Body Motion and Early Algebra

Ricardo Nemirovsky[1] Natividad Adamuz-Povedano[2] Francesca Ferrara[3] and Giulia Ferrari[3] [a]

[1]Manchester Metropolitan University, ESRI, Manchester, UK; r.nemirovsky@mmu.ac.uk

[2] Universidad de Córdoba, Spain; nadamuz@uco.es

[3] Università di Torino, Italy; francesca.ferrara@unito.it, giulia.ferrari@unito.it

This paper focuses on the emergence of abstraction through the use of a new kind of motion detector — WiiGraph — with 11-year old children. In the selected episodes, the children used the sensor to create three simultaneous graphs of position vs. time: two graphs for the motion of each hand and a third one corresponding to their difference. They explored relationships that can be ascribed to an equation of the type A – B = C. We propose two distinct paths for the attainment of abstraction, one focused on working with unknowns lacking sensible qualities, and another that involves navigating a surplus of sensible qualities. This study is a case study for the latter, which we portray as a process of opening channels of flow and exchange among sensible qualities, such that these cease to be self-enclosed and start to configure a plane of unity, which, far from denying their differences, brings them into mutual circulation.

Keywords: Body motion, sensors, early algebra, abstraction.

Introduction

Learning mathematics is often seen as a progression from the concrete to the abstract. This progression amounts to a passage across emphases, from the sensible to the intelligible. An archetypal example is that of the straight line. Out of countless acts of drawing, touching straight edges, tracing on the sand, or using tools, a sense grows for physical straightness. There is still a major gap between the latter and a geometric straight line involving a massive drawing out of sensible qualities, such as color, length, material, and thickness, to envision an entity that is intelligible but not sensible. Hence abstraction is depicted as a subtractive process, along which more and more qualities are taken out until a spectral remainder is left that is not amenable to being touched, seen, or heard, and is devoid of causal powers, whose presence is only indirectly evoked by diagrams and formulae. Numerous researchers in mathematics education have questioned this traditional image for the attainment of abstraction (Clements, 2000; de Freitas, 2016; Dreyfus, 2014; Hershkowitz, Schwarz, & Dreyfus, 2001; Noss, Hoyles, & Pozzi, 2002; Roth & Hwang, 2006).

Concluding his commentaries about multiple mythical narratives, such as the one of Thales measuring the height of an Egyptian pyramid by the shadow of a stick, or the use of the gnomon in ancient Babylonia, Serres (2017) insists: “Yes, its abstraction is a sum and not a subtraction” (p. 210), and introduces the image of white light: “Geometry integrates all our practical or ideal

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[a] All authors contributed equally to the article

habitats the way white light sums up all the colors, in transparency or translucency" (p. 210). This remark has inspired us to distinguish paths for the realization of abstraction corresponding to white and black light. Whereas the path of black light is abstraction by means of subtraction of sensible qualities, the path of white light meanders in the midst of a *surplus* of sensible qualities. In this paper, we aim at investigating a particular case of the pursuit of abstraction along a path of white light.

Generals and Unknowns

To illustrate the difference between abstracting paths of white and black light we invoke Peirce's distinction between a general and an unknown. Let us start with the notion of a general:

> A sign is objectively general, in so far as, leaving its effective interpretation indeterminate, it surrenders to the interpreter the right of completing the determination for himself. "Man is mortal." "What man?" "Any man you like." (Peirce, 1994)

A theorem proving a property of triangles, for example, deals with triangles as a general. A general is genuinely indeterminate, which makes the logic principle of the excluded middle invalid: Is the triangle isosceles?: no; is the triangle not-isosceles?: no. In contrast to generals, Peirce characterized unknowns — particulars with certain but unspecified traits — as "vague." We are uncertain whether the eye color of a friend is green or brown, but we know that it is not, say, red. The vagueness of her eye color includes infinite shades of brown and green and excludes redness. Together with such vague sense of eye color, we may also presume that her eyes are of a particular color, which is the key character of an unknown: its traits are determined but we know them only vaguely.

