefore, one has to find conditions which led to a situation where the $R_{\mu\nu}R^{\mu\nu}$ and $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ terms are not measurable. We will put forward a proposal to remove the Ostrogradski instability in section 4.4.

1.6 Weinberg's no go theorem for translationally invariant vacuum states

Weinberg has proven an interesting theorem in [38] that with a cosmological constant present one can, in classical general relativity, not have a translationally invariant solution of the Einstein field equations. The proof goes as follows:

If we add fields ψ_i, translational invariance of the fields and the metric implies

$$\frac{\partial \mathcal{L}}{\partial \psi_i} = 0 \tag{1.79}$$

and

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = 0. \tag{1.80}$$

With N fields ψ_i and a symmetric metric, one has $N+6$ equations and $N+6$ unknowns. Under a $GL(4)$ transformation

$$g'_{\mu\nu} = A^{\rho}{}_{\mu} A^{\sigma}{}_{nu} g_{\rho\sigma} \tag{1.81}$$

and

$$\psi'_i = D_{ij}(A)\psi_j. \tag{1.82}$$

Therefore, the Lagrangian becomes as

$$\mathcal{L}' = det(A)\mathcal{L}. \tag{1.83}$$

The equations of motion then have the unique solution

$$\mathcal{L} = c \cdot det(g) \tag{1.84}$$

where c is some constant. Thus

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = 0 \tag{1.85}$$

can only be satisfied for vanishing c, or $\mathcal{L} = 0$, but this implies a vanishing Lagrangian and a vanishing cosmological constant.

There exist various relaxed versions of this proof. For example, in his article, Weinberg later relaxes his equations of motion to

$$g_{\mu\nu} \frac{\partial \mathcal{L}(g,\psi)}{\partial g_{\mu\nu}} = \sum_{n}^{N} \frac{\partial \mathcal{L}(g,\psi)}{\partial \psi_n} f_n(\psi), \tag{1.86}$$

where f_n are some coefficient functions and $g_{\mu\nu}$ and ψ_i constant fields, such that

$$\frac{\partial \mathcal{L}}{\partial \psi_i} = 0, \tag{1.87}$$

which again implies

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = 0, \tag{1.88}$$

from which a similar proof can be derived.

The theorem can be translated into quantum gravity [39]. The $GL(4)$ symmetry is broken in quantum gravity but it can be replaced with the BRST symmetry which is a residual global symmetry of diffeomorphisms emerging after gauge fixing. Then the proof works similarly.

Weinberg's theorem searches for constant, translationally invariant solutions of the metric. Constant metrics are not observed in cosmology. Nevertheless, Weinberg's result is still interesting. Usually, the vacuum is the ground state of a theory. In order to circumvent Weinberg's theorem, it appears that one must find a model for quantum gravity where the ground state is not a translationally invariant constant metric like Minkowski space. In the later sections of this thesis we will argue that this is exactly what happens for the vacuum state of quantum gravity.

Chapter 2

Quantum gravity

2.1 Introduction

Most of our later calculations are concerned with the amplitude of Euclidean quantum gravity that was regulated by the zeta function technique. Therefore, we start in section 2.2 with a review how to regularize gravity amplitudes with zeta functions. Some of our later calculations need the Hamiltonian of gravity. Therefore, we also review the non-perturbative canonical quantisation of gravity that is based on the Wheeler DeWitt equation in section 2.3. We review how the canonical and covariant formalism of quantum gravity are related and argue that one can derive one formalism from the other. We note in section 2.4 that the Hamiltonian constraint of gravity vanishes and that this leads to a frozen formalism with no dynamical observables. As an example for concrete calculations, we review the computation of the black-hole entropy from perturbative quantum gravity in section 3.2. We close this section by reviewing Gibbons' and Hawking's proof that matter can not cause a curvature of the gravitational field that makes the gravitational action contribute to the entropy unless the curvature is such that a boundary term is created in Euclidean quantum gravity.

This section is review material. Sections 2.2, 2.3 and 3.2 are adapted from my Diploma thesis [17] with some changes (shortened derivations, added references, different text). Other sources from which material was taken are indicated in the text.

2.2 Zeta function renormalization of gravity amplitudes

In the second chapter of this thesis, we will work with the gravitational amplitude in a form where it is regulated with zeta functions. Therefore, we will review work of Gibbons and Hawking [40] in this section and show how this amplitude is derived. Some attention is paid to scaling properties of the amplitude that will be used later.

Perturbative quantum gravity can also be done with dimensional regularization. For this we merely want to point the reader to the excellent articles by 't Hooft [41], Veltman [42], Hamber [43]. For a general overview of quantum gravity, we point the reader to the excellent articles of DeWitt [3, 22, 26, 44–46].

When discussing these amplitudes, one should note that quantum gravity is a perturbatively non-renormalizable theory [45]. This implies that one would have to add more and more additional renormalization parameters with higher orders of perturbation theory, which would make the theory unpredictive if this procedure were carried out up to all orders. Unfortunately, for pure gravity, there is no symmetry that prevents these divergences to occur at higher orders. While the one loop amplitude of pure gravity is free of divergences [41], the amplitudes become divergent again at two loop order [47, 48].

Additionally, divergences of the amplitudes have been shown to exist for combined gravity and matter one loop amplitudes [49–51]. However, these results were calculated by treating the interaction of the matter fields with the graviton as in usual Feynman diagrams by introducing a vertex functions.

Usually, the interaction of a matter field with the gravitational field is modelled with the energy momentum tensor, which has to be renormalized in curved space-times. The author of this thesis thinks that it is more appropriate to quantize the gravitational Lagrangian at one loop order along with a computation of the renormalized energy momentum tensor of the matter fields in an arbitrarily curved space-time. From this, one gets modified field equations for the gravitational field and an energy momentum tensor that couples to it. In the opinion of the author of this thesis, this approach is more useful than working with vertex functions. The strong self interactions of the gravitons make perturbative computations of individual particle interactions with gravitons not too meaningful.

The action of Euclidean gravity with metric $\tilde{g}_{\mu\nu}$ and without cosmological constant is given by

$$I = \int d^4x \mathcal{L} = -\frac{1}{16\pi} \int \sqrt{\tilde{g}} R d^4x \tag{2.1}$$

with a Lagrangian density $\mathcal{L} = -\frac{1}{16\pi}\sqrt{\tilde{g}}R$ and we want to evaluate the path integral

$$Z = \int \mathcal{D}\tilde{g}_{\mu\nu} e^{-I} \tag{2.2}$$

perturbatively.

We decompose the metric $\tilde{g}_{\mu\nu}$ into a background field that is supposed to fulfil the classical equations of motion and a fluctuating quantum field $h_{\mu\nu}$.

Then we expand the action around the background $g_{\mu\nu}$ in a series [40]

$$I = -\int d^4x \left(\frac{1}{16\pi}\sqrt{g}R + \underline{\mathcal{L}} + \underline{\underline{\mathcal{L}}} \right), \tag{2.3}$$

where g and R are constructed from $g_{\mu\nu}$. The expansion of the Lagrangian density $\underline{\mathcal{L}}$ is linear in $h_{\mu\nu}$ and $\underline{\underline{\mathcal{L}}}$ is quadratic in $h_{\mu\nu}$. The path integral of Eq. (2.2) is then given by

$$Z = e^{-\frac{1}{16\pi}\int d^4x \sqrt{g}R} \int \mathcal{D}h_{\mu\nu} e^{-\int d^4x \left(\underline{\mathcal{L}} + \underline{\underline{\mathcal{L}}} + \ldots\right)} = e^{-I(g_{\mu\nu})} \int \mathcal{D}h_{\mu\nu} e^{-\underline{\underline{I}}(h_{\mu\nu})}, \tag{2.4}$$

since $\underline{\mathcal{L}} = 0$ by the equations of motion.

The equations of motion of general relativity are invariant under diffeomorphisms. In order to avoid over counting in the path integral, one must consider the addition of gauge fixing and ghost terms. Historically, they have been found for first orders by Feynman [52]. DeWitt then generalized this procedure to all orders [26] and shortly later a short article by Faddeev and Popov arrived [53] that was simpler to understand (which gave the procedure the name Faddeev-Popov ghosts). The procedure outlined by these authors was not mathematically rigorous because they contain an integration over the gauge group. Years later, DeWitt succeeded to find a rigorous description of the procedure with the mathematical theory of fibre bundles [54]. This leads to the same amplitude, that can be converted into a form with determinants and whose regularization we will discuss below.

In their article, Gibbons, Hawking and Perry [40] find that one can express the amplitude as

$$ln(Z) = -I(g_{\mu\nu}) - \frac{1}{2} ln \left(det \left(\frac{1}{2}\pi^{-1}\mu^{-2}(-F+G) \right) \right) + ln \left(det \left(\frac{1}{2}\pi^{-1}\mu^{-2}C \right) \right), \tag{2.5}$$

where μ is the undetermined normalization scale, C is the operator for the ghosts and $-F+G$ is an operator that describes the perturbation expansion of the Euclidean action and the gauge fixing. The determinants can be expressed with zeta functions. Denoting the zeta function for the eigenvalues of an operator A as ζ_A, the amplitude becomes

$$ln(Z) = -I(g_{\mu\nu}) + \frac{1}{2}\zeta'_F(0) + \frac{1}{2}\zeta'_G(0) - \zeta'_C(0) + \frac{1}{2}ln(2\pi\mu^2)(\zeta_F(0) + \zeta_G(0) - 2\zeta_C(0)). \quad (2.6)$$

One can generalize this procedure to gravity with a cosmological constant. From a heath kernel calculation, Gibbons, Perry, and Hawking compute a formula for the zeta functions of a space-time with cosmological constant. This formula was later corrected by Christensen and Duff [55]:

$$\zeta_F(0) + \zeta_G(0) - 2\zeta_C(0) = \frac{53}{720\pi^2}\int d^4x\sqrt{g}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - \frac{87}{120\pi^2}\Lambda^2 V \quad (2.7)$$

$$\equiv \gamma. \quad (2.8)$$

Using the generalized Gauss Bonnet theorem, the Euler characteristic for a compact manifold of dimension 4 can be computed by

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right). \quad (2.9)$$

which is equal to $\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ if Einstein's field equations hold in a vacuum or with cosmological constant Λ. The factor γ is then

$$\gamma = \frac{106}{45}\chi - \frac{87}{120\pi^2}\Lambda^2 V. \quad (2.10)$$

Zeta function renormalization of gravity amplitudes is especially interesting because it delivers the following result: If we rescale the action according to

$$I(\tilde{g}_{ab}) = kI(g_{ab}), \quad (2.11)$$

the eigenvalues of the operators F, G, C will get multiplied by k^{-1}. So we get a new amplitude of the form

$$ln(\tilde{Z}) = ln(Z) + (1-k)I(g_{\mu\nu}) + \frac{1}{2}\gamma ln(k). \quad (2.12)$$

The Euclidean action of gravity with cosmological constant is given by

$$I = \frac{-1}{16\pi}\int d^4x\sqrt{g}(R - 2\Lambda) = \frac{-\Lambda V}{8\pi}, \quad (2.13)$$

where Einstein's equations $R = 4\Lambda$ have been used. By dimensional considerations, one finds that

$$\Lambda^2 V = f^2, \tag{2.14}$$

where f is some scalar factor. This means that one can write

$$V(\Lambda) = \frac{f^2}{\Lambda^2} \tag{2.15}$$

or

$$I = -\frac{f^2}{8\pi\Lambda}. \tag{2.16}$$

From this, the cosmological constant can, in some sense, be interpreted as a rescaling of the action. One can use this to derive a useful formula that shows how the amplitude changes as a function of the cosmological constant.

2.3 The Wheeler-DeWitt equation of canonical quantum gravity

This section develops the non-perturbative formalism of canonical quantum gravity that is based on the Hamiltonian formulation of general relativity. This formulation of quantum gravity provides an excellent tool for the quantisation of closed space-times, e.g. the closed Friedmann Robertson Walker universe.

This section is merely a review of parts of the first article [3] from DeWitt's famous trilogy of three fundamental papers on quantum gravity [3, 44, 45]. Some parts of the text in this section review a famous article of Hartle and Hawking [4]. This section is adapted from my earlier Diploma thesis [17] with some changes (shortened derivations, added references, different text).

That the Hamiltonian of gauge theories like gravity needed to be written in the form of constraints was found by Dirac in 1958 [56]. One year later, Arnowit, Deser and Misner found the Hamiltonian of general relativity that is used today [57]. In 1967, DeWitt used their Hamiltonian to derive a quantum mechanical equation for relativistic space-times [3], which we will review shortly in this section. We will argue that for closed space-times without boundary it leads to a frozen formalism with no dynamics. We end this section by reviewing Hartle and Hawking's work [4] that connect the covariant formalism of quantum gravity with the canonical one.

If we assume that space-time that is globally hyperbolic, we can find a time function t such that each surface $t = const$ is a Cauchy surface Σ and there exists a time flow vector field t^μ satisfying $t^\mu \nabla_\mu t = 1$. The metric tensor $g_{\mu\nu}$ induces a spatial metric γ_{ij} on Σ_t and one can write $g_{\mu\nu}$ as:

$$g_{\mu\nu} = \begin{pmatrix} -N^2 + \beta_k \beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}. \tag{2.17}$$

The function

$$N = \frac{1}{n^\mu \nabla_\mu t} \tag{2.18}$$

is called lapse function, and n^μ is the unit normal to Σ_t. The three dimensional vector β_k is the component of t^μ that is tangential to Σ_t and is called shift vector. The spatial indices are raised and lowered using the 3 metric γ_{ij} and its inverse.

In this section, we neglect the factor $1/16\pi$ in the Lagrangian for simplicity. One can write the Lagrangian of general relativity in the form

$$L = \int d^3x \sqrt{-g} R = \int d^3x N \sqrt{\gamma} \left(K_{ij} K^{ij} - K^2 +^{(3)} R \right), \tag{2.19}$$

The ADM Hamiltonian of general relativity is then given by, see [3, 57, 58]

$$\begin{aligned} H &= \int d^3x (\pi \partial_0 N + \pi^i \partial_0 \beta_i + \pi^{ij} \partial_0 \gamma_{ij}) - L \\ &= \int d^3x \left(N \mathcal{H}_{\mathcal{G}} + \beta_i \chi^i \right) + 2 D_i (\gamma^{-1/2} \beta_j \pi^{ij}) \big), \end{aligned} \tag{2.20}$$

with canonical momenta

$$\pi = \frac{\delta L}{\delta \partial_0 N} = \pi^i = \frac{\delta L}{\delta \partial_0 \beta_i} = 0, \pi^{kl} = \sqrt{\gamma} \left(\gamma^{kl} K - K^{kl} \right). \tag{2.21}$$

In the expression for the Hamiltonian,

$$K_{ij} = \frac{1}{2} N^{-1} (D_j \beta_i + D_i \beta_j - \partial_0 \gamma_{ij}) \tag{2.22}$$

is the extrinsic curvature of the hyper-surface $x^0 = const$, D_i denotes covariant derivation with respect to the i-th direction based on the three metric γ_{ij} and ${}^{(3)}R$ is the curvature scalar with respect to γ_{ij}.

Furthermore

$$\chi^i = 2 D_j \left(\gamma^{-1/2} \pi^{ij} \right) \tag{2.23}$$

and

$$\begin{aligned}\mathcal{H}_{\mathcal{G}} &= \sqrt{\gamma}K_{ij}K^{ij} - \sqrt{\gamma}K^2 - \sqrt{\gamma}^{(3)}R \\ &= \mathcal{G}_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{\gamma}\,^{(3)}R,\end{aligned} \tag{2.24}$$

with the co-called DeWitt "metric"

$$\mathcal{G}_{ijkl} = \frac{1}{2}\gamma^{-1/2}(\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}). \tag{2.25}$$

After integration, the term $2D_i(\gamma^{-1/2}\beta_j\pi^{ij})$ turns out to be a boundary term and can be dropped for closed space-times.

For asymptotically flat space-times, the boundary term is not vanishing but can be shown to be, see [3], or [58], p. 469

$$E_\infty = \int_\Sigma N\sqrt{\gamma}\gamma^{ij}(\gamma_{ik,j} - \gamma_{ij,k}). \tag{2.26}$$

Since $\pi = 0$, the Poisson bracket yields for a closed space-time

$$\{\pi, H\} = \partial_0\pi = 0 = \frac{\partial H}{\partial N} = \mathcal{H}_g. \tag{2.27}$$

Similarly, $\pi^i = 0$, which implies

$$\left\{\pi^i, H\right\} = \partial_0\pi^i = 0 = \frac{\partial H}{\partial \beta_i} = \chi^i. \tag{2.28}$$

Eq. (2.28) is associated with spatial diffeomorphism invariance and therefore called diffeomorphism constraint.

The three metric γ_{ij} and π^{ij} are canonical coordinates and therefore one finds the following Poisson bracket:

$$\left\{\gamma_{ij}(x), \pi^{kl}(x')\right\} = \delta^k_{(i}\delta^l_{j)}\delta(x, x'). \tag{2.29}$$

In the quantum theory, this becomes a commutator relation:

$$\left[\hat{\gamma}_{ij}(x), \hat{\pi}^{kl}(x')\right] = i\delta^k_{(i}\delta^l_{j)}\delta(x, x'), \tag{2.30}$$

with operators $\hat{\gamma}_{ij}$ and $\hat{\pi}^{ij}$ that act on a state functional Ψ which depends on the three metric γ_{ij}. The relation (2.30) is fulfilled if

$$\widehat{\gamma}_{ij}\Psi(\gamma_{ij}) = \gamma_{ij}\Psi(\gamma_{ij}) \tag{2.31}$$

and

$$\hat{\pi}^{ij}\Psi(\gamma_{ij}) = \frac{1}{i}\frac{\delta}{\delta\gamma_{ij}}\Psi(\gamma_{ij}). \tag{2.32}$$

The Hamiltonian constraint of Eq. (2.27) then becomes the so-called Wheeler-DeWitt equation:

$$\hat{\mathcal{H}}_{\mathcal{G}}\Psi(\gamma_{ij}) = \left(\mathcal{G}_{ijkl}\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{kl}} + \sqrt{\gamma}\,^{(3)}R\right)\Psi(\gamma_{ij}) = 0. \tag{2.33}$$

The diffeomorphism constraint also becomes a constraint on the state functional in the quantum theory

$$2D_j\left(\gamma^{-1/2}\frac{1}{i}\frac{\delta}{\delta\gamma_{ij}}\Psi(\gamma_{ij})\right) = 0. \tag{2.34}$$

When the articles of DeWitt appeared, the relation between the perturbative and the canonical versions of quantum gravity were unknown. This changed with the work of Hartle and Hawking [4].

For a quantum theory with a scalar field, the path integral fulfils a Schrödinger equation see, e.g [59]. In analogy to this, the path integral for gravity should satisfy the Wheeler-DeWitt equation. Hartle and Hawking [4] showed that this is the case.

The path integral of quantum gravity an be used to define the expectation value of an operator $\hat{F}$ by

$$\langle\hat{F}(g)\rangle \;=\; \int \mathcal{D}g_{\mu\nu}F(g)e^{iS}. \tag{2.35}$$

One finds

$$-i\frac{\delta}{\delta\gamma_{ij}}\int\mathcal{D}g_{\mu\nu}e^{iS} = \int\mathcal{D}g_{\mu\nu}\pi^{ij}e^{iS} \tag{2.36}$$

from which one can get the operator replacements of the canonical coordinates. Using $\mathcal{H}_{\mathcal{G}} = \frac{\delta S}{\delta N}$, one can derive the Hamiltonian constraint

$$\int\mathcal{D}g_{\mu\nu}\frac{\delta S}{\delta N}e^{iS} = \hat{\mathcal{H}}_{\mathcal{G}}\int\mathcal{D}g_{\mu\nu}e^{iS} = 0. \tag{2.37}$$

Similarly, one may derive the diffeomorphism constraint.

DeWitt analysed the relation between functional integral and the Wheeler-DeWitt equation in more detail [46]. He could show that for a mini-superspace model with a matter field and no minimal coupling, one gets a non-local equation and the local Wheeler-DeWitt equation only holds approximately.

The Hamiltonian constraint in this form is therefore not equivalent to the amplitude in all possible cases. It should therefore not be used as a starting point for quantisation methods that aim towards constructing a correct theory of quantum gravity.

Recently, Feng and Matzner gave a very precise description [60] of the relationship between the Wheeler-DeWitt equation and the gravitational amplitude. They also showed that one has to work with non-local terms in order to make the transition from an amplitude to an equation with a Hamiltonian precisely.

2.4 The problem of time in quantum gravity

The Hamiltonian constraint of general relativity is numerically zero. For the time development of an operator $\hat{A}$ this means

$$\hat{A}(t) = e^{-itH} A e^{itH} = A(0), \tag{2.38}$$

which is obviously inconsistent with observation.

There are several methods to solve this problem. For example, one could try to make a canonical transformation to new canonical variables

$$(\gamma_{ab}(x), \pi^{cd}(x)) \to (X^A(x), P_B(x), \phi^r(x), p_s(x)) \tag{2.39}$$

with $A, B \in \{1, 2, 3\}$ and $r, s \in \{1, 2\}$ where the Hamiltonian constraint takes the form

$$\mathcal{H} = P_A(x) + h_A(x, X^B, \phi^r, p_s). \tag{2.40}$$

One can consider $h_A \neq 0$ as the new the new Hamiltonian of the system. Quantizing it will yield a time dependent Schrödinger equation [61].

Unfortunately, there are several problems with this approach. First, the Groenewold van-Hove theorem [62] states that the canonically transformed system is, after quantisation, in general not unitary equivalent to the quantum theory that one gets from the untransformed system. Since the canonical transformation which gives a time dependence is not unique this results in a multiple choice problem [63].

Furthermore, it was shown by Hajcek [64] that canonical transformations like the one above do not hold globally in general relativity.

Recently, it was shown by Feng and Matzner that the problem of time is there also in the amplitude of quantum gravity [60]. Therefore, this problem is not just an artefact of a "wrong" quantisation method.

DeWitt pointed out in [3] that in asymptotically Lorentzian metrics, the boundary terms one has to add to the Hamiltonian will solve the problem of time.

Quantum gravity should be consistent in closed space-times, too. A different method that yields boundary terms in a closed space-time is the introduction of worm holes, which will be discussed later in this article.

Chapter 3

Partition functions and entropy calculations in Euclidean quantum gravity

3.1 Introduction

The entropy of a thermal quantum field theory with temperature T can be computed from its the canonical partition sum. The canonical partition sum is given for a quantum field theory by the Wick rotated and Euclideanized amplitude. In quantum gravity, the canonical partition function is the path integral over all positive semi-definite metric tensors

$$Z = \int Dg_{\mu\nu} e^{-I-I_{gauge-fixing}-I_{ghosts}} \tag{3.1}$$

weighted by the Euclidean action

$$I = -\frac{1}{16\pi} \int d^4x \sqrt{g} R \tag{3.2}$$

and terms for gauge fixing and ghosts. Z can be represented as a sum over "microstates"

$$Z = \sum_n \langle g_n | e^{-\beta \hat{H}} | g_n \rangle. \tag{3.3}$$

In Eq. (3.3), $\beta = 1/T$ is the inverse temperature which is also a periodic Euclidean time parameter and $\hat{H}$ is the quantized form of a classical constraint Hamiltonian.

Unfortunately, there is some confusion in the recent literature, with some authors claiming one would not know what these "micro-states" in the partition function are and other authors even arguing the micro states would be some "constituents" or "particles" like gravitons [65] and others [66] claim there would not be a concrete picture of them. To these claims I just want to mention that at least in the articles of Hawking, the micro states $|g_n\rangle$ are defined as eigenstates of the Hamiltonian constraint $\hat{H}$.

Because gravity is invariant under diffeomorphisms, the Hamiltonian operator consists, as we have seen in subsection 2.3 of a diffeomorphism constraint $\hat{H}_{diff}$ and a gravitational constraint $\hat{H}_{\mathcal{G}}$ and potentially some contributions from boundary terms $\hat{H}_{\partial M}$:

$$\hat{H} = \hat{H}_{\mathcal{G}} + \hat{H}_{diff} + \hat{H}_{\partial M}, \tag{3.4}$$

where

$$\hat{\mathcal{H}}_{\mathcal{G}} = \left(\mathcal{G}_{ijkl} \frac{\delta}{\delta\gamma_{ij}} \frac{\delta}{\delta\gamma_{kl}} + \sqrt{\gamma}\,{}^{(3)}R \right). \tag{3.5}$$

In Eq. (3.5), γ_{ij} is a 3 metric, ${}^{(3)}R$ is a Ricci scalar computed from γ_{ij} and $\mathcal{G}_{ijkl} = (\gamma_{ik}\gamma_{jl} + \gamma_{il}\gamma_{jk} - \gamma ij \gamma_{kl})$ is the DeWitt metric as we have seen in subsection 2.3.

A quantum system described by Eq. (3.4) fulfils the time dependent Wheeler-DeWitt equation [3]

$$i\hbar\partial_t|\Psi(\gamma_{ij}, t)\rangle = \hat{H}|\Psi(\gamma_{ij}, t)\rangle. \tag{3.6}$$

with states $\Psi(\gamma_{ij}, t) = e^{-i\hat{H}t}|g_n(\gamma_{ij})\rangle$ that depend on γ_{ij} and time. In order to shorten the notation, we will abbreviate $|g_n\rangle \equiv |g_n(\gamma_{ij})\rangle$ for the time independent states from now.

As we have seen in subsection 2.3, The Wheeler-DeWitt equation is formally solved by the amplitude of Lorentzian quantum gravity[4]

$$|\Psi(\gamma_{ij}, t)\rangle = \int Dg_{\mu\nu} e^{iS + iS_{gauge-fixing} + iS_{ghosts}} \tag{3.7}$$

where

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} R \tag{3.8}$$

is the Lorentzian action of the gravitational field and one has to add ghost and gauge fixing terms to the action in the path integral.

The amplitude in Eq. (3.7) is one loop finite [41] but it diverges at higher orders in perturbation theory [47]. Yet, it is currently unknown whether it is non-perturbatively finite or not. There are some indications that it is finite. The program that investigates this question is called asymptotic safety [67, 68].

For pure gravity amplitudes, the one loop finiteness of the amplitude makes it is possible to do a WKB approximation and solve the Wheeler-DeWitt equation approximately with an Ansatz Ce^{iS} where S is the gravitational action.

It is known that the Wheeler-DeWitt equation does have an algebra where the commutator of two certain operators can not be consistently computed if the observables are infinitely close in space-time [3] (i.e. the algebra does not close in this situation).

On the quantum level, one may have a space-time where wormhole production takes place at short time scales. Space-times with wormholes are not simply connected (i.e they have a hole of finite size). Thus, the impossibility of consistently computing certain observables at infinitesimally close locations may not signal an inconsistency. The problem is then whether quantum gravity can consistently describe the production of wormholes. In section 5.4, we will investigate this and find that inconsistencies emerge.

As we have seen in subsection 2.4, the classical Hamiltonian is usually taken to be zero if no boundary terms $H_{\partial M}$ are added to H. In that case, the Wheeler-DeWitt equation becomes

$$\hat{H}|\Psi(\gamma_{ij}, t)\rangle = 0 \tag{3.9}$$

with time independent states $|\Psi(\gamma_{ij}, t)\rangle = |g_n\rangle$.

Even if this would lead to all expectation values being stationary, [3, 63, 64], which disagrees with experiment, one still can do thermodynamics in this situation. The canonical ensemble then simply becomes the microcanonical ensemble. The partition sum of the microcanonical ensemble goes over all states in the energy shell $E = 0$ and one has

$$Z_m = \sum_n \langle g_n|g_n\rangle \tag{3.10}$$

If quantized boundary constraints $\hat{H}_{\partial M}$ are added to $\hat{H}$ (like, for example, in asymptotically flat space-times, see [3]), then one finds the usual time development of the state functionals $|\Psi(\gamma_{ij}, t)\rangle$ in the time dependent Wheeler-DeWitt equation. If there several energetic configurations available, one can then define a useful canonical ensemble

$$Z = \sum_n \langle g_n|e^{-\beta\hat{H}}|g_n\rangle \tag{3.11}$$

since in the presence of boundary terms, the time independent Wheeler-DeWitt equation becomes

$$\hat{H}|g_n\rangle = \hat{H}_{\partial M}|g_n\rangle. \tag{3.12}$$

The microstates $|g_n\rangle$ in the partition sum of Eq. (3.3) are the eigenstates of $\hat{H}$ after a Wick rotation, i.e. solutions of Eq. (3.12) for an Euclideanized space-time

If one wants to compute the partition sum of a space-time, one needs to compute the path integral over the Euclidean action, whose finiteness properties are usually a bit different from path integrals with Lorentzian actions.

For the path integral over asymptotically Euclidean space-times, one can show a positive mass theorem and with this one can show a positive action theorem. This was used by Schleich to define convergent path integrals of Euclidean quantum gravity for asymptotically flat backgrounds [69].

For other backgrounds, Gibbons, Hawking and Perry considered the path integral over the Euclidean action, but without gauge fixing and ghost terms. They found that the action is not bounded from below, see [40]. More recently, Dasgupta argued if one includes the gauge fixing and ghost terms, they would render the effective action of Euclidean quantum gravity finite [5, 6].

3.2 Calculation of the black-hole entropy from the path integral

This section reviews the computation of the black-hole entropy from Euclidean quantum gravity by Gibbons and Hawking in [7]. It is also adapted from my earlier diploma thesis [17] with some changes (shortened derivations, different text)

Usually, the action of the Euclideanized gravitational field is taken to be $16\pi I = -\int d^4x\sqrt{g}R$. However a boundary term is omitted in this action that usually does not vanish for space-times which are not closed or have boundaries

Varying the gravitational action yields

$$\begin{aligned} 16\pi\delta I &= -\int d^4x\,(\sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{g}R_{\mu\nu}\delta g^{\mu\nu} + R\delta\sqrt{g}) \\ &= -\int d^4x\left(\sqrt{g}g^{\mu\nu}\delta R_{\mu\nu} + \sqrt{g}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}\right). \end{aligned} \tag{3.13}$$

where we have used that $R\delta\sqrt{g} = -\frac{1}{2}R\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu}$. Using

$$\delta R_{\mu\nu} = \nabla_\lambda \delta\Gamma^\lambda_{\nu\mu} - \nabla_\nu \delta\Gamma^\lambda_{\lambda\mu}, \tag{3.14}$$

the first term can be converted into a surface integral and further evaluation shows that it is equal to $2\sqrt{\gamma}K$, see [58, 70], where K is the trace of the extrinsic curvature and γ is the induced three metric on the boundary.

