Coalgebras of topological types

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Coalgebras of topological types

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Abstract

In this work, we focus on developing the basic theory of coalgebras over the category *Top* (the category of topological spaces with continuous maps). We argue that, besides *Set*, the category *Top* is an interesting base category for coalgebras.
In order to provide a proper framework to study coalgebras over *Top*, we study some endofunctors on *Top*, in particular, Vietoris functor $\mathbb{V}$ and P-Vietoris functor $\mathbb{V}_P$ (where P is a set of propositional letters) that due to [42] can be considered as the topological versions of the powerset functor $\mathbb{P}$ and the Kripke functor $\mathbb{P}_P$, respectively. We will familiarize with the notions of extension (up to isomorphism) and lifting (up to isomorphism) of a *Set*-endofunctor to *Top*. These notions were introduced in [9], where the authors investigated how a finitary functor on *Set* can be extended or lifted to the categories *Preord* and *Poset*. We prove that a *Top*-endofunctor F is a lifting of a *Set*-endofunctor T (up to isomorphism) if and only if F preserves monos and epis. Following this, we give a strategy to lift a special class of the *Set*-endofunctors to the category *Top*. As an application, we obtained a *Top*-endofunctor $\bar{T}$ as a lifting of the *Set*-endofunctor $T := (-)^2 - (-) + 1$ that helped us to provide some counterexamples required in this work. Building on the fact that every inverse limit in *Set* can be considered as a complete ultrametric space and also by showing that each complete ultrametric spaces is an inverse limit for some inverse system in *Set*, we give a strategy to extend the power-set functor $\mathbb{P}$ and finite power-set functor $\mathbb{P}_\omega$ to CUM^1 (the category of complete 1-bounded ultrametric spaces with non-expansive maps).
We define the notion of compact Kripke structures and we prove that Kripke homomorphisms preserve compactness. Our definition of compact Kripke structure coincides with the notion of modally saturated structures introduced in Fine [27]. We prove that the class of compact Kripke structures has Hennessy-Milner property (i.e., the notion of the Kripke bisimilarity coincides with the notion of modal equivalence). As a consequence, we show that in this class of Kripke structures, behavioral equivalence, modal equivalence and Kripke bisimilarity all coincide.
Next, we discuss some basic definitions and theorems about coalgebras in $\mathbb{C}_F$ under certain conditions on the base category $\mathbb{C}$ and $\mathbb{C}$-endofunctor F. These concepts have already been discussed in articles [31], [33] and [62] for coalgebras on the Set category. We define the concept of union of $\mathcal{M}$-subcoalgebras and we show that the union of a family of $\mathcal{M}$-subcoalgebras need not be always an $\mathcal{M}$-subcoalgebra. As one of our main results, we proved that if the base category $\mathbb{C}$ is $\mathcal{M}$-well powered with sums then the preservations of $\mathcal{M}$-morphisms by a $\mathbb{C}$-endofunctor F gives rise to the existence of equalizers in $\mathbb{C}_F$. In this case, we constructed the equalizers of two morphisms f, g in $\mathbb{C}_F$ via union of a special family of $\mathcal{M}$-subcoalgebras of their domains. Based on the notion of A-M bisimulation known by Aczel and Mendler in [2], we define a concept of the largest

A-M bisimulation, and by giving an example from [11], we will show that the largest A-M bisimulation need not always exist. We explain two strategies to find the largest A-M bisimulation. As an application of the second strategy, we obtain a way to check whether a $\mathbb{C}$-endofunctor F weakly preserves pullbacks or not. We briefly generalize the notion of modal logic for the coalgebras over Top by defining a language for a Top-endofunctor F via a modal similarity type Λ for F, that is a set of clopen subsets of $F(2)$ where $2 := \{0, 1\}$ is a discrete space.
It will be shown that if Σ is a set and $\Sigma^\star$ is the set of all finite words over Σ, then in any category $\mathbb{C}$ with object D and product, a terminal coalgebra for the functor $D \times (-)^\Sigma$ exists, and it is based on $D^{\Sigma^\star}$ ($\Sigma^\star$-fold product of D in $\mathbb{C}$).
Furthermore, we generalize the notion of descriptive structures defined in [11] by introducing a notion of Vietoris structures. We identify Vietoris frames and models as coalgebras for the functors $\mathbb{V}$ (the Vietoris functor) and $\mathbb{V}_P$ (the P-Vietoris functor) on the category Top, respectively. One can see that each compact Kripke model can be modified to a Vietoris model. This yields an adjunction between the categories VS (the category of Vietoris structure) and CKS (the category of compact Kripke structures). Moreover, we will prove that the category of Vietoris models has a terminal object. We study the concept of a Vietoris bisimulation between Vietoris models. We provide some characterizations of Vietoris homomorphisms and Vietoris bisimulations between Vietoris models on compact Hausdorff spaces. We will prove that the closure of a Kripke bisimulation between underlying Kripke models of two Vietoris models is a Vietoris bisimulation. In the end, it will be shown that in the class of Vietoris structures, Vietoris bisimilarity, behavioral equivalence, modal equivalence, all coincide.

Zusammenfassung

In dieser Arbeit konzentrieren wir uns auf die Entwicklung der grundlegenden Theorie der Coalgebren über der Kategorie Top (topologische Räume und stetige Abbildungen). Wir argumentieren, dass neben Set die Kategorie Top eine interessante Basiskategorie für Coalgebren darstellt.
Um einen geeigneten Rahmen für das Studium von Coalgebren über Top zu bieten, untersuchen wir einige Endofunktoren über Top, insbesondere den Vietoris-Funktor $\mathbb{V}$ und den P-Vietoris funktor $\mathbb{V}_P$ (wobei P eine Menge von atomaren Aussagen ist), die aufgrund von [42] als topologische Versionen des Potenzmengenfunktors $\mathbb{P}$ bzw. des Kripke Funktors $\mathbb{P}_P$ betrachtet werden können. Durch die Einführung der Begriffe Extension (bis auf Isomorphismus) und Lifting (bis auf Isomorphismus) von Funktoren zeigen wir Beziehungen zwischen Set-Endofunktoren und Top-Endofunktoren. Diese Konzepte erscheinen bereits im Artikel [9]. Wir schlagen eine Strategie vor, um eine spezielle Klasse von Set-Endofunktoren auf die Kategorie Top hoch zu ziehen. Als Anwendung erhalten wir einen Top-Endofunktor $\bar{T}$ als Lifting des Set-Endofunktors $T := (-)^2 - (-) + 1$, mit dessen Hilfe wir einige für diese Arbeit erforderliche Gegenbeispiele erstellen können. Wir beweisen, dass ein Top-Endofunktor F genau dann ein Lifting eines Set-Endofunktors T (bis auf Isomorphismus) ist, wenn F Monos und Epis erhält. Aufbauend auf der Tatsache, dass jeder inverse Limes in Set als vollständiger ultrametrischer Raum betrachtet werden kann und auch indem gezeigt wird, dass jeder vollständige ultrametrische Raum ein inverser Limes für ein bestimmtes inverses System in Set ist, geben wir eine Strategie, um Set-Endofunktoren auf CUM^1 (die Kategorie der vollständigen 1-beschränkten ultrametrischen Räume und non-expansive Abbildungen) zu erweitern. Als Beispiel untersuchen wir die Extensionen des Potenzmengenfunktors $\mathbb{P}$ und des endlichen Potenzmengenfunktors $\mathbb{P}_\omega$ auf CUM^1.
Um eine Motivation für das Studium von Coalgebren über der Kategorie Top, insbesondere Vietoris Coalgebren, zu geben, wir definieren den Begriff der kompakten Kripke-Strukturen. Unsere Definition "der kompakten Kripke-Struktur" stimmt mit der Definition "der modally saturated Strukturen" in [27] überein. Zunächst beweisen wir, dass Kripke-Homomorphismen die Kompaktheit erhalten. Wir zeigen, dass die Klasse der kompakten Kripke-Strukturen die Hennessy-Milner-Eigenschaft hat (d. h. der Begriff der Kripke-Bisimilarität stimmt mit dem Begriff der Modal-Äquivalenz überein). Es folgt, dass in dieser Klasse von Kripke-Strukturen Beobachtungs-Äquivalenz, Modal-Äquivalenz und Kripke-Bisimilarität zusammenfallen.
Wir tragen weiter zur Theorie der Coalgebren über die Kategorie der topologischen Räume bei, indem wir einige grundlegende Definitionen, Beispiele und Theoreme für Coalgebren auf eine Basiskategorie $\mathbb{C}$ mit ähnlichen Eigenschaften wie die Kategorie Top untersuchen. Diese Konzepte wurden bereits in den Artikeln [31], [33] und [62]

für Coalgebren auf der Kategorie Set erörtert. Das Konzept der Vereinigung von $\mathcal{M}$-Untercoalgebren wurde beschrieben und es wurde gezeigt, dass die Vereinigung einer Familie von $\mathcal{M}$-Untercoalgebren keine $\mathcal{M}$-Untercoalgebra sein muss. Als eines unserer wichtigsten Ergebnisse in diesem Schritt, beweisen wir, dass, wenn die Basiskategorie $\mathbb{C}$ well-powered mit Summen ist, die Erhaltung von $\mathcal{M}$-Morphismen durch einen $\mathbb{C}$-Endofunktor F die Existenz von Equalizern in der Kategorie $\mathbb{C}_F$ garantiert. In diesem Fall konstruierten wir den Equalizer zweier Morphismen f und g in $\mathbb{C}_F$ durch Vereinigung einer speziellen Familie von $\mathcal{M}$-Untercoalgebren ihrer Domänen. Als Beispiel zeigen wir, dass, wenn wir Top als eine (*epi*, *regular mono*)-Kategorie betrachten, die Equalizer in den Kategorien $Top_{\mathbb{V}}$ und $Top_{\mathbb{V}_P}$ existieren.
Darüber hinaus, basiert auf der Definition der A-M Bisimulation von Aczel und Mendler in [2] definieren wir ein Konzept der größten A-M Bisimulation. Ein Beispiel aus [11] zeigt, dass die größte A-M Bisimulation nicht immer existieren muss. Wir erklären zwei Strategien, um die größte A-M Bisimulation zu finden. Als Anwendung der zweiten Strategie erhalten wir eine Möglichkeit zu überprüfen, ob ein $\mathbb{C}$-Endofunktor F Pullbacks schwach erhält oder nicht. Wir verallgemeiner kurz den Begriff der Modallogik für Coalgebren über Top, indem wir eine Sprache für einen Top-Endofunktor F über einen modalen Ähnlichkeitstyp Λ für F definieren, der eine abgeschlossene offene Teilmenge von $F(2)$ ist, wobei $2 := \{0, 1\}$ als diskreter Raum aufgefasst wird.
Als nächstes diskutieren wir die Existenz und den Aufbau von terminalen Objekten in den Kategorien der Coalgebren für die $\mathbb{C}$-Endofunktoren $D \times (-)$ (Black-boxes) und $D \times (-)^{\Sigma}$ (automata) wobei $\mathbb{C}$ eine Kategorie mit Objekt D und Produkten ist.
Außerdem verallgemeinern wir den Begriff deskriptiver Strukturen in Venema et. al. [11] durch Einführung eines Konzepts von Vietoris-Strukturen. Wir präsentieren Vietoris Frames und Modelle als Coalgebren für die Top-Endounktoren $\mathbb{V}$ bzw. $\mathbb{V}_P$. Man sieht, dass jedes kompakte Kripke-Modell zu einem Vietoris-Modell umgebaut werden kann. Dies ergibt eine Adjunction zwischen den Kategorien VS (Kategorie der Vietoris Strukturen) und CKS (Kategorie der kompakten Kripke-Strukturen). Darüber hinaus weisen wir nach, dass die Kategorie der Vietoris-Modellen ein Terminalobjekt hat. Wir untersuchen das Konzept der Vietoris-Bisimulation zwischen Vietoris-Modellen, das erstmals in Venema et. al. [11] zwischen deskriptiven Modellen vorgestellt wurde. Wir geben einige Charakterisierungen von Vietoris-Homomorphismen und Vietoris-Bisimulationen zwischen Vietoris-Modellen auf kompakten Hausdorff-Räumen. Wir sehen, dass die abgeschlossene Hülle einer Kripke-Bisimulation zwischen den zugrunde liegenden Kripke-Modellen zweier Vietoris-Modellen eine Vietoris-Bisimulation ist. Als eine Konsequenz zeigen wir, dass die größte Vietoris-Bisimulation (in Bezug auf die Einbeziehung von Teilmengen) zwischen zwei deskriptiven Modellen existiert und und die abgeschlossene Hülle der größten Kripke-Bisimulation zwischen den zugrunde liegenden Kripke-Modellen ist. Am Ende sehen wir, dass in der Klasse der Vietoris-Strukturen Vietoris-Bisimilarität, Beobachtungs-Äquivalenz und Modal-Äquivalenz zusammenfallen.

Contents

Introduction

In the last years, coalgebras and their applications to computer science have received much attention. One of the reasons is that the theory of coalgebras provides a general framework to study and develop the general theory of transition systems, bisimulations, modal logics, etc, see [2,30,31,38,45,46,55,57,62]. The research on the coalgebraic foundation of Kripke structures (i.e., Kripke frames and models), where the powerset functor has a central role, are evident examples for this claim, see [38,62]. The relation between modal logic and coalgebras is rather tight. One can generalize the concept of classical modal logic defined in terms of Kripke structures to arbitrary coalgebras by considering coalgebras as models for the generalized logic, see [45,46]. The work on modal logics and coalgebras started with Barwise and Moss, see [8]. Then Moss [55] developed coalgebraic logic which can be understood as a generalization of modal logic to a large class of coalgebras over *Set*.
Formally, every coalgebra is based on a carrier which itself is an object in the base category. In most of the literature on coalgebras, the category *Set* has been considered as the base category. One of the aims of this work is to show that, besides *Set*, the category *Top* is also an interesting category as a base category. In this work we try to give a number of reasons to believe that coalgebras on *Top* are of interest.
The starting observation is that the category *Top* like the category *Set* has many interesting properties (see Adámek et. al. [3]), for example it is complete and cocomplete. Moreover, the category *Top* is an $(\mathcal{E}, \mathcal{M})-$category and it is $\mathcal{M}$-well powered (see section 4.17 in [3]). Moreover, there are very interesting categories that are subcategories and full subcategories of *Top*. For instance, *Stone* (i.e. the category of Stone spaces with continuous maps) and CUM^1 (i.e. the category of complete 1-bounded ultrametric spaces with non-expansive maps) are subcategories of *Top*. These advantages of *Top* motivate us to look for functors on different subcategories of *Top*, amongst them CUM^1.

The point that descriptive structures (i.e., descriptive frames and models) can be seen as coalgebras of the Vietoris functor over topological spaces is the second reason for believing that coalgebras over topological spaces are of interest, see Venema et. al. [11]. Descriptive structures are formed by Kripke structures, in the sense that a descriptive frame is a pair (X, R) where X is a Stone space and R is a binary relation on X such that for each $x \in X$ the set $\{y \in X \mid x\,R\,y\}$ is compact and for every clopen subset $U \subseteq X$ the set $\{x \in X \mid \exists y \in U.\, x\,R\,y\}$ is a clopen subset of X; a descriptive model arises by adding a binary relation $\models\, \subseteq X \times P$ (where P is a set of propositional letters) that interprets the proposition letters as clopen subset of X (i.e., for each $p \in P$ the set $\{x \in X \mid x \vDash p\}$ is a clopen subset of X). These notions were introduced for the first time by Esaki in [24]. He found that there is a connection between the Vietoris topology

and modal logic and presented his definition of descriptive structures by using this connection. Our presentation in this chapter is based on [11], where Venema, Fontaine and Bezhanishvili proved that descriptive frames and models are, respectively, coalgebras for the functors $\mathbb{V}$ (Vietoris functor) and $\mathbb{V}_P$ (P-Vietoris functor) over the category *Stone*. This connection of descriptive structures with coalgebras over topological spaces encourages us to study other topological structures with the same property, amongst them topological automata (see example 7.1.5) and Vietoris structures (see definition 9.0.2). These examples provide a motivation for us to verify coalgebras over the base categories different from *Set* (see part III).
In order to give another motivation to study coalgebras over *Top*, in particular Vietoris coalgebras, we should again refer to the results in [11]. In this article, the authors introduced a new notion of bisimulation between two descriptive models called Vietoris bisimulation. They proved that Vietoris bisimilarity coincides with Kripke bisimilarity, with behavioral equivalence and with modal equivalence, but not with A-M bisimilarity. To find more motivations to study coalgebras over topological spaces, see Kupke et. al. [47], Hofmann et. al. [42] and Viglizzo [70].

We finish this introduction with an overview of the chapters:

Chapter 1 is allocated to study the basic concepts about topological spaces, nets, metric and ultrametric spaces which will be used in this work.

In chapter 2, we introduce the most fundamental concepts of category theory, as well as some lemmas, theorems and examples that we will find useful in the remainder of this work. We will see that the category *Top*, like the category *Set*, has many interesting properties (see [3]). For example it is complete and cocomplete, meaning that it has all small limits and small colimits (see example 2.18.5 and remark 2.18.6). Moreover, one can see that the category *Top* is an $(\mathcal{E}, \mathcal{M})-$category (see example 2.11.6). It is also an $\mathcal{M}$-well powered category, i.e. the collection of $\mathcal{M}$- subobjects is a set (see section 4.17 in [3]). One of the properties of *Top* is that there are interesting categories that are subcategory and full subcategory of *Top* (see definition 2.1.6). For instance, *Stone* and CUM^1 are subcategories of *Top* (see example 2.1.7). Aside from these advantages of *Top*, there are some issues that create some problems to work with *Top* as a base category. For instance, *Top* is not cartesian closed (see section 2.14).

In chapter 3, we study some endofunctors on *Top*, in particular, the Vietoris functor $\mathbb{V}$ and the P-Vietoris functor $\mathbb{V}_P$ (where P is a set of propositional letters) that can be considered as the topological versions of the powerset functor $\mathbb{P}$ and the Kripke functor $\mathbb{P}_P$, respectively. In order to compare *Top*-endofunctors to *Set*-endofunctors, we check some properties of the endofunctors on *Top*, amongst them monos and regular monos-preservation, and also epis and regular epis-preservation.

In chapter 4, we try to find an answer to the question what are the relationships between *Set*-endofunctors and *Top*-endofunctors. To find an answer for this question, we will familiarize with the notions of extension (up to isomorphism) and lifting (up to isomorphism) of functors. In lemma 4.1.1, we prove that a Top-endofunctor F is a lifting of a Set-endofunctor T up to isomorphism if and only if the endofunctor F preserves monos and epis. We give a strategy to lift a special class of the Set-endofunctors to the category Top.

In chapter 5, Building on the fact that every inverse limit in Set can be considered as a complete ultrametric space and also by showing that each complete ultrametric spaces is an inverse limit for some inverse system in Set, we give a strategy to extend Set-endofunctors to CUM[1].

We have allocated chapter 6 to the notion of Kripke structures (see also Rutten [62]) which is one of the main motivations to study the notion of coalgebras and modal logic. We study the concept of compact Kripke structures and we will prove that Kripke homomorphisms preserve compactness. It will be shown that the class of compact Kripke structures has the Hennessy-Milner property (i.e., the notion of Kripke bisimilarity coincides with the notion of modal equivalence). As a conclusion of this chapter we find that in the class of compact Kripke structures, the notions of behavioral equivalence, modal equivalence and Kripke bisimilarity all coincide.

In chapter 7, we focus on developing the basic theory of coalgebras. We discuss some basic definitions and theorems about coalgebras for the $\mathbb{C}$-endofunctor F under certain conditions on the base category $\mathbb{C}$ and $\mathbb{C}$-endofunctor F. Throughout this chapter, our base category $\mathbb{C}$ is an $(\mathcal{E}, \mathcal{M})$-category such that $\mathcal{E} \subseteq epis$ and $\mathcal{M} \subseteq monos$. We introduce the notion of topological automata and we will show that these kinds of objects can be presented as coalgebras for the Top-endofunctor $D \times Hom_{Top}(\Sigma, -)$ (product of the constant functor D and the covariant functor $Hom_{Top}(\Sigma, -)$). By defining the notion of homomorphism between coalgebras, it is immediately concluded that the collection of coalgebras for the $\mathbb{C}$-endofunctor F and their homomorphisms forms a category, denoted by $\mathbb{C}_F$. After proving some theorems about $\mathcal{M}$-subcoalgebras and factorization systems of $\mathbb{C}_F$ (see theorems 7.1.14 and 7.2.5), we define the concept of union of $\mathcal{M}$-subcoalgebras and in example 7.3.8, one can see that the union of a family of $\mathcal{M}$-subcoalgebras need not be always an $\mathcal{M}$-subcoalgebra. We prove that if the base category $\mathbb{C}$ is an $\mathcal{M}$-well powered category with coproducts then the preservation of $\mathcal{M}$-morphisms by $\mathbb{C}$-endofunctor F gives rise to the existence of equalizers in $\mathbb{C}_F$ (see theorem 7.3.6). We study the notion of A-M bisimulation defined by Aczel and Mendler [2], and in theorem 7.4.6 we will prove that a $\mathbb{C}$-morphism $f : A_1 \longrightarrow A_2$ is a homomorphism between coalgebras $\mathcal{A}_1 = (A_1, \alpha_1)$ and $\mathcal{A}_2 = (A_2, \alpha_2)$ in $\mathbb{C}_F$ iff the $\mathcal{M}$-graph of f (see definition 7.4.5) is an A-M bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$. After defining the notion of supremum of a family of A-M bisimulations, by giving an example from [11], we will see that the supremum of a family of A-M bisimulations is not always an A-M bisimulation (see example 7.4.12). As a consequence, the largest A-M bisimulation need not exist. In theorems 7.4.13 and 7.4.15, we present two strategies to obtain the largest A-M bisimulation and the second one provides us with a way to check whether a $\mathbb{C}$-endofunctor F weakly preserves pullbacks or not (see remark 7.4.17). Based on some assumptions for the base categories $\mathbb{C}$ and $\mathbb{C}'$, $\mathbb{C}$-endofunctor F and $\mathbb{C}'$-endofunctor G, in section 7.4.3 we discuss the connection between A-M bisimulations in $\mathbb{C}_F$ with these structures in $\mathbb{C}'_G$. In section 7.5, we briefly generalize the notion of modal logic for coalgebras over Top. Next, in chapter 8, we discuss the existence and the construction of terminal objects in the categories of coalgebras for the $\mathbb{C}$-endofunctors $D \times (-)$ (black-boxes) and $D \times (-)^{\Sigma}$ (automata) where $\mathbb{C}$ is a category with object D and products.

Chapter 9 of this work introduces a concept of Vietoris structure as a generalization of the notion of descriptive structures defined in [11]. We prove that Vietoris frames and models are coalgebras for the functors $\mathbb{V}$ (Vietoris functor) and $\mathbb{V}_P$ (P-Vietoris functor) on the category Top, respectively. Besides, we define the concept of Vietoris homomorphisms. The collection of all Vietoris structures together with Vietoris homomorphisms forms a category which we shall call VS. In theorem 9.3.7, we will show that there is an adjunction between the categories VS and CKS (the category of compact Kripke structures).

In chapter 10, we will prove that the category of Vietoris models has a terminal object (see lemma 10.6.1).

In chapter 11, we generalize the notion of Vietoris bisimulation for Vietoris models. In this chapter, we will show that in the category of Vietoris models over compact Hausdorff spaces (i.e., the category of all Vietoris models $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ in which X is compact Hausdorff) the composition of two Vietoris bisimulations and the diagonal $\triangle_X$ are Vietoris bisimulations (see lemmas 11.2.5 and 11.2.2 and their corollaries). Moreover, we will prove that in the category of Vietoris models over compact Hausdorff spaces a map $f : X \longrightarrow Y$ is a Vietoris homomorphism between Vietoris models $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \models_{\mathcal{Y}})$ if and only if its graph is a Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Y}$ (see theorem 11.3.1). Next, we will give a characterization of Vietoris bisimulations between Vietoris models over compact Hausdorff spaces (see theorem 11.3.2, Canonical Vietoris bisimulation Theorem). In section 11.4, we will prove that the closure of a Kripke bisimulation between underlying Kripke models of two Vietoris models is a Vietoris bisimulation which is the main result of chapter 11. As a corollary we show that the largest Vietoris bisimulation between two Vietoris models with respect to the inclusion of subsets exists and it is the largest Kripke bisimulation between the underlying Kripke models. In the end, we will see that the Vietoris structures over Top with the notion of Vietoris bisimulation provide a complete semantic for modal logic, in the sense that Vietoris bisimilarity, behavioral equivalence, modal equivalence, all coincide.

Part I.

Foundations

1. General topology

Before starting the concepts related to the category of topological spaces, we first recall some basic notions about topology, topological spaces and metric spaces which can be found in Kelley [49], Munkres [56]. Our main references to study the basic notions of ultrametric spaces are Crampe and Ribenboim [21, 22] and Ribenboim [59].
First, we should define some auxiliary notations used in this work. Formally, for a map $f : X \longrightarrow Y$, a subset $O \subseteq X$ and a subset $V \subseteq Y$ we define $ker\, f$, $f(O)$, $im\, f$ and $f^{-1}(V)$ as

$$\begin{aligned} ker\, f &:= \{(x, x') \in X \times X \mid f(x) = f(x')\}, \\ f(O) &:= \{y \in Y \mid \exists x \in O.\, f(x) = y\}, \\ im\, f &:= f(X), \\ f^{-1}(V) &:= \{x \in X \mid f(x) \in V\}. \end{aligned}$$

If X and Y are sets, we define $X - Y$ as

$$X - Y \quad := \quad \{x \in X \mid x \notin Y\}.$$

For every subset $A \subseteq X$, the set $X - A$ is called the complement of A in X (or, the complement of A, if it is clear from the context). The complement of a set A is usually denoted by A^c.

1.1. Topological spaces

Definition 1.1.1. A **topological space** is a pair (X, τ) where X is a set (called the underlying set) and τ a collection of subsets of X satisfying the following axioms:

1) the empty set and X are in τ.

2) the union of any collection of sets in τ is also in τ (i.e., τ is closed under arbitrary unions), and

3) finite intersection of sets in τ is also in τ (i.e., τ is closed under finite intersections).

The collection τ is called the **topology** on X and its elements are named **open subsets** of X. The complements of the open sets are called **closed**.
In this work, we usually denote a topological space (X, τ) simply by its underlying set. A subset of X may be neither closed nor open. A subset that is both closed and open is called a **clopen** subset.

A given set can be equipped with many different topologies. As an example, any set X can be equipped with the **discrete topology** in which every subset is open. In this case, the topological space X is called a discrete space. We denote it by X_D. Also, any set X can be provided by the **trivial topology** (or **indiscrete topology**) in which the empty set and the whole space are the only open subsets. In this case, the topological space X is called a trivial space (or an indiscrete space) and we denote it by X_I.
If τ and δ are two topologies on a set X such that $\tau \subseteq \delta$, then τ is said to be **smaller** (or **coarser**) **topology** than δ, and δ is said to be **larger** (or **finer**) **topology** than τ. Notice that for every set X, the discrete topology is the **largest** (or **finest**) **topology** which can be defined on X. The **smallest** (or **coarsest**) **topology** on X is the indiscrete topology.

Definition 1.1.2. Given a topological space X, a subset $A \subseteq X$ and an element $x \in A$, the element x is said to be an **interior point** of the set A and the subset A is called a **neighborhood** of x, if there exists an open subset U of X such that $x \in U \subseteq A$. In the case that A is an open subset of X, we say that A is an open neighborhood of x. We denote by $\mathfrak{N}(x)$, the set of all neighborhoods of the element $x \in X$. Also, we denote by $\mathfrak{N}_O(x)$ the set of all open neighborhoods of the element $x \in X$.
The interior of a subset $A \subseteq X$ is denoted by A° and is defined to be the union of all open sets contained in A. Due to the definition, if A and B are subsets of X with $A \subseteq B$, then $A^\circ \subseteq B^\circ$. It is well-known that a subset A is open in X if and only if $A^\circ = A$.
A point $x \in X$ is called a **limit point** of a subset $A \subseteq X$, if every neighborhood of x contains at least one point of A different from x itself. A space X is a discrete space if and only if no subset of X has a limit point.
The **closure** of a subset $A \subseteq X$ consists of all elements in A together with all limit points of A. The closure of a subset $A \subseteq X$ is denoted by $\overline{A}$. If A and B are subsets of X with $A \subseteq B$, then $\overline{A} \subseteq \overline{B}$. It is well-known that a subset A is closed in X if and only if $\overline{A} = A$. A subset A of a topological space X is called **dense** in X iff $\overline{A} = X$.

1.2. Continuous functions

A function $f : X \longrightarrow Y$ between two topological spaces is called continuous if for every open subset $V \subseteq Y$ the inverse image $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is an open subset of X. This is equivalent to the condition that the preimages of the closed subsets in Y are closed in X.

Example 1.2.1. ([49], chapter 3) Every function from a topological space to an indiscrete space is continuous. Also, each function from a discrete space to an arbitrary space is continuous. The only continuous functions from an indiscrete space to a discrete space are constant functions.

Lemma 1.2.2. *([56], chapter 2, section 18) A function $f : X \longrightarrow Y$ is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for each subset $A \subseteq X$.*

Proof. Let $f : X \longrightarrow Y$ be continuous and A be an arbitrary subset of X. Since $\overline{f(A)}$ is a closed subset of Y, we conclude that $f^{-1}(\overline{f(A)})$ is a closed subset of X (because f is continuous). Since $f(A) \subseteq \overline{f(A)}$, we have $A \subseteq f^{-1}(\overline{f(A)})$. Consequently, $\overline{A} \subseteq f^{-1}(\overline{f(A)})$ (because $f^{-1}(\overline{f(A)})$ is closed). Hence $f(\overline{A}) \subseteq \overline{f(A)}$. Conversely, let $C \subseteq Y$ be a closed subset of Y. We need to show that $f^{-1}(C)$ is a closed subset of X. Consider $A := f^{-1}(C)$. Then $f(A) \subseteq C$. So $\overline{f(A)} \subseteq \overline{C} \subseteq C$. Therefore, $f(\overline{A}) \subseteq C$ (because $f(\overline{A}) \subseteq \overline{f(A)}$). Hence, we have $\overline{A} \subseteq f^{-1}(C) = A$. Then $\overline{A} = A$ and consequently $A = f^{-1}(C)$ is a closed subset of X. □

Definition 1.2.3. (Dense function) A continuous function $f : X \longrightarrow Y$ is called a dense if and only if $im\, f$ is a dense subset of Y (i.e $\overline{f(X)} = Y$).

Lemma 1.2.4. *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are dense functions, then $g \circ f$ is also a dense function.*

Proof. It suffices to show that $\overline{(g \circ f)(X)} = Z$. We know that $\overline{(g \circ f)(X)} \subseteq Z$. On the other hand, we have

$$\begin{aligned}
\overline{(g \circ f)(X)} &= \overline{g(f(X))} \\
&= \overline{\overline{g(f(X))}} \\
&\overset{\text{lemma 1.2.2}}{\supseteq} \overline{g(\overline{f(X)})} \\
&\overset{f \text{ is a dense function}}{=} \overline{g(Y)} \\
&\overset{\text{g is a dense function}}{=} Z.
\end{aligned}$$

. □

Definition 1.2.5. (Open and closed function) A function $f : X \longrightarrow Y$ between topological spaces is called **open** (resp. **closed**) if given any open (resp. closed) subset $U \subseteq X$, then $f(U)$ is an open (resp. a closed) subset of Y.

Definition 1.2.6. (homeomorphism) A function $f : X \longrightarrow Y$ between two topological spaces is called a **homeomorphism** if it is a bijective, continuous and open map. Alternatively, the function f is called a homeomorphism if f is bijective and continuous and its inverse is also continuous. In this case, X and Y are said homeomorphic spaces and we write $X \cong Y$.

1.3. Base and subbase for a topology

Base for a topology

Let (X, τ) be a topological space. A base B for the topology τ is a collection of open sets in τ such that:

- every open set in τ can be written as a union of elements of B.

We say that the base B generates the topology τ. Many topologies are most easily defined in terms of a base which generates them.

Example 1.3.1. ([49], chapter 1) Here we introduce some well-known examples.

1. Single element sets are a base for the discrete topology.
2. The collection of all open intervals of the form (a, b) (where a, $b \in \mathbb{R}$) is a base for the standard topology on $\mathbb{R}$.

A base is not unique. Many bases, even of different sizes, may generate the same topology. For example, the collection of open intervals with rational endpoints is a base for the standard topology on the real numbers $\mathbb{R}$. On the other hand, the set of open intervals with irrational-endpoints is also a base for the standard topology on $\mathbb{R}$. But these two sets are completely disjoint and both properly contained in the base of all open intervals.

Subbase for topology

Let (X, τ) be a topological space. A subbase B for the topology τ is a sub-collection of τ such that every open set in τ can be written as a union of finite intersections of elements in B. We say that τ is generated by B.

Example 1.3.2. ([49], chapter 1) The set

$$\{(-\infty, r) \mid r \in \mathbb{R}\} \bigcup \{(r, +\infty) \mid r \in \mathbb{R}\}$$

is a subbase for the standard topology on $\mathbb{R}$.

Subbases are useful, because many properties of topologies can be reduced to statements about a subbase generating that topology, remark 1.3.3 and lemma 1.3.4 are good evidence for this claim.

Remark 1.3.3. ([56], chapter 2, section 18) To check the continuity of a function, it is enough to verify the condition of continuity for the open sets in the subbase of the topology on the codomain. More clearly, let (X, τ_X) and (Y, τ_Y) be topological spaces, B_Y a subbase for τ_Y, and $f : X \longrightarrow Y$ a map. Then f is continuous if and only if for each $V \in B_Y$ the set $f^{-1}(V)$ is an open subset of X. To see this, notice that every open set in τ_Y can be written as an union of finite intersections of elements in B_Y. Now, we just need to use the following properties of the functions:

$$f^{-1}(V_1 \cap V_2) = f^{-1}(V_1) \cap f^{-1}(V_2) \tag{1.3.1}$$

$$\bigcup_{i\in I} f^{-1}(V_i) = f^{-1}(\bigcup_{i\in I} V_i) \tag{1.3.2}$$

where $\{V_i\}_{i\in I}$ is a family of subsets of Y.

Lemma 1.3.4. *Given topological spaces (X, τ_X) and (Y, τ_Y), let B_X and B_Y be subbases for the topologies τ_X and τ_Y, respectively. If $f : X \longrightarrow Y$ is a continuous function satisfying the following statement,*

$$\forall O \in B_X . \exists V \in B_Y . O = f^{-1}(V),$$

then

$$\forall O \in \tau_X . \exists V \in \tau_Y . O = f^{-1}(V).$$

Proof. Let $O \subseteq X$ be an arbitrary open subset of X. Then by the definition of subbase $O = \bigcup_{i\in I}(\bigcap_{j\in J_i} S_{ij})$ where $S_{ij} \in B_X$ and J_i is a finite set (i.e., $| J_i | < \aleph_0$, where $\aleph_0$ is the smallest infinite cardinal) for each $i \in I$. By assumption for each $i \in I$ and $j \in J_i$ there is an element $U_{ij} \in B_Y$ such that $S_{ij} = f^{-1}(U_{ij})$. Now consider $V := \bigcup_{i\in I}(\bigcap_{j\in J_i} U_{ij})$. Clearly V is an open subset of Y. According to equations 1.3.1 and 1.3.2 in the previous remark, we have $O = f^{-1}(V)$. □

1.4. Initial and Final topology

Initial topology

Suppose $\{f_i : X \longrightarrow Y_i\}_{i\in I}$ is a source[1] in which X is a set and $\{(Y_i, \tau_{Y_i})\}_{i\in I}$ is an indexed family of topological spaces. The initial topology on X generated by the source

[1] A source is a family of morphisms with a common domain. A 2-source is a source consisting of two morphisms.

$\{f_i\}_{i\in I}$ is the smallest topology on X for which $f_i : X \longrightarrow Y_i$ is continuous for each $i \in I$. A subbase for the initial topology may be described as follows,

$$B = \{f_i^{-1}(U_i) \mid i \in I, U_i \in \tau_{Y_i}\}. \tag{1.4.1}$$

Remark 1.4.1. Given a set X and a topological space (Y, τ_Y). If $f : X \longrightarrow Y$ is a map, then due to equations 1.3.1 and 1.3.2 the initial topology on X generated by f is just

$$\{f^{-1}(U) \mid U \in \tau_Y\}$$

Example 1.4.2. (Subspace topology) ([56], chapter 2, section 16) The subspace topology is the initial topology on a subset with respect to the inclusion map. More exactly, given a topological space (X, τ_X), a subset $S \subseteq X$ and the inclusion map $\iota : S \longrightarrow X$ (i.e., $\iota(x) := x$ for each $x \in S$). Then the open sets in the subspace topology on S are precisely the ones of the form $\iota^{-1}(U)$ where U changes over open sets in τ_X. It means the set

$$\tau_S = \{U \cap S \mid U \in \tau_X\}. \tag{1.4.2}$$

is the subspace topology on S. We call the topological space (S, τ_S) a subspace of the topological space (X, τ_X). Notice that the inclusion map ι from the subspace (S, τ_S) to the topological space (X, τ_X) is a continuous map called a topological embedding (or subspace inclusion).
A closed embedding is a topological embedding $\iota : S \longrightarrow X$ such that its image (i.e., S) is a closed subset of X.

Remark 1.4.3. Clearly, if $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are topological embedding, then $g \circ f$ is also a topological embedding (to see this, notice that

$$(U \cap Y) \cap X = U \cap (Y \cap X) = U \cap X$$

for every open subset $U \subseteq Z$).

We give the following lemma without proof:

Lemma 1.4.4. *([56], chapter 2, section 17) If (S, τ_S) is a subspace of the topological space (X, τ_X), then $C \subseteq S$ is a closed subset of S iff there is a closed subset F of X such that $C = S \cap F$.*

As a corollary of the previous lemma we can say:

Corollary 1.4.5. *If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are closed embeddings, then $g \circ f$ is also a closed embedding.*

Example 1.4.6. (Product topology) ([56], chapter 2, section 19) Let $\{(X_i,\tau_{X_i})\}_{i\in I}$ be an indexed family of topological spaces. The product topology is the initial topology on the set $X := \prod_{i\in I} X_i$ (i.e, the cartesian product of the underlying sets X_i) generated by the source of the projection maps $\{\pi_i : X \longrightarrow X_i\}_{i\in I}$. The product topology on X is the smallest topology on X such that for each $i \in I$ the projection map $\pi_i : X \longrightarrow X_i$ is continuous. By 1.4.1, the set $\{\pi_i^{-1}(U_i) \mid i \in I, U_i \in \tau_{X_i}\}$ is a subbase for the product topology on X. Then, the open sets in the product topology on X are unions (finite or infinite) of sets of the form $\prod_{i\in I} U_i$ where each U_i is open in X_i and $U_i \neq X_i$ for only finitely many $i \in I$. In particular, for a finite products of topological spaces (i.e., $\mid I \mid < \aleph_0$ where $\aleph_0$ is the smallest infinite cardinal), the set

$$B = \{\prod_{i\in I} U_i \mid U_i \in \tau_{X_i} \text{ for each } i \in I\}$$

is a base for the product topology on X.

Final topology

The dual concept of the initial topology is final topology. Suppose $\{f_i : X_i \longrightarrow Y\}_{i\in I}$ is a sink [2] in which Y is a set and $\{(X_i, \tau_{X_i})\}_{i\in I}$ a family of topological spaces. The final topology τ on Y induced by the sink $\{f_i\}_{i\in I}$ is the largest topology on Y such that for each $i \in I$ the map $f_i : X_i \longrightarrow Y$ is continuous. Alternatively, the final topology on Y can be described as follows,

$$\tau = \{G \subseteq Y \mid \forall i \in I. f_i^{-1}(G) \in \tau_{X_i}\}. \tag{1.4.3}$$

(Clearly, $\emptyset$ and X are in τ. By equations 1.3.1 and 1.3.2, the set τ is closed under finite intersections and arbitrary unions. This means the set τ mentioned in equation 1.4.3 is a topology on Y. Also, due to equation 1.4.3, it is obvious that for each $i \in I$ the map $f_i : X_i \longrightarrow Y$ is continuous. To see that τ is the largest topology by which $f_i : X_i \longrightarrow Y$ is a continuous map (for each $i \in I$), assume δ is another topology on Y such that for each $i \in I$ the map $f_i : X_i \longrightarrow Y$ is continuous. So for each subset $O \subseteq Y$ such that $O \in \delta$ we have $f_i^{-1}(O) \in \tau_{X_i}$, for every $i \in I$. Therefore by 1.4.3, we conclude that $O \in \tau$. Hence, $\delta \subseteq \tau$).

[2] A sink is a family of morphisms with a common codomain. A 2-sink is a sink consists of two morphisms.

Example 1.4.7. ([56], chapter 2, section 22) If (X, τ) is a topological space, Y a set and $f : X \longrightarrow Y$ a surjective map, then the final topology on Y is called quotient topology on Y. Explicitly, we can define the quotient topology on Y induced by f (In symbol:Q_f) as the collection of all subsets of Y with an open preimage under the surjective map f. So, Q_f can be described as follows

$$Q_f = \{G \subseteq Y \mid f^{-1}(G) \in \tau\}. \tag{1.4.4}$$

In this case the map f with the quotient topology on Y induced by f is a continuous map called quotient map.

Remark 1.4.8. If $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ are quotient maps, then $g \circ f$ is also a quotient map. To see this, note that for each subset $U \subseteq Z$,

$$\begin{aligned} U \text{ is open in } Z &\overset{g \text{ is a quotient map}}{\Longleftrightarrow} g^{-1}(U) \text{ is open in } Y \\ &\overset{f \text{ is a quotient map}}{\Longleftrightarrow} f^{-1}(g^{-1}(U)) \text{ is open in } X. \end{aligned}$$

By considering the definition of Q_f in example 1.4.7, we have the following lemma:

Lemma 1.4.9. *Let (X, τ) be a topological space, Y a set and $f : X \longrightarrow Y$ a surjective map. Then $Q_f = \{f(U) \mid U \underset{open}{\subseteq} X,\ f^{-1}(f(U)) = U\}$.*

Proof. Let $U \subseteq X$ be an open subset of X such that $f^{-1}(f(U)) = U$. By the definition of Q_f in equation 1.4.4, we conclude that $f(U) \in Q_f$. To prove the other direction of this equality, let $G \subseteq Y$ be a subset of Y such that $G \in Q_f$. Then $f^{-1}(G)$ is an open subset of X. Consider $U := f^{-1}(G)$. Since f is surjective, we have $f(U) = f(f^{-1}(G)) = G$. Now, it suffices to show that $f^{-1}(f(U)) = U$. Notice that

$$f^{-1}(f(U)) = f^{-1}f(f^{-1}(G)) = f^{-1}(G) = U.$$

□

1.5. Hausdorff spaces

Definition 1.5.1. (Hausdorff space) A topological space X is called a **Hausdorff space** (in symbol: T_2) if for every two elements x, $y \in X$ with $x \neq y$, there are open neighborhoods $U \in \mathfrak{N}_O(x)$ and $V \in \mathfrak{N}_O(y)$ such that U and V are disjoint (i.e., $U \cap V = \emptyset$).

Example 1.5.2. The set of real numbers $\mathbb{R}$ with the standard topology is a Hausdorff space. Every discrete space is a Hausdorff space.

Lemma 1.5.3. *Given a topological space Y. The following statements are equivalent:*

1. Y is a Hausdorff space.
2. For each continuous function $f : X \longrightarrow Y$, the graph of f, i.e. the set
$$G(f) := \{(x, f(x)) \mid x \in X\}$$
is a closed subset of the product space $X \times Y$ (i.e., $X \times Y$ carries the product topology).
3. $G(id_Y)$ (where id_Y is the identity map on Y) is a closed subset of the product space $Y \times Y$.

Proof. Let Y be a fixed topological space.
(1)⇒(2): Let Y be Hausdorff and $f : X \longrightarrow Y$ continuous. Given $(x, y) \notin G(f)$, we must find an open neighborhood of (x, y) disjoint with $G(f)$. Since Y is Hausdorff, we find $U \in \mathfrak{N}_O(f(x))$ and $V \in \mathfrak{N}_O(y)$ with $U \cap V = \emptyset$. Since f is continuous $f^{-1}(U) \times V$ is an open set in $X \times Y$ and $(x, y) \in f^{-1}(U) \times V$. For each $a \in f^{-1}(U)$ we have $f(a) \in U$, so $f(a) \notin V$, which shows that $(f^{-1}(U) \times V) \cap G(f) = \emptyset$.
(2)⇒(3): This is trivial.
(3)⇒(1): Given $x \neq y$ then $(x, y) \notin G(id_Y)$, thus there is a basic open set $U \times V$ with $(x, y) \in U \times V$ and $(U \times V) \cap G(id_Y) = \emptyset$. Hence $x \in U$, $y \in V$ and $U \cap V = \emptyset$ (see [56], page 101, exercise 13). □

Remark 1.5.4. Notice that the condition "being Hausdorff" for the space Y plays a key role to prove the implication (1)⇒(2) in the previous lemma. By giving an example, we make this issue more clear. Consider the set $\mathbb{R}$ (the set of real numbers) with the trivial topology[3]. The product topology on $\mathbb{R} \times \mathbb{R}$ is trivial topology too. It is clear that the identity function $f : \mathbb{R} \longrightarrow \mathbb{R}$ with $f(x) := x$ is a continuous function whereas its graph, i.e. $G(f) = \{(x, x) \mid x \in \mathbb{R}\}$ is not closed in the trivial space $\mathbb{R} \times \mathbb{R}$ (see also [25]).

Corollary 1.5.5. *A topological space X is a Hausdorff space iff the diagonal of X, i.e. the set $\triangle_X := \{(x, x) \mid x \in X\}$ is closed in $X \times X$.*

Proof. Notice that $\triangle_X = G(id_X)$ for each topological space X. Then by 1.5.3, X is Hausdorff iff $\triangle_X$ is closed. □

[3] In the trivial topology the only open subsets are $\emptyset$ and $\mathbb{R}$. Consequently the only closed subsets are $\emptyset$ and $\mathbb{R}$.

1.6. Compact and locally compact spaces

Definition 1.6.1. (Compact set) A subset K of a topological space X is called **compact**, if for every collection $\mathcal{A} = \{U_\alpha\}_{\alpha \in I}$ of the open subsets of X such that

$$K \subseteq \bigcup_{\alpha \in I} U_\alpha$$

there is a finite subset J of I such that

$$K \subseteq \bigcup_{\alpha \in J} U_\alpha.$$

If in definition 1.6.1, the subset K is replaced by the whole space X and the inclusion symbol is changed to the equality then the topological space X is called compact.

Example. ([56], chapter 3, section 26) According to definition 1.6.1,

- every finite subset of a topological space is compact,
- a subset of a discrete space is compact iff it is finite, and
- a discrete space is a compact space if and only if it is finite.

Example 1.6.2. ([56], chapter 3, section 26) The set of real numbers, i.e. $\mathbb{R}$ with the standard topology is not compact. The open covering

$$\mathcal{A} = \{(n, n+2) \mid n \in \mathbb{Z}\}$$

contains no finite subcollection covering $\mathbb{R}$.

Example 1.6.3. ([56], chapter 3, section 27) Consider the set of real numbers $\mathbb{R}$ with the standard topology. Then the closed interval $[a, b]$ in $\mathbb{R}$ is compact.

Let X be a set and $\mathcal{A} = \{A_i\}_{i \in I}$ be a family of subsets of X indexed by an arbitrary set I. We say that the family $\mathcal{A}$ has the **finite intersection property** (in short: $F.I.P$) if for every finite subset $J \subseteq I$, the set $\bigcap_{i \in J} A_i$ is non-empty. The following theorem shows the connection between this notion and the concept of compactness.

Theorem 1.6.4. *([56], chapter 3, section 26) A topological space X is compact if and only if each family of the closed subsets of X which has the finite intersection property has a non-empty intersection.*

Proof. If $\mathcal{A} = \{A_i\}_{i\in I}$ is a family of subsets of a topological space X, then according to De Morgan's laws[4] $(\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} A_i^c$. Hence $\mathcal{A}$ is a cover of X iff the intersection of the complements of members of $\mathcal{A}$ is empty. The space X is compact iff each family of open sets such that no finite subfamily of it covers X, fails to be a cover, and this is true iff each family of closed sets satisfying the finite intersection property has a non-empty intersection.

□

The following remark has been proven as a theorem in [56], (see chapter 3, section 26, theorem 26.5.).

Remark 1.6.5. The Compactness is a topological property. It means, if $f : X \longrightarrow Y$ is a continuous map then compactness of X results in the compactness of $f(X)$.

A **compact Hausdorff space** is a topological space that is compact and Hausdorff. A **Stone space** is a compact Hausdorff space whose topology has a basis of clopen sets.

All parts of the following theorem have been proven in section 26 of chapter 3 in [56] (see theorems 26.2. and 26.3.).

Theorem 1.6.6. *([56], chapter 3, section 26)*

1. Every closed subset of a compact space is compact.
2. Each compact subset of a Hausdorff space is closed.
3. In a compact Hausdorff space, closed subsets coincide with compact ones.

Lemma 1.6.7. *Let $f : X \longrightarrow Y$ be a continuous map from a compact space X to a Hausdorff space Y. Then f is closed (i.e., if $C \subseteq X$ is a closed subset then $f(C)$ is a closed subset of Y).*

Proof. Suppose $C \subseteq X$ is a closed subset. Since X is compact, C is compact too (by part (1) in theorem 1.6.6). Then $f(C)$ is a compact subset of Y (by remark 1.6.5). As Y is Hausdorff, each compact subset of Y is closed (by part (2) in theorem 1.6.6). So $f(C)$ is a closed subset of Y. □

[4]De Morgan's laws: if $\mathcal{A} = \{A_i\}_{i\in I}$ is a family of subsets of a set X, then $(\bigcap_{i\in I} A_i)^c = \bigcup_{i\in I} A_i^c$ and $(\bigcup_{i\in I} A_i)^c = \bigcap_{i\in I} A_i^c$.

Recall that by lemma 1.5.3, if Y is Hausdorff then for each continuous map $f : X \longrightarrow Y$ the graph of f (i.e., $G(f)$) is a closed subset of $X \times Y$. The following theorem known as "the closed graph theorem in topology" establishes the converse when Y is a compact space (see [17]). This theorem can be also found in [56] (see [56], page 171, exercise 8).

Theorem 1.6.8. ***(The closed graph theorem in topology)*** *Let $f : X \longrightarrow Y$ be a map between topological spaces X and Y and let Y be compact. If $G(f)$ (the graph of f) is a closed subset of $X \times Y$, then f is continuous.*

Proof. Let C be an arbitrary closed subset of Y. To prove that $f^{-1}(C)$ is a closed subset of X, it suffices to show that for each $a \in (f^{-1}(C))^c$, there is an open neighborhood $U \in \mathfrak{N}(a)$ such that $U \cap f^{-1}(C) = \emptyset$. Let a be a fixed element in $(f^{-1}(C))^c$. Then $f(a) \notin C$. Hence $(a, c) \notin G(f)$ for each $c \in C$. Then for each $c \in C$, there is an open neighborhood $U_c \times V_c \in\in \mathfrak{N}((a, c))$ such that $(U_c \times V_c) \cap G(f) = \emptyset$ (because $G(f)$ is closed). Notice that $C \subseteq \bigcup\limits_{c\in C} V_c$, so $\{V_c\}_{c\in C}$ is a collection of the open subsets of Y covering C. Since Y is a compact space, C is compact (by part (1) of theorem 1.6.6). Then there is a finite subset $C_0 \subseteq C$ such that $C \subseteq \bigcup\limits_{c\in C_0} V_c$. Consider $U := \bigcap\limits_{c\in C_0} U_c$. Notice that $a \in U$ and U is open (since C_0 is finite). Consequently, U is an open neighborhood of a. It remains to show that U has an empty intersection with $f^{-1}(C)$. We show this by contradiction. Suppose there exists an element $x \in U \cap f^{-1}(C)$. Then $f(x) \in C$. So there is an element $c_0 \in C_0$ such that $f(x) \in V_{c_0}$ (because $C \subseteq \bigcup\limits_{c\in C_0} V_c$). Therefore, $(x, f(x)) \in U_{c_0} \times V_{c_0}$ (since $x \in \bigcap\limits_{c\in C_0} U_c$). Then $(x, f(x)) \in (U_{c_0} \times V_{c_0}) \cap G(f)$. This gives a contradiction (because $(U_c \times V_c) \cap G(f) = \emptyset$ for each $c \in C$).

$\square$

Remark 1.6.9. One can see that the condition "compactness" for the space Y play a key role to prove the previous theorem. By giving an example, we make this issue more clear. Consider the set of the real numbers $\mathbb{R}$ with the standard topology. Define $f : \mathbb{R} \longrightarrow \mathbb{R}$ as $f(x) := \begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Its graph, i.e. $G(f) = \{(x, \frac{1}{x}) \mid x \in \mathbb{R} - \{0\}\} \cup \{(0, 0)\}$, is a closed subset of the product space $\mathbb{R} \times \mathbb{R}$. However, f is not continuous.

Corollary 1.6.10. *[17] Let X and Y be compact Hausdorff spaces and $f : X \longrightarrow Y$ a map. Then f is continuous iff $G(f) := \{(x, f(x)) \mid x \in X\}$ is a closed subset of $X \times Y$ (with respect to the product topology).*

Proof. Let $f : X \longrightarrow Y$ be a map between compact Hausdorff spaces X and Y. If f is continuous, then by lemma 1.5.3, $G(f)$ is is a closed subset of $X \times Y$. The converse direction is obtained immediately form theorem 1.6.8. $\square$

The next theorem states that the product of every collection of compact spaces is a compact space with respect to the product topology. Since this theorem is well-known, we ignore its proof in this work.

Theorem 1.6.11. ***(Tychonoff's Theorem)*** *([49], chapter 5) Let $\{(X_i, \tau_{X_i})\}_{i\in I}$ be a family of the topological spaces. The cartesian product $\prod_{i\in I} X_i$ is a compact space if and only if each factor X_i is compact.*

Definition 1.6.12. A topological space X is called **locally compact** if for each element $x \in X$ and every open neighborhood $U \in \mathfrak{N}_O(x)$ there is a compact subset $K \subseteq X$ such that $x \in K \subseteq U$ (i.e., for each element $x \in X$ every open neighborhood $U \in \mathfrak{N}_O(x)$ contains a compact neighborhood of x).

Example 1.6.13. ([56], chapter 3, section 29) The Euclidean spaces $\mathbb{R}^n$ (and in particular the real line $\mathbb{R}$) are locally compact. All discrete spaces are locally compact and Hausdorff.

1.7. Nets and convergence

Our presentation in this section is based on chapters 2, 3 and 5 in [49].

Definition 1.7.1. A **directed set** $\mathcal{D} = (D, \geq)$ is a nonempty set D with a binary relation $\geq$ satisfying,

(1) $\forall x \in D.\, x \geq x$ (reflexivity)
(2) $\forall x, y, z \in D.\, x \geq y \wedge y \geq z \implies x \geq z$ (transitivity)
(3) $\forall x, y \in D.\, \exists z \in D.\, (z \geq x) \wedge (z \geq y)$ (directedness)

For $d \in D$ let $D_d := \{d' \in D \mid d' \geq d\}$. Then $\mathcal{D}_d = (D_d, \geq)$ with the ordering inherited from $\mathcal{D}$ is a directed set, too.

Example 1.7.2. Given a topological space (X, τ) and a point x in X, then $\mathfrak{N}(x)$ (the set of all neighborhoods containing x) is a directed set. The binary relation $\geq$ on $\mathfrak{N}(x)$ is given by reverse inclusion, so that $S \geq T$ if and only if S is contained in T.

Definition 1.7.3. Given directed sets $\mathcal{D} = (D, \geq)$ and $\mathcal{E} = (E, \geq)$, a map $\varphi : D \longrightarrow E$ is said to be

- **monotonic**: $\forall d, d' \in D.\, d \geq d' \implies \varphi(d) \geq \varphi(d')$;
- **cofinal**: $\forall e \in E.\, \exists d \in D.\, \varphi(d) \geq e$;
- **converging**: $\forall e \in E.\, \exists d \in D.\, \forall d' \geq d.\, \varphi(d') \geq e$.

Obviously, every converging map is cofinal and every monotonic and cofinal map is converging. However, there are converging maps which are not monotonic, for instance, the map $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ given by $\varphi(n) := if\,(odd\,n)\,n-1\,else\,n+1$.

Lemma 1.7.4. *If $\varphi : D \longrightarrow E$ and $\lambda : E \longrightarrow J$ are converging maps, then $\lambda \circ \varphi$ is also a converging map.*

Proof. Let j be a fixed element in J. We should find an element $d \in D$ such that $\lambda \circ \varphi(d') \geq j$ for each $d' \geq d$. Since λ is a converging map, there is an element $e_j \in E$ such that $\lambda(e') \geq j$ for each $e' \geq e_j$. Besides, since φ is a converging map, there is an element $d_{e_j} \in D$ such that $\varphi(d') \geq e_j$ for each $d' \geq d_{e_j}$. Now, we need just to take $d := d_{e_j}$. $\square$

Definition 1.7.5. Given a set X, a **net** in X is a map $\phi : D \longrightarrow X$, where $\mathcal{D} = (D, \geq)$ is some directed set. Usually, we denote ϕ as $(x_d)_{d\in D}$. Sometimes we write $\phi(d)$ when we want to speak about the element x_d in the net ϕ.

Example 1.7.6. Every non-empty totally ordered [5] set is a directed set. Therefore, every function on such sets is a net. In particular, the natural numbers with the usual order forms a directed set, and a **sequence** in a set X is a function from the natural numbers $\mathbb{N}$ to the set X, so every sequence is a net (see chapter 2 in [49]). The length of a sequence $f : \mathbb{N} \longrightarrow X$ is defined as the number of terms in $im\, f$ (i.e., the image of f). A sequence of a finite length n is called a finite sequence. A sequence is called an infinite sequence if it is not finite one.

Definition 1.7.7. Given two nets $\phi : D \longrightarrow X$ and $\psi : E \longrightarrow X$ in X, we say that ϕ is a **subnet** of ψ, if there exists some converging map $\varphi : D \longrightarrow E$ with $\phi = \psi \circ \varphi$. In this case we shall use the notation $\phi = (x_{\varphi(d)})_{d\in D}$.

$$\begin{tikzcd} E \arrow[rr, "\psi"] & & X \\ & D \arrow[ul, "\varphi"] \arrow[ur, "\phi"'] & \end{tikzcd}$$

According to lemma 1.7.4, if ϕ is a subnet of ψ, and if ψ is a subnet of κ, then ϕ is a subnet of κ.

[5] A non-empty totally ordered set is a nonempty set D with a binary relation $\geq$ in which for any two elements a and b either $a \geq b$ or $b \geq a$.

Remark 1.7.8. Let X be a set. If $\psi = (x_n)_{n\in\mathbb{N}}$ is a sequence of points of X, and if $n_1 < n_2 < \ldots < n_k < \ldots$ is an increasing sequence of positive integers, then the sequence $\phi = (y_k)_{k\in\mathbb{N}}$ defined by setting $y_k = x_{n_k}$ is called a subsequence of the sequence ψ. Note that each subsequence $\phi = (x_{n_k})_{k\in\mathbb{N}}$ of a sequence $\psi = (x_n)_{n\in\mathbb{N}}$ becomes a subnet of the net ψ (define $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ by $\varphi(k) := n_k$ for each $k \in N$). Note however that a subnet of a sequence need not be a subsequence in general, e.g. it is possible to define a subnet $\phi = (x_d)_{d\in D}$ of a sequence $\psi = (x_n)_{n\in\mathbb{N}}$ such that D is uncountable.

Definition 1.7.9. ([49], chapter 2) Let X be a topological space, and let $\phi := (x_d)_{d\in D}$ be a net in X.

- Given a point $x \in X$, we say that the net ϕ is **convergent to** x, if
$$\forall U \in \mathfrak{N}(x).\, \exists d_U \in D.\, \forall d \geq d_U.\, x_d \in U.$$
In this case we say that x is a **limit** of ϕ and we write $\phi \longrightarrow x$. The set of all limits of the net ϕ, is denoted by $lim\,\phi$.
- We say the net ϕ is **eventually** in $A \subseteq X$ if: $\exists d \in D.\, \forall d' \geq d.\, x_{d'} \in A$.
- We say the net ϕ is **frequently** in $A \subseteq X$ if:$\forall d \in D.\, \exists d' \geq d.\, x_{d'} \in A$.
- A point x is said to be an **accumulation point** of the net ϕ if and only if for every neighborhood $U \in \mathfrak{N}(x)$, the net ϕ is frequently in U. We write $\phi \dashrightarrow x$ if x is an accumulation point of ϕ.

Remark 1.7.10. ([49], chapter 2) Due to definition 1.7.9, one can see that

- x is a limit of a net ϕ if for every open neighborhood $U \in \mathfrak{N}_O(x)$, the net ϕ is eventually in U, and
- if ϕ is a subnet of ψ then for all $A \subseteq X$: if ψ is eventually in A then ϕ is eventually in A, too.

Lemma 1.7.11. *([49], chapter 2) A point x is an accumulation point of a net ψ if and only if ψ has a subnet converging to x.*

Proof. Let x be an accumulation point of a net $\psi := (x_e)_{e\in E}$. We define a convergent subnet ϕ in three steps. Firstly, since x is an accumulation point of ψ, for each $U \in \mathfrak{N}(x)$ we can choose an element $e \in E$ such that $x_e \in U$. Define a directed set $(D, \geq)$ as
$$D := \{(e, U) \in E \times \mathfrak{N}(x) \mid x_e \in U\},$$
$$(e', U') \geq (e, U) :\Longleftrightarrow e' \geq e \text{ and } U' \subseteq U.$$

In the second step, we will define a converging map from D to E. Defines a function $\varphi : D \longrightarrow E$ by $\varphi((e,U)) := e$. To see that φ is a converging map, notice that since x is an accumulation point of ψ, for each $e \in E$ and each neighborhood $U \in \mathfrak{N}(x)$, there is an element $e_U \in E$ with $e_U \geq e$ such that $x_{e_U} \in U$. Now, consider the pair $(e_U, U) \in D$. If (e', U') is an element in D with $(e', U') \geq (e_U, U)$, then $\varphi(e', U') = e' \geq e$. Hence φ is a converging map and consequently $\phi := \{x_{\varphi(e,U)}\}_{(e,U)\in D}$ is a subnet of ψ. In the third step, we show that ϕ converges to x. For every open neighborhood $U \in \mathfrak{N}(x)$, we can choose an arbitrary $e \in E$ such that $x_e \in U$ (since x is an accumulation point of ψ). Then $(e, U) \in D$ and for each element $(e', U') \in D$ with $(e', U') \geq (e, U)$ we have $x_{\varphi(e',U')} = x_{e'} \in U' \subseteq U$. Thus ϕ converges to x.
Conversely, suppose $\psi := (x_e)_{e\in E}$ has a subnet converging to $x \in X$. If x is not an accumulation point of ψ, then there is a neighborhood U of x such that ψ is not frequently in U, and therefore ψ is eventually in the complement of U. Then each subnet of ψ is eventually in the complement of U and hence ψ can not converge to x. $\square$

Theorem 1.7.12. *([49], chapters 2, 3 and 5) Let X and Y be topological spaces.*

1. If X is a Hausdorff space, then the limit of each net in X (if it exists) is unique.
2. If ψ is a net in X, then ψ converges to a point $x \in X$ iff every subnet of ψ converges to x.
3. If S is a subset in X, then x is in the closure of S if and only if there exists a net ψ in S such that ψ converges to x.
4. A function $f : X \longrightarrow Y$ is continuous if and only if for each net ψ in X such that ψ converges to x, then the net $f \circ \psi$ converges to $f(x)$.
5. The topological space X is compact if and only if every net in X has a subnet converging to some point of X.
6. If $(x_d, y_d)_{d\in D}$ is a net in the product space $X \times Y$, then $(x_d, y_d)_{d\in D}$ converges to a point (x, y) iff the nets $(x_d)_{d\in D}$ and $(y_d)_{d\in D}$ converge to x and y, respectively.

Proof. Let X and Y be topological spaces.
(1) Let X be a Hausdorff space and ψ a net in X converging to elements $x \neq y \in X$. Since X is Hausdorff, there are two disjoint open subsets U, $O \subseteq X$ (i.e., $U \cap O \neq \emptyset$) such that $x \in U$ and $y \in O$. Since the net ψ can not be eventually in both disjoint subsets U and O, the net ψ does not converge to both x and y.
(2) This follows directly from definitions 1.7.7 and 1.7.9.
(3) Let ψ be a net in S converging to $x \in X$. Then each neighborhood of x contains some points of S. Hence, x is in $\overline{S}$. Conversely, let $x \in \overline{S}$. By example 1.7.2, the set $\mathfrak{N}(x)$ (the set of all neighborhoods of x) can be directed by reverse inclusion (i.e., $O \geq V$ if and only if $O \subseteq V$). Then we can find a net $(x_O)_{O\in\mathfrak{N}(x)}$ such that $x_O \in O \cap S$ for each $O \in \mathfrak{N}(x)$. Then $(x_O)_{O\in\mathfrak{N}(x)}$ is a net in S. It is easy to see that the net $(x_O)_{O\in\mathfrak{N}(x)}$ is

eventually in every neighborhood of x (i.e., for every element $O \in \mathfrak{N}(x)$ and for every $V \in \mathfrak{N}(x)$ with $O \subseteq V$, we have $x_O \in V$). Then $(x_O)_{O\in\mathfrak{N}(x)}$ converges to x.
(4) Assume f is continuous and let ψ be a net in the topological space X that converges to a point $x \in X$. Let U be a neighborhood of $f(x)$, then $f^{-1}(U)$ is a neighborhood of x. Since ψ is eventually in $f^{-1}(U)$, the net $f \circ \psi$ is eventually in U. Conversely, let $f : X \longrightarrow Y$ be a map and for every net ψ in X if ψ converges to x, then the net $f \circ \psi$ converges to $f(x)$. We want to show that f is continuous. We have to show that for each open subset $U \subseteq Y$, the set $f^{-1}(U)$ is open in X. So it suffices to prove that for each open subset $U \subseteq Y$ and each $x \in f^{-1}(U)$ there is an open neighborhood $O \in \mathfrak{N}(x)$ such that $O \subseteq f^{-1}(U)$. We prove this claim by contradiction. So assume that there are an open subset $U \subseteq Y$ and an element $x \in f^{-1}(U)$ such that $O \cap (X - f^{-1}(U)) \neq \emptyset$ for each $O \in \mathfrak{N}(x)$. By example 1.7.2, the neighborhood system $\mathfrak{N}(x)$ with the reverse inclusion (i.e., $O \geq V$ if and only if $O \subseteq V$) is a directed set. Then we can find a net $(x_O)_{O\in\mathfrak{N}(x)}$ such that $x_O \in O \cap (X - f^{-1}(U))$ for each $O \in \mathfrak{N}(x)$. Hence, for every element $O \in \mathfrak{N}(x)$ and for every $V \in \mathfrak{N}(x)$ with $O \subseteq V$, we have $x_O \in V$. Therefore $(x_O)_{O\in\mathfrak{N}(x)}$ converges to x but $(f(x_O))_{O\in\mathfrak{N}(x)}$ does not converges $f(x)$. This gives a contradiction with the assumption.
(5) Let X be a compact space and $(x_d)_{d\in D}$ be a net in the X. For each $d \in D$ let A_d be the set of all points $x_{d'}$ for $d' \geq d$. Then the family of all sets A_d has the finite intersection property (because D is directed by $\geq$). Consequently the family $\{\overline{A_d}\}_{d\in D}$ ($\overline{A_d}$ is the topological closure of A_d) has the finite intersection property. Then since X is compact, by theorem 1.6.4 we have $\bigcap\limits_{d\in D} \overline{A_d} \neq \emptyset$. Now, let $x \in \bigcap\limits_{d\in D} \overline{A_d}$, then according to the construction of the sets A_d $(d \in D)$ the point x is an accumulation point of the net $(x_d)_{d\in D}$ (see definition 1.7.9). Then by lemma 1.7.11, the net $(x_d)_{d\in D}$ has a subnet converging to x. Conversely, suppose that every net in X has a convergent subnet. For the sake of contradiction, let $\{U_i \mid i \in I\}$ be an open cover of X with no finite subcover. Consider $D = \{J \subseteq I \mid\, \mid J \mid < \aleph_0\}$. Observe that D is a directed set under inclusion and for each $J \in D$, there exists an $x_J \in X$ such that $x_J \notin U_j$ for all $j \in J$. Consider the net $(x_J)_{J\in D}$. This net can not have a convergent subnet, because for each $x \in X$ there exist $i \in I$ and a neighborhood U_i of x such that $(x_J)_{J\in D}$ is not eventually in U_i (to see this, notice that for all $J \in D$ with $\{i\} \subseteq J$, we have $x_J \notin U_i$). This is a contradiction and completes the proof.
(6) Let $(x_d, y_d)_{d\in D}$ be a net in $X \times Y$ that converges to a point (x, y). Since the projection maps $\pi_X : X \times Y \longrightarrow X$ and $\pi_X : X \times Y \longrightarrow X$ are continuous (see example 1.4.6), by part (4) of this theorem we conclude that the nets $(x_d)_{d\in D}$ and $(y_d)_{d\in D}$ converge to x and y, respectively. To show the converse, let $(x_d, y_d)_{d\in D}$ be a net in $X \times Y$ such that the nets $(x_d)_{d\in D}$ and $(y_d)_{d\in D}$ converge to x and y, respectively. Then for each open neighborhood U of x the net $(x_d)_{d\in D}$ is eventually in U and similarly for each open neighborhood V of y the net $(y_d)_{d\in D}$ is eventually in V. Hence the net $(x_d, y_d)_{d\in D}$ is eventually in $U \times V$. Since each open neighborhood of (x, y) is an union (finite or infinite) of sets of the form $U \times V$, we conclude that $(x_d, y_d)_{d\in D}$ converges to (x, y). $\square$

In the following, we prove some lemmas that will be used as the auxiliary lemmas to prove some properties of the Vietoris bisimulations in chapter 9.

Lemma 1.7.13. *Given topological spaces X and Y. Let $R \subseteq X \times Y$ be a binary relation which is closed in $X \times Y$ (with respect to the product topology). Then*

$$R^{-1} := \{(y, x) \in Y \times X \mid (x, y) \in R\}$$

is a closed subset of $Y \times X$.

Proof. We have

$$\begin{aligned} R \underset{open}{\subseteq} X \times Y &\iff R = \bigcup \{U \times V \mid U \underset{open}{\subseteq} X, V \underset{open}{\subseteq} Y\} \\ &\iff R^{-1} = \bigcup \{V \times U \mid V \underset{open}{\subseteq} Y, U \underset{open}{\subseteq} X\} \\ &\iff R^{-1} \underset{open}{\subseteq} Y \times X. \end{aligned}$$

□

Notice that the relation composition[6] of two closed relations need not to be closed, see the following example.

Example 1.7.14. Consider the set of real numbers $\mathbb{R}$ along with the standard topology and the set of natural numbers $\mathbb{N}$ with the discrete topology. Let $R = \{(\frac{1}{n}, n) \mid n \in \mathbb{N}^+\}$ and $S = \{(n, \frac{1}{n}) \mid n \in \mathbb{N}^+\}$ be binary relations between $\mathbb{R}$ and $\mathbb{N}$. Hence

$$R \circ S = \{(\frac{1}{n}, \frac{1}{n}) \mid n \in \mathbb{N}^+\}.$$

It is easy to check that the binary relations R and S are closed in $\mathbb{R} \times \mathbb{N}$ and $\mathbb{N} \times \mathbb{R}$, respectively. However, $R \circ S$ is not a closed subset of $\mathbb{R} \times \mathbb{R}$.

Lemma 1.7.15. *Given compact spaces X, Y and Z. Let the binary relations $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are closed subsets (with respect to the product topology). Then $R \circ S$ is a closed subset of $X \times Z$.*

[6] If $R \subseteq X \times Y$ and $S \subseteq Y \times Z$ are two binary relations, then their relation composition $R \circ S$ is the relation

$$\{(x, z) \in X \times Z \mid \exists y \in Y. (x, y) \in R \wedge (y, z) \in S\}.$$

Proof. We need to show that $\overline{R \circ S} \subseteq R \circ S$. Assume $(x, z) \in \overline{R \circ S}$, then according to part (3) of theorem 1.7.12, there is a net $(x_d, z_d)_{d \in D}$ in $R \circ S$ which converges to (x, z) and then by continuity of the projection maps (see example 1.4.6) the nets $(x_d)_{d \in D}$ and $(z_d)_{d \in D}$ converge to x and z, respectively (see part (4) in theorem 1.7.12). Now, since for each $d \in D$ we have $(x_d, z_d) \in R \circ S$, we can find a net $(y_d)_{d \in D}$ in Y such that $(x_d, y_d)_{d \in D}$ is a net in R and $(y_d, z_d)_{d \in D}$ is a net in S. Since Y is compact, by part (5) in theorem 1.7.12 there is a converging map $\varphi : E \longrightarrow D$ and an element $y \in Y$ such that the subnet $(y_{\varphi(e)})_{e \in E}$ converges to y. On the other hand, by part (2) in theorem 1.7.12, the subnet $(x_{\varphi(e)})_{e \in E}$ converges to x (because $(x_d)_{d \in D}$ converges to x). Then by part (6) in theorem 1.7.12, the subnet $(x_{\varphi(e)}, y_{\varphi(e)})_{e \in E}$ converges to (x, y). Note that R is closed, then $(x, y) \in R$.
Now, it suffices to show that $(y, z) \in S$. Consider the subnet $(y_{\varphi(e)}, z_{\varphi(e)})_{e \in E}$ of the net $(y_d, z_d)_{d \in D}$ in S. We know that $(y_{\varphi(e)})_{e \in E}$ converges to y, besides by part (2) in theorem 1.7.12, $(z_{\varphi(e)})_{e \in E}$ converges to z (because $(z_d)_{d \in D}$ converges to z). Then by part (6) in theorem 1.7.12, the subnet $(y_{\varphi(e)}, z_{\varphi(e)})_{e \in E}$ converges to (y, z). Since S is closed, $(y, z) \in S$.
Hence from $(x, y) \in R$ and $(y, z) \in S$ we have $(x, z) \in R \circ S$. □

1.8. Metric and Ultrametric spaces

Definition 1.8.1. A **metric space** is a pair (X, d) where X is a set (called the underlying set) and d (called metric) is a map from $X \times X$ to $\mathbb{R}$ such that for any $x, y, z \in X$, the following hold,

(1) $d(x, y) \geq 0$,
(2) $d(x, y) = 0 \iff x = y$,
(3) $d(x, y) = d(x, y)$, and
(4) $d(x, z) \leq d(x, y) + d(y, z)$.