For open space-times or space-times with boundary, the new action therefore reads

$$16\pi I = -\int d^4x\sqrt{g}R - 2\int_{\partial M} d^3x\sqrt{\gamma}K - C, \tag{3.15}$$

where C is a constant that is independent of g. In asymptotically flat space a natural choice for C is such that $I = 0$ for the Minkowski metric $\eta_{\mu\nu}$, or

$$C = -2\int_{\partial M} d^3x\sqrt{\gamma}K^0, \tag{3.16}$$

with K^0 as the extrinsic curvature at the boundary embedded in flat space.

The Euclidean path integral

$$Z = e^{-I(g_{\mu\nu})}\int \mathcal{D}h_{\mu\nu}e^{-\underline{\underline{I}}(h_{\mu\nu})} \tag{3.17}$$

can be used in the case where the background metric $g_{\mu\nu}$ is a black-hole.

We now assume that $g_{\mu\nu}$ is given by the Schwarzschild metric

$$ds^2 = \frac{32M^3}{r}e^{\frac{-r}{2M}}(-dz + dy) + r^2 d\Omega^2 \tag{3.18}$$

in Kruskal coordinates, where

$$-z^2 + y^2 = \left(\frac{r}{2M} - 1\right)e^{\frac{r}{2M}} \tag{3.19}$$

and

$$\frac{(y+z)}{(y-z)} = e^{\frac{t}{2M}}. \tag{3.20}$$

The singularity lies at $-z^2 + y^2 = -1$. Setting $\zeta = iz$, the metric becomes positive definite:

$$ds^2 = \frac{32M^3}{r}e^{\frac{-r}{2M}}(d\zeta + dy) + r^2 d\Omega^2 \tag{3.21}$$

With

$$\zeta^2 + y^2 = \left(\frac{r}{2M} - 1\right)e^{\frac{r}{2M}}, \tag{3.22}$$

the coordinate r will be real and greater than or equal to $2M$ as long as y and ζ are real. Eq. (3.20) shows that setting $t = -i\tau$ implies τ has a period of $8\pi M$.

The Euclidean path integral is a canonical partition sum. For a field φ at two different times t_1 and t_2 with a Hamiltonian H and an action S

$$\langle\varphi, t_2|\varphi, t_1\rangle = \int \mathcal{D}\varphi e^{iS} = \langle\varphi, t_2|e^{-iH(t_2-t_1)}|\varphi, t_1\rangle \tag{3.23}$$

where the path integral is over all field configurations that take the value $|\varphi, t_1\rangle$ at t_1 and $|\varphi, t_2\rangle$ at t_2. If we set $t_2 - t_1 = -i\beta$ and $\varphi_1 = \varphi_2$ and sum over all $|\varphi, t_i\rangle$, the canonical partition sum turns out to be equal to the Euclidean path integral with an Euclidean action I

$$tr(e^{-\beta H}) = \int \mathcal{D}\varphi e^{-I} = Z, \tag{3.24}$$

where the path integral is taken over all fields with period in β in imaginary time.

Since the Euclidean section of the Schwarzschild metric has $R = 0$, the non-zero part of the action comes from the boundary term

$$16\pi I = -\int d^4x\sqrt{g}R - 2\int_{\partial M} d^3x\sqrt{\gamma}(K - K^0). \tag{3.25}$$

Evaluation of the integral over the intrinsic curvature yields, see [7]:

$$I = 4\pi M^2 = \frac{\beta^2}{16\pi} \tag{3.26}$$

From statistical mechanics, we have

$$\langle E\rangle = -\frac{\partial}{\partial\beta} ln(Z) = \frac{\beta}{8\pi}. \tag{3.27}$$

The entropy of the canonical ensemble is then given by

$$S = \beta\langle E\rangle + ln(Z) = \frac{\beta^2}{8\pi} - \frac{\beta^2}{16\pi} = \frac{\beta^2}{16\pi} = 4\pi M^2 = \frac{1}{4}A, \tag{3.28}$$

where A is the area of the event horizon.

3.3 Gibbons' and Hawking's proof that the gravitational action can only contribute to the entropy if the action has a boundary term

In the Euclideanized space-time, the curvature scalar R, the extrinsic curvature K and the tensor of the electromagnetic field F_{ab} are holomorphic functions.

Therefore, the integral over the space-time in the action is a contour integral that has the same value on any section of the complexified space-time which is homologous to the Euclidean section, even if the induced metric of this section can be complex. Gibbons and Hawking argue in their work [7] that this allows them to consider metrics which have no real Euclidean section. As an example, they study the Kerr metric that describes an electrically charged rotating black-hole.

In a further step, they then discuss a black-hole that is surrounded by a perfect fluid which rotates rigidly with an angular velocity. Gibbons and Hawking find an action

$$I = 2\pi\kappa\left(M - \Omega_H J_H - \Omega_M J_M - \frac{A}{8\pi} + \int \rho K^a d\Sigma_a\right), \tag{3.29}$$

where $\kappa = \frac{1}{4M}$ and M is the total mass of the system. $\Omega_{H/M}$ and $J_{H/M}$ are the angular velocity and angular momentum of the black-hole/matter disk and A is the black-hole area. As we have seen in section 3.2, this area is computed from a boundary term. Similar boundaries also occur in worm holes, even if they have no event horizon. In fact, Gibbons and Hawking argue that even apparent horizons, which can occur in DeSitter spaces give rise to such boundary terms and thus a contribution to the gravitational entropy. K^a is the time translation Killing vector and ρ is the energy density of the matter. Σ is a surface that connects the boundary of a large sphere around the entire configuration with the bifurcation surface of the horizon of the black-hole.

The total mass M of the system is given by

$$M = M_H + 2\int_\Sigma (T_{ab} - \frac{1}{2}T)K^a d\Sigma^b \tag{3.30}$$

where M_H as the mass of the black-hole. The energy momentum tensor is given by

$$T_{ab} = (p+\rho)u_a u_b + p g_{ab} \tag{3.31}$$

where p is the pressure of the matter and ρ is its the energy density. u_a is a four velocity that can be expressed as

$$\lambda u^a = K^a + \Omega_M m^a, \tag{3.32}$$

where λ is a normalization constant and m^a is the axial Killing vector.

The calculation of canonical partition sums with quantum field theory can be extended to include partition sums of the grand canonical ensemble.

$$Z_g = Tre^{-\beta\left(H - \sum_i \mu_i C_i\right)}, \tag{3.33}$$

where μ_i are chemical potentials associated with conserved quantities C_i.

For example, for a system with angular angular momentum $C_i = J$, the associated chemical potential is the angular velocity $\mu_i = \Omega$.

For a grand canonical partition sum, one has

$$TlnZ_g = -W \tag{3.34}$$

with

$$W = M - TS - \sum_i \mu_i C_i \tag{3.35}$$

as grand potential. Whether the amplitude of the quantum field theory yields a grand canonical or just a canonical ensemble depends on the form of the action I since the partition sum is approximated by

$$Z_g \approx e^{-I}. \tag{3.36}$$

With a temperature of

$$T = \frac{\kappa}{2\pi}, \tag{3.37}$$

the leading gravitational contribution to the grand potential for a rotating black-hole surrounded by a matter disk becomes

$$W_g = -Tln(Z_g) = M - \Omega_H J_H - \Omega_M J_M - \frac{A}{8\pi} + \int \rho K^a d\Sigma_a \tag{3.38}$$

If there is no boundary in the space-time, one just has a rotating gas disk and $J_H = A = 0$. The gravitational contribution to its grand potential is

$$W_g = M - \Omega_M J_M + \int_\Sigma \rho K^a d\Sigma_a \tag{3.39}$$

Since the matter disk can be described as a gas with temperature T, there will be an additional contribution W_M to the thermodynamical potential from quantized matter fields. Gibbons and Hawking compute an approximate contribution from matter fields in thermal equilibrium to the partition sum as

$$\frac{W_M}{T} = \frac{\int pK^a d\Sigma_a}{T}. \tag{3.40}$$

This implies a grand potential

$$W = W_g + W_M = M - \Omega_M J_M + \int_\Sigma (p+\rho) K^a d\Sigma_a. \tag{3.41}$$

On the other hand, one has from thermodynamics

$$dU + pV = TS + \sum_i \mu_i N_i \tag{3.42}$$

where U is the energy, T the temperature, S is the entropy and N_i is the particle number of species i. This implies for the corresponding densities that

$$p + \rho = \overline{T}s + \sum_i \mu_i n_i \tag{3.43}$$

with $\overline{T}$ as the local temperature and s as the entropy density of the fluid. $\overline{\mu}_i$ are the local chemical potentials and n_i are the number densities of the i-th particle species. As a result Gibbons and Hawking can write the grand potential as

$$W = M - \Omega_m J_m + \int_\Sigma \left(\overline{T}s + \sum_i \overline{\mu}_i n_i \right) K^a d\Sigma_a. \tag{3.44}$$

Gibbons and Hawking argue that in thermal equilibrium, one has

$$\overline{T} = \frac{T}{\lambda},\ \overline{\mu}_i = \frac{\mu_i}{\lambda} \tag{3.45}$$

with λ as normalization factor, and T and μ_i as values of $\overline{T}$ and $\overline{\mu}_i$ at infinity. Using

$$\lambda u^a = K^a + \Omega_m J_m, \tag{3.46}$$

they find for the entropy

$$S = -\int s u^a d\Sigma_a \tag{3.47}$$

This expression just contains the entropy density of the fluid s and the 4 velocity u^a with which it rotates.

From this, Gibbons and Hawking conclude that if there is no boundary term that may be associated with a worm hole or the event horizon of a black-hole or an apparent horizon, the action of the gravitational field does not contribute to the entropy.

Part III

The space-time at the Planck-scale

Chapter 4

Space-time foam

4.1 Introduction

In the next sections, we will argue that the cosmological constant should be associated with boundary terms since it implies a contribution of the action to the gravitational entropy. We argue that observed state of the quantum gravitational field for a given volume should then be found by maximizing this gravitational entropy.

In this context, we will consider the space-time foam model of Hawking. We will modify it and remove an approximation that holds only for negative Λ. We also use a renormalization scale from the current literature. Finally, we use the corrected coefficients from Christensen and Duff [55] in the perturbative expansion of the amplitude.

We find that all these modifications imply that $\Lambda \geq 0$ and $\Lambda \approx H_0^2$ in thermal equilibrium, where H_0 is Hubble's constant. Furthermore, Λ was very large at the beginning of the universe and runs to $\Lambda \to 0$ during the expansion of the universe.

Hawking's evaluation of the amplitude is a non-perturbative path integration over classical metrics followed by a perturbative expansion.

We consider a comment of Christensen and Duff who argued that the non-perturbative evaluation does not always converge. Christensen and Duff propose to use matter particles to save the model. We argue that a proper choice of the renormalization scale suffices to render the integration finite. This would exclude certain universes as backgrounds, but only ones which already disagree with observation.

The modified space-time foam model predicts a negative Euler characteristic for the universe. We conclude that this suggests that the quantum fluctuations give rise to a gas of worm holes. As this is not a translationally invariant ground state, the space-time foam model evades Weinberg's no-go result.

Finally, we add matter terms to the model. We find that it solves the cosmological constant problem and the coincidence problem at first order in perturbation theory.

It is known that the effective matter action in higher orders of perturbation theory generates an f(R) gravity. We find that these higher order terms change the cosmological constant only slightly.

We argue that the effective matter action can be used to define a gravity action where one can avoid the Ostrogradski instability if the masses of fermions and bosons do have some non-trivial relationship.

Worm holes are space-times with a boundary. We argue that a space-time foam in form of a worm hole gas solves the problem of time.

This chapter of the thesis has appeared in the publication [8] of the author. I want to close this introduction by mentioning various references which try to solve the cosmological constant problem similarly by using black- or worm holes. For example, there are early articles from Coleman [71], Klebanov, Suesskind and Banks [72] or Preskill [73] which attempt to solve the cosmological constant problem with worm holes. Unfortunately, they do not get its numerically correct value. More recent articles proposing similar ideas are from Carlip [74], Padmanabhan [75], Xue [76] or Cyriac [77]. These papers, however, do not solve the problem of time or the coincidence problem. Furthermore, they often use assumptions that do not entirely come from the quantum gravity amplitude alone. Finally, there are articles like [78] or the recent proposal [79], which suggest the cosmological constant is related to the problem of time but these authors to not get the value of the cosmological constant.

4.2 The space-time foam model of Hawking and why it seems necessary to modify it

When DeWitt quantized the Friedmann Robertson Walker space-time in [3], he included the cosmological constant term into the vanishing Hamiltonian constraint.

As we have seen, a vanishing Hamiltonian implies the problem of time, i.e. that the observables to not have dynamics.

The cosmological constant was introduced by Einstein in [80, 81] who set $\Lambda = 4\pi\rho$, where ρ is the matter density. This was done in an attempt to model a static universe. For a static universe, one could consider it acceptable if the microstates and observables do not have dynamics. With a Hamiltonian constraint $\hat{H} = 0$, the partition sum is then over microstates in the energy shell for this vanishing Hamiltonian. This is a suitable condition for a microcanonical ensemble.

However, many space-times with cosmological constant describe an expanding or contracting universe. An expanding universe is a dynamical system. Having static microstates and observable appears to be inconsistent with a dynamical macrostate.

Furthermore, in units, one has an energy density

$$\frac{c^2}{8\pi G}\Lambda = \rho_{vac}. \tag{4.1}$$

It is therefore natural to consider

$$\frac{\Lambda^{(3)}V}{8\pi}, \tag{4.2}$$

where ${}^{(3)}V$ is the three volume, as the energy of a closed universe.

For a non-zero Hamiltonian, one would need boundary terms, which a closed space-time is assumed not to have. In the following sections, we will argue that the gravitational amplitude leads to a universe filled with worm holes. Space-times with non-traversable worm holes provide such boundaries, with which the cosmological constant can be associated. In section 4.5, we will propose to describe the non-zero part of the gravitational Hamiltonian by a number of time-like boundary terms.

The canonical partition sum of gravity with a non-vanishing Hamiltonian is given by

$$Z = \langle g_n | e^{-\beta H} | g_n \rangle = \int dg_{\mu\nu} e^{-I - I_{gauge-fixing} - I_{ghosts}}, \tag{4.3}$$

which one can approximate in the one loop or WKB expansion as

$$Z \approx C e^{-I}, \tag{4.4}$$

where C is a one-loop correction factor and

$$I = \frac{-1}{16\pi} \int d^4x \sqrt{g}(R - 2\Lambda) \tag{4.5}$$

is the gravitational action with curvature scalar R and cosmological constant Λ.

If we set the one loop correction $C = 1$, the partition function of quantum gravity in the WKB approximation is given by

$$Z = e^{\frac{1}{16\pi}\int d^4x\sqrt{g}R - \frac{1}{8\pi}\int d^4x\sqrt{g}\Lambda} \tag{4.6}$$

$$= e^{\frac{1}{16\pi}\int d^4x\sqrt{g}R - \frac{\Lambda V}{8\pi}} \tag{4.7}$$

$$\tag{4.8}$$

The free energy F is defined by

$$F = U - TS \tag{4.9}$$

where U is the inner energy of the system, T is the temperature and S the entropy.

Using $\beta = \frac{1}{k_b T}$, the free energy F is related to the canonical partition sum as

$$Z = e^{-\beta F} = e^{-\beta U + \beta TS} = e^{-\beta U + \frac{1}{k_b} S} \tag{4.10}$$

This suggests at first to identify $\frac{1}{16\pi}\int d^4x\sqrt{g}R$ with $\frac{S}{k_b}$ and $\frac{^{(3)}V\Lambda}{8\pi}$ with U as the time coordinate plays the role of β in an Euclidean thermal field theory.

In contrast to ordinary thermodynamics, there appear some differences in closed space-times.

In cosmology, one often has a certain space, e.g. a DeSitter space with positive cosmological constant that one wants to consider. Hawking writes in [82] that for positive Λ, the Euclidean solutions are necessarily compact, but he appears to give no proof for the general case.

In his work on space-time foam [9], Hawking considers the Euclidean action in the form $I = -\frac{\Lambda V}{8\pi}$ where Einstein's equations $R = 4\Lambda$ have been used and a compact Euclidean space-time with 4 volume V was assumed. With help of dimensional arguments, one can write

$$\sqrt{V} = -f/\Lambda, \tag{4.11}$$

where f is some dimensionless factor which can be positive or negative. The action then becomes

$$I = -\frac{f^2}{8\pi\Lambda}. \tag{4.12}$$

For classical metric, $R = 4\Lambda$. Then, using $\Lambda^2 V = f^2$, one gets $\frac{1}{16\pi}\int d^4x\sqrt{g}R = \frac{f^2}{4\pi\Lambda}$.

In Lorentzian gravity, a DeSitter space is an R $\times$ S^3 space. In Euclidean quantum gravity, time becomes periodical. The Euclidean version of DeSitter space is a compact S^4 space with a certain 4 volume V.

The DeSitter space has $f = -4\sqrt{6}\pi$ and the action I contributes

$$S = -I = \frac{12\pi}{\Lambda} = \frac{f^2}{8\pi} \tag{4.13}$$

to the entropy [7] for an observer inside an apparent horizon. This entropy can be verified independently of quantum gravity with a computation of an ordinary bosonic or fermionic quantum field in a classical DeSitter background. It is obviously not equal to $\frac{f^2}{4\pi\Lambda}$ which one would get from the canonical ensemble in Eq. (4.10). In order to get the interpretation with the canonical ensemble correct, one could argue Eq. (4.10) would describe a larger system than a local observer can see in one patch. For example, if this action would describe a universe with pairs of Kerr black-holes in space-like separated regions whose inner regions are connected by an Einstein-Rosen bridge, then one could argue that the entropy of the entire system would be twice the entropy that a local observer could see. Another possibility would be to consider an entropy that would be computed as $S = \frac{1}{2}\frac{1}{16\pi}\int d^4x\sqrt{g}R$ and not as in Eq. (4.10).

We have learned in section 1.4 that the cosmological constant is given by

$$\Lambda = \Lambda_g + \Lambda_m, \tag{4.14}$$

where Λ_m is a contribution to the energy density that is due to matter. Λ_m is negative and very large. The contribution Λ_g is due to gravity and must be such that Λ is very small.

We also have learned from Hawking's calculation in section 3.3 that the curvature which matter can cause in the space-time does not led to a gravitational field whose gravitational action contributes to the entropy, unless the matter causes a curvature which is so strong that the space-time develops a boundary.

As we have seen above, the action of general relativity with cosmological constant contributes to the entropy. As a result, one has to conclude that Λ_g must indeed be associated with boundary terms.

In [9], Hawking writes the Euclidean action I of gravity with cosmological constant as

$$I = -\frac{f^2}{8\pi\Lambda}. \tag{4.15}$$

In his computation, Hawking makes the approximation $f^2 = \Lambda^2 V \approx d\chi$, where d is some scalar factor and the Euler characteristic is given by

$$\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{g}(C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \Lambda^2). \tag{4.16}$$

where $C_{\alpha\beta\gamma\delta}$ is the Weyl tensor.

This approximation is based on the assumption that the Weyl tensor should not contribute to χ because if it were large, it would lead to conjugate points that one finds in black-holes. However, the Weyl tensor appears to be connected to a measure for the entropy of a space-time, see [83, 84].

Furthermore, Hawking writes in [9] that his space-time foam model describes a gas of black-holes and his computation proceeds by defining a partition sum. In a computation of the gravitational entropy, removing a quantity that describes the entropy of a space-time seems to be problematic to the author.

In order to avoid this problem, we will not write the action in the form

$$I = -\frac{d^2\chi}{8\pi\Lambda} \tag{4.17}$$

that Hawking uses, but we will repeat some of his calculations with the usual Euclidean action

$$I = -\frac{\Lambda V}{8\pi} = -\frac{f^2}{8\pi\Lambda}. \tag{4.18}$$

Using

$$\Lambda^2 V = f^2, \tag{4.19}$$

the cosmological constant can be interpreted as a factor that rescales the action from Eq. (4.15) according to

$$I = -\frac{\Lambda V}{8\pi} = \frac{1}{\Lambda} I_0, \tag{4.20}$$

where

$$I_0 = -\frac{f^2}{8\pi}. \tag{4.21}$$

The works of Gibbons Hawking and Perry [40] that we have reviewed in section 2.2 show that if the action is rescaled by $1/\Lambda$ then, the eigenvalues of the determinants in the zeta function renormalization scheme are multiplied by Λ. The amplitude of quantum gravity in the WKB approximation $Z = Ce^{-I}$ then scales according to

$$\tilde{Z} = Ze^{\left(1-\frac{1}{\Lambda}\right)I_0(g_{\mu\nu})} \left(\frac{1}{\Lambda}\right)^{\frac{1}{2}\gamma}. \tag{4.22}$$

As we have argued in section 2.2, the factor γ was first given in [40] but then corrected in [55] for space-times with cosmological constant. The result from [55] is

$$\gamma = \frac{106}{45}\chi - \frac{87}{120\pi^2}\Lambda^2 V. \tag{4.23}$$

With an unrescaled amplitude

$$Z=\left(2\pi\mu^2\right)^{\frac{1}{2}\gamma}e^{-I_0+\frac{1}{2}\zeta_F'(0)+\frac{1}{2}\zeta_G'(0)-\zeta_C'(0)} \tag{4.24}$$

one has

$$\tilde{Z}=e^{-\frac{1}{\Lambda}I_0}e^{\frac{1}{2}\zeta_F'(0)+\frac{1}{2}\zeta_G'(0)-\zeta_C'(0)}\left(\frac{\Lambda}{2\pi\mu^2}\right)^{-\frac{1}{2}\gamma} \tag{4.25}$$

If we neglect the term $\frac{1}{2}\zeta_F'(0)+\frac{1}{2}\zeta_G'(0)-\zeta_C'(0)$ in the amplitude then the factor C in the WKB approximation $Z=Ce^{-I}$ can then be estimated as

$$C\propto\left(\frac{\Lambda}{2\pi\mu^2}\right)^{-\frac{1}{2}\gamma} \tag{4.26}$$

Finally, the amplitude $Z=Ce^{-I}$ on a classical background that fulfils $R=4\Lambda$ can be written as

$$Z=\left(\frac{\Lambda}{2\pi\mu^2}\right)^{-\frac{1}{2}\left(\frac{106}{45}\chi-\frac{87}{120\pi^2}\Lambda^2V\right)}e^{\frac{\Lambda V}{8\pi}}. \tag{4.27}$$

From Gibbons, Hawking and Perry [40] comes the following attempt to evaluate the gravitational path integral: Let an ensemble of matter fields generate a cosmological constant Λ_m. Then, by dividing the metric $\tilde{g}_{\mu\nu}$ into conformal equivalence classes $\{g\}$ where

$$\tilde{g}_{\mu\nu}=\Omega^2g_{\mu\nu} \tag{4.28}$$

the amplitude is then given by

$$Z=\int D\left\{g_{\mu\nu}\right\}DY(\{g_{\mu\nu}\}) \tag{4.29}$$

where

$$Y(\{g_{\mu\nu}\})=\int d\Omega e^{\frac{1}{8\pi}\int d^4x\sqrt{\tilde{g}}\left(3\Omega A\Omega-\Omega^4\Lambda_m\right)} \tag{4.30}$$

and

$$A=\Box+\frac{1}{6}R. \tag{4.31}$$

Let us assume that $\Lambda_m<0$ is negative (as it is the case if we let all matter fields in the standard model generate a cosmological constant in curved space-time). The action in a single conformal equivalence class has stationary points $R=4\Lambda$. By using an imaginary conformal factor, one would then get, due to the $3\Omega A\Omega$ term in the action, $\Omega\frac{3}{6}R\Omega = \frac{1}{2}\Omega 4\Lambda_m\Omega > 0$ which is larger than $\Omega^4\Lambda_m$ if $|\Omega^2| < 1$. This mechanism can turn any initial negative energy density into an effective positive cosmological constant $\Lambda_{eff} > 0$ with a positive action. Usually, the integration over conformal factors is believed to lead to a divergent path integral since the action is not bounded from below. However, Dasgupta has shown in [5, 6], that the action becomes bounded if one adds a ghost and gauge breaking term. Therefore, we assume that we can do this integration.

Hawking notes in [9] that for a usual thermal field theory, the canonical partition function can be written as

$$Z = \int_0^\infty dE N(E) e^{-\beta E}, \tag{4.32}$$

where $N(E)$ is the density of states between E and $E + dE$ and $\beta = 1/k_bT$ is a Lagerange multiplier.

In analogy to this, Hawking postulates that the entire amplitude of quantum gravity can be represented by a partition function that he calls "volume canonical ensemble"

$$Z(\Lambda) = \int_0^\infty dV N(V) e^{-\frac{\Lambda V}{8\pi}}. \tag{4.33}$$

In analogy to the theory of thermodynamics, the function $N(V)$ is then interpreted as the number of states between V and $V + dV$. Hawking then computes $N(V)$ as in ordinary thermodynamics by an inverse Laplace transform [85] of $Z(\Lambda)$. He inserts the amplitude of Eq. (4.27) into

$$N(V) = \int_C Z(\Lambda) e^{\frac{\Lambda V}{8\pi}} d\Lambda \approx Z(\Lambda_s) e^{\frac{\Lambda_s V}{8\pi}}, \tag{4.34}$$

where C is a contour running parallel to and to the right of the imaginary axis,

Here, I want to comment on an article by Christensen and Duff [55]. They corrected the factor γ in the perturbative expansion that was calculated by Hawking, Gibbons and Perry. Then they noted that with the corrected factor, the inverse Laplace transform does not converge if $\gamma < 0$.

If $\gamma < 0$, the integral diverges because the inverse Laplace transform of a function $f(z)$ only exists if $\lim_{z\to\infty} f(z) = 0$ and the amplitude of Eq. (4.27) is similar to $f(\Lambda) = \Lambda^{-\gamma}e^{1/\Lambda}$, which diverges at $\Lambda \to \infty$ for $\gamma < 0$.

Christensen and Duff propose to cure this problem by adding matter terms. Here, we want to note that problem has a simpler solution. By setting $\mu^2 = \sigma\Lambda$, with a proportionality constant $\sigma \geq \frac{1}{2\pi}$, the factor $C(\Lambda)$ is given by

$$C(\Lambda) \propto \left(\frac{1}{2\pi\sigma}\right)^{-\gamma} \leq 1 \tag{4.35}$$

for $\gamma < 0$. This implies that $Z(\Lambda)$ does not diverge any more at $\Lambda \to \infty$ and the Laplace transform should converge.

In his original article, Hawking sets the renormalization scale μ^2 to the Planck length l_p since he assumes that quantum gravity breaks down at this scale. However, this choice is physically problematic.

As we have seen in section 1.4 of this thesis, by deriving the matter contribution of the cosmological constant, one gets

$$\Lambda_m = n_i \frac{m^4}{8\pi} ln\left(\frac{m^2}{\mu^2}\right) \tag{4.36}$$

after a renormalization scheme is applied. The parameter μ^2 in the amplitudes for the massive fields is the same as the parameter μ^2 in the gravity amplitude. Eq. (4.36) is the result of a complete regularization of a renormalizable field theory, where any cut-offs have been sent to infinity and all divergent terms have been removed.

Therefore, the parameter μ is not a cut-off. In Hawking's article on zeta function renormalization, it is described as a renormalization scale that has to be found by experiments.

Interestingly, Hawking presented his space-time foam model later in several conferences. There he correctly describes μ as a renormalization scale parameter that should be determined by measurement. Without having any measurement results from quantum gravity, he refuses to set a concrete value for this scale in these later conference contributions [25].

In section 1.4, we have reviewed arguments from the current literature on the scale parameter. These arguments imply that $\mu = \sqrt{T^{\mu}_{\mu}}$, where T^{μ}_{μ} is the energy momentum tensor, which, for a Friedmann universe, is approximately given by H_0^2. Thus one should set

$$\mu \approx H_0 \tag{4.37}$$

Here, we want to comment on another problem with the approach of Hawking. After evaluation of the inverse Laplace transformation by a saddle point computation, Hawking gets

$$N(V) \approx Z(\Lambda_s) e^{\frac{\Lambda_s V}{8\pi}}. \tag{4.38}$$

where the saddle point Λ_s is given by

$$\frac{\partial}{\partial \Lambda}\left(ln(Z) + \frac{\Lambda_s V}{8\pi}\right)\bigg|_{\Lambda_s} = 0. \tag{4.39}$$

The entropy is then defined as

$$S/k_b = ln(Z(\Lambda_s)) + \frac{\Lambda_s V}{8\pi} \tag{4.40}$$

The amplitude of Eq. (4.27) has already a factor $e^{\frac{\Lambda_s V}{8\pi}} = e^{\frac{f^2}{8\pi\Lambda}}$. Hence, in order to make the entropy that results from this method equal to the expression that one gets from the usual microcanonical ensemble, one has to choose a different partition function, namely

$$Z_{VC} = C(\Lambda) \tag{4.41}$$

that one inserts into the inverse Laplace transform:

$$N(V) = \int_C Z_{VC}(\Lambda) e^{\frac{\Lambda V}{8\pi}} d\Lambda \approx Z_{VC}(\Lambda_s) e^{\frac{\Lambda_s V}{8\pi}}. \tag{4.42}$$

In general, the dominant contributions to a path integral should come from classical paths. If we use a partition sum $Z_{VC} = C$, one gets from Equation (4.42)

$$S = k_b \left(\frac{\Lambda V}{8\pi} + \ln(C) \right). \tag{4.43}$$

The term $\ln(C)$ comes from a perturbative expansion of the metric up to one loop and can be seen as a correction of quantized gravitational waves of low energy to the entropy that one gets from the classical background. Such corrections should be small.

In order for the entropy not to be affected much by this quantum correction, one should have

$$C \approx 1. \tag{4.44}$$

Since

$$C = \left(\frac{\Lambda}{2\pi\mu^2} \right)^{-\frac{106}{90}\chi + \frac{87}{240\pi^2}\Lambda^2 V}, \tag{4.45}$$

setting

$$C \approx 1 \tag{4.46}$$

is consistent with

$$\Lambda \approx 2\pi\mu^2. \tag{4.47}$$

The fact that the inverse Laplace transform only converges for $\mu^2 = H_0^2 = \sigma\Lambda$, with a proportionality constant $\sigma \geq \frac{1}{2\pi}$ seems to exclude some spaces that are classically possible.

For our universe with $H_0^2 \approx 1.18 \cdot 10^{61}$ and $\Lambda = 5.6 \cdot 10^{-122}$ one has $H_0^2/\Lambda = 0.24$. This is larger than $\frac{1}{2\pi} \approx 0.1$ and thus in the appropriate parameter space for the integral to converge. Unfortunately, one can not call it predictive, if quantum gravity excludes space-times that are already ruled out by experiment.

Unfortunately, there are still problems remaining: In ordinary thermodynamics, the partition sums are given by Eq. (4.32). The parameters β and E are treated as functionally independent in the sense that for any given β, one can integrate over all E.

In gravity, this is not so. The 4 volume V of an Euclideanized space-time with positive cosmological constant depends on Λ by $V(\Lambda) = \frac{f^2}{\Lambda^2}$. Hawking notes in [82] that V is bounded from below by the 4 sphere of radius $\sqrt{\frac{3}{\Lambda}}$. This implies that the Euclidean action $I = -\frac{\Lambda V}{8\pi} = -\frac{f^2}{8\pi\Lambda}$ of a classical background where $R = 4\Lambda$, would be bounded from below.

A factor $\frac{\Lambda V}{8\pi}$ that is bounded is in contradiction to Eq. (4.42), where one assumes that for any given V one can integrate over all Λ from $-\infty$ to ∞. If $\frac{\Lambda V}{8\pi}$ has to be bounded, then the contour is not at ∞ but over some finite values and this increases the convergence of the integral.

Unfortunately, with V as function of Λ, it seems one can not bring the integral into the form of an inverse Laplace transform any more. One could try to write

$$N(f^2) = \int_C Z_{VC}(\Lambda) e^{\frac{f^2}{8\pi\Lambda}} d\Lambda. \tag{4.48}$$

and one may substitute $\tilde{\beta} = \frac{1}{8\pi\Lambda}$ to get

$$N(f^2) = \int_{C(w)} Z_{VC}(\tilde{\beta}) \frac{-1}{8\pi\tilde{\beta}} e^{\tilde{\beta} f^2} d\tilde{\beta} \tag{4.49}$$

.

While the integrand would have the form of an inverse Laplace transform, the contour $C(w)$ is problematic. In the original inverse Laplace transform it goes from $-i\infty$ to $i\infty$. With the substitution $\tilde{\beta} = \frac{1}{8\pi\Lambda}$, this contour would not be defined any more.

Form a physical viewpoint, one has of course to incorporate the fact that V is a function of Λ into the calculation. One possible solution could simply be to use $V(\Lambda) = \frac{f^2}{8\pi\Lambda}$ in the computation of the saddle points and the entropy.

From the integration over conformal factors, we have seen that we get Λ as a variable field. We have noted that it was shown in [5, 6] that the integral converges since the effective action is bounded from below. We have argued above that a universe expanding in time would need a non-vanishing Hamiltonian because it has time dependent observables. For a state with non-vanishing Hamiltonian and a well defined energy, the microcanonical partition sum is a function of the energy. In section 4.4, we will associate a number of boundary terms to the cosmological constant which provides a non-vanishing Hamiltonian. We will argue that one therefore may associate to the energy density given by the cosmological constant Λ an energy $\frac{^{(3)}V\Lambda}{8\pi}$, where $^{(3)}V$ is the three volume.

If we regard $Z_m = C(\Lambda)e^{\frac{\Lambda V}{8\pi}}$ as microcanonical partition sum, and want to compute the maximum of $S = ln(Z_m)$ for a given V, then the value Λ_s where this maximum occurs can be computed by setting the derivative of $\frac{dln(Z_m)}{d\Lambda}$ to zero. Since V is a function of Λ, one has to substitute $V(\Lambda) = \frac{f^2}{8\pi\Lambda}$, before one computes the saddle point, i.e one has to compute the maximum of $Z_m = C(\Lambda)e^{\frac{f^2}{8\pi\Lambda}}$.

If one is sceptical with the association of $Z_m = C(\Lambda)e^{\frac{f^2}{8\pi\Lambda}}$ with a microcanonical ensemble, one can also argue as follows:

The amplitude

$$Z = \int_{\{\varphi_1,t_1\}}^{\{\varphi_2,t_2\}} d\varphi e^{-I(\varphi)} \tag{4.50}$$

of a field φ with an Euclidean action I that goes over all paths starting from field configurations φ_1 at t_1 to φ_2 at t_2 is proportional to the probability amplitude $\langle\phi_1, t_1|\phi_2, t_2\rangle$ of finding the configuration at t_2 in a state φ_2. Furthermore, if $t_2 \to \infty$, then the path integral represents the ground state [4, 82].

If we have associated the cosmological constant with a non-vanishing Hamiltonian, one has time dependent states. Given an initial state and a certain 4-volume V as an end configuration, the probability of finding the state at this final configuration can then be computed from the maximum of the amplitude. With Λ as a continuous field, the maximum should be computed by setting the derivative of the amplitude with respect to Λ to zero. The observed value of Λ is then where this maximum occurs.

One should note that this procedure only seems to work if one has a non-vanishing Hamiltonian $\hat{H}$. If $\hat{H} = 0$, then one has no time dependent states and there are no amplitudes evolving from an initial time and a starting configuration to a different configuration in a given interval. We will propose a method how the cosmological constant can be associated with a non-vanishing energy in section 4.4.

The approach to maximize the amplitude in order to find the ground state was first proposed by Hawking in [82]. There, Hawking also mentions briefly that one can get the cosmological constant as a a variable field by summing the gravitational path integral over all classical metrics. However, in [82], he sets the factor C in the amplitude to 1. The one loop correction C contains topological terms. Without them, the article [82] does not discuss any results that may be related to the topology.

In [82], Hawking also notes that one can get the variable cosmological constant from a matter field that makes no other contribution to the equations of motion. This approach was criticised by Duff in [86], who noted that there would arise an inconsistency. The equations of motion would yield a cosmological constant with a different sign than in the action. The problem was cured by Wu in [87] who noted that one needs to add a boundary term to the matter field in order to get a consistent model.

Hawking's idea to threat Λ as a variable field and to maximize the amplitude with respect to that field is similar to the proposal of Barrow and Shaw [88]. A main difference is that the calculations of Barrow and Shaw need a universe with an apparent horizon that can be described with a GHY boundary term. This is not necessary in Hawking's model.

4.3 Calculations with the modified space-time foam model

Before we go into detailed computations, we want to summarize again the differences between Hawking's model and the one presented in this thesis:

- In contrast to the computation of Hawking's article [82], we do not neglect the parameter C in the amplitude $Z = Ce^{-I}$.

- In contrast to Hawking's article [9], we use a different renormalization scale μ^2. We also use a different action I in the amplitude $Z = Ce^{-I}$ without an approximation that only works for $\Lambda < 0$. Furthermore, we do not compute the saddle point from from $Ze^{\frac{\Lambda V}{8\pi}}$ but from the amplitude $Z = Ce^{-I}$, where we always consider that V is a function of Λ.

- In section 4.4 we will discuss what happens when we include the topological contributions of matter terms. We will show under which conditions these terms become ghost free. We will describe how their dependence on the Euler characteristic has to be interpreted physically and how they change the cosmological constant. In [89], Hawking also analysed matter terms. In this article, he presents an argument without a precise calculation which would imply that the topological terms from the matter amplitudes would lead to $\Lambda = 0$. We will use a simple calculation to show that this is not the case.

If we compute the saddle point Λ_{s1} from

$$\left.\frac{\partial\left(\ln Z_{VC}(\Lambda) + \frac{f^2}{8\pi\Lambda}\right)}{\partial\Lambda}\right|_{\Lambda_{s1}} = 0 \tag{4.51}$$

we get the following expression

$$\Lambda_{s1} = -\frac{90\pi f^2}{848\pi^2\chi - 261 f^2}. \tag{4.52}$$

The observed cosmological constant is positive. Therefore, if $\chi > 0$, one must have $\chi < \frac{261}{848\pi^2} f^2$.

The Euler characteristic is given by the alternating sum

$$\chi = \sum_{i=0}^{4} (-1)^i b_i, \tag{4.53}$$

where b_0 is the number of connected components and b_1, b_2, b_3, b_4 are the numbers of $i+1$ dimensional cavities. For a compact space-time,

$$b_0 = b_4 = 1 \tag{4.54}$$

and if the manifold is simply connected then [9]

$$b_1 = b_3 = 0. \tag{4.55}$$

A negative Euler characteristic can therefore only occur for space-times which are not simply connected [9]. This is the case for worm holes, for example.

Substituting $f^2 = \Lambda^2 V$ back into Eq. (4.52) yields

$$\Lambda_{s1} = \frac{5\pi}{29} \pm \frac{\sqrt{225\pi^2 V^2 + 24592\pi^2 V\chi}}{87V}. \tag{4.56}$$

This formula for the stationary point is a bit different from the result of Hawking due to the various changes that we have made.

The different signs in front of the square roots from Eq. (4.56) come from the Λ^2 contribution in the amplitude. They are a result of the usual R^2 corrections from quantum field theory in curved space-times [22].

With the stationary point from Eq. (4.52), one can compute the entropy

$$S/k_b = \ln Z_{VC}(\Lambda_{s1}) + \frac{f^2}{8\pi\Lambda_{s1}}. \tag{4.57}$$

Inserting Eq. (4.52), one sees that for large V the Λ value with the negative sign leads to a larger entropy.

The formula with in Eq. (4.56) with the negative sign in front of the square root yields $\Lambda_{s1} = 0$ in the limit $V = \infty$ for negative χ or for $\chi = 0$ and arbitrary V. Since for our universe, the cosmological constant is very small positive, this solution appears to be physical. Because the contour integration was right to the complex axis, the formula must be used such that it leads to a positive Λ. For $\chi \geq 0$, one would get $\Lambda < 0$ which is forbidden.

The formula with the positive sign has $\Lambda_{s1} = \frac{10\pi}{29}$ in the limit $V \to \infty$ and can never approach smaller values. Since the observed $\Lambda \approx 10^{-122}$ the solution with the positive sign therefore appears to be unphysical.

We observe from Eq. (4.56) that Λ_{s1} is not defined at all for negative χ if $\chi < -\frac{225V}{24592} \approx -0.009V$. This implies that for a finite volume V only a certain number of wormholes can fit, which suggests they have some minimum size.

From the form of Eq. (4.56) and the fact that one has to chose the negative sign, one can make the prediction that $\Lambda \geq 0$ for all volumes V.

If we solve Eq. (4.56) for χ, we get:

$$\chi = 9\Lambda \left(\frac{29\Lambda - 10\pi}{848\pi^2} \right) V = |c|V. \tag{4.58}$$

If we insert the observed value of the cosmological constant of 10^{-122} in Eq. (4.58), we note that χ for our universe has to be negative.

Despite some differences to Hawking's formulas, Eq. (4.58) still describes a space-time filled with one cavity per $|c|^{-1}$ unit Planck volumes. However, the cavity density $|c|$ is now much smaller than the density we get from Hawking's article. In one Planck 4 volume, Hawking gets one gravitational cavity. With Eq. (4.58) and $\Lambda = 5.6 \cdot 10^{-122}$ one gets a number density of only $\approx 1.89 \cdot 10^{-123}$ cavities per unit Planck 4 volume. On macroscopic scales, this is still a large number density. If one considers a 3 Volume of a single cubic meter for a time of one second (in si units), then, $|\chi| \approx 8.56 \cdot 10^{24}$ cavities should have been produced during that second in this 3 volume.

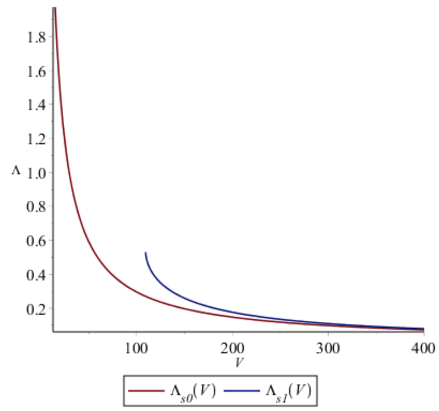

Figure 4.1: Plot of the functions $\Lambda_{s1}(V)$ and $\Lambda_{s0}(V)$ for $\chi = -1$. For smaller χ, both curves fall slower to their limit 0.

If we would do the computation of the stationary point of Λ with an amplitude where the $\Lambda^2 V$ term is omitted, we would get

$$\Lambda_{s0} = -\frac{424\pi\chi}{45V}. \tag{4.59}$$

Plotting Λ_{s1} and Λ_{s0} shows that

$$\Lambda_{s1} > \Lambda_{s0} \text{ for } V \text{ where } \Lambda_{s1} \text{ is defined} \tag{4.60}$$

if we use the same χ as parameter. For the amplitude with the $\Lambda^2 V$ term, the cosmological constant strives to zero much slower for large volumes, see fig. 4.1.

We can calculate the Euler characteristic that maximizes the entropy from

$$\frac{\partial S}{\partial \chi} = -\frac{106}{45} ln(3) - \frac{53}{45} ln(5) - \frac{53}{45} \ln\left(\frac{f^2}{\left(-848\pi^2\chi + 261f^2\right)\mu^2}\right) = 0. \tag{4.61}$$

Solving this equation for μ implies $2\pi\mu^2 = \Lambda_{s1}$ and one easily checks that

$$\frac{\partial^2 S}{\partial \chi^2} = \frac{44944}{45} \frac{\pi}{848\pi^2\chi - 261f^2}, \tag{4.62}$$

Hence $\Lambda_{s1} = 2\pi\mu^2$ can be a maximum only if

$$\chi < \frac{261f^2}{848\pi^2}. \tag{4.63}$$

which holds especially for negative Euler characteristic.

If we go a different route by solving Eq. (4.52) for χ and inserting this in the amplitude as a function of Λ_{s1}, we get

$$\frac{\partial S}{\partial \Lambda_{s1}} = \frac{f^2}{8\pi\Lambda_{s1}^2}\left(ln(2) + ln(\pi) - \ln\left(\frac{\Lambda_{s1}}{\mu^2}\right)\right), \tag{4.64}$$

Setting $\frac{\partial S}{\partial \Lambda_{s1}} = 0$ also shows that $\Lambda_{s1} = 2\pi\mu^2$, which is a maximum since

$$\frac{\partial^2 S}{\partial \Lambda_{s1}^2} = -\frac{f^2}{8\pi\Lambda_{s1}^3}\left(2ln(2) + 2ln(\pi) - 2\ln\left(\frac{\Lambda_{s1}}{\mu^2}\right) + 1\right) \tag{4.65}$$

which is

$$\frac{\partial^2 S}{\partial \Lambda_{s1}^2} = -\frac{f^2}{8\pi\Lambda_{s1}^3} < 0 \tag{4.66}$$

for a positive $\Lambda_{s1} = 2\pi\mu^2$.

One should note that in Hawking's original article, Eq. (4.47) does not follow conclusively as the only possibility because of the different action that he is using.

The coincidence problem of the value of the cosmological constant is now translated into the problem of a choice for the scale parameter μ. In sections 1.4 we have reviewed arguments from the literature and in section 4.2 we have added own additional arguments which imply that one has to set

$$\mu \approx H_0 \tag{4.67}$$

Using Eq. (1.69) in Eq. (4.47) that we computed from the amplitude implies

$$\Lambda \approx 2\pi H_0^2, \tag{4.68}$$

which is approximately what is observed.

The cosmological constant that one gets from the amplitude of quantum gravity therefore seems to be compatible with the relation $H_0^2 \approx \Lambda$ that emerges from the classical Eq. (1.63) for a universe where k and ρ can be neglected. This may be a step to solve the coincidence problem in quantum gravity. However, according to quantum field theory, the value of the scale parameter must ultimately be determined with input from measurement. Showing that $\Lambda \approx 2\pi\mu^2 \approx H_0^2$ for a certain Friedmann universe just makes it necessary to measure H_0.

For precision calculations, one should note that the measured Λ is not exactly equal to $2\pi H_0^2$. Also, setting $\Lambda = 2\pi\mu^2$ exactly would imply that $C = \left(\frac{\Lambda}{2\pi\mu^2}\right)^{-\frac{106}{90}\chi + \frac{87}{240\pi^2}\Lambda^2 V} = 1$. One has $1^x = 1$ for any x, and therefore, any terms with respect to the Euler characteristic would become trivial for $C = 1$. Furthermore, as we have seen, setting $C = 1$ implies that the amplitude is maximized at $\Lambda = 0$.

Thus, one can not have $C = 1$ exactly if the considerations about the topological above should work and if Λ should not be exactly zero.

In the following we will add matter terms and discuss how they change the considerations above.

4.4 Adding matter terms to the modified space-time foam model

In the following, we will describe what happens when we add matter corrections. It is known that in curved space-times, after the application of a suitable renormalization method, the effective Euclidean matter action for a field with mass m becomes

$$I_{m;eff} = A \int d^4x \sqrt{g} \left(\frac{1}{2} m^4 a_0 - m^2 a_1 + a_2 \right), \tag{4.69}$$

where a_i are the Seley-DeWitt coefficients that depend on the matter fields and on the topology and

$$A = \frac{1}{32\pi^2} \ln \left(\frac{m^2}{\mu^2} \right) \tag{4.70}$$

has been computed with some renormalization method of choice. We can then use this effective matter action to write a combined matter gravity amplitude

$$\tilde{Z} = Z_{VC} Z_{matter} \tag{4.71}$$

where Z_{VC} is given by the gravity amplitude $Z_{VC} = C$ of Eq. (4.41) and

$$Z_{matter} = e^{I_{m,eff}}. \tag{4.72}$$

We have worked here with Euclidean quantum gravity, which is derived by a Wick rotation. For fermions, there are several issues when one attempts to construct spinors on an Euclideanized space-time. Several well known attempts are from Osterwalder and Schrader [90], which doubles the number of spinor fields in the Euclidean Space-time. There are other approaches, e.g. from Nicolai [91], whose Euclidean action is not hermitian. Recently, there is a new approach from Nieuwenhuizen and Waldron [92], where the spinors are not doubled and the hermiticity is maintained. In this thesis, we will just assume that we can derive the amplitude in Lorentzian space-time. The effective action does not contain the spinorial fields any more. It only contains terms like the Ricci scalar, Ricci tensor, the Riemann curvature tensor and derivatives. For these terms, the Wick rotation is unproblematic. Therefore, we assume that we can make the Wick rotation after we have derived the effective action.

For fermions, the Seley-DeWitt coefficients are given by a trace over matrices $\tilde{a}_0, \tilde{a}_1, \tilde{a}_2$. One has [23, 93]

$$a_{0Fermion} = tr(\tilde{a}_0) = 4, \tag{4.73}$$

$$a_{1Fermion} = tr(\tilde{a}_1) = -\frac{1}{3}R, \tag{4.74}$$

and

In general, $a_0 = \pm 1$ for each bosonic (fermionic) degree of freedom and $a_1 \propto R$. Therefore, the first term in $I_{m,eff}$ can be written as

$$\frac{1}{8\pi}\int d^4x\sqrt{g}\Lambda_m \tag{4.75}$$

with

$$\Lambda_m = n\frac{m^4}{8\pi}\ln\left(\frac{m^2}{\mu^2}\right), \tag{4.76}$$

where n is a numerical factor that is $+1$ for each bosonic and -1 for each fermionic degree of freedom (for fermions, one degree of freedom means one dimension of the gamma matrices, so $n = -4$ for an electron, whereas one has $n = 1$ for a Higgs Boson).

Λ_m is a topology independent renormalization of the cosmological constant, which then becomes a sum

$$\Lambda_{eff} = \Lambda_g + \Lambda_m, \tag{4.77}$$

where Λ_g is the cosmological constant that we get from gravity alone.

If we set $a_2 = 0$, then the effective action will only correspond to an additional cosmological constant term and a renormalization of Newton's constant. The scaling behaviour of the gravity amplitude was derived in [55, 94] for solutions with an arbitrary cosmological constant. These results still hold if we set $a_2 = 0$ in the matter amplitude, but one has to replace Λ by Λ_{eff} everywhere. Before we compute the stationary point, we have to substitute $\Lambda_{eff}V = f^2/\Lambda_{eff}$ in the amplitude. Since the contribution Λ_m is not a variable field, we can compute the new stationary value $\Lambda_{eff,s0}$ of the cosmological constant from solving

$$\frac{d}{d\Lambda_g}\left(\frac{f^2}{8\pi\Lambda_{eff}}+\ln\left(\tilde{Z}(\Lambda_{eff})\right)\right)=0 \tag{4.78}$$

for Λ_{eff}, from which we get the relation between the Euler characteristic and Λ_{eff}.

Doing this computation shows that the relations between $\Lambda_{eff,s0}$ and the Euler characteristic and the volume are again given by Eq. (4.52).

Similarly, we also get the value

$$\Lambda_{eff,s0}=2\pi\mu^2 \tag{4.79}$$

from solving

$$\frac{d}{d\chi}\tilde{S}(\Lambda_{eff,s0})=0 \tag{4.80}$$

for μ^2. From this result we can conclude that the contribution of $\Lambda_m=-9.7\cdot 10^8 GeV$ from the standard model matter has no observable effect at all.

This becomes different if we include the contributions of terms $\propto -\int d^4x\sqrt{g}a_2$ in the matter action. For example, for a scalar boson field, one has [23, 93]

$$a_{2Boson}=\frac{1}{180}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-\frac{1}{180}R^{\mu\nu}R_{\mu\nu} \tag{4.81}$$

$$+\frac{1}{2}\left(\frac{1}{6}-\zeta\right)^2R^2-\frac{1}{6}\left(\frac{1}{5}-\zeta\right)\Box R, \tag{4.82}$$

where ζ is an undetermined coupling parameter that has to be measured by experiment.

For fermions, one has

$$a_{2Fermion}=tr(\tilde{a}_2)=-\frac{7}{360}R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}-\frac{1}{45}R^{\mu\nu}R_{\mu\nu} \tag{4.83}$$

$$+\frac{1}{72}R^2+\frac{1}{30}\Box R. \tag{4.84}$$

Obviously the effective matter action is proportional to the Euler number:

$$I_{m,eff}\propto\int d^4x\sqrt{g}a_2\propto-\eta\chi, \tag{4.85}$$

The energy momentum tensor $T_{\mu\nu}$ can be computed from the variation of the matter action. Its trace T gets into Einstein's field equations and leads to

$$R = 4\Lambda - 8\pi\tilde{T}. \tag{4.86}$$

Where it is understood that $\tilde{T}_{\mu\nu}$ only describes the part of the energy momentum tensor that does not contain the renormalizations of the gravitational and cosmological constants from a_0 and a_1. The effects of these terms were discussed above and are assumed to have been handled already.

For a conformal massless model, one can neglect the contributions from a_0 and a_1. It is known [23] that one then gets

$$\tilde{T} = -\frac{a_2}{16\pi}. \tag{4.87}$$

If one assumes that the R^2 corrections modify the equations of motion by a back reaction, then the procedure becomes more difficult.

The variation of the effective matter action with an a_2 term is given by:

$$\frac{\langle out,0|\tilde{T}_{\mu\nu}|0,in\rangle}{\langle out,0|0,in\rangle} = a\ {}^{(1)}H_{\mu\nu} + b\ {}^{(2)}H_{\mu\nu} + c\ {}^{(3)}H_{\mu\nu} \tag{4.88}$$

In Eq. (4.88), a, b, c are numerical coefficients that depend on the renormalization scale μ^2 and the particle mass. They must hence be determined by measurement. One has [23]

$${}^{(1)}H_{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{g}R^2, \tag{4.89}$$

$${}^{(2)}H_{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{g}R^{\alpha\beta}R_{\alpha\beta} \tag{4.90}$$

and

$${}^{(3)}H_{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{\mu\nu}}\int d^4x\sqrt{g}R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta}. \tag{4.91}$$

The Euler characteristic is a topological invariant whose variation vanishes. Therefore,

$$0 =^{(3)} H_{\mu\nu} +^{(1)} H_{\mu\nu} - 4\ {}^{(2)}H_{\mu\nu}. \tag{4.92}$$

One of the three tensors in the first line of Eq. (4.88) can thus be absorbed into the other two with new coefficients. For example, we could chose

$$a\ {}^{(1)}H_{\mu\nu} + b\ {}^{(2)}H_{\mu\nu} + c\ {}^{(3)}H_{\mu\nu} = \tilde{a}\ {}^{(1)}H_{\mu\nu} + \tilde{b}\ {}^{(2)}H_{\mu\nu} \tag{4.93}$$

with

$$\tilde{a} = a - c, \tilde{b} = b + 4c. \tag{4.94}$$

Another choice would be

$$a\ ^{(1)}H_{\mu\nu} + b\ ^{(2)}H_{\mu\nu} + c\ ^{(3)}H_{\mu\nu} = \hat{a}\ ^{(1)}H_{\mu\nu} + \hat{c}\ ^{(3)}H_{\mu\nu} \tag{4.95}$$

where

$$\hat{a} = \frac{4a+b}{4}, \hat{c} = \frac{b+4c}{4}. \tag{4.96}$$

Using Eq. (4.88) and

$$R = 4\Lambda_{eff} - 8\pi\tilde{T} \tag{4.97}$$

as well as the Einstein Hilbert action

$$I = -\frac{1}{16\pi}\int d^4x\sqrt{g}(R - 2\Lambda_{eff}). \tag{4.98}$$

We could try to compute an amplitude $Z = Ce^{-I}$ with the corrections $^{(1)}H_{\mu\nu}$, $^{(2)}H_{\mu\nu}$ and $^{(3)}H_{\mu\nu}$.

The tensors $^{(1)}H_{\mu\nu}$, $^{(2)}H_{\mu\nu}$ and $^{(3)}H_{\mu\nu}$ are computed in [23]. They are given by

$$\begin{aligned}
^{(1)}H_{\mu\nu} &= 2R_{;\mu\nu} - 2g_{\mu\nu}\Box R - \frac{1}{2}g_{\mu\nu}R^2 + 2RR_{\mu\nu}, \\
^{(2)}H_{\mu\nu} &= R_{;\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Box R - \Box R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta} + 2R^{\alpha}R_{\alpha\beta\mu\nu}, \\
^{(3)}H_{\mu\nu} &= -\frac{1}{2}g_{\mu\nu}R^{\alpha\beta\gamma}R_{\alpha\beta\gamma} + 2R_{\mu\alpha\beta\gamma}R_{\nu}{}^{\alpha\beta\gamma} \\
&\quad -4\Box R_{\mu\nu} + 2R_{;\mu\nu} - 4R_{\mu\alpha}R^{\alpha}{}_{\nu} + 4R^{\alpha\beta}R_{\alpha\mu\beta\nu},
\end{aligned} \tag{4.99}$$

where $;\mu$ denotes covariant derivation with respect to μ.

As $\propto R^2$ corrections to R they would make it necessary to quantize a gravity action with higher order derivatives.

It was proven by Stelle that gravity with additional terms of the form $R^2 + R_{\mu\nu}R^{\mu\nu}$ or $R^2 + R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ is renormalizable [30, 31].

As we have seen in section 1.5, theories with higher derivatives often suffer from the Ostrogradski instability [2, 32].

It is known that the Ostrogradski instability does not occur for the $\kappa^2 R + \hat{a}R^2$ model because the latter avoids the non-degeneracy assumption [32] of the Ostrogradski theorem.

Unfortunately, the Gauss Bonnet theorem allows only to remove one of the three higher curvature terms in a_2. If we remove.e.g ${}^{(3)}H_{\mu\nu}$, we still have the ${}^{(2)}H_{\mu\nu}$ term left. It contains a variation of $R_{\mu\nu}R^{\mu\nu}$ terms that lead to the Ostrogradski instability.

In the following, we will propose a method to remove the unfriendly ghosts from the coupled gravity plus effective matter amplitude.

A typical a_2 term of the effective action with several bosons and fermions consists of terms of the form

$$I_{eff} = A_{fermion}\left(a_{fermion}\int d^4xR^2 + b_{fermion}\int d^4xR_{\mu\nu}R^{\mu\nu} + c_{fermion}\int d^4xR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right) + A_{boson}\left(a_{boson}\int d^4xR^2 + b_{boson}\int d^4xR_{\mu\nu}R^{\mu\nu} + c_{boson}\int d^4xR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}\right) \quad (4.100)$$

$A_{fermion}$ and A_{boson} are sums of the factors from Eq. (4.70) for the different particles. For N bosons with masses m_i, one has

$$A_{boson} = \sum_{i=1}^{N}\frac{1}{32\pi^2}\ln\left(\frac{m_i^2}{\mu^2}\right) \quad (4.101)$$

and we have from [22, 23, 93] that $a_{fermion} = \frac{1}{72}$, $b_{fermion} = \frac{1}{45}$, $a_{fermion} = \frac{1}{72}$, $c_{fermion} = -\frac{7}{360}$ and $a_{boson} = \frac{1}{2}\left(\frac{1}{6}-\zeta\right)^2$, $b_{boson} = -\frac{1}{180}$, $c_{boson} = \frac{1}{180}$.

Eq. (4.100) is equal to

$$\begin{aligned} I_{eff} =& (A_{fermion}a_{fermion} + A_{boson}a_{boson})\int d^4xR^2 \\ &+ (A_{fermion}b_{fermion} + A_{boson}b_{boson})\int d^4xR_{\mu\nu}R^{\mu\nu} \\ &+ (A_{fermion}c_{fermion} + A_{boson}c_{boson})\int d^4xR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} \\ \equiv& u\int d^4xR^2 + v\int d^4xR_{\mu\nu}R^{\mu\nu} + w\int d^4xR_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}, \end{aligned} \quad (4.102)$$

where

$$u = A_{fermion}a_{fermion} + A_{boson}a_{boson}, \quad (4.103)$$

$$v = A_{fermion}b_{fermion} + A_{boson}b_{boson}, \quad (4.104)$$

$$w = A_{fermion}c_{fermion} + A_{boson}c_{boson} \quad (4.105)$$

If we remove the $R_{\mu\nu}R^{\mu\nu}$ term from the effective action with the Gauss Bonnet theorem, then one gets

$$I_{eff} = \hat{c}\int d^4x R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta} + \hat{a}\int d^4x R^2 + \omega\chi, \tag{4.106}$$

where ω is some numerical factor, and

$$\hat{a} = \frac{4a+b}{4} \tag{4.107}$$

and

$$\hat{c} = \frac{b+4c}{4}. \tag{4.108}$$

If we $\hat{c} = 0$, the effective action then contains only the $\kappa^2 R + \hat{a}R^2$ dependent terms in addition to a contribution which is proportional to the Euler characteristic χ. The latter has a vanishing variation. Therefore, it does not contribute to the equations of motion and it also does not give rise to particles that may propagate energy. The disastrous $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ and $R_{\mu\nu}R^{\mu\nu}$ terms that lead to ghosts have been removed. However, in contrast to the original Starobinski model from which inflation was derived, this theory contains a Λ term.