For any point x in a metric space X we define the open ball of radius $r > 0$ ($r \in \mathbb{R}$) around x as the set $B(x, r) = \{y \in X \mid d(x, y) < r\}$. We usually write $B_r(x)$ instead of $B(x, r)$. The collection of such open balls is a subbase for a topology on X. This topology is called the **metric topology** induced by d. Explicitly, a subset U of X is open if for every $x \in U$ there exists a real number $r > 0$ such that $B(x, r)$ is contained in U. A neighborhood of a point x is any subset of X that contains an open ball around x as a subset. A topological space which can arise in this way forms a metric space called a **metrizable space**. It is not hard to see that the non-empty metric spaces are Hausdorff (for every two point x, $y \in X$, consider the open balls $B(x, r)$ and $B(y, r)$ where $r := \frac{1}{2} d(x, y)$).
Two metric spaces (X, d_X) and (Y, d_Y) are called **isomorphic** if there is a bijective, continuous and open function between them (with respect to the metric topologies induced by d_X and d_Y).

In what follows we shall always assume (without loss of generality) that the metric space (X, d) is not empty.

Example 1.8.2. ([56], chapter 2, section 20) The set of real numbers $\mathbb{R}$ along with the distance function $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $d(x, y) :=\mid x - y \mid$ is a metric space.

Remark 1.8.3. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in a metric space (X, d), then according to definition 1.7.9, we say that a point $x \in X$ is a limit of $(x_n)_{n\in\mathbb{N}}$ (in symbol: $\underset{n\longrightarrow\infty}{lim} x_n = x$) if the statement below holds,

$$\forall r > 0. \exists m \in \mathbb{N}. \forall n \geq m. d(x_n, x) < r.$$

Consequently, if $x \in X$ is a limit of the sequence $(x_n)_{n\in\mathbb{N}}$, then for each radius $r > 0$,

$$\mid \{n \in \mathbb{N} \mid d(x_n, x) < r\} \mid\geq \aleph_0,$$

i.e., $\{n \in \mathbb{N} \mid d(x_n, x) < r\}$ is an infinite set.

Definition 1.8.4. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in a metric space (X, d). We say that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if

$$\forall \varepsilon > 0. \exists M \in \mathbb{N}. \forall i, j \geq M. d(x_i, x_j) < \varepsilon.$$

A metric space (X, d) is complete if every Cauchy sequence in X has a limit.

Lemma 1.8.5. *([56], chapter 7, section 43) A metric space (X, d) is complete if every Cauchy sequence in X has a convergent subsequence.*

Proof. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in (X, d). We show that if $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to a point $x \in X$, then the sequence itself, i.e. $(x_n)_{n\in\mathbb{N}}$ converges to x. Given $\varepsilon < 0$, then there is an element $M \in \mathbb{N}$ such that for all n, $m \geq M$ we have $d(x_n, x_m) < \varepsilon/2$ (because $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence). On the other hand, since $(x_{n_k})_{k\in\mathbb{N}}$ converges to x, we can choose an element $k \in \mathbb{N}$ such that $n_k \geq M$ and $d(x_{n_k}, x) < \varepsilon/2$. Putting these facts together, we have the desired result that for $n \geq M$,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon.$$

□

Definition 1.8.6. A metric space (X, d) is totally bounded if and only if for every real number $\epsilon > 0$, there exists a finite collection of open balls of radius ϵ in X whose union contains X.

The next lemma is a part of the proof of theorem 45.1 in [56] (see [56], page 276).

Lemma 1.8.7. *([56], chapter 7, section 45) Let (X, d) be a totally bounded metric space, then every sequence in X has a Cauchy subsequence.*

Proof. Assume that (X, d) is totally bounded and $(x_n)_{n\in\mathbb{N}}$ is a sequence in X. Since X is totally bounded, it can be covered by finitely many balls of radius $1/k$ for each $k \in \mathbb{N}$. Then X can be covered by finitely many balls of radius 1. Therefore at least one of those balls must contain infinitely many terms of $(x_n)_{n\in\mathbb{N}}$. Call that ball B_1, and let S_1 be the set of all indexes $n \in \mathbb{N}$ for which $x_n \in B_1$.
Now, cover X by finitely many balls of radius $1/2$. Because S_1 is infinite, at least one of these balls, say B_2, must contain x_n for infinitely many values of n in S_1. Choose S_2 to be the set of those indexes n for which $n \in S_1$ and $x_n \in B_2$. In general, given an infinite set S_k of positive integers, choose S_{k+1} to be an infinite subset of S_k such that there is a ball B_{k+1} of radius $1/k+1$ that contains x_n for all $n \in S_{k+1}$. Then for each $k \in \mathbb{N}$ the set S_k is infinite and $S_{k+1} \subseteq S_k$.
Choose an element $n_1 \in S_1$. Given n_k, choose $n_{k+1} \in S_{k+1}$ such that $n_k < n_{k+1}$ (this we can do because S_{k+1} is an infinite set). Now, we have a sequence $(n_k)_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$ we have $n_k \in S_k$ and $n_k < n_{k+1}$. One can see that whenever $i \geq k$, then $n_i \in S_k$ (because for each $k \in \mathbb{N}$ we have $S_k \subseteq S_{k-1}$). Thus for all i, $j \geq k$, the term x_{n_i} and x_{n_j} are both contained in a ball of radius $1/k$. Hence the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ is Cauchy. $\square$

In the following, we show that the notion of compactness for metric spaces involves the notions of completeness and total boundedness. To show this we first prove an auxiliary statement which is a part of the proof of the implication (3)$\Longrightarrow$(1) in theorem 28.2 in [56] (see [56], page 180).

Lemma 1.8.8. *Let (X, d) be a metric space in which every sequence has a subsequence converging to some point of X. Suppose we are given an infinite open cover[7] $\{U_i\}_{i\in I}$ of X. Then there exists an $\varepsilon > 0$ so that every ball of radius ε is contained in one of the (open) sets U_i.*

Proof. We prove this claim by contradiction. Assume $\{U_i\}_{i\in I}$ is an infinite open cover of X for which there is no $\varepsilon > 0$ so that every ball of radius ε is contained in one of the (open) sets U_i. Then for each $n \in \mathbb{N}$ there is a ball B_n of radius $1/n$ which is not contained in any of the sets U_i. Let x_n be the center of B_n. Since X is sequentially compact, the sequence of the centers i.e., $(x_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(x_{n_k})_{k\in\mathbb{N}}$ with limit $x \in X$. Since $\{U_i\}_{i\in I}$ is a cover for X, there exists an index $i_0 \in I$ such that $x \in U_{i_0}$. Since U_{i_0} is open, x is an interior point of it. Then we can choose an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq U_{i_0}$. If k is large enough that $1/n_k < \varepsilon/2$, then the set $B_{n_k}(x_{n_k})$ lies in $B_{\varepsilon/2}(x_{n_k})$,

[7] An open cover of a topological space X is a collection $\mathcal{A} = \{U_i\}_{i\in I}$ of the open subsets of X such that

$$X \subseteq \bigcup_{i\in I} U_i$$

if k is also chosen large enough that $d(x_{n_k}, x) < \varepsilon/2$, then $B_{n_k}(x_{n_k})$ lies in $B_\varepsilon(x)$. But this means that $B_{n_k}(x_{n_k})$ contrary to hypothesis. $\square$

The following is a combination of theorems 28.2 and 45.1 in [56] (see [56], pages 179 and 276, respectively):

Theorem 1.8.9. *Let (X, d) be a metric space. Then the following are equivalent:*

1. *(X, d) is complete and totally bounded.*
2. *Every sequence in X has a subsequence that converges to some point of X.*
3. *X is a compact space (with respect to the metric topology induced by d).*

Proof. ($1 \implies 2$) Since (X, d) totally bounded, by lemma 1.8.7 every sequence in X has a Cauchy subsequence. Since (X, d) is complete, every Cauchy sequence in X has a limit. Then every sequence in X has a subsequence that converges to some point of X.
($2 \implies 1$) By lemma 1.8.5. We proceed by contradiction to show that (X, d) is totally bounded. Assume that there exists an $\varepsilon > 0$ such that X cannot be covered by finitely many balls with radius ε. Construct a sequence of points of X as follows: First, choose x_1 to be any point of X. Notice that the ball $B_\varepsilon(x_1)$ is not all of X (otherwise X could be covered by a single ball with radius ε). Choose x_2 to be a point of X not in $B_\varepsilon(x_1)$. In general, given $x_1, \ldots, x_n$, choose x_{n+1} to be a point not in the union $\bigcup\limits_{i=1}^{n} B_\varepsilon(x_i)$ (using the fact that these balls do not cover X). Note that by construction $d(x_{n+1}, x_i) > \varepsilon$ for $i \in \{1, \ldots, n\}$. Therefore, the sequence $(x_n)_{n\in\mathbb{N}}$ can have no convergent subsequence. In fact, each ball of radius $\varepsilon/2$ can contains x_n for at most one value of n.
($2 \implies 3$) Let $\{U_i\}_{i\in I}$ be an infinite open cover of X. We need to show that there is a finite subset $J \subsetneq I$ such that $X \subseteq \bigcup\limits_{i\in J} U_i$. By lemma 1.8.8 there exists an $\varepsilon > 0$ so that every ball of radius ε is contained in one of the (open) sets U_i. From implication ($2 \implies 1$), we know that (X, d) is totally bounded, thus there exist a finite subset $\{x_1, \ldots, x_n\}$ of X so that $X \subseteq \bigcup\limits_{k=1}^{n} B_\varepsilon(x_k)$. As for each integer $k \in \{1, \ldots, n\}$ there is an element $i_k \in I$ such that the ball $B_\varepsilon(x_k) \subseteq U_{i_k}$, by setting $J := \{i_k, \ldots, i_n\}$ we have a finite subset $J \subsetneq I$ such that $X \subseteq \bigcup\limits_{i\in J} U_i$.
($3 \implies 2$) Let X be a compact space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in the X. For each $n \in \mathbb{N}$, let A_n be the set of all points x_m for $m \geq n$. Then the family of all sets $\{A_n\}_{n\in\mathbb{N}}$ has the finite intersection property. Consequently, the family $\{\overline{A_n}\}_{n\in\mathbb{N}}$ ($\overline{A_n}$ is the topological closure of A_n) has the finite intersection property. Then since X is compact, by theorem 1.6.4 we have $\bigcap\limits_{n\in\mathbb{N}} \overline{A_n} \neq \emptyset$. Now, let $x \in \bigcap\limits_{n\in\mathbb{N}} \overline{A_n}$. By the construction of the sets A_n, the point x is an accumulation point of the sequence $(x_n)_{n\in\mathbb{N}}$. Then for each $k \geq 1$ the ball $B_{1/k}(x)$ contains infinitely many terms of $(x_n)_{n\in\mathbb{N}}$. Hence, we can choose a sequence $(n_k)_{k\in\mathbb{N}}$ such that for each $k \in \mathbb{N}$ we have $n_k < n_{k+1}$ and $x_{n_k} \in B_{1/k}(x)$. It is easy to see that the subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converges to x. $\square$

Ultrametric spaces

Definition 1.8.10. An **ultrametric space** (X, d) is a metric space in which the metric d satisfies the strong triangle inequality, i.e.,

$$d(x, z) \leq Max\{d((x, y), d(y, z)\}$$

for each $x, y, z \in X$. In this case d is called an ultrametric.
An 1-bounded ultrametric space is an ultrametric space (X, d) where $d : X \times X \longrightarrow [0, 1]$.

In order to give a motivation to study the notion of ultrametric spaces, we continue this part with an well-known example which has many applications in mathematics and computer science (see [10], [12] and [50]).

Let X be an arbitrary set. Consider X^ω as the set of all words[8] over X. For each $p,\ q \in X^\omega$ with $p \neq q$, define

$$m(p, q) := Inf\,\{k \in \mathbb{N} \mid p(k) \neq q(k)\}. \tag{1.8.1}$$

Define a distance function $d : X \times X \longrightarrow [0, 1]$ by

$$d(p,\ q) := \begin{cases} 0 & p = q \\ 2^{-m(p,q)} & otherwise \end{cases} \tag{1.8.2}$$

We can prove the the following lemmas for d :

Lemma 1.8.11. *For each $p, q \in X^\omega$ and $n \in \mathbb{N}$,*

$$d(p, q) < 2^{-n} \Longleftrightarrow d(p, q) \leq 2^{-(n+1)}$$

Proof. Given different $p, q \in X^\omega$ and $n \in \mathbb{N}$,

$$\begin{aligned} d(p, q) < 2^{-n} &\Longleftrightarrow 2^{-m(p,q)} < 2^{-n} \\ &\Longleftrightarrow m(p, q) > n \\ &\Longleftrightarrow \forall k \in \mathbb{N}.\, p(k) \neq q(k) \Longrightarrow k > n \\ &\Longleftrightarrow \forall k \in \mathbb{N}.\, p(k) \neq q(k) \Longrightarrow k \geq n + 1 \\ &\Longleftrightarrow m(p, q) \geq n + 1 \\ &\Longleftrightarrow d(p, q) \leq 2^{-(n+1)}. \end{aligned}$$

□

[8] Every sequence in X (i.e, every map from $\mathbb{N}$ to X) is called a word over X. We denote by X^ω the set of all words over X.

Lemma 1.8.12. *For each $p, q \in X^\omega$ and $n \in \mathbb{N}$,*

$$d(p,q) \leq 2^{-n} \iff \forall k < n.\, p(k) = q(k).$$

Proof. Given different $p, q \in X^\omega$ and $n \in \mathbb{N}$,

$$\begin{aligned} d(p,q) \leq 2^{-n} &\iff 2^{-m(p,q)} \leq 2^{-n} \\ &\iff m(p,q) \geq n \\ &\iff \forall k \in \mathbb{N}.\, p(k) \neq q(k) \implies k \geq n \\ &\iff \forall k < n.\, p(k) = q(k). \end{aligned}$$

□

Corollary 1.8.13. *For each $p, q \in X^\omega$ and $n \in \mathbb{N}$,*

$$d(p,q) < 2^{-n} \iff \forall k \leq n.\, p(k) = q(k).$$

Proof. Given $p, q \in X^\omega$ (naturally different) and $n \in \mathbb{N}$,

$$\begin{aligned} d(p,q) < 2^{-n} &\overset{lemma\ 1.8.11}{\iff} d(p,q) \leq 2^{-(n+1)} \\ &\overset{lemma\ 1.8.12}{\iff} \forall k < n+1.\, p(k) = q(k) \\ &\iff \forall k \leq n.\, p(k) = q(k). \end{aligned}$$

□

Lemma 1.8.14. *d is an ultrametric.*

Proof. We need to check that for every $p, q \in X^\omega$

(1) $d(p,q) = 0 \iff p = q$,
(2) $d(p,q) = d(q,p)$, and
(3) $d(p,q) \leq Max\{d((p,r),d(r,q)\}$.

By the definition of d, the conditions (1) and (2) are trivial. Let $d(p,q) = 0$. So, for each $n \in \mathbb{N}$, we have $d(p,q) < 2^{-n}$. Therefore, due to lemma 1.8.12 for each $n \in \mathbb{N}$ we have $p(n) = q(n)$ and consequently $p = q$. Regarding (3), given p,q,$r\in X^\omega$ (naturally different), then for each $k \in \mathbb{N}$ such that $k < m(p,r)$ and $k < m(r,q)$, we conclude that $p(k) = q(k) = r(k)$. Thus

$$m(p,q) \geq Min\,\{m(p,r),\, m(r,q)\}.$$

Hence

$$d(p, q) \le Max\,\{d(p, r),\, d(r, q)\}.$$

If $p = r$ or $q = r$ or $p = q$ the claim is trivial. □

Example 1.8.15. Consider the pair (X^ω, d) (where d is the distance function defined in equation 1.8.2). According to lemma 1.8.14, the distance function d is an ultrametric and then (X^ω, d) is an ultrametric space.

In the following we will show that (X^ω, d) is a complete ultrametric space.

Lemma 1.8.16. *(X^ω, d) is a complete ultrametric space.*

Proof. We need to prove that every Cauchy sequence in X^ω converges to an element in X^ω. Given a Cauchy sequence $(q_n)_{n\in\mathbb{N}}$ in (X^ω, d). Then, by the definition of Cauchy sequences,

$$\forall n \in \mathbb{N}.\, \exists M_n \in \mathbb{N}.\, \forall i, j \ge M_n.\, d(q_i, q_j) < 2^{-n}. \tag{1.8.3}$$

So, in particular for $j = M_n$ we have

$$\forall n \in \mathbb{N}.\, \exists M_n \in \mathbb{N}.\, \forall i \ge M_n.\, d(q_i, q_{M_n}) < 2^{-n}. \tag{1.8.4}$$

For each $n \in \mathbb{N}$ let M_n be the smallest element in $\mathbb{N}$ satisfying equation 1.8.4. Then for all n, $k \in \mathbb{N}$ with $k \le n$ we have $M_k \le M_n$. Thus, by equation 1.8.4 we have

$$\forall n, k \in \mathbb{N}.\, k \le n \Longrightarrow d(q_{M_n}, q_{M_k}) < 2^{-k}. \tag{1.8.5}$$

Consequently by corollary 1.8.13,

$$\forall n, k \in \mathbb{N}.\, k \le n \Longrightarrow \forall r \le k.\, q_{M_n}(r) = q_{M_k}(r). \tag{1.8.6}$$

So, in particular for $r = k$ we have

$$\forall n, k \in \mathbb{N}.\, k \le n \Longrightarrow q_{M_n}(k) = q_{M_k}(k). \tag{1.8.7}$$

Define $q : \mathbb{N} \longrightarrow X$ by $q(n) := q_{M_n}(n)$ for each $n \in \mathbb{N}$ (where M_n is the smallest natural number satisfies equation 1.8.4). Hence by equation 1.8.7 we have

$$\forall n, k \in \mathbb{N}.\, k \le n \Longrightarrow q_{M_n}(k) = q(k). \tag{1.8.8}$$

We claim that $\lim\limits_{n\longrightarrow\infty} q_n = q$. Since, $\lim\limits_{n\longrightarrow\infty} 2^{-n} = 0$, it is enough to show that

$$\forall n \in \mathbb{N}. \exists K_n \in \mathbb{N}. \forall i \geq K_n. d(q_i, q) < 2^{-n} \tag{1.8.9}$$

Let $n \in \mathbb{N}$ be a fixed element of $\mathbb{N}$. Consider $K_n := M_n$ where M_n is the smallest natural number satisfies equation 1.8.4. Then by equation 1.8.4, we know that

$$\forall i \geq K_n. d(q_i, q_{K_n}) < 2^{-n}. \tag{1.8.10}$$

Thus by corollary 1.8.13,

$$\forall i \geq K_n. \forall k \leq n. q_i(k) = q_{K_n}(k) = q_{M_n}(k).$$

So, according to equation 1.8.8, we have

$$\forall i \geq K_n. \forall k \leq n. q_i(k) = q(k)$$

Consequently by corollary 1.8.13,

$$\forall i \geq K_n. d(q_i, q) < 2^{-n}.$$

□

Non-expansive maps

Definition 1.8.17. A map $f : X \longrightarrow Y$ between ultrametric spaces (X, d_X) and (Y, d_Y) is **non-expansive** when it is non-distance-increasing, i.e.

$$\forall x, y \in X. d_Y(f(x), f(y)) \leq d_X(x, y).$$

A map $f : X \longrightarrow Y$ between ultrametric spaces (X, d_X) and (Y, d_Y) is **contractive** when it shrinks the distance between any two points by a non-unit factor, i.e.

$$\exists \lambda \in [0, 1). \forall x, y \in X. d_Y(f(x), f(y)) \leq \lambda. d_X(x, y).$$

1.9. Properties of the ultrametric spaces

In the following, we prove some technical lemmas which will be needed in chapter 5. None of these lemmas are original and they can be found in most standard texts on the theory of ultrametric spaces, including [21], [22], [52], [59] and [67].

Lemma 1.9.1. *Given an ultrametric space (X, d). For all x, $y \in X$ and all ε, $\delta > 0$ such that $\varepsilon \leq \delta$,*

- either $B_\varepsilon(x) \cap B_\delta(y) = \emptyset$ or $B_\varepsilon(x) \subseteq B_\delta(y)$.

Proof. Suppose $B_\varepsilon(x) \cap B_\delta(y) \neq \emptyset$, then there exists $z \in X$ with $d(x, z) < \varepsilon$ and $d(z, y) < \delta$. Now, we show $B_\varepsilon(x) \subseteq B_\delta(y)$. Let $a \in B_\varepsilon(x)$, then

$$\begin{array}{rcl} d(a,y) & \leq & Max\,\{d(a,z), d(z,y)\} \\ & < & Max\,(\varepsilon, \delta) \\ & = & \delta \end{array}$$

So $d(a,y) < \delta$ and consequently $a \in B_\delta(y)$.

□

Corollary 1.9.2. *Given an ultrametric space (X, d). For all x, $y \in X$ and each $\varepsilon > 0$,*

- either $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$ or $B_\varepsilon(x) = B_\varepsilon(y)$.

Lemma 1.9.3. *Given an ultrametric space (X, d). For each $\varepsilon > 0$, the set*

$$X_\varepsilon := \{B_\varepsilon(x) \mid x \in X\}$$

forms a partition[9] *of X.*

Proof. It is obvious that, $X = \bigcup\limits_{x \in X} B_\varepsilon(x)$. In addition, by corollary 1.9.2, we have either $B_\varepsilon(x) \cap B_\varepsilon(y) = \emptyset$ or $B_\varepsilon(x) = B_\varepsilon(y)$.

□

Lemma 1.9.4. *Let (X, d) be an ultrametric space. Every ball $B_\varepsilon(x)$ where $x \in X$ and $\varepsilon > 0$, is a closed subset of X.*

Proof. Let $a \in X$ and $\varepsilon > 0$.

$$\begin{array}{rcl} (B_\varepsilon(a))^c & = & X - B_\varepsilon(a) \\ & \overset{lemma\ 1.9.3}{=} & (\bigcup\limits_{x \in X} B_\varepsilon(x)) - B_\varepsilon(a) \\ & \overset{corollary\ 1.9.2}{=} & (\bigcup\limits_{x \in X - \{a\}} B_\varepsilon(x)). \end{array}$$

This means $(B_\varepsilon(a))^c$ is open (arbitrary union of open sets is open). Hence, $B_\varepsilon(a)$ is closed.

□

Lemma 1.9.5. *Let (X, d) be an ultrametric space. For every $x, y \in X$ and $\varepsilon > 0$,*

$$d(x,y) < \varepsilon \Longleftrightarrow B_\varepsilon(x) = B_\varepsilon(y).$$

[9] A partition of a set X is a set P of non-empty subsets of X such that every element x in X is exactly in one of these subsets (i.e., P does not contain the empty set; The union of the sets in P is equal to X; The intersection of every two distinct sets in P is empty).

Proof. Given $x, y \in X$ and $\varepsilon > 0$, then

$$\begin{aligned} d(x,y) < \varepsilon \quad &\iff \quad y \in B_\varepsilon(x) \\ &\iff \quad y \in B_\varepsilon(x),\, y \in B_\varepsilon(y) \\ &\iff \quad y \in B_\varepsilon(x) \cap B_\varepsilon(y) \\ &\overset{\text{corollary } 1.9.2}{\iff} \quad B_\varepsilon(x) = B_\varepsilon(y). \end{aligned}$$

□

Lemma 1.9.6. *Let (X, d) be a complete ultrametric space. Given a family of balls $\{B_{2^{-n}}(x_n)\}_{n\in\mathbb{N}}$ such that*

$$B_{2^{-n}}(x_n) = B_{2^{-n}}(x_m) \tag{1.9.1}$$

for all $m \geq n$. Then $\bigcap\limits_{n\in\mathbb{N}} B_{2^{-n}}(x_n)$ is a singleton.

Proof. Given a family of balls $\{B_{2^{-n}}(x_n)\}_{n\in\mathbb{N}}$ that satisfies equation 1.9.1 for all $m \geq n$. Then

$$\begin{aligned} & \forall n \in \mathbb{N}.\, \forall m \geq n.\, B_{2^{-n}}(x_n) = B_{2^{-n}}(x_m) \\ \overset{\text{lemma } 1.9.5}{\iff} \quad & \forall n \in \mathbb{N}.\, \forall m \geq n.\, d(x_n, x_m) < 2^{-n} \\ \iff \quad & \forall n \in \mathbb{N}.\, \forall m_1, m_2 \geq n.\, d(x_n, x_{m_1}) < 2^{-n},\, d(x_n, x_{m_2}) < 2^{-n} \\ \overset{d \text{ is an ultrametric}}{\iff} \quad & \forall n \in \mathbb{N}.\, \forall m_1, m_2 \geq n.\, d(x_{m_1}, x_{m_2}) < 2^{-n}. \end{aligned}$$

Then the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence (see definition 1.8.4). By assumption (X, d) is a complete metric space (see definition 1.8.4), then the Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ has a limit in X. Since every metric space is Hausdorff, this limit is unique (see part (1) in theorem 1.7.12).

□

2. Category Theory

In this chapter we study the most fundamental concepts of category theory, as well as some examples that we will find useful in the remainder of this work. All of these concepts can be found in Adamek et al. [3–5]. See also Awodey [7], Gumm [30], Mac Lane [53] and Mavoungou and Nkuimi-Jugnia [54].

2.1. Categories and subcategory

Definition 2.1.1. (Category) A category $\mathbb{C}$ consists of a class $\mathcal{O}$ of objects and a class Mor of arrows between those objects. Each arrow $f \in Mor$ has a start object called the **domain** of f (in symbol: $dom(f)$) and a target object called **codomain** of f (in symbol: $cod(f)$). If f is an arrow from A to B, we shall denote this as $f : A \longrightarrow B$ (or $A \xrightarrow{f} B$, if necessary). Arrows are often called morphisms. For all objects $A \in \mathcal{O}$ and all pairs of morphisms $f, g \in Mor$ the following axioms are satisfied:

1. for each object A there exists a morphism id_A starting and ending in A. We call id_A the $\mathbb{C}$-identity on A, and
2. morphisms $f : A \longrightarrow B$ and $g : B \longrightarrow C$ can be composed to a morphism $g \circ f : A \longrightarrow C$ so that the following equations hold,
   a) $f \circ id_A = f = id_B \circ f$,
   b) $(h \circ g) \circ f = h \circ (g \circ f)$ whenever $f : A \longrightarrow B$, $g : B \longrightarrow C$ and $h : C \longrightarrow D$,

(see [3], chapter I, section 3).

Remark 2.1.2. If $\mathbb{C}$ is a category, then

1. every object in $\mathbb{C}$ (i.e., each element in $\mathcal{O}$) is called a $\mathbb{C}$-object,
2. the class $\mathcal{O}$ of $\mathbb{C}$-objects is usually denoted by $Ob(\mathbb{C})$,
3. each morphism in $\mathbb{C}$ is called $\mathbb{C}$-morphism,
4. the class Mor of $\mathbb{C}$-morphisms is usually denoted by $Mor(\mathbb{C})$, and
5. for each pair (A, B) of $\mathbb{C}$-objects, we use the notation $Hom_{\mathbb{C}}(A, B)$ for the class of all morphisms in $\mathbb{C}$ with domain A and codomain B.

We should emphasize that the objects in a category do not have to be sets and the morphisms need not be functions.

Example 2.1.3. The following are examples of some categories.

1. EC is the empty category. It has no objects and no arrows.
2. The category Set is the class of all sets with set functions.
3. The category Top is the class of all topological spaces with continuous functions between them.
4. $Stone$ is the category of Stone spaces and continuous functions between them.
5. CUM is the category of complete ultrametric spaces with non-expansive maps (see definition 1.8.17). The category of complete 1-bounded ultrametric spaces with non-expansive maps is shown by CUM^1.
6. The class of all preordered sets[1] with monotone maps (see definition 1.7.1) between them forms a category denoted by $Preord$. The category $Poset$ is the class of all posets [2] with monotone maps.

Definition 2.1.4. A category $\mathbb{C}$ is called small if both $Ob(\mathbb{C})$ and $Mor(\mathbb{C})$ are actually sets and not proper classes.

Example 2.1.5. [7] Here are some simple examples of small categories.

1. The diagram $\circ \longrightarrow \circ$ is a small category with two objects, their identity morphisms, and exactly one non-identity morphism between objects.
2. Every directed set $\mathcal{D} = (D, \geq)$ (see definition 1.7.1) can be regarded as a small category by taking the objects to be the elements of D and taking a unique morphism

$$a \longrightarrow b \text{ if and only if } b \geq a.$$

   The reflexive and transitive condition on $\geq$ ensure that this is indeed a category.

[1] A preordered set is a set P equipped with a binary relation $p \leq q$ that is both reflexive and transitive.
[2] A poset is a preordered set satisfying the additional condition of antisymmetry.

Definition 2.1.6. (Subcategory and full subcategory) A category $\mathbb{S}$ is said to be a subcategory of a category $\mathbb{C}$ provided that the following conditions are satisfied:

1. $Ob(\mathbb{S}) \subseteq Ob(\mathbb{C})$,
2. for each A, $B \in Ob(\mathbb{S})$, we have $Hom_{\mathbb{S}}(A, B) \subseteq Hom_{\mathbb{C}}(A, B)$,
3. For each $\mathbb{S}$-object A, the $\mathbb{C}$-identity on A is the $\mathbb{S}$-identity on A,
4. the composition law in $\mathbb{S}$ is the restriction of the composition law in $\mathbb{C}$ to the morphisms of $\mathbb{S}$.

The category $\mathbb{S}$ is called a full subcategory of $\mathbb{C}$ if, in addition to the conditions above, for each A, $B\in Ob(\mathbb{S})$ we have $Hom_{\mathbb{S}}(A, B) = Hom_{\mathbb{C}}(A, B)$.

Example 2.1.7. The following categories are subcategories of Top.

1. The class of all Hausdorff spaces with continuous functions specifies the subcategory $Haus$ of Top.
2. The class of all compact Hausdorff spaces with continuous functions specifies the subcategory $CHTop$ of Top.
3. The class of all complete ultrametric spaces with non-expansive maps (i.e., CUM) is a subcategory of Top.

Definition 2.1.8. Given a category $\mathbb{C}$, one can form the **dual category** $\mathbb{C}^{op}$ which has the same objects as $\mathbb{C}$ but has an arrow $f^{op} : A \longrightarrow B$ for each arrow $f : B \longrightarrow A$ in $\mathbb{C}$. Composition for $f^{op} : A \longrightarrow B$ and $g^{op} : B \longrightarrow C$ is defined as $g^{op} \circ f^{op} = (f \circ g)^{op}$.

To any purely category theoretical notion (called category theoretical property), we can form its dual notion which is obtained by

- reversing the arrows (replacing domain by codomain and vice versa),
- reversing the order of composition.

Obviously, $\mathbb{C} = (\mathbb{C}^{op})^{op}$ for each category $\mathbb{C}$. Moreover, for each category theoretical property $\mathcal{P}$ true in a category $\mathbb{C}$, its dual property[3] $\mathcal{P}^{op}$ is true in $\mathbb{C}^{op}$. Then we have the important duality principle:

[3]The dual property of a category theoretical property $\mathcal{P}$ can be obtained by reversing arrows and compositions in $\mathcal{P}$. More clearly, let $\mathcal{P}$ be any category theoretical property. We can forms the dual property $\mathcal{P}^{op}$ as follows:

1. Interchange the domain of each morphism in $\mathcal{P}$ with its codomain.
2. Interchange the order of composing morphisms, i.e., replace each occurrence of $f \circ g$ by $g \circ f$.

Theorem 2.1.9. *([3], chapter I, section 3) For each category theoretical property true in all categories, its dual property is also true in all categories.*

Proof. The proof of this theorem follows immediately from the facts that for all categories $\mathbb{C}$ and properties $\mathcal{P}$

1. $\mathbb{C} = (\mathbb{C}^{op})^{op}$, and

2. $\mathcal{P}^{op}(\mathbb{C})$ holds if and only if $\mathcal{P}(\mathbb{C}^{op})$ holds.

□

2.2. Special morphisms

Isomorphisms

Definition 2.2.1. A morphism $f : A \longrightarrow B$ in a category $\mathbb{C}$ is called an **isomorphism** provided that there exists a morphism $g : B \longrightarrow A$ with $g \circ f = id_A$ and $f \circ g = id_B$. The morphism g is called an inverse of f. Sometimes, we denote g by f^{-1}.

Example 2.2.2. ([3], chapter I, section 3) In *Set*, a morphism f with non-empty domain is an isomorphism iff f is bijective (i.e., injective and surjective[4]).

Example 2.2.3. ([3], chapter I, section 3) In *Top*, a morphism f with non-empty domain is an isomorphism iff f is a homeomorphism in *Top*.

Sections, retractions

Definition 2.2.4. In every category $\mathbb{C}$,

1. a morphism $f : A \longrightarrow B$ is called a **section** (or left invertable) provided that there exists some morphism $g : B \longrightarrow A$ such that $g \circ f = id_A$, and

2. a morphism $f : A \longrightarrow B$ is called a **retraction** (or right invertable) provided that there exists some morphism $g : B \longrightarrow A$ such that $f \circ g = id_B$. If there exists such a retraction, then B will be called a retract of A.

[4]A map $f : X \longrightarrow Y$ is said to be

- injective: $\forall x, x' \in X. f(x) = f(x') \implies x = x'$;
- surjective: $\forall y \in Y. \exists x \in X. y = f(x)$.

Since the composition of morphisms is associative (definition 2.1.1, condition 2-b), it is clear that the composition of sections (resp. retractions) is again a section (resp. a retraction).

According to definition 2.2.1, in every category $\mathbb{C}$, a morphism $f : A \longrightarrow B$ is an isomorphism iff it is section and retraction.

In the following, we discuss about these two kind of morphisms in the categories Set and Top.

Example 2.2.5. ([3], chapter II, section 7) In Set, a morphism is a section iff it is an injective function with non-empty domain (if $f : X \longrightarrow Y$ is a section in Set then there exists a morphism $h : Y \longrightarrow X$ such that $h \circ f = id_X$ and then $ker\, f \subseteq ker\,(h \circ f) = \triangle_X$ that means f is injective, and the existence of h tell us that X must be non-empty, conversely, if $f : X \longrightarrow Y$ is an injective map with $X \neq \emptyset$, define $h : Y \longrightarrow X$ as $h(y) := if\,(y \in im\, f)\, f^{-1}(y)\, else\, x_0$ where x_0 is a fixed element in X). On the other hand in Set the retractions are precisely the surjective maps (if $f : X \longrightarrow Y$ is a retraction in Set then there exists a morphism $h : Y \longrightarrow X$ such that $id_Y = f \circ h$ and so $Y = im\,(f \circ h) \subseteq im\, f$ that means f is surjective, conversely, by using the axiom of choice we can show that the surjective functions are retraction).

Remark 2.2.6. Notice that the injective maps are (up to isomorphism) exactly the inclusion of subsets. To see this, let $f : X \longrightarrow Y$ be an injective map. For $X = \emptyset$ this claim is clear, so let $X \neq \emptyset$. Note that $f : X \longrightarrow Y$ can be factored through its image, i.e. written as $A \xrightarrow{f} B = A \xrightarrow{e} im\, f \overset{m}{\hookrightarrow} B$ where $e : A \longrightarrow im\, f$ is the codomain-restriction of f and $m : im\, f \longrightarrow B$ is the inclusion, and so $ker\, e \subseteq ker\, f$. Thus, if f is injective then e is injective too. Therefore e is an isomorphism in Set (because e is also surjective). As a consequence, we obtain that in the category Set, the sections are (up to isomorphisms) precisely the inclusions of the non-empty subsets.

Before studying the notion of retractions in Top, we should get familiar with the concept of topological retractions in Top.

Definition 2.2.7. (Topological retraction) Let X be a topological space and A a subspace of X. Then a continuous map $r : X \longrightarrow A$ is a topological retraction if $r \circ \iota = id_A$, where $\iota : A \longrightarrow X$ is a topological embedding.
Note that, by definition, a topological retraction $r : X \longrightarrow A$ maps X onto A. A subspace A is called a retract of X if such a topological retraction exists. For instance, any topological space retracts to a point in the obvious way (the constant map yields a topological retraction).

Remark 2.2.8. ([3], chapter II, section 7) According to definition 2.2.7, every topological retraction $r : X \longrightarrow A$ is a retraction in Top (in fact the inclusion map $\iota : A \longrightarrow X$ with $r \circ \iota = id_A$ is a right inverse for r).

Example 2.2.9. ([3], chapter II, section 7) The retractions in Top are (up to isomorphism) exactly the topological retractions. Let $f : X \longrightarrow Y$ be a continuous map between topological space. If f is a topological retraction, by remark 2.2.8, f is a retraction in Top. To check the other direction, let $f : X \longrightarrow Y$ be a retraction in Top with g as its right inverse. Then $fog = id_Y$. Hence, g is injective. Since in Set, the injective maps are (up to isomorphism) the inclusion of subsets, g is an inclusion of subsets too. To show that f is a topological retraction, we need to show that for each open subset $U \subseteq Y$, there is an open O in X such that $U = g^{-1}(O)$ (i.e., g is a subspace inclusion). By the equality $fog = id_Y$ we have $g^{-1}(f^{-1}(U)) = U$. Now set $O := f^{-1}(U)$.

Monomorphisms and epimorphisms

Definition 2.2.10. In every category $\mathbb{C}$, a morphism $f : A \longrightarrow B$ is called

- **monomorphism** (in short: mono) if for all morphism g_1, $g_2 : C \longrightarrow A$,
$$f \circ g_1 = f \circ g_2 \Longrightarrow g_1 = g_2.$$
- **epimorphism** (in short: epi) if for all morphism g_1, $g_2 : B \longrightarrow C$,
$$g_1 \circ f = g_2 \circ f \Longrightarrow g_1 = g_2.$$

Remark 2.2.11. Since the composition of morphisms is associative, we can see that the composition of monos (resp. epis) is also mono (resp. epi).

Example 2.2.12. In every category, sections are mono. To see this, let $f : A \longrightarrow B$ be a section with a left inverse $r : B \longrightarrow A$ (i.e., $r \circ f = id_A$). Suppose g_1, $g_2 : C \longrightarrow A$ are two morphisms with $f \circ g_1 = f \circ g_2$. Then
$$g_1 = id_A \circ g_1 = r \circ f \circ g_1 = r \circ f \circ g_2 = id_A \circ g_2 = g_2.$$
Dually, in every category, retractions are epi.

Lemma 2.2.13. *([3], chapter II, section 7) Suppose $g : A \longrightarrow C$, $f : B \longrightarrow C$ and $h : A \longrightarrow B$ are three morphisms in the category $\mathbb{C}$ such that $f \circ h = g$. Then*

1. if f and h are monos (resp. epis) then g is mono (resp. epi),
2. if f and g are monos then h is mono, and
3. if g is epi then f is epi.

Proof. All parts of this lemma follow immediately from definition 2.2.10. □

Lemma 2.2.14. *([3], chapter II, section 7) In the category Set,*

1. *a morphism f is mono iff it is injective, and*
2. *a morphism f is epi iff it is surjective.*

Proof. Let $f : A \longrightarrow B$ be a morphism in *Set*.

1. Suppose f is mono in *Set* and it is not injective. So there are two elements $p, q \in A$ such that $f(p) = f(q)$ but $p \neq q$. Consider the constant maps $p, q : C \longrightarrow A$ (where C is an arbitrary set). Then $f \circ p = f \circ q$ but $p \neq q$. It gives us a contradiction with the assumption. The converse direction is clear.
2. Suppose f is epi in *Set* and it is not surjective. Consider two functions p and q from B to $\{0, 1\}$. The function p maps every point of B to 0 and q maps precisely the points of $f(A)$ to 0. So there are two maps $p, q : B \longrightarrow \{0, 1\}$ such that $p \circ f = q \circ f$ and $p \neq q$. It gives us a contradiction with the assumption. The converse direction is clear.

□

Remark 2.2.15. According to the previous lemma and remark 2.2.6, in the category *Set* the monomorphisms are (up to isomorphism) exactly the inclusions of subsets.

Lemma 2.2.16. *([3], chapter II, section 7) In the category Top,*

1. a morphism f is mono iff it is injective, and
2. a morphism f is epi iff it is surjective.

Proof. Let $f : A \longrightarrow B$ be a continuous map between topological spaces.

1. Suppose f is mono in *Top*. If f is not injective, by a similar way used in part (1) of lemma 2.2.14, we can find two maps $p, q : C \longrightarrow A$ such that $f \circ p = f \circ q$ and $p \neq q$. Equip C by discrete topology. Then p and q are continuous maps such that $f \circ p = f \circ q$ but $p \neq q$. It gives us a contradiction with assumption. The converse direction is clear.

2. Suppose f is epi in Top but it is not surjective. In the proof's of part (2) in lemma 2.2.14, consider the set $\{0, 1\}$ as an indiscrete space. Suppose f is epi in Top and it is not surjective. Then the functions p, $q : B \longrightarrow \{0, 1\}$ are continuous maps such that $p \circ f = q \circ f$ and $p \neq q$. It gives us a contradiction with assumption. The converse direction is clear.

$\square$

Lemma 2.2.17. *Let $\mathbb{C}$ be category , then for each morphism $f : A \longrightarrow B$ in $\mathbb{C}$ the following are equivalent:*

1. *f is an isomorphism.*

2. *f is epi and section.*

3. *f is mono and retraction.*

Proof. Let $f : A \longrightarrow B$ be a fixed morphism in $\mathbb{C}$. ($1 \Longrightarrow 2$) and ($1 \Longrightarrow 3$) are trivial (due to the definition of isomorphisms and example 2.2.12). To prove ($2 \Longrightarrow 1$) let f be epi and section in $\mathbb{C}$. Then f has a left inverse $r : B \longrightarrow A$ (i.e., $r \circ f = id_A$). Now we have

$$f \circ r \circ f = f \circ id_A = f = id_B \circ f.$$

Since f is epi, we conclude that $f \circ r = id_B$, and consequently f is an isomorphism in $\mathbb{C}$ (because f is section and retraction). To Show ($3 \Longrightarrow 1$) we use a similar strategy to ($2 \Longrightarrow 1$) (here if r is a right inverse for f then we obtain that $f \circ r \circ f = f \circ id_A$ and since f is mono we conclude that $r \circ f = id_A$). $\square$

2.3. Diagram lemma

Lemma 2.3.1. *[30] (Diagram lemma in Set)*

1. Let $f : X \longrightarrow Y$ be a surjective map and $g : X \longrightarrow Z$ arbitrary. There is a map $h : Y \longrightarrow Z$ with $h \circ f = g$, if and only if $ker\, f \subseteq ker\, g$. Such an h is uniquely determined.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g"'] & Y \arrow[d, dashed, "h"] \\ & Z \end{tikzcd}$$

2. Let Let $f : Y \longrightarrow X$ be an injective map and $g : Z \longrightarrow X$ arbitrary. There is a map $h : Z \longrightarrow Y$ with $f \circ h = g$, if and only if $im\, f \subseteq im\, g$. Such an h is uniquely determined.

$$\begin{tikzcd} X & Y \arrow[l, "f"'] \\ & Z \arrow[ul, "g"] \arrow[u, dashed, "h"'] \end{tikzcd}$$

Proof. For the first statement, let us assume that $g = h \circ f$, then

$$ker\, f \subseteq ker\,(h \circ f) = ker\, g.$$

So the necessity of the condition is clear. Conversely, since f is surjective, for each $y \in Y$ there is some $x \in X$ such that $y = f(x)$. Then we can easily define a map $h : Y \longrightarrow Z$ by $h(y) := g(x)$ where $x \in f^{-1}(y)$. It is clear that $g = f \circ h$. Since f is epi, h must be unique.
For the second part of the lemma, assuming $g = f \circ h$, we obtain the necessary condition $im\, g = im\, f \circ h \subseteq im\, f$. Conversely, this condition along with the injectivity of f guarantees that

$$h := \{(z, y) \mid g(z) = f(y)\}.$$

Defines a map $h : Z \longrightarrow Y$ with $f \circ h = g$. Uniqueness of h follows as f is mono. □

2.4. Terminal and initial objects

Definition 2.4.1. (Terminal object) An object T in a category $\mathbb{C}$ is called **terminal** provided that for each object A in $\mathbb{C}$ there is exactly one morphism from A to T.

Example 2.4.2. ([3], chapter II, section 7) Every one element set is a terminal object in *Set*. The terminal object in the category *Top* are all one element topological spaces.

Lemma 2.4.3. *([3], chapter II, section 7) Terminal objects, provided they exists, are uniquely determined up to isomorphism.*

Proof. Suppose T_1 and T_2 are terminal objects in the category $\mathbb{C}$. Then we would have precisely one morphism $f_1 : T_1 \longrightarrow T_2$ and also precisely one morphism $f_2 : T_2 \longrightarrow T_1$. From T_1 to T_1 we then have both id_{T_1} and $f_2 \circ f_1$. Hence $f_2 \circ f_1 = id_{T_1}$. Analogously, we obtain $f_1 \circ f_2 = id_{T_2}$. Thus f_1 and f_2 are isomorphisms. □

The dual notion of terminal object is an initial object, that is:

Definition 2.4.4. (Initial object) An object I in a category $\mathbb{C}$ is called **initial** provided that for each object A in $\mathbb{C}$ there is exactly one morphism from I to A.

Example 2.4.5. ([3], chapter II, section 7) In the category Set, the empty set $\emptyset$ is the only initial object. From empty to every set X we have the unique map $\emptyset_X : \emptyset \longrightarrow X$. In the category Top, the empty space is the only initial object.

Lemma 2.4.6. *([3], chapter II, section 7) Initial objects, provided they exist, are uniquely determined up to isomorphism.*

2.5. Products and sums

Definition 2.5.1. (Product) Let $\{A_i\}_{i\in I}$ be a family of objects in a category $\mathbb{C}$. An object P together with morphisms $\{\pi_i : P \longrightarrow A_i\}_{i\in I}$ is called a **product** of the family $\{A_i\}_{i\in I}$ in $\mathbb{C}$, if for each other object Q with morphisms $\{q_i : Q \longrightarrow A_i\}_{i\in I}$ there exists precisely one morphism $h : Q \longrightarrow P$, so that $q_i = \pi_i \circ h$ for all $i \in I$.

$$\begin{tikzcd}
Q \arrow[rr, dotted, "\exists!\, h"] \arrow[dr, dashed, "q_i"'] \arrow[drrr, dashed, "q_j"] & & P \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] & \\
& A_i & & A_j
\end{tikzcd}$$

The morphisms $\{\pi_i : P \longrightarrow A_i\}_{i\in I}$ are called the canonical projections. Some authors call $(Q, \{q_i\}_{i\in I})$ a competitor to the real product. If the product $(P, \{\pi_i\}_{i\in I})$ exists, it is unique up to isomorphisms. The proof for the uniqueness can be obtained by uniqueness of the morphism h. We often denote P by $\prod\limits_{i\in I} A_i$.

From now on we shall write $X \times Y$ for the product of two objects X and Y in a category $\mathbb{C}$.

The dual notion of product is Sum. Its definition is therefore:

Definition 2.5.2. (Sum) Let $\{A_i\}_{i\in I}$ be a family of objects in a category $\mathbb{C}$. An object S together with morphisms $\{e_i : A_i \longrightarrow S\}_{i\in I}$ is called **sum** (coproduct) of $\{A_i\}_{i\in I}$ in $\mathbb{C}$, if for each other object Q with morphisms $\{q_i : A_i \longrightarrow Q\}_{i\in I}$ there exists precisely one morphism $h : S \longrightarrow Q$, so that $q_i = h \circ e_i$ for all $i \in I$.

$$\begin{tikzcd}
& S \arrow[rr, dotted, "\exists!\, h"] & & Q \\
A_i \arrow[ur, "e_i"] \arrow[urrr, dashed, "q_i"] & & A_j \arrow[ul, "e_j"] \arrow[ur, dashed, "q_j"'] &
\end{tikzcd}$$

The morphisms $\{e_i : A_i \longrightarrow S\}_{i\in I}$ are called the canonical injections. Some authors call the pair $(Q, \{q_i\}_{i\in I})$ a competitor to the real sum. If such a sum $(S, \{e_i\}_{i\in I})$ exists, it will be unique up to isomorphism. The proof for the uniqueness can be obtained by uniqueness of the morphism h. We often denote S by $\sum\limits_{i\in I} A_i$.
From now on we shall write $X + Y$ for the sum of two objects X and Y in a category $\mathbb{C}$.

Example 2.5.3. (Product and Sum in Set) ([3], chapter III, section 10) Let $\{X_i\}_{i\in I}$ be a family of sets . The cartesian product $P := \prod\limits_{i\in I} X_i$ with projections $\{\pi_i : P \longrightarrow X_i\}_{i\in I}$ is a product of the family $\{X_i\}_{i\in I}$ in the category *Set*. To see this, let Q with maps $\{q_i : Q \longrightarrow X_i\}_{i\in I}$ be a competitor. The map $h : Q \longrightarrow P$ given by $h(q) := (q_i(q))_{i\in I}$ is a unique map such that $\pi_i \circ h = q_i$ for all $i \in I$.
A sum of the family $\{X_i\}_{i\in I}$ is given by the disjoint union $S := \biguplus\limits_{i\in I} X_i$ which is formally defined as

$$S = \biguplus_{i\in I} X_i := \bigcup_{i\in I} \{(i, x) \mid x \in X_i\} \tag{2.5.1}$$

with maps $\{e_i : X_i \longrightarrow S\}_{i\in I}$ defined by $e_i(x) := (i, x)$ (for each $i \in I$ and each $x \in X_i$). Let Q be a competitor, that is a set with maps $q_i : X_i \longrightarrow Q$, then there is precisely one map $h : S \longrightarrow Q$ with $h \circ e_i = q_i$. It is defined by $h(i, x) := q_i(x)$.

Example 2.5.4. (Product and Sum in CUM^1) [12] In the category CUM^1, the binary products are defined in the natural way:

$$(X_1, d_1) \times (X_2, d_2) = (X_1 \times X_2, d_{X_1\times X_2})$$

where $X_1 \times X_2$ is the cartesian product in *Set* and

$$d_{X_1\times X_2}((x_1, x_2), (y_1, y_2)) = max(d_1(x_1, y_1), d_2(x_2, y_2)).$$

General products are defined in the same way as binary ones, except that the distance function on an infinite product space is given by a supremum instead of a maximum.
The sum of a family $\{(X_i, d_i)\}_{i\in I}$ of objects in CUM^1 is given by $(S,\, d)$ where $S := \biguplus\limits_{i\in I} X_i$ is the disjoint union of the underlying sets $\{X_i\}_{i\in I}$ (see equation 2.5.1) and the distance function d is defined as follows:

$$d(x, y) := \begin{cases} d_i(x, y) & \text{if } x \in X_i \text{ and } y \in X_i \text{ for some } i \in I \\ 1 & \text{otherwise.} \end{cases}$$

For more details see [12] and [63].

Example 2.5.5. (Product in Top) ([3], chapter III, section 10) Let $\{(X_i, \tau_{X_i})\}_{i\in I}$ be a family of topological spaces. The pair $(P, \{\pi_i : P \longrightarrow X_i\}_{i\in I})$ where P is the cartesian product of the underlying sets with the initial topology generated by the projection maps $\{\pi_i : P \longrightarrow X_i\}_{i\in I}$ is known as the product of this family in the category *Top*. Let topological space (Y, τ_Y) along with the continuous morphisms $\{f_i : Y \longrightarrow X_i\}_{i\in I}$ be a competitor $(P, \{\pi_i\}_{i\in I})$. Since the cartesian product P is a product of the underlying sets in *Set*, there is exactly one map $f : Y \longrightarrow P$ such that $\pi_i \circ f = f_i$ for each $i \in I$. Then for each subset $U_i \subseteq X_i$, we have $f^{-1}(\pi_i^{-1}(U_i)) = f_i^{-1}(U_i)$. Notice that by example 1.4.6, the set $\{\pi_i^{-1}(U_i) \mid i \in I, U_i \in \tau_{X_i}\}$ is a subbase for the initial topology on P. So, according to remark 1.3.3, to show the continuity of f, it is enough to check that $f^{-1}(\pi_i^{-1}(U_i))$ is an open subset of Y, where $i \in I$ and U_i is an open subset of X_i. Fix $i \in I$ and choose an open subset $U_i \subseteq X_i$, due to the continuity of f_i, the set $f_i^{-1}(U_i)$ is an open subset of Y. Hence by the equality $f^{-1}(\pi_i^{-1}(U_i)) = f_i^{-1}(U_i)$, we conclude that $f^{-1}(\pi_i^{-1}(U_i))$ is open and consequently $f : Y \longrightarrow P$ is continuous.

Remark 2.5.6. The product of finitely many discrete spaces is a discrete space. More clearly, let $\{(X_i, \tau_{X_i})\}_{i\in I}$ be a family of discrete spaces (i.e., τ_{X_i} is the discrete topology on X_i for each $i \in I$). According to the previous example the product of this family is the cartesian product $\prod\limits_{i\in I} X_i$ of underlying sets with the product topology (see example 1.4.6). Then, the open sets in $\prod\limits_{i\in I} X_i$ are unions (finite or infinite) of sets of the form $\prod\limits_{i\in I} U_i$ where each U_i is open in X_i and $U_i \neq X_i$ for only finitely many $i \in I$. So, if $\mid I \mid < \aleph_0$ (where $\aleph_0$ is the smallest infinite cardinal), then for every element $(x_i)_{i\in I} \in \prod\limits_{i\in I} X_i$ the set $\{(x_i)_{i\in I}\}$ is an open subset of $\prod\limits_{i\in I} X_i$ (because $\{(x_i)_{i\in I}\} = \prod\limits_{i\in I}\{x_i\}$ and for each $i \in I$ the one element set $\{x_i\}$ is an open subset of X_i), and then every subset of $\prod\limits_{i\in I} X_i$ is open (the union of any collection of open sets is open). This means, the set $\prod\limits_{i\in I} X_i$ carries the discrete topology. However, the product of infinitely many discrete spaces each of which has at least two points is not a discrete space (because in this case the set $\{(x_i)_{i\in I}\} = \prod\limits_{i\in I}\{x_i\}$ is not an open subset of $\prod\limits_{i\in I} X_i$).
Clearly, the product of indiscrete spaces is an indiscrete space (because if $\{(X_i, \tau_{X_i})\}_{i\in I}$ is a family of indiscrete spaces, the only open sets in the product of this family are $\emptyset$ and $\prod\limits_{i\in I} X_i$).

Example 2.5.7. (Sum in Top) ([3], chapter III, section 10) Let $\{(X_i, \tau_{X_i})\}_{i\in I}$ be a family of topological spaces. A sum of this family in *Top* is the pair $(S, \{e_i : X_i \longrightarrow S\}_{i\in I})$ where S is the disjoint union of the underlying sets, (i.e, $S = \biguplus\limits_{i\in I} X_i$, see equation 2.5.1) together with the final topology generated by the canonical injection $\{e_i : X_i \longrightarrow S\}_{i\in I}$. To see this, let topological space (Y, τ_Y) with continuous morphisms $\{f_i : X_i \longrightarrow Y\}_{i\in I}$ be a competitor to $(S, \{e_i\}_{i\in I})$. Since the set S with the canonical injections $\{e_i\}_{i\in I}$ is a sum of the underlying sets in the category *Set*, there is a unique map $f : S \longrightarrow Y$ such that $f \circ e_i = f_i$. Then for each subset $U \subseteq Y$, we have $e_i^{-1}(f^{-1}(U)) = f_i^{-1}(U)$. Now, to check the continuity of f, suppose U is an arbitrary open subset of Y. Since for each $i \in I$, the map f_i is continuous, it is concluded that the set $f_i^{-1}(U)$ is an open subset of X_i for each $i \in I$. Then $e_i^{-1}(f^{-1}(U))$ is open in X_i for each $i \in I$ (because $e_i^{-1}(f^{-1}(U)) = f_i^{-1}(U))$). Now since S carries the final topology generated by the canonical injections $\{e_i : X_i \longrightarrow S\}_{i \in I}$, by equation 1.4.3 we obtain that $f^{-1}(U)$ is an open subset of S. So $f : S \longrightarrow Y$ is continuous.

Remark 2.5.8. The sum of discrete spaces is a discrete space. Let $\{(X_i, \tau_{X_i})\}_{i \in I}$ be a family of discrete spaces, and let S together with the morphisms $\{e_i : X_i \longrightarrow S\}_{i \in I}$ be a sum of this family in *Top*. By the previous example S carries the final topology generated by maps $\{e_i\}_{i \in I}$, and then for each $x \in S$ the set $\{x\}$ is an open subset of S (because for each $i \in I$ the set $e_i(\{x\})$ is an open subset of X_i). However, the sum of indiscrete spaces is not an indiscrete space (because for each $i \in I$ the set $e_i(X_i)$ is an open subset of S different from S and $\emptyset$)

Lemma 2.5.9. *Let X_1 and X_2 be subspaces of the topological spaces Y_1 and Y_2, respectively. Then $X_1 \times X_2$ (resp. $X_1 + X_2$) is a subspace of $Y_1 \times Y_2$ (resp. $Y_1 + Y_2$).*

Proof. We know that $X_1 \times X_2 \subseteq Y_1 \times Y_2$ (because $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$). By lemma 1.3.4, it suffices to check that for each $U = U_1 \times U_2$ (where U_1 and U_2 are open subsets of X_1 and X_2, respectively) there are open subsets $O_1 \subseteq Y_1$ and $O_2 \subseteq Y_2$ such that $U = (O_1 \times O_2) \cap (X_1 \times X_2)$. According to the assumption, X_1 and X_2 are subspaces of Y_1 and Y_2, respectively. So

$$\exists O_1 \underset{open}{\subseteq} Y_1 . \exists O_2 \underset{open}{\subseteq} Y_2 . (U_1 = O_1 \cap X_1 \wedge U_2 = O_2 \cap X_2).$$

Hence

$$\begin{aligned} U &= U_1 \times U_2 \\ &= (O_1 \cap X_1) \times (O_2 \cap X_2) \\ &= (O_1 \times O_2) \cap (X_1 \times X_2). \end{aligned}$$

In a similar way, we can prove that $X_1 + X_2$ is a subspace of $Y_1 + Y_2$. $\square$

2.6. Inverse limits

Let $\mathcal{I} = (I, \leq)$ be a directed poset[5] and $\mathbb{C}$ a category. Let $\{A_i\}_{i\in I}$ be a family of objects in $\mathbb{C}$ and suppose we have a family of $\mathbb{C}$-morphisms $\{f_{ij} : A_j \longrightarrow A_i\}_{i\leq j\in I}$, with the following properties:

1. f_{ii} is the identity morphism on A_i (for all $i \in I$),

2. $f_{ik} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$.

Then the pair $(\{A_i\}_{i\in I}, \{f_{ij} : A_j \longrightarrow A_i\}_{i\leq j\in I})$ is called an **inverse system**.
An **inverse limit** of an inverse system $(\{A_i\}_{i\in I}, \{f_{ij} : A_j \longrightarrow A_i\}_{i\leq j\in I})$ in $\mathbb{C}$ is an object A in $\mathbb{C}$ together with a family of morphisms $\{\pi_i : A \longrightarrow A_i\}_{i\in I}$ in $\mathbb{C}$ (called projections) satisfying

- $\pi_i = f_{ij} \circ \pi_j$ for all $i \leq j$ and

- for each $(Q, \{q_i : Q \longrightarrow A_i\}_{i\in I})$ such that $q_i = f_{ij} \circ q_j$ for all $i \leq j$, there is precisely one morphism $h : Q \longrightarrow A$ such that $q_i = \pi_i \circ h$ for all $i \in I$, i.e., the following diagram commutes.

$$
\begin{tikzcd}
& Q \arrow[ddl, "q_j"'] \arrow[d, dashed, "h"] \arrow[ddr, "q_i"] & \\
& A \arrow[dl, "\pi_j"] \arrow[dr, "\pi_i"'] & \\
A_j \arrow[rr, "f_{ij}"'] & & A_i
\end{tikzcd}
$$

Some authors call $(Q, \{q_i : Q \longrightarrow A_i\}_{i\in I})$ a competitor to the real inverse limit. In some categories, the inverse limit does not exist. If it exists, it is unique up to isomorphism, (i.e., given any other inverse limit $(A', \{\pi'_i\}_{i\in I})$, there exists a unique isomorphism $\varepsilon : A' \longrightarrow A$ such that $\pi_i \circ \varepsilon = \pi'_i$ for all $i \in I$) and is denoted by $\varprojlim A_i$.

Remark 2.6.1. The notion of inverse system makes sense even if the poset I is not assumed to be directed. However many important results only hold when I is directed. As an example, in lemma 2.6.7, we will prove that if $(\{(X_i, \tau_i)\}_{i\in I}, \{f_{ij} : X_j \longrightarrow X_i\}_{i\leq j\in I})$ is an inverse system of non-empty compact Hausdorff spaces over the directed set I, then its inverse limit is non-empty. The fact that the indexing set I is directed is essential for this proof.

[5] A directed poset is a directed set $\mathcal{I} = (I, \leq)$ in which the binary relation $\leq$ is antisymmetry (i.e., for each $i, j \in I$ if $i \leq j \leq i$ then $i = j$).

Example 2.6.2. (Inverse limits in Set) Let $(\{X_i\}_{i\in I}, \{f_{ij} : X_j \longrightarrow X_i\}_{i\leq j\in I})$ be an inverse system over I in the category *Set*. An inverse limit of this system (i.e., $\varprojlim X_i$) is a pair $(X, \{\pi_i\}_{i\in I})$ where X is a set as follows:

$$X := \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid f_{ij}(x_j) = x_i \text{ for all } i \leq j\} \tag{2.6.1}$$

and $\{\pi_i : X \longrightarrow X_i\}_{i\in I}$ are natural projections (for each $i \in I$, the map $\pi_i : X \longrightarrow X_i$ pick out the i^{th} component of the elements in X). To prove this, suppose for a set Q with morphisms $\{q_i : Q \longrightarrow X_i\}_{i\in I}$ we have $q_i = f_{ij} \circ q_j$ for all $i \leq j$. Define $h : Q \longrightarrow \prod_{i\in I} X_i$ by $h(q) := (q_i(q))_{i\in I}$ for each $q \in Q$. It is easy to see that h is unique and $\pi_i \circ h = q_i$ for all $i \in I$ (see [3], part III, section 11).

Remark 2.6.3. If $(\{X_n\}_{n\in\mathbb{N}}, \{f_{kn} : X_n \longrightarrow X_k\}_{k\leq n\in\mathbb{N}})$ is an inverse system in the category *Set* over $(\mathbb{N}, \leq)$ (the natural numbers $\mathbb{N}$ with the ordinary order $\leq$), we usually picture it as follows:

$$X_0 \xleftarrow{f_{01}} X_1 \xleftarrow{f_{12}} X_2 \longleftarrow \cdots$$

Remark 2.6.4. **(Inverse limit as complete ultrametric space)** Notice that an inverse limit of an inverse system in *Set* over $(\mathbb{N}, \leq)$ (the natural numbers $\mathbb{N}$ with the ordinary order $\leq$) can be considered as a complete 1-bounded ultrametric space. More exactly, let $(\{X_n\}_{n\in\mathbb{N}}, \{f_{kn} : X_n \longrightarrow X_k\}_{k\leq n\in\mathbb{N}})$ be an inverse system in the category *Set* over $(\mathbb{N}, \leq)$. By the previous example, its inverse limit is the set

$$X := \{(x_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} X_n \mid f_{kn}(x_n) = x_k \text{ for all } k \leq n\}$$

with the natural projections $\{\pi_n : X \longrightarrow X_n\}_{n\in\mathbb{N}}$. Consider the map $d_X : X \times X \to [0, 1]$ as the distance function d defined in equation 1.8.2, i.e. for each $p,\ q \in X$,

$$d_X(p,\ q) := \begin{cases} 0 & p = q \\ 2^{-m(p,q)} & otherwise \end{cases}$$

where $m(p, q) := Inf\,\{k \in \mathbb{N} \mid p(k) \neq q(k)\}$ (here $p(k)$ and $q(k)$ are $\pi_k(p)$ and $\pi_k(q)$, respectively). By lemma 1.8.14 d_X is an ultrametric and consequently the set X with the map d_X forms an 1-bounded ultrametric space. Moreover, by the same strategy used in lemma 1.8.16 we can prove that $(X,\ d_X)$ is a complete ultrametric space. In fact, we can see that every Cauchy sequence $(q_n)_{n\in\mathbb{N}}$ in X converges to the sequence $q := (q_{M_n}(n))_{n\in\mathbb{N}}$, where the natural number M_n is the smallest element in $\mathbb{N}$ satisfying equation 1.8.4. Note that q is an element in X. To see this we need to check that $f_{kn}(q_{M_n}(n)) = q_{M_k}(k)$ for all $k \leq n$. So, let $n \in \mathbb{N}$ be a fixed element. Since $q_{M_n} \in X$, we have $f_{kn}(q_{M_n}(n)) = q_{M_n}(k)$ for all $k \leq n$. Also due to equation 1.8.8 in lemma 1.8.16 we have $q_{M_n}(k) = q(k) = q_{M_k}(k)$ for all $k \leq n$.

Example 2.6.5. (Inverse limits in Top) If $(\{(X_i, \tau_i)\}_{i\in I}, \{f_{ij} : X_j \longrightarrow X_i\}_{i\leq j\in I})$ is an inverse system in the category *Top*, then the set X mentioned in equation 2.6.1 (X is a particular subset of the cartesian product $\prod_{i\in I} X_i$ of the underlying sets) with the initial topology generated by the projection maps $\{\pi_i : X \longrightarrow X_i \mid i \in I\}$ is known as an inverse limit of this system in *Top*. One can check this issue by the same way done for the product in *Top* (see example 2.5.5).
Then one can say that the inverse limits in the category *Top* are given by placing the initial topology generated by projections on the underlying set-theoretic inverse limit.

Lemma 2.6.6. *[60] If $(\{(X_i, \tau_i)\}_{i\in I}, \{f_{ij} : X_j \longrightarrow X_i\}_{i\leq j\in I})$ is an inverse system of Hausdorff topological spaces, then its inverse limit (i.e., $\varprojlim (X_i, \tau_i)$) is a closed in $\prod_{i\in I} X_i$ (with respect to the product topology).*

Proof. Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i - (\varprojlim (X_i, \tau_i))$. Then there exist $r, s \in I$ with $s \leq r$ and $f_{sr}(x_r) \neq x_s$. Choose open disjoint neighborhoods U and V of $f_{sr}(x_r)$ and x_s in X_s, respectively. Let U' be an open neighborhood of x_r in X_r, such that $f_{sr}(U') \subseteq U$. Consider the open subset $W = \prod_{i\in I} V_i$ of $\prod_{i\in I} X_i$ where $V_r = U'$, $V_s = V$ and $V_i = X_i$ for $i \neq r, s$. Then W is a open neighborhood of $(x_i)_{i\in I}$ in $\prod_{i\in I} X_i$, disjoint from $\varprojlim (X_i, \tau_i)$. This shows that $\varprojlim (X_i, \tau_i)$ is closed. □

Lemma 2.6.7. *[60] Let $(\{(X_i, \tau_i)\}_{i\in I}, \{f_{ij} : X_j \longrightarrow X_i\}_{i\leq j\in I})$ be an inverse system of non-empty compact Hausdorff spaces over the directed set I. Then its inverse limit (i.e., $\varprojlim (X_i, \tau_i)$) is non-empty. In particular, the inverse limit of an inverse system of non-empty finite sets is non-empty.*

Proof. For each $j \in I$, define a subset Y_j of $\prod_{i\in I} X_i$ to consist of those $(x_i)_{i\in I}$ with the property $f_{ij}(x_j) = x_i$ whenever $i \leq j$. Using the axiom of choice and an argument similar to the one used in lemma 2.6.6, one easily checks that each Y_j is a non-empty closed subset of $\prod_{i\in I} X_i$. Observe that if $j \leq j'$, then $Y_{j'} \subseteq Y_j$. Now, since I is a directed poset, it follows that the collection of subsets $\{Y_j \mid j \in I\}$ has the finite intersection property (i.e., any intersection of finitely many Y_i is nonempty). Then since $\prod_{i\in I} X_i$ is a compact space (see Tychonoff's theorem), we conclude that $\bigcap_{j\in I} Y_j$ is nonempty (by theorem 1.6.4). Notice that

$$\bigcap_{j\in I} Y_j = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid f_{ij}(x_j) = x_i \text{ for all } i \leq j\}.$$

Then $\varprojlim (X_i, \tau_i) = \bigcap_{j\in I} Y_j$ (see example 2.6.5), and we obtain the desired result. □

2.7. Equalizers and coequalizers

The notions of equalizers and coequalizers are only meaningful for morphisms with same domain and codomain (or parallel morphisms). The category theoretical definitions are formulated for a whole family of parallel morphisms:

Definition 2.7.1. (Equalizer) Let $\{f_i : A \longrightarrow B\}_{i \in I}$ be a family of parallel morphisms. A morphism $e : E \longrightarrow A$ is an **equalizer** of the family $\{f_i\}_{i \in I}$, provided that

- $f_i \circ e = f_j \circ e$ for all $i, j \in I$, and
- for each object Q and for each morphism $q : Q \longrightarrow A$ such that $f_i \circ q = f_j \circ q$ holds for all $i, j \in I$, there is precisely one $h : Q \longrightarrow E$ with $q = e \circ h$.

$$
\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f_i"] \arrow[r, shift right, "f_j"'] & B \\
Q \arrow[u, dashed, "\exists!\, h"] \arrow[ru, "q"'] & &
\end{tikzcd}
$$

Some authors call $q : Q \longrightarrow A$ a competitor to the real equalizer. If an equalizer for the family $\{f_i : A \longrightarrow B\}_{i \in I}$ exists, then it is unique up to isomorphisms. It is easy to see that equalizers must be mono, for given morphisms h_1, h_2 with $e \circ h_1 = e \circ h_2$ we get a conflict with the uniqueness condition of the above definition (i.e, uniqueness of h), unless $h_1 = h_2$.

Definition 2.7.2. (Coequalizer) Let $\{f_i : A \longrightarrow B\}_{i \in I}$ be a family of parallel morphisms. A morphism $g : B \longrightarrow C$ is a **coequalizer** of the family $\{f_i\}_{i \in I}$, provided that

- $g \circ f_i = g \circ f_j$ for all $i, j \in I$, and
- for each object Q and for each morphism $q : B \longrightarrow Q$ such that $q \circ f_i = q \circ f_j$ holds for all $i, j \in I$, there is precisely one $h : C \longrightarrow Q$ with $q = h \circ g$.

$$
\begin{tikzcd}
A \arrow[r, shift left, "f_i"] \arrow[r, shift right, "f_j"'] & B \arrow[r, "g"] \arrow[rd, "q"'] & C \arrow[d, dotted, "\exists!\, h"] \\
& & Q
\end{tikzcd}
$$

Some authors call $q : B \longrightarrow Q$ as competitor to the real coequalizer. If a coequalizer for the family $\{f_i : A \longrightarrow B\}_{i \in I}$ exists, then it is unique up to isomorphisms. By invoking duality, we obtain that in any category $\mathbb{C}$, coequalizers are epi.

Example 2.7.3. (Equalizer and coequalizer in Set) ([3], chapter II, section 7) Given a family of maps $\{f_i : X \longrightarrow Y\}_{i\in I}$ in *Set*. Their equalizer is given by the inclusion map $\iota : E \longrightarrow X$, where $E = \{x \in X \mid \forall i, j \in I, f_i(x) = f_j(x)\}$. To construct coequalizer of the family $\{f_i : X \longrightarrow Y\}_{i\in I}$, let θ be the equivalence relation on Y generated by the set of pairs

$$R := \{(f_i(x), f_j(x)) \mid x \in X, i, j \in I\}. \tag{2.7.1}$$

The factor projection map $\pi_\theta : Y \longrightarrow {}^{Y}\!/\!{}_{\theta}$ defined as $\pi_\theta(y) = [y]_\theta$ is a coequalizer of the $\{f_i\}_{i\in I}$. Obviously, for all $i, j \in I$ we have $\pi_\theta \circ f_i = \pi_\theta \circ f_j$. Let $q : Y \longrightarrow Z$ be a map such that $q \circ f_i = q \circ f_j$ for all $i, j \in I$. Consequently $\theta \subseteq ker\, q$. So by 2.3.1, there is a unique map $h : {}^{Y}\!/\!{}_{\theta} \longrightarrow Z$ such that $h \circ \pi_\theta = q$.

Example 2.7.4. (Equalizer and coequalizer in Top) ([3], chapter II, section 7) An equalizer of a family of parallel morphisms in *Top* is given by considering the subspace topology on the set theoretical equalizer. More precisely, let $\{f_i : X \longrightarrow Y\}_{i\in I}$ be a family of continuous maps between topological spaces. Their equalizer is the inclusion map $\iota : E \longrightarrow X$, where E is the set $\{x \in X \mid \forall i, j \in I, f_i(x) = f_j(x)\}$ with the subspace topology generated by ι. To see this, let $e : Q \longrightarrow X$ be a continuous map such that $f_i \circ e = f_j \circ e$ $(i, j \in I)$, then since ι is an equalizer of $\{f_i : X \longrightarrow Y\}_{i\in I}$ in *Set*, there is a unique map $f : Q \longrightarrow E$ such that $\iota \circ f = e$. To prove the continuity of f, it suffices to show that for each open subset U of X, the set $f^{-1}(U \cap E) = f^{-1}(\iota^{-1}(U))$ is open in Q (because E is a subspace of X). Notice that by the continuity of e we know that $e^{-1}(U)$ (where U is a subset of X) is open in Q. So from $f^{-1}(\iota^{-1}(U)) = e^{-1}(U)$, we obtain that for each open subset U of X, the set $f^{-1}(\iota^{-1}(U))$ is open in Q.
Dually, the coequalizer is defined by considering the final topology on the *Set* theoretic coequalizer. To construct coequalizer of a family of continuous maps $\{f_i : X \longrightarrow Y\}_{i\in I}$, let θ be the equivalence relation on Y generated by the set of pairs defined in equation 2.7.1. The factor projection map $\pi_\theta : Y \longrightarrow {}^{Y}\!/\!{}_{\theta}$ defined as $\pi_\theta(y) = [y]_\theta$ is a continuous map by equipping ${}^{Y}\!/\!{}_{\theta}$ with the final topology generated by π_θ. Obviously, for all $i, j \in I$ we have $\pi_\theta \circ f_i = \pi_\theta \circ f_j$. If $q : Y \longrightarrow Z$ is a continuous map with $q \circ f_i = q \circ f_j$, then $\theta \subseteq ker\, q$. So by the diagram lemma, there is a unique map $h : {}^{Y}\!/\!{}_{\theta} \longrightarrow Z$ such that $h \circ \pi_\theta = q$. Now to check the continuity of h, suppose U is an open subset of Z. Since q is continuous, it is concluded that the set $q^{-1}(U)$ is an open subset of Y. Then $\pi_\theta^{-1}(h^{-1}(U))$ is open in Y (by $\pi_\theta^{-1}(h^{-1}(U)) = q^{-1}(U)$). Now, since ${}^{Y}\!/\!{}_{\theta}$ carries the final topology generated by π_θ, we obtain that $h^{-1}(U)$ is open in ${}^{Y}\!/\!{}_{\theta}$. So $h : {}^{Y}\!/\!{}_{\theta} \longrightarrow Z$ is continuous and consequently is a coequalizer of the family $\{f_i : X \longrightarrow Y\}_{i\in I}$ in *Top*.