Here, we just want to note which conditions we need to have for ghost freedom. If we use Eqs. (4.104-4.105) in Eq. (4.108),

$$4\hat{c} = b + 4c \equiv 0, \tag{4.109}$$

one finds that

$$A_{boson}/60 - A_{fermion}/3 \equiv 0. \tag{4.110}$$

From the definition of A_{boson} and $A_{fermion}$ in Eq. (4.101) it then follows that the system of gravity and massive bosonic and fermionic matter is only physical for a specific relationship between the sums of the logarithms of the masses of bosonic and fermionic fields.

For massless fields, $A = \frac{\tilde{\gamma}}{32\pi^2}$ where $\tilde{\gamma}$ is the Euler Mascheroni constant and the first Seley-DeWitt coefficients vanish $a_0 = a_1 = 0$. For arbitrary fields, the coefficients a, b, c that the higher derivative terms are multiplied with may be read off from [93]. If massless fields are present, then their contributions have to be included if one aims to remove the $R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$ term from the effective matter action.

The calculation of Hawking needs the equations of motion $R = 4\Lambda$ in order to express the value of the gravity action I for a given 4 volume V as a function of Λ,

V, and some numerical coefficients only. This is possible for the classical Einstein action whose value is given by $-\frac{\Lambda V}{8\pi}$. However, with the equations of motion for $R+R^2$ gravity, this is not so easily possible.

One can make the argument that $\hat{a}, \hat{c}$ must be very small as we do not observe corrections from ${}^{(1)}H_{\mu\nu}$, ${}^{(2)}H_{\mu\nu}$ or ${}^{(3)}H_{\mu\nu}$ in Einstein's equations in the macroscopic world.

One can therefore make the assumption to neglect the back-reaction of the ${}^{(1)}H_{\mu\nu}$, ${}^{(2)}H_{\mu\nu}$ or ${}^{(3)}H_{\mu\nu}$ corrections to the Einstein equations in a first step.

Then the gravitational field fulfils Einstein's equation $R = 4\Lambda$ and $R_{\mu\nu} = \Lambda g_{\mu\nu}$ and one has to compute the effective matter action with these relations.

This would yield corrections proportional to $\Lambda_m^2 V = f^2$ and the Euler characteristic χ in the effective matter action.

If we ignore the back-reaction, then the theory becomes essentially pure gravity with a cosmological constant

$$\tilde{\Lambda} = \Lambda + \hat{a}\Lambda^2. \tag{4.111}$$

With the result of one-loop finiteness of pure gravity, one can continue to use the WKB approximation.

If one uses that $\Lambda^2 V = f$ is some dimensionless factor, then the Λ^2 correction of the effective action does not lead to a rescaling of the eigenvalues of the zeta functions. Therefore, the $\Lambda^2 V$ correction does not affect the factor $C = \left(\frac{\Lambda}{2\pi\mu^2}\right)^{-\frac{1}{2}\gamma}$ in the one loop amplitude $Z = Ce^{-I}$. The Λ^2 correction just appears in the effective action, but since it is multiplied with V, the additional $\propto e^{f^2}$ in the amplitude does not affect the saddle points.

However, even if one sets the coefficients $\hat{a}$, $\hat{c}$ both to zero (which would be problematic, since quantum gravity also deals with small corrections), one has terms proportional to the Euler characteristic in the amplitude. These terms are given by

$$\int d^4x\sqrt{g}a_2 \approx \frac{8\pi^2}{45}\chi \tag{4.112}$$

for bosons, and for fermions one arrives at

$$\int d^4x\sqrt{g}a_2 \approx -\frac{28\pi^2}{45}\chi, \tag{4.113}$$

where we have neglected the $\Lambda^2 V$ term because it does not change the saddle points.

The variation of these terms is zero so they do not result in a a correction to the classical equations of motion. They also do not result in a correction from perturbative quantum gravity.

However, Hawking's space-time foam model includes a non-perturbative summation over all classical metrics.

Ignoring potential effects of $\hat{a}$, $\hat{c}$, the combined matter gravity action yields an amplitude of the form

$$\overline{Z}(\Lambda_{eff},\chi) = \left(\frac{\Lambda_{eff}}{2\pi\mu^2}\right)^{-\frac{106}{90}\chi+\frac{87}{240\pi^2}\Lambda_{eff}^2 V} e^{\eta\chi+\theta f^2} \tag{4.114}$$

where η and θ are some numerical factor that depends on particle masses. One notes that η has a different sign for bosons and fermions.

We have already noted in section 4.2 that the gravitational amplitude we have used in the calculation above was simplified and the entire amplitude of quantum gravity reads

$$\tilde{Z}_{VC} = e^{\frac{1}{2}\zeta'_F(0)+\frac{1}{2}\zeta'_G(0)-\zeta'_C(0)} \left(\frac{\Lambda}{2\pi\mu^2}\right)^{-\frac{1}{2}\gamma}. \tag{4.115}$$

In general, the derivatives of the Euclidean zeta functions contain integrals over the Seley-DeWitt coefficients according to Eq. (1.60). If one would include these terms in the gravity amplitude, they would change the numerical coefficients in the higher derivative terms a bit.

If we are substituting

$$V(\Lambda_{eff}) = f^2/\Lambda_{eff}^2 \tag{4.116}$$

in this amplitude and compute the saddle point $\Lambda_{eff,s1}$ with the same method as in Eq. (4.78), we get the same relations between $\Lambda_{eff,s1}$ and the volume and Euler characteristic as we computed them in Eq. (4.52) from the amplitude without the $e^{\eta\chi}$ term.

With $\Lambda_{eff,s1}$ given by Eq. (4.52), we can write the entropy as

$$\overline{S}(\Lambda_{eff,s1}) \approx \ln\left(\overline{Z}(\Lambda_{eff,s1})\right) + \frac{f^2}{8\pi\Lambda_{eff,s1}}. \tag{4.117}$$

If we solve

$$\frac{d\overline{S}}{d\chi} = 0 \tag{4.118}$$

for μ^2 we get

$$\Lambda_{eff,s1} = 2\pi e^{\frac{45}{53}\eta}\mu^2. \tag{4.119}$$

and one checks that this is a maximum for large negative Euler characteristics as before.

The factor η has different signs for fermions and bosons and depends on their masses. Thereby, with the addition of appropriate matter particles, one may e.g. correct the stationary value such that it becomes $\Lambda = 2.13H_0^2$ if we set $\mu = H_0$. For our universe, another important correction may come from a boundary term at the apparent horizon. Such a term may also change the scaling behaviour of the amplitude and contribute to the gravitational entropy.

We note that this result is different from an argument of Hawking who, without a detailed calculation, argues in [89] that the topological terms would lead to $\Lambda = 0$.

We have argued that for a universe where we have added an effective matter action $I_{m,eff} \propto \chi$, the equation for the saddle point of $\Lambda_{eff,s1}$ remains the same as in Eq. (4.52).

With a positive cosmological constant, one usually gets an accelerated expansion of the volume of the universe in time. For the universe with the added matter amplitude, we may wait until some time t_1 at which the volume V becomes the same as in the universe where the matter was not added at some time t_0.

Then, Eqs. (4.119) and (4.79), or

$$\Lambda_{eff,s1} \neq \Lambda_{eff,s0}, \tag{4.120}$$

would imply that the Euler characteristic between the two universes would be different, as there are no other quantities in Eq. (4.56) that could be responsible for a change of the saddle point of Λ.

The Euler characteristic does not change the equations of motion nor does it introduce new propagating degrees of freedom. Despite this, its appearance in the matter amplitude implies by Eqs. (4.117) and (4.119), that it changes the entropy in Hawking's non-perturbative summation of the path integral over classical metrics.

This suggests that the physical interpretation of the $e^{\eta\chi}$ dependent terms in the effective matter action is simply to describe effects from black or worm hole formation that one gets from energy fluctuations of the matter.

With this perspective, the fact that the perturbation expansion breaks down in quantum gravity should not come as surprising. The usual Feynman rules only describe the scattering processes with particles and not with black-holes. Therefore, one should probably augment them with additional rules for black-hole scattering before one goes to even higher energies beyond the one-loop order. A first attempt for such a model is given in section 5.3.

For researchers working in supergravity, it may be an interesting check whether the cosmological constant would turn out to be exactly zero if matter terms are inserted such that the theory is supersymmetric. That the vacuum energy vanishes in theories with exact supersymmetry has been proven using perturbation theory in curved space-times but there are also more general grounds that the vacuum energy of supersymmetric theories is vanishing which are based on the Hamiltonian [27].

In order to see that the vacuum energy vanishes with supersymmetry one has to write the matter amplitudes with zeta function renormalization. The graviton is massless and in an exactly supersymmetric theory, the gravitino would be massless too. In a supersymmetric theory, there may be additional massless particles. Their amplitudes would have a similar scaling factor $\propto \frac{\Lambda}{2\pi\mu^2}$.

The vanishing of the sum of all vacuum diagrams in supersymmetric theories was proven for perturbation theory at all orders in curved space-times by Zumino [95]. Hence the amplitudes of the massless particles must lead to a γ factor with opposite sign than the amplitudes of their massless superpartners. Therefore

$$Z_{massless-particles} \propto \left(\frac{\Lambda}{2\pi\mu^2}\right)^{-\gamma} \tag{4.121}$$

and

$$Z_{massless-superpartners} \propto \left(\frac{\Lambda}{2\pi\mu^2}\right)^{\gamma}. \tag{4.122}$$

Both amplitudes would be multiplied. This cancels the scaling terms of the one loop action that have led to $\Lambda \approx 2\pi\mu^2$ in the space-time foam calculation. One is led to $C = 1$ and an amplitude

$$Z \propto e^{\frac{f^2}{8\pi\Lambda}} \tag{4.123}$$

which only contains the background contribution. If we compute the entropy maximum of this amplitude, we find that $\Lambda = 0$. So with exact supersymmetry, the vacuum energy also vanishes in the space-time foam model.

4.5 The modified space-time foam model and the problem of time

We have argued in section 4.2 that the approach of Hawking needs the assumption of a well defined notion of energy for a compact space-time. Space-times with cosmological constant usually lead to an accelerated expansion of the universe. This is a dynamical process for which one would need time dependent observables. As we have seen in section 2.4, such processes also need a non-vanishing Hamiltonian.

Below, we will give arguments from quantum gravity which suggest that the cosmological constant is connected to some kind of quasi-local energy.

We have already noted in section 4.2 that the entropy is proportional to

$$S \propto \frac{\Lambda_{eff} V}{8\pi} = \frac{f^2}{8\pi\Lambda_{eff}}, \tag{4.124}$$

where $\Lambda_{eff} = \Lambda_g + \Lambda_m$ and Λ_g is a contribution from gravity. In section 3.3, we have reviewed a calculation from Hawking which implies that the curvature of space-time must be strong enough to form a boundary if the gravitational action I makes a contribution to the entropy.

In the case of single black-holes, one has a boundary at infinity and, since the Euclidean space-time can not describe the space-time within the black-hole, one also gets a boundary at the horizon. In the space-time foam model, one assumes that the space-time is compact. So, there is no boundary at infinity. However, we have found that for a given volume, the state of maximum entropy has a background space-time filled with $N = |c|V$ cavities. In order to have a contribution of the action to the entropy, each of these N cavities should be associated to a boundary. Around the i-th cavity, we add the following Euclidean GHY boundary term [96] of the form

$$I_{GHY,i} = \frac{1}{8\pi}\int_{\partial M_i} d^3x\sqrt{h}K_i^{ab}h_{ab} \tag{4.125}$$

to the action

$$I_0 = -\frac{1}{16\pi}\int d^4x\sqrt{g}R \tag{4.126}$$

where h_{ab} is the induced metric on the i-th boundary ∂M_i and K_i is the extrinsic curvature there.

This proposal is actually quite similar to a proposals made in [78] and [76]. The authors of [76] show that this mechanism would lead to inflationary behaviour, and that it would even provide an exit from inflation. But they do not observe that if the value of the cosmological constant is given by the masses or areas of instantons, then the cosmological constant should be found by a principle that determines the most probable configuration for them.

From Eq. (4.58), one gets

$$N = |\chi| = |c|V = |c|\int d^4x\sqrt{g} \tag{4.127}$$

boundaries and the mean value

$$\hat{I}_{GHY} = \frac{1}{N}\sum_{i=1}^{N} I_{GHY,i} \tag{4.128}$$

an action

$$\begin{aligned} I &= -\frac{1}{16\pi}\int d^4x\sqrt{g}R + \sum_{i=1}^{N} I_{GHY,i} \\ &= -\frac{1}{16\pi}\int d^4x\sqrt{g}R + N\hat{I}_{GHY} \\ &= -\frac{1}{16\pi}\int d^4x\sqrt{g}R + \hat{I}_{GHY}|c|\int d^4x\sqrt{g} \\ &= -\frac{1}{16\pi}\left(\int d^4x\sqrt{g}R - 2\Lambda V\right), \end{aligned} \tag{4.129}$$

where we have defined

$$\Lambda \equiv 8\pi|c|\hat{I}_{GHY}. \tag{4.130}$$

Using Eqs. (4.58 and (4.130) one finds that

$$\hat{I}_{GHY} = \frac{106\pi}{9|(29\Lambda - 10\pi)|}. \tag{4.131}$$

If the boundaries of the cavities are similar to the boundaries of event horizons from black-holes in Euclidean quantum gravity, then one would expect something like

$$\hat{I}_{GHY} = \frac{A}{4} \tag{4.132}$$

where A is the area of each cavity. With $\Lambda \approx 5.6 \cdot 10^{-122}$ one gets $A = 4\hat{I}_{GHY} \approx 4.7$ Planck areas.

In order to derive equations of motion, one writes the variation δI as

$$
\begin{aligned}
\delta I &= \frac{-1}{16\pi}\delta\int d^4x\sqrt{g}R + \delta(|c|V\hat{I}_{GHY}) \\
&= \frac{-1}{16\pi}\delta\int d^4x\sqrt{g}R \\
&\quad +|c|\hat{I}_{GHY}\int d^4x\delta\sqrt{g} + |c|V\delta\hat{I}_{GHY} \\
&= \frac{-1}{16\pi}\delta\int d^4x\sqrt{g}R + \frac{\int d^4x\delta\sqrt{g}}{8\pi}\Lambda \\
&\quad +\sum_{i=1}^{N}\int_{\partial M_i} d^3x\sqrt{h}\left(\delta K_i^{ab}h_{ab}\right) \\
&= \frac{-1}{16\pi}\delta\int d^4x\sqrt{g}(R-2\Lambda),
\end{aligned} \tag{4.133}
$$

where we have used the known fact that the sum of the boundary terms

$$
\int_{\partial M_i} d^3x\sqrt{h}\left(\delta K_i^{ab}h_{ab}\right) \tag{4.134}
$$

cancel another boundary term that one gets from the variation of

$$
\frac{-1}{16\pi}\int d^4x\sqrt{g}g^{\mu\nu}\delta R_{\mu\nu}. \tag{4.135}
$$

We observe that a contribution to the equations of motion which is similar to the cosmological constant does arise because one has the the term $\hat{I}_{GHY}$ for

$$
N = |c|V = |c|\int d^4x\sqrt{g} \tag{4.136}
$$

times in the action.

We want to note that for a manifold M with boundary, the Euler characteristic

$$
\chi = \frac{1}{32\pi^2}\int d^4x\sqrt{|g_\mu\nu|}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} + R^2 - 4R_{\mu\nu}R^{\mu\nu} \tag{4.137}
$$

usually contains an additional term that has to be added to χ. This term is given by

$$
\chi_{\partial M} = \frac{1}{128\pi^2}\int_{\partial M} d^3x\sqrt{|h|}(R_{abcd}K^{ac}n^bn^d + 64det(K_b^a)), \tag{4.138}
$$

where n^a is the outward directed normalt to the boundary ∂M, K_{ab} is the extrinsic curvature and $h_{ab} = g_{ab} - n_a n_b$ is the induced metric on ∂M.

In the calculation of the space-time foam model, we used only Eq. (4.137) for the Euler characteristic and we omitted the contribution of $\chi_{\partial M}$. The reason why we think this is allowed in our case is that according to the computation above, the $N = |c|V$ boundaries lead to an action that resembles a gravity action without boundary terms but with cosmological constant term.

Eq. (4.130) also makes clear why the gravitational contribution to the cosmological constant should be seen as a continuous field. The cosmological constant

$$\Lambda = |c| 8\pi \hat{I}_{GHY} \tag{4.139}$$

is related to the mean of the area (given by $\hat{I}_{GHY}$) and density (given by $|c|$) of the cavities in the space-time that are due to quantum fluctuations. In general, the cavities can have any area and density per volume, which would lead to different possible values of Λ. If we have to sum the path integral over all possible metrics with different boundaries or cavity densities, the system, if it is at its ground state, should then be observed at the value of Λ where the ground state wave-function is at its maximum.

The difference between our cavities from the ordinary black-holes from Euclidean quantum gravity appears to be mostly that the boundary is not due to the Wick rotation since one should get a cosmological constant even when one is not transforming the metric into an Euclidean one. It therefore can not be an artefact of Euclideanization and one has to assume that the boundaries are still there in the Lorentzian theory.

The space-time of Hawking's model allows the use of the Hamiltonian of gravity, which implies that space-time is globally hyperbolic and can be written in the form of $\Omega = \mathbb{R} \times \Sigma$, where Σ is a space-like three manifold. Furthermore, Σ should contain N boundaries $\partial\Sigma$ and these boundaries should be there even in the Lorentzian space-time after Wick rotation. Now according to a definition given by Visser in [97], if a Lorentzian space-time has a compact region $\Omega = \mathbb{R} \times \Sigma$ where Σ has a non-trivial topology with boundary $\partial\Sigma \sim S^2$, then Ω contains an intra universe worm hole. Therefore, the cavities in Hawking's model are certainly compatible with worm holes, as the latter also imply that space-time is not simply connected. The assumption that space-time is globally hyperbolic is apparently needed in order to define the Hamiltonian. With traversable worm holes this would certainly be incompatible, but not with non-traversable wormholes [98].

We remark that this solution to the cosmological constant problem also appears to circumvent Weinberg's no go theorem that we have described in section 1.6.

The ground state of quantum gravity appears to be a space-time filled with worm holes. This does certainly not fulfil the requirement of a constant, translationally invariant metric and thus it appears that Weinberg's no-go result can not be applied to this solution of the cosmological constant problem.

The action in Lorentzian space-time is given by $S = \int dt\mathcal{L}$ where $\mathcal{L}$ is the Lagrangian density. After a Wick rotation to Lorentzian space-time, the term

$$\Lambda V = |c|\hat{I}_{GHY}\int d^4x\sqrt{g} \tag{4.140}$$

that was added to the Euclidean action gets a minus sign. If such a term is subtracted from the Lorentzian action, then a term

$$\mathcal{L}_{\partial\mathcal{M}} = -|c|\hat{I}_{GHY}\int d^3x\sqrt{g} = -\frac{\Lambda V^{(3)}}{8\pi}, \tag{4.141}$$

where $V^{(3)}$ is the 3 volume, is subtracted from the Lagrangian density. The $\mathcal{L}_{\partial M}$ term is a boundary term because $\Lambda = 8\pi|c|\hat{I}_{GHY}$, where $\hat{I}_{GHY}$ is the average of an area that comes from a boundary. For generalized coordinates q, p, the Lagrangian density is given by

$$\mathcal{L} = \dot{q}^i p_i - H. \tag{4.142}$$

Therefore, $\mathcal{L}_{\partial M}$ has to be added to the gravitational Hamiltonian. As the Hamiltonian of general relativity without boundary term is zero, the new Hamiltonian becomes:

$$H = \frac{\Lambda V^{(3)}}{8\pi} \tag{4.143}$$

or $\tau H = \frac{\Lambda V}{8\pi}$. This is a well-defined Hamiltonian that allows the use of canonical partition sums from thermal field theory, like

$$Z = tr\left(e^{-\tau H}\right) = tr\left(e^{-\frac{V\Lambda}{8\pi}}\right) \tag{4.144}$$

with τ as a time coordinate in Euclidean space. A non zero Hamiltonian leads, in accordance with the observations in [3, 60] to a solution of the problem of time in quantum gravity.

By saying this, one should note that even if one can now write time dependent operators like

$$A(t) = e^{-itH}Ae^{itH}, \tag{4.145}$$

DeWitt argues in [3] that one only has time evolution if the amplitude is not an Eigenstate of H and one must construct wave packets with different energy.

Here we want to note parts of an article of Page and Wooters [99]. They begin their letter on the problem of time by stating their unproven assumption that there might exist a super-selection rule for energy in quantum gravity, similar to the super selection rules that exist for charge in quantum electrodynamics. Such a rule would mean that there were no superpositions with different energy and one would still be left with a problem of time.

Hawking's model of the cosmological constant [4, 82] rests on the fact that one can write the state of quantum gravity as $\Psi = \sum_n \psi_n \overline{\psi}_n e^{iE_n t}$. Only then one can consider the amplitude of Euclidean quantum gravity as proportional to the probability of a state with some initial configuration at a time t_0 to develop into a certain different configuration at a later time $t_1 > t_0$.

Similarly, Hawking's volume canonical ensemble from [9] depends on the assumption that one can construct wave packets of different energy.

The set of states $|g_n\rangle$ forms a complete orthonormal base of energy eigenstates according to Hawking. Since one can define a trace over these states, it appears valid to build superpositions

$$|\Psi\rangle = \sum_n c_n |g_n\rangle \tag{4.146}$$

with them. Due to the equality of Eqs. (4.144) and (4.33) some of the states $|g_n\rangle$ must be associated to different 4 volumes, i.e. they are eigenstates of the Wheeler-DeWitt equation for different three volumes and thus solve the Wheeler-DeWitt equation at different times. Using

$$H = \frac{\Lambda V^{(3)}}{8\pi}, \tag{4.147}$$

one sees that eigenstates of the Wheeler-DeWitt equation for different 3 volumes have different energy. The superposition $|\Psi\rangle$ then corresponds to a wave packet of eigenstates with different energy, as envisaged by DeWitt.

Chapter 5

How matter behaves in the modified space-time foam model

5.1 Introduction

Quantum fluctuations of gravity might have experimentally observable consequences if they describe a space-time foam. After Hawking wrote his first article on this idea, Hawking, Page, Pope and Warner immediately published works where they investigated how quantum mechanical particles might change their trajectory if they were flying close to a virtual black-hole from space-time foam [100–102].

In their calculation, Hawking, Page and Pope restrict themselves to simply connected space-times and note that by using topological sums of a certain number of copies of CP^2 and $\overline{CP^2}$ (the bar means opposite orientation), one can construct a simply connected closed manifold of arbitrary signature τ and Euler characteristic χ, with an odd and definite intersection form. Similarly, by using certain numbers of copies of $S^2 \times S^2$ and K^3 if $\tau > 0$ or $\overline{K^3}$ if $\tau < 0$, one can construct a simply connected closed manifold with even and indefinite intersection form, arbitrary signature and Euler characteristic, see [103] p. 26. By Freedman's theorem, the topology of the simply connected space-times from these construction would then, up to homeomorphy, be equivalent to an arbitrary simply connected space-time with the same Euler characteristic and signature (Note, however that this equivalence just holds for the topology, and not the metric).

With the argument above, the topology of this simply connected space-time can be build out of building blocks like CP^2, $\overline{CP^2}$, $S^2 \times S^2$, K^3,$\overline{K^3}$. Hawking et al then proceed to calculate the scattering amplitudes of particles that are moving in these building blocks.

They conclude that the amplitudes are of order

$$A \propto \left(\frac{k_1 k_2}{m_p}\right)^s \tag{5.1}$$

where k_1 and k_2 are the momenta of the in and out states, m_p is the Planck mass and s is the spin. For a scalar particle, like the Higgs field, we have $s = 0$ and therefore the amplitudes would be of order one.

Hawking writes that this would suggest that the Higgs particle is of composite nature. However, in 2012, the Higgs particle has been found at the Large Hadron Collider in Genf, and further analysis of the data provided evidence for the Higgs field to be indeed a scalar particle [104]. This puts the approximations of Hawking et al severely into question. Additionally, Warner [102] has analysed scattering amplitudes of Spin 1 fields with Hawking's model, and he also found large amplitudes that are in disagreement with observation.

In the following, we argue that the modifications that we have made to the space-time foam model in section 4.2 and 4.3 may be able to cure this problem at least for slow particles at low energy.

We argue that for observers who can not measure processes that occur at short time intervals or at high energy, the details of a micro black-hole capturing a particle and releasing it during its decay can not be observed. Nevertheless, the particle is in a heath bath of very hot thermal radiation. We argue that if we insert a slow moving particle into such a heath bath, we will derive Schrödinger's equation.

Quantum mechanics also describes entangled states which generate highly correlated outcomes at space-like separated measurement stations. Using these correlations, one can show Bell's no-go theorem that excludes the outcomes to be predetermined prior to measurement if no instantaneous signalling between the two sites occurs. We review Bell's theorem in mathematical form. We also argue that the stochastic model of quantum mechanics is not a hidden variable theory in the sense of Bell's theorem because its results are not predetermined. We suggest that there may be ways do describe entangled states with this model even though we did not undertake the complicated computation of these correlations in the random model (They would involve a statistical mean of infinitely many correlated stochastic processes).

For an observer who can detect processes at short timescales and high energy, we assume that the details of the scattering process where the particle falls into the black-hole and interacts with the outgoing particle become available. For this case, 't Hooft has found that one gets a scattering matrix which has the form of a

Wick rotated Polyakov action.

We will give an argument which indicates that this s-Matrix should not be valid for very large black-holes but that it applies more to small black-holes.

We note that the space-time foam model implies topology changes during the expansion of the universe. A result of Geroch [12] shows that topology changes in Lorentzian manifolds come with the emergence of singularities in the space-time.

DeWitt has given two arguments which show that quantum field theory in curved space-time and the theory of quantum gravity becomes inconsistent when singularities emerge.

The second argument of DeWitt against topological changes was made as a short statement in a non-technical article [13] and without much mathematical details. We explain DeWitt's argument in detail for conventional quantum gravity and make some additional considerations.

DeWitt argues in [13] to investigate topological changes with the string theory framework, because it is sometimes written that the theory could be defined exactly on singular spaces like orbifolds, which are stratified spaces that consist of a manifold and singularities as strata. Specifically, DeWitt claims without proof that string theory could be defined exactly even on Lorentzian orbifolds, which would make topology changes possible.

A famous article of Dixon et al. [14] tries to define string theory on orbifolds by additional boundary conditions for twisted sectors of the string theory. The authors write that their amplitude yields results that are equal to that of strings on a manifold resulting from a blow-up in the limit of an infinitesimally small ϵ neighbourhood around the singularity.

Using the mathematically rigorous construction of the path integral of section 1.2 and arguments from constructive quantum field theory, we argue that without a previous blow up of the orbifold singularity, string theory is affected from similar problems and inconsistencies as conventional quantum gravity whenever a singularity appears in the target space.

Citing recent mathematical results [15, 16], we argue that Dixon et al. can use their boundary conditions because the exponential map is well behaved in the ϵ neighbourhood of the orbifold if its singularity was blown up. If the blow up occurs in an epsilon neighbourhood and one takes the limit $\epsilon \to 0$, one just removes a point from the orbifold. For the string theory, this means one removes an end point from each curve over which the path integral goes. Since this point has no measure, the implicit assumption of a blow-up is often not noticed in discussions of these amplitudes.

For topology changes in Lorentzian manifolds, this is a problem. Because the theorem of Geroch implies singularities if a topological change takes place, one can not, e.g. desingularize a target space into two spaces A and B of separate topology and then have a non-singular path from A to B.

The incompatibilities of quantum field theory with singular space-times seem to be quite general, since the field operators are always assumed to be tempered distributions which are defined by using smooth test functions.

We close the thesis by arguing that the stochastic model of quantum mechanics developed in section 5.2.1 appears to be be consistent in singular space-times.

Section 5.2.1 has been published in [8]. The sections 5.2.2.1 and 5.3.1 are review material. Sections 5.2.2.2, 5.3.2, 5.4 and 5.5 have not yet been send to a journal.

5.2 The low energy case

5.2.1 The Schrödinger equation

In sections 4.2 and 4.4, we noted that boundary terms have to be associated with the gravitational instantons of space-time foam. The effects of boundary terms on the particle behaviour were not investigated by Hawking and coworkers.

A model where the cavities of space-time foam are associated with boundaries may cure some of the problems from [100–102] and may lead to amplitudes which are more well behaved.

One effect of the boundary terms is that the particle can not be observed coming close to a region of very large curvature or and a singularity as this region is hidden behind the event horizon. Furthermore, it has been shown that in order to cross the black-hole horizon, a particle needs to be accelerated to high kinetic energy relative to the black-hole, as otherwise it would be in an inertial system where it would see a Poynting-Robertson effect dragging it away from the black-hole [105].

Especially for micro black-holes it appears difficult to observe a particle falling into them. The lifetime τ of an isolated black-hole was calculated by DeWitt to be approximately

$$\tau \propto M^3, \tag{5.2}$$

see [22]. With $M = 2R$ and R being the Schwarzschild radius, τ is therefore very short for a black-hole if R is approximately around one Planck length. Setting $\tau = \Delta t$ into

$$\Delta E \Delta t \geq \hbar/2 \tag{5.3}$$

(where we have used the Energy time uncertainty relation from Wigner [106]), shows that one would need to detect processes of very high energy if one would be able to observe the process of a point particle flying into a Planck sized black-hole and getting replaced by outgoing radiation.

In this section, we will work on a low energy scale and a large time interval where we have no details about the in-falling process at relativistic speeds. From far away of the sources, the gravitational field of a gas of randomly distributed black-holes has a metric which is not different from the gravitational field of a gas of randomly distributed stars and one can use Newtonian mechanics at large distances approximately in both cases.

The author thinks it is reasonable that under these circumstances the following assumptions hold on which Chandrasekhar [10] bases his article on dynamical friction in gravity:

There is "a function $W(\boldsymbol{F})$ which governs the probability of occurrence of a force $\boldsymbol{F}$ per unit mass acting on the star and a function $\tau(|\boldsymbol{F}|)$ which gives the average time duration in which such a force acts.[..] The star may be assumed to suffer a large number of discrete increments in velocity of amounts $|\boldsymbol{F}|\tau(|\boldsymbol{F}|)$ occurring in random directions."

From these assumptions, Chandrasekhar derives the result that the motion of the particle, or star of mass m and velocity $\dot{\boldsymbol{\sigma}}(t)$ in a randomly fluctuating Newtonian gravitational field is, on long time scales t, affected by a dynamical friction according to Stokes law

$$\boldsymbol{F}_v = m\ddot{\boldsymbol{\sigma}}(t) = -\gamma\dot{\boldsymbol{\sigma}}(t). \tag{5.4}$$

Chandrasekhar also found that the friction coefficient γ is of the order of the reciprocal time τ of relaxation. The relaxation time is the time how long it takes for cumulative effects to have an influence over the 2 body interaction. If one assumes that the cavities of space-time foam have a Schwarzschild radius $R = 2M$ of approximately Planck length and one could use the lifetime $\tau \propto M^3$ of a black-hole of Planck size as relaxation time, then one would have to expect a very large dynamical friction

$$\gamma \propto 1/\tau. \tag{5.5}$$

It was noted by Liberati and Maccione in [107] that the naive assumption of a vacuum with a large friction coefficient would have severe consequences for matter particles that should be experimentally measurable. In the following we will try to argue why the effects described in [107] are not observed.

The boundary terms of the cavities create a severe additional problem as they should lead to Hawking radiation. Far away from a single black hole of mass

$M = 2R$, where R is the Schwarzschild radius, an observer should see a photon gas of a temperature

$$T_H = \frac{hc^3}{8\pi G M k_b} \tag{5.6}$$

with an infinite number of photons. A space-time where a high number of Planck sized black or worm holes with boundaries and event horizons are spontaneously produced from quantum fluctuations would therefore create a photon gas of high temperature and particle number. In such a photon gas, matter particles should behave differently as in vacuum due to repeated Compton scattering with the photons. This is what we will investigate below.

At first, one could think that one would have to work with a thermal field theory. However, these theories differ from usual quantum field theory, which is defined at $T = 0$. For example, in a thermal field theory of temperature T, one finds that in the absence of interactions, a scalar field with mass m has a propagator [108]

$$D_T^0(k) = \frac{i}{k^2 - m^2 + i\epsilon} + 2\pi\delta(k^2 - m^2) f(|k^0|) \tag{5.7}$$

where

$$f(|k^0|) = \frac{1}{e^{\beta |k^0|} - 1} \tag{5.8}$$

is the Bose distribution and $\beta = 1/T$.

This propagator differs from the usual formula

$$D_T^0(k) = \frac{i}{k^2 - m^2 + i\epsilon}. \tag{5.9}$$

If one has a space-time where millions of black- or worm holes are releasing photons of extremely high temperature, the approach with a thermal field theory would imply differences to the observed equations so we now try a different approach.

In his article [10], Chandrasekhar computes an explicit formula for the friction coefficient γ based on the assumptions that a particle or star would be in purely gravitational interaction with a random field of stars. Unfortunately, this computation for γ is not entirely suitable for the case of micro black- or worm holes as it does not take effects of their radiation into account. Therefore, we have to adapt our model for this situation.

Hawking radiation consists of photons of all frequencies ν. On classical matter particles, N photons of frequency ν induce a radiative pressure with a force of magnitude

$$|\boldsymbol{F}| = \nu \frac{dN(t)}{dt}. \tag{5.10}$$

For a black-hole with Schwarzschild radius R_0, the number of photons that pass a far away spherical surface of radius $r > R_0$ is given by

$$< N >= \frac{1}{2\pi} \frac{\Gamma(\omega)}{e^{\omega/k_b T} - 1}, \tag{5.11}$$

where ω is the angular frequency of the radiation per unit time and $\Gamma(\omega)$ is an absorptive coefficient. If we use the temperature of Eq. (5.6) in Eq. (5.11), then with

$$\frac{dM}{dt} = \propto -\frac{1}{M^2} \tag{5.12}$$

from [22] one can try to get an estimate of (5.10) for a particle that orbits an isolated black-hole far away.

From this, one would have to expect that the photon gas produced by micro black-holes that emerge and explode at random places in space-time creates a random force $\boldsymbol{F}_r(t)$ that acts at every point in space on a traversing particle.

We assume that the black-holes of space-time foam are equally distributed in space. This means that if at a certain time, Hawking radiation of a black-hole induces a force $\nu\frac{dN(t)}{dt}$ in $\vec{x}$ direction on a particle, another black-hole could later induce a force $-\nu\frac{dN(t)}{dt}$ on the same particle in the same direction. For all directions, this would imply that the average of $\boldsymbol{F}_r$, or its expectation value vanishes:

$$\langle \boldsymbol{F}_r \rangle = 0 \tag{5.13}$$

Furthermore, we assume for now that the appearance and explosion of several of these black-holes should not be correlated events in time. Hence, $\boldsymbol{F}_r(t)$ should be uncorrelated with $\boldsymbol{F}_r(t-1)$. All this implies by a standard argument invoking the central limit theorem, see [109], that $\boldsymbol{F}_r(t)$ should be a Wiener process with a Gaussian distribution.

With the random force term and the dynamical friction term acting on the particle, we get Langevin's famous equation:

$$m\ddot{\boldsymbol{\sigma}}(t) + \gamma\dot{\boldsymbol{\sigma}}(t) \quad = \quad \boldsymbol{F}_r(t) \tag{5.14}$$

and since $\boldsymbol{F}_r(t)$ is Gaussian, we can use all the results from the classical theory of Brownian motion.

If we want to compute e.g. a probability density of the particle to be at a certain time and position in a gas of black-holes, it may help to use the ergodic hypothesis and consider instead of a single trajectory a statistical ensemble of infinitely many trajectories. We label an individual trajectory with an index j and write $\dot{\boldsymbol{\sigma}}_j$. By considering the motion of the system without $\boldsymbol{F}_{rj}(t)$, one gets $\gamma \propto \frac{m}{\tau}$ with τ as relaxation time. One may additionally note that the experiment may consist of additional forces $\boldsymbol{F}_j^{ext}$ as well. One then gets

$$m\ddot{\boldsymbol{\sigma}}_j(t) + \frac{m}{\tau}\dot{\boldsymbol{\sigma}}_j(t) \quad = \quad \boldsymbol{F}_{rj}(t) + \boldsymbol{F}_j^{ext} \tag{5.15}$$

for a single trajectory.

From this, one may compute average velocities for the entire ensemble that either describe the outcomes of single particle observed over a long time or an experiment where the same configuration is repeatedly measured and then expectation values are calculated.

It is known that gravitational potentials of a star can reduce the entropy of a surrounding gas of particles by compressing the gas and slowing it down (the compression in the gravitational field also heats the gas but this effect is smaller than the entropy reduction from the reduction of the particle motion, see [110]. The matter that is slowed down then gets on a trajectory where it falls into the star whose entropy increases). In our case, the dynamical friction induced by the gravitational fields collectively reduces the entropy of a particle ensemble. Thereby, Eq. (5.15) contains all the ingredients of the black-hole information paradox: A term that reduces the entropy of a particle system, and a random noise term that is purely thermal and does not suffice to restore the original state of the particles, as it does not contain any information about it.

The black-hole entropy generated by Hawking radiation is much larger than the entropy of the surrounding matter. A proposal by DeWitt [111] is therefore that the in-falling particles interact with the outgoing Hawking radiation and modify its Bogoliubov modes. This interaction may restore the state of the surrounding matter.

If the information is preserved upon black-hole decay, one has to expect that the friction term in Eq. (5.15), which reduces the entropy of the particles, gets reversed into $-\gamma\dot{\boldsymbol{\sigma}}_j(t)$ at a later stage of the process by the radiation. A corrected radiation term

$$\tilde{\boldsymbol{F}}_{rj}(t) = 2\frac{m}{\tau}\dot{\boldsymbol{\sigma}}_j(t) + \boldsymbol{F}_{rj}(t) \tag{5.16}$$

should then restore the original state of the matter. One gets the following equation:

$$m\ddot{\boldsymbol{\sigma}}_j(t) - \frac{m}{\tau}\dot{\boldsymbol{\sigma}}_j(t) = \boldsymbol{F}_{rj}(t) + \boldsymbol{F}_j^{ext}. \tag{5.17}$$

Although they did not connect Eqs. (5.15) and (5.17) to the behaviour of a particle ensemble that is put into a gas of black hole like objects, these equations were first proposed by Fritsche and Haugk in [112] as a starting point to derive Schrödinger's equation.

In their calculation, they separate the $\dot{\boldsymbol{\sigma}}_j$ into $\dot{\sigma}_j(t) = \dot{\boldsymbol{\sigma}}_{rj}(t) + \dot{\boldsymbol{\sigma}}_{cj}(t)$, where $\dot{\boldsymbol{\sigma}}_{cj}(t)$ is the convective velocity that would occur if $\boldsymbol{F}_{rj}$ would be absent, and $\dot{\boldsymbol{\sigma}}_{rj}(t)$ is caused by the random force.

Fritsche and Haugk then write the components of $\boldsymbol{\sigma}_{rj}(t)$, $\boldsymbol{\sigma}_{cj}(t)$, $\boldsymbol{F}_{rj}(t)$, and $\boldsymbol{F}_j^{ext}(t)$ as $\sigma_{rjk}(t)$, $\sigma_{cjk}(t)$, $F_{rjk}(t)$, and $F_{jk}^{ext}(t)$,

They assume that there are no correlations between σ_{rjl} and F_{rjk} including $l = k$. Furthermore, for $l \neq k$, σ_{rjl} and σ_{rjk} are assumed to be uncorrelated. Finally, it is assumed that there are no correlations between σ_{cjl} and σ_{rjk}

From the computation of the average velocities for the ensemble of all j trajectories described by Eqs. (5.15) and (5.17) under these assumptions, Fritsche and Haugk get for long time intervals $\Delta t >> \tau$

$$\partial_t(\boldsymbol{v} - \boldsymbol{u}) + (\boldsymbol{v} + \boldsymbol{u})\nabla(\boldsymbol{v} - \boldsymbol{u}) - \nu\Delta(\boldsymbol{v} - \boldsymbol{u}) = \frac{1}{m}\boldsymbol{F}^{ext} \tag{5.18}$$

and

$$\partial_t(\boldsymbol{v} + \boldsymbol{u}) + (\boldsymbol{v} - \boldsymbol{u})\nabla(\boldsymbol{v} + \boldsymbol{u}) + \nu\Delta(\boldsymbol{v} + \boldsymbol{u}) = \frac{1}{m}\boldsymbol{F}^{ext}. \tag{5.19}$$

In the equations above,

$$\boldsymbol{v} = \boldsymbol{v}_c + \boldsymbol{u} \tag{5.20}$$

is the ensemble average of $\dot{\boldsymbol{\sigma}}_j$ over j and $\boldsymbol{v}_c$ is the ensemble average of $\dot{\boldsymbol{\sigma}}_{jc}$ over j. Furthermore,

$$\boldsymbol{u} = -\nu\frac{1}{\rho}\nabla\rho \tag{5.21}$$

where ρ is the probability density of the particle and

$$\nu = \frac{k_b T\tau}{m} \tag{5.22}$$

is a diffusion coefficient. T is the effective temperature of the heath bath and one may set

$$\nu \equiv \frac{\hbar}{2m}. \tag{5.23}$$

Computing then the average of Eqs. (5.18) and (5.19), one arrives at

$$\frac{d}{dt}\boldsymbol{v} - (\boldsymbol{u}\nabla)\boldsymbol{u} + \nu\Delta\boldsymbol{u} = \frac{1}{m}\boldsymbol{F}_{ext}. \tag{5.24}$$

Setting $\hbar = 0$ results in $\nu = 0$ and Newton's second law $\boldsymbol{F} = m\dot{\boldsymbol{v}}$. On the other hand, with $\hbar \neq 0$, one can derive the one particle Schrödinger equation

$$i\hbar\partial_t\psi = \left(\frac{-\hbar^2\nabla^2}{2m} + V_{ext}\right)\psi \tag{5.25}$$

from Eq. (5.24).

If we could use Eq. (5.6) in

$$\hbar = 2k_bT\tau, \tag{5.26}$$

it would imply that

$$\tau \;=\; \frac{\hbar}{2k_bT} = \frac{4\pi GM}{c^3}. \tag{5.27}$$

For a Schwarzschild black-hole

$$M = \frac{R_0c^2}{2G}, \tag{5.28}$$

and with a Schwarzschild radius of Planck length $R_0 = l_p$, this would mean a relaxation time of $\tau = \frac{2\pi l_p}{c} = 3.35 \cdot 10^{-43}s$, which is shortly above Planck time of $5.3 \cdot 10^{-44}s$.

We note that with these assumptions, neither τ nor $\hbar$ do depend on the gravitational constant. Hence, no matter if we live in a universe where the gravitational constant G is small or not, if the heath bath is produced by Hawking radiation of a gas of black-holes of Planck size, one would always get the same relaxation time and Planck's constant.

However, one should note that Eq. (5.6), the temperature one sees for a single black-hole at infinity can, strictly speaking, probably not be used to compute the temperature in Eq. (5.26). The temperature in Eq. (5.26) contains the average temperature that a particle sees in a gas of randomly occurring black-holes that are coupled to a heath bath of their own radiation. This effective temperature is defined in the Gaussian probability distribution of $\boldsymbol{F}_j^r(t)$ which is given by

$$P(\boldsymbol{F}_j^r) = \frac{1}{\sqrt{\pi}\frac{1}{\tau_{coll}}\sqrt{mk_bT}}e^{-\left(\boldsymbol{F}_j^r/\left(\frac{1}{\tau_{coll}}\sqrt{mk_bT}\right)\right)^2}$$

where τ_{coll} is a mean time of momentum transfer from the encounters with thermal photons.

Certainly, the effective average temperature that comes from random encounters with the radiation of many black-holes does not correspond exactly to the temperature that an observer sees for a single isolated black-hole at infinity. But perhaps one can use the latter temperature as an approximation for particles which are never observed to come close to the black-holes of space-time foam. To the author, the reversible diffusion process outlined above appears to be a possible explanation for why we do not observe any of the effects mentioned in [107] when we are dealing with a gas of Planck sized black-holes that would, according to Chandrasekhar's calculations, give rise to a highly viscous medium with viscosity coefficient $\gamma \propto \frac{1}{\tau}$ for all particles immersed in it.

This is certainly only a first idea to solve the problems that space-time foam poses for the behaviour of matter particles. It is, for example still unclear, how one should derive the correlations observed in entangled states from such a model. How to extend these ideas relativistically is also unclear at the moment.

5.2.2 A note on entangled states

This section only aims to show that the treatment of entangled states with the model described in section 5.2.1 is not entirely impossible.

A well known no go result that concerns the inability of theories to model entangled states is Bell's theorem. In subsection 5.2.2.1 we will review this no-go result in a mathematically rigorous way. In subsection 5.2.2.2, we will then argue how it may be possible for the stochastic theory above, to circumvent it.

5.2.2.1 Bell's inequalities

This subsection is a review on Bell's work and it is essentially similar to the old unpublished preprint [18] which the author wrote when he accidentally came across the article [112] of Fritsche and Haugk and then the works of Nelson [113–115] and Faris [116]. The articles of the mathematicians Nelson and Faris are largely unknown but they contain a mathematically rigorous analysis of Bell's theorem.

It turns out that Bell's work on the hidden variable question can actually be divided into two theorems.

The first is published in [117] and starts by defining two random variables $A(\boldsymbol{a}, \lambda) = \pm 1$ and $B(\boldsymbol{b}, \lambda) = \pm 1$, where $\boldsymbol{a}$ is the setting or axis at detector A, $\boldsymbol{b}$ is the setting at detector B and λ is some parameter over which one integrates.

In order to describe a theory with exact anti-correlations at the detectors, the random variables are defined to fulfil:

$$A(\boldsymbol{b},\lambda) = -B(\boldsymbol{b},\lambda). \tag{5.29}$$

Assuming $\rho(\lambda)$ to be the probability distribution of λ, Bell then writes the expectation

$$\mathrm{E} = \int d\lambda \rho(\lambda) A(\boldsymbol{a},\lambda) B(\boldsymbol{b},\lambda) \tag{5.30}$$

and using Eq. (5.29), Bell gets

$$\mathrm{E} = -\int d\lambda \rho(\lambda) A(\boldsymbol{a},\lambda) A(\boldsymbol{b},\lambda). \tag{5.31}$$

From this starting point, Bell then derives his inequality.

However, already at this point the model is in severe disagreement with quantum physics. In an EPRB experiment, the outcomes at A and B can be the results of spin measurements, or of position or momentum measurements. In each case, the observables for different detector settings do not commute. Spin observables fulfill an angular momentum commutator

$$[\hat{s}_x, \hat{s}_y] = i\hbar \hat{s}_z \tag{5.32}$$

which leads to an uncertainty relation for different axes.

This means that upon measuring axis $\boldsymbol{a}$ at Station A, the measurement result for axis $\boldsymbol{b} \neq \boldsymbol{a}$ at the same detector A may be disturbed. As a result, one can not assume that Eq. (5.29) would hold for the unobserved events $A(\boldsymbol{b},\lambda)$ for axis $\boldsymbol{b}$ at A, if one measures $A(\boldsymbol{a},\lambda)$ with axis $\boldsymbol{a}$ at the same time at A. One therefore can not insert Eq. (5.29) into Eq.(5.30).

That there is in fact no locality assumption behind this first derivation of Bell's inequality can be seen when reformulating the same theorem in the form given by Faris in [116]:

Definition 1. Two events A and B are called equivalent with respect to P if

$$\mathrm{P}(A) = \mathrm{P}(B) = \mathrm{P}(A \cap B) \tag{5.33}$$

Definition 2. We denote the event that corresponds to a spin up result at detector A for an axis $\boldsymbol{A}$ as A_A, for an axis $\boldsymbol{B}$ as B_A and for an axis $\boldsymbol{C}$ as C_A. Similarly, the event that corresponds to a spin down result at detector B is denoted by A_B for axis $\boldsymbol{A}$, B_B for axis $\boldsymbol{B}$ and C_B for axis $\boldsymbol{C}$. The events A_A, B_A, C_A, and A_B, B_B, C_B are defined on a probability space with a measure P

One furthermore has the following relations between these events:

Definition 3. According to quantum mechanics, the pairs A_A and A_B are equivalent events if both experimenters choose to measure axis $\boldsymbol{A}$, B_A and B_B are equivalent if axis $\boldsymbol{B}$ is chosen and C_A and C_B are equivalent for a measurement on axis $\boldsymbol{C}$. A similar statement holds for the complements $(A_A)^c$, $(A_B)^c$ as well as $(B_A)^c$, $(B_B)^c$ and $(C_A)^c$, $(C_B)^c$.

If we assume that this equivalence holds even for axes which are not chosen for an actual measurement, one gets the following theorem:

Theorem 1. *(Bell's first theorem) We assume that the definitions (1-3) hold. If the equivalences of definition (3) are fulfilled even for axes which are not chosen for measurement, then Bell's inequality holds*

Proof. With the probability measure P, we can write the expression

$$\mathrm{P}\left(A_1 \bigcap (B_2)^c\right) + \mathrm{P}\left(B_1 \bigcap (C_2)^c\right) + \mathrm{P}\left(C_1 \bigcap (A_2)^c\right)$$

Since we assumed that the equivalence of A_1^c, A_2^c and B_1^c, B_2^c as well as C_1^c, C_2^c holds even if we measure different axes at the detectors, we can substitute the equivalent events and arrive at a form of Bell's inequality:

$$\begin{aligned} &\mathrm{P}\left(A_1 \bigcap (B_2)^c\right) + \mathrm{P}\left(B_1 \bigcap (C_2)^c\right) + \mathrm{P}\left(C_1 \bigcap (A_2)^c\right) \\ &= \mathrm{P}\left(A_1 \bigcap (B_1)^c\right) + \mathrm{P}\left(B_1 \bigcap (C_1)^c\right) + \mathrm{P}\left(C_1 \bigcap (A_1)^c\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq 1, \end{aligned}$$

where we have used that the events whose probabilities are computed are exclusive □

One clearly sees that in this naive proof of Bell's inequality, there is no locality argument. Instead, we just have ignored the spin uncertainty relation.

However, a few years later, Bell gave a new proof of his inequality in [118]. This time, the proof was more involved and has a clear implication for locality. We review Bell's second theorem below, in a version that goes along the lines of Nelson [113, 114] with corrections in [115], who analysed Bell's second theorem with mathematical rigour.

We want to describe the EPR experiment with a probability space $(\Omega, \mathcal{F}, P)$, with outcomes Ω, sigma algebra $\mathcal{F} = \mathcal{P}(\Omega)$, where $\mathcal{P}$ denotes the power set, and P as the probability measure.

The observables can, for example, be results of spin measurements. Hence we define a measurable space $(E, \mathcal{P}(E))$ with $E = \{\uparrow, \downarrow\}$

The outcomes are measured at different points in space-time. Therefore, we have to use random fields. The outcomes can also depend on the settings of the measurement devices which can be chosen by the experimenters at will.

We let the random field

$$\phi_{\boldsymbol{\mu}}(\boldsymbol{x}, \omega) : (M, \Omega) \to E \tag{5.34}$$

where $\omega \in \Omega$, describe the outcome measured by a detector with setting $\boldsymbol{\mu}$ at point $\boldsymbol{x} \in M$ in the space-time M.

In order to make contact with Nelson's notation, we define the following notation for the outcomes at the detectors: $\{\sigma_A =\uparrow\} \equiv \{\uparrow \times E\}$, $\{\sigma_A =\downarrow\} \equiv \{\downarrow \times E\}$, $\{\sigma_B =\downarrow\} \equiv \{E \times \downarrow\}$, $\{\sigma_B =\uparrow\} \equiv \{E \times \uparrow\}$.

The events

$$\{\phi_{\boldsymbol{\mu}}(A, \omega) \otimes \phi_{\boldsymbol{\nu}}(B, \omega) \in \{\sigma_A =\uparrow\}\} \tag{5.35}$$
$$\{\phi_{\boldsymbol{\mu}}(A, \omega) \otimes \phi_{\boldsymbol{\nu}}(B, \omega) \in \{\sigma_A =\downarrow\}\} \tag{5.36}$$

give information about a spin up/down outcome detector at a point A. We put all events that give information about an outcome at A into a sigma algebra $\mathcal{F}_A \subset \mathcal{F}$. Similarly, the events

$$\{\phi_{\boldsymbol{\mu}}(A, \omega) \otimes \phi_{\boldsymbol{\nu}}(B, \omega) \in \{\sigma_B =\uparrow\}\} \tag{5.37}$$
$$\{\phi_{\boldsymbol{\mu}}(A, \omega) \otimes \phi_{\boldsymbol{\nu}}(B, \omega) \in \{\sigma_B =\downarrow\}\} \tag{5.38}$$

give information about a spin up/down outcome at B and are put into a sigma algebra $\mathcal{F}_B \subset \mathcal{F}$.

Furthermore, we define a family of axis dependent probability measures $\mathrm{P}_{\phi_{A\boldsymbol{\mu}} \otimes \phi_{B\boldsymbol{\nu}}}$ as follows:

$$\mathrm{P}\left(\left\{\phi_{\boldsymbol{\mu}}(A, \omega) \otimes \phi_{\boldsymbol{\nu}}(B, \omega) \in \left(\sigma_A \bigcap \sigma_B\right)\right\}\right) \equiv \mathrm{P}_{A_{\boldsymbol{\mu}} B_{\boldsymbol{\nu}}}\left(\sigma_A \bigcap \sigma_B\right). \tag{5.39}$$

The EPR experiment consists two stages. A measurement stage, where the outcomes are measured at two spatially separated detectors located at A and B and a preparation stage. The events happening at preparation stage take place at a region S which is in the overlap of the past light cones of A and B. The events happening in S take place before the measurement is done.

We put the events happening at S into a sigma algebra $\mathcal{F}_S$. The conditional probability of an event A given the events in $\mathcal{F}_S$ is then the conditional expectation

$$\mathrm{P}\left(A|\,\mathcal{F}_S\right)(\omega) \equiv \mathrm{EX}\left[1_A|\,\mathcal{F}_S\right](\omega) \tag{5.40}$$

with 1_A as the indicator function of A.

Then one can define the condition of passive locality.

Definition 4. (Passive locality, Nelson) We call a theory passively local if

$$\mathrm{P}_{A_{\boldsymbol{\mu}}B_{\boldsymbol{\nu}}}\left(\sigma_A \bigcap \sigma_B \middle|\, \mathcal{F}_S\right) = \mathrm{P}_{A_{\boldsymbol{\mu}}B_{\boldsymbol{\nu}}}\left(\sigma_A|\,\mathcal{F}_S\right)\mathrm{P}_{A_{\boldsymbol{\mu}}B_{\boldsymbol{\nu}}}\left(\sigma_B|\,\mathcal{F}_S\right), \tag{5.41}$$

for every pair of axes $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$.

The violation of passive locality would imply that a dependence of the outcomes at A and B does not originate from the events in $\mathcal{F}_S$.

Now we will define an additional locality condition that forbids instantaneous signaling.

Definition 5. (Active locality, Nelson) We call the random fields $\phi_{\boldsymbol{\mu}}$ and $\phi_{\boldsymbol{\mu}'}$ actively local if, whenever $\boldsymbol{\mu}$ and $\boldsymbol{\mu}'$ agree except on a region B in space-time, then $\phi_{\boldsymbol{\mu}'}$ and $\phi_{\boldsymbol{\mu}'}$ agree, except on the future cone of B.

This implies than an experimenter at B can not send a signal outside of the future cone of B.

Consider the products of random fields $\phi_{\boldsymbol{\mu}}(A,\omega) \otimes \phi_{\boldsymbol{\nu}}(B,\omega)$ and $\phi_{\boldsymbol{\mu}'}(A,\omega) \otimes \phi_{\boldsymbol{\nu}'}(B,\omega)$, where A and B are space-like separated points in the space-time where A is outside the future cone of B. Let $\boldsymbol{\mu},\boldsymbol{\mu}'$ be settings at A and $\boldsymbol{\nu},\boldsymbol{\nu}'$ are settings at B.

A is outside the future cone of B. If the random fields are actively local, then if $\mu = \mu'$ in A, one has

$$\phi_{\mu}(A,\omega) = \phi_{\mu'}(A,\omega) \tag{5.42}$$

even if we have selected different axes $\nu \neq \nu'$ at B.

An event in $\mathcal{F}_A$ only contains information about the outcome at A, with the result for B being the sure event. Active locality implies that an event $\{\phi_{\boldsymbol{\mu}}(A,\omega) \otimes \phi_{\boldsymbol{\nu}}(B,\omega) \in \{\sigma_A = \uparrow\}\} \in \mathcal{F}_A$ whose probability is computed by $P_{A_{\boldsymbol{\mu}}B_{\boldsymbol{\nu}}}$ is equivalent to $\{\phi_{\boldsymbol{\mu}}(A,\omega) \otimes \phi_{\boldsymbol{\nu}'}(B,\omega) \in \{\sigma_A = \uparrow\}\} \in \mathcal{F}_A$ whose probability is computed with with respect to $P_{A_{\boldsymbol{\mu}}B_{\boldsymbol{\nu}'}}$.

In the proof of Bell's inequality below, we will only use the axis dependent family of probability measures $P_{A_{\boldsymbol{\mu}}B_{\boldsymbol{\nu}}}$. With this measure, the probability of a spin up outcome for different axes at the same detector can not be computed like in the proof above with $P(A_1 \cap B_1)$.

Nelson's proof of Bell's inequality also uses equivalent events but this is only done in the active locality condition. There, one only has equivalences of events that give information about the outcome at a single detector whose settings stay the same. The outcomes of the detector whose settings are changed are not specified. This different from the assumptions of Bell's first theorem.

The proof below does not explicitly assume that one would have equivalences between events for specific up and down outcomes at the detectors for an axis ν even if one measures a different axis $\mu \neq \nu$ at one detector. Thereby, the local effects of the spin uncertainty relation, which may disturb the exact correlations for the unobserved outcomes are at not invalidated in the assumptions of the proof below.

By theory and experimental observation, one has

$$\begin{aligned} \mathrm{P}_{A_\mu B_\mu}(\sigma_A =\uparrow) = \mathrm{P}_{A_\mu B_\mu}\left(\sigma_A =\uparrow \bigcap \sigma_B =\downarrow\right) = \mathrm{P}_{A_\mu B_\mu}(\sigma_B =\downarrow) = \frac{1}{2}, \\ \mathrm{P}_{A_\mu B_\mu}(\sigma_A =\downarrow) = \mathrm{P}_{A_\mu B_\mu}\left(\sigma_A =\downarrow \bigcap \sigma_B =\uparrow\right) = \mathrm{P}_{A_\mu B_\mu}(\sigma_B =\uparrow) = \frac{1}{2} \end{aligned} \tag{5.43}$$

for the spin measurements in an EPR experiment with an arbitrary axis $\boldsymbol{\mu}$.

With these definitions, Bell's second theorem states:

Theorem 2. *(Bell's second theorem) Let Eq. (5.43) and active and passive locality hold. Then the Clauser-Holt-Shimony-Horne (CHSH) inequality [119] holds*

$$|\mathrm{E}(\boldsymbol{\mu},\boldsymbol{\nu}) - \mathrm{E}(\boldsymbol{\mu},\boldsymbol{\nu}') + \mathrm{E}(\boldsymbol{\mu}',\boldsymbol{\nu}) + \mathrm{E}(\boldsymbol{\mu}',\boldsymbol{\nu}')| \leq 2, \tag{5.44}$$

where the function

$$\begin{aligned} \mathrm{E}(\boldsymbol{\mu},\boldsymbol{\nu}) \equiv\ & \mathrm{P}_{A_\mu B_\nu}\left(\sigma_A =\uparrow \bigcap \sigma_B =\uparrow\right) + \mathrm{P}_{A_\mu B_\nu}\left(\sigma_A =\downarrow \bigcap \sigma_B =\downarrow\right) \\ & - \mathrm{P}_{A_\mu B_\nu}\left(\sigma_A =\uparrow \bigcap \sigma_B =\downarrow\right) - \mathrm{P}_{A_\mu B_\nu}\left(\sigma_A =\downarrow \bigcap \sigma_B =\uparrow\right). \end{aligned} \tag{5.45}$$

is called correlation coefficient

Proof. In his work [113–115] Nelson proves an inequality which is a bit different. Here, the CHSH [119] inequality, a variant of Bell's inequality is proven which is tested in EPR experiments.