2.8. Pullbacks and Pushouts

Definition 2.8.1. (Pullback) Let $\{f_i : A_i \longrightarrow B\}_{i\in I}$ be a sink. An object P together with a family of morphisms $\{p_i : P \longrightarrow A_i\}_{i\in I}$ is called a **pullback** of the family $\{f_i\}_{i\in I}$, provided that

- $\forall i, j \in I.\ f_i \circ p_i = f_j \circ p_j$, and
- to each other object Q, with source $\{q_i : Q \longrightarrow A_i\}_{i \in I}$ satisfying $f_i \circ q_i = f_j \circ q_j$ (for all $i, j \in I$), there exists a unique morphism $h : Q \longrightarrow P$ with $p_i \circ h = q_i$ for all $i \in I$.

$$\begin{tikzcd}
 & & A_i \arrow[dr, "f_i"] & \\
Q \arrow[urr, dashed, "q_i"] \arrow[r, dotted, "\exists!\, h"] \arrow[drr, dashed, "q_j"'] & P \arrow[ur, "p_i"'] \arrow[dr, "p_j"] & & B \\
 & & A_j \arrow[ur, "f_j"'] &
\end{tikzcd}$$

Some authors call $(Q, \{q_i : Q \longrightarrow A_i\}_{i \in I})$ as competitor to the real pullback. If a pullback for a sink $(f_i : A_i \longrightarrow B)_{i \in I}$ exists, then it is unique up to isomorphisms.

Definition 2.8.2. (Pushout) Let $\{f_i : A \longrightarrow B_i\}_{i \in I}$ be a source. An object P together with a family of morphisms $\{p_i : B_i \longrightarrow P\}_{i \in I}$ is called a **pushout** of the family $\{f_i\}_{i \in I}$, provided that

- $\forall i, j \in I.\ p_i \circ f_i = p_j \circ f_j$, and
- to each other object Q with sink $\{q_i : B_i \longrightarrow Q\}_{i \in I}$ satisfying $q_i \circ f_i = q_j \circ f_j$ for all $i, j \in I$, there exists a unique morphism $h : P \longrightarrow Q$ with $h \circ p_i = q_i$ for all $i \in I$.

$$\begin{tikzcd}
 & B_i \arrow[dr, "p_i"'] \arrow[drr, dashed, "q_i"] & & \\
A \arrow[ur, "f_i"] \arrow[dr, "f_j"'] & & P \arrow[r, dotted, "\exists!\, h"] & Q \\
 & B_j \arrow[ur, "p_j"] \arrow[urr, dashed, "q_j"'] & &
\end{tikzcd}$$

Some authors call $(Q, \{q_i : B_i \longrightarrow Q\}_{i\in I})$ as competitor to the real pushout. If a pushout for a source $\{f_i : A \longrightarrow B_i\}_{i\in I}$ exists, then it is unique up to isomorphisms.

Example 2.8.3. (Pullbacks and Pushouts in Set) ([3], chapter III, section 11) Let $\{f_i : X_i \longrightarrow Y\}_{i\in I}$ be a sink in *Set*. Let $P := \prod\limits_{i\in I} X_i$ with the projections $\pi_i : P \longrightarrow X_i$ be the product of the family $\{X_i\}_{i\in I}$. The pullback of the family $\{f_i\}_{i\in I}$ is constructed by the equalizer of the maps $\{f_i \circ \pi_i\}_{i\in I}$ that is the inclusion map $\iota : Pb \longrightarrow P$ where

$$Pb := \{(x_i)_{i\in I} \in P \mid \forall i,\, j \in I,\, f_i(x_i) = f_j(x_j)\} \tag{2.8.1}$$

More clearly, the set Pb with the family of morphisms $\{\pi_i \circ \iota\}_{i\in I}$ is a pullback of the family $\{f_i\}_{i\in I}$. To see this, let $\{q_i : Q \longrightarrow X_i\}_{i\in I}$ be a source satisfying $f_i \circ q_i = f_j \circ q_j$ for each $i,\, j \in I$. Then Q with the family $\{q_i\}_{i\in I}$ is a competitor for the product P. So there is a unique morphism $k : Q \longrightarrow P$ such that $\pi_i \circ k = q_i$ for each $i \in I$. Then for each $i \in I$ we have $f_i \circ \pi_i \circ k = f_i \circ q_i$. So for each $i,\, j \in I$ we conclude that $f_i \circ \pi_i \circ k = f_j \circ \pi_j \circ k$ (because $f_i \circ q_i = f_j \circ q_j$). Therefore, $k : Q \longrightarrow P$ is a competitor for the equalizer $\iota : Pb \longrightarrow P$ in *Set*. Then there is a unique map $h : Q \longrightarrow Pb$ such that $\iota \circ h = k$. So h is a unique map with $\pi_i \circ \iota \circ h = \pi_i \circ k = q_i$ for each $i \in I$.
Dually, if $\{f_i : X \longrightarrow Y_i\}_{i\in I}$ is a family of morphisms in *Set*, then their pushout is obtained as the coequalizer of the family $\{e_i \circ f_i\}_{i\in I}$, where $\{e_i : Y_i \longrightarrow S\}_{i\in I}$ are the canonical injections to the sum S of the family $\{Y_i\}_{i\in I}$.

Example 2.8.4. (Pullback and pushouts in Top) A pullback of a sink in *Top* is given by considering the initial topology on the set theoretical pullback. More clearly, let $\{f_i : X_i \longrightarrow Y\}_{i\in I}$ be a family of continuous maps. Let the pair $(X, \{p_i : X \longrightarrow X_i\}_{i\in I})$ be a pullback of the underlying sink of $\{f_i\}_{i\in I}$ in *Set* (the sink obtained by forgetting topologies on domains and codomain). By the previous example $(X, \{p_i : X \longrightarrow X_i\}_{i\in I})$ exists. We claim that the set X with the initial topology generated by the source $\{p_i\}_{i\in I}$ is a pullback of the sink $\{f_i\}_{i\in I}$ in *Top*. Let the topological space Q with the continuous morphisms $\{q_i : Q \longrightarrow X_i\}_{i\in I}$ be a competitor. Then $f_i \circ q_i = f_j \circ q_j$ for each $i,\, j \in I$. Since $(X, \{p_i : X \longrightarrow X_i\}_{i\in I})$ is a pullback of the underlying sink in *Set*, there is exactly a unique function $k : Q \longrightarrow X$ such that $p_i \circ k = q_i$ for each $i \in I$. It remains to shoe that k is continuous. Notice that by example 1.4.6, the set $\{p_i^{-1}(U_i) \mid i \in I,\, U_i \underset{open}{\subseteq} X_i\}$ is a subbase for the initial topology on X. So, according to remark 1.3.3, to show the continuity of k, it is enough to check that $k^{-1}(p_i^{-1}(U_i))$ is an open subset of Q, where $i \in I$ and U_i is an open subset of X_i. Due to the continuity of q_i, for each open subset $U_i \subseteq X_i$, the set $q_i^{-1}(U_i)$ is an open subset of Q. Hence by the equality $k^{-1}(p_i^{-1}(U_i)) = q_i^{-1}(U_i)$ $(i \in I)$, we conclude that $k : Q \longrightarrow X$ is continuous.
Dually, the pushout is defined by considering the final topology on the *Set* theoretic pushout (see also [3], chapter III, section 11).

2.9. Regular morphisms

Definition 2.9.1. In every category $\mathbb{C}$, a morphism is called **regular monomorphism** (or regular mono) if it is an equalizer of a pair of parallel morphisms and **regular epimorphism** (or regular epi) if it is a coequalizer.

Example 2.9.2. ([3], chapter II, section 7) In *Set*,

1. the regular monomorphisms are the injective functions, i.e. they are (up to isomorphism) exactly the inclusions of subsets, and

2. the regular epimorphisms are the surjective functions.

Lemma 2.9.3. *([3], chapter II, section 7) In Top,*

1. the regular monomorphisms are (up to isomorphism) precisely the topological embeddings (subspace inclusion).

2. the regular epimorphisms are (up to isomorphism) precisely the quotient maps (surjective and continuous maps onto spaces with the final topology).

Proof. Suppose $f : A \longrightarrow B$ is a continuous map.

1. Let $f : A \longrightarrow B$ be a topological embedding. Suppose $\{0, 1\}$ carries indiscrete topology. Consider function g_1 and g_2 from B to $\{0, 1\}$ such that g_1 maps every point of B to 1 and g_2 maps just points of A to 1 and the rest to 0. Then f is an equalizer of g_1 and g_2 (because A is the equalizer of g_1 and g_2 in *Set* with the initial topology on its domain). So every topological embedding in *Top* is a regular mono. Conversely, if $f : A \longrightarrow B$ is a regular mono in *Top*, according to example 2.7.4 it is up to isomorphisms an topological embedding.

2. Let $f : A \longrightarrow B$ be a surjective and continuous map in which the topological space B carries the final topology generated by f. Provide

$$ker\, f = \{(a, a') \in A \times A \mid f(a) = f(a')\}$$

with the initial topology generated by the projection maps from $ker\, f$ to A. Then the projection maps from $\pi_1, \pi_2 : ker\, f \longrightarrow A$ are continuous maps such that $f \circ \pi_1 = f \circ \pi_2$. On the other hand, according to example 2.7.4, the quotient map $\pi_\theta : A \longrightarrow A/_{ker\, f}$ is a coequalizer of π_1, π_2. Then there is a unique continuous morphism $h : A/_{ker\, f} \longrightarrow B$ such that $h \circ \pi_\theta = f$. It is easy see that h is a bijective map. Then h is an isomorphism in *Top*. The other direction obtains by the construction of coequalizers on *Top* (see example 2.7.4).

□

Remark 2.9.4. Notice that in every category, regular monos are monos (because equalizers are mono) and regular epis are epis (because coequalizers are epi). However, in general not every mono is a regular mono and not every epi is a regular epi. For example the identity function $id_A : A_D \longrightarrow A_I$ is both mono and epi in Top whereas it is neither regular mono nor regular epi.

Remark 2.9.5. In every category, each retraction is a regular epimorphism (let the morphism $f : A \longrightarrow B$ be a retraction with a right inverse $r : B \longrightarrow A$, then f is a coequalizer of $r \circ f$ and id_A). Dually, in every category, sections are regular monomorphisms.

2.10. Mono sources and epi sinks

Recall that a source is a family of morphisms with common domain and a sink is a family of morphisms with common codomain.

Definition 2.10.1. (**Mono source**) A source $\{f_i : A \longrightarrow B_i\}_{i\in I}$ in a category $\mathbb{C}$ is called **mono source** provided that it can be cancelled from the left, i.e. for any pair $C \overset{s}{\underset{r}{\rightrightarrows}} A$ of morphisms, if $f_i \circ r = f_i \circ s$ for each $i \in I$, then we have $r = s$.

Example 2.10.2. ([3], chapter III, section 10) Let $\mathbb{C}$ be a category with products, and let $\{A_i\}_{i\in I}$ be a family of objects in $\mathbb{C}$. If P with the family $\{\pi_i : P \longrightarrow A_i\}_{i\in I}$ is a product of $\{A_i\}_{i\in I}$ in $\mathbb{C}$, then $\{\pi_i\}_{i\in I}$ is a mono source in $\mathbb{C}$.

Remark 2.10.3. ([3], chapter III, section 10) In Set and Top, mono sources are precisely point-separating sources, i.e., a source $\{f_i : A \longrightarrow B_i\}_{i\in I}$ is a mono source if for every two different elements a and a' in A there exists some $i \in I$ with $f_i(a) \neq f_i(a')$.

Lemma 2.10.4. *([3], chapter III, section 10) Given mono sources* $\{f_i : A \longrightarrow B_i\}_{i\in I}$ *and* $\{f'_i : A' \longrightarrow B_i\}_{i\in I}$ *and morphisms* $f : A \longrightarrow A'$ *and* $g : S \longrightarrow A$ *in a category* $\mathbb{C}$*, then*

1. if $g : S \longrightarrow A$ is mono, then $\{f_i \circ g\}_{i\in I}$ is a mono source, and
2. if $f'_i \circ f = f_i$ (for each $i \in I$), then f is mono.

Definition 2.10.5. (**Epi sink**) A sink $\{f_i : A_i \longrightarrow B\}_{i\in I}$ in a category $\mathbb{C}$ is called **epi sink** provided that it can be cancelled from the right, i.e. for any pair $B \underset{r}{\overset{s}{\rightrightarrows}} D$ of morphisms, if $r \circ f_i = s \circ f_i$ for each $i \in I$, then we have $r = s$.

Example 2.10.6. ([3], chapter III, section 10) Let $\mathbb{C}$ be a category with sums, and let $\{A_i\}_{i\in I}$ be a family of objects in $\mathbb{C}$. If S with the family $\{e_i : A_i \longrightarrow S\}_{i\in I}$ is a sum of the family $\{A_i\}_{i\in I}$ in $\mathbb{C}$, then $\{e_i\}_{i\in I}$ is an epi sink in $\mathbb{C}$.

Remark 2.10.7. ([3], chapter III, section 10) In Set and Top, epi sinks are precisely jointly surjective sinks, i.e. a sink $\{f_i : A_i \longrightarrow B\}_{i\in I}$ is an epi sink if and only if $B = \bigcup\limits_{i\in I} f_i(A_i)$.

Lemma 2.10.8. *([3], chapter III, section 10) Given epi sinks $\{f_i : A_i \longrightarrow B\}_{i\in I}$ and $\{f'_i : A_i \longrightarrow B'\}_{i\in I}$ and morphisms $f : B \longrightarrow B'$ and $g : B \longrightarrow C$ in a category $\mathbb{C}$, then*

1. if $g : B \longrightarrow C$ is epi, then $\{g \circ f_i\}_{i\in I}$ is an epi sink, and
2. if $f_i \circ f = f'_i$ (for each $i \in I$), then f is epi.

2.11. Factorization systems

Before starting, we should mention that all concepts in this subsection are originally from section 14 of chapter IV in [3].

Definition 2.11.1. Let $\mathcal{E}$ and $\mathcal{M}$ be classes of morphisms in a category $\mathbb{C}$. Then $(\mathcal{E}, \mathcal{M})$ is called a **factorization system** for morphisms in $\mathbb{C}$ and $\mathbb{C}$ is called **$(\mathcal{E}, \mathcal{M})$-category**, provided that:

1. each of $\mathcal{E}$ and $\mathcal{M}$ is closed under composition,
2. $\mathbb{C}$ has $(\mathcal{E}, \mathcal{M})$ -factorization of morphisms, i.e. each morphism f in $\mathbb{C}$ has a factorization $f = m \circ e$, with $e \in E$ and $m \in \mathcal{M}$, and
3. $\mathbb{C}$ has the unique $(\mathcal{E}, \mathcal{M})$ -diagonalization property, i.e. for each commutative square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, there exists a unique diagonal, i.e. a unique morphism d such that the following diagram commutes,

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \arrow[dl, "d"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

(i.e., $d \circ e = f$ and $m \circ d = g$). In this case, we call the morphisms e and m **orthogonal** and we write $e \perp m$.
The morphisms in $\mathcal{E}$ are called $\mathcal{E}$-morphisms and those in $\mathcal{M}$ are called $\mathcal{M}$-morphisms.

Remark 2.11.2. Let f can be factored as $f = m \circ e$ where $e \in \mathcal{E}$ and $m \in \mathcal{M}$. Then $m \circ e$ is said to be an $(\mathcal{E}, \mathcal{M})$-factorization of f.

Theorem 2.11.3. *([3], chapter IV, section 14) Let $\mathbb{C}$ be an $(\mathcal{E}, \mathcal{M})$-category. The following facts on factorization systems are well known and we omit their proofs.*

1. If a morphism f can be factored as $f = m \circ e$ where $e \in \mathcal{E}$ and $m \in \mathcal{M}$, then this factorization is unique up to isomorphism.

2. $\mathcal{E} \cap \mathcal{M} = Iso$, where Iso is the class of isomorphisms in the corresponding category.

3. The classes $\mathcal{E}$ and $\mathcal{M}$ are determined one by the other, in the sense that $\mathcal{E}$ is exactly the class of morphisms which are orthogonal to every $\mathcal{M}$-morphisms and $\mathcal{M}$ is exactly the class of morphisms which are orthogonal to every $\mathcal{E}$-morphism.

4. If $f \circ g \in \mathcal{M}$ and $f \in \mathcal{M}$, then $g \in \mathcal{M}$.

Lemma 2.11.4. *[31] In every category $\mathbb{C}$,*

1. if e is epi and m regular mono, alternatively, if e is regular epi and m mono then $e \perp m$, and

2. if a morphism f can be factored as $f = m \circ e$ where e is epi (resp. regular epi) and m is regular mono (resp. mono), then this factorization is unique up to isomorphism.

Proof. (1) Assume that there are f and g with $f \circ e = m \circ g$ and that m is regular mono, i.e. the equalizer of some pair p_1, p_2. Then $p_1 \circ m \circ g = p_2 \circ m \circ g$ hence $p_1 \circ f \circ e = p_2 \circ f \circ e$, too. As e is epi, it follows $p_1 \circ f = p_2 \circ f$, which reveals f as a competitor to the equalizer m. This yields a unique map d with $m \circ d = f$. Then $m \circ d \circ e = f \circ e = m \circ g$. Since m is mono, we have $d \circ e = g$. The rest follows by duality.
(2) Let $m_i \circ e_i$ be an (epi, regular mono)-factorization of $f : A \longrightarrow B$, for $i \in \{1, 2\}$. By the definition of factorization system there are unique diagonals $d : C_1 \longrightarrow C_2$ and $d' : C_2 \longrightarrow C_1$ such that the diagram below is commutative for both d and d'.

$$\begin{tikzcd}
A \arrow[r, "e_1"] \arrow[d, "e_2"'] & C_1 \arrow[d, "m_1"] \arrow[dl, shift left, "d"] \\
C_2 \arrow[r, "m_2"'] \arrow[ur, shift left, "d'"] & B
\end{tikzcd}$$

Also, by the third property of factorization system, $d \circ d' = id_{C_2}$ and $d' \circ d = id_{C_1}$. $\square$

Example 2.11.5. ([3], chapter IV, section 14) The category *Set* is an (*epi*, *mono*)-category. Recall that in *Set* every function $f : A \longrightarrow B$ can be factored through its image, i.e. the following diagram commutes,

$$\begin{tikzcd}
A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\
& im\, f \arrow[ur, "m"'] &
\end{tikzcd} \tag{2.11.1}$$

where $e : A \longrightarrow im\, f$ is the codomain-restriction of f and $m : im\, f \longrightarrow B$ is the inclusion map.

Example 2.11.6. ([3], chapter IV, section 14) In *Top*, the pair $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E} = epis$ and $\mathcal{M} = mono$ can not be considered as a factorization system (because the intersection of the classes of the epis and monos is not a subclass of isomorphisms). However, by factoring each continuous function $f : A \longrightarrow B$ through its image (see diagram 2.11.1), we will obtain two factorization systems for the category *Top* as follows:

- if we equip $im\, f$ with the subspace topology generated by m, then according to remarks 2.2.11 and 1.4.3 and lemma 2.11.4, the pair $(\mathcal{E}, \mathcal{M})$ where $\mathcal{E} = epis$ and $\mathcal{M} = regular\, monos$ is a factorization system in *Top*, and
- in the case that $im\, f$ carries the quotient topology generated by e, then by remarks 2.2.11 and 1.4.8 and lemma 2.11.4, the pair $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E} = regular\, epis$ and $\mathcal{M} = mono$ is a factorization system in *Top*.

Lemma 2.11.7. *In Top, the pair $(\mathcal{E}, \mathcal{M})$, where $\mathcal{M}$ is the class of closed embeddings, and $\mathcal{E}$ is the class of dense functions, is a factorization system.*

Proof. Notice that

- every continuous map can be factored through the closure of its image, i.e. written as $A \xrightarrow{f} B = A \xrightarrow{e} \overline{im\,f} \overset{m}{\hookrightarrow} B$ where $e : A \longrightarrow \overline{im\,f}$ is the codomain-restriction of f and $m : \overline{im\,f} \longrightarrow B$ is the topological embedding,
- the class of dense functions is closed under composition (lemma 1.2.2), and
- by corollary 1.4.5, the class of closed embeddings is closed under composition.

Regarding the condition (3) of definition 2.11.1, let $e : A \longrightarrow B$ be a dense function and $m : C \longrightarrow D$ an closed embedding. Assume that there are f and g with $f \circ e = m \circ g$. Since m is regular mono (see lemma 2.9.3), it is an equalizer of some pair p_1, p_2 i.e., $p_1 \circ m = p_2 \circ m$. Then $p_1 \circ m \circ g = p_2 \circ m \circ g$ and hence $p_1 \circ f \circ e = p_2 \circ f \circ e$, too. This means $p_1 \circ f(b) = p_2 \circ f(b)$ for each $b \in e(A)$. We claim that $p_1 \circ f(e(A)) = p_2 \circ f(e(A))$ for each $b \in B$. To see this, assume b is an arbitrary element in $B - e(A)$. Since e is a dense function, $\overline{e(A)} = B$. Then we can find a net ψ in A such that the net $e \circ \psi$ converges to b. Since f is continuous, the net $f \circ e \circ \psi$ converges to $f(b)$. Then the net $m \circ g \circ \psi$ converges to $f(b)$. It means $f(b)$ is in the closure of the image of m. Then $f(b)$ is in the image of m (since m is a closed embedding, its image is a closed subset of its codomain). So $p_1(f(b)) = p_2(f(b))$ (becausem m is an equalizer of p_1, p_2). Therefore we conclude that $p_1(f(b)) = p_2(f(b))$ for each $b \in B$. It means $p_1 \circ f = p_2 \circ f$ which reveals f as a competitor to the equalizer m. This yields a unique map d with $m \circ d = f$. Then $m \circ d \circ e = f \circ e = m \circ g$. Since m is mono, we have $d \circ e = g$ (see [3], chapter 14). $\square$

Remark 2.11.8. Take $\mathcal{E}$ as the class of dense functions and $\mathcal{M}$ as the class of closed embeddings. Note that the pair $(\mathcal{E}, \mathcal{M})$ is a factorization system in *Top* where $\mathcal{E}$ does not consist of epimorphisms (i.e., $\mathcal{E} \nsubseteq$ *epis*). The reason is that: in *Top* there are dense functions that are not epi. As an example, consider the function f as the constant function from the indiscrete space $\{0, 1\}$ to itself with value 1. Clearly, f is a dense function whereas it is not epi.

Definition 2.11.9. ($\mathcal{M}$-Image and $\mathcal{E}$-coimage of a morphism) Suppose $\mathbb{C}$ is a $(\mathcal{E}, \mathcal{M})$-category and $f : A \longrightarrow B$ is a morphism in $\mathbb{C}$. By factoring f in the factorization system $(\mathcal{E}, \mathcal{M})$ we have a diagram as follows:

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "e"'] & & B \\ & E \arrow[ur, "m"'] & \end{tikzcd}$$

The $\mathcal{M}$-morphism $m : E \longrightarrow B$ and the $\mathcal{E}$-morphism $e : A \longrightarrow E$ are called $\mathcal{M}$-image of and $\mathcal{E}$-coimage of f, respectively (see [54]).

2.12. $\mathcal{M}$-subobjects and $\mathcal{M}$-union

Let $\mathbb{C}$ be a $(\mathcal{E}, \mathcal{M})-$category and let $m_1 : M_1 \longrightarrow X$ and $m_2 : M_2 \longrightarrow X$ be two $\mathcal{M}-$morphisms in $\mathbb{C}$. We write $m_1 \subseteq m_2$, if there is a morphism $f : M_1 \longrightarrow M_2$ with $m_1 = m_2 \circ f$. We say that m_1 and m_2 are equivalent (in symbol: $m_1 \sim m_2$) iff $m_1 \subseteq m_2$ and $m_2 \subseteq m_1$. An $\mathcal{M}$- subobject of an object X in $\mathbb{C}$ is an equivalence class of some $\mathcal{M}$-morphism $m : M \longrightarrow X$. We usually identify the equivalence class of an $\mathcal{M}$- morphism $m : M \longrightarrow X$ with m itself, as an abuse of language, (for more details, see [54] or section 5.1 in [7]).

Definition 2.12.1. ($\mathcal{M}$-well powered category) A category $\mathbb{C}$ is said to be $\mathcal{M}$-well powered, if for each object A in $\mathbb{C}$, the collection of $\mathcal{M}$- subobjects of A is a set.

Definition 2.12.2. Let A be an object in an $\mathcal{M}$-well powered category $\mathbb{C}$. The $\mathcal{M}$-union of a family $\{m_i : S_i \longrightarrow A\}_{i\in I}$ of $\mathcal{M}$- subobjects of A is an $\mathcal{M}$- subobject $m : S \longrightarrow A$ satisfying two conditions as follows:

1. $m_i \subseteq m$, for all $i \in I$, and
2. $m \subseteq m'$ for any $\mathcal{M}$- subobject $m' : S' \longrightarrow A$ with $m_i \subseteq m'$ (for all $i \in I$),

see [54].

The following theorem gives us a construction of $\mathcal{M}$-unions. This construction will be often used in part 3 of this work:

Theorem 2.12.3. *[54] Let $\mathbb{C}$ be a category with $(\mathcal{E}, \mathcal{M})-$factorization system and coproducts and let $\mathbb{C}$ be $\mathcal{M}$-well powered. Suppose $\{m_i : S_i \longrightarrow A\}_{i\in I}$ is a family of $\mathcal{M}$-subobjects of A. If the object S with morphisms $\{e_i : S_i \longrightarrow S\}_{i\in I}$ is the sum of the objects $\{S_i\}_{i\in I}$ in $\mathbb{C}$, then the $\mathcal{M}$-union of the family $\{m_i\}_{i\in I}$ exists and it is the $\mathcal{M}$-image of the unique morphism $q : S \longrightarrow A$ with $m_i = q \circ e_i$.*

$$\begin{tikzcd}
S_i \arrow[r, "e_i"] \arrow[rrr, bend left, "m_i"] & S \arrow[rr, "q"] \arrow[dr, "e"'] & & A \\
& & E \arrow[ur, "m"'] &
\end{tikzcd}$$

Lemma 2.12.4. *Let $\mathbb{C}$ be a category with $(\mathcal{E}, \mathcal{M})$−factorization system and coproducts and let $\mathbb{C}$ be $\mathcal{M}$-well powered. Given $\mathbb{C}$-morphisms $f, g : A \longrightarrow B$. If $\{m_i : S_i \longrightarrow A\}_{i \in I}$ is a sink of $\mathcal{M}$-morphisms in $\mathbb{C}$ such that $f \circ m_i = g \circ m_i$ (for each $i \in I$), then $f \circ m = g \circ m$ where $m : E \longrightarrow A$ is the $\mathcal{M}$-union of $\{m_i\}_{i \in I}$.*

Proof. Let the object S with morphisms $\{e_i : S_i \longrightarrow S\}_{i \in I}$ be the sum of the objects $\{S_i\}_{i \in I}$ in $\mathbb{C}$. Then according to theorem 2.12.3, the morphism $m : E \longrightarrow A$ (i.e., $\mathcal{M}$-union of collection $\{m_i\}_{i \in I}$) is the $\mathcal{M}$-image of the unique morphism $q : S \longrightarrow A$ in the following commutative diagram.

$$\begin{tikzcd}
S_i \arrow[r, "e_i"] \arrow[rr, bend left, "m_i"] & S \arrow[r, "q"] \arrow[dr, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
& & E \arrow[u, "m"'] &
\end{tikzcd}$$

If $f \circ m_i = g \circ m_i$ (for each $i \in I$), then we have

$$\begin{aligned}
f \circ m \circ e \circ e_i &= f \circ q \circ e_i \\
&= f \circ m_i \\
&= g \circ m_i \\
&= g \circ q \circ e_i \\
&= g \circ m \circ e \circ e_i
\end{aligned}$$

Since $\{e \circ e_i\}_{i \in I}$ is an epi sink, it is concluded that $f \circ m = g \circ m$.

□

2.13. Exponential objects

Definition 2.13.1. Let $\mathbb{C}$ be a category with binary products and let Σ and Z be objects of $\mathbb{C}$. An object Z^{Σ} together with a morphism

$$ev : Z^{\Sigma} \times \Sigma \longrightarrow Z$$

is an exponential object (for two objects Z and Σ), if for any object X and each morphism $g : X \times \Sigma \longrightarrow Z$ there is a unique morphism

$$\begin{tikzcd}
X \arrow[d, "\lambda g"] \\
Z^{\Sigma}
\end{tikzcd}$$

such that the following diagram commutes:

$$\begin{tikzcd} X \times \Sigma \arrow[d, "\lambda g \times id_\Sigma"'] \arrow[dr, "g"] & \\ Z^\Sigma \times \Sigma \arrow[r, "ev"'] & Z \end{tikzcd}$$

Here is some terminology:

- $ev : Z^\Sigma \times \Sigma \longrightarrow Z$ is called evaluation map, and

- $\lambda g : X \longrightarrow Z^\Sigma$ is called the curried form of g.

Example 2.13.2. In the category *Set*, the exponential object Z^Σ (for two sets Z and Σ) is the set of all functions from Σ to Z (i.e., $Z^\Sigma := Hom_{Set}(\Sigma, Z)$). The evaluation map $ev : (Z^\Sigma \times \Sigma) \longrightarrow Z$ is just a map sending a pair (f, y) to $f(y)$ for each $f \in Z^\Sigma$ and $y \in \Sigma$, (for more details see section 6.1 in [7]). For every map $g : (X \times \Sigma) \longrightarrow Z$, the morphism $\lambda g : X \longrightarrow Z^\Sigma$ called the curried form of g is defined by

$$\lambda g(x)(y) = g(x, y).$$

Definition 2.13.3. A category $\mathbb{C}$ is called cartesian closed if

- products of every finite families of objects exists, and

- for every two objects Z and Σ in $\mathbb{C}$, there is the exponential object Z^Σ.

Example 2.13.4. [7] According to the examples 2.5.3 and 2.13.2, the category *Set* is a cartesian closed category.

Example 2.13.5. [12] The category CUM^1 is a cartesian closed category. The exponential object A^B (for two complete ultrametric spaces A and B), has the set of non-expansive functions from B to A as the underlying set, and the metric d defined by $d(f, g) = sup\{d_B(f(x), g(x)) \mid x \in A\}$ as distance function (for more details see [12] and [67]).

2.14. Exponential objects in Top

Definition 2.14.1. (Compact-open topology) Let Σ and Z be two topological spaces, and let Z^{Σ} denotes the set of all continuous maps from Σ to Z. Given a compact subset K of Σ and an open subset U of Z, let $[K, U]$ be the set of all continuous functions $f \in Z^{\Sigma}$ such that $f(K) \subset U$. Then the collection of all such $[K, U]$ forms a subbase for the compact-open topology on Z^{Σ}.

In the category of topological spaces, the exponential object Z^{Σ} (for two topological spaces Z and Σ) exists provided that Σ is a locally compact space. In that case, the space Z^{Σ} is the set of all continuous functions from Σ to Z together with the compact-open topology. The evaluation map is defined the same as this map in the category of Set.

Lemma 2.14.2. *([16], chapter 7) Let Σ be a locally compact space and Z be an arbitrary space. Let Z^{Σ} be the set of all continuous maps from Σ to Z equipped with the compact-open topology. Then the map $ev : Z^{\Sigma} \times \Sigma \longrightarrow Z$ defined as $ev(f, y) = f(y)$ (for each $f \in Z^{\Sigma}$ and $y \in \Sigma$) is continuous.*

Proof. Let U be an open subset of Z. By the definition of ev,

$$ev^{-1}(U) = \{(f, y) \,|\, y \in f^{-1}(U),\, f \in Z^{\Sigma},\, y \in \mathit{\Sigma}\}.$$

It is enough to show that the set $ev^{-1}(U)$ is open. It means we need to show that every element (f, y) in $ev^{-1}(U)$ is an interior point of $ev^{-1}(U)$. Fix $(f, y) \in ev^{-1}(U)$. Then since $f^{-1}(U)$ is an open neighborhood of $y \in \mathit{\Sigma}$ and since $\mathit{\Sigma}$ is a locally compact space, there is a compact neighborhood $K_y \subseteq \mathit{\Sigma}$ such that $y \in K_y \subseteq f^{-1}(U)$. Since $K_y \subseteq \mathit{\Sigma}$ is a neighborhood of y, it contains an open subset V_y such that $y \in V_y \subseteq K_y \subseteq f^{-1}(U)$. So, $f(y) \in f(V_y) \subseteq f(K_y) \subseteq U$. Consequently $(f, y) \in [K_y, U] \times V_y$. Now, it suffices to show that $([K_y, U] \times V_y) \subseteq ev^{-1}(U)$. Let $x \in [K_y, U] \times V_y$. Thus, there exist $\varepsilon \in [K_y, U]$ and $d \in V_y$ with $x = (\varepsilon, d)$. Since, $d \in V_y \underset{open}{\subseteq} K_y$ and $\varepsilon \in [K_y, U]$, we have $\varepsilon(d) \in U$. Therefore, $x = (\varepsilon, d) \in ev^{-1}(U)$. □

Corollary 2.14.3. *Let Σ be a locally compact space and Z be an arbitrary space. Then the pair $(Z^{\Sigma}, ev : Z^{\Sigma} \times \Sigma \longrightarrow Z)$ where Z^{Σ} is the set of all continuous maps from Σ to Z equipped with the compact-open topology and $ev : Z^{\Sigma} \times \Sigma \longrightarrow Z$ is defined as $ev(f, y) = f(y)$, (for each $f \in Z^{\Sigma}$ and $y \in \Sigma$), is an exponential object in Top.*

If Σ is not locally compact, the exponential object might not exist (the space Z^{Σ} still exists, but it may fail to be an exponential object since the evaluation function ev need not be continuous). As an example, if Σ is $\mathbb{Q}$ and Z is the closed interval $[0, 1]$ in $\mathbb{R}$, the evaluation map $ev : Z^{\Sigma} \times \Sigma \longrightarrow Z$ is not continuous (see chapter 7 in [16], for more details). For this reason the category of topological spaces fails to be cartesian closed.

2.15. Functors

Functors relate different categories. If we consider categories as structured objects, then functors can be considered as morphisms between them that preserve their structure.

Definition 2.15.1. (Covariant functor) Let $\mathbb{C}$ and $\mathbb{D}$ be categories. A covariant functor F from $\mathbb{C}$ to $\mathbb{D}$ (in symbol: $F : \mathbb{C} \longrightarrow \mathbb{D}$) associates:

- to each object $A \in \mathbb{C}$ an object $F(A) \in \mathbb{D}$,
- to each morphism $f : A \longrightarrow B$ in $\mathbb{C}$, a morphism $Ff : FA \longrightarrow FB$ in $\mathbb{D}$ such that the following identities hold:

$$\begin{array}{rcl} F(id_A) & = & id_{FA}, \\ F(f \circ g) & = & Ff \circ Fg. \end{array}$$

The covariant functors are simply called functors. An endofunctor is a functor whose domain and codomain are equal. If F is an endofunctor over a category $\mathbb{C}$ (i.e., $F : \mathbb{C} \longrightarrow \mathbb{C}$), then it is called a $\mathbb{C}$-endofunctor.

Remark 2.15.2. ([3], chapter I, section 3) Notice that

1. by definition, a functor behaves like a pair of maps. The first one (called the object-map) is defined between objects and the second one between morphisms, and
2. since a category may have a proper class of objects and a proper class of morphisms between them, the domains of functors may be proper classes. So they are not really "maps". However, we are writing $F : \mathbb{C} \longrightarrow \mathbb{D}$.

Example 2.15.3. ([3], chapter I, section 3) In any category $\mathbb{C}$,

1. there is the **identity functor** $id_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$ (or $(-)_{\mathbb{C}} : \mathbb{C} \longrightarrow \mathbb{C}$, if necessary) defined by $id_{\mathbb{C}}(f : A \longrightarrow B) = A \xrightarrow{f} B$, and
2. for any object $C \in \mathbb{C}$, there is the **constant functor** $C : \mathbb{C} \longrightarrow \mathbb{C}$ with value C, defined by $C(f : A \longrightarrow B) = C \xrightarrow{id_C} C$.

Example 2.15.4. ([3], chapter I, section 3) The **discrete functor** $D : Set \longrightarrow Top$ (resp. **indiscrete functor** $I : Set \longrightarrow Top$) is defined by $D(f : A \longrightarrow B) = A_D \xrightarrow{f} B_D$ (resp. $I(f : A \longrightarrow B) = A_I \xrightarrow{f} B_I$).

Example 2.15.5. (Power functor) If $\mathbb{C}$ has product, for any set Σ, there is the **power functor** $(-)^{\Sigma} : \mathbb{C} \longrightarrow \mathbb{C}$ which associates to each object $A \in \mathbb{C}$ the object $\prod_{i\in\Sigma} A$ (i.e., Σ-fold product of A in $\mathbb{C}$), and to each morphism $f : A \longrightarrow B$ the obvious morphism f^{Σ} with $\pi_i^B \circ f^{\Sigma} = \pi_i^A$ for each $i \in \Sigma$ (π_i^B and π_i^A are canonical projections).

Example 2.15.6. (Covariant Hom_{Set} functor) ([3], chapter I, section 3) Let Σ be a fixed set, the construction $F(-) := Hom_{Set}(\Sigma, -)$ which associates to each set X the set of all maps from Σ to X and to each morphism $f : X \longrightarrow Y$ the map

$$F(f) : Hom_{Set}(\Sigma, X) \longrightarrow Hom_{Set}(\Sigma, Y)$$

defined by $(Ff)(\delta) := f \circ \delta$, is an endofunctor on the category Set.

Example 2.15.7. Let X be an arbitrary set and $n \in \mathbb{N}$ be a fixed element in $\mathbb{N}$. The distance function $d_n^X : X \times X \longrightarrow [0, 1]$ defined by

$$d_n^X(x,\, y) := \begin{cases} 2^{-n} & if\, x \neq y \\ 0 & if\, x = y. \end{cases}$$

is an ultrametric. A sequence $(x_i)_{i\in\mathbb{N}}$ in (X, d_n^X) is Cauchy iff

$$\exists M \in \mathbb{N}. \forall i \geq M.\, d_n^X(x_i,\, x_{i+1}) = 0.$$

and consequently (X, d_n^X) is a complete 1-bounded ultrametric space. We can easily see that each map $f : X \longrightarrow Y$ is a non-expansive map from (X, d_n^X) to (Y, d_n^Y). Then, for each $n \in \mathbb{N}$, the construction $D_n : Set \longrightarrow CUM^1$ defined by

$$D_n(f : X \longrightarrow Y) = (X, d_n^X) \xrightarrow{f} (Y, d_n^Y)$$

is a functor from Set to CUM^1.

Lemma 2.15.8. *Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ are categories with sums and products. Let F_1, $F_2 : \mathbb{C}_1 \longrightarrow \mathbb{C}_2$ be functors, then so are $F_1 + F_2$, $F_1 \times F_2$ and $F_1 \circ F_2$ if for an arbitrary object X and an arbitrary $\mathbb{C}_1$-morphism $f : X \longrightarrow Y$ we define:*

- $(F_1 \circ F_2)(X) := F_1(F_2(X))$ *and* $(F_1 \circ F_2)(f) := F_1(F_2(f))$,

- *$(F_1 \times F_2)(X) := F_1(X) \times F_2(X)$ and $(F_1 \times F_2)(f) := q$ where q is the unique morphism in the following commutative diagram*

$$\begin{tikzcd}
F_1(X) \arrow[r, "F_1(f)"] & F_1(Y) \\
F_1(X) \times F_2(X) \arrow[u, "\pi_1^{F_1(X)}"] \arrow[d, "\pi_2^{F_2(X)}"'] \arrow[r, "\exists! q"] & F_1(Y) \times F_2(Y) \arrow[u, "\pi_1^{F_1(Y)}"'] \arrow[d, "\pi_2^{F_2(Y)}"] \\
F_2(X) \arrow[r, "F_2(f)"] & F_2(Y)
\end{tikzcd} \tag{2.15.1}$$

- *$(F_1 + F_2)(X) := F_1(X) + F_2(X)$ and $(F_1 + F_2)(f) := r$ where r is the unique morphism in the following commutative diagram*

$$\begin{tikzcd}
F_1(X) \arrow[r, "F_1(f)"] \arrow[d, "e_1^{F_1(X)}"'] & F_1(Y) \arrow[d, "e_1^{F_1(Y)}"] \\
F_1(X) + F_2(X) \arrow[r, "\exists! r"] & F_1(Y) + F_2(Y) \\
F_2(X) \arrow[r, "F_2(f)"] \arrow[u, "e_2^{F_2(X)}"] & F_2(Y) \arrow[u, "e_2^{F_2(Y)}"']
\end{tikzcd} \tag{2.15.2}$$

Proof. Let F_1, $F_2 : \mathbb{C}_1 \longrightarrow \mathbb{C}_2$ be functors. We prove that $F_1 + F_2$ is also a functor from $\mathbb{C}_1$ to $\mathbb{C}_2$. We need to show that the following equalities hold (see definition 2.15.1):

- $(F_1 + F_2)(id_X) = id_{(F_1+F_2)(X)}$, and
- $(F_1 + F_2)(g \circ f) = (F_1 + F_2)(g) \circ (F_1 + F_2)(f)$.

To prove the first equality, in diagram 2.15.2 replace the morphism f by id_X. Clearly, $id_{F_1(X)+F_2(X)}$ is the unique map such that $id_{F_1(X)+F_2(X)} \circ e_i^{F_i(X)} = e_i^{F_i(X)} \circ id_{F_i(X)}$ for each $i \in \{1, 2\}$. Hence $(F_1 + F_2)(id_X) = id_{F_1(X)+F_2(X)} = id_{(F_1+F_2)(X)}$.
To prove the second one, consider the following diagram

$$\begin{tikzcd}
F_1(X) \arrow[r, "F_1(f)"] \arrow[d, "e_1^{F_1(X)}"'] & F_1(Y) \arrow[r, "F_1(g)"] \arrow[d, "e_1^{F_1(Y)}"] & F_1(Z) \arrow[d, "e_1^{F_1(Z)}"] \\
F_1(X) + F_2(X) \arrow[r, "\exists! r"] & F_1(Y) + F_2(Y) \arrow[r, "\exists! h"] & F_1(Z) + F_2(Z) \\
F_2(X) \arrow[r, "F_2(f)"'] \arrow[u, "e_2^{F_2(X)}"] & F_2(Y) \arrow[r, "F_2(g)"'] \arrow[u, "e_2^{F_2(Y)}"'] & F_2(Z) \arrow[u, "e_2^{F_2(Z)}"']
\end{tikzcd} \tag{2.15.3}$$

where h (resp. r) is the unique morphism which makes the right (resp. the left) rectangle of diagram 2.15.3 in to a commutative diagram. Then we have $(F_1 + F_2)(g) = h$ (resp.

$(F_1 + F_2)(f) = r)$. It is easy to see that $h \circ r$ is the unique morphism which makes diagram 2.15.3 commutative. Since $\{e_i^{F_i(X)}\}_{i\in\{1,2\}}$ is an epi sink, the morphism $h \circ r$ is unique. Then we have

$$(F_1 + F_2)(g \circ f) \quad = h \circ r = \quad (F_1 + F_2)(g) \circ (F_1 + F_2)(f).$$

By using a similar strategy for diagram 2.15.1, we can prove that $F_1 \times F_2$ is also a functor from $\mathbb{C}_1$ to $\mathbb{C}_2$. □

Definition 2.15.9. (Polynomial functor) [30] Let $\mathbb{C}$ be a category with sums and products. The class of polynomial functors over $\mathbb{C}$ is inductively defined as follows:

$$F ::= C \mid id_{\mathbb{C}} \mid F_1 + F_2 \mid F_1 \times F_2 \mid F^D$$

Here $id_{\mathbb{C}}$ is the identity functor on the category $\mathbb{C}$. C denotes the constant functor (for an arbitrary object C). $+$ and $\times$ are sum and product in $\mathbb{C}$, respectively; and for every set D, we consider F^D as the functor sending an object X to the D-fold product $(F(X))^D$ in $\mathbb{C}$ (i.e $F^D := (-)^D \circ F$, the composition of the functor F and the power functor $(-)^D$on $\mathbb{C}$).

Example 2.15.10. For fixed sets C and D, the functor $F(-) := C \times (-)^D$ (i.e., the product of the constant functor C with the power functor $(-)^D$) is a type of the polynomial functor on Set.

Concrete category

Generally speaking, a category $\mathbb{C}$ is called concrete if its objects are sets with some additional structure, and the arrows are structure preserving maps between those objects such that

- id_A is the identity map on the base set of A, and
- composition of arrows is function composition.

To have a categorical definition of concrete categories we need to define faithful functors as follows:

Definition 2.15.11. Let $F : \mathbb{C} \longrightarrow \mathbb{D}$ be a functor. F is called **faithful** provided that for all $A, B \in \mathbb{C}$, the map

$$F_{A,B} : Hom_{\mathbb{C}}(A, B) \longrightarrow Hom_{\mathbb{D}}(FA, FB)$$

defined by $f \longmapsto F(f)$ is injective. Similarly, F is **full** if $F_{A,B}$ is always surjective.

Now we give a categorical definition of the concrete categories used in this work:

Definition 2.15.12. (Concrete category) A category $\mathbb{C}$ is a concrete category if there is a faithful functor $U : \mathbb{C} \longrightarrow Set$. Sometimes U is called forgetful (or underlying) functor of the category $\mathbb{C}$ and the category Set is called the base category.

Example 2.15.13. The category Top with the forgetful functor $U : Top \longrightarrow Set$ which associates each topological space (X, τ) to its underlying set X and each continuous map $f : X \longrightarrow Y$ to the same morphism in Set is a concrete category. In fact all categories mentioned in example 2.1.3 are concrete categories.

Remark 2.15.14. Let $\mathbb{C}$ be a concrete category and $U : \mathbb{C} \longrightarrow Set$ be the forgetful functor of $\mathbb{C}$, then each case $U(A)$ is the underlying set of the object $A \in \mathbb{C}$, and $U(f)$ is the underlying function of the morphism f.

Remark 2.15.15. If the categories $\mathbb{C}_1$ and $\mathbb{C}_2$ mentioned in lemma 2.15.8 are concrete categories with sums and products, then for an arbitrary object X and an arbitrary morphism $f : X \longrightarrow Y$ we define the functors $F_1 + F_2$, $F_1 \times F_2$ and $F_1 \circ F_2$ as follows:

- $(F_1 \circ F_2)(X) := F_1(F_2(X))$ and $(F_1 \circ F_2)(f) := F_1(F_2(f))$,
- $(F_1 \times F_2)(X) := F_1(X) \times F_2(X)$ and $(F_1 \times F_2)(f)(x, y) := (F_1(f)(x), F_2(f)(y))$,
- $(F_1 + F_2)(X) := F_1(X) + F_2(X)$ and

$$(F_1 + F_2)(f)(u) := \begin{cases} (F_1(f))(u) & if\, u \in F_1(X) \\ (F_2(f))(u) & if\, u \in F_2(X). \end{cases}$$

Contravariant functors

Definition 2.15.16. (Contravariant functor) A contravariant functor F from a category $\mathbb{C}$ to a category $\mathbb{D}$ is simply a functor $F : \mathbb{C} \longrightarrow \mathbb{D}^{op}$. It means F associates each object X of $\mathbb{C}$ to an object $F(X)$ of $\mathbb{D}$ and each $\mathbb{C}$-morphism $f : X \longrightarrow Y$ to a $\mathbb{D}$-morphism $Ff : F(Y) \longrightarrow F(X)$ so that $F(id_X) = id_{F(X)}$ and $F(f \circ g) = Fg \circ Ff$. Obviously, the composition of two contravariant functors yields a covariant functor.

Example 2.15.17. (Contravariant Hom_{Set} functor) ([3], chapter I, section 3) Given a fixed set C, the set of all maps to C yields a contravariant functor $F(-) := Hom_{Set}(-, C)$. For each map $f : X \longrightarrow Y$, the morphism $F(f) : Hom_{Set}(Y, C) \longrightarrow Hom_{Set}(X, C)$ sends each $\delta : Y \longrightarrow C$ to $\delta \circ f : X \longrightarrow C$.

2.16. More functors on *Set*

Example 2.16.1. ([3], chapter I, section 3) The **powerset functor** $\mathbb{P}$ associates to each set X the set $\mathbb{P}(X)$ of all subsets of X. To map $f : X \longrightarrow Y$ between arbitrary sets X and Y we associate the image map $\mathbb{P}(f) : \mathbb{P}(X) \longrightarrow \mathbb{P}(Y)$ with $(\mathbb{P}(f))(U) ::= f(U)$. A variation is the **finite powerset functor** $\mathbb{P}_\omega$ where $\mathbb{P}_\omega(X)$ is the set of all finite subsets of X. On maps it acts exactly the same as $\mathbb{P}(-)$.

Example 2.16.2. [15] The class KPF of **Kripke polynomial functors** over *Set* is inductively defined as follows:

$$F ::= C \mid id_{Set} \mid \mathbb{P}_\omega \mid F_1 + F_2 \mid F_1 \times F_2 \mid F^D.$$

Notice the Kripke polynomial functors obtain by conjucting the powerset functor with the polynomial functors.

Example 2.16.3. [31] For an arbitrary set X define $\mathfrak{P}(X) := \mathbb{P}(X)$, but for each map $f : X \longrightarrow Y$ define $\mathfrak{P}(f) : \mathbb{P}(Y) \longrightarrow \mathbb{P}(X)$ by $(\mathfrak{P}(f))(V) := \{x \in X \mid f(x) \in V\}$ for each $V \subseteq Y$. This construction does not result in a functor, since the direction of maps is reversed. However, This construction is a contravariant functor called contravariant powerset functor. The correspondence of subsets with characteristic functions suggests an alternate notation of $\mathfrak{P}$ as $2^{(-)}$. It is easy to see that the composition $\mathfrak{P} \circ \mathfrak{P}$ (or $2^{2^{(-)}}$) is a functor. This functor will be called the **neighbourhood functor**. It is instructive to watch the neighborhood functor in action. Elements of 2^{2^X} are collections of subsets of X. Given a map $f : X \longrightarrow Y$, then 2^{2^f} associates each collection $\sigma \subseteq \mathbb{P}(X)$ to the collection $2^{2^f}(\sigma) = \{V \subseteq Y \mid f^{-1}(V) \in \sigma\} \subseteq \mathbb{P}(Y)$.

The following *Set*-endofunctor is used in some example of this work:

Example 2.16.4. [2] The *Set*-endofunctor $T := (-)^2 - (-) + 1$ associates each set X to the set

$$TX = \{(x, x') \in X^2 \mid x \neq x'\} \cup \{\bot\}$$

and every map $f : X \longrightarrow Y$ to the map $Tf : TX \longrightarrow TY$ defined by

$$Tf(x, x') := \begin{cases} (f(x), f(x')) & f(x) \neq f(x') \\ \bot & f(x) = f(x') \\ \bot & \bot \end{cases}$$

2.17. Functors and morphisms

Definition 2.17.1. A functor $F : \mathbb{C}_1 \longrightarrow \mathbb{C}_2$ is said to preserve

- monos (resp. epis) if: for every mono (resp. epi) f in $\mathbb{C}_1$, Ff is also a mono (resp. epi) in $\mathbb{C}_2$,
- sections (resp. retractions) if: for every section (resp. retraction) f in $\mathbb{C}_1$, Ff is also a section (resp. retraction) in $\mathbb{C}_2$, and
- isomorphisms if: for every isomorphism f in $\mathbb{C}_1$, Ff is also an isomorphism in $\mathbb{C}_2$.

Example 2.17.2. ([3], chapter 2, section 7) All functors preserve sections and retractions. As a consequence, all functors preserve isomorphisms.

Example 2.17.3. The forgetful functor $U : Top \longrightarrow Set$ and the discrete and indiscrete finctors D, $I : Set \longrightarrow Top$ preserve all monos and epis.

Example 2.17.4. [31] Every Set-endofunctor preserves monos with non-empty domains. Dually, every Set-endofunctor preserves epis. To check this, just use this fact that in the category Set, monos with non-empty domains (resp. epis) are exactly sections (resp. retractions).

Example 2.17.5. If $\mathbb{C}$ has products, then for any set Σ, the power functor $(-)^\Sigma$ preserves monos. To see this notice that for each morphism $f : A \longrightarrow B$ and each $i \in \Sigma$ we have $\pi_i^B \circ f^\Sigma = \pi_i^A$, where π_i^B and π_i^A are canonical projections. By part (2) of lemma 2.10.4, f^Σ is mono. Moreover, it follows from lemma 2.5.9 that the power functor $(-)^\Sigma$ preserves regular monos.

Lemma 2.17.6. *Suppose $\mathbb{C}_1$ and $\mathbb{C}_2$ are categories with sums and products. Let F_1, $F_2 : \mathbb{C}_1 \longrightarrow \mathbb{C}_2$ be functors preserving monos. Then $F_1 + F_2$, $F_1 \times F_2$ and $F_1 \circ F_2$ preserve monos too.*

Proof. Suppose $f : X \longrightarrow Y$ is a monomorphism in $\mathbb{C}_1$. By assumption $F_1(f)$ and $F_2(f)$ are monomorphisms in $\mathbb{C}_2$. To prove that $(F_1 \times F_2)(f)$ is mono, consider the following diagram:

$$\begin{tikzcd}
& F_1(X) \arrow[r, "F_1(f)"] & F_1(Y) \\
Z \arrow[r, shift left, "g"] \arrow[r, shift right, "h"'] & F_1(X) \times F_2(X) \arrow[u, "\pi_1^{F_1(X)}"] \arrow[d, "\pi_2^{F_2(X)}"'] \arrow[r, "(F_1 \times F_2)(f)"] & F_1(Y) \times F_2(Y) \arrow[u, "\pi_1^{F_1(Y)}"] \arrow[d, "\pi_2^{F_2(Y)}"] \\
& F_2(X) \arrow[r, "F_2(f)"] & F_2(Y)
\end{tikzcd}$$

Let $(F_1 \times F_2)(f) \circ h = (F_1 \times F_2)(f) \circ g$, then

$$(F_1 \times F_2)(f) \circ h \circ \pi_1^{F_1(Y)} = (F_1 \times F_2)(f) \circ g \circ \pi_1^{F_1(Y)}.$$

Since this diagram commutes, we have $F_1(f) \circ \pi_1^{F_1(X)} \circ h = F_1(f) \circ \pi_1^{F_1(X)} \circ g$. Since $F_1(f)$ is mono we have $\pi_1^{F_1(X)} \circ h = \pi_1^{F_1(X)} \circ g$. Similarly $\pi_2^{F_2(X)} \circ h = \pi_2^{F_2(X)} \circ g$. As the family $(\pi_i^{F_i(X)})_{i \in \{1,2\}}$ is a mono source, we have $h = g$. In a similar way, we can prove this claim for $F_1 + F_2$ and $F_1 \circ F_2$.

$\square$

Definition 2.17.7. Let $\mathbb{C}$ be a category, $\mathcal{M}$ a class of $\mathbb{C}-$morphisms and $F : \mathbb{C} \longrightarrow \mathbb{C}$ a $\mathbb{C}$-endofunctor. We say that the $\mathbb{C}$-endofunctor F preserves $\mathcal{M}$-morphisms if $Ff \in \mathcal{M}$ for each morphism $f \in \mathcal{M}$.

Definition 2.17.8. A functor $F : \mathbb{C}_1 \longrightarrow \mathbb{C}_2$ is said to **reflect monos** (resp. epis) if for every $\mathbb{C}_1$-morphism f, whenever Ff is a mono (resp. epi) in $\mathbb{C}_2$, then f is also a mono (resp. epi) in $\mathbb{C}_1$.

Example 2.17.9. The forgetful functor $U : Top \longrightarrow Set$ and the discrete and indiscrete finctors D, $I : Set \longrightarrow Top$ reflect all monos and epis.

2.18. Limits and colimits in general

Let $\mathbb{I}$ and $\mathbb{C}$ be categories. A diagram of type $\mathbb{I}$ in $\mathbb{C}$ is a functor $D : \mathbb{I} \longrightarrow \mathbb{C}$ with codomain $\mathbb{C}$. The category $\mathbb{I}$ is called the index category of the diagram D. We often denote the image of an object $i \in Ob(\mathbb{I})$ under a diagram D by D_i rather than $D(i)$. A diagram is said to be **small** whenever its index category is a small category.

A **cone** over a diagram D consists of an object P in $\mathbb{C}$ and a family of morphisms $\{p_i : P \longrightarrow D_i\}_{i \in Ob(\mathbb{I})}$ in $\mathbb{C}$ such that for each arrow $\alpha : i \longrightarrow j$ in $\mathbb{I}$ we have $D(\alpha) \circ p_i = p_j$. A cone $\mathcal{L} = (L, \{l_i : L \longrightarrow D_i\}_{i \in Ob(\mathbb{I})})$ over D is called a **limit** (or **limit source**) of the diagram D, provided that for every cone $\mathcal{Q} = (Q, \{q_i : Q \longrightarrow D_i\}_{i \in Ob(\mathbb{I})})$ over D there exists precisely one $\mathbb{C}$-morphism $m : Q \longrightarrow L$ such that $l_i \circ m = q_i$ for all $i \in Ob(I)$. The cone $\mathcal{Q}$ is called a competitor for the cone $\mathcal{L}$. The cone $\mathcal{L} = (L, \{l_i : L \longrightarrow D_i\}_{i \in Ob(\mathbb{I})})$ is called an **weak limit** of the diagram D, if we remove the uniqueness condition of the morphism m (i.e, for every cone $\mathcal{Q} = (Q, \{q_i : Q \longrightarrow D_i\}_{i \in Ob(\mathbb{I})})$ there exists at least one $\mathbb{C}$-morphism $m : Q \longrightarrow L$ such that $l_i \circ m = q_i$ for all $i \in Ob(\mathbb{I})$).

Example 2.18.1. ([3], chapter III, section 11)

1. Terminal objects are limits of the empty diagrams (i.e., diagrams of type the empty category[6]). One should note that a cone over an empty diagram is just an object in the target category).

2. Products are limits of diagrams of type discrete categories[7] (note that if $\mathbb{I}$ is a discrete category, then a diagram of type $\mathbb{I}$ in a category $\mathbb{C}$ is just a family of $\mathbb{C}$-objects).

3. Consider $\mathbb{I}$ as the category $\circ \rightrightarrows \circ$ (i.e., $\mathbb{I}$ is a category with two objects, their required identity morphisms and two parallel non-identity morphisms). A diagram of type $\mathbb{I}$ in a category $\mathbb{C}$ is simply a pair of parallel morphisms $f,\ g : A \longrightarrow B$ in $\mathbb{C}$. As a consequence we can say that equalizers of pairs of parallel morphisms are limits of diagrams of type $\mathbb{I}$. If, in the category $\mathbb{I}$, the two parallel non-identity morphisms are replaced by a set of parallel non-identity morphisms with cardinality $\lambda > 2$, then limits of diagrams of type $\mathbb{I}$ are equalizers of families of parallel morphisms indexed by sets with cardinality λ.

4. Take $\mathbb{I}$ to be the category $\circ \longrightarrow \circ \longleftarrow \circ$ (i.e., $\mathbb{I}$ is a category with three objects; their required identity morphisms and two non-identity morphisms with common codomain). As a diagram of type $\mathbb{I}$ in a category $\mathbb{C}$ is a 2-sink $A \xrightarrow{f} C \xleftarrow{g} B$ in $\mathbb{C}$), we can say that pullbacks of 2-sinks[8] are limits of the diagrams of type $\mathbb{I}$.

Colimits are defined dually. Explicitly, a **cocone** over the diagram D consists of an object C and morphisms $\{c_i : D_i \longrightarrow C\}_{i \in Ob(\mathbb{I})}$ such that for each arrow $\alpha : i \longrightarrow j$ in $\mathbb{I}$ we have $c_i \circ D(\alpha) = c_j$. A cocone $\mathcal{C} = (C, \{c_i : D_i \longrightarrow C\}_{i \in Ob(\mathbb{I})})$ is called a **colimit** (or colimit sink) of the diagram D, provided that for every cocone $\mathcal{R} = (R, \{r_i : D_i \longrightarrow R\}_{i \in Ob(\mathbb{I})})$ there exists precisely one morphism $r : C \longrightarrow R$ such that $r \circ c_i = r_i$ for all $i \in Ob(\mathbb{I})$. The cocone $\mathcal{R}$ mentioned above is called a competitor for the cocone $\mathcal{C}$.

[6] Recall that the empty category is the category whose class of objects is the empty set (see example 2.1.3).

[7] A discrete category $\mathbb{C}$ is a category whose only morphisms are the identity morphisms, i.e.

- $Hom_{\mathbb{C}}(A, A) = \{id_A\}$ for each $A \in Ob(\mathbb{C})$, and
- $.Hom_{\mathbb{C}}(A, B) = \emptyset$ for all $A \neq B$.

[8] A 2-sink is a sink consists of two morphisms.

Examples of colimits are given by the dual versions of the notions in example 2.18.1.

Example 2.18.2. ([3], chapter III, section 11) Initial objects are colimits of the empty diagrams. Sums correspond to colimits of diagrams of type discrete categories. Coequalizers of pairs of parallel morphisms correspond to colimits of diagrams of type $\circ \rightrightarrows \circ$. Pushouts of 2-sources sre colimits of diagrams of type $\circ \longleftarrow \circ \longrightarrow \circ$.

Remark 2.18.3. ([3], chapter III, section 10) It is easy to see that limit sources must be mono sources. To show this let $D : \mathbb{I} \longrightarrow \mathbb{C}$ be a diagram of type $\mathbb{I}$ in $\mathbb{C}$ and let the cone $\mathcal{L} = (L, \{l_i : L \longrightarrow D_i\}_{i \in Ob(\mathbb{I})})$ is a limit source of D. If $m_1, m_2 : Q \longrightarrow L$ are two $\mathbb{C}$-morphisms with $l_i \circ m_1 = l_i \circ m_2$ (for each $i \in Ob(\mathbb{I})$) then the cone $\mathcal{Q} = (Q, \{l_i \circ m_1\}_{i \in Ob(\mathbb{I})}) = (Q, \{l_i \circ m_2\}_{i \in Ob(\mathbb{I})})$ is a competitor for $\mathcal{L}$.The uniqueness requirement in the definition of in the definition of limit sources implies that $m_1 = m_2$. By invoking duality, we obtain that in any category $\mathbb{C}$, colimit sinks are epi sinks.

A limit is said to be small if it is a limit of a small diagram. Dually, small colimit are colimits of small diagrams. We say, a category $\mathbb{C}$ is **complete** if all small limits in $\mathbb{C}$ exist, and **cocomplete** if all small colimits in $\mathbb{C}$ exist.

As the following theorem is well known, we omit its proof.

Theorem 2.18.4. *([3], chapter III, section 12) Given a category $\mathbb{C}$. Then*

1. $\mathbb{C}$ is complete iff small products[9] and small equalizers[10] exist, and
2. $\mathbb{C}$ is cocomplete iff small sums and small coequalizers exist.

Example 2.18.5. The category *Top* has small products and small equalizers (see examples 2.5.5 and 2.5.7). Then according to the previous theorem, *Top* is both complete and cocomplete. Similarly, we can conclude that the category *Set* is complete and cocomplete (see example 2.5.3)

Remark 2.18.6. If K is a diagram in *Top*, then $U \circ K$ (where U is the forgetful functor from *Top* to *Set*) is called the underlying diagram of K in *Set*. One can easily see that the limit of K in *Top* is obtained by defining the initial topology on the limit of the diagram $U \circ K$. Specifically, let $K : \mathbb{I} \longrightarrow Top$ be a diagram of type $\mathbb{I}$ in *Top* with values denoted $K(i) = (X_i, \tau_i)$ for each $i \in Ob(\mathbb{I})$ (then the underlying diagram $U \circ K$ is a

[9] A small product (resp. sum) in a category $\mathbb{C}$ is a product (resp. sum) of a family of objects in $\mathbb{C}$ indexed by a set.

[10] A small equalizer (resp. coequalizer) in a category $\mathbb{C}$ is an equalizer (resp. a coequalizer) of a family of parallel morphisms in $\mathbb{C}$ indexed by a set.

diagram of type $\mathbb{I}$ in Set with values $U \circ K(i) = X_i$ for each $i \in Ob(\mathbb{I})$). Let the cone $(L, \{l_i : L \longrightarrow X_i\}_{i \in Ob(\mathbb{I})})$ be a limit of the underlying diagram $U \circ K$ in Set. Equip L with the initial topology τ_{int} generated by the source $\{l_i : L \longrightarrow X_i\}_{i \in Ob(\mathbb{I})}$. For any cone $((Q, \delta), \{q_i : Q \longrightarrow X_i\}_{i \in Ob(\mathbb{I})})$ over K there is a unique function $m : Q \longrightarrow L$ such that $l_i \circ m = q_i$ for all $i \in Ob(\mathbb{I})$ (because $(Q, \{q_i\}_{i \in Ob(\mathbb{I})})$ is a competitor for $(L, \{l_i\}_{i \in Ob(\mathbb{I})})$ in Set). Now, we require to show that the unique function m is continuous. Note that the set $\{l_i^{-1}(U_i) \mid i \in Ob(\mathbb{I}), U_i \in \tau_i\}$ is a subbase for the initial topology τ_{int} on L. Then according to remark 1.3.3, to show the continuity of m, we just need to check that $m^{-1}(l_i^{-1}(U_i))$ is an open subset of Q, where $i \in Ob(\mathbb{I})$ and U_i is an open subset of X_i. Fix $i \in Ob(\mathbb{I})$ and choose an open subset $U_i \subseteq X_i$, then due to the continuity of q_i we conclude that the set $q_i^{-1}(U_i)$ is an open subset of Q. Hence by the equality $m^{-1}(l_i^{-1}(U_i)) = q_i^{-1}(U_i)$ we conclude that $m^{-1}(l_i^{-1}(U_i))$ is open and so m is continuous. Dually, colimits in Top are obtained by replacing the final topology on the colimit of the underlying diagrams in Set (see also examples 2.5.5, 2.5.7,2.6.5, 2.7.4 and 2.8.4).

2.19. Functors and limits

Definition 2.19.1. Given a limit $\mathcal{L} = (L, \{L \xrightarrow{f_i} D_i\}_{i \in Ob(I)})$ of a diagram $D : I \longrightarrow \mathbb{C}_1$ in a category $\mathbb{C}_1$. A functor $F : \mathbb{C}_1 \longrightarrow \mathbb{C}_2$ is said to preserve the limit $\mathcal{L}$ provided that $F\mathcal{L} = (FL, \{FL \xrightarrow{Ff_i} FD_i\}_{i \in Ob(I)})$ is a limit of the diagram $F \circ D : I \longrightarrow \mathbb{C}_2$ in $\mathbb{C}_2$. We say that F weakly preserves the limit $\mathcal{L}$ if $F\mathcal{L} = (FL, \{FL \xrightarrow{Ff_i} FD_i\}_{i \in Ob(I)})$ is a weak limit of the diagram $F \circ D : I \longrightarrow \mathbb{C}_2$ in $\mathbb{C}_2$.

Notice that colimits-preservation is a dual notion.

Example 2.19.2. According to example , the forgetful functor $U : Top \longrightarrow Set$ preserves all limits and colimits.

Example 2.19.3. Accordind to remarks 2.5.6 the discrete functor $D : Set \longrightarrow Top$ preserves finite products[11]. Besides, due to the remark 2.5.8 this functor preserves all sums.

2.20. Natural transformations

Definition 2.20.1. Let $F, G : \mathbb{C} \longrightarrow \mathbb{D}$ be functors. A **natural transformation** η from F to G associates to each object $X \in \mathbb{C}$ a $\mathbb{D}-$morphism $\eta_X : F(X) \longrightarrow G(X)$, such that for each $\mathbb{C}-$morphism $f : X \longrightarrow Y$ we have

$$Gf \circ \eta_X = \eta_Y \circ Ff$$

[11] A finite product in a category $\mathbb{C}$ is a product of a finite set of objects in $\mathbb{C}$.

that is, the following diagram commutes.

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "Ff"'] & G(X) \arrow[d, "Gf"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

We write $\eta : F \longrightarrow G$ if η is a natural transformation from F to G. We denote the set of all natural transformation from F to G by $Nat\,(F, G)$.

Definition 2.20.2. Suppose F, $G : \mathbb{C} \longrightarrow \mathbb{D}$ are functors and $\eta : F \longrightarrow G$ is a natural transformation from F to G. we call η a **natural isomorphism**, if each component η_X ($X \in \mathbb{C}$) is an isomorphism in $\mathbb{D}$. In this case we write $F \cong G$. More generally, we call η as an $\mathcal{M}$-**transformation** (where $\mathcal{M}$ is a special class of $\mathbb{D}$-morphisms), if $\eta_X \in \mathcal{M}$ for each $X \in \mathbb{C}$ (see section 6 of chapter I in [3]).

Definition 2.20.3. Suppose F, $G : \mathbb{C} \longrightarrow \mathbb{C}$ are $\mathbb{C}-$endofunctors and $\mathbb{C}$ is a $(\mathcal{E}, \mathcal{M})$-category. We say that G is ***factor** of* F, if there is an $\mathcal{E}$- transformation $\eta : F \longrightarrow G$.

Lemma 2.20.4. *Let $\eta : F \longrightarrow G$ be a natural transformation between two Functors F and G.*

1. If η is an epi-transformation, then : if F preserves epis then G preserves epis, too.
2. If η is a mono-transformation, then : if G preserves monos then F preserve monos, too.

Proof. Let $f : X \longrightarrow Y$ be an epimorphism and $P_1, P_2 : G(Y) \rightrightarrows Z$ two morphism such that $P_1 \circ Gf = P_2 \circ Gf$. We have to show $P_1 = P_2$. Since $\eta : F \longrightarrow G$ is a natural transformation, the diagram below commutes.

$$\begin{tikzcd} F(X) \arrow[r, "\eta_X"] \arrow[d, "Ff"'] & G(X) \arrow[d, "Gf"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

So, we have

$$\begin{aligned} P_1 \circ \eta_Y \circ Ff &= P_1 \circ Gf \circ \eta_X \\ &= P_2 \circ Gf \circ \eta_X \\ &= P_2 \circ \eta_Y \circ Ff. \end{aligned}$$

By assumption Ff is epi, so $P_1 \circ \eta_Y = P_2 \circ \eta_Y$. Since η_Y is epi, we have $P_1 = P_2$. The second part of this lemma will be prove in a similar way. □

Lemma 2.20.5. *Let $\mathbb{C}$ and $\mathbb{C}'$ are two categories and U, F and G are functors shown in the following diagram,*

$$\begin{tikzcd} \mathbb{C}' \arrow[loop left, "G"] & & \mathbb{C} \arrow[ll, bend left, "U"] \arrow[loop right, "F"] \end{tikzcd}$$

Assume, there is a mono-transformations $\mu : UF \to GU$. If G preserves monos and U preserves and reflects monos, then F preserves monos too.

Proof. Let $f : A \longrightarrow B$ be a monomorphism in $\mathbb{C}$. Since U reflects monos, it suffices to show that UFf is also a monomorphism in $\mathbb{C}'$. Suppose $P_1, P_2 : C \rightrightarrows UF(A)$ are two morphisms such that $UFf \circ P_1 = UFf \circ P_2$. We have to show $P_1 = P_2$. Since $\eta : UF \longrightarrow GU$ is a natural transformation, the following diagram is commutative.

$$\begin{tikzcd} UFA \arrow[r, "UF(f)"] \arrow[d, "\mu_A"'] & UFB \arrow[d, "\mu_B"] \\ GUA \arrow[r, "GU(f)"'] & GUB \end{tikzcd}$$

So, we have

$$\begin{aligned} GU(f) \circ \mu_A \circ P_1 &= \mu_B \circ UF(f) \circ P_1 \\ &= \mu_B \circ UF(f) \circ P_2 \\ &= GU(f) \circ \mu_A \circ P_2 \end{aligned}$$

By assumption $GU(f)$ is mono, so $\mu_A \circ P_1 = \mu_A \circ P_2$. Since η_A is mono, we have $P_1 = P_2$. □

2.21. Adjunction

Definition 2.21.1. An adjunction between categories $\mathbb{C}$ and $\mathbb{D}$ consists of:

1. a functor $F : \mathbb{D} \longrightarrow \mathbb{C}$ called the **left adjoint**,
2. a functor $G : \mathbb{C} \longrightarrow \mathbb{D}$ called the **right adjoint**,
3. a natural transformation $\xi : FG \longrightarrow 1_{\mathbb{C}}$ called **counit**, and
4. a natural transformation $\eta : 1_{\mathbb{D}} \longrightarrow GF$ called **unit**,

such that satisfying the following equations:

$$1_{FX} = \xi_{FX} \circ F(\eta_X)$$

$$1_{GY} = G(\xi_Y) \circ \eta_{GY}$$

Note that $1_{\mathbb{C}}$ (resp. $1_{\mathbb{D}}$) denotes the identity functor on the category $\mathbb{C}$ (resp. $\mathbb{D}$) and 1_{FX} (resp. 1_{GY}) denotes the identity morphism of the object FX (resp. GY).

Example 2.21.2. ([3], chapter V, section 18) The discrete functor $D : Set \longrightarrow Top$ is a left adjoint for the forgetful functor $U : Top \longrightarrow Set$. The indiscrete functor $I : Set \longrightarrow Top$ is a right adjoint for the forgetful functor U.