If $\boldsymbol{\mu} = \boldsymbol{\nu}$, then passive locality implies:

$$\mathrm{P}_{A_\mu B_\mu}\left(\sigma_A \bigcap \sigma_B \middle| \mathcal{F}_S\right) = \mathrm{P}_{A_\mu B_\mu}(\sigma_A | \mathcal{F}_S)\, \mathrm{P}_{A_\mu B_\mu}(\sigma_B | \mathcal{F}_S). \tag{5.46}$$

and because $0 \leq \mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_B | \mathcal{F}_S\right) \leq 1$, we have

$$\mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_A \bigcap \sigma_B \Big| \mathcal{F}_S\right) \leq \mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_A | \mathcal{F}_S\right). \tag{5.47}$$

By Eq. (5.40),

$$\mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_A\right) = \mathrm{EX}\left[\mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_A | \mathcal{F}_S\right)\right] \tag{5.48}$$

and

$$\mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_A \bigcap \sigma_B\right) = \mathrm{EX}\left[\mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_A \bigcap \sigma_B \Big| \mathcal{F}_S\right)\right]. \tag{5.49}$$

With Eq. (5.43), the expectation values in Eqs. (5.48) and (5.49) must be equal. It then follows from Eq. (5.47) that

$$\mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_A \bigcap \sigma_B \Big| \mathcal{F}_S\right) = \mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_A | \mathcal{F}_S\right). \tag{5.50}$$

We can condition the probabilities of the events $\{\phi_{A\boldsymbol{\mu}} \otimes \phi_{B\boldsymbol{\nu}} \in \sigma_A =\uparrow\} \in \mathcal{F}_1$ and $\{\phi_{A\boldsymbol{\mu}} \otimes \phi_{B\boldsymbol{\nu}} \in \sigma_B =\downarrow\} \in \mathcal{F}_2$ with respect to $\mathcal{F}_S$ and get from active locality

$$\mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_A =\uparrow | \mathcal{F}_S\right) \;=\; \mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_A =\uparrow | \mathcal{F}_S\right) \equiv P_{\boldsymbol{\mu}}. \tag{5.51}$$

and similarly

$$\mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_B =\downarrow | \mathcal{F}_S\right) \;=\; \mathrm{P}_{A_{\nu}B_{\nu}}\left(\sigma_B =\downarrow | \mathcal{F}_S\right) \equiv P_{\boldsymbol{\nu}}. \tag{5.52}$$

where in the last line, we have used Eq. (5.43)

Plugging Eq. (5.51) and Eq. (5.52) in the passive locality condition yields

$$\mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_A =\uparrow \bigcap \sigma_B =\downarrow \Big| \mathcal{F}_S\right) \;=\; P_{\boldsymbol{\mu}}P_{\boldsymbol{\nu}}. \tag{5.53}$$

The events $\{\phi_{A\boldsymbol{\nu}} \otimes \phi_{B\boldsymbol{\nu}} \in \sigma_B =\uparrow\} \in \mathcal{F}_2$ and $\{\phi_{A\boldsymbol{\nu}} \otimes \phi_{A\boldsymbol{\nu}} \in \sigma_B =\downarrow\} \in \mathcal{F}_2$ are disjoint and their union is the sure event. The sum of the conditional probabilities with respect to a sigma algebra is equal to unity for such events. Using Eq. (5.52), we can write

$$\mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_B =\uparrow | \mathcal{F}_S\right) = \mathrm{P}_{A_{\nu}B_{\nu}}\left(\sigma_B =\uparrow | \mathcal{F}_S\right) = 1 - P_{\boldsymbol{\nu}}, \tag{5.54}$$

and similarly,

$$\mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_A =\downarrow | \mathcal{F}_S\right) \;=\; \mathrm{P}_{A_{\mu}B_{\mu}}\left(\sigma_A =\downarrow | \mathcal{F}_S\right) = 1 - P_{\boldsymbol{\mu}}. \tag{5.55}$$

Applying the passive locality condition to Eqs. (5.55) yields:

$$\mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_A = \downarrow \bigcap \sigma_B = \uparrow \middle| \mathcal{F}_S\right) \;=\; \left(1 - P_{\boldsymbol{\mu}}\right)\left(1 - P_{\boldsymbol{\nu}}\right). \tag{5.56}$$

In the same way, one can compute

$$\mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_A = \uparrow \bigcap \sigma_B = \uparrow \middle| \mathcal{F}_S\right) = P_{\boldsymbol{\mu}}\left(1 - P_{\boldsymbol{\nu}}\right) \tag{5.57}$$

and

$$\mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_A = \downarrow \bigcap \sigma_B = \downarrow \middle| \mathcal{F}_S\right) = \left(1 - P_{\boldsymbol{\mu}}\right) P_{\boldsymbol{\nu}}. \tag{5.58}$$

Using the Eqs. (5.57), (5.58), (5.53) and (5.56), we may define the expression

$$\begin{aligned}\mathrm{E}\left(\boldsymbol{\mu},\boldsymbol{\nu}\middle|\, \mathcal{F}_S\right) \equiv\; & \mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_A = \uparrow \bigcap \sigma_B = \uparrow \middle| \mathcal{F}_S\right) + \mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_A = \downarrow \bigcap \sigma_B = \downarrow \middle| \mathcal{F}_S\right) \\ & -\mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_A = \uparrow \bigcap \sigma_B = \downarrow \middle| \mathcal{F}_S\right) - \mathrm{P}_{A_{\mu}B_{\nu}}\left(\sigma_A = \downarrow \bigcap \sigma_B = \uparrow \middle| \mathcal{F}_S\right) \\ & = P_{\boldsymbol{\mu}}\left(1 - P_{\boldsymbol{\nu}}\right) + \left(1 - P_{\boldsymbol{\mu}}\right) P_{\boldsymbol{\nu}} - P_{\boldsymbol{\mu}} P_{\boldsymbol{\nu}} - \left(1 - P_{\boldsymbol{\mu}}\right)\left(1 - P_{\boldsymbol{\nu}}\right).\end{aligned} \tag{5.59}$$

The conditional probabilities in Eq. (5.59) are all in the interval $[0, 1]$. For this reason, one can compute the following inequality with four arbitrary axes $\boldsymbol{\mu}, \boldsymbol{\mu}'$ and $\boldsymbol{\nu}, \boldsymbol{\nu}'$ at 1 and 2:

$$\left|\mathrm{E}\left(\boldsymbol{\mu},\boldsymbol{\nu}\middle|\, \mathcal{F}_S\right) - \mathrm{E}\left(\boldsymbol{\mu},\boldsymbol{\nu}'\middle|\, \mathcal{F}_S\right) + \mathrm{E}\left(\boldsymbol{\mu}',\boldsymbol{\nu}\middle|\, \mathcal{F}_S\right) + \mathrm{E}\left(\boldsymbol{\mu}',\boldsymbol{\nu}'\middle|\, \mathcal{F}_S\right)\right| \leq 2. \tag{5.60}$$

The unconditional probabilities are given by the expectation values of the conditional probabilities. For random variables U, V, W, Y on a probability space, where $W = U + V$, one has $\mathrm{EX}[W] = \mathrm{EX}[U] + \mathrm{EX}[V]$. Furthermore, if $W \leq Y$, then $\mathrm{EX}[W] \leq \mathrm{EX}[Y]$, almost surely, and if $Y = c$, where c is a constant, then $\mathrm{EX}[Y] = c$. In turn, since $\mathrm{EX}[W] \leq \mathrm{EX}[Y]$, one has $\mathrm{EX}[W] < c$. Hence, an inequality analogous to Eq. (5.60) must be true for the probabilities:

$$\left|\mathrm{E}\left(\boldsymbol{\mu},\boldsymbol{\nu}\right) - \mathrm{E}\left(\boldsymbol{\mu},\boldsymbol{\nu}'\right) + \mathrm{E}\left(\boldsymbol{\mu}',\boldsymbol{\nu}\right) + \mathrm{E}\left(\boldsymbol{\mu}',\boldsymbol{\nu}'\right)\right| \leq 2. \tag{5.61}$$

□

Naively, Bell's first and second theorem seem to look quite different. However, Faris has shown in [116], that passive locality together with the observed exact anticorrelations at the measurement stations implies that all events at the detectors are equivalent to events in $\mathcal{F}_S$, which means the events at the detectors are determined by events happening at the earlier preparation stage. By active locality, the settings of the instruments which may be chosen later can have no influence on the events in $\mathcal{F}_S$. This can be used to rewrite Bell's inequality from Bell's second theorem in the form of Bell's first theorem, which shows a clear connection between both theorems.

One should note that a the same conclusion can be obtained by a variant of the Kochen-Specker theorem. The so-called "Free-Will Theorem" from Conway and Kochen [120] also implies that if active locality and exact anti-correlations hold in quantum mechanics, then the outcomes at the detectors can not predetermined

5.2.2.2 What Bell's inequality implies for the stochastic model

The stochastic model from section 5.2.1 has several features which open up the possibility that one might describe entangled states with it, and one might do this without needing any mechanism for instantaneous signalling between separated measurement stations.

The outcomes in the stochastic theory at some time t_2 are never entirely predetermined by events that happen at earlier times $t_2 - t_2$, where $t_2 > t_1$. One assumption that goes into proofs of Bell's inequality is that the outcomes at the detectors are determined by events happening before measurement. As this is not the case in the stochastic model, it does not constitute a "hidden variable" theory in the sense of Bell.

Furthermore, the "velocities"

$$\boldsymbol{u} = -\frac{\hbar}{2m}\frac{1}{\rho}\nabla\rho \tag{5.62}$$

and

$$\boldsymbol{v} = \frac{\hbar}{2m}\nabla\varphi, \tag{5.63}$$

where ρ is the probability density and φ is the phase of the wave function, are, in this model, just ensemble averages. This is different from other theories. Models like Bohmian mechanics [121] for example, at first look similar. But there, the functions $\boldsymbol{u}, \boldsymbol{v}$ are some kind of "particle velocities". For a many particle system in an entangled state

$$|\Psi(r_1, r_2, t)\rangle = \frac{1}{\sqrt{2}}\left(|\psi_1(r_1, t)\rangle\psi_2(r_2, t)\rangle - |\psi_2(r_1, t)\rangle\psi_1(r_2, t)\rangle\right) \tag{5.64}$$

measuring one system and finding it in a state $|\psi_1\rangle$ implies that the second system is in state $|\psi_2\rangle$. Associating the states with "particle velocities" like it is done in

Bohmian mechanics, would mean that a measurement in one system can change the velocities in a space-like separated system, see [121].

If, however, $\boldsymbol{u}$ and $\boldsymbol{v}$ are just statistical expressions from ensemble averages, then a change of $\boldsymbol{u}$ and $\boldsymbol{v}$ in a distant system can occur just because our knowledge of this system has been altered since we know there are correlations with an outcome that we have measured locally.

Finally, the theory discriminates between influences that might be brought in by experimenters via a term $\boldsymbol{F}_j^{ext}$ and an intrinsic randomness that comes from a term $\boldsymbol{F}_{rj}$. As $\boldsymbol{F}_j^{ext}$ can be defined to be local in a many particle system that stretches over space-like separated regions, this can be used to make the model fulfil the active locality condition.

The intrinsically random term like $\boldsymbol{F}_{rj}$ can generate exact correlations between distant systems if experimenters chose to make the settings of their measurement devices equal. Since the random field would be out of the control of the experimenters, this kind of non locality could not be used to send signals between space-like separated locations. Non local correlations that can not be used to send signals are exactly the behaviour that one observes in quantum mechanics.

For example, consider two separate systems

$$m\ddot{\boldsymbol{\sigma}}_{j1}(t) + \gamma\dot{\boldsymbol{\sigma}}_{1j}(t) = \boldsymbol{F}_{rj1}(t) + \boldsymbol{F}_{j1}^{ext} \tag{5.65}$$
$$m\ddot{\boldsymbol{\sigma}}_{j1}(t) - \gamma\dot{\boldsymbol{\sigma}}_{j1}(t) = \boldsymbol{F}_{rj1}(t) + \boldsymbol{F}_{j1}^{ext} \tag{5.66}$$

and

$$m\ddot{\boldsymbol{\sigma}}_{j2}(t) + \gamma\dot{\boldsymbol{\sigma}}_{j2}(t) = \boldsymbol{F}_{rj2}(t) + \boldsymbol{F}_{j2}^{ext} \tag{5.67}$$
$$m\ddot{\boldsymbol{\sigma}}_{j2}(t) - \gamma\dot{\boldsymbol{\sigma}}_{j2}(t) = \boldsymbol{F}_{rj2}(t) + \boldsymbol{F}_{j1}^{ext} \tag{5.68}$$

where $\boldsymbol{\sigma}_{1/2}(t)$ describe separate trajectory ensembles that represent two particles at different locations.

If we imagine that $\boldsymbol{F}_{j1}^{ext}, \boldsymbol{F}_{j2}^{ext}$ are describing the influences of measurement devices, then if both devices have the same setting, one has

$$\boldsymbol{F}_{j1}^{ext} = \boldsymbol{F}_{j2}^{ext} \tag{5.69}$$

If we assume that there are correlations between $\boldsymbol{F}_{rj1}$, and $\boldsymbol{F}_{rj2}$, then one can certainly expect correlations between $\boldsymbol{\sigma}_{j2}(t)$ and $\boldsymbol{\sigma}_{j2}(t)$.

If we make two ensemble averages over the j trajectories and associate average velocities $\boldsymbol{u}_{1/2}$, $\boldsymbol{v}_{1/2}$, and quantum states $\psi_{1/2}$ to these systems and want the entire ensemble to be described with one quantum state, then one has to this in such a way that the information about the correlations between the trajectory ensembles $\boldsymbol{\sigma}_{j2}(t)$ and $\boldsymbol{\sigma}_{j2}(t)$ is not lost.

In [112], Fritsche and Haugk derived the N particle Schrödinger equation for independent particles by expanding the vector space of the system to a $3N$ dimensional space.

We denote the components of $\boldsymbol{\sigma}_{rjm}(t)$, $\boldsymbol{\sigma}_{cjm}(t)$, $\boldsymbol{F}_{rjm}(t)$, and $\boldsymbol{F}^{ext}_{jm}(t)$, where $1 \leq m \leq N$, as $\sigma_{rjkm}(t)$, $\sigma_{cjkm}(t)$, $F_{rjkm}(t)$, and $F^{ext}_{jkm}(t)$,

During their derivation of the many particle Schrödinger equation for independent particles, Fritsche and Haugk proceeded along the same lines as in the single particle case. They assume that there are no correlations between σ_{rjlm} and F_{rjkn} including $l = k$, $m = n$. Furthermore, for $l \neq k$, $m \neq n$, σ_{rjlm} and σ_{rjkn} are assumed to be uncorrelated. Finally, it is assumed that there are no correlations between σ_{cjlm} and σ_{rjkn}.

In order to describe entangled states with this model, one may just loose this restriction and assume that some of these correlations are not vanishing and then one may try to derive the many particle Schrödinger equation for an entangled state.

The author of this thesis has, however, not done this calculation yet. One has to compute ensemble averages of infinitely many correlated Brownian motion processes. It appears that the correlation terms that would emerge would make the computation very difficult.

This section therefore therefore just has the purpose to show that while the description of entangled states with this model is an open problem, it is not something that seems to be entirely forbidden by some no-go theorem.

A description of entangled states with this model would, however, imply a correlation between the terms $\boldsymbol{F}_{rj1}, \boldsymbol{F}_{rj2}$ that are responsible for the intrinsic randomness in the theory.

In subsection 5.2.1, we have associated these terms with a heath bath that is caused by Hawking radiation of microscopic black- or worm holes.

The AdS-CFT conjecture contains an equivalence between conformal field theories and asymptotically anti DeSitter spaces, including black-hole DeSitter spaces. The AdS-CFT conjecture can be used to show that a maximally entangled state of two thermal conformal field theories is dual to an Einstein Rosen bridge [122]. This reasoning was extrapolated by Suesskind and Maldacena who proposed in [123] a conjecture called "ER=EPR" which says that a non traversable wormhole is equivalent to a pair of maximally entangled black-holes.

We have argued in chapter 4 that space-time contains of a gas of worm holes. Furthermore, we have argued that their collective gravitational field and the thermal heath bath of their Hawking radiation would imply that a classical particle would be be governed by the Schrödinger equation. Finally, we have noted in this subsection that in order to be able to describe an entangled state with such a model, one would need to have correlations with the photon gas emerging from the Hawking radiation at spatially separated space-time points. If the wormholes of space-time foam were were equivalent to pairs of entangled black-holes, as proposed in the ER=EPR conjecture, then one would have to expect correlations in the photon gas produced by these black-holes.

5.3 The high energy case

5.3.1 't Hooft's derivation of the s-matrix of string theory from particles scattering with black-holes at high energy

In this section we discuss the scattering of particles with black-holes at high energy. Unfortunately, the usual Feynman rules of quantum field theory only include the scattering of particles with themselves. They do not to include rules for scattering processes of particles with the event horizon of a black-hole or a worm-hole with some boundary or event horizon.

From a result of 't Hooft, one has to expect dominant gravitational interactions in such a process. Let a particle fall into a black-hole at time t_0 and an outgoing particle arrive at the observer outside at time t_1. When the particles meet, their centre of mass energy is boosted by [124]

$$\sqrt{g_{00}} \propto e^{\pi(t_1-t_0)T_H}, \tag{5.70}$$

where T_H is the Hawking temperature. This means that for large $t_1 - t_0$ and for large T_H i.e. small black-hole masses, the gravitational interactions between ingoing and outgoing particles are important.

In 't Hooft's calculation from [41, 124–128], a particle's gravitational field is described by a Schwarzschild metric. Upon falling into a black-hole, the particle accelerates to relativistic speed and generates a shock-wave [129, 130]. This modifies the motion of the outgoing particles.

't Hooft's assumption is that the entire process can be described with the usual rules for quantum mechanical scattering. This means that the black-holes with which the particle is scattered must be large enough and exist long enough that the length and time scales where the infalling particle becomes ultra relativistic are such that the flight and the interaction between the in and out going particles can be described by the Schroedinger equation.

The scattering amplitude of this process has a form that is similar to the amplitude over the Polyakov action from string theory (in Wick rotated form) [41, 124–128].

A particle that flies through a space-time foam made out of a gas of worm- or black-holes may encounter many of these scattering processes. These processes also occur for gravitons. 't Hooft only derived his S-matrix only for a scalar particle. If a similar S-matrix could be derived for gravitons scattering with a black-hole, this could give a scientific reason for the hypothesis that one should use string theoretic corrections for high energy graviton scattering.

String theory amplitudes are perturbatively renormalizable. Based on the previous results, where Euler characteristic appears in the energy momentum tensor of a field in curved space-time, one could assume that the scattering with black- or wormholes is the dominating process at high energies.

Below we will give a short review of 't Hooft's derivation of the Path integral over the Polyakov action action. The calculation adopts Kruskal coordinates x, y for a black-hole of mass M, where

$$xy = -\left(\frac{r}{2M} - 1\right) e^{\frac{r}{2M}} \tag{5.71}$$

and

$$x/y = e^{\frac{t-t_0}{2M}} \tag{5.72}$$

with t_0 as some reference time and $\Omega = (\varphi, \vartheta)$ are the angular coordinates.

The ingoing particles that enter the hole and cross the future horizon and are represented by states $|in\rangle = |p_{in}\rangle$ where p_{in} is the momentum. The particles that leave the hole and cross the past horizon are represented by states $|out\rangle = |p_{out}\rangle$.

A particle that falls into a black-hole of mass M is accelerated to relativistic speed and can be described with the Aichelburg Sexl metric [131]. The $|in\rangle$ state of the ingoing particle causes a gravitational shock wave. Using Kruskal coordinates, this shock wave implies a coordinate shift

$$\delta y = \kappa p_{in} f(\Omega, \Omega') \tag{5.73}$$

where

$$\kappa = 4\pi 2^8 G M^4 e^{-1} \tag{5.74}$$

and f is the Green's function fulfilling

$$\Delta f - f = -\delta^2(\Omega, \Omega'). \tag{5.75}$$

Outgoing wave-functions $e^{-ip_{out}y}$ are then confronted with a shift $y - \delta y$ which implies that the outgoing states are multiplied by

$$|p_{out}\rangle = e^{i\int d^2\Omega p_{out}\delta y}|p_{out}\rangle = e^{i\kappa\int d^2\Omega p_{out}\cdot p_{in}f(\Omega,\Omega')}|p_{out}\rangle. \tag{5.76}$$

We are assuming that information is not lost when the particle goes into the black-hole. Therefore, the s-matrix $\langle p_{out}|p_{in}\rangle$ should be unitary and invariant under time reversals. The phase shift

$$|p_{in}\rangle = |p_{in} + \delta p_{in}\delta(\Omega,\Omega')\rangle \tag{5.77}$$

of the in going state can be ignored for now as it should not depend on the the details of the outgoing states. With a Fourier transformation

$$|p_{out}\rangle = C\int \mathcal{D}u^- e^{-\int d^2\Omega p_{out}u^-}|u^-\rangle, \tag{5.78}$$

$$|p_{in}\rangle = C\int \mathcal{D}u^+ e^{-\int d^2\Omega p_{in}u^+}|u^+\rangle, \tag{5.79}$$

where $u^- = y$ and $u^+ = x$ and C is some normalization constant, one finds

$$\langle u^-|u^+\rangle = C'e^{-\int d^2\Omega' \frac{i}{\kappa}f^{-1}(\Omega,\Omega')u^+u^-}, \tag{5.80}$$

where

$$f^{-1} = 1/f = 1 - \nabla/2\pi. \tag{5.81}$$

Using $\langle u^-|u^+\rangle$ and the two equations above, one gets

$$\langle p_{out}|p_{in}\rangle = \int \mathcal{D}u^- \int \mathcal{D}u^+ e^{\int d^2\Omega\left(\frac{i}{2\pi\kappa}(u^+u^- + \partial_\Omega u^+\partial_\Omega u^-) + iu^-p_{out} - iu^+p_{in}\right)} \tag{5.82}$$

or, with membrane coordinates x^0, x^3

$$u^\pm = x^0 \pm x^3, \tag{5.83}$$

and the external momenta

$$p_{ext} = (p_{in} - p_{out}, 0, 0, -p_{in} - p_{out}), \tag{5.84}$$

one finds

$$\langle p_{out}|p_{in}\rangle = \int \mathcal{D}x e^{\int d^2\Omega\left(\frac{-i}{2\pi\kappa}(x^2 + \partial_\Omega x^2) + ixp_{ext}\right)}, \tag{5.85}$$

where
$$x^2 = \boldsymbol{x}^2 - (x^0)^2. \tag{5.86}$$

Finally, the x^2 term gets ignored as it is due to the curvature of the horizon and one may write the amplitude covariantly as path integral over the Wick rotated Polyakov action:

$$\langle p_{out}|p_{in}\rangle = \int \mathcal{D}x^\mu \mathcal{D}g^{ab} e^{\int d^2\Omega \left(\frac{-i}{2\pi\kappa}\sqrt{g}g^{ab}\partial_a x^\mu \partial_b x_\mu + ix^\mu p_\mu\right)} \tag{5.87}$$

One should note that the similarity to string theory may be only formal. In string theory, one has the following Poisson bracket

$$\{x^\mu(\sigma,\tau), x^\nu(\sigma',\tau)\} = 0 \tag{5.88}$$

whereas 't Hooft computes, after transforming the system into Rindler coordinates [128]
$$\{x^\mu(\sigma,\tau), x^\nu(\sigma',\tau)\} = -T\epsilon^{\mu\nu 12} f(\sigma,\sigma') \tag{5.89}$$
where $\epsilon^{0123} = -i$ and the string constant $T = 8\pi G$ in Rindler coordinates. This should prevent the derivation of a Virasoro algebra. However, the result appears to hold only for Rindler coordinates.

5.3.2 On a critical argument against the black-hole scattering matrix

Some aspects of the construction of this s-matrix have been refined by Itzhaki [132], Arcioni [133] and Polchinski [134] on which I want to comment. We adopt Kruskal coordinates
$$u = t - r* \tag{5.90}$$
and
$$v = t + r* \tag{5.91}$$
with a tortoise coordinate
$$r* = r + 2Mln\left|\frac{r}{2M} - 1\right| \tag{5.92}$$

If a particle with Schwarzschild energy δE falls into a black-hole at v_1, the formation time of the horizon v_0 is shifted to $v_0 + \delta v_0$ with

$$\delta v_0 = -4\delta E e^{\frac{v_0 - v_1}{4M}} \tag{5.93}$$

and a light ray that would arrive at some time u_1 will arrive at a shifted time

$$\delta u = -4Mln\left(1 + \frac{\delta v_0}{4M} exp \frac{u - v_0}{4M}\right). \tag{5.94}$$

This shift diverges at a finite time

$$u_1 - v_0 \approx -4Mln\left(\frac{|\delta v_0|}{4M}\right) = 4Mln\left(\frac{4M}{4\delta E}\right) - v_0 + v_1. \tag{5.95}$$

If

$$u_1 - v_1 \geq 4Mln\left(\frac{4M}{4\delta E}\right) \tag{5.96}$$

then the outgoing particle is shifted so much that it does not escape the black-hole, which renders the S-matrix approach invalid [133]. The validity of the black-hole S-matrix was criticised by Itzhaki [132] especially because it depends not only on the black-hole mass but also on the energy of the ingoing particle.

I therefore want to make the following comment on the limitations of the black-hole S-matrix.

In order for this formalism to generate a unitary S-matrix, one should have time reversal invariance. Time reversal invariance means that there exists an anti unitary time reversal operator T which, if applied to a state yields

$$T|\psi_{p,\sigma}\rangle = \psi_{\mathcal{P}p,-\sigma} \tag{5.97}$$

where p is the four momentum, σ is the particle spin and $\mathcal{P}$ is the parity operator which in this case reverses the space components of the four momentum.

For the S-matrix, this means

$$\langle\psi_{out,-p}|\psi_{in,p}\rangle = T\langle\psi_{in,p}|T\psi_{out,-p}\rangle. \tag{5.98}$$

This condition implies for the scattering process that if one has an ingoing photon with momentum $p = \hbar k = \hbar\nu$, the outgoing photon must have the momentum $p = -\hbar k = -\hbar\nu$, and thus their energy $\hbar\nu$ must be equal at measurement stations that are positioned far away from the place of the scattering.

For a Schwarzschild black-hole, one has the following evaporation law

$$M^3(t) = M_0^3 - \frac{27}{10\pi \cdot 8^4}\Delta t \tag{5.99}$$

where we define M_0 as the black-hole mass at time v_1 from before [22]. This means that if one starts with a black-hole at mass $M_0 + \delta E$ at time v_1, it takes a time of

$$\Delta t \equiv u_1 - v_1 = \frac{40906\pi\delta E(M^2 + M\delta E + \frac{1}{3}\delta E^2)}{9} \tag{5.100}$$

until at some time u_1 the entire mass δE was released by the black-hole.

If we have ingoing particles, we know that the metric gets perturbed. In order to derive the evaporation law for this case, one could indeed make a similar computation as in DeWitt's review article but with the modified Bogoliubov coefficients from Arcioni's work [133]. However, we already know that the time interval $\Delta t_{perturbed}$ until outgoing matter reaches an observer for the perturbed black-hole gets longer, so

$$\Delta t_{perturbed} > \Delta t = u_1 - v_1. \tag{5.101}$$

Putting everything together, we now have the statement that in the case

$$\Delta t_{perturbed} > u_1 - v_1 = \frac{40906\pi\delta E(M^2 + M\delta E + \frac{1}{3}\delta E^2)}{9} \geq 4Mln\left(\frac{M}{\delta E}\right) \tag{5.102}$$

the S-matrix becomes invalid. One can make this more concrete by putting in a few examples for the black-hole mass.

Assume for example, a black-hole with 1 Solar mass, or $M = 9.1 \cdot 10^{37}$ Planck masses. From the above inequality, according to a calculation with Wolfrahm alpha, it would follow that the ingoing particle must have a Schwarzschild energy of $0 < \delta E < 5.47 \cdot 10^{-40}$ Planck masses if the black-hole S-matrix were to be valid. An electron has a mass of $4.1 \cdot 10^{-23}$ Planck masses. The S-matrix would thus not hold for most observable particles (apart from Neutrinos perhaps).

On the other hand, if the black-hole has a mass of $M = 1$, then $0 < \delta E < 0.0017$ Planck masses before the black-hole s-matrix breaks down, according to a calculation with Maple 2015, which means that especially for small black-holes, the S-matrix should be valid.

For a large macroscopic black-hole, this implies that most if not all particles with energy δE before they fall into the black-hole are outside of the so called scrambling time and their scattering with black-holes can not be described by the 't Hooft's S-matrix. The same should hold for ingoing particles with large energy. Such a configuration appears to be out of the scrambling time, too.

It therefore appears reasonable to set the mass in the constant κ very small, perhaps close to the Planck scale.

The black-hole S-matrix of 't Hooft has additional interesting properties. 't Hooft has developed his theory further and included electromagnetic interactions. These interactions yield a large extra dimension with an S^1 topology and radius $r = 2\pi/e$. Non-abelian gauge theories and fermions have not been added to this model yet.

String theory usually allows the definition of a Virasoro algebra, from which a no-ghost theorem can be derived. This result implies that the theory is only consistent with 26 dimensions in the purely bosonic case or with 10 dimensions in the supersymmetric case. 't Hooft has tried to derive commutator rules for his model in the Rindler frame. Unfortunately, he arrived at a different algebra. It would be interesting whether one can derive a similar restriction for the number of extra dimensions in the black-hole scattering matrix of 't Hooft.

Then one may add additional interactions that result in extra dimensions. Finally, one may ask how many extra-dimensions one would get if one adds the known standard model interactions to this theory. This could help to deduce whether there must exist additional interactions other than those that are in the known standard model. This may be a problem that the author wants to investigate in the future.

Saying this, one should be clear that this program would differ very much from usual methods in which string theory phenomenology is done. There, one usually wants to create string theory backgrounds where one then can find an effective action that contains the standard model. Compared to that, the approach by 't Hooft is a very different one.

The Polyakov action emerges from the standard model scattering with black-holes. This process changes the physics so much that one can not automatically expect the effective action of this to contain the standard model in a subgroup.

The classical limit with the standard model would be only obtained by recognizing that on energy scales where quantum field theory of the standard model is investigated at present, the black-hole s-matrix of 't Hooft is simply not there. For macroscopic black-holes, much of the matter emerges after scrambling time. For microscopic black-holes, an observer that can only see low energy processes on a large time scale, would not be able to observe the details of the scattering process at high energy. Instead, he would only see an ingoing state and after a time interval too small to get measured, an outgoing state would be there that contains the same information and entropy.

However, at low energy levels, some residual collective effect of the gravitational field of the black-hole gas of space-time foam should be seen, since gravitation is a long range force. What this process leads to was described in the section 5.2.1.

5.4 Problems with matter amplitudes and topology changes in the space-time foam

5.4.1 Changes of the Euler characteristic during the expansion of the universe

The cosmological constant was computed in the space-time foam model by maximizing the entropy that is given by the amplitude of Euclidean quantum gravity. Usually, there is no time evolution in a system that has reached a state of thermodynamical equilibrium and maximum entropy. However, we have computed this equilibrium state only for a fixed given volume.

If the system with a fixed volume V reaches a state of maximum gravitational entropy, it will have a positive non vanishing cosmological constant of $\Lambda \approx H_0^2$. On an infinitesimally small timescale, the latter implies an expansion of the volume V of the universe to $V + dV$.