Part II.

Topological functors

3. Functors on Top

In this part we study some Top-endofunctors. We show that not every endofunctor on Top preserves monos or regular monos and similarly, not every endofunctor on Top preserves epis or regular epis. We explain that if F is a Top-endofunctor which preserves all monos with non-empty domain, then we can modify F to construct a Top-endofunctor F^+ (called the positive functor of F) that preserves all monos. The idea of the construction of F^+ is from Barr [10].

3.1. Polynomial functors on Top

The class of polynomial functors on Top is a topological analogue of the polynomial functors on Set (see definition 2.15.9), and they will be inductively defined as follows:

$$F ::= C \mid id_{Top} \mid F_1 + F_2 \mid F_1 \times F_2 \mid F^D.$$

Here id_{Top} is the identity functor on the category Top; C denote the constant functor (for an arbitrary topological space C); $+$ and $\times$ are sum and binary product in Top, respectively; and for every set D, we consider F^D as the functor sending a topological space X to the D-fold product $(F(X))^D$ in Top (i.e, $F^D := (-)^D \circ F$, the composition of functor F and power functor $(-)^D$on Top).

Example 3.1.1. Given a fixed topological space C and a fixed set D, then the construction $F(-) := C \times (-)^D$ (the product of the constant functor C with the power functor $(-)^D$) is a polynomial functor on Top.

Lemma 3.1.2. *Polynomial functors on Top preserve monos.*

Proof. It is known that the identity functor, the constant functor and the power functor $(-)^D$ preserve monos (see also example 2.17.5). The rest of this proof follows immediately from lemma 2.17.6. □

Lemma 3.1.3. *If F_1 and F_2 are two Top-endofunctors preserving topological embeddings, then $F_1 + F_2$, $F_1 \times F_2$ and $F_1 \circ F_2$ preserve topological embeddings too.*

Proof. We check this claim for $F_1 \times F_2$. Assume $\iota : S \longrightarrow X$ is a topological embedding (see example 1.4.2). By assumption the morphisms $F_1(\iota) : F_1(S) \longrightarrow F_1(X)$ and $F_2(\iota)$:

$F_2(S) \longrightarrow F_2(X)$ are topological embeddings. So $F_1(S)$ and $F_2(S)$ are subspaces of $F_1(X)$ and $F_2(X)$, respectively. Then by lemma 2.5.9, $F_1(S) \times F_2(S)$ is a subspace of $F_1(X) \times F_2(X)$. Now by the definition of $F_1 \times F_2$ (see remark 2.15.15),

$$(F_1 \times F_2)(\iota)(x_1, x_2) := (F_1(\iota)(x_1), F_2(\iota)(x_2)) = (x_1, x_2)$$

where $(x_1, x_2) \in F_1(S) \times F_2(S)$. It means $(F_1 \times F_2)(\iota)$ is a topological embedding. We can prove this lemma for $F_1 + F_2$ and $F_1 \circ F_2$ in a similar way. □

Corollary 3.1.4. *If F_1 and F_2 are two Top-endofunctors preserving regular monos, then $F_1 + F_2$, $F_1 \times F_2$ and $F_1 \circ F_2$ preserve regular monos too.*

Proof. Since by lemma 2.9.3, in the category *Top* regular monos are (up to isomorphism) precisely the topological embeddings, this corollary follows from the previous lemma. □

Corollary 3.1.5. *Polynomial functors on Top preserve regular monos.*

3.2. Vietoris functor

For every topological space X, the Vietoris space $\mathbb{V}(X)$ has as the base set the set of all compact[1] subsets $K \subseteq X$. The Vietoris topology on $\mathbb{V}(X)$ is generated by a subbase consisting of all sets

- $[U] := \{K \in \mathbb{V}(X) | K \subseteq U\}$, and
- $\langle U \rangle := \{K \in \mathbb{V}(X) | K \cap U \neq \emptyset\}$,

where U ranges over all open subsets of X.

Note that for every subset U of X, we can define $[U]$ and $\langle U \rangle$ as they are defined above for the open subsets. So for each subset U of X, we have $[U]^c = \langle U^c \rangle$ (where U^c is the complement of U).

By using the fact that the image $f(K)$ of a compact set K by a continuous map f is compact, we can extend the Vietoris construction to a functor $\mathbb{V} : Top \longrightarrow Top$ which associates to each topological space X, its Vietoris space $\mathbb{V}(X)$ and to each continuous map $f : X \longrightarrow Y$ the continuous map $\mathbb{V}f : \mathbb{V}(X) \longrightarrow \mathbb{V}(Y)$ given by $(\mathbb{V}f)(K) = f[K]$ (for all compact subset $K \subseteq X$). To check that $\mathbb{V}f$ is continuous, we only need to check

[1] A subset K of a topological space (X, τ) is compact if every open cover of K has a finite subcover.

that inverse images of subbase members are open (see remark 1.3.3). Thus, let U be an open subset of Y, then

$$\begin{aligned}(\mathbb{V}f)^{-1}([U]) &= \{K \in \mathbb{V}(X) \mid f(K) \in [U]\} \\ &= \{K \in \mathbb{V}(X) \mid f(K) \subseteq U\} \\ &= \{K \in \mathbb{V}(X) \mid K \subseteq f^{-1}(U)\} \\ &= [f^{-1}(U)]\end{aligned}$$

and

$$\begin{aligned}(\mathbb{V}f)^{-1}(\langle U\rangle) &= \{K \in \mathbb{V}(X) \mid f(K) \in \langle U\rangle\} \\ &= \{K \in \mathbb{V}(X) \mid f(K) \cap U \neq \emptyset\} \\ &= \{K \in \mathbb{V}(X) \mid K \cap f^{-1}(U) \neq \emptyset\} \\ &= \langle f^{-1}(U)\rangle\end{aligned}$$

In each case, the result is open in $\mathbb{V}(X)$, in fact, $(\mathbb{V}f)^{-1}$ takes the subbase of $\mathbb{V}(Y)$, to the subbase of $\mathbb{V}(X)$ (see also [69]). To see that this construction is also an endofunctor on *Stone*, refer to [47].

Definition 3.2.1. [11] **(P -Vietoris functor)** Let P be a set, consider $\mathbb{P}(P)$ as the set of all subsets of P equipped with the topology generated by a subbase containing all clopen sets of the form $\uparrow p := \{u \subseteq P \mid p \in u\}$, where $p \in P$. The P-Vietoris functor (in symbol: $\mathbb{V}_P$) is the product of Vietoris functor $\mathbb{V}$ with the constant functor with value $\mathbb{P}(P)$ on *Top* (i.e., $\mathbb{V}(-) \times \mathbb{P}(P)$).

Lemma 3.2.2. *The Vietoris functor preserves mono.*

Proof. Let $f : X \longrightarrow Y$ be an arbitrary monomorphism in *Top*. If $X = \emptyset$ then $\mathbb{V}f$ is a map from the one element space $\{\emptyset\}$ to $\mathbb{V}(Y)$ and consequently it is mono (because it is injective). Now suppose $X \neq \emptyset$. It suffices to show that $\mathbb{V}f : \mathbb{V}(X) \longrightarrow \mathbb{V}(Y)$ is injective. Assume $K_1, K_2 \in \mathbb{V}(X)$ such that $\mathbb{V}f(K_1) = \mathbb{V}f(K_2)$. So $f(K_1) = f(K_2)$. Since f is mono, it is injective. Thus from $f(K_1) = f(K_2)$ we obtain that $K_1 = K_2$. □

Remark 3.2.3. Notice that if S and X are topological spaces, then a continuous map $f : S \longrightarrow X$ is a regular monomorphism in *Top* iff f is mono and S carries the initial topology generated by f. To see this issue, suppose $f : S \longrightarrow X$ is a monomorphism such that S carries the initial topology generated by f. Let $f = m \circ r$ be a factorization of f in the (*epi*, *regular mono*)-system on *Top*. It suffices to prove that r is an isomorphism in

Top. Since f and m are monos, r is mono and consequently injective. Also r is surjective (because it is epi). Then r is a bijective and continuous map. Now we show that r is an open map. Since S carries the initial topology generated by f, to show that r is an open map we just need to prove that for each open subset $O \subseteq X$ the set $r(f^{-1}(O))$ is open in $cod(r)$. Notice that from $f = m \circ r$ we obtain $f^{-1}(O) = r^{-1}(m^{-1}(O))$ for each open subset $O \subseteq X$. Then since r is bijective, we have $r(f^{-1}(O)) = m^{-1}(O)$. As m is continuous, we know that $m^{-1}(O)$ is an open subset of $cod(r)$ and consequently $r(f^{-1}(O))$ is open in $cod(r)$. To prove the converse direction, recall that in *Top* the regular monos are exactly the topological embeddings (see lemma 2.9.3). So, if f is regular mono then $f = \iota \circ r$ where $\iota : A \longrightarrow X$ is a topological embedding and $r : S \longrightarrow A$ is an isomorphism in *Top*. It is easy to see that f is mono and S carries the initial topology generated by f.

Lemma 3.2.4. *The Vietoris functor preserves regular monos.*

Proof. Suppose $f : S \longrightarrow X$ is a regular mono in *Top*. If $S = \emptyset$ then $\mathbb{V}f$ is a map from the one element space $\{\emptyset\}$ to $\mathbb{V}(Y)$ and consequently it is regular mono (because $\mathbb{V}f$ is mono and $\{\emptyset\}$ carries the initial topology generated by $\mathbb{V}f$, see the previous remark). Now suppose $S \neq \emptyset$. Since f is regular mono, it is mono. Then by lemma 3.2.2, $\mathbb{V}f$ is mono . Now, it suffices to check that

$$\forall U \underset{open}{\subseteq} S. \exists O \underset{open}{\subseteq} X. (\mathbb{V}(f))^{-1}[O] = [U].$$

Let U be an arbitrary open subset of S. Since f is a regular mono in *Top*, by remark 3.2.3, S carries the initial topology generated by f i.e.

$$\exists O_U \underset{open}{\subseteq} X. U = f^{-1}(O_U).$$

So,

$$\begin{aligned}
(\mathbb{V}f)^{-1}([O_U]) &= \{K \underset{com}{\subseteq} S \mid f(K) \in [O_U]\} \\
&= \{K \underset{com}{\subseteq} S \mid f(K) \subseteq O_U\} \\
&= \{K \underset{com}{\subseteq} S \mid K \subseteq f^{-1}(O_U)\} \\
&= [f^{-1}(O_U)] \\
&= [U].
\end{aligned}$$

and similarly we can show that $(\mathbb{V}f)^{-1}(\langle O_U \rangle) = \langle U \rangle$. So by lemma 1.3.4, $\mathbb{V}(S)$ carries the initial topology generated by $\mathbb{V}f$. Hence by remark 3.2.3, $\mathbb{V}f$ is regular mono.

$\square$

Remark 3.2.5. If X and Y are two fixed topological space then according to theorem 1.6.11 (Tychonoff's Theorem) we have $\mathbb{V}(X)\times\mathbb{V}(Y) \subseteq \mathbb{V}(X\times Y)$. In general the converse of this inclusion does not hold. Notice that $\mathbb{V}(X) \times \mathbb{V}(Y) \supseteq \mathbb{V}(X \times Y)$ iff

$$\forall K \underset{com}{\subseteq} X \times Y. K = \pi_X(K) \times \pi_Y(K) \tag{3.2.1}$$

(since the canonical projections $\pi_X : X \times Y \longrightarrow X$ and $\pi_Y : X \times Y \longrightarrow Y$ are continuous, by remark 1.6.5 the sets $\pi_X(K)$ and $\pi_Y(K)$ are the compact subsets of X and Y, respectively). By giving an easy example we can see that statement 3.2.1 in general does not hold. Consider the topological space X and Y as the two element discrete space $\{1, 2\}$. The set $K := \{(1, 2),(2, 1)\}$ is a compact subset of $X \times Y$ (finite subsets are compact) such that $K \neq \pi_X(K) \times \pi_Y(K)$. Therefore $\mathbb{V}(X) \times \mathbb{V}(Y) \not\supseteq \mathbb{V}(X \times Y)$. As a consequence, we can say that the Vietoris functor does not preserve products.

3.3. Vietoris polynomial functors

The class VPF of Vietoris polynomial functors over Top is inductively defined as follows:

$$F ::= C \mid id_{Top} \mid \mathbb{V} \mid F_1 + F_2 \mid F_1 \times F_2 \mid F^D.$$

Notice that the Vietoris polynomial functors are topological version of the Kripke polynomial functors (see example 2.16.2) and they will be obtained by adding the Vietoris functor to the grammar of the polynomial functors on Top.

Remark 3.3.1. Since the Vietoris functor and the polynomial functors over Top preserve monos (see lemmas 3.2.2 and 3.1.2), by lemma 2.17.6 we conclude that the Vietoris polynomial functors preserve monos. Similarly, since the Vietoris functor and the polynomial functors over Top preserve regular monos (see lemma 3.2.4 and corollary 3.1.5), according to corollary 3.1.4 the Vietoris polynomial functors preserve regular monos.

3.4. Path Functor

Definition 3.4.1. A path in a topological space X is a continuous map p from the unit interval $I = [0, 1]$ to X. A path from $x \in X$ to $y \in X$ is a continuous map $p : I \longrightarrow X$ with $p(0) = x$ and $p(1) = y$. We denote by $\overrightarrow{P}(X)(x, y)$ the set of all paths in X between elements x and y. There are some paths started from a point $x \in X$ and never leave it. These paths are called the constant path. A loop based at $x \in X$ is a path from x to x. Suppose p is a path from x to y and q is a path from y to z. The (in symbol: $p \star q$) is a path defined by first traversing p and then traversing q:

$$p \star q(s) := \begin{cases} p(2s) & 0 \leq s \leq 1/2 \\ q(2s-1) & 1/2 \leq s \leq 1 \end{cases} \tag{3.4.1}$$

We say that two elements $x, y \in X$ are path-connected, if there exists a path from x to y. A topological space X is called path-connected, if any two elements $x, y \in X$ are path-connected. Now let us to define the relation $\rightleftarrows$ on the topological space X as follows:

$$x \rightleftarrows y \iff x \text{ and } y \text{ are path-connected}$$

$\rightleftarrows$ is an equivalence relation and its equivalence classes are called the path-components of X. For each $x \in X$, consider $[x]_{\rightleftarrows}$ as the path-component of X containing x (that is, the set of points $y \in X$ with $x \rightleftarrows y$). We denote by $\Pi_0(X)$, the set of all path-components of X (i.e., $\Pi_0(X) = \{[x]_{\rightleftarrows} \mid x \in X\}$). Then the map $q_X : X \longrightarrow \Pi_0(X)$ defined as $q_X(x) = [x]_{\rightleftarrows}$ (for each $x \in X$), is well-defined and surjective.
Provide $\Pi_0(X)$ with the quotient topology generated by the map q_X. If $f : X \longrightarrow Y$ is a continuous map between two topological spaces, then $f([x]_{\rightleftarrows}) \subseteq [f(x)]_{\rightleftarrows}$ for each $x \in X$. Hence the construction Π_0 can be extended to an endofunctor on the category Top, if for each continuous map $f : X \longrightarrow Y$ we define $\Pi_0(f) : \Pi_0(X) \longrightarrow \Pi_0(Y)$ by

$$\Pi_0(f)([x]_{\rightleftarrows}) := [f(x)]_{\rightleftarrows}.$$

To check the continuity of $\Pi_0(f)$, we only need to check that inverse images of every element in the quotient topology over $\Pi_0(Y)$ is open in $\Pi_0(X)$. Consider the following diagram.

$$\begin{tikzcd} X \arrow[r, "q_X"] \arrow[d, "f"'] & \Pi_0(X) \arrow[d, "\Pi_0(f)"] \\ Y \arrow[r, "q_Y"'] & \Pi_0(Y) \end{tikzcd}$$

This diagram is commutative, i.e. $\Pi_0(f) \circ q_X = q_Y \circ f$. Now, let U be an element in the quotient topology over $\Pi_0(Y)$, then $q_Y^{-1}(U)$ is an open subset of Y, and consequently by the continuity of f, it is concluded that $f^{-1}(q_Y^{-1}(U))$ is an open subset of X. Since $f^{-1}(q_Y^{-1}(U)) = q_X^{-1}(\Pi_0(f)^{-1}(U))$, we have that $q_X^{-1}(\Pi_0(f)^{-1}(U))$ is an open subset of X. So as $\Pi_0(X)$ carries the quotient topology generated by q_X, we obtain that $\Pi_0(f)^{-1}(U)$ is open in $\Pi_0(X)$, see [18].

3.5. covariant Hom_{Top} functors

The covariant Hom_{Top} functor is a topological version of the covariant Hom_{Set} functor (see example 2.15.6) and it is defined as follows:

Lemma 3.5.1. *For a fixed topological space Σ, the construction $F(-) := Hom_{Top}(\Sigma, -)$ which associates every topological space X to the set of all continuous functions from Σ to X with the compact-open topology (see definition 2.14.1) and every continuous function $f : X \longrightarrow Y$ to the function $F(f) : F(X) \longrightarrow F(Y)$ defined as $(Ff)(\delta) := f \circ \delta$, is an endofunctor on the category Top.*

Proof. For each topological space X we have $id_X \circ \delta = \delta$ and so $F(id_X) = id_{FX}$. Besides, since the composition of maps is associative, we conclude that $F(f \circ g) = Ff \circ Fg$. Then to prove this lemma, we just need to show that for each continuous function $f : X \longrightarrow Y$ the function $F(f) : Hom_{Top}(\Sigma, X) \longrightarrow Hom_{Top}(\Sigma, Y)$ is continuous. Let $[K, U]$ be an element in the sub-base of the compact-open topology on $Hom_{Top}(\Sigma, Y)$.

$$\begin{aligned}
(Ff)^{-1}([K,U]) &= \{\delta \in F(X) \mid f \circ \delta \in [K,U]\} \\
&= \{\delta \in F(X) \mid f(\delta(K)) \subseteq U\} \\
&= \{\delta \in F(X) \mid \delta(K) \subseteq f^{-1}(U)\} \\
&= \{\delta \in F(X) \mid \delta \in [K,\, f^{-1}(U)]\} \\
&= [K,\, f^{-1}(U)].
\end{aligned}$$

□

Lemma 3.5.2. *For a fixed topological space Σ, the functor $F(-) := Hom_{Top}(\Sigma, -)$ preserves monos.*

Proof. Assume $f : X \longrightarrow Y$ is a monomorphism in Top. Let $h_1, h_2 : C \longrightarrow F(X)$ be two arbitrary continuous functions such that $Ff \circ h_1 = Ff \circ h_2$. So $(Ff \circ h_1)(c) = (Ff \circ h_2)(c)$, for each $c \in C$. Thus, $Ff(h_1(c)) = Ff(h_2(c))$ and then by the definition of F on morphisms, $f \circ (h_1(c)) = f \circ (h_2(c))$. Since f is mono, we have $h_1(c) = h_2(c)$. Now, since $c \in C$ is arbitrary, we conclude that $h_1 = h_2$.

□

Lemma 3.5.3. *The covariant functor $F(-) := Hom_{Top}(\Sigma, -)$ preserves regular monos.*

Proof. Suppose $f : S \longrightarrow X$ is a regular monomorphism in Top. By the previous lemmas, F preserves monos and so Ff is mono. Now, we show that the following statement holds:

$$\forall K \underset{com}{\subseteq} \Sigma.\, \forall U \underset{open}{\subseteq} S.\, \exists O \underset{open}{\subseteq} X.\, [K, U] = (Ff)^{-1}([K, O]).$$

Let $K \subseteq \Sigma$ be a fixed compact subset of Σ and $U \subseteq S$ be a fixed open subset of S. Since f is a regular monomorphism in Top, by remark 3.2.3 we conclude that S carries the initial topology generated by f, i.e.

$$\exists O_U \underset{open}{\subseteq} X.\, U = f^{-1}(O_U).$$

So

$$\begin{aligned}
(Ff)^{-1}([K,O_U]) &= \{\delta \in F(S) \mid f \circ \delta \in [K,O_U]\} \\
&= \{\delta \in F(S) \mid (f \circ \delta)(K) \subseteq O_U\} \\
&= \{\delta \in F(S) \mid \delta(K) \subseteq f^{-1}(O_U)\} \\
&= \{\delta \in F(S) \mid \delta(K) \subseteq U\} \\
&= [K,U].
\end{aligned}$$

So by lemma 1.3.4, $F(S)$ carries the initial topology generated by Ff. Hence by remark 3.2.3, Ff is regular mono.

$\square$

3.6. Contravariant Hom_{Top} functor and neighborhood functors

Lemma 3.6.1. *For a fixed topological space Σ, the construction $F(-) := Hom_{Top}(-, \Sigma)$ on Top which associates every topological space X to the set of all continuous functions from X to Σ with the compact-open topology and every continuous function $f : X \longrightarrow Y$ to the function $F(f) : F(Y) \longrightarrow F(X)$ defined as $(Ff)(\delta) := \delta \circ f$ is a contravariant endofunctor on Top.*

Proof. For each topological space X we have $id_X \circ \delta = \delta$ and then $F(id_X) = id_{FX}$. Besides since the composition of functions is associative, we have $F(f \circ g) = Ff \circ Fg$. Hence to prove this lemma, we just need to show that for each continuous function $f : X \longrightarrow Y$, the morphism $F(f) : F(Y) \longrightarrow F(X)$ is continuous. Let $[K, U]$ be an element in the subbase of the compact-open topology on $F(X)$. Then

$$\begin{aligned}(Ff)^{-1}([K, U]) &= \{\delta \in F(Y) \mid \delta \circ f \in [K, U]\}\\ &= \{\delta \in F(Y) \mid \delta(f(K)) \subseteq U\}\\ &= \{\delta \in F(Y) \mid \delta \in [f(K), U]\}\\ &= [f(K), U].\end{aligned}$$

$\square$

Example 3.6.2. Consider the two elements set $2 := \{0, 1\}$. In fact, there are only three inequivalent[2] topologies on the set 2. In this example we want to verify the contravariant functor $F(-) := Hom_{Top}(-, 2)$, for all three cases. Suppose X is an arbitrary topological space.

a) If 2 is an indiscrete space, then the topological space $F(X)$ is the set $\mathbb{P}(X)$ with the indiscrete topology.

b) In case that we provide 2 with the Sierpinski topology (i.e., the only open sets are $\emptyset$, $\{1\}$ and 2), then there are two possibility for the topological space $F(X)$ as follows:

  b-1) $F(X)$ is the set $\{U \subseteq X \mid U$ is an open subset of $X\}$ with the topology generated by a subbase consisting all elements as

$$[K] = \{U \underset{open}{\subseteq} X \mid K \subseteq U\} \tag{3.6.1}$$

  where K ranges over all compact subsets of X.

[2]We say that two topologies τ and δ on a set X are equivalent iff the topological spaces (X, τ) and (X, δ) are homeomorphic spaces (i.e., $(X, \tau) \cong (X, \delta)$).

b-2) $F(X)$ is the set $\{U \subseteq X \mid U \text{ is a closed subset of } X\}$ with the topology generated by a subbase consisting all elements as

$$[K] \quad = \quad \{U \underset{closed}{\subseteq} X \mid K \cap U = \emptyset\} \tag{3.6.2}$$

where K ranges over all compact subsets of X.

c) Whenever the set 2 carries the discrete topology, the topological space $F(X)$ is the set $\{U \subseteq X \mid U \text{ is a clopen subset of } X\}$ which carries a topology similar to what mentioned in part (b) (notice that in this case $[K] = \{U \underset{clopen}{\subseteq} X \mid K \subseteq U\}$ for each compact subsets K of X).

For all cases, if $f : X \longrightarrow Y$ is a continuous map, then $F(f)(V) = f^{-1}(V)$ for each $V \in F(Y)$.
As it has been mentioned in lemma 3.6.1, $F(-)$ is a contravariant endofunctor on Top. Therefore, $F \circ F$ is a covariant endofunctor on Top. This functor will be called

- **Full neighborhood functor:** in case that the set 2 is equipped by trivial topology;
- **Open neighborhood functor:** if 2 carries the Sierpinski topology and $F(X)$ (for each topological space X) is the topological space in **(b-1)**;
- **Closed neighborhood functor:** if we povide 2 with Sierpinski topology and $F(X)$ (for each topological space X) is the topological space in **(b-2)**;
- **Clopen neighborhood functor:** whenever 2 is a discrete space.

3.7. Properties of Top-Endofunctors

By lemma 2.17.1, every endofunctor on Top preserves isomorphisms. However, there are Top-endofunctors that do not preserve monos or regular monos. We have a similar problem for epis and regular epis. It means, not every endofunctors on Top preserves epis or regular epis. In this part we try to clear this issue with some examples.

Epi-preservation

In general, not every endofunctor on Top preserves epis. The following two examples make this claim more clear. In the first one we will see that the covariant functor $F(-) = Hom_{Top}(\Sigma, -)$ does not preserve epimorphisms which are not right invertible. The second example presents an epimorphism f in the category Top for which $\mathbb{V}(f)$ ($\mathbb{V}$ is the Vietoris functor) is not epi.

Example 3.7.1. Consider the covariant functor $F(-) = Hom_{Top}(\Sigma, -)$. Suppose the Top-morphism $e : X \longrightarrow \Sigma$ is an epimorphism which is not right invertible. We claim that the morphism $F(e) : F(X) \longrightarrow F(\Sigma)$ is not surjective and consequently not epi. Consider the identity map $id_\Sigma : \Sigma \longrightarrow \Sigma$. Since e is not right invertible, there is no morphism $\tau : \Sigma \longrightarrow X$ in Top such that $e \circ \tau = id_\Sigma$. Then id_Σ does not belong to the image of the morphism $Hom(\Sigma, -)(e)$.

Example 3.7.2. Let X be an infinite set. Define the map $f : X_D \longrightarrow X_I$ as the identity map id_X. It is easy to see that f is continuous and surjective map and thus an epimorphism in the category Top. Since a discrete space is compact iff X is finite, we have $X \notin \mathbb{V}(X_D)$. On the other hand every indiscrete space is compact, so $X \in \mathbb{V}(X_I)$. Hence the continuous map $\mathbb{V}(f) : \mathbb{V}(X_D) \longrightarrow \mathbb{V}(X_I)$ is not surjective and consequently it is not epi.

Regular epi-preservation

In the sequel we will show that the Vietoris functor does not preserve regular epis.

Example 3.7.3. Define the map $e : \mathbb{R} \longrightarrow [a, b]$ as

$$e(x) := \begin{cases} x & a \lneqq x \lneqq b \\ a & x = b \\ b & else \end{cases}$$

where $\mathbb{R}$ is the set of real numbers with the standard topology. The map e is a regular epimorphism in Top, if we provide the closed interval $[a, b]$ with the quotient topology generated by e (in short: $[a, b]_e$). Now we prove that $\mathbb{V}(e)$ is not a regular epimorphism in Top. To show this claim we need to prove the following two claims:

Claim. The half open interval $[a, b)$ is a compact subset of the closed interval $[a, b]_e$.

Proof. Notice that the only open subset of $[a, b]_e$ containing the element a is the whole space $[a, b]_e$. If $\bigcup_{i \in I} O_i$ is an open cover of the interval $[a, b) \subseteq [a, b]_e$, then there is $i \in I$ such that $O_i = [a, b]_e$. This yields that $[a, b)$ is compact in $[a, b]_e$. □

In fact it can be seen that every subset of $[a, b]_e$ which contains a is compact.

Now apply the Vietoris functor $\mathbb{V}$ on the regular epi e.

Claim. $\mathbb{V}(e)$ is not surjective.

Proof. We want to show that there is no compact subset $C \subseteq \mathbb{R}$ with $e(C) = [a, b)$. We do this by contradiction. Assume there is a compact subset $C \subseteq \mathbb{R}$ with $e(C) = [a, b)$. Then we have $C \subseteq (a, b]$ (by definition of e). Hence $C \subsetneqq (a, b]$ (since C is compact and $(a, b]$ is not compact in $\mathbb{R}$). Then $e(C) \subsetneqq e(a, b] = [a, b)$ (since $e \mid_{(a,b]}$ (restriction of e) is an injective map). This contradicts the assumption. $\square$

Mono-preservation

Note that there are endofunctors on the category Top which do not preserve monos, amongst them the path-functor $\Pi_0 : Top \longrightarrow Top$. The following example bring this matter to light.

Example 3.7.4. Let $S = \{1, 3\}$ be a subspace of the set of real numbers $\mathbb{R}$ with the standard topology. So, S is the discrete space and then the only continuous maps from the unit interval $I = [0, 1]$ to S are the constant maps. Hence $\Pi_0(S)$ is the set $\{\{1\}, \{3\}\}$ with the discrete topology (recall that $\Pi_0(S)$ carries the quotient topology generated by $q_X : S \longrightarrow \Pi_0(S)$, and note that the quotient topology generated by a surjective map from a discrete space is the discrete topology). On the other hand, since $\mathbb{R}$ is a path-connected space, we have $\Pi_0(\mathbb{R}) = 1$. Now let $\iota : S \longrightarrow \mathbb{R}$ be the subspace inclusion, it is obvious that ι is mono but $\Pi_0(\iota)$ is not mono (because it is not injective).

Example 3.7.5. The following functors preserve monos:

- polynomial functors,
- Vietoris functor,
- Vietoris polynomial functors, and
- covariant Hom_{Top} functor.

Regular mono-preservation

Recall from lemma 2.9.3 that a morphism in Top is regular mono iff it is a topological embedding. Generally, not every endofunctor on Top preserves regular monos. For instance, in example 3.7.4 we have seen that the path-functor $\Pi_0 : Top \longrightarrow Top$ does not preserve regular monos. In this example, we have introduced a regular mono ι such that its image under the functor Π_0 (i.e., $\Pi_0(\iota)$) is not mono and consequently it is not reqular mono.

Example 3.7.6. The following functors preserve regular monos:

- Polynomial functors,
- Vietoris functor,
- Vietoris polynomial functors, and
- Covariant Hom_{Top} functor.

3.8. Positive functors

As it has been mentioned by Barr in [10], if T is a Set-endofunctor with $T(\emptyset) \neq \emptyset$, then the morphism $T(\emptyset_X)$ (where $\emptyset_X$ is the unique morphism from the empty set $\emptyset$ to some set X) is not necessarily injective and consequently it need not be mono. The same thing may happen when we work with the endofunctors on the category Top, i.e. if F is a Top-endofunctor with $F(\emptyset) \neq \emptyset$, then the morphism $F(\emptyset_X)$ (where $\emptyset_X$ is the unique morphism from the empty space $\emptyset$ to some topological space X) need not be mono. In order to remove this exception about the morphisms with empty domains, we modify F and define a new Top-endofunctor F^+ called the positive functor of F. Our definition of F^+ has been borrowed from Barr [10].

To start, let F be a Top-endofunctor with $F\emptyset \neq \emptyset$. Suppose F preserves all monos with non-empty domains. The proviso about the empty space $\emptyset$ can be discarded by modifying the functor F on this space and on all morphisms with empty domains. To this end, consider the coproduct $1+1$ (where 1 is terminal object in Top) with canonical injections e_0, $e_1 : 1 \longrightarrow 1+1$. Let $e : P \longrightarrow F(1)$ be an equalizer of $F(e_0)$ and $F(e_1)$.

$$\begin{tikzcd} P \arrow[r, "e"] & F(1) \arrow[r, bend left, "Fe_0"] \arrow[r, bend right, "Fe_1"'] & F(1+1) \end{tikzcd}$$

Now, define a construction $F^+ : Top \longrightarrow Top$ on objects by

$$F^+(A) := \begin{cases} P & \text{if } A = \emptyset \\ F(A) & \text{else} \end{cases}$$

where A changes over all topological spaces. For each topological space B, by identifying any $b \in B$ with the constant morphism $C_b : 1 \longrightarrow B$ with value b, we have the morphism $Fb : F(1) \longrightarrow F(B)$, and we can define for any morphism $f : A \longrightarrow B$ in Top:

$$F^+(f) := \begin{cases} (Fb) \circ e & \text{if } A = \emptyset,\ b \in B \\ F(f) & \text{else} \end{cases}$$

Due to the construction of e as equalizer, one easily checks that the definition of $F^+(f)$ does not depend on the choice of $b \in B$. Since composition of continuous maps are continuous and since F is a Top-endofunctor, for each continuous map $f : A \longrightarrow B$, the morphism $F^+(f)$ is continuous. Notice that for every morphism $f : A \longrightarrow B$ in Top we can choose an element $a \in A$ such that $F^+(f) \circ F^+(\emptyset_A) = F(f) \circ (Fa) \circ e$ (where $\emptyset_A$ is the unique morphism from the empty space to the topological space A). Then

$$\begin{aligned} F^+(f) \circ F^+(\emptyset_A) &= F(f) \circ (Fa) \circ e \\ &= F(f \circ a) \circ e \\ &= F(f(a)) \circ e \\ &= F^+(\emptyset_B) \\ &= F^+(f \circ \emptyset_A). \end{aligned}$$

Hence, for every morphism $f : A \longrightarrow B$ in Top we have $F^+(f) \circ F^+(\emptyset_A) = F^+(f \circ \emptyset_A)$. Since there is no morphism to $\emptyset$, the construction F^+ (called the positive functor of F) is a Top-endofunctor. Then the following lemma can be verified:

Lemma 3.8.1. *F^+ is a Top-endofunctor which preserves all monos.*

The idea of the construction of F^+ comes from [10] (see, pages 308 and 309).

4. Lifting and extending of Set-endofunctors to Top

The aim of this section is to find a connection between Set-endofunctors and Top-endofunctors. In order to achieve this goal, we study the notions of extension and lifting of a $\mathbb{C}_1$-endofunctor to a category $\mathbb{C}_2$ along the functors $D : \mathbb{C}_1 \longrightarrow \mathbb{C}_2$ and $U : \mathbb{C}_2 \longrightarrow \mathbb{C}_1$, respectively (here we assume that D and U exist). We also define the notions of lifting and extension up to isomorphism. Our presentation in this section is based on [9], where the authors investigated how a finitary functor on Set can be extended or lifted to the categories $Preord$ and $Poset$.
In this section, we replace $F \circ G$ (composition of the functors F and G) by FG.

Definition 4.0.1. Assume $D : \mathbb{C}_1 \longrightarrow \mathbb{C}_2$ and $U : \mathbb{C}_2 \longrightarrow \mathbb{C}_1$ are two functors between the categories $\mathbb{C}_1$ and $\mathbb{C}_2$. Given $\mathbb{C}_1$-endofunctor T and $\mathbb{C}_2$-endofunctor F.

- **Lifting:** F is called a lifting of T to $\mathbb{C}_2$ along U if $UF = TU$.
- **Extension:** F is called an extension of T to $\mathbb{C}_2$ along D if $FD = DT$.
- **Lifting up to isomorphism:** F is called a lifting of T up to isomorphism along U if $UF \cong TU$, i.e. there is a natural isomorphism between functors UF and TU (UF and TU are both functors from $\mathbb{C}_2$ to $\mathbb{C}_1$).
- **Extension up to isomorphism:** F is called an extension of T up to isomorphism along D if $FD \cong DT$, i.e. there is a natural isomorphism between functors FD and DT (FD and DT are both functors from $\mathbb{C}_1$ to $\mathbb{C}_2$).

Remark 4.0.2. Let $D : Set \longrightarrow Top$ and $U : Top \longrightarrow Set$ be the discrete and the forgetful functors between Top and Set, respectively. Given Top-endofunctor F and Set-endofunctor T. Then according to definition 4.0.1,

1. if F is a lifting (resp. an extension) of T along U (resp. along D), then F satisfies the equation $T = UFD$,
2. if F is a lifting (resp. an extension) of T up to isomorphism along U (resp. along D), then T and UFD are naturally isomorphic functors,

3. if F is a lifting of T along U, then for each set X and all topologies τ and σ on X we have $F(X,\tau) = F(X,\sigma)$, (we replace the given equality by $F(X,\tau) \cong F(X,\sigma)$ if F is a lifting of T up to isomorphism),

4. if F is an extension of T along D, then for each discrete space X_D[1] we have $F(X_D) = (TX)_D$, (we replace the given equality by $F(X_D) \cong (TX)_D$ if F is an extension of T up to isomorphism), and

5. each lifting of T along U is a lifting of T up to isomorphism along U.

6. each extension of T along D is an extension of T up to isomorphism along D.

Remark 4.0.3. In order to define the notion of extension of a *Set*-endofunctor F to the category *Top* along the indiscrete functor $I : Set \longrightarrow Top$, we should just in remark 4.0.2 replace the indiscrete functor I with the discrete functor D.

Lemma 4.0.4. *If Top-endofunctors F_1 and F_2 are liftings (resp. extensions) of the Set-endofunctors T_1 and T_2 along the forgetful functor $U : Top \longrightarrow Set$ (resp. the discrete functor $D : Set \longrightarrow Top$), respectively, then the Top-endofunctors $F_1 + F_2$, $F_1 \times F_2$ and $F_1 \circ F_2$ are also liftings (resp. ectensions) of the Set-endofunctors $T_1 + T_2$, $T_1 \times T_2$ and $T_1 \circ T_2$ along the forgetful functor U (resp. the discrete functor D), respectively.*

Proof. Let F_1 and F_2 be liftings of the *Set*-endofunctors T_1 and T_2, respectively. Then the following identities hold:

$$\begin{array}{rcl} UF_1 & = & T_1U \\ UF_2 & = & T_2U \end{array} \tag{4.0.1}$$

So

$$\begin{array}{rcll} U \circ (F_1 + F_2)(X) & = & U(F_1(X) + F_2(X)) & \text{Sum of functors} \\ & = & UF_1(X) + UF_2(X) & U \text{ preserves sums} \\ & = & T_1U(X) + T_2U(X) & By\ 4.0.1 \\ & = & (T_1 + T_2) \circ U(X) & By\ 2.15.15 \end{array}$$

Product is same (notice that U preserves products). To prove the claim for composition we have to show

$$U \circ (F_1 \circ F_2) = (T_1 \circ T_2) \circ U.$$

So,

[1] X_D and X_I are the set X with discrete and indiscrete topologies, respectively.

$$
\begin{array}{rcll}
U \circ (F_1 \circ F_2)(X) & = & U(F_1(F_2(X))) & \text{Composition of functors} \\
& = & T_1(UF_2(X)) & UF_1 = T_1U \\
& = & T_1(T_2U(X)) & UF_2 = T_2U \\
& = & (T_1 \circ T_2) \circ U(X) & \mathit{By}\ 2.15.15
\end{array}
$$

To prove this lemma for extension, use the equalities below instead of the equalities in 4.0.1.

$$
\begin{array}{rcl}
F_1D & = & DT_1 \\
F_2D & = & DT_2
\end{array}
$$

In this case, we use the fact that the discrete functor preserves sums and finite products (see example 2.19.3). □

Remark 4.0.5. Note that lemma 4.0.4 dose not hold when we replace the notion of lifting with extension along the indiscrete functor from Set to Top. In fact, if the Top-endofunctors F_1 and F_2 are extensions of the Set-endofunctors T_1 and T_2 with respect to the indiscrete and forgetful functors I and U, then $F_1 + F_2$ is not an extension of $T_1 + T_2$. The reason is that the sum of two indiscrete spaces is not an indiscrete space (see remark 2.5.8).

Example 4.0.6. The following functors are some examples of liftings and extensions of some Set-endofunctors to Top:

1. The Vietoris functor $\mathbb{V} : Top \longrightarrow Top$ is an extension of the finite powerset functor $\mathbb{P}_\omega : Set \longrightarrow Set$ to Top along the discrete functor $D : Set \longrightarrow Top$. To see this let X be a fixed set. By example 1.6 the underlying set of $\mathbb{V}D(X)$ is the set of all finite subsets of X (i.e., $\mathbb{P}_\omega(X)$). Notice that for each finite subset $K \subseteq X$ the one element set $\{K\}$ is an open subset of $\mathbb{V}D(X)$ (because $\{K\} = (\bigcap_{x \in K} \langle\{x\}\rangle) \cap [K]$ for each finite subset $K \subseteq X$). Then $\mathbb{V}D(X)$ carries the discrete topology, i.e., $\mathbb{V}D(X) = D\mathbb{P}_\omega(X)$

2. The path endofunctor $\Pi_0 : Top \longrightarrow Top$ is an extension of the identity functor $id_{Set} : Set \longrightarrow Set$ up to isomorphism along the discrete functor D. To show this let X be a fixed set. Since the only continuous maps from the unit interval $I = [0,1]$ to a discrete space are the constant maps, by definition of Path functor Π_0 (see 3.4) we conclude that

   $$\Pi_0(D(X)) = \{\{x\} \mid x \in X\}.$$

   Recall that $\Pi_0(D(X))$ carries the quotient topology generated by the function $q_X : D(X) \longrightarrow \Pi_0(D(X))$, then the set $\Pi_0(D(X))$ carries the discrete topology (because the quotient topology generated by a surjective map from a discrete space is the discrete topology). Now, it is obvious that $\eta : D \circ id_{Set} \longrightarrow \Pi_0 \circ D$ such that for each set X the morphism $\eta_X : D(id_{Set}(X)) \longrightarrow \Pi_0(D(X))$ is defined by $\eta_X(x) := \{x\}$ is a natural isomorphism.

3. It is easy to see that the power functor $(-)^{\Sigma} : Top \longrightarrow Top$ is a lifting of the power functor $(-)^{\Sigma} : Set \longrightarrow Set$ along the forgetful functor U (because the forgetful functor preserves products).

4. The power functor $(-)^{\Sigma} : Top \longrightarrow Top$ is an extension of the power functor $(-)^{\Sigma} : Set \longrightarrow Set$ along the indiscrete functor $I : Set \longrightarrow Top$ (note that the product of indiscrete spaces is an indiscrete space).

5. For each topological space (C, τ), the constant functor $(C, \tau) : Top \longrightarrow Top$ is a lifting of the constant functor $C : Set \longrightarrow Set$ along the forgetful functor U.

6. For each topological space (C, τ), the functor $(C, \tau) \times (-)^{\Sigma} : Top \longrightarrow Top$ is a lifting of the functor $C \times (-)^{\Sigma} : Set \longrightarrow Set$ along the forgetful functor U (by lemma 4.0.4).

4.1. Lifting lemma

Lemma 4.1.1. *Let $F : Top \longrightarrow Top$ be an endofunctor on Top. The following statements are equivalent:*

1. *F preserve monos and epis,*
2. *F is a lifting of a Set-endofunctor T up to isomorphism along the forgetful functor $U : Top \longrightarrow Set$,*
3. *F preserves monos and there is a Top- endofunctor G and a natural transformatian $\eta : G \longrightarrow F$ with the following properties:*
   a) *G preserves epis,*
   b) *η is surjective.*

Proof. Let $F : Top \longrightarrow Top$ be an endofunctor on Top.
(1)⇒(2): Suppose F preserves epis and monos in Top. Define $\delta : DU \longrightarrow id_{Top}$ as

$$\begin{aligned} \delta_X : DU(X) &\longrightarrow X \\ x &\longmapsto x \end{aligned}$$

where X is an arbitrary topological space. It is easy to check that δ is a bijective natural transformation between Top-endofunctors DU and id_{Top}. Since F preserves epis and monos, for each topological space X, the morphism $F\delta_X : FDU(X) \longrightarrow F(X)$ is a

bijective continuous map. So $UF\delta_X : UFDU(X) \longrightarrow UF(X)$ is a bijective map. Now by setting $T = UFD$, the functor TU is naturally isomorphism with UF.
(2)⇒(1): Let F be a lifting up to isomorphism of a Set -endofunctor T along the forgetful functor $U : Top \longrightarrow Set$. So, there is a natural isomorphism $\varphi : UF \longrightarrow TU$. Suppose $f : X \longrightarrow Y$ is an epi (mono) in Top. Since in the category Top the epimorphisms are exactly surjective and continuous maps (the monomorphisms are exactly injective and continuous maps), it is enough to show that UFf is surjective (injective). Consider the diagram below:

$$\begin{tikzcd} UFX \arrow[r, "\varphi_X"] \arrow[d, "UFf"'] & TUX \arrow[d, "TUf"] \\ UFY \arrow[r, "\varphi_Y"'] & TUY \end{tikzcd}$$

Since φ is a natural transformation, this diagram is commutative. Also, since T as a set functor preserves surjective (injective) maps TUf is surjective (injective). Now by commutativity of this diagram and since φ is a natural isomorphism, it is obtained that UFf is surjective (injective).

(3)⇒(1): It is enough to show that F preserves epis. By lemma 2.20.4, it is clear.

(1) ⇒(3): By taking F as Top- endofunctor G, the result will be obtained. □

4.2. A strategy to lift $Set-$endofunctors to Top

Let $U : Top \longrightarrow Set$ be the forgetful functor. Suppose F and T are Top-endofunctor and Set-endofunctor, respectively. Assume, there is a surjective natural transformation $\eta : UF \longrightarrow TU$. Now, for any topological space X, provide $TU(X)$ with the topology Q_{η_X} defined as follows:

$$V \in Q_{\eta_X} \text{ iff } \eta_X^{-1}(V) \text{ is an open subset of } F(X). \tag{4.2.1}$$

where $V \subseteq TU(X)$. Generally speaking, we can say that Q_{η_X} is the quotient topology generated by the surjective map $\eta_X : UF(X) \longrightarrow TU(X)$.

Remark 4.2.1. Notice that by lemma 1.4.9, we can describe the topology Q_{η_X} defined in equation 4.2.1 as follows:

$$Q_{\eta_X} = \{\eta_X(O) \mid O \underset{open}{\subseteq} UF(X), \, \eta_X^{-1}(\eta_X(O)) = O\}.$$

Now, consider the construction $\bar{T} : Top \longrightarrow Top$ to associate each topological space X to the topological space $(TU(X), Q_{\eta_X})$ and each continuous map $f : X \longrightarrow Y$ to the map $\bar{T}(f) : TU(X) \longrightarrow TU(Y)$ defined as $\bar{T}(f)(x) := TU(f)(x)$. Then we have the following lemma:

Lemma 4.2.2. *$\bar{T} : Top \longrightarrow Top$ is a Top-endofunctor.*

Proof. It is enough to show that for each continuous map $f : X \longrightarrow Y$, the map TUf is also continuous. Consider the following diagram,

$$\begin{tikzcd} UF(X) \arrow[r, "\eta_X"] \arrow[d, "UFf"'] & TU(X) \arrow[d, "TUf"'] \\ UF(Y) \arrow[r, "\eta_Y"] & TU(Y) \end{tikzcd} \tag{4.2.2}$$

Since η is a natural transformation, this diagram is commutative. So, by remark 4.2.1, it suffices to show that for each open subset $O \subseteq UF(Y)$ such that $\eta_Y^{-1}(\eta_Y(O)) = O$ the following equation holds.

$$\eta_X^{-1}(\eta_X((UFf)^{-1}(O))) = (UFf)^{-1}(O). \tag{4.2.3}$$

By the properties of functions we know

$$\eta_X^{-1}(\eta_X((UFf)^{-1}(O))) \supseteq (UFf)^{-1}(O).$$

We prove the inverse direction of equation 4.2.3 by contradiction.
Let $a \in \eta_X^{-1}(\eta_X((UFf)^{-1}(O)))$ and $a \notin (UFf)^{-1}(O)$. So

$$\eta_X(a) \in \eta_X((UFf)^{-1}(O)).$$

Hence, there exist an element $b \in (UFf)^{-1}(O)$ such that

$$\eta_X(a) = \eta_X(b)$$

and consequently $(a, b) \in Ker\, \eta_X$. Now, since $a \notin (UFf)^{-1}(O)$, we have $(UFf)(a) \notin O$. So

$$\eta_Y((UFf)(a)) \notin \eta_Y(O) \tag{4.2.4}$$

(because, $\eta_Y^{-1}(\eta_Y(O)) = O$). On the other hand, since $b \in (UFf)^{-1}(O)$,

$$(UFf)(b) \in O.$$

Therefore,

$$\eta_Y((UFf)(b)) \in \eta_Y(O). \tag{4.2.5}$$

By equations 4.2.4 and 4.2.5, it is concluded that $(a, b) \notin Ker\, \eta_Y \circ (UFf)$. Hence

$$Ker\, \eta_X \nsubseteq Ker\, \eta_Y \circ (UFf). \tag{4.2.6}$$

But equation 4.2.6 is a contradiction with the commutativity of diagram 4.2.2. So we conclude that $a \in (UFf)^{-1}(O)$. $\square$

Example 4.2.3. Let T be the Set-endofunctor $(-)^2 - (-) + 1$ introduced in example 2.16.4. Consider the Top-endofunctor F as the power functor $(-)^2$. For each topological space X and for each $(x, x') \in X^2$, define $\eta_X : UF(X) \longrightarrow TU(X)$ as

$$\eta_X(x, x') := \begin{cases} (x, x') & x \neq x' \\ \perp & x = x' \end{cases}$$

η_X is a surjective map. It is easy to check that $\eta : UF \longrightarrow TU$ is a natural transformation. For any topological space X, we provide the set $TUX = \{(x, x') \in X^2 \mid x \neq x'\} \cup \{\perp\}$ with the topology $Q_{\eta_{\mathbf{X}}}$ defined in equation 4.2.1. Then by remark 4.2.1, the topology $Q_{\eta_{\mathbf{X}}}$ on TUX is as follows:

$$Q_{\eta_X} = \{O \mid O \underset{open}{\subseteq} X^2, O \cap \triangle_X = \emptyset\} \cup \{(O - \triangle_X) \cup \{\perp\} \mid O \underset{open}{\subseteq} X^2, \triangle_X \subseteq O\} \quad (4.2.7)$$

where O is an open subset of X^2 with respect to the product topology.
By lemma 4.2.2, the construction $\bar{T} : Top \longrightarrow Top$ which associates to each topological space X the topological space (TUX, Q_{η_X}), and to each continuous map $f : X \longrightarrow Y$ the continuous map $\bar{T}(f) : \bar{T}X \longrightarrow \bar{T}Y$ defined as $\bar{T}(f)(x) := TU(f)(x)$ is a lifting of the Set-endofunctor T to Top along the forgetful functor $U : Top \longrightarrow Set$.

Recall that every Set-endofunctor preserves regular monos. However, there are endofunctors on Top which are liftings of some Set-endofunctors along the forgetful functor but do not preserve regular monos. The following example shows this issue for the functor $\bar{T} : Top \longrightarrow Top$ introduced in the previous example.

Example 4.2.4. Consider topological space Y as the set $\{1, 2, 3, 4, 5\}$ with the topology generated by the following subbase :

$$B_Y = \{\{1, 4, 5\}, \{2, 5\}, \{3, 5\}, \{5\}, \{2, 3, 4, 5\}\}$$

Notice that for each open subset O of Y^2 (with respect to the product topology on Y^2), we have $O \cap \triangle_Y \neq \emptyset$. So, by equation 4.2.7, we have :

$$Q_{\eta_Y} = \{(O - \triangle_Y) \cup \{\perp\} \mid O \underset{open}{\subseteq} Y^2, \triangle_Y \subseteq O\} \bigcup \{\emptyset\} \quad (4.2.8)$$

Let $S = \{1, 2, 3, 4\}$ be a subspace of Y. We claim that $\bar{T}S$ is not a subspace of $\bar{T}Y$. Notice that $\{(2, 3)\}$ is an open subset of S^2 (by definition of the product topology). Since $\{(2, 3)\} \cap \triangle_S = \emptyset$, by equation 4.2.7, $\{(2, 3)\}$ is open in $\bar{T}S$. But, according to equation 4.2.8 for each open subset $V \subseteq \bar{T}Y$ with $(2, 3) \in V$, we obtain that with $V \cap \bar{T}S \neq \{(2, 3)\}$ (because $\perp \in V \cap \bar{T}S$).

5. Extending $Set-$endofunctors to CUM^1

The theory of ultrametric spaces is closely connected with various branches of mathematics, amongst them general topology, category theory (see remark 2.6.4) and so on. The properties of the ultrametric spaces have many applications in computer science, see [50] and [51]. As a well-known ultrametric space which has many applications in computer science, we can point to the set X^ω (the set of all words over some alphabet X) in which the distance between two different words is 2^n, where n is the first place at which the words differ (see example 1.8.15). Moreover due to the obtained results in [12] and [63], the category CUM^1 (i.e., the category of complete 1-bounded ultrametric spaces with non-expansive maps) is complete and cocomplete (see example 2.5.4 for product in CUM^1). According to [12] and [67], this category is a cartesian closed category (see example 2.13.5). So, because of all these advantages, the category CUM^1 can be a good candidate as a base category for coalgebras. The purpose of this part is to describe a few properties of complete ultrametric spaces and to give a strategy to extend Set-endofuctors to CUM^1 by using these properties. To achieve this goal we will show that each complete ultrametric space is an inverse limit for some inverse system in Set. Besides, we used this fact that every inverse limit in Set can be considered as an complete ultrametric space (see remark 2.6.4). We close this chapter by extending the power-set functor $\mathbb{P}$ and finite power-set functor $\mathbb{P}_\omega$ on CUM^1 (see section 5.2). One should notice that all results which will be discussed in this chapter have been originally worked out by Worrel in [71, 72].

5.1. Complete ultrametric spaces as limits of inverse systems in Set

Before proving the following theorem we should recall that if (X, d) is an ultrametric space, then for each $n \in \mathbb{N}$ the set $X_n := \{B_{2^{-n}}(x) \mid x \in X\}$ forms a partition for X (see lemma 1.9.3). Define $g_{n,m} : X_m \longrightarrow X_n$ by $g_{n,m}(B_{2^{-m}}(x)) := B_{2^{-n}}(x)$ for each $m, n \in \mathbb{N}$ with $m \geq n$. Notice that $g_{n,m}$ $(m \geq n)$ is well-defined. Because for each $x, y \in X$ if $B_{2^{-m}}(x) = B_{2^{-m}}(y)$ then $y \in B_{2^{-m}}(x)$ and consequently $d(x, y) \leq 2^{-m} \leq 2^{-n}$. Hence by lemma 1.9.5 we have $B_{2^{-n}}(x) = B_{2^{-n}}(y)$.

Now we have the following theorem:

Theorem 5.1.1. *If (X,d) is a complete ultrametric space , then the set X can be recovered as a limit of the following inverse system in the category Set,*

$$X_0 \xleftarrow{g_{0,1}} X_1 \xleftarrow{g_{1,2}} X_2 \xleftarrow{g_{2,3}} \cdots \tag{5.1.1}$$

where $X_n := \{B_{2^{-n}}(x) \mid x \in X\}$ for each $n \in \mathbb{N}$, and $g_{n,m} : X_m \longrightarrow X_n$ is defined by $g_{n,m}(B_{2^{-m}}(x)) := B_{2^{-n}}(x)$ for $m \geq n$.

Proof. For each $n \in \mathbb{N}$, define map $g_n : X \longrightarrow X_n$ by $g_n(x) := B_{2^{-n}}(x)$. We claim that the set X with the family of maps $\{g_n\}_{n\in\mathbb{N}}$ is a limit for diagram 5.1.1 in Set. Let $(Y, (\varphi_n)_{n\in\mathbb{N}})$ be a competitor for $(X, (g_n)_{n\in\mathbb{N}})$. Then $g_{n,m} \circ \varphi_m = \varphi_n$ for each $m, n \in \mathbb{N}$ with $m \geq n$. Then for each $y \in Y$ the family $\{\varphi_n(y)\}_{n\in\mathbb{N}}$ is a family of balls $\{B_{2^{-n}}(x_n)\}_{n\in\mathbb{N}}$ satisfying equation 1.9.1 in lemma 1.9.6. Then by lemma 1.9.6, for each $y \in Y$ the set $\bigcap_{n\in\mathbb{N}} \varphi_n(y)$ is a singleton, i.e. for each $y \in Y$ there is an element $x \in X$ such that $\bigcap_{n\in\mathbb{N}} \varphi_n(y) = \{x\}$. Define $f : Y \longrightarrow X$ as $f(y) = x$ (where $\bigcap_{n\in\mathbb{N}} \varphi_n(y) = \{x\}$). So for each $n \in \mathbb{N}$,

$$\begin{aligned} g_n(f(y)) &= B_{2^{-n}}(f(y)) \\ &= B_{2^{-n}}(x) \\ &\overset{corollary\ 1.9.2}{=} B_{2^{-n}}(x_n) \\ &= \varphi_n(y). \end{aligned}$$

Since for each $y \in Y$ the set $\bigcap_{n\in\mathbb{N}} \varphi_n(y)$ is a singleton, f is unique.

$\square$

5.2. A strategy to extend Set-endofunctors to CUM^1

Assume T is an endofunctor on Set. Our purpose is to extend the Set-endofunctor T to CUM^1 along the functor $D_1 : Set \longrightarrow CUM^1$ (see example 2.15.7). We try to do this step by step. Before starting, we should recall that if (X,d) is a complete ultrametric space then by theorem 5.1.1, the set X can be recovered as a limit of the inverse system 5.1.1 given in theorem 5.1.1.

Step 1: Extending T to objects (1-bounded ultrametric space)

Let (X,d) be a complete 1-bounded ultrametric space. By applying the functor T on diagram 5.1.1, we obtain the following diagram,

$$TX_0 \xleftarrow{Tg_{0,1}} TX_1 \xleftarrow{Tg_{1,2}} TX_2 \qquad \cdots \qquad TX_n \xleftarrow{Tg_{n,n+1}} TX_{n+1} \qquad \cdots \tag{5.2.1}$$

By properties of Set-endofunctors, this diagram is also an inverse system in Set.

Remark 5.2.1. According to remark 2.6.4, if $(L_X, (\pi_n : L_X \longrightarrow TX_n)_{n\in\mathbb{N}})$ is a limit of the inverse system given in diagram 5.2.1, then L_X with the map $d_{L_X} : L_X \times L_X \to [0,1]$ defined by

$$d_{L_X}(p,\, q) := \begin{cases} 0 & p = q \\ 2^{-m(p,q)} & otherwise \end{cases}$$

(where $m(p,q) := Inf\,\{n \in \mathbb{N} \mid \pi_n(p) \neq \pi_n(q)\}$ for each $p, q \in L_X$) is a complete 1-bounded ultrametric space.

Step 2: Extending T to morphisms (non-expansive maps)

Lemma 5.2.2. *Let $f : X \longrightarrow Y$ be a non-expansive map between two complete 1-bounded ultrametric spaces (X, d) and (Y, d'). Then for every $n \in \mathbb{N}$ there is an unique map $f_n : X_n \longrightarrow Y_n$ with $f_n \circ g_n = h_n \circ f$ where g_n and h_n are maps in the following diagram:*

$$\begin{tikzcd}
X_0 \arrow[d, "f_0"'] & X_1 \arrow[l, "g_{0,1}"'] \arrow[d, "f_1"] & \cdots & X_n \arrow[d, "f_n"'] & X_{n+1} \arrow[l, "g_{n,n+1}"'] \arrow[d, "f_{n+1}"'] & \cdots & X \arrow[d, "f"] \arrow[ll, bend right, "g_{n+1}"'] \\
Y_0 & Y_1 \arrow[l, "h_{0,1}"] & \cdots & Y_n & Y_{n+1} \arrow[l, "h_{n+1}"] & \cdots & Y \arrow[ll, bend left, "h_{n+1}"]
\end{tikzcd}$$

(i.e., $g_n(x) := B_{2^{-n}}(x)$ for each $x \in X$, and $h_n(y) := B_{2^{-n}}(y)$ for each $y \in Y$)

Proof. For every $n \in \mathbb{N}$, define $f_n : X_n \longrightarrow Y_n$ as

$$f_n(B_{2^{-n}}(x)) := B_{2^{-n}}(f(x)).$$

First we claim that for all $n \in \mathbb{N}$ the map f_n is well-defined. Let $B_{2^{-n}}(x_1) = B_{2^{-n}}(x_2)$, then by lemma 1.9.5, $d(x_1, x_2) < 2^{-n}$. Since $f : X \longrightarrow Y$ is a non-expansive map, $d'(f(x_1), f(x_2)) < 2^{-n}$ and again by lemma 1.9.5, $B_{2^{-n}}(f(x_1)) = B_{2^{-n}}(f(x_2))$. It remains to show that $f_n \circ g_n = h_n \circ f$ for all $n \in \mathbb{N}$. Let $x \in X$ be a fixed element, then

$$f_n \circ g_n(x) = f_n(g_n((x)) = f_n(B_{2^{-n}}(x)) = B_{2^{-n}}(f(x)) = h_n(f(x)) = h_n \circ f(x).$$

□

Lemma 5.2.3. *Given two complete 1-bounded ultrametric spaces (X, d) and (Y, d'). Consider $(L_X, (\pi_n : L_X \longrightarrow TX_n)_{n\in\mathbb{N}})$ and $(L_Y, (\pi'_n : L_Y \longrightarrow TY_n)_{n\in\mathbb{N}})$ as inverse limits of the corresponding inverse systems mentioned in 5.2.1, then for every non-expansive map $f : X \longrightarrow Y$, there exists an unique non-expansive map $\tilde{f} : L_X \longrightarrow L_Y$* such that the following diagram commutes, i.e. $\pi'_n \circ \tilde{f} = (Tf_n) \circ \pi_n$ for each $n \in \mathbb{N}$.

$$\begin{tikzcd}
TX_0 \arrow[d, "Tf_0"'] & TX_1 \arrow[l, "Tg_{0,1}"'] \arrow[d, "Tf_1"'] & \cdots & TX_n & TX_{n+1} \arrow[l, "Tg_{n,n+1}"'] \arrow[d, "Tf_{n+1}"'] & \cdots & L_X \arrow[ll, bend right, "\pi_{n+1}"'] \\
TY_0 & TY_1 \arrow[l, "Th_{0,1}"] & \cdots & TY_n & TY_{n+1} \arrow[l, "Th_{n,n+1}"] & \cdots & L_Y \arrow[ll, bend left, "\pi'_{n+1}"]
\end{tikzcd}$$

(here $\{f_n : X_n \longrightarrow Y_n\}_{n\in\mathbb{N}}$ are the unique maps introduced in lemma 5.2.2)

Proof. By the diagram above, $(L_X, ((Tf_n) \circ \pi_n)_{n\in\mathbb{N}})$ is a competitor for $(L_Y, (\pi'_n)_{n\in\mathbb{N}})$. Thus, there is an unique map $\tilde{f} : L_X \longrightarrow L_Y$ such that $\pi'_n \circ \tilde{f} = (Tf_n) \circ \pi_n$ for each $n \in \mathbb{N}$. It remains to show $\tilde{f}$ is a non-expansive map between ultrametric spaces (L_X, d_{L_X}) and (L_Y, d_{L_Y}), where d_{L_X} and d_{L_Y} are ultrametrics defined in remark 5.2.1. We need to prove that $d_{L_Y}(\tilde{f}(p), \tilde{f}(q)) \leq d_{L_X}(p, q)$ for each $p, q \in L_X$ (naturally diferent). To achieve this goal, it suffices to show that $m(\tilde{f}(p), \tilde{f}(q)) \geq m(p, q)$ for each $p, q \in L_X$ (naturally diferent).

$$\begin{aligned}
m(\tilde{f}(p), \tilde{f}(q)) &= Inf\,\{n \in \mathbb{N} \mid \pi'_n(\tilde{f}(p)) \neq \pi'_n(\tilde{f}(q))\} \\
&= Inf\,\{n \in \mathbb{N} \mid (\pi'_n \circ \tilde{f})(p) \neq (\pi'_n \circ \tilde{f})(q)\} \\
&= Inf\,\{n \in \mathbb{N} \mid ((Tf_n) \circ \pi_n)(p) \neq ((Tf_n) \circ \pi_n)(q)\} \\
&= Inf\,\{n \in \mathbb{N} \mid (Tf_n)(\pi_n(p)) \neq (Tf_n)(\pi_n(q))\} \\
&\geq Inf\,\{n \in \mathbb{N} \mid \pi_n(p) \neq \pi_n(q)\} \\
&= m(p, q).
\end{aligned}$$

□

As a consequence of this part we have the following theorem:

Theorem 5.2.4. *Foe each Set-endofunctor T, define $T^\star : CUM^1 \longrightarrow CUM^1$ on objects as $T^\star(X, d) := (L_X, d_{L_X})$ and on non-expansive maps by $T^\star(f : X \longrightarrow Y) := \tilde{f}$. Then $T^\star$ is an endofunctor on CUM^1.*

Proof. Let (X, d), (Y, d') and (Z, d'') be complete 1-bounded ultrametric spaces and let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be non-expansive maps. We need to show that the following equalities hold:

- $T^\star(id_{(X,d)}) = id_{T^\star(X,d)}$, and
- $T^\star(g \circ f) = T^\star(g) \circ T^\star(f)$.

Before starting our proof, let $(L_X, (\pi_n : L_X \longrightarrow TX_n)_{n\in\mathbb{N}})$, $(L_Y, (\pi'_n : L_Y \longrightarrow TY_n)_{n\in\mathbb{N}})$ and $(L_Z, (\pi''_n : L_Z \longrightarrow TZ_n)_{n\in\mathbb{N}})$ be inverse limits of the corresponding inverse systems mentioned in 5.2.1.
To prove the first equation, note that by lemma 5.2.2 for each $n \in \mathbb{N}$ the identity map id_{X_n} is the only map such that $id_{X_n} \circ g_n = g_n \circ id_X$. Then according to lemma 5.2.3, $\pi_n \circ \widetilde{id}_{(X,d)} = T(id_{X_n}) \circ \pi_n$. Besides $\pi_n \circ id_{L_X} = T(id_{X_n}) \circ \pi_n$. So by uniqueness of $\widetilde{id}_{(X,d)}$, we conclude that $\widetilde{id}_{(X,d)} = id_{(L_X, d_{L_X})}$, and then

$$T^\star(id_{(X,d)}) = \widetilde{id}_{(X,d)} = id_{(L_X, d_{L_X})} = id_{T^\star(X,d)}.$$

Regarding the second equation, note that according to lemma 5.2.3, there are the unique non-expansive maps $\tilde{f} : L_X \longrightarrow L_Y$ and $\tilde{g} : L_Y \longrightarrow L_Z$ such that

- $\pi'_n \circ \tilde{f} = (Tf_n) \circ \pi_n$ (where $\{f_n : X_n \longrightarrow Y_n\}_{n\in\mathbb{N}}$ are the unique maps introduced in lemma 5.2.2), and
- $\pi''_n \circ \tilde{g} = (Tg_n) \circ \pi'_n$ (where $\{g_n : Y_n \longrightarrow Z_n\}_{n\in\mathbb{N}}$ are the unique maps introduced in lemma 5.2.2).

Therefore

$$\begin{aligned}\pi''_n \circ (\tilde{g} \circ \tilde{f}) &= (Tg_n) \circ (Tf_n) \circ \pi_n \\ &= T(g_n \circ f_n) \circ \pi_n.\end{aligned}$$

On the other hand, due to lemma 5.2.3, the morphism $\widetilde{f \circ g}$ is the unique morphism such that $\pi''_n \circ (\widetilde{f \circ g}) = T(g_n \circ f_n) \circ \pi_n$, then we have $\widetilde{f \circ g} = \tilde{g} \circ \tilde{f}$ and so

$$T^\star(f \circ g) = \widetilde{f \circ g} = \tilde{g} \circ \tilde{f} = T^\star(f) \circ T^\star(g).$$

□

Lemma 5.2.5. *Let T be an arbitrary Set-endofunctor, then the CUM^1-endofunctor $T^\star$ (defined in the previous theorem) is an extension of T up to isomorphism along the functor $D_1 : Set \longrightarrow CUM^1$ (see example 2.15.7), i.e. for each set X the metric spaces $T^\star D_1(X)$ and $D_1T(X)$ are isomorphic (see definition 1.8.1).*

Proof. Let X be a fixed set. Then $D_1(X)$ is the complete 1-bounded ultrametric space (X, d_1^X) where $d_1^X : X \times X \longrightarrow [0,1]$ is defined by $d_1^X(x,y) := if\,(x = y)\,0\,else\,1$. So, by theorem 5.1.1 the set X with maps $\{g_n : X \longrightarrow X_n\}_{n\in\mathbb{N}}$ (defined by $g_n(x) = B_{2^{-n}}(x)$, for each $n \in \mathbb{N}$) is a limit for diagram 5.1.1. We claim that the set TX with maps $\{Tg_n : TX \longrightarrow TX_n\}_{n\in\mathbb{N}}$ is a limit of diagram 5.2.1. To show this first note that all maps $\{g_n : X \longrightarrow X_n\}_{n\geq 1}$ are surjective (by the definition of g_n), and injective (let $n \geq 1$ be a fixed element, then for each $x, x' \in X$ if $B_{2^n}(x) = B_{2^n}(x')$ then by lemma 1.9.5, $d(x, x') < 2^n < 1$ and so by the definition of the metric d_1^X we have $x = x'$), and consequently they are isomorphisms. Similarly, we can see that all maps $\{g_{n,m} : X_m \longrightarrow X_n\}_{m \geq n > 1}$ in diagram 5.1.1 (defined by $g_{n,m}(B_{2^{-m}}(x)) = B_{2^{-n}}(x)$ for $m \geq n$) are isomorphisms, and then since the functor T preserves isomorphisms, the maps $\{Tg_n : TX \longrightarrow TX_n\}_{n>1}$ and $\{Tg_{n,m} : TX_m \longrightarrow TX_n\}_{m \geq n > 1}$ are also isomorphisms. Now, let $(Q, \{q_n\}_{n \in \mathbb{N}})$ be a competitor to $(TX, \{g_n\}_{n \in \mathbb{N}})$, i.e. $(Tg_{n,m}) \circ q_m = q_n$ for all $m, n \in \mathbb{N}$ with $m \geq n$. Since $\{Tg_n\}_{n \geq 1}$ are isomorphisms, for each $n \geq 1$ the set $(Tg_n)^{-1}(q_n(y))$ is a singleton. Besides, for each $y \in Q$ and each $n \geq 1$ we have

$$
\begin{array}{rcl}
(Tg_1)^{-1}(q_1(y)) & \overset{Tg_1 = (Tg_{1,n}) \circ (Tg_n)}{=} & ((Tg_{1,n}) \circ (Tg_n))^{-1}(q_1(y)) \\
& = & ((Tg_n)^{-1} \circ (Tg_{1,n})^{-1})(q_1(y)) \\
& = & (Tg_n)^{-1}((Tg_{1,n})^{-1}(q_1(y))) \\
& \overset{(Tg_{1,n}) \circ q_n = q_1}{=} & (Tg_n)^{-1}(q_n(y)).
\end{array}
$$

Then for each $y \in Q$ there is a unique element $z_y \in TX$ such that $(Tg_n)^{-1}(q_n(y)) = z_y$ for each $n \geq 1$. Define $f : Q \longrightarrow TX$ as $f(y) = z_y$ (where $(Tg_n)^{-1}(q_n(y)) = z_y$ for each $n \geq 1$). For each $n \geq 1$,

$$
\begin{array}{rcl}
Tg_n(f(y)) & = & Tg_n(z_y) \\
& = & Tg_n((Tg_n)^{-1}(q_n(y))) \\
& = & (Tg_n) \circ (Tg_n)^{-1}(q_n(y)) \\
& = & q_n(y).
\end{array}
$$

Besides for $n = 0$ we have

$$
\begin{array}{rcl}
Tg_0(f(y)) & \overset{Tg_0 = (Tg_{0,1}) \circ (Tg_1)}{=} & (Tg_{0,1}) \circ (Tg_1)(f(y)) \\
& = & Tg_{0,1}(Tg_1(f(y))) \\
& \overset{Tg_1(f(y)) = q_1(y)}{=} & Tg_{0,1}(q_1(y)) \\
& \overset{(Tg_{0,1}) \circ q_1 = q_0}{=} & q_0(y).
\end{array}
$$

Since for each $y \in Q$ the element $z_y \in TX$ is unique, f is unique.
Then by the previous theorem $T^\star D_1(X) = T^\star(X, d_1^X) = (TX, d_{TX})$ where d_{TX} is defined by

$$d_{TX}(p, q) := if\,(p = q)\,0\,else\,2^{-m(p,q)}$$

where $m(p, q) := Inf\,\{n \in \mathbb{N} \mid Tg_n(p) \neq Tg_n(q)\}$ for each $p, q \in TX$. Notice that $D_1T(X) = (TX, d_1^{TX})$. Now since the both metrics d_{TX} and d_1^{TX} induce the discrete topology on TX, we conclude that the metric spaces $T^\star D_1(X)$ and $D_1T(X)$ are isomorphic. □

5.3. Extending power-set functor and finite power-set functor on CUM[1]

Let (X, d_X) be a complete ultrametric space. Now by applying the powerset functor $\mathbb{P}$ on the inverse system given in diagram 5.1.1, we obtain an inverse system as follows:

$$\mathbb{P}X_0 \xleftarrow{\mathbb{P}(g_{0,1})} \mathbb{P}X_1 \xleftarrow{\mathbb{P}(g_{1,2})} \mathbb{P}X_2 \qquad \cdots \qquad \mathbb{P}X_n \xleftarrow{\mathbb{P}(g_{n,n+1})} \mathbb{P}X_{n+1} \qquad \cdots \tag{5.3.1}$$

Now for an arbitrary complete ultrametric space (X, d_X) we have the following lemmas:

Lemma 5.3.1. *Assume $C := \{U \subseteq X \mid U = \overline{U}\}$ (where $\overline{U}$ is the closure of U with respect to the metric topology obtained by the open balls). Then the family of maps $\{\psi_n : C \longrightarrow \mathbb{P}(X_n)\}_{n\in\mathbb{N}}$ defined by $\psi_n(U) := \{B_{2^{-n}}(x) \mid x \in U\}$ is a mono-source.*

Proof. Assume $U \neq V$ are two closed subsets of X. We may assume that, there is $q \in U$ such that $q \notin V$. It suffices to show that there is $n \in \mathbb{N}$ such that $\psi_n(U) \neq \psi_n(V)$. Suppose, for every $n \in \mathbb{N}$, we have $\psi_n(U) = \psi_n(V)$. Hence,

$$\forall n \in \mathbb{N}.\, \exists q_n \in V.\, B_{2^{-n}}(q) = B_{2^{-n}}(q_n).$$

Then by lemma 1.9.5,

$$\forall n \in \mathbb{N}.\, \exists q_n \in V.\, d_X(q, q_n) < 2^{-n}. \tag{5.3.2}$$

Now, consider the sequence $(q_n)_{n\in\mathbb{N}}$ in V. We claim that $(q_n)_{n\in\mathbb{N}}$ converges to q. Since, $\lim\limits_{n\longrightarrow\infty} 2^{-n} = 0$, it is enough to show that

$$\forall n \in \mathbb{N}.\, \exists K_n \in \mathbb{N}.\, \forall m \geq K_n.\, d_X(q, q_m) < 2^{-n}$$

Let $n \in \mathbb{N}$ be a fixed element. Consider $K_n := n$, So for each $m \geq n$,

$$\begin{aligned} d_X(q, q_m) &\overset{\text{equation } 5.3.2}{<} 2^{-m} \\ &\leq 2^{-n}. \end{aligned}$$

Therefore, $\lim\limits_{n\longrightarrow\infty} q_n = q$. Since V is a closed subset in X, we have $q \in V$. This gives a contradiction with $q \notin V$.

□

Remark 5.3.2. The lemma 5.3.1 still holds, if we replace $\mathbb{P}$ (the power-set functor) with $\mathbb{P}_\omega$ (the finite power-set functor). It means the family of maps $\{\phi_n : C \longrightarrow \mathbb{P}_\omega(X_n)\}_{n\in\mathbb{N}}$ defined by $\phi_n(U) := \{B_{2^{-n}}(x) \mid x \in U\}$ is a mono-source too (notice that C is the set $\{U \subseteq X \mid U = \overline{U}\}$).

Lemma 5.3.3. *$(C, \{\psi_n : C \longrightarrow \mathbb{P}(X_n)\}_{n\in\mathbb{N}})$ is an inverse limit of the inverse system in diagram 5.3.1.*

Proof. Let $(Y, \{\varphi_n : Y \longrightarrow \mathbb{P}(X_n)\}_{n\in\mathbb{N}})$ be a competitor for $(C, \{\psi_n : C \longrightarrow \mathbb{P}(X_n)\}_{n\in\mathbb{N}})$. Define $f : Y \longrightarrow C$ as $f(y) = \{x \in X \mid \forall n \in \mathbb{N}.\, B_{2^{-n}}(x) \in \varphi_n(y)\}$. To show that f is well-defined, we need to prove that $f(y) = \overline{f(y)}$ (i.e., $f(y)$ is a closed subset of X). It suffices to show that $\overline{f(y)} \subseteq f(y)$. Let $a \in \overline{f(y)}$. Then $B_{2^{-n}}(a) \cap f(y) \neq \emptyset$ for each $n \in \mathbb{N}$. Hence

$$\forall n \in \mathbb{N}.\, \exists a_n \in f(y).\, a_n \in B_{2^{-n}}(a).$$

Then

$$\forall n \in \mathbb{N}.\, \exists a_n \in f(y).\, d_X(a, a_n) < 2^{-n}.$$

Therefore by lemma 1.9.5,

$$\forall n \in \mathbb{N}.\, \exists a_n \in f(y).\, B_{2^{-n}}(a_n) = B_{2^{-n}}(a). \tag{5.3.3}$$

We know that $a_n \in f(y)$ for each $n \in \mathbb{N}$. Then $B_{2^{-n}}(a_n) \in \varphi_n(y)$ for each $n \in \mathbb{N}$. Hence by equation 5.3.3, for each $n \in \mathbb{N}$ we have $B_{2^{-n}}(a) \in \varphi_n(y)$ and consequently $a \in f(y)$. It is easy to see that $\psi_n \circ f = \varphi_n$ for each $n \in \mathbb{N}$. Since $\{\psi_n : C \longrightarrow \mathbb{P}(X_n)\}_{n\in\mathbb{N}}$ is mono-source, f is unique. ☐

Corollary 5.3.4. ***(Extension of the power-set functor on** CUM^1**)*** *The endofunctor $\mathbb{P}^\star : CUM^1 \longrightarrow CUM^1$ maps a complete 1-bounded ultrametric space (X, d_X) to the set $C = \{U \subseteq X \mid U = \overline{U}\}$ (the set of all closed subsets of X with respect to the metric topology obtained by the open balls) equipped with the metric as follows*

$$d(U, V) := \begin{cases} 0 & U = V \\ 2^{-m(U,V)} & otherwise \end{cases}$$

where $m(U, V) := Inf\, \{n \in \mathbb{N} \mid \psi_n(U) \neq \psi_n(V)\}$ for all closed subsets U, $V \subseteq X$.

Remark 5.3.5. As an application of remark 5.3.2 we can show that lemma 5.3.3 is also true for the finite power-set functor $\mathbb{P}_\omega$. It means the set $C = \{U \subseteq X \mid U = \overline{U}\}$ with the morphisms $\{\phi_n : C \longrightarrow \mathbb{P}_\omega(X_n)\}_{n\in\mathbb{N}}$ defined in remark 5.3.2 is a limit of the following inverse system:

$$\mathbb{P}_\omega(X_0) \xleftarrow{\mathbb{P}_\omega(g_{0,1})} \mathbb{P}_\omega(X_1) \qquad \cdots \qquad \mathbb{P}_\omega(X_n) \xleftarrow{\mathbb{P}_\omega(g_{n,n+1})} \mathbb{P}_\omega(X_{n+1}) \qquad \cdots \tag{5.3.4}$$

Consequently, if (X, d_X) is a complete 1-bounded ultrametric space then the underlying set of $\mathbb{P}^\star_\omega(X, d_X)$ is the set $C = \{U \subseteq X \mid U = \overline{U}\}$ (the set of all closed subsets of X).

In the following, we try to present the CUM^1-endofunctor $\mathbb{P}^\star_\omega$ in terms of compact subsets. In other words, we will to show that if (X, d_X) is a complete ultrametric space then

$$\mathbb{P}^\star_\omega(X, d_X) = \{U \subseteq X \mid U \text{ is a compact subset of } X\}. \tag{5.3.5}$$

Recall that a topological space is compact if and only if it is complete and totally bounded (see theorem 1.8.9). Since every closed subset of a complete metric space is complete, each $U \in \mathbb{P}^\star_\omega(X, d_X)$ is also complete. Then in order to prove equation 5.3.5, we just need to show that each $U \in \mathbb{P}^\star_\omega(X, d_X)$ is totally bounded (see definition 1.8.6).

Suppose $V \in \mathbb{P}^\star_\omega(X, d_X)$ is a fixed element. According to remark 5.3.5, V is also an element in the set $C = \{U \subseteq X \mid U = \overline{U}\}$. Note that the set C with the morphisms $\{\phi_n : C \longrightarrow \mathbb{P}_\omega(X_n)\}_{n\in\mathbb{N}}$ defined by $\phi_n(U) := \{B_{2^{-n}}(x) \mid x \in U\}$ (for each $n \in \mathbb{N}$) is a limit of the inverse system given in diagram 5.3.4 (remark 5.3.5). Then the set $\phi_n(V)$ (for each $n \in \mathbb{N}$) is finite. Hence, for every $n \in \mathbb{N}$, we can find a finite subset $V_n \subseteq V$ such that $V \subseteq \bigcup\limits_{x\in V_n} B_{2^{-n}}(x)$. As a consequence of the above mentioned evidences, we conclude that each $U \in \mathbb{P}^\star_\omega(X, d_X)$ is totally bounded. So we have:

Corollary 5.3.6. ***(Extension of the finite power-set functor on** CUM^1**)*** *The functor $P^\star_\omega : CUM \longrightarrow CUM$ maps a complete 1-bounded ultrametric space (X, d_X) to the set of all compact subsets of X with the following metric*

$$d(U, V) := \begin{cases} 0 & U = V \\ 2^{-m(U,V)} & \textit{otherwise} \end{cases}$$

where $m(U, V) := Inf\,\{n \in \mathbb{N} \mid \phi_n(U) \neq \phi_n(V)\}$ for all compact subsets $U, V \subseteq X$.

Part III.

Coalgebras

6. Kripke Structures

The concept of Kripke structures is the main motivation to study and develop the theory of coalgebras and modal logic. As an example, in the next section one can see that Kripke structures can be presented as coalgebras for the Set-endofunctor $\mathbb{P}_P$ (Kripke functor). The aim of this chapter is to review the notion of Kripke structures and its connection with modal logic. The main references of section 6.1 are Rutten [62], Gumm [30, 37, 38] and Hennessey and Milner [41]. The works of Fine [26, 27], Goldblatt [28], Goldblatt and Thomason [29], Areces and Goldblatt [6] and Hollenberg [43] are the other references for the notions discussed in this chapter.

6.1. Classical modal logic

Throughout this section, let P be a fixed set of propositional letters.

6.1.1. Kripke models and Kripke frames

Definition 6.1.1. (Kripke model) A Kripke model is a triple $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ where

- X is a set,
- $R_{\mathcal{X}} \subseteq X \times X$ is a binary relation called transition,
- $\models_{\mathcal{X}} \subseteq X \times P$ is a binary relation called validity.

In this section, we replace $x\, R_{\mathcal{X}}\, y$ by $x \longrightarrow_{R_{\mathcal{X}}} y$.
Kripke frames are Kripke models with $P = \emptyset$ (i.e., a Kripke frame is a pair $\mathcal{X} = (X, R_{\mathcal{X}})$ where X is a set and $R_{\mathcal{X}}$ is a binary relation on X).
The validity relation $\models_{\mathcal{X}}$ can be coded by a validity (or valuation) map $\vartheta_{\mathcal{X}} : X \longrightarrow \mathbb{P}(P)$ via $\vartheta_{\mathcal{X}}(x) := \{p \in P \mid x \models_{\mathcal{X}} p\}$.
We can also consider the transition relation $R_{\mathcal{X}}$ as a transition map $R_{\mathcal{X}} : X \longrightarrow \mathbb{P}(X)$ defined by $R_{\mathcal{X}}(x) := \{y \in X \mid x \longrightarrow_{R_{\mathcal{X}}} y\}$.
We can present a Kripke model $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ as a triple $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$.

6.1.2. Modal formula

Definition 6.1.2. Modal formulas over P are generated inductively as follows:

$$
\begin{aligned}
\varphi \quad ::= \quad & \top \\
| \quad & p\ for\ each\ p \in P \\
| \quad & \varphi_1 \wedge \varphi_2 \\
| \quad & \neg\varphi \\
| \quad & \Box\varphi
\end{aligned}
$$

The truth functional connectives $\vee$ ("or"), $\longrightarrow$ ("implication") and also the modal operation $\Diamond\varphi$ ("possibility of φ") are defined in the usual way:

1. $\varphi_1 \longrightarrow \varphi_2 := \neg(\varphi_1 \wedge \neg\varphi_2)$
2. $\varphi_1 \vee \varphi_2 := \neg(\neg\varphi_1 \wedge \neg\varphi_2)$
3. $\Diamond\varphi := \neg\Box\neg\varphi$

In addition, $\bigvee\limits_{i\in I_0} \varphi_i$ and $\bigwedge\limits_{i\in I_0} \varphi_i$ can be considered as modal formulas, whenever I_0 is finite and each φ_i is a formula. We denote the set of modal formulas over P by L_P.

6.1.3. Validity

Definition 6.1.3. If $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ is a Kripke model, we extend the validity relation $\models_{\mathcal{X}}$ from $p \in P$ to $\varphi \in L_P$ as follows:

$$
\begin{aligned}
x \models_{\mathcal{X}} \top \quad &:\Longleftrightarrow \quad true \\
x \models_{\mathcal{X}} \varphi_1 \wedge \varphi_2 \quad &:\Longleftrightarrow \quad x \models_{\mathcal{X}} \varphi_1\, and\, x \models_{\mathcal{X}} \varphi_2 \\
x \models_{\mathcal{X}} \neg\varphi \quad &:\Longleftrightarrow \quad x \not\models_{\mathcal{X}} \varphi \\
x \models_{\mathcal{X}} \Box\varphi \quad &: \Longleftrightarrow \quad \forall y.\, x \to y \implies y \models_{\mathcal{X}} \varphi
\end{aligned}
$$

So the semantics of the modal formulas constructed by the other logical connections can be defined as:

$$
\begin{aligned}
x \models_{\mathcal{X}} \varphi_1 \vee \varphi_2 \quad &: \Longleftrightarrow \quad x \models_{\mathcal{X}} \varphi_1\, or\, x \models_{\mathcal{X}} \varphi_2 \\
x \models_{\mathcal{X}} \varphi_1 \longrightarrow \varphi_2 \quad &: \Longleftrightarrow \quad x \not\models_{\mathcal{X}} \varphi_1\, or\, x \models_{\mathcal{X}} \varphi_2 \\
x \models_{\mathcal{X}} \Diamond\varphi \quad &: \Longleftrightarrow \quad x \not\models_{\mathcal{X}} \Box\neg\varphi
\end{aligned}
$$

A formula φ is valid in a Kripke model $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ (in symbols: $\models_{\mathcal{X}} \varphi$) iff $x \models_{\mathcal{X}} \varphi$, for each $x \in X$. As an example $\models_{\mathcal{X}} \top$, for every Kripke model $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$. Let Σ be a subset of L_P, we say that Σ is valid in a Kripke model $\mathcal{X}$ (in symbols: $\models_{\mathcal{X}} \Sigma$) iff $\models_{\mathcal{X}} \varphi$, for each $\varphi \in \Sigma$.

6.1.4. Negation normal form

Definition 6.1.4. In negation normal form each formula is defined by the following grammar:

$$\begin{array}{rcl} \varphi & := & \top \mid \bot \\ & \mid & p \mid \neg p \text{ for each } p \in P \\ & \mid & \varphi_1 \wedge \varphi_2 \mid \varphi_1 \vee \varphi_2 \\ & \mid & \Box\varphi \mid \Diamond\varphi \end{array}$$

Lemma 6.1.5. *Each modal formula is equivalent to a formula in the negation normal form.*

Proof. By using induction and the following validities, this claim can be proven.

$$\begin{array}{rcl} \neg\neg\varphi & = & \varphi \\ \neg(\varphi_1 \wedge \varphi_2) & = & \neg\varphi_1 \vee \neg\varphi_2 \\ \neg(\varphi_1 \vee \varphi_2) & = & \neg\varphi_1 \wedge \neg\varphi_2 \\ \neg\Box\varphi & = & \Diamond\neg\varphi \\ \neg\Diamond\varphi & = & \Box\neg\varphi \end{array}$$

□

6.1.5. Semantic map and modal equivalence

Definition 6.1.6. Given a Kripke model $\mathcal{X} = (X, R_{\mathcal{X}}, \vDash_{\mathcal{X}})$. For each $x \in X$, we define

$$\| x \| \quad := \quad \{\varphi \in L_P \mid x \vDash_{\mathcal{X}} \varphi\}. \tag{6.1.1}$$

Also, for each $\varphi \in L_P$ we denote

$$\| \varphi \|^{\mathcal{X}} \quad := \quad \{x \in X \mid x \vDash_{\mathcal{X}} \varphi\}. \tag{6.1.2}$$

We can just write $\| \varphi \|$, if it is clear from the context.

Remark 6.1.7. We can define a binary relation $\equiv_{L_P} \subseteq L_P \times L_P$ by

$$\varphi \equiv_{L_P} \psi \text{ iff } \| \varphi \|^{\mathcal{X}} = \| \psi \|^{\mathcal{X}} \text{ for each Kripke model } \mathcal{X}.$$

It is easy to see that $\equiv_{L_P}$ is an equivalence relation. We say two formulas φ and ψ are equivalent if $\varphi \equiv_{L_P} \psi$.

In the rest of this section, let $\mathcal{X} = (X, R_{\mathcal{X}}, \vDash_{\mathcal{X}})$, $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vDash_{\mathcal{Y}})$ and $\mathcal{Z} = (Z, R_{\mathcal{Z}}, \vDash_{\mathcal{Z}})$ be fixed Kripke models.

Definition 6.1.8. Let $x \in X$ and $y \in Y$, then we say that x and y are **modally equivalent** (in symbols: $x \approx_{\mathcal{X},\mathcal{Y}} y$), if for each modal formula $\varphi \in L_P$,

$$x \vDash_{\mathcal{X}} \varphi \iff y \vDash_{\mathcal{Y}} \varphi$$

We call the binary relation $\approx_{\mathcal{X},\mathcal{Y}} \subseteq X \times Y$ the modal equivalence relation between $\mathcal{X}$ and $\mathcal{Y}$. It is clear that $\approx_{\mathcal{X},\mathcal{Y}}$ is an equivalence relation. We drop the index and write $\approx$, if it is clear from the context.

6.1.6. Kripke bisimulation

Definition 6.1.9. A binary relation $B \subseteq X \times Y$ is called a **Kripke bisimulation** between $\mathcal{X}$ and $\mathcal{Y}$, if for each two elements $x \in X$ and $y \in Y$ with $x\,B\,y$, we have

1. $\forall p \in P.\, x \vDash_{\mathcal{X}} p \iff y \vDash_{\mathcal{Y}} p$;
2. $\forall x' \in X.\, x \longrightarrow_{R_{\mathcal{X}}} x' \Longrightarrow \exists y' \in Y.\, y \longrightarrow_{R_{\mathcal{Y}}} y' \wedge x'\,B\,y'$;
3. $\forall y' \in Y.\, y \longrightarrow_{R_{\mathcal{Y}}} y' \Longrightarrow \exists x' \in X.\, x \longrightarrow_{R_{\mathcal{X}}} x' \wedge x'\,B\,y'$.

Remark 6.1.10. [62] Notice that each Kripke bisimulation can be made into a Kripke structure. To see that, let B be a Kripke bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. We can define a transition relation $R_B \subseteq B \times B$ and a validity relation $\models_B \subseteq B \times P$ as follows:

- $(x, y) \longrightarrow_{R_B} (x', y')$: $\iff x \longrightarrow_{R_{\mathcal{X}}} x'$ and $y \longrightarrow_{R_{\mathcal{Y}}} y'$;
- $\forall p \in P.\, (x, y) \models_B p :\iff x \models_{\mathcal{X}} p$ and $y \models_{\mathcal{Y}} p$.

Then $(B, R_B, \models_B)$ is a Kripke model. Notice that R_B is not uniquely determined.

We have the following well-known facts about the Kripke bisimulations:

Lemma 6.1.11. *[62]:*

1. The empty relation $\emptyset \subseteq X \times Y$ is a Kripke bisimulation.
2. The diagonal $\triangle_X := \{(x, x) \mid x \in X\}$ is a Kripke bisimulation.
3. The converse of a Kripke bisimulation is a Kripke bisimulation too.

4. If B_1 and B_2 are Kripke bisimulations, then their relation composition $B_1 \circ B_2$ is also a Kripke bisimulation.

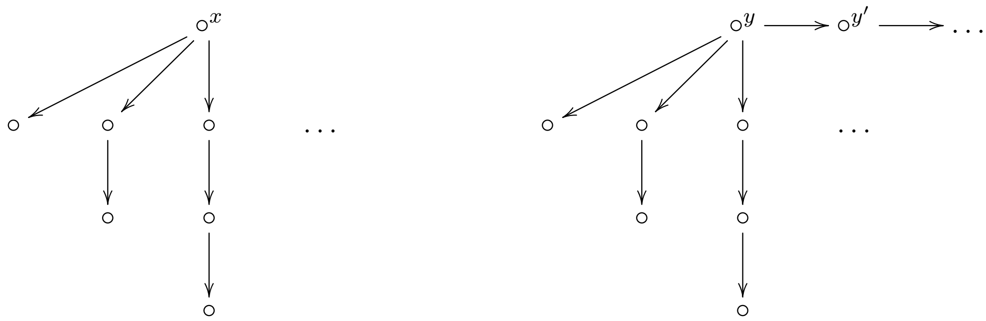
5. The union of a family of Kripke bisimulations between $\mathcal{X}$ and $\mathcal{Y}$ is again a Kripke bisimulation.

Remark 6.1.12. As a consequence of (1) and (5), the **largest Kripke bisimulation** between $\mathcal{X}$ and $\mathcal{Y}$ exists and it is denoted by $\sim_{\mathcal{X},\mathcal{Y}}$. We say B is a Kripke bisimulation on $\mathcal{X}$ if B is a Kripke bisimulation between $\mathcal{X}$ and $\mathcal{X}$. The largest Kripke bisimulation on $\mathcal{X}$ is denoted by $\sim_{\mathcal{X}}$ or simply $\sim$, when $\mathcal{X}$ is clear from the context. According to part (2), (3) and (4), the largest Kripke bisimulation on $\mathcal{X}$ is an equivalence relation. We say that two points $x \in X$ and $y \in Y$ are **Kripke bisimilar** if there is a Kripke bisimulation B between $\mathcal{X}$ and $\mathcal{Y}$ with $x \, B \, y$. Consequently $x \in X$ and $y \in Y$ are Kripke bisimilar iff $x \sim_{\mathcal{X},\mathcal{Y}} y$.

In the following, we will investigate the existence relationships between Kripke bisimilarity and modal equivalence. A straightforward induction over the construction of the modal formulas shows that bisimilar states are modally equivalent, i.e.

Theorem 6.1.13. *[41] Let $x \in X$ and $y \in Y$ be Kripke bisimilar elements. Then $x \approx_{\mathcal{X},\mathcal{Y}} y$.*

An example clearly demostrating that the converse of theorem 6.1.13 does not hold, is given by the infinite systems displayed below. In both systems, the root notes have countably many immediate successors from which branches of increasing length emanate. In the left structure, all branches are finite, whereas in the right structure an infinite branch (shown horizontally) is added. In both structures, we consider $P = \emptyset$.



It is easy to see that y' can not be Kripke bisimilar to any successor of x in the left structure. There for x and y are not bisimilar. However, they are modally equivalent. This is due to the fact that the modal depth[1] of modalities limits the scope of a modal formula (taken from [39]).

[1] The modal depth of a Modal formula φ (in symbol: $MD(\varphi)$) is the deepest nesting of modal operators ($\Box$ and $\Diamond$). Note that

- modal formulas without modal operators have a modal depth of zero, The following theorem shows that for the class of the image finite Kripke models, the converse of theorem 6.1.13 does hold. Since this theorem is an well-known result of Hennessey and Milner [41], we ignore its proof. We should mention that a Kripke model $\mathcal{X}$ is called image finite if $R_{\mathcal{X}}(x)$ is a finite set for each $x \in X$.

Theorem 6.1.14. *[41] (**Hennessy-Milner theorem**) Let $\mathcal{X}$ and $\mathcal{Y}$ be image finite Kripke models. Then the modal equivalence relation $\approx_{\mathcal{X},\mathcal{Y}} \subseteq X \times Y$ is a Kripke bisimulation between $\mathcal{X}$ and $\mathcal{Y}$.*

Corollary 6.1.15. *Let $\mathcal{X}$ and $\mathcal{Y}$ be image finite Kripke models. Then the equivalence relation $\approx_{\mathcal{X},\mathcal{Y}} \subseteq X \times Y$ is the largest Kripke bisimulation between $\mathcal{X}$ and $\mathcal{Y}$.*

6.1.7. Kripke homomorphisms and canonical bisimulations

Definition 6.1.16. A map $f : X \longrightarrow Y$ is called a **Kripke homomorphism**, if its graph, (i.e., the set $G(f) := \{(x, f(x)) \mid x \in X\}$) is a Kripke bisimulation. It means for every $x \in X$, we have

1. $\forall p \in P. x \models_{\mathcal{X}} p \Longleftrightarrow f(x) \models_{\mathcal{Y}} p$;
2. $\forall x' \in X. x \longrightarrow_{R_{\mathcal{X}}} x' \Longrightarrow f(x) \longrightarrow_{R_{\mathcal{Y}}} f(x')$;
3. $\forall y \in Y.\ f(x) \longrightarrow_{R_{\mathcal{Y}}} y \Longrightarrow \exists x' \in X. x \longrightarrow_{R_{\mathcal{X}}} x' \wedge f(x') = y$.

Lemma 6.1.17. *[37] Kripke homomorphisms preserve and reflect modal formulas, in the sense that $x \models_{\mathcal{X}} \varphi$ iff $f(x) \models_{\mathcal{Y}} \varphi$ (where $f : X \longrightarrow Y$ is a Kripke homomorphism between Kripke models $\mathcal{X}$ and $\mathcal{Y}$).*

Proof. Suppose $f : X \longrightarrow Y$ is a Kripke homomorphism between Kripke models $\mathcal{X}$ and $\mathcal{Y}$ and $x \in X$, then since Kripke bisimulations preserve modal formulas, for each formula φ we have

$$x \models_{\mathcal{X}} \varphi \iff f(x) \models_{\mathcal{Y}} \varphi.$$

$\square$

Corollary 6.1.18. *Suppose $f : X \longrightarrow Y$ is a Kripke homomorphism. Then for each modal formula $\varphi \in L_P$, the following conditions hold:*

- $MD(\varphi \oplus \psi) = max(MD(\varphi), MD(\psi))$, where $\oplus \in \{\wedge, \vee, \rightarrow\}$,
- $MD(\neg\varphi) = MD(\varphi)$, and
- $MD(\Box\varphi) = MD(\Diamond\varphi) = 1 + MD(\varphi)$.

As an exampel, $MD(\Box(\Diamond p \longrightarrow \Diamond\Box p)) = 3$.

1. $f(\parallel \varphi \parallel^{\mathcal{X}}) \subseteq \parallel \varphi \parallel^{\mathcal{Y}}$;
2. $f^{-1}(\parallel \varphi \parallel^{\mathcal{Y}}) = \parallel \varphi \parallel^{\mathcal{X}}$.

Lemma 6.1.19. *(**Canonical bisimulation theorem**) [62] Given Kripke homomorphisms $\varphi_X : Z \longrightarrow X$ and $\varphi_Y : Z \longrightarrow Y$, then the set*

$$(\varphi_X, \varphi_Y)[Z] := \{(\varphi_X(z), \varphi_Y(z)) \mid z \in Z\}$$

is a Kripke bisimulation between $\mathcal{X}$ and $\mathcal{Y}$, and each Kripke bisimulation is of this shape.

Proof. Since $(\varphi_X, \varphi_Y)[Z] = G(\varphi_X)^{-1} \circ G(\varphi_Y)$, the result follows from parts (3) and (4) of lemma 6.1.11. To prove the rest of this lemma let B be a Kripke bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. We know that $(\pi_X, \pi_Y)[B] = B$, where $\pi_X : B \longrightarrow X$ and $\pi_Y : B \longrightarrow Y$ are projection maps. Then it is enough to show that the projections π_X and π_Y are Kripke homomorphisms. Let $R_B \subseteq B \times B$ and $\models_B \subseteq P \times B$ be the transition relation and the validity relation defined by B, respectively (see remark 6.1.10). Then $(B, R_B, \models_B)$ is a Kripke model. Now, it is easy to see that the projections $\pi_X : B \longrightarrow X$ and $\pi_Y : B \longrightarrow Y$ are Kripke homomorphisms (see definition 6.1.16). $\square$

Remark 6.1.20. Kripke structures together with the Kripke homomorphisms form a category denoted by KS.

6.1.8. Congruence

Definition 6.1.21. If $f : X \longrightarrow Y$ is a Kripke homomorphism, then its kernel

$$ker\, f := \{(x, x') \in X \times X \mid f(x) = f(x')\}$$

is called a **congruence relation**. This is clearly an equivalence relation and a Kripke bisimulation as well, since we can write it as a relation composition of $G(f)$ (the graph of f) with its converse as

$$ker\, f = G(f) \circ G(f)^{-1}.$$

6.2. Compactness and modal saturation

In this subsection we introduce the notion of compactness for Kripke structures. We will prove that this notion coincides with the notion of modal saturation introduced in Fine [27] for the class of Kripke models. The notion of modal saturation has been also studied by Goldblatt in [28], Goldblatt and Thomason in [29], Goranko and Otto in [40] and Hollenberg in [43]. Both notions are used to answer this question what connection is between the notions of modally equivalence and bisimilarity equivalence.
First we should notice that in general, infinitary disjunctions or conjunctions are not considered as a formula, but we informally use them. As an instance according to [39],

for every element x in a Kripke model $\mathcal{X}$ we write $x \models_{\mathcal{X}} \bigvee_{i\in I} \varphi_i$ iff $(\exists i \in I.\, x \models_{\mathcal{X}} \varphi_i)$. Consequently, we write

$$x \models_{\mathcal{X}} \Box \bigvee_{i\in I} \varphi_i$$

to mean $\forall x'.(x \longrightarrow_{R_{\mathcal{X}}} x' \implies \exists i \in I.\, x' \models_{\mathcal{X}} \varphi_i)$.
We also use the notation $x \models_{\mathcal{X}} \bigwedge_{i\in I} \varphi_i$ iff $(\forall i \in I.\, x \models_{\mathcal{X}} \varphi_i)$. We write

$$x \models_{\mathcal{X}} \Diamond \bigwedge_{i\in I} \varphi_i$$

to mean $\exists x'.((x \longrightarrow_{R_{\mathcal{X}}} x') \wedge (\forall i \in I.\, x' \models_{\mathcal{X}} \varphi_i))$. If I is finite, the above coincides with the standard formula semantics.

The next denition can be found in [40] and elsewhere.

Definition 6.2.1. An element x in a Kripke model $\mathcal{X}$ is called **modally saturated**, if for every family $(\varphi_i)_{i\in I}$ of formulas the following condition holds,

- if $x \models_{\mathcal{X}} \Diamond \bigwedge_{i\in I_0} \varphi_i$ for each finite subset $I_0 \subseteq I$, then $x \models_{\mathcal{X}} \Diamond \bigwedge_{i\in I} \varphi_i$.

Now, we introduce the notion of compactness in terms of the modal operator $\Box$ (box), (defined by Gumm in [39]):

Definition 6.2.2. An element x in the Kripke model $\mathcal{X}$ is called **compact**, if for each family $(\varphi_i)_{i\in I}$ with $x \models_{\mathcal{X}} \Box \bigvee_{i\in I} \varphi_i$ we can find a finite subset $I_0 \subseteq I$ such that $x \models_{\mathcal{X}} \Box \bigvee_{i\in I_0} \varphi_i$. A Kripke structure is called compact (resp. modally saturated) if each of its elements is compact (resp. modally saturated).

The following lemma shows that the notions of compactness and modal saturation coincide.

Lemma 6.2.3. *An element x in the Kripke model $\mathcal{X}$ is compact if and only if it is modally saturated.*

Proof. Let x be a compact element such that $x \models_{\mathcal{X}} \Diamond \bigwedge_{i\in I_0} \varphi_i$ for each finite subset $I_0 \subseteq I$. We want to prove that x is modally saturated. We show this claim by contradiction. Suppose $x \not\models_{\mathcal{X}} \Diamond \bigwedge_{i\in I} \varphi_i$, then $x' \not\models_{\mathcal{X}} \bigwedge_{i\in I} \varphi_i$ for every element x' with $x \longrightarrow_{R_{\mathcal{X}}} x'$. So, $x' \models_{\mathcal{X}} \bigvee_{i\in I} \neg\varphi_i$ for every element x' with $x \longrightarrow_{R_{\mathcal{X}}} x'$ and consequently $x \models_{\mathcal{X}} \Box \bigvee_{i\in I} \neg\varphi_i$.

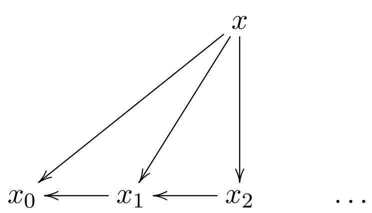
Now, by the compactness of x, there is a finite subset $I_0 \subseteq I$ such that $x \models_{\mathcal{X}} \Box \bigvee_{i\in I_0} \neg\varphi_i$. Hence $x \models_{\mathcal{X}} \neg\Diamond\neg(\bigvee_{i\in I_0} \neg\varphi_i)$ and so $x \models_{\mathcal{X}} \neg\Diamond \bigwedge_{i\in I_0} \varphi_i$. It means $x \not\models_{\mathcal{X}} \Diamond \bigwedge_{i\in I_0} \varphi_i$ which gives a contradiction with the assumption. The other direction can be proven in a similar way. □

According to definition 6.2.2, the image finite elements are clearly compact, but they are not the only ones. We make this issue more clear by giving an example discovered by Gumm in [39]. In this example, there are two Kripke structures with an image infinite element x, such that in the first structure, x is a non-compact element and in the second one it is compact. In both structures, we consider $P = \emptyset$.

Example 6.2.4. Given the set $X = \{x_i \mid i \in \mathbb{N}\} \cup \{x\}$, define the binary relation R on X as

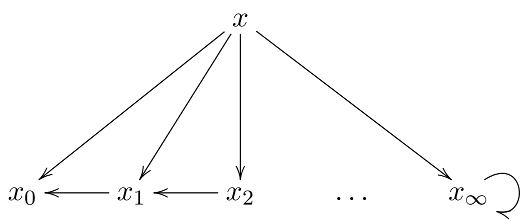
$$R \quad := \quad \{(x,\, x_i) \mid i \in \mathbb{N}\} \cup \{(x_{i+1},\, x_i) \mid i \in \mathbb{N}\}.$$

Then for each x_i we have that $x_i \models \Box^{i+1} \mathit{false}$, but $x_i \not\models \Box^j \mathit{false}$ for $j \leq i$. Therefore, (X, R) is not compact, because $x \models \Box \bigvee_{i\in\mathbb{N}} (\Box^{i+1}\mathit{false})$, but there is no finite $I_0 \subseteq \mathbb{N}$ such that $x \models \Box \bigvee_{i\in I_0} (\Box^{i+1}\mathit{false})$.



(6.2.1)

We now modify the structure given in picture 6.2.1 by adding a limit point x_∞ together with a self-loop $(x_\infty,\, x_\infty)$ to obtain the following structure:



(6.2.2)

Now, we claim that:

Lemma 6.2.5. *The Kripke structure in picture 6.2.2 is compact.*

Proof. We can see that for the point at infinity (i.e., x_∞) we have:

$$x_\infty \models \Box\varphi \iff x_\infty \models \varphi \iff x_\infty \models \Diamond\varphi.$$

To continue this proof, we need to show that for each nnf-formula φ, the following claim holds.

Claim. If $x_\infty \models \varphi$, then there is some $k \in \mathbb{N}$ such that $x_i \models \varphi$ for each $i \geq k$.

We prove the claim by induction over the construction of formulas in negation normal form. For $\varphi = true$ and $\varphi = false$ the claim is obviously true. Suppose the claim is true for negation normal formulas φ_1 and φ_2. For $\varphi = \varphi_1 \wedge \varphi_2$, from $x_\infty \models \varphi_1 \wedge \varphi_2$ the hypothesis yields $k_1, k_2 \in \mathbb{N}$ such that $x_i \models \varphi_1$ for each $i \geq k_1$ and $x_i \models \varphi_2$ for each $i \geq k_2$. With $k = max(k_1, k_2)$ we obtain $x_i \models \varphi_1 \wedge \varphi_2$ for each $i \geq k$. For $\varphi = \varphi_1 \vee \varphi_2$ we could similarly choose $k = min(k_1, k_2)$.
Regarding the formulas $\varphi = \Box\varphi_1$, it should be observed that we have: $x_\infty \models \varphi$ iff $x_\infty \models \varphi_1$. By assumption, there is some k such that $x_i \models \varphi_1$ for each $i \geq k$. It follows that $x_i \models \Box\varphi_1$ for $i \geq k+1$. Similarly we argue for $\varphi = \Diamond\varphi_1$.
Now to show that x in picture 6.2.2, is compact, suppose that $x \models \Box \bigvee\limits_{i \in I} \varphi_i$ then there is some $i_\infty \in I$ such that $x_\infty \models \varphi_{i_\infty}$. The claim above provides a k such that for each $j \geq k$ we have $x_j \models \varphi_{i_\infty}$, and for each $j \lneqq k$ there is $i_j \in I$ with $x_j \models \varphi_{i_j}$. Altogether then with $I_0 := \{i_0, i_1, \ldots, i_{k-1}\} \cup \{i_\infty\}$ we have $x \models \Box \bigvee\limits_{i \in I_0} \varphi_i$. Thus x is compact. All other points in picture 6.2.2 are image finite, hence compact, too. $\Box$

6.2.1. Compactness and Kripke bisimilarity

Points may be modally equivalent without being bisimilar. In [28], Goldblatt defined that a class $\mathbb{C}$ of Kripke structures has the Hennessy-Milner property, if modally equivalent elements are Kripke bisimilar. For instance, as it is mentioned in theorem 6.1.14 the class of all image finite Kripke structures has the Hennessy-Milner property (i.e., the notions of Kripke bisimilarity and modal equivalence coincide). Also, Goranko and Otto in [40] have shown that the class of the saturated models[2] has the Hennessy-Milner property. Here we replace the concept of saturation by the notion of compactness and we will prove that the class of the compact Kripke structures has Hennessy-Milner property. None of lemmas and proofs in this subsection are original and they are straightforward consequences of the known results of Goldblatt [28] for the class of saturated structures.

We know that Kripke bisimilar elements satisfy same formulas. We extend this well-known fact to infinitary formulas in the following sense:

Lemma 6.2.6. ***(Kripke bisimulations preserve compactness)*** *Let $B \subseteq X \times Y$ be a Kripke bisimulation between $\mathcal{X}$ and $\mathcal{Y}$ and let $x\,B\,y$. Then x is compact iff y is compact.*

Proof. Suppose $x\,B\,y$ and x is compact. If $y \models_{\mathcal{Y}} \Box \bigvee\limits_{i \in I} \varphi_i$, then each y' with $y \longrightarrow_{R_{\mathcal{Y}}} y'$ satisfies one of the formulas φ_i. By the definition of Kripke bisimulation, each x' with

[2] A Kripke model $\mathcal{X} = (X, R, \vartheta)$ is called saturated if each $x \in X$ is a modally saturated elements.

$x \longrightarrow_{R_{\mathcal{X}}} x'$ is Kripke bisimilar to some y' with $y \longrightarrow_{R_{\mathcal{Y}}} y'$. Then since Kripke bisimilar elements are modally equivalent, each x' satisfies one of the φ_i, i.e. $x \models_{\mathcal{X}} \Box \bigvee_{i \in I} \varphi_i$. Since x is compact, there is a finite subset $I_0 \subseteq I$ with $x \models_{\mathcal{X}} \Box \bigvee_{i \in I_0} \varphi_i$. Again, since Kripke bisimulations preserve modal formulas, we have $y \models_{\mathcal{Y}} \Box \bigvee_{i \in I_0} \varphi_i$. □

Corollary 6.2.7. *Kripke homomorphisms preserve and reflect compactness, in the sense that x is compact iff $f(x)$ is compact (where $f : X \longrightarrow Y$ is a Kripke homomorphism between Kripke models $\mathcal{X}$ and $\mathcal{Y}$).*

Proof. Since f is a Kripke homomorphism, its graph is Kripke bisimulation (definition 6.1.16). Then by lemma 6.2.6, x is compact iff $f(x)$ is compact.

□

Corollary 6.2.8. *If $f : X \longrightarrow Y$ is a surjective Kripke homomorphism, then $\mathcal{X}$ is compact iff $\mathcal{Y}$ is compact.*

Remark 6.2.9. Notice that compact Kripke structures with Kripke homomorphisms form a category called CKS.

The following theorem shows us that the class of compact Kripke structures has the Hennessy-Milner property.

Theorem 6.2.10. ***(Compact Hennessy-Milner theorem)*** *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact Kripke models. Then the modal equivalence relation $\approx_{\mathcal{X},\mathcal{Y}} \subseteq X \times Y$ is a Kripke bisimulation between $\mathcal{X}$ and $\mathcal{Y}$.*

Proof. Suppose $x \approx_{\mathcal{X},\mathcal{Y}} y$. Then obviously x and y satisfy the same propositional variables. Now, assume x' is an element in X with $x \longrightarrow_{R_{\mathcal{X}}} x'$ and suppose there is no element $y' \in Y$ such that $y \longrightarrow_{R_{\mathcal{Y}}} y'$ and $x' \approx_{\mathcal{X},\mathcal{Y}} y'$. This means that for every element $y' \in Y$ with $y \longrightarrow_{R_{\mathcal{Y}}} y'$ there exists a formula $\varphi_{y'}$ such that $x' \not\models_{\mathcal{X}} \varphi_{y'}$ and $y' \models_{\mathcal{Y}} \varphi_{y'}$. So $y \models_{\mathcal{Y}} \Box \bigvee_{y' \in R_{\mathcal{Y}}(y)} \varphi_{y'}$. Moreover, y is compact. Therefore, there exists a finite subset $\{y'_1, \ldots, y'_n\} \subseteq R_{\mathcal{Y}}(y)$ with $y \models_{\mathcal{Y}} \Box(\varphi_{y'_1} \vee \ldots \vee \varphi_{y'_n})$ and $x' \not\models_{\mathcal{X}} (\varphi_{y'_1} \vee \ldots \vee \varphi_{y'_n})$. Finally, since x and y are modally equivalent, we obtain that $x \models_{\mathcal{X}} \Box(\varphi_{y'_1} \vee \ldots \vee \varphi_{y'_n})$, which together with $x \longrightarrow_{R_{\mathcal{X}}} x'$ gives us a contradiction. The proof of the third condition of Kripke bisimulation is similar. □

As an easy corollary of the previous theorem, we have:

Corollary 6.2.11. *Let $\mathcal{X}$ and $\mathcal{Y}$ be compact Kripke models. Then the modally equivalence relation $\approx_{\mathcal{X},\mathcal{Y}} \subseteq X \times Y$ is the largest Kripke bisimulation between $\mathcal{X}$ and $\mathcal{Y}$.*

Proof. By the previous theorem, $\approx_{\mathcal{X},\mathcal{Y}} \subseteq X \times Y$ is a Kripke bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. Suppose $B \subseteq X \times Y$ is an arbitrary Kripke bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. Due to lemma 6.1.13, if $x\, B\, y$ then $x \approx_{\mathcal{X},\mathcal{Y}} y$ and consequently $B \subseteq \approx_{\mathcal{X},\mathcal{Y}}$. □

6.2.2. Modally equivalent and behaviorally equivalent

Definition 6.2.12. Let $x \in X$ and $y \in Y$. We say that x and y are **behaviorally equivalent** (in symbols: $x \,\nabla_{\mathcal{X},\mathcal{Y}}\, y$), if there exists a Kripke model $\mathcal{K} = (K, R_{\mathcal{K}}, \models_{\mathcal{K}})$ and Kripke homomorphisms $f : X \longrightarrow K$ and $g : Y \longrightarrow K$ such that $f(x) = g(y)$. We drop the index and write $x \nabla y$, if it is clear from the context.

Remark 6.2.13. It is well known that in the class of Kripke structures, behaviorally equivalent implies modally equivalent (if $x \in X$ and $y \in Y$ are two elements such that $x \nabla_{\mathcal{X},\mathcal{Y}}\, y$, then there is a Kripke model $\mathcal{K} = (K, R_{\mathcal{K}}, \models_{\mathcal{K}})$ and Kripke homomorphisms $f : X \longrightarrow K$ and $g : Y \longrightarrow K$ such that for some element $z \in Z$ we have $f(x) = z$ and $f(y) = z$, and consequently by lemma 6.1.17, $x \approx_{\mathcal{X},\mathcal{Z}} z$ and $y \approx_{\mathcal{Y},\mathcal{Z}} z$, and hence $x \approx_{\mathcal{X},\mathcal{Y}} y$).

In the sequel we will show that in the class of the compact Kripke structures modally equivalent elements are behaviorally equivalent.

It is well-known from Aczel and Mendler [2] that:

Lemma 6.2.14. *[2] Every Kripke bisimulation $B \subseteq X \times X$ which is an equivalence relation is a congruence relation on X.*

Proof. Let $B \subseteq X \times X$ be a Kripke bisimulation which is also an equivalence relation. Consider the factor set X/B, consisting of all equivalence classes $[x]_B$ with $x \in X$. Define a transition relation $R_B \subseteq X/B \times X/B$ and a validity relation $\models_B \subseteq X/B \times P$ as follows:

- $[x]_B \longrightarrow_{R_B} [y]_B :\Longleftrightarrow$ there exist $x' B\, x$ and $y' B\, y$ such that $x' \longrightarrow_{R_{\mathcal{X}}} y'$;
- $\forall p \in P.\, [x]_B \models_B p :\Longleftrightarrow \exists x' B\, x.\, x' \models_{\mathcal{X}} p$.

Then $\mathcal{X}/B = (X/B, R_B, \models_B)$ is a Kripke model.
Now, we show that the canonical map $f : X \longrightarrow X/B$ where $f(x) := [x]_B$ is a Kripke homomorphism.
According to definition 6.1.16, it suffices to show that its graph is a Kripke bisimulation.

- Firstly, by the definition of $\models_B$ on X/B we have: $x \models_{\mathcal{X}} p$ iff $[x]_B \models_B p$.

- Secondly, according to the definition of R_B on ${}^{X}\!/_{B}$ we have: if $x \longrightarrow_{R_{\mathcal{X}}} x'$ then $[x]_B \longrightarrow_{R_B} [x']_B$.
- Finally, assume $[x]_B \longrightarrow_{R_B} [y]_B$, we need to find some y'' such that $x \longrightarrow_{R_{\mathcal{X}}} y''$ and $f(y'') = [y]_B$. The definition of R_B on ${}^{X}\!/_{B}$ yields there are $x' B\, x$ and $y' B\, y$ with $x' \longrightarrow_{R_{\mathcal{X}}} y'$. Since B is a Kripke bisimulation, we conclude that there exists y'' with $x \longrightarrow_{R_{\mathcal{X}}} y''$ and $y' B\, y''$. Since $yB\, y''$ and by transitivity, we have $f(y'') = [y]_B$.

Then $f : X \longrightarrow {}^{X}\!/_{B}$ is a Kripke homomorphism with kernel B.

$\square$

Lemma 6.2.15. *If $\mathcal{X}$ is a compact Kripke model, then $\approx$ is a congruence relation.*

Proof. We know that the modal equivalence relation $\approx$ on X is an equivalence relation. Besides, due to theorem 6.2.10 the relation $\approx$ is a Kripke bisimulation. Then by the previous lemma, $\approx$ is a congruence relation on $\mathcal{X}$. $\square$

As a consequence of lemma 6.2.15, we have:

Corollary 6.2.16. *For all x, $y \in X$, if $x \approx_{\mathcal{X},\mathcal{Y}} y$, then $x \,\nabla_{\mathcal{X},\mathcal{Y}}\, y$.*

Proof. By the previous lemma there is a Kripke model $\mathcal{K} = (K, R_{\mathcal{K}}, \models_{\mathcal{K}})$ and a Kripke homomorphism $f : X \longrightarrow K$ such that $\approx_{\mathcal{X},\mathcal{Y}} = \mathit{ker}\, f$. Then from $x \approx_{\mathcal{X},\mathcal{Y}} y$, we have $f(x) = f(y)$ and this yields $x \,\nabla_{\mathcal{X},\mathcal{Y}}\, y$. $\square$

As a conclusion of this section we find that in the class of compact Kripke structures, the notions of behavioral equivalence, modal equivalence and Kripke bisimilarity all coincide.

Theorem 6.2.17. *Let x and y be elements in the compact Kripke structures $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then the following are equivalent:*

1. x and y are behaviorally equivalent,
2. x and y are modally equivalent,
3. x and y are Kripke bisimilar.

Proof. We prove this theorem step by step.

***Step* 1 :** The implication ' 1$\Longrightarrow$2' is concluded by remark 6.2.13 and the implication ' 2$\Longrightarrow$1' follows from lemma 6.2.16.

***Step* 2 :** The implication ' 2$\Longrightarrow$3' follows from theorem 6.2.10 and the implication '3$\Longrightarrow$2' is concluded by lemma 6.1.13.

$\square$

7. Coalgebras over $(\mathcal{E}, \mathcal{M})$-categories

In this chapter, We study some basic definitions, examples and theorems for coalgebras over a base category $\mathbb{C}$ with the following property:

A1: $\mathbb{C}$ has a factorization system $(\mathcal{E}, \mathcal{M})$ such that $\mathcal{E} \subseteq epis$ and $\mathcal{M} \subseteq monos$.

We use the categories Set and Top as the base categories in our examples. Our main references to introduce the theory of coalgebras include Rutten [62], Jacobs and Rutten [44], Gumm [30–32], Gumm and Schroeder [33, 34, 36] and Venema [68].
Throughout this chapter we assume that F is an arbitrary endofunctor on the category $\mathbb{C}$.
Also, in this chapter, if A and B are two sets and $b \in B$, we denote by C_b^A the constant map from A to B that sends each element of A to b.

7.1. Coalgebras and subcoalgebras

Definition 7.1.1. (Coalgebra) An F-coalgebra on $\mathbb{C}$ is a pair $\mathcal{A} =(A, \alpha_A)$ consisting of an object A in $\mathbb{C}$ and a morphism $\alpha_A : A \longrightarrow F(A)$ in $\mathbb{C}$ called the structure morphism (or F-coalgebra structure) of A. We shall often drop the index to the structure map α when it is clear from the context.

Example 7.1.2. Let $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ be a Kripke model and $R_{\mathcal{X}}$ and $\vartheta_{\mathcal{X}}$ the transition and the validity maps obtained by the binary relations $R_{\mathcal{X}}$ and $\models_{\mathcal{X}}$, respectively. It is well-known that Kripke model $\mathcal{X}$ can be written as a pair (X, α) where α is a map from X to $\mathbb{P}(X) \times \mathbb{P}(P)$ defined by $\alpha(x) := (R_{\mathcal{X}}(x), \vartheta_{\mathcal{X}}(x))$. Recall that the Kripke functor $\mathbb{P}_P$ maps every set X to the set $\mathbb{P}(X) \times \mathbb{P}(P)$ and each function $f : X \longrightarrow Y$ to the function $\mathbb{P}f \times id_{\mathbb{P}(P)}$ given by $(\mathbb{P}f \times id)(U, M) = (f[U], M)$ (where $U \subseteq X$ and $M \subseteq P$). Therefore, we can say that Kripke models are coalgebras for the Kripke functor $\mathbb{P}_P$. Kripke frames (i.e., Kripke models with $P = \emptyset$) are coalgebra for the powerset functor $\mathbb{P}$ on the category of sets.

Example 7.1.3. Let C be an arbitrary topological space. Consider the endofunctor $F(-) := C \times (-)_{Top}$ (i.e the polynomial functor obtained by the product of the constant functor C with the identity functor $(-)_{Top}$). The coalgebras of the Top-endofunctor F correspond to the black boxes on Top which can be described by a triple (S, h, t) consisting of a topological space S as input states and a pair of continuous maps $h : S \longrightarrow C$ and $t : S \longrightarrow S$.

Example 7.1.4. Let Σ be a set of symbols and D an arbitrary set of data. Recall that an automaton $\mathcal{S}$ over Σ with output in D consists of a set S of states, a transition function δ and an output function γ where

$$\delta \quad : \quad S \times \Sigma \longrightarrow S,$$

$$\gamma \quad : \quad S \longrightarrow D.$$

We show the automaton $\mathcal{S}$ by $\mathcal{S} = (S, \Sigma, D, \delta, \gamma)$. To explain how the automaton $\mathcal{S}$ can be seen as a coalgebras for the Set-endofunctor $D \times (-)^{\Sigma}$, we shall make use of the existence of the curried form of the transition function δ (see section 2.13). Due to the existence of the exponential objects in the category Set, there is a morphism $\tilde{\delta} : S \longrightarrow S^{\Sigma}$ defined by

$$\tilde{\delta}(x)(y) = \delta(x, y) \tag{7.1.1}$$

where $x \in S$ and $y \in \Sigma$. A coalgebra structure $\alpha : S \longrightarrow D \times S^{\Sigma}$ which is corresponded to the automaton $\mathcal{S}$ can be defined as $\alpha(x) := (\gamma(x), \tilde{\delta}(x))$ for each $x \in S$.

Example 7.1.5. A **topological automaton** is an automaton $\mathcal{S} = (S, \Sigma, D, \delta, \gamma)$ such that:

- S, Σ and D are topological spaces,
- Σ is a locally compact space, and
- $\delta : S \times \Sigma \longrightarrow S$ and $\gamma : S \longrightarrow D$ are continuous maps.

We repeat the same way used in the previous example to show that the topological automata are coalgebras for the Top-endofunctor $D \times Hom_{Top}(\Sigma, -)$ (where $Hom_{Top}(\Sigma, -)$ is the covariant functor defined in lemma 3.5.1). Since Σ is a locally compact space, by lemma 2.14.2 the map $ev : S^{\Sigma} \times \Sigma \longrightarrow S$ is continuous. Thus, there is the curried form of δ, i.e. the unique continuous map $\tilde{\delta} : S \longrightarrow S^{\Sigma}$ defined as equation 7.1.1. Define a coalgebra structure $\alpha : S \longrightarrow D \times Hom_{Top}(\Sigma, S)$ by $\alpha(x) := (\gamma(x), \tilde{\delta}(x))$ for each $x \in S$. Therefore, one can see that the topological automaton $\mathcal{S}$ can be presented as coalgebra (S, α).

Definition 7.1.6. (Homomorphism) Suppose $\mathcal{A} = (A, \alpha_A)$ and $\mathcal{B} = (B, \alpha_B)$ are F-coalgebras over $\mathbb{C}$. An arrow $\varphi : A \longrightarrow B$ in $\mathbb{C}$ is called a homomorphism from $\mathcal{A}$ to $\mathcal{B}$ if

$$\alpha_B \circ \varphi = F\varphi \circ \alpha_A.$$

That is, if the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, dashed, "\varphi"] \arrow[d, "\alpha_A"'] & B \arrow[d, "\alpha_B"] \\
F(A) \arrow[r, dashed, "F\varphi"'] & F(B)
\end{tikzcd}$$

They are easy to check that id_A is always a homomorphism and homomorphisms are stable under composition. Thus F-coalgebras together with their homomorphisms form a category denoted by $\mathbb{C}_F$.

Remark 7.1.7. There is a forgetful functor $U_{\mathbb{C}} : \mathbb{C}_F \longrightarrow \mathbb{C}$ which associates each coalgebra $\mathcal{A}$ =(A, α_A) in $\mathbb{C}_F$ to the underlying object A in $\mathbb{C}$ and each homomorphism $f : A \longrightarrow B$ (between coalgebras $\mathcal{A}$ =(A, α_A) and $\mathcal{B}$ =(B, α_B)) to the same morphism (called the underlying morphism of f) in $\mathbb{C}$.

Example 7.1.8. Every Kripke homomorphism between Kripke models is a homomorphism between corresponding $\mathbb{P}_P$-coalgebras and vice versa. Let $\mathcal{A} = (A, R_{\mathcal{A}}, \vartheta_{\mathcal{A}})$ and $\mathcal{B}$ =$(B, R_{\mathcal{B}}, \vartheta_{\mathcal{B}})$ be two Kripke models and let $\mathcal{A}$ =(A, α_B) and $\mathcal{B}$ =(B, α_B) be the corresponding $\mathbb{P}_P$-coalgebras. A function $\varphi : A \longrightarrow B$ is a homomorphism between $\mathbb{P}_P$-coalgebras $\mathcal{A}$ and $\mathcal{B}$ iff the following diagram commutes (see definition 7.1.6).

$$\begin{tikzcd}
\mathbb{P}(P) \arrow[r, "id_{\mathbb{P}(P)}"] & \mathbb{P}(P) \\
A \arrow[u, "\vartheta_{\mathcal{A}}"] \arrow[r, "\varphi"] \arrow[d, "R_{\mathcal{A}}"'] & B \arrow[u, "\vartheta_{\mathcal{B}}"'] \arrow[d, "R_{\mathcal{B}}"] \\
\mathbb{P}(A) \arrow[r, "\mathbb{P}(\varphi)"'] & \mathbb{P}(B)
\end{tikzcd} \tag{7.1.2}$$

Notice that the upper square in diagram 7.1.2 is commutative (i.e., $\vartheta_{\mathcal{A}}(a) = \vartheta_{\mathcal{B}}(\varphi(a))$ for each $a \in A$) iff for each $a \in A$ and each $p \in P$ we have $a \models_{\mathcal{A}} p \iff \varphi(a) \models_{\mathcal{B}} p$. Moreover, the equality $R_{\mathcal{B}} \circ \varphi = \mathbb{P}(\varphi) \circ R_{\mathcal{A}}$ is equivalent to conditions (2) and (3) in definition 6.1.16. Then $\varphi : A \longrightarrow B$ is a homomorphism between $\mathbb{P}_P$-coalgebras $\mathcal{A}$ and $\mathcal{B}$ iff φ is a Kripke homomorphism between Kripke models $\mathcal{A}$ and $\mathcal{B}$ (see [62]).

Lemma 7.1.9. *[62] Given F-coalgebras $\{\mathcal{A}_i = (A_i, \alpha_i)\}_{i \in I}$, $\mathcal{B}$ =(B, β) and $\mathcal{C} = (C, \gamma)$ in $\mathbb{C}_F$. Let $\{f_i : A_i \longrightarrow C\}_{i \in I}$ be a sink of homomorphisms and let $\{g_i : A_i \longrightarrow B\}_{i \in I}$ and $h : B \longrightarrow C$ be morphisms such that $h \circ g_i = f_i$ (for each $i \in I$), then*

1. if $\{g_i\}_{i \in I}$ is an epi sink in $\mathbb{C}$ and for each $i \in I$ the morphism g_i is a homomorphism, then h is a homomorphism, and

2. if h is a homomorphism with Fh mono, then for each $i \in I$ the morphism g_i is a homomorphism.

Proof. For both parts of this lemma, consider the following diagram:

$$\begin{tikzcd}
A_i \arrow[rr, "f_i"] \arrow[dr, "g_i"'] \arrow[dd, "\alpha_i"'] & & C \arrow[dd, "\gamma"] \\
& B \arrow[ur, "h"'] \arrow[dd, bend left, "\beta"] & \\
F(A) \arrow[rr, "Ff_i"] \arrow[dr, "Fg_i"'] & & F(C) \\
& F(B) \arrow[ur, "Fh"'] &
\end{tikzcd}$$

The rest of this proof is similar to the proof of lemma 2.4 in [62], where instead of the sink $\{g_i\}_{i \in I}$ we have a morphism g. ☐

Remark 7.1.10. Additional to the assumption **A1** (i.e., $\mathbb{C}$ is an $(\mathcal{E}, \mathcal{M})$-category with $\mathcal{E} \subseteq$ *epis* and $\mathcal{M} \subseteq$ *monos*), in the rest of this chapter we use the following assumptions:

A2: $\mathbb{C}$ is a cocomplete category.

A3: $\mathbb{C}$ is $\mathcal{M}$-well-powered.

A4: $\mathbb{C}$ has binary products.

A5: $\mathbb{C}$-endofunctor F preserves $\mathcal{M}$-morphisms.

Subcoalgebras

Consider the category $\mathbb{C}_F$ where the base category $\mathbb{C}$ satisfies the assumption **A1**. We have the following definition:

Definition 7.1.11. Let $\mathcal{A} =(A, \alpha_A)$ be an $F-$coalgebra in $\mathbb{C}_F$ and let $m : S \longrightarrow A$ be an $\mathcal{M}$-subobject (see section 2.12) of A. Then $m : S \longrightarrow A$ is called an $\mathcal{M}$-**subcoalgebra** of $\mathcal{A}$ if there is a $\mathbb{C}$-morphism $\alpha_S : S \longrightarrow F(S)$ (called $\mathcal{M}$-subcoalgebra structure) such that the morphism $m : S \longrightarrow A$ is a homomorphism in $\mathbb{C}_F$.
Here, we call $\mathcal{M}$-subcoalgebras of $\mathcal{A}$ simply as a subcoalgebras of $\mathcal{A}$, if the class $\mathcal{M}$ is equal to the class of regular monos in $\mathbb{C}$.

Remark 7.1.12. Recall that in Top the regular monomorphisms are (up to isomorphisms) exactly the topological embedding (see lamma 2.9.3). Then a subcoalgebra of a coalgebra $\mathcal{A}=(A, \alpha_A)$ in Top_F (where Top is an $(epi, regular\, mono)$- category) is a topological embedding $\iota : S \longrightarrow A$ such that there exists a continuous map $\rho : S \longrightarrow F(S)$ with $\alpha_A \circ \iota = F\iota \circ \rho$ (i.e., ι is a homomorphism in Top_F).

Definition 7.1.11 does not uniquely determine the $\mathcal{M}$-subcoalgebra structure α_S. The next example shows this issue clearly. It shows that in the category of Π_0-coalgebras over Top (where Top is an $(epi, regular\, mono)$- category), the $\mathcal{M}$-subcoalgebras structures are not uniquely determined. The reason is that Π_0 does not preserve monos.

Example 7.1.13. Recall example 3.7.4 where $S = \{1, 3\}$ is a subspace of the real numbers $\mathbb{R}$ with the standard topology. We have the following diagram,

$$\begin{tikzcd} S \arrow[r, hook, "\iota"] & \mathbb{R} \arrow[d, "\beta"] \\ \Pi_0(S) \arrow[r, "\Pi_0(\iota)"'] & 1 \end{tikzcd}$$

where $\iota : S \longrightarrow \mathbb{R}$ is the subspace inclusion. Since 1 (i.e., one element space $\{1\}$) is a terminal object in Top, every continuous map $\alpha_S : S \longrightarrow \Pi_0(S)$ makes diagram above commutative (recall that $\Pi_0(S)$ is the set $\{\{1\}, \{3\}\}$ with the discrete topology).

The previous example suggests the following theorem:

Theorem 7.1.14. *Let $\mathbb{C}$ be a concrete category (see definition 2.15.12) such that*

1. *monos in $\mathbb{C}$ are precisely those morphisms with the injective underlying functions in Set, and*
2. *for each A, $B \in \mathbb{C}$ and for each constant map $C_b^{U(A)} : U(A) \longrightarrow U(B)$ (where U is the forgetful functor from $\mathbb{C}$ to Set and $b \in U(B)$), there is an unique morphism $\alpha_b : A \longrightarrow B$ in $\mathbb{C}$ such that $U\alpha_b = C_b^{U(A)}$.*

Then in $\mathbb{C}_F$, each $\mathcal{M}$-subcoalgebra structure is unique iff F preserves monos.

Proof. Let $\mathcal{M}$-subobject $m : S \longrightarrow A$ be an $\mathcal{M}$-subcoalgebra of $\mathcal{A}=(A, \alpha_A)$, i.e. there is a structure $\alpha_S : S \longrightarrow F(S)$ such that the morphism $m : S \longrightarrow A$ is a homomorphism in $\mathbb{C}_F$. If F preserves monos, Fm is also mono (because by assumption **A1**, we have $\mathcal{M} \subseteq monos$) and then α_S is the unique morphism from S to $F(S)$ that satisfies the equation $\alpha_A \circ m = Fm \circ \alpha_S$ (by the definition of monos).

Conversely, assume that each $\mathcal{M}$-subcoalgebra structure is unique and assume there is a monomorphism $m : S \longrightarrow A$ such that Fm is not mono. By assumption in (1), the morphism UFm (where U is the forgetful functor from $\mathbb{C}$ to Set) is not injective. So

$$\exists x, y \in UF(S),\ \exists c \in UF(A).\ (x \neq y) \wedge (UFm(x) = UFm(y) = c).$$

According to the given assumption in (2), there are $\mathbb{C}$-morphisms $\alpha_c : A \longrightarrow F(A)$ and $\alpha_x, \alpha_y : S \longrightarrow F(S)$ such that $U\alpha_c = C_c^{U(A)}$, $U\alpha_x = C_x^{U(S)}$ and $U\alpha_y = C_y^{U(S)}$ (where $C_c^{U(A)} : U(A) \longrightarrow UF(A)$ is the constant map with value c, and the morphisms $C_x^{U(S)}$, $C_y^{U(S)} : U(S) \longrightarrow UF(S)$ are, respectively, the constant maps with values x and y). It is easy to see that the following diagram commutes.

$$\begin{tikzcd} U(S) \arrow[r, hook, "Um"] \arrow[d, shift right, "U\alpha_x"'] \arrow[d, shift left, "U\alpha_y"] & U(A) \arrow[d, "U\alpha_c"] \\ UF(S) \arrow[r, "UFm"'] & UF(A) \end{tikzcd}$$

Then

$$\begin{aligned} U((Fm) \circ \alpha_x) &= U(\alpha_c \circ m), \\ U((Fm) \circ \alpha_y) &= U(\alpha_c \circ m). \end{aligned}$$

Since U is faithful (because $\mathbb{C}$ is a concrete category), we have

$$\begin{aligned} (Fm) \circ \alpha_x &= \alpha_c \circ m, \\ (Fm) \circ \alpha_y &= \alpha_c \circ m. \end{aligned}$$

But this is a contradiction with our assumption, (because α_x and α_y are two $\mathcal{M}$-subcoalgebra structures on S making m into a homomorphism). $\square$

Example 7.1.15. Consider Top as an $(epi, regular\,mono)$-category.

1. Each subcoalgebra structure in $Top_{\mathbb{V}}$ is unique (because the Vietoris functor $\mathbb{V}$ preserves monos, see lemma 3.2.2).

2. If F is a lifting (or a lifting up to isomorphism) of a Set-endofunctor T to Top along the forgetful functor $U : Top \longrightarrow Set$, then each subcoalgebra structure in Top_F is unique.

7.2. Factorization system in $\mathbb{C}_F$

Assume F is an arbitrary $\mathbb{C}$-endofunctor where the category $\mathbb{C}$ satisfies assumption **A1** (see remark 7.1.10). The aim of this section is to verify the following statement:

- $\mathbb{C}_F$ is a $(\mathcal{E}_F, \mathcal{M}_F)$-category where $\mathcal{E}_F$ is the class of homomorphisms such that the underlying morphisms in $\mathbb{C}$ belong to $\mathcal{E}$, and $\mathcal{M}_F$ the class of homomorphisms such that the underlying morphisms in $\mathbb{C}$ are in $\mathcal{M}$.

In general, this statement is not true. By giving an example, we make this issue clear. In this example, we consider Top as an (*epi*, *regular mono*)-category and our objects are coalgebras of the Top-endofunctor $\overline{T}$ introduced in example 4.2.3, i.e. if X is an arbitrary topological space then $\bar{T}X$ is the set $(X)^2 - (X) + 1$ that carries the followig topology

$$\{O \mid O \underset{open}{\subseteq} X^2, O \cap \triangle_X = \emptyset\} \cup \{(O - \triangle_X) \cup \{\perp\} \mid O \underset{open}{\subseteq} X^2, \triangle_X \subseteq O\} \tag{7.2.1}$$

(where O is an open subset of X^2 with respect to the product topology), and for each continuous map $f : X \longrightarrow Y$ the continuous map $\bar{T}(f) : \bar{T}X \longrightarrow \bar{T}Y$ is defined by $\bar{T}(f)(x) := TUf(x)$ for every $x \in X$ (notice that U is the forgetful functor from Top to Set). Recall that $\overline{T}$ does not preserves regular monos (see example 4.2.4). We define a homomorphism φ in $Top_{\bar{T}}$ and we show that there are no homomorphisms ε and m in $Top_{\bar{T}}$ with $\varphi = m \circ \varepsilon$, where the underlying morphisms of ε and m are, respectively, epi and regular mono in Top.

Example 7.2.1. Let $X = \{1, 2, 3, 4\}$ be a discrete space. Define a map $\alpha : X \longrightarrow \bar{T}X$ by $\alpha(x) := (1, 2)$ if $x = 1$ else $\perp$ (for each $x \in X$). Since X is a discrete space, α is continuous. Then $\mathcal{X} = (X, \alpha)$ is a $\bar{T}$-coalgebra. Now, assume Y is the topological space defined in example 4.2.4. Recall that since $O \cap \triangle_Y \neq \emptyset$ for each open subset O of Y^2(with respect to the product topology on Y^2), by equation 7.2.1 $\bar{T}Y$ carries the followig topology :

$$\{(O - \triangle_Y) \cup \{\perp\} \mid O \underset{open}{\subseteq} Y^2, \triangle_Y \subseteq O\} \bigcup \{\emptyset\}.$$

Define $\beta : Y \longrightarrow \bar{T}Y$ as

$$\beta(y) := \begin{cases} (1, 2) & y = 1 \\ \perp & else \end{cases} \tag{7.2.2}$$

β is continuous. To see this, suppose $O \subseteq \bar{T}Y$ is an arbitrary non-empty open subset. Consider two cases:
Case1: If $(1, 2) \in O$, then $\beta^{-1}(O) = Y$.
Case2: If $(1, 2) \notin O$, then $\beta^{-1}(O) = \{2, 3, 4, 5\}$ is an open subset of Y.

Then $\mathcal{Y} = (Y, \beta)$ is a $\bar{T}$-coalgebra.
Consider $\varphi : X \longrightarrow Y$ as an inclusion of subsets. Since X is a discrete space, φ is continuous. One can easily check that φ is a homomorphism in $Top_{\bar{T}}$. Notice that $\varphi = \iota \circ \varepsilon$, where $\varepsilon : X \longrightarrow im\,\varphi$ is the codomain-restriction of φ and $\iota : im\,\varphi \longrightarrow Y$ is the subspace inclusion, is a factorization of φ in the corresponding factorization system on Top (see example 2.11.6), then we have a diagram as follows:

$$\begin{tikzcd}
X \arrow[r, two heads, "e"] \arrow[rr, bend left, "\varphi"] \arrow[d, "\alpha"'] & im\,\varphi \arrow[r, hook, "\iota"] & Y \arrow[d, "\beta"] \\
\bar{T}X \arrow[r, "\bar{T}e"] \arrow[rr, bend right, "\bar{T}\varphi"'] & \bar{T}(im\,\varphi) \arrow[r, "\bar{T}\iota"] & \bar{T}Y
\end{tikzcd}$$

We claim that there is no continuous map $\rho : im\,\varphi \longrightarrow \bar{T}(im\,\varphi)$ such that ε and ι are homomorphism $Top_{\bar{T}}$. We show this claim by contradiction. Suppose there exists a continuous map ρ such that this diagram is commutative. Since the right square commutes, we conclude that $\rho(y) = \beta(y)$ for each $y \in im\,\varphi$. The definition of ρ implies that it is not continuous. Because $\{(1,2),(4,2)\}$ is an open subset of $\bar{T}(im\,\varphi)$, but $\rho^{-1}(\{(1,2),(4,2)\}) = \{1\}$ is not open in $im\,\varphi$. This give us a contradiction (because we had assumed that ρ is continuous). Now, since the factorization $\varphi = \iota \circ \varepsilon$ is unique up to isomorphism (see theorem 2.11.3, part 1), we conclude that there are no homomorphisms ε and m in $Top_{\bar{T}}$ with $\varphi = m \circ \varepsilon$ where the underlying morphisms of ε and m are, respectively, epi and regular mono in Top.

Theorem 7.2.3 shows that if the $\mathbb{C}$-endofunctor F preserves $\mathcal{M}$-morphisms, then $\mathbb{C}_F$ is an $(\mathcal{E}_F, \mathcal{M}_F)$-category. We first prove the following technical lemma.

Lemma 7.2.2. ***(factorization)*** *Suppose F is an endofunctor on $\mathbb{C}$ which preserves $\mathcal{M}$-morphisms. Given a homomorphism $f : A \longrightarrow B$ in $\mathbb{C}_F$. Let $f = m \circ e$ be a decomposition of f into $\mathcal{E}$-morphism $e : A \longrightarrow E$, followed by an $\mathcal{M}$-morphism $m : E \longrightarrow B$. Then there is a unique F-coalgebra structure ρ on E such that e and m become homomorphism.*

Proof. The idea of this proof comes from [31]. Suppose $f : A \longrightarrow B$ is a homomorphism between coalgebras $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ in $\mathbb{C}_F$. Then by factoring f in $(\mathcal{E}, \mathcal{M})$-

system we have a diagram as follows:

$$\begin{tikzcd}
A \arrow[r, two heads, "e"] \arrow[rr, bend left, "f"] \arrow[d, "\alpha"'] & E \arrow[r, hook, "m"] & B \arrow[d, "\beta"] \\
F(A) \arrow[r, "Fe"] \arrow[rr, bend right, "Ff"'] & F(E) \arrow[r, "F(m)"] & F(B)
\end{tikzcd}$$

Since F preserves $\mathcal{M}$-morphisms, Fm is an $\mathcal{M}$-morphism too. Hence e is orthogonal to Fm, so there is a unique diagonal $\rho : E \longrightarrow F(E)$ such that this diagram is commutative. $\square$

Theorem 7.2.3. *[54] If $\mathbb{C}$-endofunctor F preserves $\mathcal{M}$-morphisms, then the category $\mathbb{C}_F$ is a $(\mathcal{E}_F, \mathcal{M}_F)$-category where $\mathcal{E}_F$ is the class of homomorphisms such that the underlying morphisms in $\mathbb{C}$ belong to $\mathcal{E}$, and $\mathcal{M}_F$ the class of homomorphisms such that the underlying morphisms in $\mathbb{C}$ are in $\mathcal{M}$.*

Proof. By lemma 7.2.2, every homomorphism φ in $\mathbb{C}_F$ can be written as the composition of two homomorphism ε and m in $\mathcal{E}_F$ and $\mathcal{M}_F$, respectively. The others conditions in definition 2.11.1 are obvious. For more details see proposition 4.1 in [54]. $\square$

Now, this question naturally arises whether the converse of theorem 7.2.3 holds?

In theorem 7.2.5, we prove that if $\mathbb{C}_F$ (where $\mathbb{C}$ is a concrete category with some additional property) is an $(\mathcal{E}_F, \mathcal{M}_F)$-category, then for every $\mathcal{M}$-morphism $m : S \longrightarrow A$ with $U(S) \neq \emptyset$ the morphism $F(m)$ is mono. To prove it, we use the following technical remark.

Remark 7.2.4. Let S be an arbitrary object in the category $\mathbb{C}$. Consider $S + S$ as sum (coproduct) in $\mathbb{C}$. The object S with the identity map $id_S : S \longrightarrow S$ is a competitor to the sum $S + S$ in $\mathbb{C}$. Hence there is a unique morphism $\varepsilon : S + S \longrightarrow S$ (codiagonal) in $\mathbb{C}$ such that $\varepsilon \circ e_i = id_S$ for each $i \in \{1, 2\}$, and then ε is a retraction in $\mathbb{C}$.

Theorem 7.2.5. *Let $\mathbb{C}$ be a concrete category (see definition 2.15.12) such that*

1. *monos in $\mathbb{C}$ are precisely those morphisms with the injective underlying functions in Set,*

2. *epis in $\mathbb{C}$ are precisely those morphisms with the surjective underlying functions in Set,*

3. *each retraction in $\mathbb{C}$ is an $\mathcal{E}$-morphism, and*

4. *for each $A, B \in \mathbb{C}$ and for each constant map $C_b^{U(A)} : U(A) \longrightarrow U(B)$ (where U is the forgetful functor from $\mathbb{C}$ to Set and $b \in U(B)$), there is a unique morphism $\alpha_b : A \longrightarrow B$ in $\mathbb{C}$ such that $U\alpha_b = C_b^{U(A)}$.*

If for every homomorphism φ in $\mathbb{C}_F$, the $\mathcal{E}$-coimage and $\mathcal{M}$-image of the underlying morphism of φ in $\mathbb{C}$ are homomorphisms in $\mathbb{C}_F$, then for every $\mathcal{M}$-morphism $m : S \longrightarrow A$ with $U(S) \neq \emptyset$, the morphism $Fm : F(S) \longrightarrow F(A)$ is mono.