In order to reach a consistent description, we should be able to find out whether the entropy of the universe increases if its volume is increased.

Did we have only the equation (4.59), then

$$\Lambda_{s0} = -\frac{424\pi\chi}{45V} \tag{5.103}$$

and a constant Euler characteristic would make it impossible that the entropy increases with the volume, since then

$$S \propto \frac{\Lambda_{s0} V}{8\pi} = const. \tag{5.104}$$

Such a result would imply that either the universe does not expand or that the Euler characteristic changes during the expansion.

With the Λ^2 corrections from quantum gravity, the entropy is proportional to

$$S \propto -I = \frac{\Lambda_{s1} V}{8\pi} \tag{5.105}$$

where Λ_{s1} is given by the saddle point of Eq. (4.56). If we insert the expression for Λ_{s1} and simplify this, we get

$$S \propto \frac{5V}{232} - \frac{\sqrt{V(225V + 24592\chi)}}{696} \tag{5.106}$$

This expression for the entropy as a function of the volume is a bit simplified since have not included the factor $C(\Lambda)$ from Eq. (4.26).

For a given volume V, Eq. (5.106) implies that the entropy increases the smaller the Euler characteristic χ becomes. With volume V constant, the entropy is at its maximum if the term in the squareroot simply vanishes. After this has been achived, the entropy can only increase by increasing the volume. Maximizing the entropy then implies a new adjustment of χ.

From Eq. (4.58), one finds

$$\chi = \frac{9}{848\pi^2}\Lambda_{s1}(29\Lambda_{s1} - 10\pi)V = \frac{9}{848}\frac{f(10\pi\sqrt{V} + 29f)}{\pi^2} \tag{5.107}$$

after substitution of $\Lambda = \frac{-f}{\sqrt{V}}$. With $\Lambda > 0$, one has $f < 0$ and therefore, the Euler characteristic becomes smaller with

$$\chi \propto -\sqrt{V}, \tag{5.108}$$

which has the units of an area.

Unfortunately, the space-time foam calculation does not describe the transition amplitudes that relate the states of maximum entropy for different volumes and topologies to each other. There are certain issues that need to be addressed when one computes transition amplitudes in quantum gravity. For example, instead of the local Wheeler-DeWitt equation, one gets non-local equations in an appropriate treatment of the problem [13, 60].

Furthermore, DeWitt had found two arguments against topology change, whose validity for the model above will be discussed below

5.4.2 Incompatibilities with quantum field theory on curved space-times during topology change

Many arguments against topology changes in quantum field theory and quantum gravity come from the work of Geroch [12]. Based on a result from Misner:

Theorem 3. *Theorem 1 (Misner): Let S and S' be two compact 3-manifolds. Then there exists a compact geometry M whose boundary is the disjoint union of S and S', and in which S and S' are both space-like.*

and the following definition from Calabi

Definition 6. A space-time with metric of Lorentzian signature is called isochronous if a continuous choice of the forward light cone can be made.

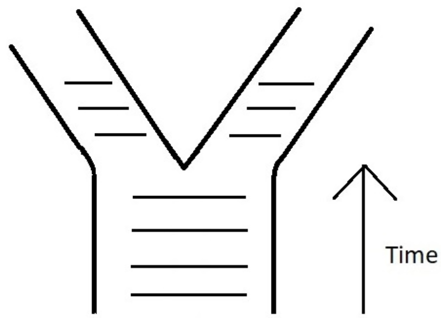

Figure 5.1: Trousers topology. The solid horizontal lines indicate field which has different boundary conditions after the topology changed at some time

Geroch was able to prove the following

Theorem 4. *(Geroch) Let M be a compact geometry whose boundary is the disjoint union of two compact space-like 3-manifolds, S and S'. Suppose M is isochronous, and has no closed time-like curve. Then S and S' are diffeomorphic, and further M is topologically $S \times [0, 1]$.*

This implies that if one starts from a certain 3 geometry embedded in a Lorentzian space-time and ends at a 3 geometry with different topology in this Lorentzian space-time, one has either to go over a 4 manifold which has closed time-like curves, or singularities.

In [135], Anderson and DeWitt used this result to show that quantum field theory on curved space-times would become inconsistent during a topology change that would create a singularity.

They begin their argument with what they call a trousers topology. This is a singular space-time with the topology of a pair of pants. At one end, it has the form of a cylinder which then undergoes a topology change at some time slice and gets cut into two cylinders, see fig. 5.1.

Anderson and DeWitt define a set of ingoing modes of a quantum field that start propagation at the side where the topology is described by one single cylinder. The field is assumed to fill the entire space and to propagate in time up into the region where the cylinder is split into two "legs" where one has outgoing modes.

Furthermore, the field is assumed to fall off at the boundary of the space-time. Then Anderson and DeWitt compute the energy operator of the in and outgoing states and find

$$E_{in}|in, vac\rangle = C|in, vac\rangle \quad (5.109)$$

and

$$E_{out}|out, vac\rangle = (C_R + C_L)|out, vac\rangle \quad (5.110)$$

where C, C_L, C_R are the Casimir energies of the trunk and the left and right leg. Computing the expectation values of in and out energies, Anderson and DeWitt find that they differ by an infinite amount. Additionally, they investigate the expectation of the renormalized energy momentum tensor. The change of the boundary conditions during the topology change introduces discontinuities in the field modes. This then results in infinite energies. Anderson and DeWitt repeat a similar computation with linearised gravity on a 3 Torus which splits up into 2 separate 3 tori. They find similar discontinuities in the mode functions.

The argument of Anderson and DeWitt against topological transitions concerns a quantum field on a space-time where it is assumed that before the topology change, the field can reach places that are unreachable after the topological transition took place.

This problem may probably be solved by simply assuming that the worm holes of space-time foam are non-traversable. With a traversable wormhole, one could always try to move the throats such that they face each other. Then, a geodesic passing through both throats would then be a closed time-like curve [98]

The requirement that wormholes are non-traversable can thus be seen to follow from a causality requirement.

If the wormholes of space-time foam are non-traversable, quantum fields on that space-time can not be cut off from regions with different boundary conditions by the singularity of a topological change. Thus an energy flash could not be observed when a non-traversable wormhole fluctuates into or out of existence.

However, even in that case, there emerges a problem for quantum gravity. The space-time during the transition is singular and the question arises whether quantum gravity is compatible with such space-times.

5.4.3 Problems with path integrals over non-simply connected configuration spaces

In his non-technical article [13], DeWitt has made a short comment where he writes that topological changes would be forbidden because there exists no fundamental group of the configuration space of quantum gravity. Since DeWitt's comment is basically just a short sentence and rather non-technical, we will add the necessary mathematical terminology for a proof of this result below.

If one were to describe the Aharonov-Bohm experiment with a path integral, the configuration space of the system would be the set $x_i(t)$ of a countably infinite number of paths indexed by i and parametrized by a time parameter t in which a particle can get from the source to the destination.

Because some paths can circle around the solenoid for several times one can not deform each path into another. The homotopy class of a certain group of paths can be defined by the number of how many times each path is winded around the solenoid.

A result from Laidlaw and DeWitt-Morette [136] shows that for path integrals over non simply connected configuration spaces, an amplitude becomes a superposition of partial amplitudes K^α where in each K^α only paths of a certain homotopy class α are considered:

$$Z = \sum_\alpha D(\alpha) K^\alpha \tag{5.111}$$

Laidlaw and DeWitt-Morette also show in [136] that the weighting factor $D(\alpha)$ must be a scalar unitary representation of the fundamental group of the configuration space. Their result holds for path integrals with arbitrary actions and configuration spaces. The only requirement is that one can define an integration measure over paths.

The labelling of a class of paths to a group element requires an arbitrarily chosen homotopy mesh. Laidlaw and DeWitt-Morette have shown in [136] that the entire amplitude is independent of the choice of that mesh.

The analysis of the configuration space of quantum gravity began with the early article from DeWitt [3] on canonical quantum gravity. Then, Fischer analysed the configuration space more closely in [137]. He found that it is not a manifold but that it is a stratified space which has strata consisting of ordinary points and boundary points. In the same conference, DeWitt [138] argued that the configuration space can be extended such that it becomes a manifold (see also [139] for an introduction). More recent articles on that topic are the ones from Giulini [140] and Anderson [141].

In the path integral of quantum gravity, one starts with a 3 manifold Σ_1 and then considers all possible paths over various 4 metrics. These paths end at another 3 manifold Σ_2 as their boundary.

When one analyses the configuration space, one usually talks about the 3 manifold (Σ_1, γ_{ij}) from which the paths start. This 3 manifold is an argument of the wave functional that solves the Wheeler-DeWitt equation $\Psi(\gamma_{ij}, t) = \int dg_{\mu\nu} e^{iS}$.

When one sums path integrals of point particles, the dimensionality of the point from which all the paths start is the same as the dimensionality of the points from which the paths in the integration are made. This can be different in quantum gravity. If the wave functional solves the Wheeler deWitt equation with a vanishing Hamiltonian, it is time independent, i.e. only a functional of γ_{ij}: $\Psi(\gamma_{ij}) = \int dg_{\mu\nu} e^{iS}$. Paths in quantum gravity start and end at 3 manifolds Σ_1 and Σ_2 but one has paths whose points describe the metrics of 4 manifolds $(M, g_{\mu\nu})$.

For the formula of DeWitt-Morette, one needs to consider the paths over the 4-manifolds $g_{\mu\nu}$ and not the 3 manifolds γ_{ij} that are usually used in articles to define the configuration space of quantum gravity.

But in other aspects, the analysis of the configuration space of the paths is similar. Let us first begin by considering only a single topological manifold M with fixed topology. The space of 4 metrics $g_{\mu\nu}$ is called $Riem(M)$ and the space $\tilde{\Theta}(M)$ over which one integrates is given by

$$\Theta(M) = Riem(M)/Diff(M), \tag{5.112}$$

where $Diff(M)$ is the diffeomorphism group.

As a quotient space, it has quotient singularities. But it can probably be desingularized in the same way as the 3 manifolds (Σ_1, γ_{ij}). Then one could do a similar analysis of this space as in [137, 140].

Let us now consider several topological manifolds M_i with different topology, and then compute the union of these configuration spaces

$$\tilde{\Theta} = \cup_i \Theta(M_i). \tag{5.113}$$

This is the configuration space if we include paths that end at a different topology than the one which they started from.

For the formula of DeWitt-Morette, we need to be able to define the fundamental group of $\tilde{\Theta}$. The fundamental group exists only if we can define a base-point $x_0 \in \tilde{\Theta}(M)$. Then we can consider various loops that start and end at x_0. For this, $\tilde{\Theta}(M)$ must be path connected.

As an example, choose a flat space and define the metric of this flat space as a base-point x_0. Due to Geroch's theorem, any path from this space to a space with different topology will go over a space that has singularities. This means that the metrics in this space will be divergent somewhere, if we do not allow metrics with closed time-like curves.

Let us define a continuous path

$$I : [0, 1] \to X \tag{5.114}$$

to a space of metrics

$$X = \cup_i Riem(M_i) \tag{5.115}$$

where $I(0) = I(1)$ is a flat space metric. Unfortunately, if I is continuous, there does not exist a $\sigma \in (0,1)$ where $I(\sigma)$ maps to a metric tensor that is divergent.

The path I simply can not continuously go to the divergent metric tensor and then back to a regular flat metric. If we divide the non-path connected space of metrics X trough the diffeomorphisms, the quotient space remains not to be path-connected.

The singular spaces of general relativity are usually not manifolds but stratified spaces. They consist of strata that contain manifolds, and two types of singularities: boundary singularities and singularities with divergences in metrics and topological invariants as strata.

In order to do homotopy theory on such a space one may think of using some kind of stratified homotopy theory [142]. One has to describe the different strata as a partially ordered set and then one can define an exit path ∞ category where 1-morphisms describe exit paths from lower to higher strata. Unfortunately, this theory only provides exit paths, which means they are not allowed to return to the strata they were starting from [142].

One can use a fundamental groupoid on stratified spaces. The fundamental groupoid does not need a base point. Assume we group the regular metrics without singularities, and the metrics with singularities into different strata and order them. Then an exit path could go from a regular metric to a singular one. For a topology change one would want to go from a regular metric, like e.g. flat space, over a singular metric back to a regular metric, e.g. a wormhole, with different topology. Unfortunately, since the exit path can not go back to the stratum where it came from, stratified homotopy theory can not be used to define topology changing paths.

This suggests that one needs a version of quantum mechanics that can work with singularities in order to realize topological changes and at the same time avoids closed time-like curves.

5.5 Problems with singularities and topological changes in string theory

In [13], DeWitt notes that sometimes [14] it is claimed that it would be possible to formulate string theory exactly on orbifolds, which are stratified spaces that consist of a manifold and a quotient singularity as strata. DeWitt adopts these claims, and adds that one could write string theory even on Lorentzian orbifolds. He argues in his non-technical paper that one may therefore use string theory in order to describe topology changes of Lorentzian space-times.

The usual string theory amplitude is a path integral of the form

$$\langle p_{out}|p_{in}\rangle = \int \mathcal{D}X^\mu \mathcal{D}\gamma_{ab} V_1 \dots V_n e^{iS_p} \tag{5.116}$$

where

$$S_p = \frac{-1}{T}\int d^2\sigma \sqrt{-\gamma}\gamma_{ab}\partial^a X^\mu(\sigma)\partial^b \eta_{\mu\nu} X^\nu(\sigma) \tag{5.117}$$

is the Polyakov action. γ_{ab} is a 2x2 metric of a world-sheet which is parametrized by two coordinates $\sigma \in [0, 2\pi], \tau \in \mathbb{R}$. The functions $X^\mu(\sigma, \tau)$ describe a mapping of the world sheet into the target space whose metric is $\eta_{\mu\nu}$ and $V_1 \dots V_n$ are so called vertex operators.

Harvey, Vafa and Witten computed the partition function of string theory on an orbifold in [14]. They set the following conditions for the embedding functions in the form of

$$X^\mu(\sigma + 2\pi, \tau) = gX^\mu(\sigma, \tau) \tag{5.118}$$

and for the string states

$$g|\Psi\rangle = |\Psi\rangle, \tag{5.119}$$

where g is some element of G. They then calculate the path integral of string theory with these boundary conditions.

The authors of [14] do not discuss what happens in the classical limit of the string theory amplitude at the singular point of the orbifold. The article [14] also seems to leave out how the quantum theory is defined at this point itself.

The equations of motion of string theory are found by minimizing the world sheet with respect to the target space. Classically, the functions $X^\mu(\sigma, \tau)$ thus describe a geodesic field, with one geodesic $\lambda^\mu(\tau) = X^\mu(y, \tau)$ for each $y \in [0, 2\pi]$. The solutions of the string equation of motion are derived from the action where $\partial_a X^\mu(\sigma, t)$ appears and are are given by

$$\partial_a\left(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu\right) = 0. \tag{5.120}$$

On the singularity of the orbifold, the derivatives of functions that map into the target space diverge. Mathematically, canonical quantisation procedures rely on a smooth Poisson manifold [143]. Therefore, it is unclear how one should define a quantum theory at the singular point of an orbifold by canonical quantisation.

The string theory amplitude contains a path integration over the world sheet metric. If one fixes a topological manifold $\Sigma^{(2)}$ for the world sheet, the configuration space over which one integrates is then the quotient space

$$\Omega(\Sigma^{(2)}) = \frac{Riem(\Sigma^{(2)})}{Diff(\Sigma^{(2)}) \ltimes Weyl(\Sigma^{(2)})} \tag{5.121}$$

of 2 metrics divided through the diffeomorphism group and the Weyl invariance group [144]. This is quite similar to the configurations space of ordinary quantum gravity.

Obviously, if we consider an enlarged space that includes singular world sheet metrics or world sheet spaces $\Sigma_i^{(2)}$ that have different topology, then the configuration space is given by

$$\tilde{\Omega} = \cup_i \Omega(\Sigma_i^{(2)}) \tag{5.122}$$

The path integral formula Eq. (5.111) of Laidlaw and DeWitt-Morette is completely general in that it is not related to a specific action or a certain topological space of paths. It thus is also applicable in the string theory case.

If we were to allow singular world-sheet metrics into the path integral of the string theory amplitude, the space of metrics would also not be path connected any more. By the same arguments as in the foregoing section, this would prevent the definition of the fundamental group of the configuration space. By Eq. (5.111), this would prevent the definition of the string theory amplitude.

The world sheet is of course not the target space. However, the string theory amplitude also contains a path integration over the embedding functions. We will see below on which spaces this can defined.

As we have seen in section 1.2, there exists a mathematically rigorous formulation of path integration [1, 19–21, 145]. Unfortunately, this theory shows that the measure $\mathcal{D}X^\mu$ in the path integral does only exist if the functions X^μ form a Banach space whose norm induces a metric on the space of functions X^μ. The reason for this is simply that integrals measure lengths and areas.

The arguments in [1] use parts of advanced measure theory.

String theorists often use the Euclidean path integral with a Wick rotated Polyakov action. This integral can be mathematically defined with the Feynman-Kac formula, see [144]. Its measure is in fact a usual Lesbegue measure, see [1].

Here we give a simplified proof which shows that the Euclidean path integral needs the embedding functions to form a metrizable space. The proof only uses simple concepts from Lesbegue theory that can be easily understood by physicists without much mathematical education.

In Lesbegue measure theory, one writes a function $f(x) : M \to N$ that maps into some space N as a limit of simple functions $f_n(x)$

$$f(x) = \lim_{n\to\infty} f_n(x) \tag{5.123}$$

where

$$f_n = \sum_{i=1}^{n} \omega_i \chi_{X_i}(x), \tag{5.124}$$

is a simple function has only finitely many function values ω_i. $\chi_{X_i}(x)$ is the characteristic function of a measurable Borel set X_i. We integrate over an interval $[a, b] = \cup_i X_i \in M$. The measure of the set X_i is given by the measure $\mu(X_i)$. The integral becomes

$$\int_a^b f d\mu = \lim_{n\to\infty} \sum_{n=1}^{\infty} \omega_i \mu(X_i). \tag{5.125}$$

If the set $\cup_i X_i$ is not measurable, then clearly, the integral does not exist. On the other hand, if such an integral exists, then one can choose to set $f = 1, \omega_i = 1$ and define $|\int_a^b d\mu|$ as length between points a and b, which yields a metric for the space $M = \cup_i X_i$.

The existence of the integral of the function $f(x)$ thereby implies that the space M whose element x is has to be metrizable.

Similarly, the existence of a path integral

$$\int \mathcal{D}X^\mu(\sigma, \tau) e^{-S_p(X^\mu(\sigma,\tau))} \tag{5.126}$$

over a function space with a Feynman pseudo measure $\mathcal{D}X^\mu(\sigma, \tau)$ implies that the function space of X^μ must be metrizable. This is simply because the functional integral measures areas in this function space.

A more formal measure theoretical proof which shows that the space of functions X^μ must be metrizable if the path integral over $\mathcal{D}X^\mu$ is an Euclidean path integral that can be described by the Feynman-Kac formula, is given in [1] on pp. 28-31 and pp. 387-390.

For ordinary path integrals with imaginary exponent in the integrand, the rigorous construction of the path measure from DeWitt-Morette in [21] also works only with a metrizable function space.

If the space of the functions $X^\mu(\sigma, \tau)$ is metrizable, there must be some way to construct this metric. The metric in the function space defines whether two functions $X_1^\mu(\sigma, \tau)$ and $X_2^\mu(\sigma, \tau)$ are regarded as "close" or "far" from each other. The only way to do so is by using the function values of $X^\mu(\sigma, \tau)$ themselves.

For this $X^\mu(\sigma,\tau) \in T$ must map into a metrizable target space T. Then one can create, for example, a supremum norm

$$||X^\mu||_\infty = sup||X^\mu(\sigma,\tau)||_T, \tag{5.127}$$

where $||\cdot||_T$ is the norm of the target space T to which $X^\mu(\sigma,t)$ map. Naturally, the norm of the target spaces also induces a metric on the target space.

In usual circumstances, one would not need to repeat such simple mathematical facts in a thesis. However, there are various articles which claim that one could write string theory even on non-metrizable spaces, see [146] and the references therein.

These articles use the so-called Buscher rules, which relate metrics by a T-duality transformation repeatedly. The works in [146] claim that several applications in different directions, one would get a non-metrizable space. During the proof of these Buscher rules, one needs to be able to define a differential form $d\lambda \wedge dX^0$, where dX^0 is a differential form made of the embedding functions that map into the original manifold, see [147], p. 7-8. The quantity λ is then identified as $\lambda \equiv \tilde{X}^0$, where $\tilde{X}^0$ are the embedding functions on the transformed coordinates.

Hence, in order to proof the Buscher rules, one has to assume that one can define differential forms $d\tilde{X}^0$ on a space that is the result of a t-duality transformation. Unfortunately, since differential forms like $d\tilde{X}^0$ only exist on differentiable manifolds, the proof of the Buscher rules requires the assumption that the transformed space is a differentiable manifold. Unless someone provides a more general proof that works for spaces which are not manifolds, this result rules out a repeated application of t-duality transformations that would lead to a non-metrizable space as a result of the t-duality transformation.

And, as we have seen above, the path integral of string theory also exists only if the target space is metrizable.

In order for an integral, including a functional integral, to exist, it does not suffice that the measure of the set over which one integrates exists. The integrand also must be finite in order to approximate the integral with simple functions.

In physics, one usually has an action where squares of derivatives appear. This means that that the functions over which the path integral is summed must be elements of the space $L^{2,1}$ of absolutely continuous functions whose derivatives are square integrable, see [1], pp. 59-77.

In case of $X^\mu \in O$, where O is an orbifold with singularities, the Polyakov action would diverge at the singular points. Thus one can not define the path integral of the string theory amplitude everywhere on singular spaces.

In string theory, the embedding functions X^μ define the field of a conformal quantum field theory. Usually, Dirac delta distributions appear in the commutators or anticommutators of these fields. Delta distributions are so-called tempered distributions for which a Fourier transformation can be shown to exist. Hence, the field operators of a quantum field theory also have to be related to, or must be tempered distributions.

In constructive quantum field theory, one defines quantum field theories purely axiomatically. By these axioms, field operators are, in fact, defined to be tempered distributions [148–151]. Tempered distributions can be non-regular. Hence, one could think that such a construction would work with singular spaces like orbifolds.

However, tempered distributions need so-called fast falling test functions. These are smooth functions that go to zero faster than any polynomial function.

In order to see how this construction works we review the following example from [151].

Consider a massive Dirac field $\psi(x)$. It is an operator valued distribution of mass m that fulfils the Dirac equation of motion

$$(i\gamma^\mu\partial_\mu - m)\psi(x). \tag{5.128}$$

One can now define an operator valued functional $\Theta(\rho)$ for a fast falling test function ρ by

$$\Theta(\rho) = \langle\psi,\rho\rangle = \int d^4y\psi(y)\rho(y) \tag{5.129}$$

and one can define a translated functional

$$\Psi(x) = T_x\Theta(\rho) = \langle T_x\psi,\rho\rangle = \int d^4y\psi(y)\rho(x-y). \tag{5.130}$$

In both equations (5.129-5.130), the integrations are only to be understood as a symbolic notation, as integrals over divergent quantities are not rigorously defined. The scalar product can be exactly defined if one writes the tempered distribution as the limit of a family of functions (which is possible with all distributions).

The function $\Psi(x)$ also fulfils the Dirac equation but one can show that it is an analytic function in x.

So in order to be able to work with the possible divergences of the field operators, one must first assume that the fields are operator valued tempered distributions. Then, with the help of fast falling smooth test functions and a representation of the distribution as the limit of a function family, one has to turn the possibly divergent field into a regular function before one can do analysis with it.

In this mechanism, the test functions must be smooth and they must be defined everywhere where the field ψ is defined.

In string theory, the field operators of the embedding functions X^{μ} would map on the orbifold. Then, the test functions would also have to map to the orbifold. At the singularity of the orbifold, it would then not be possible to define smooth fast-falling test functions.

Thereby, as in the path integral approach, the amplitudes defined by the constructive quantum field theory approach are not defined exactly at a singularity of a target space.

It therefore appears that the only way to define string theory on an orbifold is to desingularize the orbifold in a neighbourhood of the singularity. Then one can make this neighbourhood infinitesimally small and try to write the classical string theory on that space. Finally, one quantizes the classical X^{μ}.

But until recently, it was not even clear if the classical string theory could be defined in the vicinity of a singularity. As this appears not to be well known in the string theory community, we will discuss these recent mathematical developments below.

If G is a finite subgroup of $GL_n(\mathbb{C})$, one can define the orbifold $\chi = \mathbb{C}^n/G$. This is an algebraic variety that is constructed from the algebra of G invariant polynomials on $\mathbb{C}^n$ (a G invariant polynomial is a polynomial $f \in \mathbb{C}[x]$ which fulfils $f(x) =)f(\gamma(g)x)$ where γ is a representation of G).

One has the famous theorem

Theorem 5. *(Hironaka) every algebraic variety over a field with characteristic 0 has a resolution.*

A resolution is a manifold $\tilde{U}$ where one can apply a map $\beta : \tilde{U} \to V$ to get to the original space V that is described by the algebraic variety.

The characteristic of a ring is the smallest number of times how often one must add the ring's multiplicative identity in order to get the additive identity. If this number does not exist, the characteristic is zero. For example $\mathbb{R}$ and $\mathbb{C}$ are fields that are described by varieties of characteristic 0.

A simple example of an Orbifold is $\mathbb{C}/\mathbb{Z}_2$, which is a cone. Its resolution is a cylinder $\tilde{U}$ with a metric

$$ds^2 = dr^2 + r^2 d\sigma^2. \tag{5.131}$$

After the resolution is done in an ϵ neighbourhood around the singularity, one can use the recent results Grandjean and Grieser [15] or [16]. For a nice presentation of these works, see [152].

In [15, 16] it is shown that

Theorem 6. *(Grandjean, Grieser) For every $\sigma \in \partial\tilde{U}$, there is a unique geodesic starting at σ.*

This result was then used by Grandjean and Grieser in [15, 16] to define the exponential map in a neighbourhood of $\partial\tilde{U}$.

With this, a geodesic field $X^\mu(\sigma, \tau)$ starting at every point $\sigma \in \partial\tilde{U}$ can be defined on the cylinder. Since the circle described by $\partial\tilde{U}$ is periodic, one gets the condition

$$X^\mu(\sigma + 2\pi, \tau) = gX^\mu(\sigma, \tau). \tag{5.132}$$

If one goes from a string on a manifold M to an orbifold M/G, then it is clear that for the string states, one should additionally have

$$g|\Psi\rangle = |\Psi\rangle. \tag{5.133}$$

Because of theorem 6, the geodesic field $X^\mu(\sigma, \tau)$ also exists in the limit $\epsilon \to 0$ where the epsilon neighbourhood in which we made the resolution becomes vanishingly small. It appears that the string theory is defined on the orbifold only in this limit.

When computing the amplitude on the orbifold with the method of [14], the removed epsilon neighbourhood is not seen. This has the following reason:

Let all geodesics start starting at an epsilon neighbourhood around the singularity. If we go to the limit $\epsilon \to 0$, the set that was removed from the orbifold around the singularity has a measure of zero. If you remove an area with a measure of zero from an orbifold, the resulting space has the same area as the orbifold. Since the embedding functions of string theory map into this space, the area of the function space remains the same as the orbifold.

In the geodesic picture, each curve does then not begin at the tip of the cone, but at the boundary of the blow up, which, as we make the ϵ neighbourhood smaller, gets closer and closer to the singularity at the tip of the cone. Thereby, in the limit $\epsilon \to 0$, the curves which will be quantized have the same length as they would have on the orbifold. The same then also applies for the quantized strings. This is the reason why the removed ϵ neighbourhood of the orbifold is not noticed in the path integral of string theory.

That the procedure of Dixon, Harvey, Vafa and Witten implicitly assumes a blowup is also indicated by the fact that on p. 685 of their article [14], the authors first compute the Euler characteristic of a manifold that they obtain "after blowing up [the] fixed points" of the orbifold M/G where M is an arbitrary manifold and

$G = Z_3$. Then they derive the same Euler characteristic from their string theory calculation.

Dixon et al. propose that one may use their technique also in cases where the blowup manifold is not known or where it can not be computed. They make the conjecture that there exist more general methods to resolve singularities on orbifolds. However, the work of Grandjean and Grieser [16] suggests that one should not apply this technique without knowing the result of the blowup.

Whereas the geodesic field also exists for cuspidal singularities, the exponential map differs much from the conical case. For general singularities, one would need to understand the degeneracies of resolved metrics to higher order, see [152].

Some researchers have investigated the use of blowup techniques to change the topology in 6 dimensional Euclidean string theory backgrounds. Usually, they use flip and flop operations [153], where one applies a blowdown from a manifold to a singular space and then blows it up into two topologically different manifolds. This operation was also used in the open string case [154].

Separately, there are some claims from Witten [155] that one can avoid the singularity of the compact 6D space during a topology change with gauged linear sigma models by supplying a non-zero theta angle.

The authors of [153] analyse the topology changes by going to the mirror manifolds of the varieties they have blown up.

Unfortunately, flop transitions do not change the Hodge number (which are the analogue of the Betti numbers but for complex manifolds). Flop transitions are restricted to other topological indices. e.g. intersection numbers among homology cycles [154].

Sadly, it seems these results that claim the compatibility of string theory with topology changes of 6 dimensional Euclidean backgrounds can not translated to Lorentzian 4 manifolds in a mathematically rigorous way.

The theorem of Geroch does not rest on other properties than a 4D manifold of Lorentzian signature and causality. For these space-times, the theorem implies that one can not use the flop operation to blow a singular variety up in two different ways to get two causal, topologically distinct Lorentzian 4D manifolds A and B with metrics g and $\tilde{g}$ and then have a non-singular path of metrics from g to $\tilde{g}$ (as it may be possible in 6 dimensional Euclidean space).

As we have seen, the embedding functions X^μ of string theory are tied by the definitions of the path integral and the axioms of quantum field theory to the existence of smooth test functions on the target space. This condition can not be fulfilled at a singularity.

If one blows a 4 dimensional Lorentzian singular space up into two topologically distinct nonsingular spaces, one may define two different string theories on these two spaces but one can not connect them since one can not define a string theory exactly on the singular space.

Therefore, string theory, as any other conventional quantum field theory, seems to have just the same difficulties as ordinary quantum gravity when a Lorentzian 4D sub-manifold of the target space gets singular during a topological change.

5.6 Perhaps one needs a version quantum mechanics that can work with non-differentiable paths

As we have seen above, field operators of a quantum field theory necessarily have to be tempered distributions and these need a manifold where they are defined on.