Proof. Let $m : S \longrightarrow A$ be an $\mathcal{M}$-morphism in $\mathbb{C}$ with $U(S) \neq \emptyset$. Suppose that the morphism $Fm : F(S) \longrightarrow F(A)$ is not mono. So $UFm : UF(S) \longrightarrow UF(A)$ is not injective. Then

$$\exists z \in UF(A), \exists x, y \in UF(S). ((x \neq y) \wedge (UFm(x) = UFm(y) = z)). \tag{7.2.3}$$

Consider $B = S + S$ (i.e., B is the sum of S with itself in $\mathbb{C}$). By remark 7.2.4, there is a unique morphism $\varepsilon : S + S \longrightarrow S$ in $\mathbb{C}$ such that $\varepsilon \circ e_i = id_S$ for each $i \in \{1, 2\}$ (notice that $\{e_i\}_{i \in \{1,2\}}$ are the canonical injections). Now, consider the morphism $\varphi : B \longrightarrow A$ as $\varphi := m \circ \varepsilon$. By applying F on φ we have a diagram as follows,

$$\begin{tikzcd} B \arrow[r, two heads, "\varepsilon"] \arrow[rr, bend left, "\varphi"] & S \arrow[r, hook, "m"] & A \end{tikzcd}$$

$$\begin{tikzcd} F(B) \arrow[r, "F\varepsilon"] \arrow[rr, bend right, "F\varphi"'] & F(S) \arrow[r, "Fm"] & F(A) \end{tikzcd}$$

Since each functor preserves retractions, $F\varepsilon$ is a retraction and consequently it is epi. Now, if we turn back to equation 7.2.3, since $F\varepsilon$ is well-defined and surjective,

$$\exists p_1, p_2 \in UF(B) \,.\, (p_1 \neq p_2) \wedge (UF\varepsilon(p_1) = x) \wedge (UF\varepsilon(p_2) = y). \tag{7.2.4}$$

Let $\alpha_{p_1}, \alpha_{p_2} : S \longrightarrow FB$ be the unique morphisms in $\mathbb{C}$ such that $U\alpha_{p_1} = C_{p_1}^{U(S)}$ and $U\alpha_{p_2} = C_{p_2}^{U(S)}$ where for each $i \in \{1, 2\}$ the morphism $C_{p_i}^{U(S)} : U(S) \longrightarrow UF(B)$ is the constant map with value p_i (by assumption in (4), α_{p_1} and α_{p_2} exist). $F(B)$ with the morphisms $\alpha_{p_1}, \alpha_{p_2} : S \longrightarrow F(B)$ is a competitor for the sum $B = S + S$ in $\mathbb{C}$. Hence there is a unique morphism $[\alpha_{p_1}, \alpha_{p_2}] : B \longrightarrow F(B)$ in $\mathbb{C}$ such that $[\alpha_{p_1}, \alpha_{p_2}] \circ e_i = \alpha_{p_i}$, for each $i \in \{1, 2\}$.
Besides, let $\alpha_z : A \longrightarrow FA$ be the unique morphism in $\mathbb{C}$ with $U\alpha_z = C_z^{U(A)}$ (where

$C_z^{U(A)} : U(A) \longrightarrow UF(A)$ is the constant map with value p_1). Consider the following diagram.

$$\begin{tikzcd}
U(S) \arrow[r, "Ue_i"] \arrow[dr, "C_{p_i}^{U(S)}"'] & U(B) \arrow[r, two heads, "U\varepsilon"] \arrow[d, "U[\alpha_{p_1},\alpha_{p_2}]"] \arrow[rr, bend left, "U\varphi"] & U(S) \arrow[r, hook, "Um"] & U(A) \arrow[d, "U\alpha_z"] \\
& UF(B) \arrow[r, "UF\varepsilon"] \arrow[rr, bend right, "UF\varphi"'] & UF(S) \arrow[r, "UFm"] & UF(A)
\end{tikzcd}$$

If $C_z^{U(S)}$ is the constant map from $U(S)$ to $UF(A)$ with value z, then we have

$$\begin{aligned}
U(Fm \circ F\varepsilon \circ [\alpha_{p_1}, \alpha_{p_2}] \circ e_i) &= UFm \circ UF\varepsilon \circ U[\alpha_{p_1}, \alpha_{p_2}] \circ Ue_i \\
&= UFm \circ UF\varepsilon \circ C_{p_i}^{U(S)} \\
&= C_z^{U(S)} \\
&= U\alpha_z \circ Um \circ U\varepsilon \circ Ue_i \\
&= U(\alpha_z \circ m \circ \varepsilon \circ e_i).
\end{aligned}$$

Since U is faithful (because $\mathbb{C}$ is a concrete category), we have

$$Fm \circ F\varepsilon \circ [\alpha_{p_1}, \alpha_{p_2}] \circ e_i \quad = \quad \alpha_z \circ m \circ \varepsilon \circ e_i.$$

As $\{e_i\}_{i\in\{1,2\}}$ is an epi sink, we conclude that

$$Fm \circ F\varepsilon \circ [\alpha_{p_1}, \alpha_{p_2}] \quad = \quad \alpha_z \circ m \circ \varepsilon.$$

It means the following diagram commutes.

$$\begin{tikzcd}
B \arrow[r, two heads, "\varepsilon"] \arrow[d, "{[\alpha_{p_1},\alpha_{p_2}]}"'] \arrow[rr, bend left, "\varphi"] & S \arrow[r, hook, "m"] & A \arrow[d, "\alpha_z"] \\
F(B) \arrow[r, "F\varepsilon"] \arrow[rr, bend right, "F\varphi"'] & F(S) \arrow[r, "Fm"] & F(A)
\end{tikzcd}$$

The morphism $\varepsilon : S + S \longrightarrow S$ is a retraction (because $\varepsilon \circ e_i = id_S$ for each $i \in \{1, 2\}$). Then by assumption it is an $\mathcal{E}$-morphism and so $m \circ \varepsilon$ is a decomposition of φ in $(\mathcal{E}, \mathcal{M})-$category $\mathbb{C}$ (see part (1) of theorem 2.11.3). Hence, there is a unique morphism $\rho : S \longrightarrow F(S)$ in $\mathbb{C}$ such that m and ε become homomorphisms in $\mathbb{C}_F$ (by assumption). Now, let $s \in U(S)$ be an arbitrary element, then $(Ue_1(s), Ue_2(s)) \in ker\, U\varepsilon$ (because $U\varepsilon \circ Ue_i = id_{U(S)}$, for each $i \in \{1, 2\}$). On the other hand, from $U[\alpha_{p_1}, \alpha_{p_2}] \circ Ue_i = C_{p_i}^{U(S)}$ (for each $i \in \{1, 2\}$) and equation 7.2.4, we obtain that

$$U(F\varepsilon \circ [\alpha_{p_1},\alpha_{p_2}])(Ue_1(s)) = x \neq y = U(F\varepsilon \circ [\alpha_{p_1},\alpha_{p_2}])(Ue_2(s))$$

It means $(Ue_1(s),\, Ue_2(s)) \notin ker\, U(F\varepsilon \circ [\alpha_{p_1},\alpha_{p_2}])$, then $ker\, U\varepsilon \nsubseteq ker\, U(F\varepsilon \circ [\alpha_{p_1},\alpha_{p_2}])$. So by the diagram lemma in Set, $UF\varepsilon \circ U[\alpha_{p_1},\alpha_{p_2}] \neq U\rho \circ U\varepsilon$. This gives a contradiction with $F\varepsilon \circ [\alpha_{p_1},\alpha_{p_2}] = \rho \circ \varepsilon$.

□

As a consequence of theorem 7.2.5 we have:

Example 7.2.6. Consider the category Top as a $(\mathcal{E},\mathcal{M})$-category where $\mathcal{E} = regular\ epis$ and $\mathcal{M} = monos$. Then by the previous theorem a Top-endofunctor F preserves monos with non-empty domains iff Top_F is an $(\mathcal{E}_F,\mathcal{M}_F)$-category.

7.3. Limits and Colimits

This part is a generalization of Rutten's results about limits and colimits in Set_F, see [62]. We will see that all colimits in $\mathbb{C}_F$ exist and they are constructed as colimits in the base category $\mathbb{C}$, i.e. they have the same underlying objects and the canonical morphisms are homomorphisms. This is not true for limits. As an well-known example, one can not guarantee the existence of terminal objects in $Set_{\mathbb{P}}$, see [35].
However, in this section, we will show that if the $\mathbb{C}$-endofunctor F preserves $\mathcal{M}$-morphisms (where $\mathbb{C}$ is an $\mathcal{M}$-well powered category with coproducts) then an equalizer of two homorphisms $\varphi_1,\varphi_2 : A \longrightarrow B$ in $\mathbb{C}_F$ does exist and it is constructed via union of a special family of $\mathcal{M}$-subcoalgebras of their domain.

7.3.1. Colimits

Recall that $U_{\mathbb{C}}$ is the forgetful functor from $\mathbb{C}_F$ to $\mathbb{C}$ (i.e., for each coalgebra $\mathcal{A} = (A,\alpha_A)$ we have $U_{\mathbb{C}}(\mathcal{A}) = A$ and for each homomorphism $f : A \longrightarrow B$ between coalgebras $\mathcal{A} = (A,\alpha_A)$ and $\mathcal{B} = (B,\alpha_B)$, we have $U_{\mathbb{C}}(f) = f$).

Additional to the assumption **A1**, in this section we assume that the base category $\mathbb{C}$ satisfies the assumption **A2** (see remark 7.1.10).

Theorem 7.3.1. *$\mathbb{C}_F$ is a cocomplete category.*

Proof. We need to show that in the category $\mathbb{C}_F$ all small colimits exist. Given a small diagram $K : \mathbb{I} \longrightarrow \mathbb{C}_F$ of type $\mathbb{I}$ in $\mathbb{C}_F$. It means $K(i) = (X_i,\alpha_i)$ is a coalgebra in $\mathbb{C}_F$ (for each $i \in Ob(\mathbb{I})$). Let the object C with the sink $\{e_i : X_i \longrightarrow C\}_{i\in Ob(\mathbb{I})}$ be a colimit of the underlying diagram $U_{\mathbb{C}} \circ K$ in $\mathbb{C}$, then we have the following diagram,

$$\begin{tikzcd} X_i \arrow[r, "e_i"] \arrow[d, "\alpha_i"] & C \arrow[d, dashed, "\gamma"] \\ F(X_i) \arrow[r, "Fe_i"'] & F(C) \end{tikzcd}$$

We find that $F(C)$ with the family of morphisms $\{F(e_i) \circ \alpha_i\}_{i \in Ob(\mathbb{I})}$ is a competitor to the colimit $(C, \{e_i\}_{i \in Ob(\mathbb{I})})$ in the category $\mathbb{C}$. So there exists a unique structure $\gamma : C \longrightarrow F(C)$ such that $\gamma \circ e_i = F(e_i) \circ \alpha_i$, i.e. for each $i \in Ob(\mathbb{I})$ the morphism e_i is a homomorphism in $\mathbb{C}_F$. Now, let $Q = (Q, \delta)$ with homomorphism $\{\varphi_i : X_i \longrightarrow Q\}_{i \in Ob(\mathbb{I})}$ be a competitor of $((C, \gamma), \{e_i\}_{i \in Ob(\mathbb{I})})$ in $\mathbb{C}_F$. Thus the follow diagram can be obtained,

$$\begin{tikzcd} Q \arrow[d, "\delta"] & X_i \arrow[l, "\varphi_i"'] \arrow[r, "e_i"] \arrow[d, "\alpha_i"] & C \arrow[d, "\gamma"] \arrow[ll, bend right, "\sigma"'] \\ F(Q) & F(X_i) \arrow[l, "F(\varphi_i)"'] \arrow[r, "F(e_i)"] & F(C) \arrow[ll, bend left, "F(\sigma)"] \end{tikzcd}$$

The upper row of this diagram say us that Q with the family of morphisms $\{\varphi_i\}_{i \in Ob(\mathbb{I})}$ is a competitor of $(C, \{e_i\}_{i \in Ob(\mathbb{I})})$ in the category $\mathbb{C}$. Hence there is a unique morphism $\sigma : C \longrightarrow Q$ with $\varphi_i = \sigma \circ e_i$ for all $i \in Ob(\mathbb{I})$. It remains to show that σ is a homomorphism. For all $i \in Ob(\mathbb{I})$ we have

$$\begin{aligned} \delta \circ \sigma \circ e_i &= \delta \circ \varphi_i \\ &= F(\varphi_i) \circ \alpha_i \\ &= F(\sigma) \circ F(e_i) \circ \alpha_i \\ &= F(\sigma) \circ \gamma \circ e_i \end{aligned}$$

Since the $(e_i)_{i \in Ob(\mathbb{I})}$ is an epi sink, we conclude that $\delta \circ \sigma = F(\sigma) \circ \gamma$. □

Corollary 7.3.2. *In the category $\mathbb{C}_F$ all small colimits exist and they are constructed as in $\mathbb{C}$.*

Remark 7.3.3. According to the previous corollary small sums[1], small coequalizers[2] and small pushouts[3] in $\mathbb{C}_F$ exist and they are formed precisely as in $\mathbb{C}$, i.e. they have the same underlying objects and the canonical maps are homomorphisms in $\mathbb{C}_F$. More exactly:

[1] Recall that a small sum in a category $\mathbb{C}$ is a sum of a family of objects in $\mathbb{C}$ indexed by a set.

[2] Recall that a small coequalizer in a category $\mathbb{C}$ is a coequalizer of a family of parallel morphisms in $\mathbb{C}$ indexed by a set.

[3] A small pushout in a category $\mathbb{C}$ is a pushout of a source in $\mathbb{C}$ indexed by a set.

1. Suppose $\{\mathcal{A}_i = (A_i, \alpha_i)\}_{i\in I}$ is a family of coalgebras in $\mathbb{C}_F$ indexed by the set I. Let $S := \sum\limits_{i\in I} A_i$ with the canonical injection $\{e_i : A_i \longrightarrow P\}_{i\in I}$ be a sum of the underlying objects $\{A_i\}_{i\in I}$ in $\mathbb{C}$. Then $F(S)$ with the morphisms $\{F(e_i) \circ \alpha_i\}_{i\in I}$ is a competitor for the sum $(S, \{e_i\}_{i\in I})$ in $\mathbb{C}$. So a coalgebra structure $\gamma : S \longrightarrow F(S)$ can be uniquely defined on S such that e_i is a homomorphism for all $i \in I$. The coalgebra (S, γ) with $\{e_i\}_{i\in I}$ is a sum of $\{\mathcal{A}_i\}_{i\in I}$ in $\mathbb{C}_F$.

2. Let $\{\varphi_i : A \longrightarrow B\}_{i\in I}$ be a family of coalgebra homomorphisms between coalgebras $\mathcal{A} =(A, \alpha)$ and $\mathcal{B} =(B, \beta)$ in $\mathbb{C}_F$ indexed by the set I. Let $\varphi : B \longrightarrow C$ be a coequalizer of the morphisms $\{\varphi_i\}_{i\in I}$ in the category $\mathbb{C}$. Then $F(C)$ with morphism $F\varphi \circ \beta$ is a competitor for φ in $\mathbb{C}$. So, there is a unique morphism $\gamma : C \longrightarrow F(C)$ making the morphism $\varphi : B \longrightarrow C$ into a homomorphism in $\mathbb{C}_F$, and it is easy to see that φ is also a coequalizer of the family $\{\varphi_i\}_{i\in I}$ in $\mathbb{C}_F$.

3. Suppose $\{\varphi_i : A \longrightarrow B_i\}_{i\in I}$ is a family of homomorphisms between coalgebras $\mathcal{A} =(A, \alpha)$ and $\mathcal{B}_\mathrm{i} =(B_i, \beta_i)$ in $\mathbb{C}_F$. Let P with a sink $\{\psi_i : B_i \longrightarrow P\}_{i\in I}$ be a pushout of the source $\{\varphi_i\}_{i\in I}$ in $\mathbb{C}$. Then $F(P)$ with morphisms $\{F\psi_i \circ \beta_i\}_{i\in I}$ is a competitor for $(P, \{\psi_i\}_{i\in I})$ in $\mathbb{C}$. Then a coalgebra structure $\gamma : P \longrightarrow F(P)$ can be uniquely defined such that ψ_i is a homomorphism for all $i \in I$, and (P, γ) with the morphisms $\{\psi_i\}_{i\in I}$ is a pushout of the source $\{\varphi_i\}_{i\in I}$ in $\mathbb{C}_F$.

7.3.2. Union of subcoalgebras

Here we introduce the notion of union of $\mathcal{M}$-subcoalgebras. It will be used to find an equalizer of a pair of parallel homomorphisms f and g in $\mathbb{C}_F$ under an assumption like the functor F preserves $\mathcal{M}$-morphisms. This notion is also used to check the existence of the largest A-M bisimulation between F-coalgebras, if in addition to preserving $\mathcal{M}$-morphisms by F, we assume that the functor F weakly preserves pullbacks.

Additional to the assumptions **A1** and **A2**, in this subsection we assume that the conditions **A3** and **A5** (see remark 7.1.10) hold.

Definition 7.3.4. (Union of $\mathcal{M}$-subcoalgebras) Let $\{m_i : S_i \longrightarrow A\}_{i\in I}$ be an arbitrary family of $\mathcal{M}$-subcoalgebras of a coalgebra $\mathcal{A} = (A, \alpha)$ in $\mathbb{C}_F$. A union of $\mathcal{M}$-subcoalgebras $\{m_i\}_{i\in I}$ (in symbols: $\bigsqcup\limits_{i\in I} m_i$) is the $\mathcal{M}$-union of the underlying $\mathcal{M}$-morphisms of $\{m_i\}_{i\in I}$ in $\mathbb{C}$ (see section 2.12).

Remark 7.3.5. According to theorem 2.12.3, $\coprod_{i\in I} m_i$ is the $\mathcal{M}$-morphism m in the following diagram.

$$\begin{tikzcd} S_i \arrow[r, "e_i"] \arrow[rr, bend left, "m_i"] & S \arrow[r, "q"] \arrow[dr, "e"'] & A \\ & & E \arrow[u, "m"'] \end{tikzcd} \tag{7.3.1}$$

where S with the canonical injections $\{e_i : S_i \longrightarrow S\}_{i\in I}$ is a sum of the family $\{S_i\}_{i\in I}$ in $\mathbb{C}$; the morphism $q : S \longrightarrow A$ is the unique morphism such that $m_i = q \circ e_i$, and $m \circ e$ is a decomposition of q in $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathbb{C}$.

In the following theorem we will show that the union of every family of $\mathcal{M}$-subcoalgebras is an $\mathcal{M}$-subcoalgebra too.

Theorem 7.3.6. *Let $\{m_i : S_i \longrightarrow A\}_{i\in I}$ be an arbitrary family of $\mathcal{M}$-subcoalgebras of a coalgebra $\mathcal{A} = (A, \alpha)$ in $\mathbb{C}_F$. Then $\coprod_{i\in I} m_i$ is an $\mathcal{M}$-subcoalgebra of $\mathcal{A}$.*

Proof. Since $\{m_i : S_i \longrightarrow A\}_{i\in I}$ are $\mathcal{M}$-subcoalgebras of $\mathcal{A}$, by definition 7.1.11, for each $i \in I$ there is a $\mathbb{C}$-morphism $\alpha_i : S_i \longrightarrow F(S_i)$ such that $m_i : S_i \longrightarrow A$ is a homomorphism in $\mathbb{C}_F$. Now, consider the following diagram,

$$\begin{tikzcd} S_i \arrow[r, "e_i"] \arrow[rrr, bend left, "m_i"] \arrow[ddd, "\alpha_i"'] & S \arrow[rr, "q"] \arrow[dr, "e"'] & & A \arrow[ddd, "\alpha"] \\ & & E \arrow[ur, "m"'] & \\ & & & \\ F(S_i) \arrow[r, "Fe_i"'] \arrow[rrr, bend left, "Fm_i"] & F(S) \arrow[rr, "Fq"'] \arrow[dr, "Fe"'] & & F(A) \\ & & F(E) \arrow[ur, "Fm"'] & \end{tikzcd} \tag{7.3.2}$$

where S with the canonical injections $\{e_i : S_i \longrightarrow S\}_{i\in I}$ is a sum of the family $\{S_i\}_{i\in I}$ in $\mathbb{C}$; the morphism $q : S \longrightarrow A$ is the unique morphism such that $m_i = q \circ e_i$, and $m \circ e$ is a decomposition of q in $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathbb{C}$.
Due to part (1) in example 7.3.3, there is a unique structure $\gamma : S \longrightarrow F(S)$ such that for all $i \in I$ the morphism $e_i : S_i \longrightarrow S$ is a homomorphism. Then, since $q \circ e_i = m_i$ (for each $i \in I$) and since $\{e_i\}_{i\in I}$ is an epi sink in $\mathbb{C}$, by part (1) of lemma 7.1.9 the unique morphism $q : S \longrightarrow A$ is a homomorphism in $\mathbb{C}_F$. According to remark 7.3.5,

we know that $\coprod_{i\in I} m_i$ is the $\mathcal{M}$-image of the unique morphism $q : S \longrightarrow A$ (i.e., $\coprod_{i\in I} m_i$ is the $\mathcal{M}$-morphism $m : E \longrightarrow A$). Since F preserves $\mathcal{M}$-morphisms, there is a unique diagonal $\beta : E \longrightarrow F(E)$ such that $e : S \longrightarrow E$ and $m : E \longrightarrow A$ are homomorphisms in $\mathbb{C}_F$. Then $m : E \longrightarrow A$ is an $\mathcal{M}$-subcoalgebra of $\mathcal{A}$. $\square$

As an application of the previous theorem we have:

Corollary 7.3.7. *Given an arbitrary family $\{m_i : S_i \longrightarrow A\}_{i\in I}$ of $\mathcal{M}$-subcoalgebras of a coalgebra $\mathcal{A} = (A, \alpha)$ in $\mathbb{C}_F$. Then for each $i \in I$, there is a homomorphism $g_i : dom(m_i) \longrightarrow dom(\coprod_{i\in I} m_i)$ such that $(\coprod_{i\in I} m_i) \circ g_i = m_i$.*

Proof. Consider diagram 7.3.2 in the previous theorem. For every $i \in I$, define a morphism $g_i : S_i \longrightarrow E$ by $g_i := e \circ e_i$. $\square$

We have just proven that if the $\mathbb{C}$-endofunctor F preserves $\mathcal{M}$-morphisms, then the union of $\mathcal{M}$-subcoalgebras is also an $\mathcal{M}$-subcoalgebra. In general the union of $\mathcal{M}$-subcoalgebras need not be an $\mathcal{M}$-subcoalgebra. The next example shows this issue clearly. In this example we will consider *Top* as an (*epi*, *regular mono*)-category and our objects are coalgebras of the *Top*-endofunctor $\overline{T}$ introduced in example 4.2.3. Recall that $\overline{T}$ does not preserves regular monos (see example 4.2.4).

Example 7.3.8. Consider $\mathcal{Y} = (Y, \beta)$ as the coalgebra introduced in example 7.2.1. Let $\iota_1 : S_1 \longrightarrow Y$ and $\iota_2 : S_2 \longrightarrow Y$ be two topological embeddings where $S_1 = \{1, 2\}$ and $S_2 = \{3, 4\}$. Consider the following maps.

$$\begin{array}{cc} \alpha_1 : S_1 \longrightarrow \overline{T}(S_1) & \qquad \alpha_2 : S_2 \longrightarrow \overline{T}(S_2) \\ \alpha(c) := \begin{cases} (1,2) & c = 1 \\ \perp & c = 2 \end{cases} & \qquad \alpha_2(c) := \perp \end{array}$$

It is easy to see that α_1 and α_2 are continuous maps making ι_1 and ι_2 into homomorphisms in Top_F. Therefore, ι_1 and ι_1 are subcoalgebras of $\mathcal{Y} = (Y, \beta)$. However, their union $\coprod_{i\in\{1,2\}} \iota_i$ that is the topological embedding $\iota : E \longrightarrow Y$ (where $E = \{1, 2, 3, 4\}$), is not a homomorphism in Top_F. Because as it has been shown in example 7.2.1, the only structure ρ which makes the following diagram into a commutative diagram is not continuous.

$$\begin{tikzcd} E \arrow[r, "\iota"] \arrow[d, "\rho"'] & Y \arrow[d, "\beta"] \\ \overline{T}(E) \arrow[r, "\overline{T}(\iota)"'] & \overline{T}(Y) \end{tikzcd}$$

7.3.3. Equalizer

In this subsection, we assume that the assumptions **A1**, **A2**, **A3** and **A5** (see remark 7.1.10) still hold.

Theorem 7.3.9. *Let $f, g : A \longrightarrow B$ be homomorphisms between coalgebras $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ in $\mathbb{C}_F$. The equalizer of f and g in $\mathbb{C}_F$ is the union of all $\mathcal{M}$-subcoalgebras $\{m_i : S_i \longrightarrow A\}_{i \in I}$ of $\mathcal{A}$ such that $f \circ m_i = g \circ m_i$ for each $i \in I$.*

Proof. Let $\{m_i : S_i \longrightarrow A\}_{i \in I}$ be the family of all $\mathcal{M}$-subcoalgebras of $\mathcal{A}$ such that $f \circ m_i = g \circ m_i$ for each $i \in I$. Let the object S with morphisms $\{e_i : S_i \longrightarrow S\}_{i \in I}$ be a sum of the objects $\{S_i\}_{i \in I}$ in $\mathbb{C}$. By remark 7.3.5, we know that $\coprod_{i \in I} m_i$ is the $\mathcal{M}$-image of the unique morphism $q : S \longrightarrow A$ in diagram 7.3.1 (i.e., $\coprod_{i \in I} m_i$ is the $\mathcal{M}$-morphism $m : E \longrightarrow A$). Then by lemma 2.12.4, $f \circ (\coprod_{i \in I} m_i) = g \circ (\coprod_{i \in I} m_i)$. Besides, due to theorem 7.3.6, $\coprod_{i \in I} m_i$ is a homomorphism in $\mathbb{C}_F$. Now, assume $\varphi : (Q, \rho) \longrightarrow (A, \alpha)$ is a homomorphism in $\mathbb{C}_F$ with $f \circ \varphi = g \circ \varphi$. Suppose $\varphi = m' \circ e$ is a decomposition of the underlying morphism of φ in $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathbb{C}$. Consider the diagram below:

$$\begin{tikzcd}
Q \arrow[rr, "\varphi"] \arrow[dr, "e"'] \arrow[dd, "\rho"'] & & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[dd, "\alpha"'] & B \arrow[dd, "\beta"] \\
& K \arrow[ur, "m'"'] & & \\
F(Q) \arrow[rr, "F\varphi"] \arrow[dr, "Fe"'] & & F(A) \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] & F(B) \\
& F(K) \arrow[ur, "Fm'"'] & &
\end{tikzcd}$$

By theorem 7.2.2, there is a morphism $\sigma : K \longrightarrow F(K)$ such that $m' : K \longrightarrow A$ is an $\mathcal{M}$-subcoalgebra of $\mathcal{A}$ (because Fm' is an $\mathcal{M}$-morphism). Also

$$\begin{aligned}
f \circ m' \circ e &= f \circ \varphi \\
&= g \circ \varphi \\
&= g \circ m' \circ e
\end{aligned}$$

Since e is epi, we obtain $f \circ m' = g \circ m'$. Then by assumption there is an element $i \in I$ such that $m' = m_i$. So by corollary 7.3.7, there is a homomorphism $r : K \longrightarrow dom(\bigsqcup_{i \in I} m_i)$ such that $m \circ r = m'$. Now, consider $h := r \circ e$. obviously, one can see that $m \circ h = \varphi$. Since m is mono, h is unique. $\square$

Example 7.3.10. Consider the category Top as an (epi, $regular\, mono$)-category. Since the Vietoris functor $\mathbb{V}$ (resp. P-Vietoris functor $\mathbb{V}_P$) preserves regular monos (see lemma 3.2.4), an equalizer of two homorphisms $f, g : A \longrightarrow B$ in $Top_{\mathbb{V}}$ (resp. $Top_{\mathbb{V}_P}$) does exist and it is the union of all $\mathcal{M}$-subcoalgebras $\{m_i : S_i \longrightarrow A\}_{i \in I}$ such that $f \circ m_i = g \circ m_i$ for each $i \in I$.

7.3.4. $\mathcal{M}$-morphism in C_F

In this subsection we assume that the assumptions **A1**, **A2** and **A5** (see remark 7.1.10) hold, then:

Lemma 7.3.11. *Let $f, g : A \longrightarrow B$ be homomorphisms between coalgebras $\mathcal{A} = (A, \alpha)$ and $\mathcal{B} = (B, \beta)$ in $\mathbb{C}_F$. If $e : (E, \rho) \longrightarrow (A, \alpha)$ is an equalizer of f and g in $\mathbb{C}_F$, then the underlying morphism of e is an $\mathcal{M}$-morphism in $\mathbb{C}$.*

Proof. Suppose $e : (E, \rho) \longrightarrow (A, \alpha)$ is an equalizer of f and g in $\mathbb{C}_F$. Let $e = i \circ \tilde{e}$ be a decomposition of the underlying morphism of e in $(\mathcal{E}, \mathcal{M})$-system in $\mathbb{C}$, and then we have the following diagram:

$$\begin{tikzcd}
E' \arrow[rr, "e"] \arrow[dr, shift left, "\tilde{e}"] \arrow[dd, "\rho"'] & & A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[dd, "\alpha"] & B \arrow[dd, "\beta"] \\
& Q \arrow[ul, shift left, "h"] \arrow[ur, hook, "i"'] \arrow[dd, dashed, "\delta"] & & \\
F(E') \arrow[rr, "Fe"] \arrow[dr, shift left, "F\tilde{e}"] & & F(A) \arrow[r, bend left, "Ff"] \arrow[r, bend right, "Fg"'] & F(B) \\
& F(Q) \arrow[ul, shift left, "Fh"] \arrow[ur, "Fi"'] & &
\end{tikzcd}$$

It is enough to prove that $\tilde{e}$ is an isomorphism. Since F preserve $\mathcal{M}$-morphisms, $\tilde{e}$ is orthogonal to Fi. So there is a unique diagonal $\delta : Q \longrightarrow F(Q)$ such that this diagram is commutative.We have

$$\begin{aligned} f \circ i \circ \tilde{e} &= f \circ e \\ &= g \circ e \\ &= g \circ i \circ \tilde{e}. \end{aligned}$$

Since $\tilde{e}$ is epi in $\mathbb{C}$, we have $f \circ i = g \circ i$. Hence (Q, δ) with $i : (Q, \delta) \longrightarrow (A, \alpha)$ is a competitor for $e : (E, \rho) \longrightarrow (A, \alpha)$. So there is a unique homomorphism $h : Q \longrightarrow E$ such that $e \circ h = i$.

Claim. $\tilde{e}$ is a section in $\mathbb{C}$.

It suffices to show that h is a left inverse for $\tilde{e}$. From $e \circ h = i$ we have

$$\begin{aligned} e \circ id_{E'} &= e \\ &= i \circ \widetilde{e} \\ &= e \circ h \circ \widetilde{e}. \end{aligned}$$

Now, since e is an equalizer in $\mathbb{C}_F$, it is a monomorphism in $\mathbb{C}_F$. So $h \circ \widetilde{e} = id_{E'}$, i.e. $\tilde{e}$ is a section in $\mathbb{C}_F$. Hence $\tilde{e}$ is a section in $\mathbb{C}$ (because the forgetfull functor $U_{\mathbb{C}} : \mathbb{C}_F \longrightarrow \mathbb{C}$ preserves sections).
On the other hand $\tilde{e}$ is epi in $\mathbb{C}$. Then $\tilde{e}$ is an isomorphism in $\mathbb{C}$ (see lemma 2.2.17). Now, since $i \circ \widetilde{e} = e$, it is concluded that e is an $\mathcal{M}$-morphism in $\mathbb{C}$. □

In the case that in the base category $\mathbb{C}$ the class of $\mathcal{M}$-morphisms is a subclass of regular monomorphisms, the converse of lemma 7.3.11 holds. The following theorem shows this issue.

Theorem 7.3.12. *Let $\mathbb{C}$ be a category in which every $\mathcal{M}$-morphism be regular mono. A homomorphism φ is regular mono in $\mathbb{C}_F$ iff the underlying morphism of φ in $\mathbb{C}$ is an $\mathcal{M}$-morphism in $\mathbb{C}$.*

Proof. Let $\varphi : A \longrightarrow B$ be a homomorphism in $\mathbb{C}_F$ such that the underlying morphism of φ in $\mathbb{C}$ is an $\mathcal{M}$-morphism in $\mathbb{C}$. By assumption φ is a regular monomorphism in $\mathbb{C}$. So there are $\mathbb{C}$-morphisms f,$g : B \longrightarrow K$ such that φ is an equalizer of them. Now, consider (P, p_1, p_2) as a pushout of φ with itself in $\mathbb{C}_F$. Then according to corollary 7.3.2, $(U_{\mathbb{C}}P, U_{\mathbb{C}}p_1, U_{\mathbb{C}}p_2)$ (where $U_{\mathbb{C}} : \mathbb{C}_F \longrightarrow \mathbb{C}$ is the forgetful functor) is a pushout of φ with itself in $\mathbb{C}$. Hence there is a unique morphism $h : UP \longrightarrow K$ such that $h \circ p_1 = f$ and $h \circ p_2 = g$. We claim that φ is an equalizer of p_1 and p_2 in $\mathbb{C}_F$. So let $\chi : Q \longrightarrow B$ be a competitor of φ in $\mathbb{C}_F$, i.e, χ is a homomorphism with $p_1 \circ \chi = p_2 \circ \chi$.

$$\begin{tikzcd}
 & & B \arrow[dr, "p_1"] \arrow[drr, dashed, "f"] & & \\
Q \arrow[r, "\rho"] \arrow[urr, bend left, "\chi"] \arrow[drr, bend right, "\chi"'] & A \arrow[ur, "\varphi"] \arrow[dr, "\varphi"'] & & P \arrow[r, dotted, "\exists!\, h"] & K \\
 & & B \arrow[ur, "p_2"'] \arrow[urr, dashed, "g"'] & &
\end{tikzcd}$$

Then

$$\begin{aligned} f \circ \chi &= h \circ p_1 \circ \chi \\ &= h \circ p_2 \circ \chi \\ &= g \circ \chi. \end{aligned}$$

It means $\chi : Q \longrightarrow B$ is a competitor of φ in $\mathbb{C}$. Then there is a unique morphism $\rho : Q \longrightarrow A$ such that $\varphi \circ \rho = \chi$. Since F preserves $\mathcal{M}$-morphisms, it is easy to see that ρ is a homomorphism in $\mathbb{C}_F$. The converse follows from lemma 7.3.11. ☐

7.4. A-M Bisimulation

Throughout this section, we assume that the base category $\mathbb{C}$ satisfies the conditions **A1** and **A4** (see remark 7.1.10). Now we have:

Definition 7.4.1. Suppose $\mathcal{A}_1 = (A_1, \alpha_1)$ and $\mathcal{A}_2 = (A_2, \alpha_2)$ are two arbitrary coalgebras in $\mathbb{C}_F$. An $\mathcal{M}$-subobject $m : R \longrightarrow A_1 \times A_2$ of the product $A_1 \times A_2$ in $\mathbb{C}$ is called an **A-M bisimulation** between $\mathcal{A}_1$ and $\mathcal{A}_2$, if there exists a coalgebra structure $\rho : R \longrightarrow F(R)$ such that the following diagram commutes, i.e., for each $i \in \{1, 2\}$ the morphism $\pi_{A_i} \circ m : R \longrightarrow A_i$ is a homomorphism in $\mathbb{C}_F$.

$$\begin{tikzcd}
R \arrow[r, "m"] \arrow[d, "\rho"] & A_1 \times A_2 \arrow[r, "\pi_{A_i}"] & A_i \arrow[d, "\alpha_i"] \\
F(R) \arrow[r, "F(m)"] & F(A_1 \times A_2) \arrow[r, "F(\pi_{A_i})"] & F(A_i)
\end{tikzcd} \tag{7.4.1}$$

Remark 7.4.2. If $\mathbb{C}$ is a category in which the objects are sets with additional structures (as example *Set*, *Top*, CUM^1,...etc), we say that two elements $a_1 \in A_1$ and $a_2 \in A_2$ are A-M bisimilar if there is an A-M bisimulation $m : R \longrightarrow A_1 \times A_2$ between coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathbb{C}_F$ and an element $r \in R$ such that $\pi_{A_1} \circ m(r) = a_1$ and $\pi_{A_2} \circ m(r) = a_2$.

Remark 7.4.3. Recall that in the category *Top* the regular monomorphisms are (up to isomorphisms) exactly the topological embeddings (see lemma 2.9.3). Therefore, if we consider *Top* as an (*epi, regular mono*)-category (see example 2.11.6), then a topological embedding $\iota : R \longrightarrow A_1 \times A_2$ (where $A_1 \times A_2$ with the projection maps $\{\pi_{A_i}\}_{i\in\{1,2\}}$ is product of the topological spaces A_1 and A_2 in *Top*) is an A-M bisimulation between coalgebras $\mathcal{A}_1 = (A_1, \alpha_1)$ and $\mathcal{A}_2 = (A_2, \alpha_2)$ in Top_F provided that a continuous map $\rho : R \longrightarrow F(R)$ on R can be defined so that the morphisms $\{\pi_{A_i} \circ m : R \longrightarrow A_i\}_{i\in\{1,2\}}$ become homomorphisms in Top_F. Consequently we say that two elements $a_1 \in A_1$ and $a_2 \in A_2$ are A-M bisimilar if there is an A-M bisimulation $\iota : R \longrightarrow A_1 \times A_2$ between coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $(a_1, a_2) \in R$.
In general, the product $A_1 \times A_2$ need not be an A-M bisimulation.

Example 7.4.4. Recall that the category *Set* is an (*epi, mono*)-category (see example 2.11.5). Since in *Set*, the monomorphisms are (up to isomorphisms) exactly the inclusion of subsets (see remark 2.2.15), we can say that an A-M bisimulation between coalgebras $\mathcal{A}_1 = (X_1, \alpha_1)$ and $\mathcal{A}_2 = (X_2, \alpha_2)$ in Set_F is a subset $R \subseteq X_1 \times X_2$ for which there exists a map $\rho : R \longrightarrow F(R)$ that makes the projections $\{\pi_{A_i} : R \longrightarrow A_i\}_{i\in\{1,2\}}$ into homomorphisms in Set_F. It is straightforward to check that a sebset $R \subseteq X_1 \times X_2$ is a Kripke bisimulation between Kripke models $\mathcal{X}_1$ and $\mathcal{X}_2$ if and only if the inclusion map R is an A-M bisimulation between the corresponding $\mathbb{P}_P$-coalgebras to $\mathcal{X}_1$ and $\mathcal{X}_2$ (this example is covered in details in [31], [45] and [62]).

7.4.1. Some facts about A-M Bisimulations

Definition 7.4.5. Given a 2-source $\{f_i : P \longrightarrow A_i\}_{i\in\{1,2\}}$ in $\mathbb{C}$. Then P with morphism f_1 and f_2 is a competitor for $(A_1 \times A_2,\ \pi_{A_1},\ \pi_{A_2})$, i.e., the product of A_1 and A_2 in $\mathbb{C}$. So there exists a unique morphism $[f_1,\ f_2] : P \longrightarrow A_1 \times A_2$ such that $\pi_{A_1} \circ [f_1,\ f_2] = f_1$ and $\pi_{A_2} \circ [f_1,\ f_2] = f_2$. If $[f_1,\ f_2] = m \circ e$ is the decomposition of the morphism $[f_1,\ f_2]$

in $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathbb{C}$, we have the following diagram:

$$\begin{tikzcd}
 & A_1 & \\
P \arrow[ur, "f_1"] \arrow[rr, "{[f_1, f_2]}"] \arrow[dr, "e"] \arrow[ddr, "f_2"'] & & A_1 \times A_2 \arrow[ul, "\pi_{A_1}"'] \arrow[ddl, "\pi_{A_2}"] \\
 & G([f_1,\, f_2]) \arrow[ur, "m"] & \\
 & A_2 &
\end{tikzcd} \tag{7.4.2}$$

The $\mathcal{M}$-subobject $m : G([f_1,\, f_2]) \longrightarrow A_1 \times A_2$ of $A_1 \times A_2$ is called **$\mathcal{M}$-graph of the 2-source** $(f_i)_{i\in\{1,2\}}$.
The $\mathcal{M}$-graph of the 2-source $(id_{dom(f)},\, f)$ is called **$\mathcal{M}$-graph of** f and it is denoted by $m : G(f) \longrightarrow A_1 \times A_2$.
For each object A in $\mathbb{C}$, the diagonal $\triangle_A$ is the $\mathcal{M}$-graph of the identity morphism id_A.
In this work, if the class $\mathcal{M}$ is equal to the class of the regular monos, we call $\mathcal{M}$-graphs just graphs.

The following theorem gives us a characterization of the coalgebra homomorphisms.

Theorem 7.4.6. *Let $\mathcal{A}_1 = (A_1, \alpha_1)$ and $\mathcal{A}_2 = (A_2, \alpha_2)$ be arbitrary coalgebras in $\mathbb{C}_F$. A morphism $f : A_1 \longrightarrow A_2$ is a homomorphism in $\mathbb{C}_F$ iff the $\mathcal{M}$-graph of f is an A-M bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$.*

Proof. In diagram 7.4.2, replace the morphisms f_1 and f_2 by the morphisms id_{A_1} and f, respectively. Then $\pi_{A_1} \circ m \circ e = id_{A_1}$. So e is a section with left inverse $\pi_{A_1} \circ m$. It is also epi. Consequently e is an isomorphism (see lemma 2.2.17). Consider the following commutative diagram:

$$\begin{tikzcd}
 & A_1 \times A_2 \arrow[dl, "\pi_{A_1}"'] \arrow[dr, "\pi_{A_2}"] & \\
A_1 \arrow[loop left, "id_{A_1}"] \arrow[r, shift left, "e"] \arrow[d, "\alpha_1"'] \arrow[rr, bend left=60, "f"] & G(f) \arrow[u, "m"'] \arrow[l, shift left, "\pi_{A_1} \circ m"] \arrow[r, "\pi_{A_2} \circ m"] \arrow[d, dashed, "\gamma"'] & A_2 \arrow[d, "\alpha_2"] \\
F(A_1) \arrow[r, shift left, "Fe"] \arrow[rr, bend right=60, "Ff"'] & F(G(f)) \arrow[l, shift left, "F(\pi_{A_1} \circ m)"] \arrow[r, "F(\pi_{A_2} \circ m)"'] \arrow[d, "Fm"'] & F(A_2) \\
 & F(A_1 \times A_2) \arrow[ul, "F\pi_{A_1}"] \arrow[ur, "F\pi_{A_2}"'] &
\end{tikzcd}$$

Suppose $f : A_1 \longrightarrow A_2$ is a homomorphism. Define $\gamma := Fe \circ \alpha_1 \circ \pi_{A_1} \circ m$, then

$$\begin{aligned}
F(\pi_{A_1} \circ m) \circ \gamma &= F(\pi_{A_1}) \circ Fm \circ Fe \circ \alpha_1 \circ \pi_{A_1} \circ m \\
&= F(id_{A_1}) \circ \alpha_1 \circ \pi_{A_1} \circ m \\
&= \alpha_1 \circ \pi_{A_1} \circ m.
\end{aligned}$$

Also $F(\pi_{A_2} \circ m) \circ \gamma = F(\pi_{A_2}) \circ Fm \circ Fe \circ \alpha_1 \circ \pi_{A_1} \circ m$, then we have

$$\begin{aligned}
F(\pi_{A_2} \circ m) \circ \gamma \circ e &= F(\pi_{A_2}) \circ Fm \circ Fe \circ \alpha_1 \circ \pi_{A_1} \circ m \circ e \\
&= F(\pi_{A_2}) \circ Fm \circ Fe \circ \alpha_1 \circ id_{A_1} \\
&= Ff \circ \alpha_1 \\
&= \alpha_2 \circ f \\
&= \alpha_2 \circ \pi_{A_2} \circ m \circ e.
\end{aligned}$$

Since e is epi, we have $F(\pi_{A_2} \circ m) \circ \gamma = \alpha_2 \circ \pi_{A_2} \circ m$.

□

Corollary 7.4.7. *The diagonal $\triangle_A$ of a coalgebra $\mathcal{A} = (A, \alpha)$ is always an A-M bisimulation.*

Theorems 7.4.8 and 7.4.9 provide us with a characterization of A-M bisimulations between coalgebras in $\mathbb{C}_F$, whenever the base category $\mathbb{C}$ has special properties. In the first theorem, the $\mathcal{E}$-morphisms in $\mathbb{C}$ are retractions. In the second one, the $\mathcal{M}$-morphisms in $\mathbb{C}$ are exactly monos. Notice that in both theorems $\mathcal{A}_1 = (A_1, \alpha_1)$ and $\mathcal{A}_2 = (A_2, \alpha_2)$ are given as coalgebras in $\mathbb{C}_F$.

Theorem 7.4.8. *Suppose $\mathbb{C}$ is a category in which each $\mathcal{E}$-morphism is a retraction. An $\mathcal{M}$-subobject $m : R \longrightarrow A_1 \times A_2$ is an A-M bisimulation between coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathbb{C}_F$ if and only if there is a 2-source $\{f_i : P \longrightarrow A_i\}_{i\in\{1,2\}}$ in $\mathbb{C}_F$ such that m is the $\mathcal{M}$-graph of its underlying 2-source in $\mathbb{C}$.*

Proof. Let $m : R \longrightarrow A_1 \times A_2$ be an $\mathcal{M}$-subobject in $\mathbb{C}$. Suppose $(f_i : P \longrightarrow A_i)_{i\in\{1,2\}}$ is a 2-source in $\mathbb{C}_F$ such that m is the $\mathcal{M}$-graph of its underlying 2-source in $\mathbb{C}$ (i.e., there is an $\mathcal{E}$-morphism e such that $[f_1, f_2] = m \circ e$ is a decomposition of the $\mathbb{C}$-morphism $[f_1,\ f_2]$ in $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathbb{C}$, see definition 7.4.5). So according to diagram 7.4.2, we have the following diagram:

$$\begin{tikzcd}
P \arrow[r, "e"'] \arrow[rrr, bend left, "f_i"] \arrow[d, "\gamma"'] & R \arrow[r, "m"'] & A_1 \times A_2 \arrow[r, "\pi_{A_i}"'] & A_i \arrow[d, "\alpha_i"] \\
F(P) \arrow[r, "Fe"] \arrow[rrr, bend right, "Ff_i"'] & F(R) \arrow[r, "Fm"] & F(A_1 \times A_2) \arrow[r, "F\pi_{A_i}"] & F(A_i)
\end{tikzcd}$$

By assumption e is retraction and consequently it has a right inverse μ (i.e., $e \circ \mu = id_R$). Define the structure $\rho : R \longrightarrow F(R)$ as $\rho := Fe \circ \gamma \circ \mu$. We are required to show that $\pi_{A_i} \circ m$ is a homomorphism, for all $i \in \{1, 2\}$.

$$\begin{aligned}
F(\pi_{A_i}) \circ Fm \circ \rho &= F(\pi_{A_i}) \circ Fm \circ Fe \circ \gamma \circ \mu \\
&= F(\pi_{A_i} \circ m \circ e) \circ \gamma \circ \mu \\
&= F(f_i) \circ \gamma \circ \mu \\
&= \alpha_i \circ f_i \circ \mu \\
&= \alpha_i \circ \pi_{A_i} \circ m \circ e \circ \mu \\
&= \alpha_i \circ \pi_{A_i} \circ m \circ id_R \\
&= \alpha_i \circ \pi_{A_i} \circ m
\end{aligned}$$

The other direction of this theorem is already proven in [54]. □

Recall that in the following theorem, $\mathbb{C}$ is an $(\mathcal{E}, mono)$-category

Theorem 7.4.9. *Suppose $\mathbb{C}$ is a category in which $\mathcal{M}$-morphisms are exactly monos. An mono-subobject $m : R \longrightarrow A_1 \times A_2$ is an A-M bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$, if and only if there is a 2-source $\{f_i : P \longrightarrow A_i\}_{i\in\{1,2\}}$ in $\mathbb{C}_F$ such that its underlying 2-source in $\mathbb{C}$ is a mono source and m is its graph.*

Proof. Let $m : R \longrightarrow A_1 \times A_2$ be an $\mathcal{M}$-subobject in $\mathbb{C}$. Suppose $\{f_i : P \longrightarrow A_i\}_{i\in\{1,2\}}$ is a 2-source in $\mathbb{C}_F$ such that its underlying 2-source in $\mathbb{C}$ is a mono-source and m is its graph (i.e., there is an $\mathcal{E}$-morphism e such that $[f_1, f_2] = m \circ e$ is a decomposition of the $\mathbb{C}$-morphism $[f_1,\ f_2]$ in $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathbb{C}$, see definition 7.4.5). Then by diagram 7.4.2 we have $\pi_{A_i} \circ m \circ e = f_i$ for each $i \in \{1, 2\}$. Since $m : R \longrightarrow A_1 \times A_2$ is mono, by part 1 of lemma 2.10.4 $\{\pi_{A_i} \circ m\}_{i\in\{1,2\}}$ is a mono source in $\mathbb{C}$ too. So by part 2 of lemma 2.10.4, e is mono. Hence e is an isomorphism in $\mathbb{C}$. The rest of this proof is the same as what we have done in theorem 7.4.8.

□

7.4.2. Largest A-M bisimulation

Definition 7.4.10. Let $\mathcal{A}_1 = (A_1, \alpha_1)$ and $\mathcal{A}_2 = (A_2, \alpha_2)$ be two arbitrary coalgebras in $\mathbb{C}_F$. Given a family $\{m_i : R_i \longrightarrow A_1 \times A_2\}_{i\in I}$ of A-M bisimulations between $\mathcal{A}_1$ and $\mathcal{A}_2$. We define the **supremum** of the family $\{m_i\}_{i\in I}$ (in symbol: $\bigvee_{i\in I} m_i$) as the $\mathcal{M}$-union of the underlying $\mathcal{M}$-subobjects of $\{m_i\}_{i\in I}$ in $\mathbb{C}$.

Definition 7.4.11. Let $\mathcal{R} := \{m_i : R_i \longrightarrow A_1 \times A_2\}_{i\in I}$ be the family of all A-M bisimulations between coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ and let $\mathcal{R}$ be non-empty. We call $\bigvee_{i\in I} m_i$ the largest A-M bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$ iff $\bigvee_{i\in I} m_i$ is an A-M bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$. The **largest A-M bisimulation** between $\mathcal{A}_1$ and $\mathcal{A}_2$ (if it is exist, i.e. if $\bigvee_{i\in I} m_i$ is an A-M bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$) is denoted by $\sim_{\mathcal{A}_1,\mathcal{A}_2}$. We shall drop the label $\mathcal{A}_1, \mathcal{A}_2$, if it is clear from the context.

It follows from [11] that the supremum of a family $(m_i : R_i \longrightarrow A_1 \times A_2)_{i\in I}$ of A-M bisimulations need not to be an A-M bisimulation. As a consequence the largest A-M bisimulation need not exist. If it exists, it is determined up to isomorphism. In [11], the authors have concentrated on the category of coalgebras for the P-Vietoris functor $\mathbb{V}_P$ (see section 3.2) over the category of Stone spaces. The following example, which is a simplified version of example 4.6 in [11] brings this matter to light. Notice that in this example, we consider the category Top as an $(epi,\ regular\, mono)$-category.

Example 7.4.12. Given discrete spaces $T = \{t_{1i},\ t_{2i} \mid i \in \omega\}$, $U = \{u_{1i},\ u_{2i} \mid i \in \omega\}$ and $V = \{v_{1i},\ v_{2i} \mid i \in \omega\}$. Let $T_\infty := T \cup \{t_\infty\}$, $U_\infty := U \cup \{u_\infty\}$ and $V_\infty := V \cup \{v_\infty\}$ be the Alexandroff compactification of T, U and V, respectively. It means, O is an open subset of T_∞ iff O is an open subset of T or $O = (T - C) \cup \{t_\infty\}$ where C is a finite subset of T (similarly for U_∞ and V_∞). Now consider X as the topological sum $T_\infty + U_\infty + V_\infty$. Define the binary relation $R \subseteq X \times X$ as

$$R := \{(t_{1i}, u_{1i}),\ (t_{1i}, v_{1i}) \mid i \in \omega\} \cup \{(t_{2i}, u_{2i}),\ (t_{2i}, v_{2i}) \mid i \in \omega\} \cup \{(t_\infty, u_\infty),\ (t_\infty, v_\infty)\}$$

Let α be the structure map defined by R, i.e., for each x, $x' \in X$, we have

$$x' \in \alpha(x) \iff (x, x') \in R$$

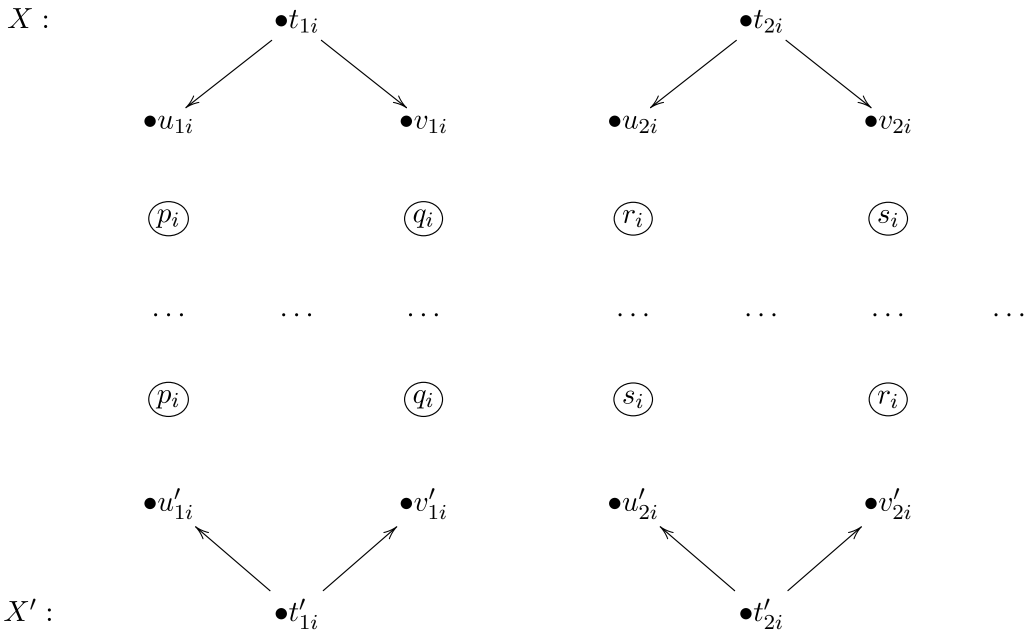
Clearly α is a well-defined map from X to $\mathbb{V}(X)$ (because $\alpha(x)$ is finite for each $x \in X$). The following diagram gives a picture of (X, α).

$$\begin{array}{ccccc} U_\infty & & T_\infty & & V_\infty \\ & & & & \\ \bullet u_{1i} & \longleftarrow & \bullet t_{1i} & \longrightarrow & \bullet v_{1i} \\ & & & & \\ \bullet u_{2i} & \longleftarrow & \bullet t_{2i} & \longrightarrow & \bullet v_{2i} \\ & & & & \\ \bullet u_\infty & \longleftarrow & \bullet t_\infty & \longrightarrow & \bullet v_\infty \end{array} \tag{7.4.3}$$

It is easy to see that α is continuous. Now let (X', α') be an isomorphic copy of (X, α). If $P = \{p_i, q_i, r_i, s_i \mid i \in \omega\}$ is the set of proposition letters, define the valuation maps $\vartheta : X \longrightarrow \mathbb{P}(P)$ and $\vartheta' : X' \longrightarrow \mathbb{P}(P)$ as

- $\vartheta(u_{1i}) = \vartheta'(u'_{1i}) := p_i$ for all $i \in \omega$,
- $\vartheta(v_{1i}) = \vartheta'(v'_{1i}) := q_i$ for all $i \in \omega$,
- $\vartheta(u_{2i}) = \vartheta'(v'_{2i}) := r_i$ for all $i \in \omega$,
- $\vartheta(v_{2i}) = \vartheta'(u'_{2i}) := s_i$ for all $i \in \omega$,
- $\vartheta(t_{1i}) = \vartheta(t_{2i}) = \vartheta'(t'_{1i}) = \vartheta'(t'_{2i}) := \emptyset$ for all $i \in \omega$,
- $\vartheta(t_\infty) = \vartheta(u_\infty) = \vartheta(v_\infty) = \vartheta'(t'_\infty) = \vartheta'(u'_\infty) = \vartheta'(v'_\infty) := \emptyset$.

The following picture shows how to the various proposition letters are satisfied for the elements of X and X'.



The valuation maps $\vartheta : X \longrightarrow \mathbb{P}(P)$ and $\vartheta' : X' \longrightarrow \mathbb{P}(P)$ are continuous maps (consider $\mathbb{P}(P)$ as the topological space mentioned in definition 3.2.1, one can see that for all $i \in \omega$, the subsets $\vartheta^{-1}(\uparrow p_i)$, $\vartheta^{-1}(\uparrow q_i)$, $\vartheta^{-1}(\uparrow r_i)$ and $\vartheta^{-1}(\uparrow s_i)$ are open in X, similarly for ϑ'). Then $\mathcal{X} = (X, \alpha_P)$ and $\mathcal{X}' = (X', \alpha'_P)$ (where $\alpha_P(x) := (\alpha(x), \vartheta(x))$ and $\alpha'_P(x') := (\alpha'(x'), \vartheta(x'))$ for any $x \in X$ and $x' \in X'$) are coalgebras for P-Vietoris functor $\mathbb{V}_P$ (because α_P is continuous iff α and ϑ are continuous, similarly for α'_P).

Claim. The supremum of a family of A-M bisimulations between $\mathcal{X}$ and $\mathcal{X}'$ does not need to be an A-M bisimulation.

Proof. Let

$$\begin{aligned} B_1 &:= \{(t_\infty, t'_\infty), (u_\infty, u'_\infty), (v_\infty, v'_\infty)\} \\ B_2 &:= \{(t_{1i}, t'_{1i}), (u_{1i}, u'_{1i}), (v_{1i}, v'_{1i}) \mid i \in \omega\} \\ B_3 &:= \{(t_{2i}, t'_{2i}), (u_{2i}, v'_{2i}), (v_{2i}, u'_{2i}) \mid i \in \omega\} \end{aligned}$$

Step 1: B_1, B_2 and B_3 are discrete spaces with respect to the subspace topology generated by the product topology on $X \times X'$. Now, we want to show that the topological embeddings $\iota_1 : B_1 \longrightarrow X \times X'$, $\iota_2 : B_2 \longrightarrow X \times X'$ and $\iota_3 : B_3 \longrightarrow X \times X'$ are A-M bisimulations. For each $j \in \{1, 2, 3\}$, define a map $\beta_j : B_j \longrightarrow \mathbb{V}(B_j)$ by

$$\beta_j(x, x') \quad := \quad (\alpha(x) \times \alpha'(x')) \cap B_j.$$

Since for every $x \in X$ and $x' \in X'$, the subsets $\alpha(x)$ and $\alpha'(x')$ are finite, β_j ($j \in \{1, 2, 3\}$) is well-defined. Regarding the continuity of β_j ($j \in \{1, 2, 3\}$), notice that β_1, β_2 and β_3

are maps from discrete spaces (we should recall that every map from a discrete space is continuous). Now, consider function $\beta^j_P : B_j \longrightarrow \mathbb{V}_P(B_j)$ as

$$\beta^j_P(x, x') \;\; := \;\; (\beta_j(x, x'), \vartheta(x))$$

where $j \in \{1, 2, 3\}$. Obviously, β^j_P $(j \in \{1, 2, 3\})$ is continuous and makes the following diagram commute,

$$\begin{tikzcd}
X \arrow[d, "\alpha"] & B_j \arrow[l, "\pi_X \circ \iota_j"'] \arrow[r, "\pi_{X'} \circ \iota_j"] \arrow[d, "\beta^i_P"] & X' \arrow[d, "\beta"] \\
\mathbb{V}_P(X) & \mathbb{V}_P(B_j) \arrow[l, "\mathbb{V}_P(\pi_X \circ \iota_j)"] \arrow[r, "\mathbb{V}_P(\pi_{X'} \circ \iota_j)"'] & \mathbb{V}_P(X')
\end{tikzcd}$$

where π_X and $\pi_{X'}$ are projections maps from $X \times X'$ to X and X', respectively). Thus for each $j \in \{1, 2, 3\}$, the map ι_j is an A-M bisimulation between $\mathcal{X}$ and $\mathcal{X}'$.
Step 2: Now consider the set $B = B_1 \cup B_2 \cup B_3$. Provide B with the subspace topology generated by the inclusion map $\iota : B \longrightarrow X \times X'$. It is easy to see that ι is the supremum of the family $\{\iota_j\}_{j \in \{1,2,3\}}$. We claim that ι is not an A-M bisimulation between $\mathcal{X}$ and $\mathcal{X}'$. Suppose for contradiction that ι is an A-M bisimulation, i.e., there exists a continuous map $\beta_P : B \longrightarrow \mathbb{V}_P(B)$ such that $\pi_X \circ \iota$ and $\pi_{X'} \circ \iota$ are homomorphisms in $Top_{\mathbb{V}_P}$. In this case β_P is unique (because the projection maps $\pi_X \circ \iota : B \longrightarrow X$ and $\pi_{X'} \circ \iota : B \longrightarrow X'$ are mono in Top and the functor $\mathbb{V}_P$ preserves monos) and if $\beta : B \longrightarrow \mathbb{V}(B)$ is the composition of $\beta_P : B \longrightarrow \mathbb{V}(B) \times \mathbb{P}(P)$ with the first projection $\pi_1 : \mathbb{V}(B) \times \mathbb{P}(P) \longrightarrow \mathbb{V}(B)$ (i.e., $\beta = \pi_1 \circ \beta_P$), then it is not hard to see that for each $(x, x') \in B$,

$$\beta(x, x') \;\; := \;\; (\alpha(x) \times \alpha'(x')) \cap B. \tag{7.4.4}$$

In order to check the continuity of β_P, it suffices to check the continuity of β. Therefore, we need to show that β is not continuous. We know that the set $U_\infty \times U'_\infty$ is an open subset of $X \times X'$, then $C := (U_\infty \times U'_\infty) \cap B$ is open in B. Obviously, $C = \{(u_{1i}, u'_{1i}) \mid i \in \omega\} \cup \{(u_\infty, u'_\infty)\}$. Now, by assumed continuity of β, the set $\beta^{-1}(\langle C \rangle)$ must be an open subset of B. However, according to equation 7.4.4 we can see $\beta^{-1}(\langle C \rangle) = \{(t_{1i}, t'_{1i}) \mid i \in \omega\} \cup \{(t_\infty, t'_\infty)\}$ which is not an open subset of B. The reason is that for every open neighborhood O of the pair (t_∞, t'_∞) we have $O \cap (B - \beta^{-1}(\langle C \rangle)) \neq \emptyset$. This gives us the desired contradiction and proves the claim.

□

One may have this question that: when does the largest A-M bisimulation between two coalgebras exist and how can we find it. As it is mentioned by Kurz in [45], there are two ways to obtain the largest A-M bisimulation between F-coalgebras in $\mathbb{C}_F$. In fact the both strategies help us to find a coalgebra structure on the supremum of the family of all A-M bisimulations between arbitrary coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$.
In the rest of this part, we try to list these strategies as some theorems and corollaries.

To convenient, in the sequel of this part we assume that $\mathcal{A}_1 = (A_1, \alpha_1)$ and $\mathcal{A}_2 = (A_2, \alpha_2)$ are the fixed coalgebras in $\mathbb{C}_F$.

First strategy: Largest bisimulation via right invertible morphisms

This strategy can be used in the case that in the base category $\mathbb{C}$ of our coalgebras the $\mathcal{E}$-morphisms are right invertible. It means if the category $\mathbb{C}$ satisfies the conditions **A2** and **A3** as well as **A1** and **A4** (see remark 7.1.10), then:

Theorem 7.4.13. *Suppose $\mathbb{C}$ is a category in which each $\mathcal{E}$-morphism is a retraction, then the supremum of each collection of A-M bisimulations between $\mathcal{A}_1$ and $\mathcal{A}_2$ is an A-M bisimulation.*

Proof. Let $\mathcal{R} = \{m_j : R_j \longrightarrow A_1 \times A_2\}_{j \in J}$ be a collection of A-M bisimulations between $\mathcal{A}_1$ and $\mathcal{A}_2$. Consider the following diagram (taken from Kurz [45], section 1.1):

$$\begin{tikzcd} S \arrow[r, "e"] \arrow[dr, "q"] & E \arrow[d, "m"] & \\ R_j \arrow[u, "e_j"] \arrow[r, "m_j"'] & A_1 \times A_2 \arrow[r, "\pi_i"'] & A_i \end{tikzcd} \tag{7.4.5}$$

where $S := \sum_{j \in J} R_j$ with canonical injections $\{e_j : R_j \longrightarrow S\}_{j \in J}$ is a sum of the family $\{R_j\}_{j \in J}$ in $\mathbb{C}$; $A_1 \times A_2$ with canonical projections $\{\pi_i : A_1 \times A_2 \longrightarrow A_i\}_{i \in \{1,2\}}$ is a product of the family $\{A_i\}_{i \in \{1,2\}}$ in $\mathbb{C}$; the morphism $q : S \longrightarrow A_1 \times A_2$ is the unique morphism such that $m_j = q \circ e_j$, and $m \circ e$ is a decomposition of q in the $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathbb{C}$. Notice that $\bigvee_{j \in J} m_j$ (i.e., supremum of $\mathcal{R}$) is the $\mathcal{M}$-morphism $m : E \longrightarrow A_1 \times A_2$ in diagram 7.4.5 (by theorem 2.12.3). We need to show for each $i \in \{1, 2\}$ the morphism $\pi_i \circ m : E \longrightarrow A_i$ is a homomorphism in $\mathbb{C}_F$. Since $\mathcal{R}$ is a collection of A-M bisimulations, the morphisms $\{\pi_i \circ m_j : R_j \longrightarrow A_i\}_{i \in \{1,2\}, j \in J}$ are homomorphisms in $\mathbb{C}_F$. Then by diagram 7.4.5, the morphisms$\{\pi_i \circ q \circ e_j\}_{i \in \{1,2\}, j \in J}$ are homomorphisms in $\mathbb{C}_F$. It means in the following digram the outer rectangle commutes.

$$\begin{tikzcd} R_j \arrow[r, "e_j"] \arrow[d, "\beta_j"'] & S \arrow[r, "e"] \arrow[d, "\gamma"'] \arrow[rr, bend left, "q"] & E \arrow[r, "m"] \arrow[d, dashed, "\rho"] & A_1 \times A_2 \arrow[r, "\pi_i"] & A_i \arrow[d, "\alpha_i"] \\ F(R_j) \arrow[r, "Fe_j"] & F(S) \arrow[r, "Fe"] \arrow[rr, bend right, "Fq"'] & F(E) \arrow[r, "Fm"] & F(A_1 \times A_2) \arrow[r, "F\pi_i"] & F(A_i) \end{tikzcd}$$

Then for each $i \in \{1,2\}$ and for each $j \in J$ we have

$$\begin{aligned} \alpha_i \circ \pi_i \circ q \circ e_j &= F(\pi_i) \circ Fm \circ Fe \circ Fe_j \circ \beta_j \\ &= F(\pi_i) \circ F(m \circ e) \circ F(e_j) \circ \beta_j \\ &= F(\pi_i) \circ F(q) \circ F(e_j) \circ \beta_j. \end{aligned}$$

Notice that there is a unique structure $\gamma : S \longrightarrow F(S)$ that makes the canonical injections $\{e_j\}_{j \in J}$ into $\mathbb{C}_F$-homomorphisms (see lemma 7.3.1). Then

$$\begin{aligned} \alpha_i \circ \pi_i \circ q \circ e_j &= F(\pi_i) \circ F(q) \circ F(e_j) \circ \beta_j \\ &= F(\pi_i) \circ F(q) \circ \gamma \circ e_j. \end{aligned}$$

Since $\{e_j\}_{j \in J}$ is an epi sink, we have $\alpha_i \circ \pi_i \circ q = F(\pi_i) \circ F(q) \circ \gamma$. So, for each $i \in \{1,2\}$ the morphisms$\{\pi_i \circ q\}_{j \in J}$ are homomorphisms in $\mathbb{C}_F$. Now we need to find a structure $\rho : E \longrightarrow F(E)$ that makes the morphisms $\{\pi_i \circ m : E \longrightarrow A_i\}_{i \in \{1,2\}}$ into the homomorphisms in $\mathbb{C}_F$. By assumption e is a retraction in $\mathbb{C}$ and consequently it has a right inverse μ (i.e., $e \circ \mu = id_E$). Define the structure ρ as $\rho := Fe \circ \gamma \circ \mu$. We are required to show that $\pi_i \circ m$ is a homomorphism, for all $i \in \{1,2\}$.

$$\begin{aligned} F(\pi_i) \circ Fm \circ \rho &= F(\pi_i) \circ Fm \circ Fe \circ \gamma \circ \mu \\ &= F(\pi_i) \circ F(m \circ e) \circ \gamma \circ \mu \\ &= F(\pi_i) \circ F(q) \circ \gamma \circ \mu \\ &= \alpha_i \circ \pi_i \circ q \circ \mu \\ &= \alpha_i \circ \pi_i \circ m \circ e \circ \mu \\ &= \alpha_i \circ \pi_i \circ m \circ id_E \\ &= \alpha_i \circ \pi_i \circ m. \end{aligned}$$

□

Corollary 7.4.14. *If in $\mathbb{C}$, $\mathcal{E}$-morphisms are retractions, then the largest A-M bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$ exists.*

To see an application of the first strategy, one should study Gumm [31]. He has used this strategy to show that in the categories of coalgebras over Set, the union of a family of A-M bisimulations (union of underlying sets) is always an A-M bisimulation, in particular, the largest A-M bisimulation between two coalgebras always exists.