Usual quantum field theory therefore appears to be mathematically incompatible with singular spaces that are needed for the description of a topological change in causal Lorentzian 4 dimensional space-times.

However, mathematicians have long noticed that stochastic systems usually have non-differentiable paths and they have developed methods to describe them by so called stochastic differential equations.

In section 5.2 we have described a stochastic model for quantum mechanics.

We argued in section 5.2 that it may be possible to derive entangled states with this stochastic model. Since the theory does not have outcomes that are predetermined by prior events, it may be that it allows to describe entangled states without needing mechanisms for instantaneous signalling. Furthermore, it may be possible to rewrite the model via Boltzmann equations in a manifestly covariant way. Then one may hope to study relativistic particles and derive effects like particle spin from the stochastic model. Thus it may be that one can describe the entire framework of quantum mechanics and quantum field theory by stochastic differential equations.

In the stochastic derivation of non-relativistic single-particle quantum mechanics, the state function

$$\psi = \pm\sqrt{\rho(x,t)}e^{i\varphi(\boldsymbol{x},t)} \tag{5.134}$$

is derived from the averages of the velocities

$$\boldsymbol{v}(\boldsymbol{x},t) = \frac{\hbar}{m}\nabla\varphi(\boldsymbol{x},t) \tag{5.135}$$

and the probability density $\rho(\boldsymbol{x},t)$. The functions $\rho(\boldsymbol{x},t)$ and $\boldsymbol{v}(\boldsymbol{x},t)$ are differentiable functions only on a large distance scale. They are derived from the arithmetic mean of two equations that are derived from the averages of

$$m\ddot{\boldsymbol{\sigma}}_j(t) + \frac{m}{\tau}\dot{\boldsymbol{\sigma}}_j(t) = \boldsymbol{F}_{rj}(t) + \boldsymbol{F}_j^{ext}. \tag{5.136}$$

and

$$m\ddot{\boldsymbol{\sigma}}_j(t) - \frac{m}{\tau}\dot{\boldsymbol{\sigma}}_j(t) = \boldsymbol{F}_{rj}(t) + \boldsymbol{F}_j^{ext}. \tag{5.137}$$

over all trajectories $\boldsymbol{\sigma}_j$.

In contrast to the averages, the individual trajectories $\boldsymbol{\sigma}_j$ are not everywhere differentiable. The velocity $\dot{\boldsymbol{\sigma}}_j$ is only defined during the times where the random force term $\boldsymbol{F}_{rj}$ does not abruptly change.

With non-differentiable trajectories, it has to be expected that the system can can be written on singular space-times. However, a smooth description of the average values in terms of the Schroedinger equation would then no longer be possible at the singularity.

Quantum gravity is defined by the Wheeler-DeWitt equation

$$\hat{\mathcal{H}}_{\mathcal{G}}\Psi(\gamma_{ij}) = \left(\mathcal{G}_{ijkl}\frac{\delta}{\delta\gamma_{ij}}\frac{\delta}{\delta\gamma_{kl}} + \sqrt{\gamma}\,^{(3)}R\right)\Psi(\gamma_{ij}) = 0, \tag{5.138}$$

which is a version of a Schroedinger equation.

If it were possible to do a similar derivation of the Wheeler-DeWitt equation from some kind of Brownian motion or stochastic differential equations, then one could perhaps hope to use the theory at the singularities that emerge during topology changes, where space-time appears to be non-smooth.

The description of quantum gravity on a smooth manifold would then just be an approximation for large distance scales that becomes invalid at small scales.

In the vicinity of the singularity during a topological change, one may then go to the non-differentiable description that is described by stochastic differential equations.

In section 5.3, we have reviewed 't Hooft's argument that the scattering of particles with black-holes, which is expected to occur at high energy in a space-time foam picture, leads to a Wick-rotated string theory amplitude.

String theory describes a 2 dimensional quantum field theory. For Euclidean quantum field theory of ordinary fields, it is known that one can describe them in terms of Markov processes and stochastic differential equations [113–115]. At the moment, one can only speculate, whether one can describe String theory by non-differentiable paths. Such a formulation could be useful to describe topological changes and other analysis in cases where the differential structure of the target space breaks down.

Part IV

Conclusion and outlook

In this thesis, we have modified Hawking's model of space-time foam by removing a problematic approximation for the action and by using a different renormalization scale. After repeating the calculations of Hawking with these modifications, it turned out that the cosmological constant that one gets from this model is of the observed order. We then added matter terms and noted that the large contributions of the first Seley-DeWitt coefficient do not affect the observable result for the expected value of Λ.

We also have made a preliminary discussion about the implications of the higher perturbative orders of the added matter amplitude.

We have argued that the one loop amplitudes of matter terms can yield small modifications of the effective cosmological constant and that some of the topological terms in the effective actions seem to be related to the number of black or wormholes in of the background space-time at equilibrium.

We have shown that the effective matter amplitude is free from the Ostrogradski instability for certain mass configurations where it leads to the effective action of a Starobinski model.

Here one could do clearly more work. Starobinski inflation in higher derivative models with a Λ term of dark energy was for example analysed in [156, 157] by Myrzakulov, Odintsov and Sebastiani. These authors used the so-called RG improved effective action where the coupling constants of the effective action are replaced by one-loop effective coupling constants. The authors then solve the equations of motion approximately for a DeSitter space and get values for the slow-roll parameter and spectral index. Our model also contains the Starobinski $R + R^2$ terms. However, we evaluate the amplitude by a different method. One may therefore look whether one can compute similar estimates for measurable parameters of the early universe from our model. A main difference is that we include a contribution from the Euler characteristic in the amplitude and then seek the maximum of the entropy. This leads to a space with high Euler characteristic, which is not a DeSitter space any more.

Last but not least we have commented on the work of Christensen and Duff, who noted that the inverse Laplace transform which Hawking uses does not always converge and who propose to save the model with matter terms.

We have argued that, with our choice of the renormalization scale, for spaces where $\mu^2 = H_0^2 = \sigma\Lambda, \sigma \geq \frac{1}{2\pi}$, the inverse Laplace transformation does converge. These are spaces whose quantum correction to the Friedmann equations is small.

At this point, one may do further work. For example, in the gravity action, the terms depending on the derivatives of the zeta functions were neglected and one could include them and repeat the calculations again.

One could also try include effects from the back-reaction of the R^2 modifications from the matter term when solving the equations of motion whose solution is then inserted into the one loop amplitude. Unfortunately, it is not clear how to do this, since including the back-reaction implies that the one-loop renormalizability of the gravity amplitude is lost.

The model in this thesis leads to a specific evolution of the cosmological constant as a function of the volume of the universe. One may do further comparison studies with other theories for dark energy. Especially interesting would be the determination of an equation of state for the dark energy given by this model. This would make it easier to compare the theory with other models for dark energy.

We have argued in section 5 that a particle in such a space-time foam within a gas of wormholes may be affected by a collective influence of the gravitational field and the Hawking radiation that comes from the boundaries that have to be associated with these holes. We have argued that this process would lead to a particle being governed by the Schrödinger equation.

We have discussed that the resulting theory is not a hidden variable theory in the sense of Bell. We noted that if one assumes that the Hawking radiation at spatially separated locations is correlated, it may be possible to derive entangled states. We have not done this calculation in this thesis because it would involve the computation of average values of a infinite number of correlated stochastic processes, which is a difficult task.

If the entanglement of particles is due to the entanglement of Hawking radiation of spatially separated black holes, this would be interesting in view of the so-called ER=EPR hypothesis. These hypothesis states that a wormhole is dual to an entangled pair of black-holes. The amplitude of quantum gravity predicts a space-time with a wormhole gas. The relations of the space-time foam model to the ER=EPR hypothesis could be further investigated.

In this thesis, we also have not made a derivation of particles with spin from a stochastic model. Furthermore, we did not discuss how the proposed stochastic process would look for a particle that moves with relativistic speed. For relativistic extension of the theory, one would need to write the equations of the stochastic process in manifestly covariant form. This may be possible by using the Boltzmann equation.

Quantum field theory only describes scattering processes among particles. With a space-time filled with a gas of black or worm holes, one has to describe scattering processes between particles and black or worm holes. In section 5.3 we have discussed a model for such a process that was invented by 't Hooft. We have argued that for large astronomical black-holes, most particles would be beyond the scrambling time and that the model is suitable mostly for small black-holes. For this argument, we made use of the known inequality $\Delta t = u_1 - v_1 \geq 4Mln\,(M/\delta E)$ where $\Delta t = u_1 - v_1$ was computed for a semi-classical black-hole.

In reality, the inequality is even stronger with $\Delta t_{perturbed} > \Delta t \geq 4MlnM/\delta E$. One may calculate $\Delta t_{perturbed}$ from the lifetime of the perturbed black-hole that can be calculated from its Hawking radiation. We have not done this but one may do this in further work.

The black-hole scattering matrix is a Wick rotated string theory amplitude. Usually, such an amplitude is only consistent in 26 or 10 dimensions. 't Hooft has shown that by adding electrodynamic interactions, one gets an additional $S1$ dimension. He suggests that adding non-abelian interactions may deliver further extra-dimensions.

One may then try to add the extra-dimensions that can be computed from the particles of the standard model. If one could derive similar restrictions in the black-hole s-matrix for the number of extra dimensions than in ordinary string theory, one may conclude how many interactions are missing in order to have a consistent black-hole s-matrix. This is something that also has not been done in this thesis.

Furthermore, the amplitude of 't Hooft only describes the scattering of an individual particle with a single black-hole. The space-time foam model suggests that one would have such scatterings repeatedly if a particle flies through the space-time. One needs to work out how the amplitude for repeated scattering with several black-holes looks like.

We have found in section 5.4 that the expansion of the universe would imply topology changes in the space-time foam model. These topology changes lead to singularities in the space-time. We have explained a non-technical argument from DeWitt in technical terms which shows that quantum gravity is incompatible with such singularities.

It is often claimed that string theory can be written exactly in singular space-times. We have given mathematical arguments that use existence theorems for path integrals and arguments from constructive quantum field theory, which show that this is unfortunately not the case. However, for conical and cuspidal singularities, one can show that string theory exists in such singular spaces in the limit of a blow-up in an infinitesimal neighbourhood of the singularity. In a future work, one may try to prove whether this result holds for singular spaces with more general singularities.

Our arguments suggest that the singularities which emerge from topological changes lead to incompatibilities with all models of quantum gravity that are based on differentiable paths or field operators which are defined as tempered distributions.

We have argued that the stochastic model from which we have derived the single particle Schroedinger equation works with non-differentiable paths and only on the average, at a large distance scale, one gets smooth functions from which one can create a state function and the usual Hamiltonian operator.

One may now try to develop the stochastic model further and one may invent other methods to compute path integrals of quantum fields from stochastic processes that can be non differentiable.

Then one may apply these techniques to quantum gravity and string theory. Perhaps it is somehow possible to define a space-time that fluctuates randomly like as a stochastic process. If a similar method works with the Wheeler DeWitt equation of quantum gravity as with the usual Schroedinger equation, the metric is then non-differentiable at a small scale but on the average, one should be able to derive the Wheeler-DeWitt equation in a continuum limit.

In section 4.4, we have argued that the non-renormalizability of perturbative quantum gravity may simply be due to black-hole production. The emergence of black-holes at high energy suggests that one has to use the black-hole scattering matrix of 't Hooft for high energies. It describes a Wick rotated string theory amplitude, which should be finite.

Since string theory, as any ordinary quantum field theory, has certain difficulties with its path integral formulation if the target space becomes singular, one may develop string theory in terms of stochastic differential equations, too. At neighbourhoods of singularities, one could then use the non-differentiable stochastic model for the consistent description in cases where the differential structure of the target space breaks down.

Bibliography

[1] M. Cartier, C. DeWitt-Morette, Functional integration Action and Symmetries (Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2010)

[2] M. Ostrogradski , Mémoires sur les équations différentielles relatives au problème des isopérimètres, Mem. Ac. St. Petersbourg VI (1850) 385.

[3] B. S. DeWitt, Phys. Rev. **160**, 1113 (1967)

[4] J. B. Hartle, S. W. Hawking, Phys. Rev. D **28**, 2960 (1983)

[5] A. Dasgupta, Gen.Rel.Grav.43:2237-2255,2011

[6] A. Dasgupta arXiv:1106.1679[gr-qc]

[7] G. W. Gibbons, S.W. Hawking, Phys. Rev. D**15**, 2752 (1977)

[8] B. Schulz, Fortschr. Phys. 2018, 66, 1800070. https://doi.org/10.1002/prop.201800070

[9] S. W. Hawking, Nucl. Phys. B**144**, 349 (1978)

[10] S. Chandrasekhar, Astrophys. J. **97**, 255 (1943)

[11] G. 't Hooft, Physica Scripta T**36**, 247 (1991)

[12] R.P. Geroch, J. of Math. Phys. **8**, 782 (1967)

[13] B. S. DeWitt, Gen. Rel. Grav. 41:413 (2009); Erratum-ibid.41:671,2009

[14] L. Dixon, J.A. Harvey, C. Vafa, E. Witten, Nucl. Phys. B**261**, 678 (1985)

[15] R. Melrose, J. Wunsch, Invent. Math. **156**, 235 (2004)

[16] V. Grandjean, D. Grieser, J. fur die reine und angewandte Mathematik **736**, 33 (2018)

[17] B. Schulz, Review on the quantisation of gravity, arXiv:1409.7977 [gr-qc]

[18] B. Schulz, Review of Nelson's analysis of Bell's theorem, arXiv:1302.4180 [quant-ph], https://arxiv.org/abs/1302.4180

[19] P. Cartier, C. DeWitt-Morette, J. of Math. Phys. **36**, 2137 (1995)

[20] P. Cartier, M. Berg, C. DeWitt-Morette, A. Wurm, Characterising Volume Forms, in Fluctuating Paths and Fields - Festschrift Dedicated to Hagen Kleinert on the Occasion of his 60th Birthday. Edited by W. Janke et al. (World Scientific Publishing Co. Pte. Ltd., 2001), p. 139

[21] C. DeWitt-Morette, Comm. Math Phys **28** 47 (1972)

[22] B. S. DeWitt, Phys. Repts. **19**, 295 (1975)

[23] N. D. Birrell, P. C. W. Davies, Quantum fields in curved space, Cambridge Monography on Mathematical Physics (Cambridge University Press 1982)

[24] V. Mukhanov, S. Winitzki, Introduction to Quantum Effects in gravity, Cambridge University Press; (2007)

[25] G. W. Gibbons, S. W. Hawking, Euclidean Quantum Gravity, World Scientific Publishing (1993)

[26] B. S. DeWitt, Dynamical theory of groups and fields in: Relativity, groups and topology, Series: Houches Lecture, Gordan and Breach (1964),

[27] J. Martin, Comptes Rendus Physique **13**, 566 (2012), arXiv:1205.3365

[28] I. L. Shapiro, J. Sola, JHEP 0202:006,2002, arXiv:hep-th/0012227v3

[29] J. F. Koksma, T. Prokopec, The cosmological constant and Lorentz invariance of the vacuum state, arXiv:1105.6296

[30] K. S. Stelle, Phys. Rev. D**16** 953 (1977).

[31] K. S. Stelle, Gen. Rel. Grav. **9** 353 (1978)

[32] R.P. Woodard, Lect.Notes Phys.720:403-433 (2007)

[33] H. Motohashi, T. Suyama, Phys. Rev. D **91** 085009 (2015)

[34] A. Strominger, Phys. Rev. D **30**, 2257 (1984)

[35] L. Alvarez-Gaume, A. Kehagias, C. Kounnas, D. Luest, A. Riotto, Fortschr. Phys., **64**: 176-189.

[36] A. Starobinsky, Phys. Lett. B **91**, 99 (1980)

[37] A. S. Koshelev, L. Modesto, L. Rachwal, A. A. Starobinsky, JHEP, 2016(11), 1-41

[38] S. Weinberg, Rev. Mod. Phys. **61**, 1(1989)

[39] I. Oida, Phys. Rev. D **96**, 124012 (2017), arXiv:1706.05804

[40] G.W. Gibbons, S. W. Hawking, M.J. Perry, Nucl. Phys. B**138**, 141 (1978)

[41] G. ‘t Hooft, M Veltman, Ann. de l’ Inst. Henri Poincare **20**, 69 (1974)

[42] M. Veltman, Quantum Theory of Gravitation, in Methods in Field Theory, Series: Les Houches Summer School Proceedings, World Scientific (1981)

[43] H. W. Hamber, Discrete and Continuum Quantum Gravity, arXiv:0704.2895

[44] B. S. DeWitt, Phys Rev. **162** 1195 (1967)

[45] B. S. DeWitt, Phys. Rev. **162**, 1239 (1967)

[46] B. S. DeWitt, The Quantum and Gravity: The Wheeler-DeWitt Equation in: Proceedings of the Eighth Marcel Grossmann Conference, The Hebrew University, Jerusalem, Israel (World Scientific, 1997).

[47] M. Goroff, A. Sagnotti, Phys Lett. B**160**, 81 (1985)

[48] A.E.M. Van deVen, Nucl.Phys. B**378**, 309 (1992)

[49] S. Deser, Hung-Sheng Tsao, P. van Nieuwenhuizen, Phys. Lett. B**50**, 491 (1974)

[50] S. Deser, P. van Nieuwenhuizen, Phys.Rev. D**10**, 401 (1974)

[51] S. Deser, P. van Nieuwenhuizen, Phys.Rev. D**10**, 411 (1974)

[52] R. P. Feynman, Quantum theory of gravitation, and: Problems in quantizing the Yang Mills and gravitational fields, in: Selected Papers of Richard Feynman, Series: World Scientific Series in 20th Century Physics, World Scientific (2000)

[53] L.D. Faddeev, V.N. Popov, Physics Letters B**25** 29 (1967)

[54] B. S. DeWitt, The Space of Gauge Fields, its Structure and Geometry, in: G. ‘t Hooft, 50 years of Yang Mills theory, World Scientific Publishing (2005)

[55] S. M. Christensen, M. J. Duff, Nucl. Phys. B**I70**480 (1980)

[56] P. A. M. Dirac, Phys Rev. **114**, 924 (1959),

[57] R. Arnowitt, S. Deser, C. Misner, Phys. Rev. **116** (5): 1322-1330 (1959)

[58] R. Wald, General Relativity, University Of Chicago Press (1984)

[59] M. E. Peskin, D. V. Schroeder, Introduction to Quantum Field Theory, Series: Frontiers in Physics, Westview Press (1995)

[60] J.C. Feng, R. A. Matzner, Phys. Rev. D**96**, 106005 (2017), arXiv: 1708.07001

[61] A. Peres, Critique of the Wheeler-deWitt equation in: On Einstein’s Path, edited by A. Harvey (Springer, 1998)

[62] H. J. Groenewold, Physica **12**, 405 (1946)

[63] K. V. Kuchar, Time and interpretations in quantum gravity in: Proc. of the 4.th Canadian Conference on General Relativity and Relativistic Astrophysics, edited by G. Kunstatter, D. Vincent, J. Williams (World Scientific, Singapore 1992)

[64] P. Hajicek, Phys. Rev. D**34**, 1040 (1986)

[65] G. Dvali, C. Gomez, Eur. Phys. J. C **74**: 2752 (2014)

[66] J. Maldacena, Black hole entropy and quantum mechanics, https://arxiv.org/abs/1810.11492

[67] Niedermaier, M., Reuter, M. Living Rev. Relativ. (2006) 9: 5. https://doi.org/10.12942/lrr-2006-5

[68] https://arxiv.org/abs/0709.3851

[69] K. Schleich Phys. Rev. D**36**, 2342 (1987)

[70] S. M. Carroll, Lecture Notes on General Relativity, arXiv:gr-qc/9712019

[71] S. Coleman, Nucl. Phys. B**310**, 643 (1988)

[72] I. Klebanov, L. Suesskind, T. Banks, Nucl. Phys. B**317**, 665 (1989)

[73] J. Preskill, Nucl. Phys. B**232**, 141 (1989)

[74] S. Carlip, Phys. Rev. Lett. **79**: 4071-4074 (1997)

[75] T. Padmanabhan, Phys. Rept. **380**:235-320 (2003)

[76] S. Xue, J.Korean Phys. Soc. **49**, 759 (2006)

[77] J. Cyriac, The Expectation Value of the Cosmological Constant, arXiv:1511.08125

[78] R. Garattini, Int. J. Mod. Phys. D4:635 (2002), arXiv:gr-qc/0003090

[79] E. Ita, C. Soo, H. Yu, Cosmic Time and General Relativity, arXiv:1711.01394

[80] A. Einstein "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie". Sitzungs. König. Preuss. Akad.: Sitzungsb. König. Preuss. Akad.142 (1917).

[81] O'Raifeartaigh; et al., Eur. Phys. J. H. **42** (3): 431 (2017)

[82] S. W. Hawking, Phys. Lett. B**134**, 403 (1984)

[83] J. W. Moffat, Entropy **17**, 8341 (2015)

[84] O. Gron, S. Hervig, The Weyl Curvature Conjecture, arXiv:gr-qc/0205026

[85] R. K. P. Zia, E.F. Redish, S.R. McKay, Am. J. Phys. **77**, 614 (2009)

[86] M. Duff, Phys. Lett. B **226**, 36 (1989)

[87] Z. C. Wu, Physics Letters B **659** 891 (2008)

[88] D. J. Shaw, J. D. Barrow, A Testable solution of the Cosmological Constant and Coincidence Problem, arXiv:1010.4262

[89] S.W. Hawking, The cosmological constant, Phil. Trans. Roy. Soc. A., Mathematical and Physical Sciences, Vol. 310, No. 1512, The Constants of Physics, pp. 303 (Dec. 20, 1983),

[90] K. Osterwalder, R. Schrader, Comm. Math. Phys. **42**281 (1975)

[91] H. Nicolai, Nucl. Phys. B **156** 153 (1979)

[92] P. v. Nieuwenhuizen, A, Waldron, Phys.Lett. B**389** 29 (1996)

[93] S. M. Christensen, M. J. Duff, Phys. Lett **76B** 571 (1978)

[94] M. J. Perry, Nucl. Phys. B**143**, 114(1978)

[95] B. Zumino, Nicl. Phys. B **89** 535 (1975) J. of Cosmology and Astropart. Phys. 02(2013) 042

[96] James W. York, Jr. Phys. Rev. Lett. 28, 1082 (1972)

[97] M. Visser, Lorentzian Wormholes (AIP-Press, 1996)

[98] S. Krasnikov, The wormhole hazard, arXiv:gr-qc/0302107

[99] D. N. Page, W. K. Wooters, Phys. Rev. D**27**, 2885 (1983)

[100] S. W. Hawking, D. N. Page, C. N. Pope, Nucl. Phys. B**170**, 283 (1981)

[101] S. W. Hawking, D. N. Page, C. N. Pope, Phys. Lett. B**86**, 178 (1979)

[102] N. P. Warner, Commun. Math. Phys. **86**, 419 (1982)

[103] R. C. Kirby, The Topology of four manifolds, Series: Lect. Notes in Mathematics 1374, Springer Verlag (1980).

[104] Atlas Collaboration, Phys. Lett. B**726**, 120 (2013), CMS Collaboration, Phys. Rev. Lett. **110**, 081803(2013)

[105] L. C. Barbado, C. Barcel?, L. J. Garay et al. J. High Energy Phys. 2016:161 (2016), arXiv:1608.02532

[106] E.P. Wigner, On the time-energy uncertainty relation, In: Aspects of Quantum Theory, ed. by A. Salam, E.P. Wigner, Cambridge University Press, 1972, 237

[107] S. Liberati, L. Maccione, Phys. Rev. Lett. **112**, 151301 (2014)

[108] P. V. Landshoff, Introduction to thermal field theory, arXiv:hep-ph/9705376

[109] Lennart Sjogren, Lecture notes Stochastic processes, chapter 6, http://physics.gu.se/~frtbm/joomla/media/mydocs/LennartSjogren/kap6.pdf

[110] J. Baez, Can Gravity Decrease Entropy? http://math.ucr.edu/home/baez/entropy.html

[111] B. S. DeWitt, The Global Approach to Quantum Field Theory, Vol. 2, International Series of Monographs on Physics (Clarendon Press, 2003)

[112] L. Fritsche, M. Haugk, Ann. Phys. (Berlin) **12**, 371 (2003)

[113] E. Nelson, Field Theory and the Future of Stochastic Mechanics in: S. Albeverio, G. Casati and D. Merlini (eds.) Stochastic Processes in Classical and Quantum Systems, Proc. Ascona, Switzerland, 1985 in: Lecture Notes in Physics **262**, p. 438, Springer, Berlin (1988)

[114] E. Nelson, Ann. N. Y. Acad. Sci. **480**, 533 (1986)

[115] E. Nelson, Stochastic Mechanics and Random Fields, in: École d'Été de Probabilités de Saint Flour XV-XVII 1985-87, Lecture Notes in Mathematics **1362**, p. 427 (Springer, Berlin, 1988)

[116] W. G. Faris, Probability in Quantum Mechanics, appendix to: D. Wick, The Infamous Boundary: Seven Decades of Controversy in Quantum Physics, Birkhauser, Boston (1995)

[117] J. S. Bell, On The Einstein Podolsky Rosen Paradox, p. 14, in J. S. Bell, Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press, Cambridge, 1987)

[118] J. S. Bell, The Theory of Local Beables, p. 52 in J. S. Bell, Speakable and Unspeakable in Quantum Mechanics (Cambridge University Press, Cambridge, 1987)

[119] J. F. Clauser, R. A. Holt, M. A. Horne, A. Shimony, Phys. Rev. Lett. **23**, 880 (1969)

[120] J. H. Conway, S. Kochen, Notices of the AMS **56.2** 226 (2009)

[121] D. Dürr. S. Teufel, Bohmian Mechanics: The Physics and Mathematics of Quantum Theory, Series: Fundamental Theories of Physics, Springer (2009)

[122] M. van Raamsdonk, Gen. Rel. Grav. **42** (14): 2323-2329 (2014). arXiv:1005.3035

[123] J. Maldacena, L. Susskind, Fortsch. Phys. **61** (9): 781-811 (2013).

[124] G. 't Hooft, On the Quantization of space and time in: Proc. of the 4th seminar on Quantum Gravity, May 25-29, 1987, Moscow, USSR. Eds. M.A. Markov, V.A. Berezin and V.P. Frolov (World Scientific 1988), p. 551-567.

[125] G. 't Hooft, Physica Scripta T**36**, 247 (1991)

[126] G. 't Hooft, Physica Scripta T**15**, 143 (1987)

[127] G. 't Hooft, Int. J. Mod. Phys. A**11**: 4623 (1996)

[128] G. 't Hooft, Nucl. Phys. B**335**, 138 (1990)

[129] G. 't Hooft, Nucl. Phys. B**253**, 173 (1985)

[130] G. 't Hooft, Commun. Math. Phys. **99**, 613 (1985)

[131] P. C. Aichelburg, R. U. Sexl, Gen Relat Gravit **2**: 303 (1971)

[132] N. Itzhaki, Some remarks on 't Hooft's S matrix for black holes, arXiv:hep-th/963067

[133] G. Arcioni, On 't Hooft's S matrix Ansatz for quantum black holes, arXiv: hep-th/0408005

[134] J. Polchinski, Chaos in the black hole s-matrix, arXiv:1505.08108

[135] A. Anderson, B. S. DeWitt, Found. Phys. **16**, 91 (1986)

[136] M. G. G. Laidlaw, C. DeWitt-Morette, Phys. Rev. D **3**, 1375 (1971)

[137] A. E. Fischer, The Theory of Superspace. In: Carmeli M., Fickler S.I., Witten L. (eds) Relativity. Springer, Boston, MA (1970)

[138] B.S. DeWitt, Spacetime as a Sheaf of Geodesics in Superspace. In: Carmeli M., Fickler S.I., Witten L. (eds) Relativity. Springer, Boston, MA (1970)

[139] B.S. DeWitt, Gen. Relativ. Gravit. **1** 181 (1970)

[140] D. Giulini, Gen. Relativ. Gravit. 41:785-815 (2009)

[141] E. Anderson, Spaces of Spaces, arXiv:1412.0239v2 [gr-qc]

[142] P. J. Haine, On the homotopy theory of stratified spaces, arXiv:1811.01119 [math.AT]

[143] M. Kontsevich, Lett. Math. Phys. 66:157, 2003

[144] Quantum Fields and Strings, Volume 2: A Course for Mathematicians, eds. P. Deligne, P. Etingof, D. S. Freed, L. C. Jeffrey, D. Kazhdan, J. W. Morgan, D. R. Morrison, E. Witten,American Mathematical Society (1999)

[145] S. Albeverio, R. H?egh-Krohn, S. Mazzucchi, Mathematical Theory of Feynman Path Integrals -An Introduction,Lecture notes in Mathematics **523**, Springer-Verlag, Berlin-Heidelberg (2008), second edition.

[146] E. Plauschinn, Phys.Rept. **798** 1-122(2019)

[147] F. Rennecke, T Duality I: Constructions, `https://www.theorie.physik.uni-muenchen.de/17ls_th_mathphys_en/seminars/lunch_seminar/archive_wise1415/felix_rennecke/tduality_1.pdf`

[148] P. Grangé, E. Werner, Functional Integration on Paracompact Manifolds, arXiv:1812.04059 [math-ph]

[149] P. Grangé, E. Werner, Quantum Fields as Operator Valued Distributions and Causality, arXiv:math-ph/0612011

[150] P. Grangé, E. Werner, Nucl. Phys. Proc. Suppl. **161**, 75 (2006) arXiv:math-ph/0510052

[151] P. Grangé, E. Werner, Few Body Syst. **36**, 103 (2005), arXiv:math-ph/0409068

[152] D. Grieser, Geodesics, singularities, and blow-ups, presentation from WE Heraeus-Seminar, Algebro-Geometric Methods in Fundamental Physics, Physikzentrum Bad Honnef September 3-7, 2012, http://www.staff.uni-oldenburg.de/daniel.grieser/wwwvortraege/geodesics-sept2012.pdf

[153] P. S. Aspinwall, B.R. Greene, D.R. Morrison, Nucl. Phys. B**416**, 414 (1994)

[154] T. Muto, D-branes on Orbifolds and Topology Change, Nucl. Phys. B**521**, 183 (1998) arXiv:hep-th/9711090v1

[155] E. Witten, Nucl.Phys. B**403**, 159 (1993)

[156] L. Sebastiani , R. Myrzakulov, Int. J. Geom. Meth. Mod. Phys. **12** no.09, 1530003 (2015)

[157] R. Myrzakulov, S. Odintsov , L. Sebastiani, Phys. Rev. D**91** 8, 083529 (2015)