Second strategy: Largest bisimuation via functors weakly preserve pullbacks

The second way can be efficient whenever the base category $\mathbb{C}$ has pullbacks and the $\mathbb{C}$-endofunctor F weakly preserves them. It means if the category $\mathbb{C}$ satisfies the conditions **A1**, **A2**, **A3** and **A4**, then:

Theorem 7.4.15. *Suppose $\mathbb{C}$ has pullbacks and the $\mathbb{C}$-endofunctor F weakly preserves them. Then the supremum of the collection of all A-M bisimulations between coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is an A-M bisimulation.*

Proof. Let $\mathcal{R} = \{m_j : R_j \longrightarrow A_1 \times A_2\}_{j \in J}$ be the collection of all A-M bisimulations between $\mathcal{A}_1$ and $\mathcal{A}_2$. Consider the following diagram

$$\begin{tikzcd}
 & & A_1 & \\
R_j \arrow[r, "e_j"'] \arrow[rrr, bend left=70, "m_j"] & S \arrow[ur, "f_1"] \arrow[r, "e"'] \arrow[dr, "f_2"'] & G[f_1, f_2] \arrow[r, "m"'] & A_1 \times A_2 \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] \\
 & & A_2 &
\end{tikzcd} \tag{7.4.6}$$

where $S := \sum_{j \in J} R_j$ with canonical injections $\{e_j : R_j \longrightarrow S\}_{j \in J}$ is a sum of the family $\{R_j\}_{j \in J}$ in $\mathbb{C}$; $A_1 \times A_2$ with canonical projections $\{\pi_i : A_1 \times A_2 \longrightarrow A_i\}_{i \in \{1,2\}}$ is a product of the family $\{A_i\}_{i \in \{1,2\}}$ in $\mathbb{C}$; for each $i \in \{1, 2\}$ the morphism $f_i : S \longrightarrow A_i$ is the unique morphism such that $f_i \circ e_j = \pi_i \circ m_j$ for each $j \in J$, and $m \circ e$ is a decomposition of the unique morphism $[f_1, f_2] : S \longrightarrow A_1 \times A_2$ in the $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathbb{C}$ (see definition 7.4.5). Notice that $\bigvee_{j \in J} m_j$ (i.e., supremum of $\mathcal{R}$) is the $\mathcal{M}$-morphism $m : G[f_1, f_2] \longrightarrow A_1 \times A_2$ in diagram 7.4.6. We need to show for each $i \in \{1, 2\}$ the morphisms $\pi_i \circ m : G[f_1, f_2] \longrightarrow A_i$ is a homomorphism in $\mathbb{C}_F$. Notice that the morphisms $\{\pi_i \circ m_j\}_{i \in \{1,2\}, j \in J}$ are homomorphisms in $\mathbb{C}_F$ (because $\{m_j : R_j \longrightarrow A_1 \times A_2\}_{j \in J}$ is a family of A-M bisimulations between $\mathcal{A}_1$ and $\mathcal{A}_2$). Also, the morphisms $\{e_j\}_{j \in J}$ are homomorphisms in $\mathbb{C}_F$ (see remark 7.3.3, part (1)). Then according to equation $f_i \circ e_j = \pi_i \circ m_j$ (where $i \in \{1, 2\}$ and $j \in J$) and part (1) of lemma 7.1.9, we conclude that $\{f_i : S \longrightarrow A_i\}_{i \in \{1,2\}}$ are homomorphisms in $\mathbb{C}_F$. Now, suppose C with morphisms $g_1 : A_1 \longrightarrow C$ and $g_2 : A_2 \longrightarrow C$ is a pushout of the 2-source $\{f_i : S \longrightarrow A_i\}_{i \in \{1,2\}}$ in $\mathbb{C}_F$ (by theorem 7.3.1, it exists). So

$$\begin{aligned}
g_1 \circ \pi_1 \circ m \circ e &= g_1 \circ f_1 \\
&= g_2 \circ f_2 \\
&= g_2 \circ \pi_2 \circ m \circ e.
\end{aligned}$$

Since e is epi, we have $g_1 \circ \pi_1 \circ m = g_2 \circ \pi_2 \circ m$. Now consider (B, p_1, p_2) as a pullback of g_1 and g_2 in $\mathbb{C}$ (by assumption, it exists). One can see that $(G[f_1, f_2], \pi_1 \circ m, \pi_2 \circ m)$ is a competitor for (B, p_1, p_2) in $\mathbb{C}$. As a consequence, there is an unigue morphism $q : G[f_1, f_2] \longrightarrow B$ such that $p_i \circ q = \pi_i \circ m$ for all $i \in \{1, 2\}$. Due to part (2) of lemma 2.10.4, q is a monomorphism in $\mathbb{C}$ (notice that by remark 2.18.3 and lemma

2.10.4, $\{p_i\}_{i\in\{1,2\}}$ and $\{\pi_i \circ m\}_{i\in\{1,2\}}$ are mono-sources). On the other hand (B, p_1, p_2) is a competitor for the product $(A_1 \times A_2, \pi_1, \pi_2)$ in $\mathbb{C}$. So, there is a unique morphism $k : B \longrightarrow A_1 \times A_2$ with $\pi_i \circ k = p_i$. By lemma 2.10.4, k is a monomorphism in $\mathbb{C}$. We claim that k is an $\mathcal{M}$-morphism in $\mathbb{C}$. To check this claim, let $m' \circ e'$ (where $e' : B \longrightarrow E$ and $m' : E \longrightarrow A_1 \times A_2$) be a decomposition of k in $(\mathcal{E}, \mathcal{M})$-factorization system of $\mathbb{C}$. It suffices to show that e' is an isomorphism in $\mathbb{C}$. Notice that $\pi_i \circ m' \circ e' = p_i$ (because $\pi_i \circ k = p_i$). Consequently, $g_1 \circ \pi_1 \circ m' \circ e' = g_2 \circ \pi_2 \circ m' \circ e'$ (because $g_1 \circ p_1 = g_2 \circ p_2$). Since e' is epi, we have $g_1 \circ \pi_1 \circ m' = g_2 \circ \pi_2 \circ m'$. Therefore, $(E, \{\pi_i \circ m'\}_{i\in\{1,2\}})$ is a competitor for (B, p_1, p_2). Hence, there is a unique map $h : E \longrightarrow B$ such that $p_i \circ h = \pi_i \circ m'$. We prove that h is a left inverse of e'. For each $i \in \{1, 2\}$, we have

$$\begin{aligned} \pi_i \circ m' \circ e' \circ h \circ e' &= p_i \circ h \circ e' \\ &= \pi_i \circ m' \circ e'. \end{aligned}$$

Since $\{\pi_i\}_{i\in\{1,2\}}$ is a monosource, we have $m' \circ e' \circ h \circ e' = m' \circ e'$. Also, since m' is mono, we conclude that $e' \circ h \circ e' = e'$. From $m' \circ e' = k$ we obtain that e' is mono (because k is mono). Then $h \circ e' = id_B$. Now, since e' is epi and a section, it is an isomorphism.
Now since F weakly preserves pullbacks, there exists a structure $\beta : B \longrightarrow FB$ that makes p_1 and p_2 into a homomorphism in $\mathbb{C}_F$. Then $k : B \longrightarrow A_1 \times A_2$ is an A-M bisimulation between F-coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathbb{C}_F$. To complete this proof, it suffices to show that the unique morphism $q : G[f_1, f_2] \longrightarrow B$ is an isomorphism in $\mathbb{C}$.
Since $k : B \longrightarrow A_1 \times A_2$ is an A-M bisimulation between F-coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$, there is a morphism $\iota : B \longrightarrow G[f_1, f_2]$ such that $m \circ \iota = k$ (because m is a supremum of all A-M bisimulations between $\mathcal{A}_1$ and $\mathcal{A}_2$). Since m is mono, ι is unique. We claim that ι is a right inverse for q. To show this claim, notice that

$$\begin{aligned} m \circ \iota \circ q \circ \iota &= k \circ q \circ \iota \\ &= m \circ \iota \\ &= k. \end{aligned}$$

So by uniqueness of ι, we have $\iota \circ q \circ \iota = \iota$. Since ι is mono (lemma 2.2.13), it is concluded that $q \circ \iota = id_B$. Now, since every right invertable monomorphisms is an isomorphism, it is concluded that q is an isomorphism in $\mathbb{C}$. We need to find an F-coalgebra structure $\rho : G[f_1, f_2] \longrightarrow F(G[f_1, f_2])$ which makes the morphisms $\{\pi_i \circ m : G[f_1, f_2] \longrightarrow A_i\}_{i\in\{1,2\}}$ into the homomorphisms in $\mathbb{C}_F$. By assumption q is an isomorphism in $\mathbb{C}$ and consequently Fq is an isomorphism in $\mathbb{C}$ too (functors preserve isomorphisms). Define the structure ρ as $\rho := (Fq)^{-1} \circ \beta \circ q$.

□

Corollary 7.4.16. *If F weakly preserves pullback, then the largest A-M bisimulation between $\mathcal{A}_1$ and $\mathcal{A}_2$ exists.*

Remark 7.4.17. Suppose $\mathbb{C}$ is a category with pullbacks. Theorem 7.4.15 and its corollary provides us with a way to check whether a $\mathbb{C}$-endofunctor F weakly preserves pullbacks or not. According to this theorem, a $\mathbb{C}$-endofunctor F does not weakly preserve pullbacks, if we can find two coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathbb{C}_F$ and a family $\{m_i\}_{i\in I}$ of A-M bisimulations between them such that its supremum is not an A-M bisimulation. Consequently, we can say that a $\mathbb{C}$-endofunctor F does not weakly preserve pullbacks, if there are two coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathbb{C}_F$ such that the largest A-M bisimulation between them does not exist.

Remark 7.4.18. Due to example 7.4.12, there are two coalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in $Top_{\mathbb{V}_P}$ and a family $\{m_i\}_{i\in I}$ of A-M bisimulations between them such that its supremum is not an A-M bisimulation. So by the previous remark, we can say that the P-Vietoris functor $\mathbb{V}_P$ over the category Top does not weakly preserve pullbacks.

7.4.3. Transforming A-M bisimulations between categories of coalgebras

Let $\mathbb{C}$ be an $(\mathcal{E},\mathcal{M})$-category and $\mathbb{C}'$ an $(\mathcal{E}',\mathcal{M}')$-category. Suppose U is a functor from $\mathbb{C}$ to $\mathbb{C}'$ and F and G are endofunctors over $\mathbb{C}$ and $\mathbb{C}'$, respectively. Assume there is a natural transformation $\mu : UF \to GU$. Now under this hypothesis, we discuss the connection between A-M bisimulations in $\mathbb{C}_F$ with these structures in $\mathbb{C}'_G$.
We have the following lemma and theorem:

Lemma 7.4.19. *Suppose $\mathcal{L} = (L,\rho)$ and $\mathcal{A} = (A,\alpha)$ are two coalgebras in $\mathbb{C}_F$. If the morphism $f : (L,\rho) \longrightarrow (A,\alpha)$ is a homomorphism in $\mathbb{C}_F$ then $Uf : (UL, \mu_L \circ U\rho) \longrightarrow (UA, \mu_A \circ U\alpha)$ is a homomorphism between coalgebras $U\mathcal{L} = (UL, \mu_L \circ U\rho)$ and $U\mathcal{A} = (UA, \mu_A \circ U\alpha)$ in $\mathbb{C}'_G$.*

Proof. Consider the diagram below:

$$\begin{tikzcd}
UL \arrow[r, "Uf"] \arrow[d, "U\rho"'] & UA \arrow[d, "U\alpha"] \\
UFL \arrow[r, "UFf"] \arrow[d, "\mu_L"'] & UFA \arrow[d, "\mu_A"] \\
GUL \arrow[r, "GUf"'] & GUA
\end{tikzcd}$$

The top part of this rectangle is commutative because f is a homomorphism and every functor preserves commutative diagrams. The bottom of this diagram is also commutative because μ is a natural transformation. □

As a consequence, we obtain that the structure $\widehat{U} : \mathbb{C}_F \longrightarrow \mathbb{C}'_G$ which associates each coalgebra $\mathcal{A} = (A, \alpha)$ in $\mathbb{C}_F$ to the coalgebra $U\mathcal{A} = (UA, \mu_A \circ U\alpha)$ in $\mathbb{C}'_G$ and each homomorphism f in $\mathbb{C}_F$ to the homomorphism Uf in $\mathbb{C}'_G$, is a functor.

Theorem 7.4.20. *Suppose $U(f) \in \mathcal{M}'$ for each $\mathcal{M}$-morphism $f \in \mathcal{M}$. Then, if the $\mathcal{M}$-subobject $j : R \longrightarrow A_1 \times A_2$ is an A-M bisimulation between coalgebras $\mathcal{A}_1 = (A_1, \alpha_1)$ and $\mathcal{A}_2 = (A_2, \alpha_2)$ in $\mathbb{C}_F$, then $Uj : UR \longrightarrow U(A_1 \times A_2)$ is an A-M bisimulation between coalgebras $(UA_1, \mu_{A_1} \circ U\alpha_1)$ and $(UA_2, \mu_{A_2} \circ U\alpha_2)$ in $\mathbb{C}'_G$.*

Proof. According to definition 7.4.1 and lemma 7.4.19, it is clear. □

7.5. Coalgebraic modal logic

We now intend to develop a modal logic for arbitrary coalgebras over Top. Our goals will be to design a language (i.e., a set of formulas) for an arbitrary Top-endofunctor F and a semantics providing a meaning for each formula with respect to the F-coalgebras over Top. The idea of this chapter comes from Pattinson [57] where the author has developed a modal logic for coalgebras constructed by a Set-endofunctor T, via a set of predicate liftings (i.e natural transformations $\lambda : 2^{(-)} \longrightarrow 2^{(-)} \circ T$ where $2^{(-)}$ is the contravariant powerset functor over the category Set). However, our presentation here is based on Gumm [37], where instead of working with predicate liftings the author has employed the Yoneda lemma[4] to look at subsets of $T(2)$, instead. We try to define a language for a Top-endofunctor F via a modal similarity type Λ for F (a set of clopen subsets of $F(2)$ where $2 := \{0, 1\}$ is a discrete space). The works of Cirstea et. al [20], Kupke and Pattinson [48], Schröder [64], Schröder and Mossakowski [65] and Schröder and Pattinson [66] are the other references for the notions discussed in this chapter.

Definition 7.5.1. Given a Top-endofunctor F. A **modal similarity type** Λ for F is a set of clopen subsets of $F(2)$ where $2 := \{0, 1\}$ is a discrete space (note that Λ can be any subset of the set $\{C \subseteq F(2) \mid C$ is a clopen subset of $F(2)\}$).

Throughout this section we assume that $F : Top \longrightarrow Top$ is a fixed endofunctor on Top and Λ is a fixed modal similarity type for F.

[4] Yoneda lemma says that for any Set-endofunctor T, there is a bijective function between the set $Nat\,(2^{(-)}, 2^{(-)} \circ T)$ (i.e, the set of all natural transformations $\lambda : 2^{(-)} \longrightarrow 2^{(-)} \circ T$, where $2^{(-)}$ is the contravariant powerset functor) and the set $(2^{(-)} \circ T)(2)$. This is denoted by

$$Nat\,(2^{(-)}, 2^{(-)} \circ T) \cong (2^{(-)} \circ T)(2).$$

7.5.1. Syntax and semantics

Definition 7.5.2. A logic for F-coalgebras with respect to Λ consists of two parts language and semantics determined as follows:

1. **(Language)** A set of Λ-formulas defined by the following grammer:

$$\begin{array}{rcl} \varphi & ::= & \top \\ & | & \neg\varphi \\ & | & \varphi_1 \wedge \varphi_2 \\ & | & [C]\varphi \text{ for } C \in \Lambda \end{array}$$

   The truth functional connectives $\vee$ ("or") and implications $\varphi \longrightarrow \psi$, and equivalences $\varphi \longleftrightarrow \psi$ are defined as usual. We denote the set of all Λ-formulas by $L(\Lambda)$.

2. **(Semantics)** For each F-coalgebra $\mathcal{X} = (X, \alpha)$ and $x \in X$, the binary relation $\models_{\mathcal{X}} \subseteq X \times L(\Lambda)$ is defined inductively as

$$\begin{array}{rcl} x \models_{\mathcal{X}} \top & :\Longleftrightarrow & \mathit{true} \\ x \models_{\mathcal{X}} \neg\varphi & :\Longleftrightarrow & x \not\models_{\mathcal{X}} \varphi \\ x \models_{\mathcal{X}} \varphi_1 \wedge \varphi_2 & :\Longleftrightarrow & x \models_{\mathcal{X}} \varphi_1 \, \mathit{and} \, x \models_{\mathcal{X}} \varphi_2 \end{array}$$

which gives the standard interpretation of the Λ-formulas obtained by the boolean conectives, and for the Λ-formula $[C]\varphi$ we put

$$x \models_{\mathcal{X}} [C]\varphi \quad :\Longleftrightarrow \quad (F\varphi^{\mathcal{X}} \circ \alpha)(x) \in C. \tag{7.5.1}$$

In order for $F\varphi^{\mathcal{X}}$ and hence the semantics of $[C]\varphi$ in equation 7.5.1 to be defined, we must verify by induction that for each Λ-formula φ the characteristic function $\varphi^{\mathcal{X}} : X \longrightarrow 2$ (defined by $\varphi^{\mathcal{X}}(x) := \mathit{if}\,(x \models_{\mathcal{X}} \varphi)\,1\,\mathit{else}\,0)$ is a continuous map, i.e. the set

$$\|\varphi\|^{\mathcal{X}} := \{x \in X \mid x \models_{\mathcal{X}} \varphi\} \tag{7.5.2}$$

is a clopen subset of X. For the base case $\varphi = \top$, and for the Boolean connectives $\neg\varphi$ and $\varphi_1 \wedge \varphi_2$, this is obvious. Suppose φ is an arbitrary element of $L(\Lambda)$ such that $\varphi^{\mathcal{X}}$ is continuous, then $([C]\varphi)^{\mathcal{X}} = F\varphi^{\mathcal{X}} \circ \alpha$ is continuous as well. Now, we use this to give a meaning to the Λ-formula $[\lambda]\varphi$:

Then for each Λ-formula φ, the set

$$\|[C]\varphi\|^{\mathcal{X}} = \{x \in X \mid x \models_{\mathcal{X}} [C]\varphi\}$$

is clopen (because $x \models_{\mathcal{X}} [C]\varphi$ iff $x \in (F\varphi^{\mathcal{X}} \circ \alpha)^{-1}(C)$ and we know that $F\varphi^{\mathcal{X}} \circ \alpha$ is continuous). Notice that we can replace $\|\varphi\|^{\mathcal{X}}$ by $\|\varphi\|$, if it is clear from the context.

Definition 7.5.3. We say that a Λ-formula φ is valid in an F-coalgebra $\mathcal{X} = (X, \alpha)$ (in symbols: $\models_{\mathcal{X}} \varphi$) iff $x \models_{\mathcal{X}} \varphi$, for each $x \in X$, i.e. if $\|\varphi\| = X$. As an example $\models_{\mathcal{X}} \top$, for every coalgebra $\mathcal{X} = (X, \alpha)$. If Σ be a subset of $L(\Lambda)$, we say that Σ is valid in a coalgebra $\mathcal{X}$ (in symbols: $\models_{\mathcal{X}} \Sigma$) iff $\models_{\mathcal{X}} \varphi$, for each $\varphi \in \Sigma$.
If $\mathcal{X} = (X, \alpha)$ is an F-coalgebra, then for each $x \in X$, we define

$$\| \, x \, \| \quad := \quad \{\varphi \in L_P \mid x \models_{\mathcal{X}} \varphi\}. \tag{7.5.3}$$

Example 7.5.4. Consider F as the Vietoris functor $\mathbb{V}$ which associates to each topological space X the set of all compact subsets $K \subseteq X$ (see definition 3.2.1). Then $F(2) = \mathbb{V}(2) = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. Notice that by definition of the Vietoris topology (see definition 3.2.1) the sets

$$C_{\Box} := [\{1\}] = \{\emptyset, \{1\}\}$$

and

$$C_{\Diamond} := \langle\{1\}\rangle = \{\{1\}, \{0,1\}\}$$

are clopen subsets of $F(2)$. Take $\Lambda := \{C_{\Box}, C_{\Diamond}\}$. According to equation 7.5.1, for each $\varphi \in L(\Lambda)$ and every F-coalgebra $\mathcal{X} = (X, \alpha)$ we have

$$x \models_{\mathcal{X}} [C_{\Box}]\varphi \iff \forall t \in \alpha(x).\, t \models_{\mathcal{X}} \varphi$$

and

$$x \models_{\mathcal{X}} [C_{\Diamond}]\varphi \iff \exists t \in \alpha(x).\, t \models_{\mathcal{X}} \varphi.$$

Remark 7.5.5. The reader will have noticed that we do not include propositional letters in the language of $L(\Lambda)$. In order to extend a given coalgebraic modal logic for F with a set P of propositional letters, consider the Top-endofunctor $F'(-) := F(-) \times \mathbb{P}(P)$ where $\mathbb{P}(P)$ is the set of all subsets of P equipped with the topology generated by a subbase containing all clopens of the form $\uparrow p = \{u \subseteq P \mid p \in u\}$ (note that F' associates each topological space X to the product space $F(X) \times \mathbb{P}(P)$ and sends every continuous function $f : X \longrightarrow Y$ to the continuous map $Ff \times id_{\mathbb{P}(P)} : F(X) \times \mathbb{P}(P) \longrightarrow F(Y) \times \mathbb{P}(P)$ defined by $(Ff \times id_{\mathbb{P}(P)})(K, M) = ((Ff)(K), M)$, for all $K \in F(X)$ and all $M \subseteq P$). Extend the modal similarity type Λ by adding the clopen subsets $\{C_p := F(2) \times \uparrow p\}_{p \in P}$. Now, if $p \in P$ is a fixed element, then for every $\varphi \in L(\Lambda)$ and each F'-coalgebra $\mathcal{X} = (X, \alpha)$ we have

$$\begin{array}{lcl}
x \models_{\mathcal{X}} [C_p]\varphi & \overset{\text{equation } 7.5.1}{\iff} & (F'\varphi^{\mathcal{X}} \circ \alpha)(x) \in C_p \\
& \iff & F'\varphi^{\mathcal{X}}(\alpha(x)) \in C_p \\
& \iff & (F\varphi^{\mathcal{X}} \times id_{\mathbb{P}(P)})(\alpha(x)) \in C_p \\
& \iff & (F\varphi^{\mathcal{X}} \times id_{\mathbb{P}(P)})(\alpha(x)) \in F(2) \times \uparrow p \\
& \overset{\text{by the definition of } F' \text{ on morphisms}}{\iff} & \pi_{\mathbb{P}(P)}(\alpha(x)) \in \uparrow p \\
& \iff & (\pi_{\mathbb{P}(P)} \circ \alpha)(x) \in \uparrow p
\end{array}$$

(here $\pi_{\mathbb{P}(P)}$ denotes the projection $F(X) \times \mathbb{P}(P) \longrightarrow \mathbb{P}(P)$). Hence we conclude that

$$x \models_{\mathcal{X}} [C_p]\varphi \iff p \in (\pi_{\mathbb{P}(P)} \circ \alpha)(x)$$

Therefore, for every φ, $\phi \in L(\Lambda)$ we have

$$\|[C_p]\varphi\|^{\mathcal{X}} = \|[C_p]\phi\|^{\mathcal{X}}$$

Hence we can write the propositional letter $p \in P$ in place of $[C_p]\varphi$, with the expected meaning (see also [64] and [65]).

Definition 7.5.6. Let $\mathcal{X} = (X, \alpha)$ and $\mathcal{Y} = (Y, \beta)$ be F-coalgebras. Elements $x \in X$ and $y \in Y$ are called **modally equivalent**, and we write $x \approx_{\mathcal{X},\mathcal{Y}} y$, if they satisfy the same Λ-formulas, i.e. if for all $\varphi \in L(\Lambda)$ we have

$$x \models_{\mathcal{X}} \varphi \quad \iff \quad y \models_{\mathcal{Y}} \varphi.$$

We drop the index and write $x \approx y$, if it is clear from the context.

Lemma 7.5.7. *Let $\mathcal{X} = (X, \alpha)$ and $\mathcal{Y} = (Y, \beta)$ be F-coalgebras. Then the relation $\approx \subseteq X \times Y$ is closed in $X \times Y$ (the product of X and Y in Top).*

Proof. It is suffices to show that the complement of $\approx$ is an open subset of $X \times Y$. Suppose $(x, y) \notin \approx$. Then there exists a Λ-formula φ such that $x \models_{\mathcal{X}} \varphi$ and $y \not\models_{\mathcal{Y}} \varphi$. Let U be the set $\| \varphi \|^{\mathcal{X}} \times \| \neg\varphi \|^{\mathcal{Y}}$. Clearly U is an open subset of $X \times Y$ such that $(x, y) \in U$. Also it is obvious that $U \cap \approx = \emptyset$. Thus we found an open neighborhood of (x, y) contained in the complement of $\approx$. This means that $\approx$ is closed. $\square$

The next lemma comes from [39]:

Lemma 7.5.8. *Let $\mathcal{X} = (X, \alpha)$ and $\mathcal{Y} = (Y, \beta)$ be F-coalgebras. If elements $x \in X$ and $y \in Y$ are A-M bisimilar then they are modally equivalent.*

Proof. Suppose elements $x \in X$ and $y \in Y$ are A-M bisimilar. Then by remark 7.4.2 there is an A-M bisimulation $\iota : R \longrightarrow X \times Y$ between coalgebras $\mathcal{X}$ and $\mathcal{Y}$ and an element $r \in R$ such that $\pi_X \circ \iota(r) = x$ and $\pi_Y \circ \iota(r) = y$ (here π_X and π_Y are respectively the canonical projection from $X \times Y$ to X and Y). We need to prove for each $\varphi \in L(\Lambda)$ the following diagram commutes.

$$\begin{tikzcd} X \arrow[dr, "\varphi^{\mathcal{X}}"'] & R \arrow[l, "\pi_X \circ \iota"'] \arrow[r, "\pi_Y \circ \iota"] & Y \arrow[dl, "\varphi^{\mathcal{Y}}"] \\ & 2 & \end{tikzcd} \tag{7.5.4}$$

We prove this, by induction over the construction of Λ-formulas. For the base case $\varphi = \top$, and for the Boolean connectives $\neg\varphi$ and $\varphi_1 \wedge \varphi_2$, this is obvious. Suppose the claim is true for φ, so the diagram 7.5.4, is commutative. Applying the functor F on diagram 7.5.4, and using the fact that $\iota : R \longrightarrow X \times Y$ is an A-M bisimulation, we obtain the following commutative diagram:

$$\begin{tikzcd}
 & 2 & \\
X \arrow[ur, "\varphi^{\mathcal{X}}"] \arrow[d, "\alpha"'] & R \arrow[l, "\pi_X \circ \iota"'] \arrow[r, "\pi_Y \circ \iota"] \arrow[d, "\rho"] & Y \arrow[ul, "\varphi^{\mathcal{Y}}"'] \arrow[d, "\beta"] \\
F(X) \arrow[dr, "F\varphi^{\mathcal{X}}"'] & F(R) \arrow[l, "F(\pi_X \circ \iota)"] \arrow[r, "F(\pi_Y \circ \iota)"'] & F(Y) \arrow[dl, "F\varphi^{\mathcal{Y}}"] \\
 & F(2) &
\end{tikzcd}$$

From this we obtain that

$$\begin{aligned}
(F\varphi^{\mathcal{X}} \circ \alpha)(x) &= (F\varphi^{\mathcal{X}} \circ \alpha)(\pi_X \circ \iota)(r) \\
&= (F\varphi^{\mathcal{X}} \circ \alpha \circ \pi_X \circ \iota)(r) \\
&= (F\varphi^{\mathcal{Y}} \circ \alpha \circ \pi_Y \circ \iota)(r) \\
&= (F\varphi^{\mathcal{Y}} \circ \alpha)(\pi_Y \circ \iota)(r) \\
&= (F\varphi^{\mathcal{Y}} \circ \alpha)(y).
\end{aligned}$$

Thus for each clopen subset $C \subseteq F(2)$ we have

$$(F\varphi^{\mathcal{X}} \circ \alpha)(x) \in C \iff (F\varphi^{\mathcal{Y}} \circ \alpha)(y) \in C$$

Then by equation 7.5.1, we conclude that $x \models_{\mathcal{X}} [C]\varphi$ iff $y \models_{\mathcal{Y}} [C]\varphi$. $\square$

In general the converse of the previous lemma does not hold. The following example borrowed from [11] makes this issue more clear. Notice that in this example we consider *Top* as an (*epi*, *regular mono*)-category.

Example 7.5.9. Let (X,α) and (X',α') be $\mathbb{V}$-coalgebras mentioned in example 7.4.12. Let Y and Y' be the topological sums $X + \{a\}$ and $X' + \{a'\}$, respectively. Define the maps $\gamma : Y \longrightarrow \mathbb{V}(Y)$ and $\gamma' : Y' \longrightarrow \mathbb{V}(Y')$ as follows

- $\gamma(x) = \alpha(x)$ and $\gamma'(x') = \alpha'(x')$ for each $x \in X$ and $x' \in X'$,
- $\gamma(a) = T_\infty$ and $\gamma'(a') = T'_\infty$.

We define the valuation maps $\mu : Y \longrightarrow \mathbb{P}(P)$ and $\mu' : Y' \longrightarrow \mathbb{P}(P)$ by

- $\mu(x) = \vartheta(x)$ and $\mu(x') = \vartheta'(x')$ for each $x \in X$ and $x' \in X'$,

- $\mu(a) = \mu'(a') = \emptyset$,

where $\vartheta : X \longrightarrow \mathbb{P}(P)$ and $\vartheta' : X' \longrightarrow \mathbb{P}(P)$ are the valuation maps defined in example 7.4.12. It is easy to see that the transition maps γ and γ' and also the valuation maps μ and μ' are continuous maps. Then $\mathcal{Y} = (Y, \gamma_P)$ and $\mathcal{Y}' = (Y', \gamma'_P)$ (where $\gamma_P(y) := (\gamma(y), \mu(y))$ and $\gamma'_P(y') := (\gamma'(y'), \mu(y'))$ for every $y \in Y$ and $y' \in Y'$) are $\mathbb{V}_P$-coalgebras.

Claim. The points a and a' are modally equivalent, but not A-M bisimilar.

Proof. We prove this claim by contradiction. Assume a and a' are A-M bisimilar. Then by remark 7.4.3, there is an inclusion of subspace $\iota' : B' \longrightarrow Y \times Y'$ such that $\pi_Y \circ \iota'$ and $\pi_{Y'} \circ \iota'$ are homomorphisms between coalgebras $\mathcal{Y}$ and $\mathcal{Y}'$ and $(a, a') \in B'$. By lemma 7.5.8, we conclude that a and a' are modal equivalence. On the other hand, by lemma 7.4.20, $U\iota' : U(B') \longrightarrow U(Y \times Y')$ (where U is the forgetful functor from *Top* to *Set*) is an A-M bisimulation between $\mathbb{P}_P$-coalgebras $(U(Y), \eta_Y \circ U\gamma_P)$ and $(U(Y'), \eta_{Y'} \circ U\gamma'_P)$ (because $\eta : U\mathbb{V} \longrightarrow \mathbb{P}U$ such that for each topological space A the morphism η_A is the inclusion of subset, is a natural transformation). Then by example 7.4.4, $U(B')$ is a Kripke bisimulation between corresponding Kripke models. Then $U(B') = \{(a, a')\} \cup B$ where B is the subset of $X \times X'$ mentioned in example 7.4.12. Now, by the same argument which has been explained for $\iota : B \longrightarrow X \times X'$ in example 7.4.12, we conclude that $\iota' : B' \longrightarrow Y \times Y'$ is not an A-M bisimulation. $\square$

We say that our logic has Hennessy-Milner property if A-M bisimilarity coincides with modal equivalence. In this case we call these kind of logics H-M logics.

Definition 7.5.10. Let $\mathcal{X} = (X, \alpha)$ and $\mathcal{Y} = (Y, \beta)$ be two coalgebras for the *Top*-endofunctor F, and let $x \in X$ and $y \in Y$. We say that x and y are behaviorally equivalent (in symbols $x \nabla_{\mathcal{X},\mathcal{Y}}\, y$), if there exists a F- coalgebra $\mathcal{Z} = (Z, \gamma)$ and homomorphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ such that $f(x) = g(y)$.

Remark 7.5.11. It follows immidiatly from lemmas 7.4.6 and 7.5.8 that bihaviorally equivalent elements are modally equivalent.

8. Terminal Coalgebras

In the theory of coalgebras, the terminal coalgebra is of special importance, provided it does exist, (see [45] and [62]). As it is mentioned in [35], one can not guarantee the existence of the terminal coalgebras. If it is exists, it is unique up to isomorphism and its elements can be interpreted as behaviors.
In this chapter we discuss the existence and the structure of terminal objects in the categories of coalgebras for the $\mathbb{C}$-endofunctors $D \times (-)$ (Black-boxes) and $D \times (-)^{\Sigma}$ (automata) where $\mathbb{C}$ is a category with object D and products. The idea of this chapter was discovered by Gumm [39].

Before starting this section, we recall that an F-coalgebra $\mathcal{T} = (T, \gamma)$ is called a terminal coalgebra in $\mathbb{C}_F$, if each F-coalgebra $\mathcal{A} = (A, \alpha)$ admits precisely one homomorphism $\tau : A \longrightarrow T$ in $\mathbb{C}_F$.

8.1. Terminal Black box

In this part we want to show that in any category $\mathbb{C}$ with object D and products, the terminal coalgebra for the $\mathbb{C}$-endofunctor $D \times (-)_{\mathbb{C}}$ (product of the constant functor D and the identity functor $(-)_{\mathbb{C}}$) exists, and it is based on D^{ω} (the $\omega-$fold product of D in $\mathbb{C}$). Recall that $X \mapsto D \times X$ is the object part of the functor $D \times (-)_{\mathbb{C}}$. On morphisms $f : X \to Y$ it is defined uniquely by $D \times f := id_D \times f$ in the following diagram,

$$\begin{tikzcd}
X \arrow[rr, "f"] & & Y \\
D \times X \arrow[u, "\pi_2"] \arrow[rr, "id_D \times f"] \arrow[dr, "\pi_1"'] & & D \times Y \arrow[u, "\pi_2"'] \arrow[dl, "\pi_1"] \\
& D &
\end{tikzcd}$$

(see lemma 2.15.8). To start with, let D^{ω} with canonical projections $(p_i)_{i \in \omega}$ be the $\omega-$fold product of D in $\mathbb{C}$, then one easily checks:

Lemma 8.1.1. *There are a unique morphism $\alpha : D^{\omega} \to D \times D^{\omega}$ such that (D^{ω}, α) is a $D \times (-)$-coalgebra.*

Proof. Assume now that D^ω with canonical projections $(p_i)_{i\in\omega}$ is the $\omega-$fold power of D. Then D^ω with the family $(p_{i+1})_{i\in\omega}$ is a competitor to the product, yielding a unique morphism $t : D^\omega \to D^\omega$ such that

$$p_{i+1} = p_i \circ t \tag{8.1.1}$$

for all $i \in \omega$. Next, D^ω with

$$h := p_0 \tag{8.1.2}$$

and t is a competitor to $D \times D^\omega$ yielding the product morphism $(h, t) : D^\omega \to D \times D^\omega$, which can be considered a $D \times (-)_{\mathbb{C}}$-coalgebra.

$$\begin{tikzcd}
 & D^\omega \arrow[r, "p_{i+1}"] \arrow[dl, dashed, "h=p_0"'] \arrow[d, dotted, "{(h,t)}"] \arrow[dr, dashed, "t"] & D \\
D & D \times D^\omega \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & D^\omega \arrow[u, "p_i"']
\end{tikzcd}$$

□

Theorem 8.1.2. *$(D^\omega, (h, t))$ is the terminal $D \times (-)$-coalgebra.*

The proof is split into two lemmas

Lemma 8.1.3. *Let (A, α) be an arbitrary $D \times (-)$ -coalgebra with $\alpha = (\alpha_0, \alpha_1)$ where $\alpha_0 : A \to D$ and $\alpha_1 : A \to A$. Any coalgebra homomorphism $\varphi : A \to D^\omega$ must satisfy for each $i \in \omega$:*

$$p_i \circ \varphi = \alpha_0 \circ (\alpha_1)^i \tag{8.1.3}$$

Proof. We show this by induction, using the homomorphism diagram:

$$\begin{tikzcd}
D \arrow[r, equal] & D \\
A \arrow[u, "\alpha_0"] \arrow[r, "\varphi"] \arrow[d, "\alpha_1"'] & D^\omega \arrow[u, "h"'] \arrow[d, "t"] \\
A \arrow[r, "\varphi"] & D^\omega
\end{tikzcd}$$

For $i = 0$ the claim is obvious, since $h = p_0$.
For the inductive step, we calculate

$$\begin{aligned}
p_{i+1} \circ \varphi &= p_i \circ t \circ \varphi \\
&= p_i \circ \varphi \circ \alpha_1 \\
&= \alpha_0 \circ (\alpha_1)^i \circ \alpha_1 \\
&= \alpha_0 \circ (\alpha_1)^{i+1}.
\end{aligned}$$

□

Lemma 8.1.4. *The equations $p_i \circ \varphi := \alpha_0 \circ (\alpha_1)^i$ define a unique coalgebra morphism $\varphi : A \to D^\omega$.*

Proof. The morphism $\alpha_0 \circ (\alpha_1)^i : A \to D$ turn A into a competitor to the product D^ω, which yields the unique morphism φ. To show that φ is indeed a homomorphism between $D \times (-)_{\mathbb{C}}$-coalgebras, we must show that this φ makes the previous diagram commute, so we calculate:

$$h \circ \varphi = p_0 \circ \varphi = \alpha_0 \circ (\alpha_1)^0 = \alpha_0$$

and for all i:

$$\begin{aligned} p_i \circ t \circ \varphi &= p_{i+1} \circ \varphi \\ &= \alpha_0 \circ (\alpha_1)^{i+1} \\ &= \alpha_0 \circ (\alpha_1)^i \circ \alpha_1 \\ &= p_i \circ \varphi \circ \alpha_1 \end{aligned}$$

from which $t \circ \varphi = \varphi \circ \alpha_1$, since the projections p_i are jointly mono. Thus φ is a coalgebra morphism, which is unique by the previous lemma. $\square$

8.2. Terminal Automaton

Let Σ be a set and $\Sigma^\star$ the set of all finite words[1] over Σ. In this part we want to show that in any category $\mathbb{C}$ with object D and products, the terminal coalgebras for the functor $D \times (-)^\Sigma$ (product of the constant functor D and the power functor $(-)^\Sigma$ on $\mathbb{C}$) exists, and it is based on $D^{\Sigma^\star}$ (the $\Sigma^\star$−fold product of D in $\mathbb{C}$). Before starting, we should recall that $X \mapsto D \times X^\Sigma$ is the object part of a functor $D \times (-)^\Sigma$. On morphisms $f : X \to Y$ it is defined uniquely by $D \times f^\Sigma := id_D \times f^\Sigma$ in the following diagram, where for $e \in \Sigma$, we let π_e be the projection to the e-th component:

$$\begin{tikzcd} X \arrow[rr, "f"] & & Y \\ D \times X^\Sigma \arrow[u, "\pi_e"] \arrow[rr, "id_D \times f"] \arrow[dr, "\pi_1"'] & & D \times Y^\Sigma \arrow[u, "\pi_e"'] \arrow[dl, "\pi_1"] \\ & D & \end{tikzcd}$$

We denote the empty word[2] by ε, and given $e \in \Sigma$ and $w \in \Sigma^\star$, we denote by $e.w$ the word obtained by prefixing e to w. Now, let $D^{\Sigma^\star}$ with projections $(p_w)_{w \in \Sigma^\star}$ be the $\Sigma^\star$−fold product of D in $\mathbb{C}$, then one easily checks:

Lemma 8.2.1. *There is a unique morphism $\alpha : D^{\Sigma^\star} \to D \times (D^{\Sigma^\star})^\Sigma$ such that $(D^{\Sigma^\star}, \alpha)$ is a $D \times (-)^\Sigma$-coalgebra.*

[1] Every finite sequence of elements from Σ is called a finite word over Σ.
[2] The empty word is the unique sequence in Σ of length 0.

Proof. Assume that $D^{\Sigma^\star}$ with projections $(p_w)_{w\in\Sigma^\star}$ is the $\Sigma^\star$−fold product of D. Then for each $e \in \Sigma$ the same object $D^{\Sigma^\star}$ but with the family $(p_{e.w})_{w\in\Sigma^\star}$ is a competitor to the product, yielding a unique morphism $t_e : D^{\Sigma^\star} \to D^{\Sigma^\star}$ such that

$$p_{e.w} = p_w \circ t_e \tag{8.2.1}$$

for all $w \in \Sigma^\star$. Next, $D^{\Sigma^\star}$ with p_ε and $t = (t_e)_{e\in\Sigma}$ is a competitor to $D \times (D^{\Sigma^\star})^\Sigma$ yielding the product morphism $(p_\varepsilon, t) : D^{\Sigma^\star} \to D \times (D^{\Sigma^\star})^\Sigma$, which can be considered a $D \times (-)^\Sigma$-coalgebra.

$$\begin{tikzcd}
 & D^{\Sigma^\star} \arrow[r, "p_{e.w}"] \arrow[dl, dashed, "p_\varepsilon"'] \arrow[d, dotted, "{(p_\varepsilon,t)}"] \arrow[dr, dashed, "t_e"] & D \\
D & D \times (D^{\Sigma^\star})^\Sigma \arrow[l, "\pi_1"] \arrow[r, "\pi_e"'] & D^{\Sigma^\star} \arrow[u, "p_w"']
\end{tikzcd}$$

□

Theorem. $(D^{\Sigma^\star}, (p_\varepsilon, t))$ *is the terminal* $D \times (-)^\Sigma$*-coalgebra.*

The proof is split into two lemmas. Given an arbitrary $D \times (-)^\Sigma$-coalgebra (A, α) with $\alpha = (\alpha_0, (\alpha_e)_{e\in\Sigma})$, where $\alpha_0 : A \to D$ and $\alpha_e : A \to A$, we can define inductively morphism $\alpha_w : A \to A$ for each word $w \in \Sigma^\star$ by

$$\begin{aligned}
\alpha_\varepsilon &:= id_A \\
\alpha_{e.w} &:= \alpha_w \circ \alpha_e
\end{aligned}$$

With this definition, we have

Lemma. *Let* (A, α) *be an arbitrary* $D \times (-)^\Sigma$ *-coalgebra with* $\alpha = (\alpha_0, (\alpha_e)_{e\in\Sigma})$*, where* $\alpha_0 : A \to D$ *and* $\alpha_e : A \to A$*. Any coalgebra homomorphism* $\varphi : A \to D^{\Sigma^\star}$ *must satisfy for each word* $w \in \Sigma^\star$*:*

$$p_w \circ \varphi = \alpha_0 \circ \alpha_w \tag{8.2.2}$$

Proof. We show this by induction, using the homomorphism diagram:

$$\begin{tikzcd}
D \arrow[r, equal] & D \\
A \arrow[r, "\varphi"] \arrow[u, "\alpha_0"] \arrow[d, "\alpha_e"'] & D^{\Sigma^\star} \arrow[u, "p_\varepsilon"'] \arrow[d, "t_e"] \\
A \arrow[r, "\varphi"] & D^{\Sigma^\star}
\end{tikzcd}$$

For $w = \varepsilon$ the claim is obvious, since $p_\varepsilon \circ \varphi = \alpha_0 = \alpha_0 \circ id_A = \alpha_0 \circ \alpha_\varepsilon$.
For the inductive step, we assume the condition for $w \in \Sigma^\star$ and calculate for an arbitrary

$e \in \Sigma$:

$$\begin{aligned} p_{e.w} \circ \varphi &= p_w \circ t_e \circ \varphi \\ &= p_w \circ \varphi \circ \alpha_e \\ &= \alpha_0 \circ \alpha_w \circ \alpha_e \\ &= \alpha_0 \circ \alpha_{e.w}. \end{aligned}$$

□

Lemma. *The equations* $\{p_w \circ \varphi = \alpha_0 \circ \alpha_w\}_{w \in \Sigma^\star}$ *define a unique coalgebra morphism* $\varphi : A \to D^{\Sigma^\star}$.

Proof. The morphisms $\alpha_0 \circ \alpha_w : A \to D$ turn A into a competitor to the product $D^{\Sigma^\star}$, which yields the unique morphism $\varphi : A \longrightarrow D^{\Sigma^\star}$satisfying the equation. To show that φ is indeed a morphism of $D \times (-)^\Sigma$-coalgebras, we must show that this φ makes the previous diagram commute for each $e \in \Sigma$, so we calculate:

$$p_\varepsilon \circ \varphi = \alpha_0 \circ \alpha_\varepsilon = \alpha_0$$

and for all $w \in \Sigma^\star$:

$$\begin{aligned} p_w \circ t_e \circ \varphi &= p_{e.w} \circ \varphi \\ &= \alpha_0 \circ \alpha_{e.w} \\ &= \alpha_0 \circ \alpha_w \circ \alpha_e \\ &= p_w \circ \varphi \circ \alpha_e \end{aligned}$$

from which $t_e \circ \varphi = \varphi \circ \alpha_e$, since the projections p_w are jointly mono. Thus φ is a coalgebra morphism, which is unique by the previous lemma. □

Part IV.

Vietoris structures

9. Vietoris models and Vietoris frames

The aim of this section is to introduce the notion of Vietoris structures. These notions were introduced for the first time by Esaki in [24]. We also discuss the coalgebraic perspective of these structures. Our presentation in this chapter is based on [11].
Throughout this section, let P be a fixed set of propositional letters.
First we give an auxiliary definition needed to study the concept of Vietoris structures and their properties.

Definition 9.0.1. Given a binary relation $R \subseteq X \times Y$ and a subset $V \subseteq Y$, define

- $R(x) = \{y \in X \mid x\, R\, y\}$,
- $\langle R\rangle(V) = \{x \in X \mid \exists y \in V.\, x\, R\, y\}$,
- $[R](V) = \{x \in X \mid \forall y \in Y.\, x\, R\, y \Rightarrow y \in V\}$.

Then

- $\langle R\rangle(V) = \{x \in X \mid R(x) \cap V \neq \emptyset\}$,
- $[R](V) = \{x \in X \mid R(x) \subseteq V\}$.

Obviously,

- $[R](V) = X - \langle R\rangle(Y - V)$, and
- $\langle R\rangle(V) = X - [R](Y - V)$.

Suppose X and Y are topological spaces and $R \subseteq X \times Y$ a binary relation. We say that R is a **compact binary relation** if for each $x \in X$ the set $R(x)$ is a compact subset of Y.

Definition 9.0.2. (Vietoris model) Generally speaking, a Vietoris model is a Kripke model $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ (called the underlying Kripke model) with a topology τ on X such that:

1. $\forall x \in X.\, R_{\mathcal{X}}(x)$ is compact (i.e., $R_{\mathcal{X}}$ is a compact binary relation).
2. $\forall U \in \tau.\, \langle R_{\mathcal{X}}\rangle(U)$ is open.
3. $\forall U \in \tau.\, [R_{\mathcal{X}}](U)$ is open.

4. $\forall p \in P. \|p\|^{\mathcal{X}} \in \tau$ and $(X - \|p\|^{\mathcal{X}}) \in \tau$, where $\|p\|^{\mathcal{X}} = \{x \mid x \models_{\mathcal{X}} p\}$.

Vietoris frames are Vietoris models with $P = \emptyset$ (i.e., a Vietoris frame is a Kripke frame $\mathcal{X} = (X, R_{\mathcal{X}})$ with a topology τ on X such that the conditions (1), (2) and (3) hold).

Lemma 9.0.3. *In a Vietoris model $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$, if C is closed in X, then so are $\langle R_{\mathcal{X}}\rangle(C)$ as well as $[R_{\mathcal{X}}](C)$.*

Proof. Suppose C is a closed subset of X. According to definition 9.0.1, we have $\langle R_{\mathcal{X}}\rangle(C) = X - [R_{\mathcal{X}}](X - C)$ and $[R_{\mathcal{X}}](C) = X - \langle R_{\mathcal{X}}\rangle(X - C)$. Since $\mathcal{X}$ is a Vietoris model and since $X - C$ is open, $[R_{\mathcal{X}}](X - C)$ and $\langle R_{\mathcal{X}}\rangle(X - C)$ are open. Consequently, $\langle R_{\mathcal{X}}\rangle(C)$ and $[R_{\mathcal{X}}](C)$ are closed.

$\square$

Lemma 9.0.4. *If $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ is a Vietoris model, then $\|\varphi\|$ is a clopen (closed and open) subset of X, for each $\varphi \in L_P$.*

Proof. We prove this claim by induction over the construction of formulas.

Base case **:** $\|p\|^{\mathcal{X}}$ is clopen, for each $p \in P$ (by part (4) in the definition of Vietoris models).

Inductive step **:** Suppose $\|\varphi_1\|^{\mathcal{X}}$ and $\|\varphi_2\|^{\mathcal{X}}$ are clopen for φ_1, $\varphi_2 \in L_P$. So

- $\|\varphi_1 \wedge \varphi_2\|^{\mathcal{X}} = \|\varphi_1\|^{\mathcal{X}} \cap \|\varphi_2\|^{\mathcal{X}}$ is clopen.
- $\|\neg\varphi_1\|^{\mathcal{X}} = X - \|\varphi_1\|^{\mathcal{X}}$ is clopen
- $\|\Box\varphi_1\|^{\mathcal{X}} = \{x \in X \mid R_{\mathcal{X}}(x) \subseteq \|\varphi_1\|^{\mathcal{X}}\} = [R_{\mathcal{X}}](\|\varphi_1\|^{\mathcal{X}})$ is clopen (by definition 9.0.2 and lemma 9.0.3).
- $\|\Diamond\varphi_1\|^{\mathcal{X}} = \|\neg\Box\neg\varphi_1\|^{\mathcal{X}} = X - \|\Box\neg\varphi_1\|^{\mathcal{X}}$.

$\square$

9.1. Vietoris homomorphisms

Definition 9.1.1. (Vietoris homomorphism) Suppose that $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \models_{\mathcal{Y}})$ are two Vietoris models, then a map $f : X \longrightarrow Y$ is called a Vietoris homomorphism from $\mathcal{X}$ to $\mathcal{Y}$, if f is a continuous Kripke homomorphism between underlying Kripke models.

Remark 9.1.2. Vietoris structures together with Vietoris homomorphisms form a category which we shall call VS.

9.2. Vietoris structures as coalgebras over Top

In the following, we will see that Vietoris structures can be presented as $\mathbb{V}_P$-coalgebras over Top. Before starting, we should recall the P-Vietoris functor $\mathbb{V}_P : Top \longrightarrow Top$ (see 3.2.1), where P is the set of propositions letters. Recall that for each topological space X, the Vietoris space $\mathbb{V}(X)$ is the set of all compact subsets of X with the topology generated by a subbase consisting all sets $[U] := \{K \in \mathbb{V}(X) \,|\, K \subseteq U\}$ and $\langle U \rangle := \{K \in \mathbb{V}(X) \,|\, K \cap U \neq \emptyset\}$ where U is any open subset of X (see 3.2). Let $\mathbb{P}(P)$ be the set of all subsets of P equipped with the topology generated by a subbase containing all clopens of the form $\uparrow p := \{u \subseteq P \mid p \in u\}$, where $p \in P$. The endofunctor $\mathbb{V}_P : Top \longrightarrow Top$ associates to each topological space X, the product space $\mathbb{V}(X) \times \mathbb{P}(P)$ and sends every continuous function $f : X \longrightarrow Y$ to the continuous function $\mathbb{V}f \times id_{\mathbb{P}(P)}$ given by $(\mathbb{V}f \times id_{\mathbb{P}(P)})(K, M) = (f[K], M)$ (for all $K \in \mathbb{V}(X)$ and all $M \subseteq P$).

Notice that if X is an arbitrary topological space and $R \subseteq X \times X$ a compact binary relation on X, then $R : X \longrightarrow \mathbb{V}(X)$ defined by $R(x) := \{y \in X \mid x\,R\,y\}$ (for each $x \in X$) is a map (because $R(x) \in \mathbb{V}(X)$ for each $x \in X$). Then, we have the following lemmas:

Lemma 9.2.1. *Let X be a fixed topological space and $R \subseteq X \times X$ a compact binary relation on X. For every open subset $U \subseteq X$, we have:*

- $R^{-1}([U]) = [R](U)$; and
- $R^{-1}(\langle U \rangle) = \langle R \rangle(U)$.

Proof. Assume $U \subseteq X$ is open. So

$$\begin{aligned} R^{-1}([U]) &= \{x \in X \mid R(x) \in [U]\} \\ &= \{x \in X \mid R(x) \subseteq U\} \\ &= [R](U). \end{aligned}$$

Also

$$\begin{aligned} R^{-1}(\langle U \rangle) &= \{x \in X \mid R(x) \in \langle U \rangle\} \\ &= \{x \in X \mid R(x) \cap U \neq \emptyset\} \\ &= \{x \in X \mid x \in \langle R \rangle(U)\} \\ &= \langle R \rangle(U). \end{aligned}$$

□

Lemma 9.2.2. *Let X be a fixed topological space and $R \subseteq X \times X$ a compact binary relation on X. The map $R : X \longrightarrow \mathbb{V}(X)$ is continuous iff for each open subset $U \subseteq X$, the subsets $[R](U)$ and $\langle R\rangle(U)$ are open subsets of X.*

Proof. By remark 1.3.3, R is continuous iff $R^{-1}([U])$ and $R^{-1}(\langle U\rangle)$ are open in X (where U is an open subset of X). According to the previous lemma, for each open subset $U \subseteq X$ the set $R^{-1}([U])$ (resp. $R^{-1}(\langle U\rangle)$) is open in X iff $[R](U)$ (resp. $\langle R\rangle(U)$) is open in X. Then the claim is trivial. $\square$

As a consequence of the previous lemma we have the following corollary:

Corollary 9.2.3. *Vietoris frames are the same as $\mathbb{V}$-coalgebras on Top.*

In order to identify Vietoris models as $\mathbb{V}_P$-coalgebras over Top, let X be a topological space and $\models \subseteq X \times P$ be a binary relation. Let $\mathbb{P}(P)$ be the set of all subsets of P equipped with the topology generated by a subbase containing all clopens of the form

$$\uparrow p = \{u \subseteq P \mid p \in u\}$$

where $p \in P$. Define a map $\vartheta : X \longrightarrow \mathbb{P}(P)$ by $\vartheta(x) = \{p \in P \mid x \vDash p\}$. Now we have:

Lemma 9.2.4. ϑ is continuous iff for each $p \in P$ the set $\{x \in X \mid x \vDash p\}$ is a clopen subset of X.

Proof. By remark 1.3.3, we know that ϑ is continuous iff $\vartheta^{-1}(\uparrow p)$ is a clopen subset of X, for each $p \in P$. On the other hand

$$\begin{aligned} \vartheta^{-1}(\uparrow p) &= \vartheta^{-1}(\{u \subseteq P \mid p \in u\}) \\ &= \{x \in X \mid p \in \vartheta(x)\} \\ &= \{x \in X \mid x \vDash p\} \end{aligned}$$

So $\vartheta^{-1}(\uparrow p)$ is a clopen subset of X, if and only if $\{x \in X \mid x \vDash p\}$ is clopen in X, for each $p \in P$. $\square$

Lemma 9.2.5. *A Kripke model $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ with a topology τ on X, is a Vietoris model iff*

1. *$R_{\mathcal{X}}(x)$ is compact, for each $x \in X$, and*
2. *$R_{\mathcal{X}} : X \longrightarrow \mathbb{V}(X)$* defined by $R_{\mathcal{X}}(x) := \{y \in X \mid x\, R_{\mathcal{X}}\, y\}$ is a continuous map, and
3. *$\vartheta_{\mathcal{X}} : X \longrightarrow \mathbb{P}(P)$* coded by $\vartheta_{\mathcal{X}}(x) = \{p \in P \mid x \models_{\mathcal{X}} p\}$ is a continuous map.

Proof. Lemmas 9.2.2 and 9.2.4 yield both directions of our claim. □

Now, suppose $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ is a Kripke model and let $R_{\mathcal{X}} : X \longrightarrow \mathbb{V}(X)$ and $\vartheta_{\mathcal{X}} : X \longrightarrow \mathbb{P}(P)$ be maps defined by the binary relations $R_{\mathcal{X}}$ and $\models_{\mathcal{X}}$, respectively (see, conditions (2) and (3) in the previous lemma). Suppose $\alpha : X \longrightarrow \mathbb{V}(X) \times \mathbb{P}(P)$ is a map defined by $\alpha(x) := (R_{\mathcal{X}}(x), \vartheta_{\mathcal{X}}(x))$ for any $x \in X$. Obviously, α is continuous iff $R_{\mathcal{X}}$ and $\vartheta_{\mathcal{X}}$ are continuous. Therefore each Vietoris model $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ can be represented as a $\mathbb{V}_P$-coalgebra (X, α) and vice versa. Our finding in this subsection can be summarized as the following theorem:

Theorem 9.2.6. *Vietoris models are the same as $\mathbb{V}_P$-coalgebras on Top.*

The following theorem shows that each Vietoris homomorphism between Vietoris models is a homomorphism between corresponding $\mathbb{V}_P$-coalgebras and vice versa.

Theorem 9.2.7. *Given Vietoris models $\mathcal{X} =(X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ and $\mathcal{Y} =(Y, R_{\mathcal{Y}}, \models_{\mathcal{Y}})$. A continuous map $f : X \longrightarrow Y$ is a Vietoris homomorphism iff f is a homomorphism between corresponding $\mathbb{V}_P$-coalgebras.*

Proof. Let $f : X \longrightarrow Y$ be a continuous map. By definition 9.1.1 f is a Vietoris homomorphism between $\mathcal{X}$ and $\mathcal{Y}$ then iff it is a Kripke homomorphism between the underlying Kripke structures which means the conditions (1), (2) and (3) in definition 6.1.16 hold. This means the following diagram commutes

$$\begin{tikzcd}
\mathbb{P}(P) \arrow[r, "id_{\mathbb{P}(P)}"] & \mathbb{P}(P) \\
X \arrow[u, "\vartheta_{\mathcal{X}}"] \arrow[r, "f"] \arrow[d, "R_{\mathcal{X}}"'] & Y \arrow[u, "\vartheta_{\mathcal{Y}}"'] \arrow[d, "R_{\mathcal{Y}}"] \\
\mathbb{V}(X) \arrow[r, "\mathbb{V}(f)"] & \mathbb{V}(Y)
\end{tikzcd} \tag{9.2.1}$$

where $R_{\mathcal{X}}$ and $\vartheta_{\mathcal{X}}$ (resp. $R_{\mathcal{Y}}$ and $\vartheta_{\mathcal{Y}}$) be the maps obtained by the binary relations $R_{\mathcal{X}}$ and $\models_{\mathcal{X}}$ (resp. $R_{\mathcal{Y}}$ and $\models_{\mathcal{Y}}$), respectively (see, conditions (2) and (3) in the previous lemma). So we can conclude that f is a homomorphism between $\mathbb{V}_P$-coalgebras corresponded to the Vietoris models $\mathcal{X}$ and $\mathcal{Y}$. □

We can combine the previous two theorms to a theorem as follows:

Theorem 9.2.8. *$\mathbb{V}_P$-coalgebras with coalgebra homomorphisms are the same as Vietoris models with continuous Kripke-homomorphisms.*

9.3. Compact Kripke models and Vietoris models

In this subsection we will discuss the connection between Vietoris structures and compact Kripke structures (see definition 6.2.2). In fact, by forgetting the topologies of Vietoris structures we will obtain an ordinary Kripke structure which is also compact. This forgetting can be formalized as a functor U_K from the category VS (the category of Vietoris structures together with Vietoris homomorphisms) to the category KS (the category of Kripke structures together with Kripke homomorphisms) which can be factored through the forgetful functor $U_C : VS \longrightarrow CKS$ and the inclusion functor $\mathfrak{I} : CKS \longrightarrow KS$ (where CKS is the category of compact Kripke structures with Kripke homomorphisms between them). On the other hand, given a Kripke model $\mathcal{X}$=$(X, R_{\mathcal{X}}, \models_{\mathcal{X}})$. By equipping the set X with a special topology $\tau_\square$ called $\mathcal{X}$-modal topology (see definition 9.3.1), we will obtain a new structure $\mathcal{X}_\square = (X_\square, R_{\mathcal{X}}, \models_{\mathcal{X}})$ called $\mathcal{X}$-modal model (see definition 9.3.1). In case that $\mathcal{X}$ is a compact Kripke model, it will be shown that $\mathcal{X}_\square$ is a Vietoris model (see theorem 9.3.5). Then, we can define a functor $\mathcal{F}_\square : CKS \longrightarrow VS$ which associates each compact Kripke model $\mathcal{X}$ to the Vietoris model $\mathcal{X}_\square$. We will see that the functor $\mathcal{F}_\square$ is a right adjoint to the functor U_C (see theorem 9.3.7). We can present these functors in one picture as follows:

$$\begin{tikzcd}
VS \arrow[rr, "U_K"] \arrow[dr, shift left, "U_C"] & & KS \\
& CKS \arrow[ul, shift left, "\mathcal{F}_\square"] \arrow[ur, "\mathfrak{I}"'] &
\end{tikzcd}$$

Definition 9.3.1. ($\mathcal{X}$-modal topology, $\mathcal{X}$-modal model) For each Kripke model $\mathcal{X}$=$(X, R_{\mathcal{X}}, \models_{\mathcal{X}})$, we define $\mathcal{X}$ -modal topology $\tau_\square$ on the set X as a topology generated by a base consisting of all opens as follows:

$$\|\varphi\|^{\mathcal{X}} = \{x \in X \mid x \models_{\mathcal{X}} \varphi\}$$

where $\varphi \in L_P$.
Denote by $X_\square$ the topological space obtained by equipping the set X with the $\mathcal{X}$-modal topology $\tau_\square$. Thus each open subset O in $X_\square$ is of the form $\bigcup\limits_{\varphi\in\Sigma} \|\varphi\|^{\mathcal{X}}$ for some subset $\Sigma \subseteq L_P$. Consequently each closed subset C in $X_\square$ is of the shape $\bigcap\limits_{\varphi\in\Sigma} \|\varphi\|^{\mathcal{X}}$ for some subset $\Sigma \subseteq L_P$.
For every Kripke model $\mathcal{X}$ =$(X, R_{\mathcal{X}}, \models_{\mathcal{X}})$, we call the triple $\mathcal{X}_\square = (X_\square, R_{\mathcal{X}}, \models_{\mathcal{X}})$ the $\mathcal{X}$-modal model.

Lemma 9.3.2. *For each compact Kripke model* $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$*, the* $\mathcal{X}$*-modal model* $\mathcal{X}_\Box = (X_\Box, R_{\mathcal{X}}, \models_{\mathcal{X}})$ *is a Vietoris model.*

Proof. Let $O = \bigcup_{\varphi \in \Sigma} \|\varphi\|^{\mathcal{X}}$ be an arbitrary open subset of $X_\Box$. Hence

$$\begin{aligned}
\langle R \rangle (O) &= \{x \in X \mid R(x) \cap O \neq \emptyset\} \\
&= \{x \in X \mid R(x) \cap (\bigcup_{\varphi \in \Sigma} \|\varphi\|^{\mathcal{X}}) \neq \emptyset\} \\
&= \{x \in X \mid \exists \varphi \in \Sigma .\, x \models_{\mathcal{X}} \Diamond \varphi\} \\
&= \bigcup_{\varphi \in \Sigma} \|\Diamond \varphi\|^{\mathcal{X}}
\end{aligned}$$

which is a union of open sets. Similarly

$$\begin{aligned}
[R](O) &= \{x \in X \mid R(x) \subseteq O\} \\
&= \{x \in X \mid R(x) \subseteq \bigcup_{\varphi \in \Sigma} \|\varphi\|^{\mathcal{X}}\} \\
&\overset{\mathcal{X} \text{ is compact}}{=} \{x \in X \mid \exists \Sigma_0 \underset{fin}{\subseteq} \Sigma .\, R(x) \subseteq \bigcup_{\varphi \in \Sigma_0} \|\varphi\|^{\mathcal{X}}\} \\
&= \{x \in X \mid \exists \Sigma_0 \underset{fin}{\subseteq} \Sigma .\, x \models_{\mathcal{X}} \Box (\bigvee_{\varphi \in \Sigma_0} \varphi)\} \\
&= \bigcup_{\Sigma_0 \in \mathbb{P}_f(\Sigma)} \|(\bigvee_{\varphi \in \Sigma_0} \varphi)\|^{\mathcal{X}}
\end{aligned}$$

which is open, as well. Clearly for each $p \in P$, the sets $\|p\|^{\mathcal{X}}$ and $X - \|p\|^{\mathcal{X}}$ are clopen subsets of $X_\Box$ (see definition 9.3.1). Next we have to show $R(x)$ is compact subset of X for each $x \in X$. Assume $\bigcup_{\varphi \in \Sigma} \|\varphi\|^{\mathcal{X}}$ is an open cover of $R(x)$. So $x \models_{\mathcal{X}} \Box(\bigvee_{\varphi \in \Sigma} \varphi)$ and by assumption there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $x \models_{\mathcal{X}} \Box(\bigvee_{\varphi \in \Sigma_0} \varphi)$. Then $R(x) \subseteq \bigcup_{\varphi \in \Sigma_0} \|\varphi\|^{\mathcal{X}}$. □

Lemma 9.3.3. *Suppose* $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ *and* $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \models_{\mathcal{Y}})$ *are compact Kripke models and* $f : X \longrightarrow Y$ *a function. Then* f *is a Kripke homomorphism iff it is a Vietoris homomorphism between Vietoris models* $\mathcal{X}_\Box$ *and* $\mathcal{Y}_\Box$.

Proof. Suppose f is a Vietoris homomorphism between Vietoris models $\mathcal{X}_\Box$ and $\mathcal{Y}_\Box$, then it is continuous and a Kripke homomorphism between underlying Kripke models $\mathcal{X}$ and $\mathcal{Y}$ (by definition 9.1.1). Conversely, let f be a Kripke homomorphism between $\mathcal{X}$ and $\mathcal{Y}$. We just need to show that f is continuous. Let $O = \bigcup_{\varphi \in \Sigma} \|\varphi\|^{\mathcal{Y}}$ be an arbitrary open subset of $Y_\Box$. Hence

$$\begin{aligned} f^{-1}(O) &= f^{-1}(\bigcup_{\varphi\in\Sigma} \|\varphi\|^{\mathcal{Y}}) \\ &= \bigcup_{\varphi\in\Sigma} f^{-1}(\|\varphi\|^{\mathcal{Y}}) \\ &\overset{\text{corollary 6.1.18}}{=} \bigcup_{\varphi\in\Sigma} \|\varphi\|^{\mathcal{X}} \end{aligned}$$

which is an union of open sets in $X_\Box$. □

Remark 9.3.4. As a corollary of the previous lemma, we can define a functor $\mathcal{F}_\Box : CKS \longrightarrow VS$ which associates to each compact Kripke structure $\mathcal{X}{=}(X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ the corresponding $\mathcal{X}$-modal model $\mathcal{X}_\Box = (X_\Box, R_{\mathcal{X}}, \models_{\mathcal{X}})$. The functor $\mathcal{F}_\Box$ is called the modal functor. If $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \models_{\mathcal{Y}})$ are arbitrary compact Kripke models, then for each Kripke homomorphism $f : X \longrightarrow Y$ we define $\mathcal{F}_\Box(f) := f$.

Theorem 9.3.5. *Suppose $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ is a Kripke model. The following conditions are equivalent:*

1. *$\mathcal{X}$ is compact.*
2. *$\mathcal{X}$ is the underlying Kripke model of a Vietoris model.*
3. *$\mathcal{X}$ is the underlying Kripke model of a $\mathbb{V}_P$-coalgebra.*

Proof. Suppose $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ is a Kripke model, then:
1⇒2 follows from lemma 9.3.2.
2⇔3 is corollary 9.2.6.
2⇒1 : Suppose $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ is the underlying Kripke model of a Vietoris model and $x \models_{\mathcal{X}} \Box(\bigvee_{\varphi\in\Sigma} \varphi)$ where Σ is a subset of L_P. Then $R(x) \subseteq \bigcup_{\varphi\in\Sigma} \|\varphi\|^{\mathcal{X}}$. By lemma 9.0.4, the right hand side is union of open sets, thus by compactness of $R(x)$ there is a finite subset $\Sigma_0 \subseteq \Sigma$ with $R(x) \subseteq \bigcup_{\varphi\in\Sigma_0} \|\varphi\|^{\mathcal{X}}$ which means $x \models_{\mathcal{X}} \Box(\bigvee_{\varphi\in\Sigma_0} \varphi)$. □

Remark 9.3.6. As a fairly direct corollary of theorem 9.3.5, we can define a forgetful functor $U_C : VS \longrightarrow CKS$ which assigns to each Vietoris model $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ its underlying compact Kripke model and to each Vietoris homomorphism $f : X \longrightarrow Y$ (between Vietoris structures $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \models_{\mathcal{Y}})$) the same morphism between the underlying Kripke structures, i.e., $U_C(f) := f$.

Lemma 9.3.7. *The modal functor $\mathcal{F}_\Box : CKS \longrightarrow VS$ is a right adjoint to the functor $U_C : VS \longrightarrow CKS$.*

Proof. For each Vietoris model $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$, the Vietoris structure $\mathcal{F}_{\Box}U_C(\mathcal{X})$ is the $\mathcal{X}$-modal model generated by its underlying compact Kripke model, i.e.

$$\mathcal{F}_{\Box}U_C(\mathcal{X}) = (X_{\Box}, R_{\mathcal{X}}, \models_{\mathcal{X}}).$$

The morphism $\eta_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathcal{F}_{\Box}U_C(\mathcal{X})$ defined as "the identity map on the underlying set X" is a Vietoris homomorphism from $\mathcal{X}$ to $\mathcal{F}_{\Box}U_C(\mathcal{X})$ (to see the continuity of $\eta_{\mathcal{X}}$ notice that since $\mathcal{X}$ is a Vietoris model, $\|\varphi\|^{\mathcal{X}}$ is clopen in X, for each modal formula φ, see lemma 9.0.4). Moreover, for each compact Kripke model $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \models_{\mathcal{Y}})$, we have $\mathcal{Y} = U_C\mathcal{F}_{\Box}(\mathcal{Y})$. Then the morphism $\xi_{\mathcal{Y}} : U_C\mathcal{F}_{\Box}(\mathcal{Y}) \longrightarrow \mathcal{Y}$ defined as "the identity map on the underlying set Y" is a Kripke homomorphism. We can easily check that $id_{U_C(\mathcal{X})} = \xi_{U_C(\mathcal{X})} \circ U_C(\eta_{\mathcal{X}})$ and $id_{\mathcal{F}_{\Box}(\mathcal{Y})} = \mathcal{F}_{\Box}(\xi_Y) \circ \eta_{\mathcal{F}_{\Box}(\mathcal{Y})}$. $\Box$

10. Terminal Vietoris model

Let P be a fixed set of propositional letters and L_P be the set of modal formulas constructed inductively over P. In this chapter, we will show that the category of Vietoris models has a terminal object. In fact the result of this chapter is a straightforward consequence of the works of Abramsky [1] and Venema et. al. [11]. In [11], the authors have determined $(X^c, R^c, \models^c)$ as a terminal object in the category of descriptive models. Here X^c is the collection of all maximal consistent sets of formulas over L_P (the set of modal formulas constructed inductively over P) equipped with a topology generated by the clopen sets of the form $\hat{\varphi} = \{u \in X^c \mid \varphi \in u\}$. The binary relation $R^c \subseteq X^c \times X^c$ is defined by $u\, R^c v$ iff $\{\Diamond\varphi \mid \varphi \in v\} \subseteq u$ and the relation $\models^c \subseteq X^c \times P$ is given by $u \models^c p \iff p \in u \cap P$.
In this work, we replace the collection of all maximal consistent sets over L_P, by the set of all Kripke-ultrafilters (in symbol: $\mathcal{U}$) over L_P, and we will show that the $\mathfrak{U}$-Modal model induced by the Kripke-Ultrafilter model $\mathfrak{U} = (\mathcal{U}, \mathcal{R}_{\mathfrak{U}}, \models_{\mathfrak{U}})$ (where the binary relation $\mathcal{R}_{\mathfrak{U}}$ on $\mathcal{U}$ is defined by $u\, \mathcal{R}_{\mathfrak{U}}\, v$ iff $\{\varphi \mid \Box\varphi \in u\} \subseteq v$ and the relation $\models_{\mathfrak{U}} \subseteq \mathcal{U} \times P$ is given by the equation $u \models_{\mathfrak{U}} p \iff p \in u \cap P$) is a terminal object in the category of Vietoris models. Throughout this section, we assume that the set of propositional letters P is non-empty and countable. The set P may be finite (in which case, $P = \{p_1, p_2, p_3, ..., p_n\}$, for some $n \in \mathbb{N}$) or countably infinite (in which case, $P = \{p_1, p_2, p_3, ...\}$). We first review the background materials needed to understand the theorems and lemmas of this chapter.

10.1. Preliminary

In this section we define the notions of tautology and deducibility in terms of Kripke structures.

Definition 10.1.1. (Kripke tautology) A modal formula $\varphi \in L_P$ is called a Kripke tautology (in symbol: $\models \varphi$) if for each Kripke model $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ we have $\models_{\mathcal{X}} \varphi$ (i.e., φ is valid in Kripke model $\mathcal{X}$). We denote by KT the set of all Kripke tautologies.

Example 10.1.2. For every φ, $\psi \in L_P$, the following formulas are Kripke tautologies.

1. $(\varphi \longrightarrow \psi) \longleftrightarrow (\neg\varphi \vee \psi)$;
2. $\top \longrightarrow \Box\top$;
3. $\Diamond\neg\varphi \longleftrightarrow \neg\Box\varphi$;

4. $\Box(\varphi \longrightarrow \psi) \longrightarrow \Box\varphi \longrightarrow \Box\psi$;

5. $(\Box\varphi \wedge \Box\psi) \longrightarrow \Box(\varphi \wedge \psi)$.

In the next remark, we give some statements which can be easily proved by using the definition of Kripke tautology and semantics of formulas defined in subsection 6.1.3. For more details we refer to [73].

Remark 10.1.3. According to the definition of Kripke tautology, the following statements hold.

1. If $\models \varphi$ and $\models \psi$ then $\models (\varphi \wedge \psi)$.
2. If $\models \varphi$ and $\models \varphi \longrightarrow \psi$ then $\models \psi$.
3. If $\models \varphi \longrightarrow \psi$ and $\models \psi \longrightarrow \phi$ then $\models \varphi \longrightarrow \phi$.
4. If $\models \varphi \longrightarrow \psi$ and $\models \chi \longrightarrow \phi$ then $\models \varphi \wedge \chi \longrightarrow \psi \wedge \phi$.
5. If $\models \varphi$ then $\models \Box\varphi$.
6. If $\models \Box(\varphi \longrightarrow \psi)$ and $\models \Box\varphi$ then $\models \Box\psi$.
7. If $\models \Box\varphi$ and $\models \Box\psi$ then $\models \Box(\varphi \wedge \psi)$.

Definition 10.1.4. (Kripke deducibility) Suppose $\varphi \in L_P$ and Σ is a subset of L_P. We say φ is deducible from Σ and we write,

$$\Sigma \vdash \varphi$$

iff there is a finite sequence $\varphi_1, \varphi_2, ..., \varphi_n$ of members of Σ such that the formula,

$$(\varphi_1 \wedge \varphi_2 \wedge ... \wedge \varphi_n) \longrightarrow \varphi$$

is a Kripke tautology.

Lemma 10.1.5. *Suppose Σ is a non-empty subset of L_P, then the following statements hold.*

1. $\Sigma \vdash \bot \Longleftrightarrow \forall\varphi \in L_P. \Sigma \vdash \varphi$.
2. $\forall\varphi \in L_P. (\Sigma \vdash \varphi \Longleftrightarrow \Sigma \cup \{\neg\varphi\} \vdash \bot)$.

Proof. Let Σ be a subset of L_P.

(1): Assume $\Sigma \vdash \bot$, then there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\models \bigwedge_{\psi \in \Sigma_0} \psi \longrightarrow \bot$ (i.e., $\bigwedge_{\psi \in \Sigma_0} \psi \longrightarrow \bot$ is a Kripke tautology). Also we know that $\models \bot \longrightarrow \varphi$ for each $\varphi \in L_P$. Then by part (3) in remark 10.1.3, $\models \bigwedge_{\psi \in \Sigma_0} \psi \longrightarrow \varphi$ for each $\varphi \in L_P$. Hence by definition 10.1.4, for each $\varphi \in L_P$ we have $\Sigma \vdash \varphi$. For the converse, set $\varphi := \bot$.

(2): Let $\varphi \in L_P$ be arbitrary.

$$
\begin{aligned}
\Sigma \cup \{\neg\varphi\} \vdash \bot &\iff \exists \Sigma_0 \underset{finite}{\subseteq} \Sigma.\, \Sigma_0 \cup \{\neg\varphi\} \vdash \bot \\
&\iff \exists \Sigma_0 \underset{finite}{\subseteq} \Sigma.\, (\bigwedge_{\psi \in \Sigma_0} \psi) \wedge (\neg\varphi) \vdash \bot \\
&\iff \exists \Sigma_0 \underset{finite}{\subseteq} \Sigma.\, \models \bigwedge_{\psi \in \Sigma_0} \psi \longrightarrow \varphi \\
&\iff \Sigma \vdash \varphi.
\end{aligned}
$$

□

10.2. Kripke-Ultrafilters

Here, we introduce the notions of Kripke-Filters and Kripke-Ultrafilters and some of their properties.

Definition 10.2.1. (Kripke-Filter) A Kripke-Filter on L_P is a subset $F \subseteq L_P$ which has the following properties:

1. $\top \in F$,
2. $\varphi \in F$ *and* $\psi \in F \implies \varphi \wedge \psi \in F$, and
3. $\varphi \in F$ *and* $\models \varphi \longrightarrow \psi \implies \psi \in F$.

Lemma 10.2.2. *For each Kripke-Filter F on L_P and each modal formula $\varphi \in L_P$, the following statement holds:*

$$F \vdash \varphi \iff \varphi \in F.$$

Proof. We have,

$$
\begin{aligned}
F \vdash \varphi &\overset{\text{def } 10.1.4}{\iff} \exists \varphi_1, \ldots, \varphi_n \in F.\, \models \varphi_1 \wedge \ldots \wedge \varphi_n \longrightarrow \varphi \\
&\overset{(2) \text{ in def } 10.2.1}{\iff} \exists \psi \in F.\, \models \psi \longrightarrow \varphi \\
&\overset{(3) \text{ in def } 10.2.1}{\iff} \varphi \in F.
\end{aligned}
$$

□

Lemma 10.2.3. *If $u \subseteq L_P$ is a Kripke-Filter, then*

$$\Box^{-1}u := \{\varphi \in L_P \mid \Box\varphi \in u\}$$

is a Kripke-Filter on L_P.

Proof. Let $u \subseteq L_P$ be a Kripke-Filter. We should prove that $\Box^{-1}u$ satisfies the three conditions in definition 10.2.1.

1. We know that $\top \in u$ and $\models \top \longrightarrow \Box\top$. Then by (3) in definition 10.2.1 we have $\Box\top \in u$. Hence $\top \in \Box^{-1}u$.

2. Suppose φ, $\psi \in \Box^{-1}u$. Then $\Box\varphi$, $\Box\psi \in u$ and consequently $\Box\varphi \wedge \Box\psi \in u$ (u is a Kripke-Filter). Since $(\Box\varphi \wedge \Box\psi) \longrightarrow \Box(\varphi \wedge \psi)$ is a Kripke tautology, by (3) in definition 10.2.1 we have $\Box(\varphi \wedge \psi) \in u$. Therefore $\varphi \wedge \psi \in \Box^{-1}u$.

3. Assume $\varphi \in \Box^{-1}u$ and $\models \varphi \longrightarrow \psi$. So $\Box\varphi \in u$ and $\models \Box\varphi \longrightarrow \Box\psi$, respectively. Then by (3) in definition 10.2.1 we have $\Box\psi \in u$. This gives $\psi \in \Box^{-1}u$.

$\Box$

Definition 10.2.4. (Kripke-Ultrafilter) A Kripke-Filter F is called a Kripke-Ultrafilter on L_P if for every element $\varphi \in L_P$, either $\varphi \in F$ or $\neg\varphi \in F$. We denote by $\mathcal{U}$ the set of all Kripke-Ultrafilters over L_P.

Example 10.2.5. Suppose $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ is a Kripke model. It is easy to see that for each $x \in X$, the set $\|x\| = \{\varphi \in L_P \mid x \models_{\mathcal{X}} \varphi\}$ is a Kripke-Ultrafilter on L_P.

10.3. Kripke-Ultrafilter lemma

The idea for the proof of the following lemma is borrowed from the proof of Lindenbaum's lemma in Zalta [73]. Before starting this subsection, recall that $\Sigma \subseteq L_P$ is called a proper subset if $\Sigma \neq L_P$.

Lemma 10.3.1. ***(Kripke-Ultrafilter lemma)*** *Let $\Sigma \subseteq L_P$ be a non-empty proper subset of L_P such that $\Sigma \nvdash \bot$. Then there is a Kripke-Ultrafilter F containing Σ.*

Proof. Let $\varphi_1, \varphi_2, \ldots$ be a listing of all modal formulas in L_P. We define the set F as the union of an infinite sequence of sets $F_0, F_1, \ldots$ of modal formulas, as follows:

$$\begin{aligned} F_0 &:= \Sigma \\ F_{n+1} &:= \begin{cases} F_n \cup \{\varphi_n\} & \text{if } F_n \vdash \varphi_n \\ F_n \cup \{\neg\varphi_n\} & \text{else} \end{cases} \\ F &:= \bigcup_{n \geq 0} F_n. \end{aligned} \tag{10.3.1}$$

Notice that by this construction we have:

$$F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n \subseteq F_{n+1} \subseteq \ldots \tag{10.3.2}$$

Moreover, the sets F and F_n $(n \in \mathbb{N})$ satisfy the following properties:
***Property* 1 :** $F_n \nvdash \bot$ for each $n \geq 0$. We prove this claim by induction over n. For $n = 0$ we have $F_0 \nvdash \bot$ (because by assumption $\Sigma \nvdash \bot$). Now, let $F_n \nvdash \bot$ for an arbitrary $n \geq 0$. We need to show that $F_{n+1} \nvdash \bot$. We show this by contradiction. So assume $F_{n+1} \vdash \bot$. Then

case 1: $F_n \vdash \varphi_n$, so $F_{n+1} = F_n \cup \{\varphi_n\} \vdash \bot$. Then $F_n \vdash \neg\varphi_n$ (by part (2) of lemma 10.1.5). Thus $F_n \vdash \bot$ which is a contradiction with the induction hypothesis.

case 2: $F_n \nvdash \varphi_n$, so $F_{n+1} = F_n \cup \{\neg\varphi_n\} \nvdash \bot$ (by part (2) of the lemma 10.1.5). This gives a contradiction with assumption $F_{n+1} \vdash \bot$.

***Property* 2 :** $F \nvdash \bot$. To see this, assume $F \vdash \bot$, then by definition 10.1.4, we have

$$\exists \Sigma_0 \underset{finite}{\subseteq} F.\, \Sigma_0 \vdash \bot \tag{10.3.3}$$

We can find an element $n \in \mathbb{N}$ such that $\Sigma_0 \subseteq F_n$ (since Σ_0 is finite, we can find a finite subset $K \subseteq \mathbb{N}$ such that $\Sigma_0 \subseteq \bigcup_{k \in K} F_k$. Let n be the largest element in K. Then by equation 10.3.2, we conclude that $\Sigma_0 \subseteq F_n$). Then, by 10.3.3, we obtain that $F_n \vdash \bot$. This is a contradiction with property 1.
Now, it suffices to show that F is a Kripke-Ultrafilter on L_P. We prove this claim step by step.
***Step* 1** : By the construction of F and claim 2, for each modal formula φ we conclude that either $\varphi \in F$ or $\neg\varphi \in F$.
***Step* 2** : By step 1 and property 2, we obtain that $\top \in F$.
***Step* 3 :** Let $\varphi, \psi \in F$, we have to show that $\varphi \wedge \psi \in F$. It is enough to show that $F_n \nvdash \neg(\varphi \wedge \psi)$ for each $n \geq 0$. We prove this by contradiction. Suppose $F_n \vdash \neg(\varphi \wedge \psi)$ for some $n \geq 0$. Then $F \vdash \neg(\varphi \wedge \psi)$. Also we have $F \vdash \varphi \wedge \psi$ (because $\varphi, \psi \in F$). Hence $F \vdash \bot$ and this is a contradiction with property 2.
***Step* 4 :** It must be proven that if $\varphi \in F$ and $\models \varphi \longrightarrow \psi$, then $\psi \in F$. We need to show that $\neg\psi \notin F$. Assume $\neg\psi \in F$, then $F \vdash \neg\psi$. On the other hand since $\varphi \in F$, we have $\varphi \in F_n$ (for some $n \geq 0$). So from $\varphi \in F_n$ and $\models \varphi \longrightarrow \psi$, we have $F_n \vdash \psi$ and consequently $F \vdash \psi$. Hence $F \vdash \psi \wedge \neg\psi$, i.e., $F \vdash \bot$, but this is a contradiction with property 2.

□

Remark 10.3.2. For each subset $\Sigma \subseteq L_P$ with $\Sigma \nvdash \bot$, we denote by $\mathcal{U}_\Sigma$ the set of all Kripke-Ultrafilters over L_P which contains Σ. According to the lemma 10.3.1, $\mathcal{U}_\Sigma \neq \emptyset$.

Now, as a direct corollary of lemma 10.3.1, we have:

Corollary 10.3.3. *For each non-empty proper subset Σ of L_P, the following statement holds:*

$$\Sigma \vdash \bot \iff \mathcal{U}_\Sigma = \emptyset.$$

Proof. Suppose $\Sigma \vdash \bot$. If $\mathcal{U}_\Sigma \neq \emptyset$, then there is a Kripke-Ultrafilter $u \subseteq L_P$ such that $\Sigma \subseteq u$. Then $u \vdash \bot$ and consequently $\bot \in u$ (by lemma 10.2.2). This gives a contradiction, because u is a Kripke-Ultrafilter. Conversely, let $\mathcal{U}_\Sigma = \emptyset$. If $\Sigma \nvdash \bot$, then by lemma 10.3.1, there is a Kripke-Ultrafilter $u \subseteq L_P$ with $\Sigma \subseteq u$. But it is a contradiction with the assumption $\mathcal{U}_\Sigma = \emptyset$. □

Corollary 10.3.4. *Let φ be an element of L_P, and Σ a non-empty proper subset of L_P such that $\Sigma \nvdash \varphi$. Then there is a Kripke-Ultrafilter u with $\Sigma \cup \{\neg\varphi\} \subseteq u$.*

Proof. Suppose $\Sigma \nvdash \varphi$, then due to lemma 10.1.5, $\Sigma \cup \{\neg\varphi\} \nvdash \bot$. So by lemma 10.3.1, there is an Kripke-Ultrafilter u with $\Sigma \cup \{\neg\varphi\} \subseteq u$. □

10.4. Kripke-Ultrafilter model

In the sequel, we will introduce a Kripke model constructed over $\mathcal{U}$ (the set of all Kripke-Ultrafilters over L_P). We call this Kripke model as **Kripke-Ultrafilter model**. The original motivation to define this construction comes from the notion of the canonical models defined for normal logics, see [13], [14], [19] and [61]. By proving Truth lemma for the Kripke-Ultrafilter model, we will show that the modal equivalence relation on this structure is same as the equality relation between subsets.

Definition 10.4.1. The Kripke-Ultrafilter model is a triple $\mathfrak{U} = (\mathcal{U}, \mathcal{R}_\mathfrak{U}, \models_\mathfrak{U})$ where,

- $\mathcal{U}$ is the collection of all Kripke-Ultrafilters over L_P.
- $\mathcal{R}_\mathfrak{U} \subseteq \mathcal{U} \times \mathcal{U}$ is a binary relation defined as

$$u\,\mathcal{R}_\mathfrak{U}\,v \iff \forall \varphi \in L_P. \Box\varphi \in u \implies \varphi \in v$$

  for each u, $v \in \mathcal{U}$.
- $\models_\mathfrak{U} \subseteq \mathcal{U} \times P$ is a binary relation defined by

$$u \models_\mathfrak{U} p \iff p \in u \cap P$$

  for each $u \in \mathcal{U}$ and $p \in P$.

We should recall that $\mathcal{R}_\mathfrak{U}(u) = \{v \in X \mid u\,\mathcal{R}_\mathfrak{U}\,v\}$ for each Kripke-Ultrafilter $u \in \mathcal{U}$. In the sequel, we usually write $v \in \mathcal{R}_\mathfrak{U}(u)$ instead of $u\,\mathcal{R}_\mathfrak{U}\,v$.

Remark 10.4.2. According to the definition of $\mathcal{R}_{\mathfrak{U}}$, one can see that

$$v \in \mathcal{R}_{\mathfrak{U}}(u) \iff \Box^{-1}u \subseteq v$$

where $\Box^{-1}u = \{\varphi \in L_P \mid \Box\varphi \in u\}$.

Lemma 10.4.3. *Given $u \in \mathcal{U}$, then for each $\varphi \in L_P$,*

$$\varphi \in \Box^{-1}u \iff \forall v \in \mathcal{R}_{\mathfrak{U}}(u).\, \varphi \in v.$$

Proof. Fix $\varphi \in L_P$:

($\Longrightarrow$): It will be obtained by remark 10.4.2.

($\Longleftarrow$): Suppose $\varphi \notin \Box^{-1}u$. Since by lemma 10.2.3, $\Box^{-1}u$ is a Kripke-Filter on L_P, it follows from lemma 10.2.2 that $\Box^{-1}u \nvdash \varphi$ and then by corollary 10.3.4 there is a Kripke-Ultrafilter v such that $\Box^{-1}u \cup \{\neg\varphi\} \subseteq v$. By remark 10.4.2, we have $v \in R_U(u)$, then by assumption $\varphi \in v$ and this gives a contradiction (because v is a Kripke-Ultrafilter) .

$\Box$

Truth lemma

Lemma 10.4.4. ***(Truth lemma)*** *For each formula $\varphi \in L_P$ and for every Kripke-Ultrafilter $u \in \mathcal{U}$:*

$$\varphi \in u \iff u \models_{\mathfrak{U}} \varphi$$

Proof. By induction over the construction of the modal formulas:
Here we ignore the inductive steps for the Boolean operations.
Base case **:** For any $p \in P$, and for any $u \in \mathcal{U}$:

$$p \in u \iff u \models_{\mathfrak{U}} p$$

Inductive step **:** It remains to show that the claim holds also for $\Box\varphi$, $\Diamond\varphi$.
Inductive hypothesis **:** Suppose the claim holds for the formula φ. It means for every Kripke-ultrafilter $u \in \mathcal{U}$, we have:

$$\varphi \in u \iff u \models_{\mathfrak{U}} \varphi$$

Case $\Box\varphi$:

$$\begin{aligned}
\Box\varphi \in u &\overset{Definition\,of\,\Box^{-1}u}{\iff} \varphi \in \Box^{-1}u \\
&\overset{lemma\ 10.4.3}{\iff} \forall v \in \mathcal{U}.\, v \in \mathcal{R}_{\mathfrak{U}}(u) \implies \varphi \in v \\
&\overset{Inductive\,hypothesis}{\iff} \forall v \in \mathcal{U}.\, v \in \mathcal{R}_{\mathfrak{U}}(u) \implies v \models_{\mathfrak{U}} \varphi \\
&\overset{\text{semantic of }\Box}{\iff} u \models_{\mathfrak{U}} \Box\varphi
\end{aligned}$$

Regarding $\Diamond\varphi$, notice that $\Diamond\varphi = \neg\Box\neg\varphi$.

$\Box$

Corollary 10.4.5. *Let* $u \in \mathcal{U}$, then *for each* $\varphi \in L_P$*,*

$$\Diamond\varphi \in u \iff \exists v \in \mathcal{R}_{\mathfrak{U}}(u).\, \varphi \in v.$$

Proof. We have

$$\begin{aligned}
\Diamond\varphi \in u &\overset{\text{Truth lemma}}{\iff} u \models_{\mathfrak{U}} \Diamond\varphi \\
&\overset{\text{semantic of }\Diamond}{\iff} \exists v \in \mathcal{R}_{\mathfrak{U}}(u).\, \varphi \in v.
\end{aligned}$$

$\Box$

Corollary 10.4.6. *Suppose* $u, v \in \mathcal{U}$*, then the following statements hold:*

1. $\mathcal{R}_{\mathfrak{U}}(u) = \bigcap_{\Box\varphi \in u} \|\varphi\|^{\mathfrak{U}}$;

2. $u \approx_{\mathfrak{U}} v \iff u = v$ (where $\approx_{\mathfrak{U}}$ is the modal equivalence relation on $\mathfrak{U}$).

Proof. (1) For each Kripke-Ultrafilter $v \in \mathcal{R}_{\mathfrak{U}}(u)$ we have

$$\begin{aligned}
v \in \mathcal{R}_{\mathfrak{U}}(u) &\overset{remark\ 10.4.2}{\iff} \{\varphi \in L_P \mid \Box\varphi \in u\} \subseteq v \\
&\overset{Truth\,lemma}{\iff} v \in \bigcap_{\Box\varphi \in u} \|\varphi\|^{\mathfrak{U}}
\end{aligned}$$

(2) It follows immediately from the Truth lemma. $\Box$

10.5. $\mathfrak{U}$-Modal model as a Vietoris Model

In this part, we try to prove that $\mathfrak{U}_\Box = (\mathcal{U}_\Box, \mathcal{R}_\mathfrak{U}, \models_\mathfrak{U})$, the $\mathfrak{U}$-modal model (see definition 9.3.1) induced by the Kripke-Ultrafilter model $\mathfrak{U} = (\mathcal{U}, \mathcal{R}_\mathfrak{U}, \models_\mathfrak{U})$, is a Vietoris model. We check this claim in three steps:

Step 1: Compactness of $\mathcal{R}_\mathfrak{U}$, i.e.

- $\forall u \in \mathcal{U}.\, \mathcal{R}_\mathfrak{U}(u)$ is a compact subset of $\mathcal{U}_\Box$.

Step 2: Continuity of $\mathcal{R}_\mathfrak{U}$, i.e.

- $\forall O \underset{open}{\subseteq} \mathcal{U}_\Box.\, \langle \mathcal{R}_\mathfrak{U} \rangle (U)$ is an open subset of $\mathcal{U}_\Box$, and
- $\forall O \underset{open}{\subseteq} \mathcal{U}_\Box.\, [\mathcal{R}_\mathfrak{U}](U)$ is an open subset of $\mathcal{U}_\Box$.

Step 3: Continuity of $\models_\mathfrak{U}$, i.e.

- $\forall p \in P.\, \|p\|^\mathfrak{U}$ is a clopen subset of $\mathcal{U}_\Box$.

Compactness of $\mathcal{R}_\mathfrak{U}$

In the first, we need to prove the following lemma:

Lemma 10.5.1. *$\mathcal{U}_\Box$ is a compact space.*

Proof. It suffices to show that every family of closed subsets in $\mathcal{U}_\Box$ which satisfies the $F.I.P$ property has a non-empty intersection (see theorem 1.6.4). Each closed subset of $\mathcal{U}_\Box$ is of the form $\bigcap_{\varphi\in\Sigma} \|\varphi\|^\mathfrak{U}$ for some $\Sigma \subseteq L_P$ (see definition 9.3.1). Hence to show that $\mathcal{U}_\Box$ is compact, we need to prove for every $\Sigma \subseteq L_P$ if $\bigcap_{\varphi\in\Sigma} \|\varphi\|^\mathfrak{U} = \emptyset$ then there is a finite subset $\Sigma_0 \subseteq \Sigma$ such that $\bigcap_{\varphi\in\Sigma_0} \|\varphi\|^\mathfrak{U} = \emptyset$. So,

$$
\begin{aligned}
\bigcap_{\varphi\in\Sigma} \|\varphi\|^\mathfrak{U} = \emptyset \quad &\overset{\text{corollary } 10.3.3}{\Longrightarrow} \quad \Sigma \vdash \bot \\
&\overset{\text{Def} \vdash}{\Longrightarrow} \quad \exists \Sigma_0 \underset{finite}{\subseteq} \Sigma.\, \Sigma_0 \vdash \bot \\
&\overset{\text{corollary } 10.3.3}{\Longrightarrow} \quad \bigcap_{\varphi\in\Sigma_0} \|\varphi\|^\mathfrak{U} = \emptyset
\end{aligned}
$$

$\Box$

Lemma 10.5.2. *Assume $u \in \mathcal{U}$, then $\mathcal{R}_{\mathfrak{U}}(u)$ is a compact subset of $\mathcal{U}_\Box$.*

Proof. It is easy to check that $\mathcal{U}_\Box$ is a Hausdorff space (if $u, v \in \mathcal{U}$ are two Kripke-ultrafilters with $u \neq v$, then there is a modal formula $\varphi \in L_P$ such that $\varphi \in u$ and $\neg\varphi \in v$, consequently $u \in \|\varphi\|^{\mathfrak{U}}$ and $v \in \|\neg\varphi\|^{\mathfrak{U}}$ and we know that $\|\varphi\|^{\mathfrak{U}} \cap \|\neg\varphi\|^{\mathfrak{U}} \neq \emptyset$). By lemma 10.5.1, $\mathcal{U}_\Box$ is a compact Hausdorff space. Since in a compact Hausdorff space a subset is compact if and only if it is closed (see theorem 1.6.6), it suffices to show that for each $u \in \mathcal{U}$ the set $\mathcal{R}_{\mathfrak{U}}(u)$ is closed in $\mathcal{U}_\Box$. According to part (1) of corollary 10.4.6, $\mathcal{R}_{\mathfrak{U}}(u)$ is the intersection of a family of closed subsets of $\mathcal{U}_\Box$. So it is closed. $\Box$

Continuity of $\mathcal{R}_{\mathfrak{U}}$

Lemma 10.5.3. *Given an open subset O of $\mathcal{U}_\Box$. The following statements:*

1. $\langle\mathcal{R}_{\mathfrak{U}}\rangle(O)$ is an open subset of $\mathcal{U}_\Box$.

2. $[\mathcal{R}_{\mathfrak{U}}](O)$ is an open subset of $\mathcal{U}_\Box$.

Proof. Suppose O is an open subset of $\mathcal{U}_\Box$, then $O = \bigcup_{\varphi\in\Sigma} \|\varphi\|^{\mathfrak{U}}$ for some $\Sigma \subseteq L_P$. So:

$$\begin{aligned}
\langle\mathcal{R}_{\mathfrak{U}}\rangle(O) &= \{u \in \mathcal{U} \mid \mathcal{R}_{\mathfrak{U}}(u) \cap O \neq \emptyset\} \\
&= \{u \in \mathcal{U} \mid \mathcal{R}_{\mathfrak{U}}(u) \cap (\bigcup_{\varphi\in\Sigma} \|\varphi\|^{\mathfrak{U}}) \neq \emptyset\} \\
&= \{u \in \mathcal{U} \mid \exists v \in \mathcal{R}_{\mathfrak{U}}(u).\, v \in \bigcup_{\varphi\in\Sigma} \|\varphi\|^{\mathfrak{U}}\} \\
&= \{u \in \mathcal{U} \mid \exists v \in \mathcal{R}_{\mathfrak{U}}(u).\, \exists\varphi \in \Sigma.\, v \in \|\varphi\|^{\mathfrak{U}}\} \\
&\overset{\text{Truth lemma}}{=} \{u \in \mathcal{U} \mid \exists v \in \mathcal{R}_{\mathfrak{U}}(u).\, \exists\varphi \in \Sigma.\, \varphi \in v\} \\
&\overset{\textit{corollary } 10.4.5}{=} \{u \in \mathcal{U} \mid \exists\varphi \in \Sigma.\, \Diamond\varphi \in u\} \\
&= \bigcup_{\varphi\in\Sigma} \|\Diamond\varphi\|^{\mathfrak{U}}
\end{aligned}$$

and

$$
\begin{aligned}
[\mathcal{R}_{\mathfrak{U}}](O) &= \{u \in \mathcal{U} \mid \mathcal{R}_{\mathfrak{U}}(u) \subseteq O\} \\
&= \{u \in \mathcal{U} \mid \mathcal{R}_{\mathfrak{U}}(u) \subseteq \bigcup_{\varphi \in \Sigma} \|\varphi\|^{\mathfrak{U}}\} \\
&\overset{\text{lemma } 10.5.2}{=} \{u \in \mathcal{U} \mid \exists \Sigma_0 \underset{finite}{\subseteq} \Sigma. \mathcal{R}_{\mathfrak{U}}(u) \subseteq \bigcup_{\varphi \in \Sigma_0} \|\varphi\|^{\mathfrak{U}}\} \\
&= \{u \in \mathcal{U} \mid \exists \Sigma_0 \underset{finite}{\subseteq} \Sigma. \mathcal{R}_{\mathfrak{U}}(u) \subseteq \| \bigvee_{\varphi \in \Sigma_0} \varphi\|^{\mathfrak{U}}\} \\
&= \{u \in \mathcal{U} \mid \exists \Sigma_0 \underset{finite}{\subseteq} \Sigma. \Box(\bigvee_{\varphi \in \Sigma_0} \varphi) \in u\} \\
&= \bigcup_{\Sigma_0 \in \mathbb{P}_f(\Sigma)} \| \Box(\bigvee_{\varphi \in \Sigma_0} \varphi)\|^{\mathfrak{U}}
\end{aligned}
$$

Regarding the trivial open sets $\mathcal{U}_\Box$ and $\emptyset$, we should check $\langle \mathcal{R}_{\mathfrak{U}} \rangle(\|\top\|^{\mathfrak{U}})$ and $\langle \mathcal{R}_{\mathfrak{U}} \rangle(\|\bot\|^{\mathfrak{U}})$, respectively. □

Continuity of $\models_{\mathfrak{U}}$

Lemma 10.5.4. *For each $p \in P$, the set $\|p\|^{\mathfrak{U}}$ is a clopen subset of $\mathcal{U}_\Box$.*

Proof. Notice that $\mathcal{U}_\Box$ is the set $\mathcal{U}$ along with the $\mathcal{U}$-Modal topology (i.e., the topology generated by a base which consists of all clopens as $\|\varphi\|^{\mathfrak{U}} = \{u \in \mathcal{U} \mid u \models_{\mathfrak{U}} \varphi\}$ where $\varphi \in L_P$, see definition 9.3.1). Then for each $p \in P$ the set $\|p\|^{\mathfrak{U}}$ is a clopen subset of $\mathcal{U}_\Box$. □

10.6. Homomorphism lemmas

In the previous subsection we already saw that $\mathfrak{U}_\Box = (\mathcal{U}_\Box, \mathcal{R}_{\mathfrak{U}}, \models_{\mathfrak{U}})$ i.e., the $\mathfrak{U}$-Modal model induced by the Kripke model $\mathfrak{U}$, is a Vietoris model. We will now prove something rather stronger, namely that $\mathfrak{U}_\Box$ is a terminal object in the category of Vietoris models.

Lemma 10.6.1. *Let $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ be an arbitrary Vietoris model. Then the map $!_{\mathcal{X}} : X \longrightarrow \mathcal{U}_\Box$ defined by $!_{\mathcal{X}}(x) := \|x\|$ (for each $x \in X$) is the unique Vietoris homomorphism from $\mathcal{X}$ to $\mathfrak{U}_\Box$.*

Proof. Regarding the continuity of $!_{\mathcal{X}}$, since for every $\varphi \in L_P$ the set $\|\varphi\|^{\mathcal{X}}$ is a clopen subset of X (see lemma 9.0.4), $!_{\mathcal{X}}$ is continuous. To prove that $!_{\mathcal{X}}$ is a Kripke homomorphism between the underlying Kripke structures of $\mathcal{X}$ and $\mathfrak{U}_\Box$, we need to show that the set

$$G(!_{\mathcal{X}}) = \{(x, \|x\|) \mid x \in X\}$$

(i.e., the graph of $!_{\mathcal{X}}$) is a Kripke bisimulation.

1. For each $x \in X$ and $p \in P$, we have

$$
\begin{aligned}
x \models_{\mathcal{X}} p &\iff p \in \|x\| \\
&\iff p \in \|x\| \cap P \\
&\overset{\text{by the definition of } \models_{\mathfrak{U}} \text{ (see, definition 10.4.1)}}{\iff} \|x\| \models_{\mathfrak{U}} p.
\end{aligned}
$$

2. It remains to check that for each $x \in X$ the following statements hold:
   a) $\forall y \in R_{\mathcal{X}}(x).\, \|y\| \in \mathcal{R}_{\mathfrak{U}}(\|x\|)$;
   b) $\forall v \in \mathcal{R}_{\mathfrak{U}}(\|x\|).\, \exists y \in R_{\mathcal{X}}(x).\, !_{\mathcal{X}}(y) = v$;

(a): Suppose $y \in R_{\mathcal{X}}(x)$. Then for each modal formula $\psi \in L_P$ such that $x \models \Box\psi$ we have $y \models \psi$. So we have $\{\psi \mid \Box\psi \in \|x\|\} \subseteq \|y\|$. Hence by remark 10.4.2, $\|x\| \,\mathcal{R}_{\mathfrak{U}}\, \|y\|$ and consequently $\|y\| \in \mathcal{R}_{\mathfrak{U}}(\|x\|)$.

(b): Assume $v \in \mathcal{R}_{\mathfrak{U}}(\|x\|)$. We have to find an element $y \in R_{\mathcal{X}}(x)$ such that $!_{\mathcal{X}}(y) = \|y\| = v$. Let $v \neq \|y\|$ for each $y \in R_{\mathcal{X}}(x)$. Hence for each $y \in R_{\mathcal{X}}(x)$, there is $\varphi_y \in L_P$ such that $\varphi_y \in v$ and $y \models_{\mathcal{X}} \neg\varphi_y$ and then $R_{\mathcal{X}}(x) \subseteq \bigcup\limits_{y \in R_{\mathcal{X}}(x)} \|\neg\varphi_y\|^{\mathcal{X}}$. Since for any $y \in R_{\mathcal{X}}(x)$, the set $\|\neg\varphi_y\|^{\mathcal{X}}$ is clopen in X and since $R_{\mathcal{X}}(x)$ is compact, there are $y_1, ..., y_n \in R_{\mathcal{X}}(x)$ such that $R_{\mathcal{X}}(x) \subseteq \bigcup\limits_{i=1}^{n} \|\neg\varphi_{y_i}\|^{\mathcal{X}}$. So for each $y \in R_{\mathcal{X}}(x)$, there exists $i \leqslant n$ such that $y \models_{\mathcal{X}} \neg\varphi_{y_i}$. Thus, for every $y \in R_{\mathcal{X}}(x)$, we have $y \models_{\mathcal{X}} \bigvee\limits_{i=1}^{n} \neg\varphi_{y_i}$. Hence $x \models_{\mathcal{X}} \Box(\bigvee\limits_{i=1}^{n} \neg\varphi_{y_i})$. Therefore, $\bigvee\limits_{i=1}^{n} \neg\varphi_{y_i} \in v$. So there is $i \leqslant n$ such that $\neg\varphi_{y_i} \in v$. This is a contradiction with $\varphi_{y_i} \in v$ for each $i \leqslant n$.

It remains to show that $!_{\mathcal{X}}$ is unique. Suppose $g : X \longrightarrow \mathcal{U}_{\Box}$ is another Vietoris homomorphism from $\mathcal{X}$ to $\mathfrak{U}_{\Box}$. It suffices to prove that $!_{\mathcal{X}}(x) = g(x)$, for each $x \in X$. Since Kripke bisimilar elements are modally equivalent, for each modal formula $\varphi \in L_P$ we have

$$!_{\mathcal{X}}(x) \models_{\mathfrak{U}} \varphi \iff x \models_{\mathcal{X}} \varphi \iff g(x) \models_{\mathfrak{U}} \varphi$$

Hence by lemma 10.4.4, for every formula $\varphi \in L_P$ we have

$$\varphi \in !_{\mathcal{X}}(x) \iff \varphi \in g(x).$$

$\Box$

Remark 10.6.2. Let $\mathcal{X} = (X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \models_{\mathcal{Y}})$ be two Vietoris models and let $x \in X$ and $y \in Y$. We say that x and y are behaviorally equivalent (in symbols $x \nabla_{\mathcal{X},\mathcal{Y}} y$), if there exists an Vietoris model $\mathcal{Z} = (Z, \gamma, \models_{\mathcal{Z}})$ and Vietoris homomorphisms $f : X \longrightarrow Z$ and $g : Y \longrightarrow Z$ such that $f(x) = g(y)$. According to lemma 10.6.1, we can see that two elements x and y in the Vietoris models $\mathcal{X}$ and $\mathcal{Y}$ are behaviorally equivalent iff $!_{\mathcal{X}}(x) = !_{\mathcal{Y}}(y)$. Therefore x and y are behaviorally equivalent iff they are modally equivalent. This remark is not a new result and has been discussed by Venema, Fontaine and Bezhanishvili for descriptive models (see [11]), and also by Kurz and Pattinson for canonical models (see [46]).

11. Vietoris bisimulation

As it has been shown in example 7.4.12, in the category of Vietoris coalgebras the supremum of a family of A-M bisimulations need not be an A-M bisimulation. As a cosequence, the largest A-M bisimulation may not exist. Besides, in example 7.5.9, we can see that A-M bisimilarity is different from modal equivalence and consequently different from behavioral equivalence. To overcome these shortcomings of A-M bisimilarity, in this section we want to study a different concept of bisimilarity between Vietoris models called Vietoris bisimulation. The notion of Vietoris bisimulation was introduced for the first time by Venema, Fontaine and Bezhanishvili in [11], between descriptive models. They proved that Vietoris bisimilarity coincides with Kripke bisimilarity, with behavioral equivalence and with modal equivalence, but not with A-M bisimilarity. In this chapter, we try to check these results for Vietoris structures. Morover, we try to find a connection between Vietoris homomorphisms and Vietoris bisimulations.
Without loss of generality, in this part we will make use of lemma 9.2.5 and we present Vietoris models as triples $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ where $R_{\mathcal{X}}$ and $\vartheta_{\mathcal{X}}$ are continuous maps from X to $\mathbb{V}(X)$ and $\mathbb{P}(P)$, respectively (i.e., $R_{\mathcal{X}} : X \longrightarrow \mathbb{V}(X)$ and $\vartheta_{\mathcal{X}} : X \longrightarrow \mathbb{P}(X)$).

11.1. The concept of Vietoris bisimulation

Definition 11.1.1. (Vietoris bisimulation) Let $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vartheta_{\mathcal{Y}})$ be Vietoris models and suppose that $B \subseteq X \times Y$. We say that B is a Vietoris bisimulation if B is a closed set in the product topology and a Kripke bisimulation of the underlying Kripke models.
We say that two points $x \in X$ and $y \in Y$ are Vietoris bisimilar if there is a Vietoris bisimulation B between Vietoris models $\mathcal{X}$ and $\mathcal{Y}$ with $(x, y) \in B$. Clearly Vietoris bisimilar elements are Kripke bisimilar.

11.2. Some properties of Vietoris bisimulations

Lemma 11.2.1. *Let $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vartheta_{\mathcal{Y}})$ be Vietoris models.*

1. The empty relation $\emptyset \subseteq X \times Y$ is a Vietoris bisimulation.

2. The converse of a Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Y}$ is a Vietoris bisimulation between $\mathcal{Y}$ and $\mathcal{X}$.

Proof. Given Vietoris models $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vartheta_{\mathcal{Y}})$.
(1) This is trivial.
(2) Suppose B is a Vietoris bisimulation between Vietoris models $\mathcal{X}$ and $\mathcal{Y}$. So B is closed in $X \times Y$ and it is a Kripke bisimulation between the underlying Kripke models. By lemma 6.1.11, B^{-1} is also a Kripke bisimulation between the underlying Kripke models, so it is enough to show that B^{-1} is closed. Due to lemma 1.7.13, it is clear. □

In general the following statements do not hold:

- The relation composition of two Vietoris bisimulations is a Vietoris bisimulation.
- The diagonal $\triangle_X$ is a Vietoris bisimulation.

To find some evidences for these negative points of the Vietoris bisimulations, we start from the second one. We can see that for each topological apace X, the diagonal $\triangle_X$ is closed in $X \times X$ iff X is an Hausdorff space (see lemma 1.5.5). The following lemma shows that in the category of Vietoris models over *HTop* (the category of Hausdorff spaces with continuous maps), the diagonal $\triangle_X$ is always a Vietoris bisimulations.

Lemma 11.2.2. *Let $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ be a Vietoris model, then the diagonal $\triangle_X$ is a Vietoris bisimulation iff X is a Hausdorff space.*

Proof. This is concluded immediately from corollary 1.5.5 and lemma 6.1.11. □

Corollary 11.2.3. *Let $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ be a Vietoris model in which X is a compact Hausdorff space. Then the diagonal $\triangle_X$ is a Vietoris bisimulation.*

To show that in general the relation composition of two Vietoris bisimulations need not be a Vietoris bisimulation, in the next example we show that the composition of two closed relations between non-compact spaces need not be closed.

Example 11.2.4. Consider the set of real numbers $\mathbb{R}$ with the standard topology and the set of natural numbers $\mathbb{N}^+$ with the discrete topology. Define $\alpha : \mathbb{R} \longrightarrow \mathbb{V}(\mathbb{R})$ as $\alpha(x) := \{x\}$.The structure α is continuous. Also define $\beta : \mathbb{N}^+ \longrightarrow \mathbb{V}(\mathbb{N}^+)$ by $\beta(n) := \{n\}$. The structure β is continuous, too. Then $\mathcal{X} := (\mathbb{R}, \alpha)$ and $\mathcal{Y} := (\mathbb{N}^+, \beta)$ are Vietoris frames. Let $R = \{(\frac{1}{n}, n) \mid n \in \mathbb{N}^+\}$ be a binary relation between $\mathbb{R}$ and $\mathbb{N}^+$. Define $\gamma : R \longrightarrow \mathbb{V}(R)$ as $\gamma(x, y) := (\alpha(x), \beta(y))$ for each $(x, y) \in R$. It is easy to see that (R, γ) is a Kripke bisimulation between corresponding underlying Kriple frames to $\mathcal{X}$ and $\mathcal{Y}$. From example 1.7.14, we know that the binary relation R is closed in $\mathbb{R} \times \mathbb{N}^+$. Hence R is a Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. Then by lemma 11.2.1, R^{-1} is a Vietoris bisimulation between $\mathcal{Y}$ and $\mathcal{X}$. Now, consider the relation composition $R \circ R^{-1} = \{(\frac{1}{n}, \frac{1}{n}) \mid n \in \mathbb{N}^+\}$. By example 1.7.14, $R \circ R^{-1}$ is not a closed subset of $\mathbb{R} \times \mathbb{R}$ and consequently it is not a Vietoris bisimulation.

The next lemma shows that in the category of Vietoris models over compact spaces the compositions of two Vietoris bisimulations is also a Vietoris bisimulation.

Lemma 11.2.5. *Let $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$, $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vartheta_{\mathcal{Y}})$ and $\mathcal{Z} = (Z, R_{\mathcal{Z}}, \vartheta_Z)$ be Vietoris models such that X, Y and Z are compact spaces. If R_1 and R_2 are Vietoris bisimulations between Vietoris models $\mathcal{X}$ and $\mathcal{Y}$ and Vietoris models $\mathcal{Y}$ and $\mathcal{Z}$, respectively, then their relation composition $R_1 \circ R_2$ is also a Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Z}$.*

Proof. We know that $R_1 \circ R_2$ is a Kripke bisimulation between the underlying Kripke models of $\mathcal{X}$ and $\mathcal{Z}$. Also, by lemma 1.7.15, $R_1 \circ R_2$ is a closed subset of $X \times Z$.

□

Corollary 11.2.6. *Let $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$, $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vartheta_{\mathcal{Y}})$ and $\mathcal{Z} = (Z, R_{\mathcal{Z}}, \vartheta_Z)$ be Vietoris models such that X, Y and Z are compact Hausdorff spaces. If R_1 and R_2 are Vietoris bisimulations between Vietoris models $\mathcal{X}$ and $\mathcal{Y}$ and Vietoris models $\mathcal{Y}$ and $\mathcal{Z}$, respectively, then their relation composition $R_1 \circ R_2$ is also a Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Z}$.*

11.3. Vietoris bisimulations and Vietoris homomorphisms

Our plan in this part is to find some characterizations of Vietoris homomorphisms and Vietoris bisimulations. In order to achieve this goal we generalize definition 6.1.16 and also we prove lemma 6.1.19 (originally stated by Rutten in [62] for Kripke models) for Vietoris homomorphisms and Vietoris bisimulations between Vietoris models over compact Hausdorff spaces (see theorems 11.3.1 and 11.3.2).

Theorem 11.3.1. ***(Characteristic Theorem for Vietoris homomorphism)*** *Let $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vartheta_{\mathcal{Y}})$ be Vietoris models in which the underlying spaces X and Y are compact Hausdorff spaces. Then a map $f : X \longrightarrow Y$ is a Vietoris homomorphism between Vietoris models $\mathcal{X}$ and $\mathcal{Y}$ if and only if its graph*

$$G(f) := \{(x, f(x)) \mid x \in X\}$$

is a Vietoris bisimulation between Vietoris models $\mathcal{X}$ and $\mathcal{Y}$.

Proof. Indeed

$$
\begin{gathered}
f \text{ is a Vietoris homomorphism} \\
\overset{\text{definition } 9.1.1}{\Longleftrightarrow} \\
f \text{ is continuous and a Kripke homomorphism between the underlying Kripke models} \\
\overset{\text{corollary } 1.6.10 \text{ and definition } 6.1.16}{\Longleftrightarrow} \\
G(f) \text{ is closed and a Kripke bisimulation between the underlying Kripke models} \\
\overset{\text{definition } 11.1.1}{\Longleftrightarrow} \\
G(f) \text{ is a Vietoris bisimulation}
\end{gathered}
$$

□

Notice that the conditions "being Hausdorff" and "compactness" for the space Y play the key roles to prove theorem 11.3.1. For more details see remarks 1.5.4 and 1.6.9.

Theorem 11.3.2. ***(Canonical Vietoris bisimulation Theorem)*** *Let $\mathcal{X} = (X, R, \vartheta_X)$, $\mathcal{Y} = (Y, S, \vartheta_Y)$ and $\mathcal{Z} = (Y, T, \vartheta_Z)$ be Vietoris models such that X, Y and Z are compact Hausdorff spaces. If $\varphi_X : Z \longrightarrow X$ and $\varphi_Y : Z \longrightarrow Y$ are Vietoris homomorphisms, then,*

$$(\varphi_X, \varphi_Y)[Z] = \{(\varphi_X(z), \varphi_Y(z)) \mid z \in Z\}$$

is a Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Y}$, and each Vietoris bisimulation is of this shape.

Proof. It is easy to see that $(\varphi_X, \varphi_Y)[Z] = G(\varphi_X)^{-1} \circ G(\varphi_Y)$. By theorem 11.3.1, $G(\varphi_Y)$ and $G(\varphi_X)$ are Vietoris bisimulations, and by parts (2) of lemma 11.2.1, $G(\varphi_X)^{-1}$ is a Vietoris bisimulation too. Now, due to corollary 11.2.6, $(\varphi_X, \varphi_Y)[Z]$ is a Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. □

Theorem 11.3.2 does not hold if we omit the condition "compactness of the spaces X and Y" from our assumptions. By giving an example, we make this issue more clear. Consider the Vietoris frames $\mathcal{X} := (\mathbb{R}, \alpha)$ and $\mathcal{Y} := (\mathbb{N}^+, \beta)$ defined in example 11.2.4. It is easy to see that the function $f : \mathbb{N}^+ \longrightarrow \mathbb{R}$ defined by $f(n) := \frac{1}{n}$ is a Vietoris homomorphism. According to theorem 11.3.1 $G(f) = \{(\frac{1}{n}, n) \mid n \in \mathbb{N}^+\}$ is a Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. Then by parts (2) of lemma 11.2.1, $G(f)^{-1}$ is a Vietoris bisimulation between $\mathcal{Y}$ and $\mathcal{X}$. Notice that $G(f)^{-1} \circ G(f) = \{(\frac{1}{n}, \frac{1}{n}) \mid n \in \mathbb{N}^+\}$ is not a closed relation and consequently it is not Vietoris bisimulation.

11.4. From Kripke bisimulation to Vietoris bisimulation

The aim of this section is to find a stronger connection between two types of bisimulations in the category of Vietoris structures. In order to achieve this goal, in theorem 11.4.4 we will prove that for every two Vietoris models, the closure of each Kripke bisimulation between the underlying Kripke models is a Vietoris bisimulation. This claim originally dates back to an article by Venema, Fontaine and Bezhanishvili, see [11]. They prove this claim as a theorem for the category of descriptive models, and as a corollary of this theorem they have shown that the largest Vietoris bisimulation (with respect to the inclusion of subsets) between two descriptive models exists and it is the closure of the largest Kripke bisimulation between the underlying Kripke models.
Our givens in theorem 11.4.4 are two Vietoris models $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vartheta_{\mathcal{Y}})$, a Kripke bisimulation B between underlying Kripke models, and a pair (x, y) in $\overline{B}$ (the topological closure of B with respect to the product topology on $X \times Y$). We need to prove that for each $a \in R_{\mathcal{X}}(x)$, there is an element $b \in R_{\mathcal{Y}}(y)$ such that $(a, b) \in \overline{B}$. What we want to do here is to find a net $\tau := (a_j, b_j)_{j \in J}$ in B which converges to (a, b). The key tools used here are topological concepts stated in theorem 1.7.12.
We will find the net τ in two steps which are proven as two auxiliary lemmas 11.4.1 and 11.4.3, respectively.

To start, let X be a topological space, then:

Lemma 11.4.1. *Given a net $\kappa : \mathcal{I} \to \mathbb{V}(X)$ converging to $K \in \mathbb{V}(X)$ and $a \in K$. Then there are a directed set D (see definition 1.7.1), a converging map $\varphi : \mathcal{D} \to \mathcal{I}$ (see definition 1.7.3) and a net $\tau : \mathcal{D} \to X$ (see definition 1.7.5) such that $\tau(d) \in (\kappa \circ \varphi)(d)$ for all $d \in D$ and $\lim \tau = a$.*

Proof. The set $\mathcal{D} = \{(i, U) \mid i \in \mathcal{I}, U \in \mathfrak{N}_O(a), \kappa(i) \in \langle U \rangle\}$ with an order defined as $(i, U) \leq (j, V) :\iff i \leq j \wedge U \supseteq V$ is directed. Indeed, if (i, U) and (j, V) are elements in D, we can choose first $W \subseteq U \cap V$ and then $\lambda \in \mathcal{I}$ such that $\lambda \geq i$, $\lambda \geq j$ and $\kappa(\lambda) \in \langle W \rangle$. So $(i, U) \leq (\lambda, W)$ and $(j, V) \leq (\lambda, W)$. Define a map $\varphi : \mathcal{D} \to \mathcal{I}$ as $\varphi(i, U) := i$. Then φ is monotone and cofinal in $\mathcal{I}$, hence $\kappa \circ \varphi : \mathcal{D} \to \mathbb{V}(X)$ is a subnet of κ and therefore converges to K.
In the next step, by the axiom of choice, we are able to find a map $\tau : \mathcal{D} \to X$ such that $\tau(i, U) \in (\kappa \circ \varphi)(i, U) \cap U$, for each $(i, U) \in \mathcal{D}$. So $\tau(i, U) \in \kappa(i) \cap U$, for each $(i, U) \in \mathcal{D}$.
We should show that the net $\tau : \mathcal{D} \to X$ converges to a. Given any $V \in \mathfrak{N}_O(a)$, we have $a \in V \cap K$, so $K \cap V \neq \emptyset$, which means that $K \in \langle V \rangle$. Since κ converges to K we have

$$\forall V \in \mathfrak{N}_O(a). \exists i_V \in \mathcal{I}. \forall i \geq i_V . \kappa(i) \in \langle V \rangle \tag{11.4.1}$$

Then for each $V \in \mathfrak{N}_O(a)$, there is an element $i_V \in \mathcal{I}$ such that $(i_V, V) \in \mathcal{D}$. For each $(i, W) \geq (i_V, V)$ we have $i \geq i_V$ and $W \subseteq V$, and so $\tau(i, W) \in \kappa(i) \cap W \subseteq V$. This asserts that τ converges to a. □

Corollary 11.4.2. *Suppose that $\alpha : X \to \mathbb{V}(X)$ is a continuous function. Given a net $\kappa : I \to X$ which converges to $x \in X$. For each $a \in \alpha(x)$, then there exists a subnet $\sigma : D \to X$ of κ and elements $a_d \in \alpha(\sigma(d))$ for all $d \in D$, such that $\sigma \to x$ and $a_d \to a$.*

Lemma 11.4.3. *Given a net $\kappa : I \to \mathbb{V}(X)$ converging to K and a net $\tau : I \to X$ such that $\tau(i) \in \kappa(i)$ for all $i \in I$. Then there is a subnet $\sigma : D \to X$ of τ with $\sigma \longrightarrow a \in K$ for some $a \in K$.*

Proof. It is enough to show that $\tau : I \to X$ has an accumulation point in K. For each $x \in K$ such that x is not an accumulation point of $\tau : I \to X$, there exists an open neighborhood U_x of x and $i_x \in I$ such that $\forall i \geq i_x . \tau(i) \notin U_x$. Assuming that no $x \in K$ is an accumulation point, the collection of all U_x, with $x \in K$ yields an open cover for the compact set K. Let $U_{x_1}, ..., U_{x_k}$ be a finite subcover and $U := \bigcup_{j=1}^{k} U_{x_j}$. Choose $i_U \geq i_{x_1}, ..., i_{x_n}$, then for every $i \geq i_U$ we have $\tau(i) \notin U \supseteq K$.
On the other hand, $K \subseteq U$, i.e. $K \in [U]$. Since $\kappa \to K$, we have that eventually $\kappa(i) \subseteq U$, from which it follows that eventually $\tau(i) \in U$. So there is $d \in I$ such that $\forall i \geq d . \tau(i) \in U$. For $i \geq i_U, d$ we have the contradiction $\tau(i) \in U$ and $\tau(i) \notin U$. □

Theorem 11.4.4. *Suppose $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vartheta_{\mathcal{Y}})$ are Vietoris models. If B is a Kripke-Bisimulation between underlying Kripke models, then its closure $\bar{B}$ is a Vietoris bisimulation.*

Proof. Assume $B \subseteq X \times Y$ is a bisimulation. Since $\bar{B}$ is closed, it suffices to show that $\bar{B}$ is a Kripke bisimulation, too. Given $(x, y) \in \bar{B}$. By part (3) in theorem 1.7.12, there exists a net $(x_i, y_i)_{i \in I}$ converging to (x, y) with $(x_i, y_i) \in B$ for each $i \in I$. It follows that $(x_i)_{i \in I}$ converges to x and $(y_i)_{i \in I}$ converges to y. We need to check the following three steps:

1. $\vartheta_{\mathcal{X}}(x) = \vartheta_{\mathcal{Y}}(y)$;
2. $\forall a \in R_{\mathcal{X}}(x). \exists b \in R_{\mathcal{Y}}(y). (a, b) \in \bar{B}$;
3. $\forall b \in R_{\mathcal{Y}}(y). \exists a \in R_{\mathcal{X}}(x). (a, b) \in \bar{B}$;

Step 1: Since $\vartheta_{\mathcal{X}} : X \to \mathbb{P}(P)$ and $\vartheta_{\mathcal{Y}} : Y \to \mathbb{P}(P)$ are continuous, $(\vartheta_{\mathcal{X}}(x_i))_{i \in I}$ converges to $\vartheta_{\mathcal{X}}(x)$ in $\mathbb{P}(P)$ and likewise $(\vartheta_{\mathcal{Y}}(y_i))_{i \in I}$ converges to $\vartheta_{\mathcal{Y}}(y)$ in $\mathbb{P}(P)$. As $(x_i, y_i) \in B$ for each $i \in I$, we have $\vartheta_{\mathcal{X}}(x_i) = \vartheta_{\mathcal{Y}}(y_i)$ for every $i \in I$. Then by uniqueness of limit in the Hausdorff space $\mathbb{P}(P)$, it is concluded that $\vartheta_{\mathcal{X}}(x) = \vartheta_{\mathcal{Y}}(y)$.

Step 2: Since $R_{\mathcal{X}} : X \to \mathbb{V}(X)$ and $R_{\mathcal{Y}} : Y \to \mathbb{V}(Y)$ are continuous, $(R_{\mathcal{X}}(x_i))_{i \in I}$ converges to $R_{\mathcal{X}}(x)$ in $\mathbb{V}(X)$ and likewise $(R_{\mathcal{Y}}(y_i))_{i \in I}$ converges to $R_{\mathcal{Y}}(y)$ in $\mathbb{V}(Y)$. Now by lemma 11.4.1 we can find a converging map $\varphi : \mathcal{D} \to \mathcal{I}$ and a net $\tau : D \to X$ such that $\tau(d) \in (R_{\mathcal{X}}(x_{\varphi(d)}))_{d\in D}$ for all $d \in D$ and $\lim \tau = a$. Since φ is converging, we also have that the subnet $(x_{\varphi(d)})_{d\in D}$ converges to x and the subnet $(y_{\varphi(d)})_{d\in D}$ converges to y.

Now just concentrating on the mentioned subnets, forgetting the φ, we can state that we have

1. a net $(x_j)_{j\in J}$ converging to x
2. a net $(y_j)_{j\in J}$ converging to y
3. each $(x_j, y_j) \in B$
4. a net $(a_j)_{j\in J}$ with $a_j \in R_{\mathcal{X}}(x_j)$ converging to a
5. the net $(R_{\mathcal{Y}}(y_j))_{j\in J}$ converging to $R_{\mathcal{Y}}(y)$.

From 3 and 4, for each $j \in J$ we find $b_j \in R_{\mathcal{Y}}(y_j)$ such that $(a_j, b_j) \in B$. With 5 and lemma 11.4.3, we find a subnet $(b_{\tau(j)})_{j\in J}$ of $(b_j)_{j\in J}$ converging to some $b \in R_{\mathcal{Y}}(y)$. Also $(a_{\tau(j)})_{j\in J}$ converges to a, hence $(a_{\tau(j)}, b_{\tau(j)})_{j\in J}$ converges to (a, b) which proves that $(a, b) \in \bar{B}$.

Step 3: It will be proven by a symmetric argument.

□

Corollary 11.4.5. *In the category of Vietoris models, Kripke bisimilar elements are Vietoris bisimilar.*

Proof. Let $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vartheta_{\mathcal{Y}})$ be Vietoris models and $x \in X$ and $y \in Y$ Kripke bisimilar elements. Then there is a Kripke bisimulation B between the underlying Kripke models with $(x, y) \in B$ and so $(x, y) \in \bar{B}$. By lemma 11.4.4, $\bar{B}$ is a Vietoris bisimulation, so $x \in X$ and $y \in Y$ are Vietoris bisimilar. □

11.5. Hennessy-milner property for the Vietoris models

Let $\mathcal{X} = (X, R_{\mathcal{X}}, \vartheta_{\mathcal{X}})$ and $\mathcal{Y} = (Y, R_{\mathcal{Y}}, \vartheta_{\mathcal{Y}})$ be Vietoris models. Then the following lemmas hold:

Lemma 11.5.1. *The modal equivalence relation $\approx \subseteq X \times Y$ between $\mathcal{X}$ and $\mathcal{Y}$, is a Kripke bisimulation between the underlying Kripke models.*

Proof. According to theorem 9.3.5, the underlying Kripke models of $\mathcal{X}$ and $\mathcal{Y}$, are compact. Then from Compact Hennessy-Milner theorem (see 6.2.10), we have that $\approx \subseteq X \times Y$ is a Kripke bisimulation between the underlying Kripke models. □

Corollary 11.5.2. ***(Vietoris Hennessy-Milner theorem)*** *The modal equivalence relation $\approx \subseteq X \times Y$ is a Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Y}$.*

Proof. According to lemmas 11.5.1 and 7.5.7, this claim holds. □

Corollary 11.5.3. *The modal equivalence relation $\approx \subseteq X \times Y$ is the largest Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Y}$ with respect to the inclusion of subsets.*

Proof. According to the Vietoris Hennessy-Milner theorem, $\approx$ is a Vietoris bisimulation between $\mathcal{X}$ and $\mathcal{Y}$. Now, Suppose B is a Vietoris bisimulation between Vietoris models $\mathcal{X}$ and $\mathcal{Y}$, then B is a Kripke bisimulation between the underlying Kripke structures. Since the underlying Kripke structures are compact structures (theorem 9.3.5), $\approx$ is the largest Kripke bisimulation (corollary 6.2.11), and then we have $B \subseteq \approx$. □

Corollary 11.5.4. *Vietoris bisimilar elements are modally equivalent elements.*

We finish this section by stating that the notions of Vietoris bisimilarity, Kripke bisimilarity, behavioral equivalence, modal equivalence, all coincide.

Theorem 11.5.5. *Let x and y be elements in the Vietoris models $\mathcal{X}$ and $\mathcal{Y}$, respectively. Then the following are equivalent:*

1. x and y are behaviorally equivalent
2. x and y are Kripke bisimilar
3. x and y are Vietoris bisimilar
4. x and y are modally equivalent

Proof. The equivalence $1 \iff 4$ is concluded by remark 10.6.2. The implication $4 \iff 3$ follows from corollaries 11.5.2 and 11.5.4. The equivalence $2 \iff 3$ will be obtained by definition of the vietoris bisimulations and corollary 11.4.5. □

Conclusion and future direction of research

In this thesis we tried to show that, besides *Set*, the category *Top* forms an interesting base category for coalgebras.

We introduced some endofunctors on *Top*, in particular, the Vietoris functor $\mathbb{V}$ and the P-Vietoris functor $\mathbb{V}_P$ (where P is a set of propositional letters) that can be considered as the topological versions of the powerset functor $\mathbb{P}$ and the Kripke functor $\mathbb{P}_P$, respectively. We proved that these two functors preserve monos and regular monos. However, there are *Top*-endofunctors that do not preserve monos and regular monos. As an example, we mentioned the path functor Π_0. It was also shown that the Vietoris functor $\mathbb{V}$ does not preserve products. We think, it will also be interesting to investigate whether the Vietoris functor preserves closed embedding.
Besides, by considering three inequivalent topologies on the two elements set $2 := \{0, 1\}$, we defined four topological versions of the neighbourhood functor $2^{2^{(-)}}$ on *Set*. Now, an open question for future research is whether the clopen neighbourhood functor preserves monos and regular monos. We also like to know whether this functor preserves products. One can pose these questions for other *Top*-endofunctors studied here.
By introducing the notions of extension (up to isomorphism) and lifting (up to isomorphism) of functors, we exhibited relationships between *Set*-endofunctors and *Top*-endofunctors. We showed that a *Top*-endofunctor F is a lifting of a *Set*-endofunctor T up to isomorphism if and only if F preserves monos and epis. We gave a strategy to lift a special class of *Set*-endofunctors to the category *Top*. As an application, we obtained a *Top*-endofunctor $\bar{T}$ as a lifting of the *Set*-endofunctor $T := (-)^2 - (-) + 1$ that helped us to provide some counterexamples required in this work. Besides, we studied a strategy to extend the powerset functor $\mathbb{P}$ and the finite powerset functor $\mathbb{P}_\omega$ to CUM^1. Now, it would be worthwhile to investigate what connections are between limits in Top_F and limits in Set_T, where F is an extension (up to isomorphism) or a lifting (up to isomorphism) of the *Set*-endofunctor T to *Top*. For some related work, see Balan et. al. [9] and Worrel [71].

In order to give a motivation to study coalgebras over *Top*, in particular Vietoris coalgebras, we introduced the notion of compact Kripke structure and we found that in the class of compact Kripke structures, the notions of behavioral equivalence, modal equivalence and Kripke bisimilarity all coincide. Our definition of compact Kripke structure coincides with the notion of modally saturated structures introduced in Fine [27].

We further contributed to the theory of coalgebras over the category of topological spaces by discussing some basic definitions, examples and theorems for coalgebras over a base category $\mathbb{C}$ with the same properties as the category Top. We used the categories Set and Top as base categories in our examples. The concept of union of $\mathcal{M}$-subcoalgebras was described and it was shown that the union of a family of $\mathcal{M}$-subcoalgebras need not be an $\mathcal{M}$-subcoalgebra. As one of our main results in this step, we proved that if the base category $\mathbb{C}$ is $\mathcal{M}$-well powered with sums then the preservations of $\mathcal{M}$-morphisms by a $\mathbb{C}$-endofunctor F gives rise to the existence of equalizers in $\mathbb{C}_F$. In that case, we constructed the equalizers of two morphisms f, g in $\mathbb{C}_F$ via union of a special family of $\mathcal{M}$-subcoalgebras of their domains. As an example, we mentioned that if we consider Top as an (*epi*, *regular mono*)-category then the equalizers of parallel morphisms in the categories $Top_{\mathbb{V}}$ and $Top_{\mathbb{V}_P}$ exist. Thus a natural question for future work is whether the categories $Top_{\mathbb{V}}$ and $Top_{\mathbb{V}_P}$ have products. It will also be interesting to investigate whether these categories are complete. For some related work, see Hofmann [42].
Based on the notion of A-M bisimulation known by Aczel and Mendler in [2], we define a concept of the largest A-M bisimulation, and by giving an example from Venema et. al. [11], it was shown that the largest A-M bisimulation need not always exist. Two strategies were proposed to answer the questions of when does the largest A-M bisimulation between two coalgebras exist and how we can find it. As an application of the second strategy, we obtain a way to check whether a $\mathbb{C}$-endofunctor F weakly preserves pullbacks or not. In this way, we found that the Vietoris functor on Top does not weakly preserve pullbacks. We briefly generalized the notion of modal logic for coalgebras over Top by defining a language for a Top-endofunctor F via a modal similarity type Λ for F, that is a set of clopen subsets of $F(2)$ where $2 := \{0,1\}$ is a discrete space.
In order to study terminal objects in the categories of coalgebras over other base categories than Set, we discussed the existence and the construction of terminal objects in the categories of coalgebras for the $\mathbb{C}$-endofunctors $D \times (-)$ (black-boxes) and $D \times (-)^{\Sigma}$ (automata) where $\mathbb{C}$ is a category with object D and products. We proved that if Σ is a set and $\Sigma^\star$ is the set of all finite words over Σ, then in any category $\mathbb{C}$ with object D and product, a terminal coalgebra for the functor $D \times (-)^{\Sigma}$ exists, and it is based on $D^{\Sigma^\star}$ ($\Sigma^\star$-fold product of D in $\mathbb{C}$).

Finally, in order to provide an interesting example of coalgebras over the category Top, we introduced the notion of Vietoris structures as a generalization of the notion of descriptive structures defined in [11]. We saw that Vietoris frames and models are respectively coalgebras for the functors $\mathbb{V}$ and $\mathbb{V}_P$ over the category Top. It was proven that each compact Kripke model $\mathcal{X}{=}(X, R_{\mathcal{X}}, \models_{\mathcal{X}})$ together with $\mathcal{X}$-modal topology over X can be seen as a Vietoris model and consequently it is a $\mathbb{V}_P$-coalgebra. This yields an adjunction between the categories VS (category of Vietoris structure) and CKS (category of compact Kripke structures). Now, two interesting questions for future research are then how to generalize the notion of compactness of a Kripke structure to a coalgebra (X, α) in Set_T with respect to the given logic for T-coalgebras in Gumm [37], and under what conditions on the Set-endofunctor T we can guarantee that each coalgebra (X, α) in Set_T is compact. Another question that would be worthwhile to investigate is whether each

compact T-coalgebra in Set_T can be modified to a coalgebra on the category Top, i.e. whether we can find a Top-endofunctor F such that each compact T-coalgebra (X, α) together with a special topology on X is a coalgebra for the Top-endofunctor F.
Looking to find a terminal object in the category of Vietoris models, we defined the notion of a Kripke-ultrafilter on L_P (the set of modal formulas over P). By using Kripke-Ultrafilter lemma and Truth lemma it was proven that $\mathfrak{U}_\Box = (\mathcal{U}_\Box, \mathcal{R}_\mathfrak{U}, \models_\mathfrak{U})$ (the $\mathfrak{U}$-Modal model induced by the Kripke-Ultrafilter model $\mathfrak{U} = (\mathcal{U}, \mathcal{R}_\mathfrak{U}, \models_\mathfrak{U})$ given in definition 10.4.1) is a Vietoris model. We argued that $\mathfrak{U}_\Box$ is a terminal object in the category of Vietoris models.
As has been shown in example 7.4.12, in the category of Vietoris-coalgebras the supremum of a family of A-M bisimulations need not be an A-M bisimulation. As a consequence, the largest A-M bisimulation need not exist. Besides, in example 7.5.9, we saw that A-M bisimilarity is different from modal equivalence and consequently different from behavioral equivalence. To overcome these shortcomings of A-M bisimilarity, we studied a different concept of bisimilarity between Vietoris models called Vietoris bisimulation. It was explained that the class of Vietoris models with the notion of Vietoris bisimulation provides a complete semantic for modal logic in the sense that Vietoris bisimilarity, behavioral equivalence, modal equivalence, all coincide. Moreover, we gave some characterizations of Vietoris homomorphisms and Vietoris bisimulations between Vietoris models over compact Hausdorff spaces. As one of our main results in this work, we proved that the closure of a Kripke bisimulation between underlying Kripke models of two Vietoris models is a Vietoris bisimulation.
Now, a general task for future work is to investigate the open questions mentioned in [11] that unfortunately, largely remained unanswered in this thesis. In particular, it would be interesting to understand the relation between bisimulation for other functor pairs than Vietoris and powerset. One starting point for such an investigation is with the work of Enqvist et. al. [23] on bisimulations for coalgebras over the category of Stone spaces. In that paper the authors associated to each Set-endofunctor T a Stone companion $\widehat{T} : Stone \longrightarrow Stone$, by making use of the Moss-style ∇-modality mentioned in [55]. They introduced a notion of bisimulation, called neighbourhood bisimulation, between coalgebras for the endofunctor $\widehat{T}$ over $Stone$. They showed that neighbourhood bisimilarity coincides with behavioral equivalence and modal equivalence. We think it is also interesting to check our results for two functors $F : Top \longrightarrow Top$ and $T : Set \longrightarrow Set$ that are related by a natural transformation $\eta : UF \longrightarrow TU$ (where U is the forgetful functor from Top to Set). We would like to introduce a notion of bisimulation, namely F-bisimulation, between F-coalgebras on Top, and show that the topological closure of an A-M bisimulation between two T-coalgebras in Set_T is always an F-bisimulation.

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List of symbols

Set theory

$\mathbb{N}$	set of natural numbers with 0
$\mathbb{R}$	set of real numbers
$[a, b]$	closed interval in $\mathbb{R}$, i.e. $[a,b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$
(a, b)	open interval in $\mathbb{R}$, i.e. $(a,b) := \{x \in \mathbb{R} \mid a < x < b\}$
$[a, b)$	half open interval in $\mathbb{R}$, i.e. $[a,b) := \{x \in \mathbb{R} \mid a \leq x < b\}$
$[x]_\theta$	equivalence class of x modulo θ, i.e. $[x]_\theta := \{y \mid (x,y) \in \theta\}$
π_θ	factor projection map $X \longrightarrow X/\theta$ defined by $\pi_\theta(x) := [x]_\theta$
$\triangle_X$	diagonal of X, i.e., $\triangle_X := \{(x,x) \mid x \in X\}$
A^c	complement of a subset $A \subseteq X$ in X, i.e., $A^c := X - A = \{x \in X \mid x \notin A\}$
$\bigcap_{i \in I} X_i$	intersection of the sets $\{X_i\}_{i \in I}$, i.e. $\bigcap_{i \in I} X_i := \{x \mid \forall i \in I.\, x \in X_i\}$
$\bigcup_{i \in I} X_i$	union of the sets $\{X_i\}_{i \in I}$, i.e. $\bigcup_{i \in I} X_i := \{x \mid \exists i \in I.\, x \in X_i\}$
$\biguplus_{i \in I} X_i$	disjoint union of the sets $\{X_i\}_{i \in I}$, i.e. $\biguplus_{i \in I} X_i := \bigcup_{i \in I} \{(i,x) \mid x \in X_i\}$
$f(O)$	image of O under f, i.e., $f(O) := \{y \in Y \mid \exists x \in O.\, f(x) = y\}$
$f^{-1}(V)$	preimage of V under f, i.e., $f^{-1}(V) := \{x \in X \mid f(x) \in V\}$
$G(f)$	graph of a function f, i.e., $G(f) := \{(x, f(x)) \mid x \in X\}$
$im\, f$	image of f, i,e., $im\, f := \{y \in Y \mid \exists x \in X.\, f(x) = y\}$
$ker\, f$	kernel of f, i.e., $ker\, f := \{(x,x') \in X \times X \mid f(x) = f(x')\}$
$R \circ S$	relation composition, i.e. $R \circ S := \{(x,z) \in X \times Z \mid \exists y \in Y.\, (x,y) \in R \wedge (y,z) \in S\}$
$x\, R\, y$	pair (x, y) is an element in the binary ralation R
$x \longrightarrow_R y$	i.e. $x\, R\, y$

Topology theory

$B_r(x)$	open ball of radius $r > 0$ around x, definition 1.8.1
$\mathfrak{N}(x)$	set of all neighborhoods of an element $x \in X$, definition 1.1.2
$\mathfrak{N}_O(x)$	set of all open neighborhoods of an element $x \in X$, definition 1.1.2
$\tau_\Box$	$\mathcal{X}$-modal topology, definition 9.3.1
Q_f	quotient topology induced by f, example 1.4.7
A°	interior of a subset $A \subseteq X$, definition 1.1.2
$\overline{A}$	closure of a subset $A \subseteq X$, definition 1.1.2
X_D	set X with the discrete topology, definition 1.1.1
X_I	set X with the indiscrete topology, definition 1.1.1

Category theory

$\mathbb{C}^{op}$	dual category, definition 2.1.8
$\mathbb{C}_F$	category of F-coalgebras over the category $\mathbb{C}$, definition 7.1.6
$CHTop$	category of compact Hausdorff spaces with continuous functions, example 2.1.7
CKS	category of compact Kripke structures with Kripke homomorphisms, remark 6.2.9
CUM	category of complete ultrametric spaces and non-expansive maps, example 2.1.3
CUM^1	category of complete 1-bounded ultrametric spaces and non-expansive maps, example 2.1.3
EC	empty category, example 2.1.3
KS	category of Kripke structures with Kripke homomorphisms, remark 6.1.20
$Poset$	category of posets with monotone maps, example 2.1.3
$Preord$	category of preordered sets with monotone maps, example 2.1.3
Set	category of sets and functions, example 2.1.3
$Stone$	category of Stone spaces and continuous functions, example 2.1.3
Top	category of topological spaces with continuous functions, example 2.1.3

VS	category of Vietoris structures with Vietoris homomorphisms, remark 9.1.2
id_X	identity morphism on an object X, definition 2.1.1
$g \circ f$	composition g after f, definition 2.1.1
$cod(f)$	codomain of f, definition 2.1.1
$dom(f)$	domain of f, definition 2.1.1
$Hom_{\mathbb{C}}(A, B)$	class of all morphisms in a category $\mathbb{C}$ with domain A and codomain B, remark 2.1.2
$Mor(\mathbb{C})$	class of morphisms in a category $\mathbb{C}$, remark 2.1.2
$Ob(\mathbb{C})$	class of objects in a category $\mathbb{C}$, remark 2.1.2
$X + Y$	sum of objects X and Y in a category $\mathbb{C}$, definition 2.5.2
$X \times Y$	product of objects X and Y in a category $\mathbb{C}$, definition 2.5.1
$\sum_{i \in I} A_i$	sum of objects in a category, definition 2.5.2
$\prod_{i \in I} A_i$	product of objects in a category, definition 2.5.1

Universal coalgebra

$(-)^{\Sigma}$	power functor, example 2.15.5
$2^{(-)}$	contravariant powerset functor, example 2.16.3
$\mathbb{P}$	powerset functor, example 2.16.1
$\mathbb{P}^{\star}$	extention of the powerset functor on CUM^1, corollary 5.3.4
$\mathbb{P}_{\omega}$	finite powerset functor, example 2.16.1
$\mathbb{P}^{\star}_{\omega}$	extention of the finite powerset functor on CUM^1, corollary 5.3.6
$\mathbb{P}_P$	Kripke functor, example 7.1.2
$\mathbb{V}(-)$	Vietoris functor, section 3.2
$\mathbb{V}_P(-)$	P-Vietoris functor, product functor $\mathbb{V}(-) \times \mathbb{P}(P)$, definition 3.2.1
$\mathcal{F}_{\Box}$	modal functor from the category CKS to the category VS, remark 9.3.4
$\Pi_0(X)$	path functor, section 3.4
$Hom_{Top}(-, \Sigma)$	contravariant Hom_{Top} functor, lemma 3.6.1

$Hom_{Top}(\Sigma, -)$	covariant Hom_{Top} functor, lemma 3.5.1
$U_{\mathbb{C}}$	forgetful functor from $\mathbb{C}_F$ to $\mathbb{C}$, remark 7.1.7
U_C	forgetful functor from VS to CKS, remark 9.3.6
F^+	positive functor of a functor F, section 3.8
$F_1 + F_2$	sum of functors F_1 and F_2, lemma 2.15.8
$F_1 \circ F_2$	composition of functors F_1 and F_2, lemma 2.15.8
$F_1 \times F_2$	product of functors F_1 and F_2, lemma 2.15.8
$\mathcal{X}_\Box$	$\mathcal{X}$-modal model, definition 9.3.1
$\mathfrak{U}$	Kripke-Ultrafilter model, definition 10.4.1
$\mathfrak{U}_\Box$	$\mathfrak{U}$-modal model, section 10.5
$x \nabla_{\mathcal{X},\mathcal{Y}} y$	elements x and y are behaviorally equivalent - w.r.t Kripke structures, definition 6.2.12 - w.r.t coalgebras, definition 7.5.10
$\approx_{\mathcal{X},\mathcal{Y}}$	modal equivalence relation between $\mathcal{X}$ and $\mathcal{Y}$ - w.r.t Kripke structures, definition 6.1.8 - w.r.t coalgebras, definition 7.5.6
$\sim_{\mathcal{X},\mathcal{Y}}$	largest A-M bisimulation between $\mathcal{X}$ and $\mathcal{Y}$ - w.r.t Kripke structures, remark 6.1.12 - w.r.t coalgebras, definition 7.4.11

Logic

$\top$	true formula (top), definition 6.1.2
$\bot$	false formula (bottom), definition 6.1.4
$\Diamond$	modal operator diamond on modal formulas, definition 6.1.2
$\Box$	modal operator box on modal formulas, definition 6.1.2
$\bigvee_{i \in I} \varphi_i$	conjunction of modal formulas, definition 6.1.2
$\bigwedge_{i \in I} \varphi_i$	disjunction of modal formulas, definition 6.1.2
$L(\Lambda)$	set of all Λ-formulas, definition 7.5.2
L_P	set of modal formulas over P, definition 6.1.2

$\Sigma \vdash \varphi$	formula φ is deducible from Σ, definition 10.1.4
$\models \varphi$	formula φ is a Kripke tautology, definition 10.1.1
$\models_{\mathcal{X}} \Sigma$	set of modal formulas Σ is valid in $\mathcal{X}$ - w.r.t Kripke structures, definition 6.1.3 - w.r.t coalgebras, definition 7.5.3
$\models_{\mathcal{X}} \varphi$	formula φ is valid in $\mathcal{X}$ - w.r.t Kripke structures, definition 6.1.3 - w.r.t coalgebras, definition 7.5.3
$x \models_{\mathcal{X}} \varphi$	formula φ is valid in the element x of $\mathcal{X}$ - w.r.t Kripke structures, definition 6.1.3 - w.r.t coalgebras, definition 7.5.3
$\parallel x \parallel$	set of all valid formulas in an element x - w.r.t Kripke structures, equation 6.1.1 - w.r.t coalgebras, equation 7.5.3
$\parallel \varphi \parallel^{\mathcal{X}}$	set of all element x in a model $\mathcal{X}$ valid in the formula φ - w.r.t Kripke structures, equation 6.1.2 - w.r.t coalgebras, equation 7.5.2
$\mathcal{U}$	set of all Kripke-Ultrafilters over L_P, definition 10.2.4
$\mathcal{U}_{\Sigma}$	set of all Kripke-Ultrafilters over L_P which contains the set Σ, remark 10.3.2

Index

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Declaration

I confirm that this is my own work and the use of all material from other sources has been properly and fully acknowledged.

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