Grappling with an unknown entails relating to an entity that lacks, perhaps only momentarily, certain sensible qualities both in itself (e.g. her eye color) or in its signs (e.g. a textual description of her eye color). On the other hand, we navigate a general, such as mortals or triangles, by immersing ourselves in a vast and familiar terrain of sensible variations and differences, such as mortals of different age, sex, species, bodies, and behaviors; or triangles differing in shape, size, angles, perimeters, and colors. The question we strive to address in this study is precisely: What kind of navigation arrives at abstraction across a *surplus* of sensible qualities, that is, of the white light type (in terms of generals)? We examine this question through selected episodes in which children explore the kinesthetic production of graphical expressions, for a general that can be named by the equation: A – B = C. We situate our study within the growing field of early algebra (Kieran, Pang, Schifter, & Fong Ng, 2016). The emphasis of the early algebra work tends to be on the logic of unknowns and on generalizing processes with respect to patterns, variables, structures and relational thinking (Blanton et al., 2016; Bodanskii, 1969/1991; Carraher, Schliemann, Brizuela, & Earnest, 2016; Kaput, 2008; Kaput, Blanton, & Moreno, 2008; Ng & Lee, 2009; Radford, 2014). While generals are different from the recursive-empirical reasoning often associated with generalizing, they are also significant in the early algebra literature (see Bodanskii, 1969/1991, which discusses Davidoff's vision of early algebraic thinking — perhaps the closest to engaging children with generals). The work described in this paper belongs to early algebra, we suggest, because algebra can be taken to be the symbolic treatment of unknowns and generals

Sensors, Kinesthesia and Method

In this paper, we attend to the kinesthetic production of graphical expressions by means of a particular mathematical instrument. By "mathematical instrument" we refer to a material implement used interactively by means of individual or collective continuous body movements, to obtain and transform mathematical expressions (Nemirovsky, Kelton, & Rhodehamel, 2013). "WiiGraph" is a mathematical instrument we have used in our study. Among its many possible settings, there is one in which the distances between two hand-held Wiimotes (remotes) and a LED sensor bar are graphed over time, while a third graph, corresponding to the differences between these two distances, is also displayed in real time. The color of each position vs. time graph corresponded to the color of the Wiimote being recorded (i.e. light blue and pink; the presence of two large dots with these colors on the screen indicates the sensor as connected to the Wiimotes), or a different one for the case of the difference graph (i.e. dark blue; see Figure 1). WiiGraph belongs to a family of mathematical instruments based on motion detection, which work at body-scale involving wide body movements like walking or overarm gestures and are responsive to two movements occurring simultaneously, whether performed by one or two people at a time.

Figure 1: A child generating two position vs. time graphs and their difference graph

In the study, we worked with a group of four children aged 11 years, who did not previously know each other, over three sessions. The children had been recruited as volunteers through a network of families practicing home schooling education. They do not attend regular lessons at school, therefore we cannot infer about their mathematical background. The participants were filmed with two fixed cameras during each session and two of them wore a head-based Go-Pro camera. During the first two sessions they explored position vs. time graphs generated by two children, each moving a Wiimote. In addition to free explorations, they engaged in diverse activities anticipating and matching body motions and graphical shapes of position vs. time. In the third session three children worked by holding both the remotes individually, one remote in each hand. As opposed to a pair of children each handling one Wiimote, the one-in-each-hand arrangement differs markedly, among other reasons because of the centrality it confers to relative arm motion (Nemirovsky, Kelton, & Rhodehamel, 2012). The instructor chose to turn on the difference graph, displayed in dark blue, as a significant way of exploring relationships between graphs symbolically, beginning the episodes we examine in the next section. We have selected these episodes because they span the students' production and exploration of the difference graph. The first two authors were both present in the episodes (respectively, RN and NA below; D, M and Z refer to the children).

Selected Episodes

Episode 1: Introducing the difference graph and trying to keep it on zero

RN: (…) the computer also generates another line [turns on the difference graph] that is, em, dark blue, [points at the dark blue graph; Figure 2] (…) so we'll investigate what this third line is doing there, what it is showing. So the first thing we will try…

M: It's called, it's called minus because that, that purple [dark blue] line, line is, is, is pink minus blue.

RN: OK, how do you know that?

M: It's real, it's quite obvious, where it says pink minus blue [points to the screen, note the area pointed at with a black arrow in Figure 2] at the top of the screen.

RN: Aha (…) So you move [showing the Wiimotes to move with], you do whatever you want, [moves alternately right and left hands] but try to keep the dark blue on zero [points to the dark blue line], on this line [left hand runs along the *x* axis].

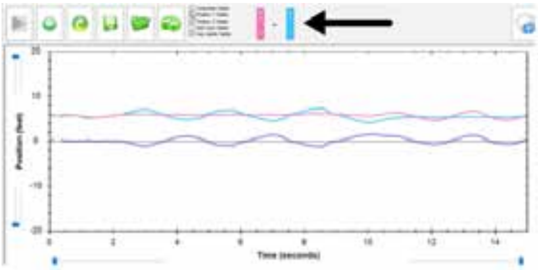

Figure 2: Graphical display in which the dark blue difference graph is displayed for the first time

M begins his first difference graph: he starts with the pink remote in his left hand and the blue one in his right hand. At the beginning of the experiment, the pink remote is kept slightly ahead of the blue one, and then the two are slowly switched in their positions. Holding the two remotes separated, he then walks forward (see Figure 3).

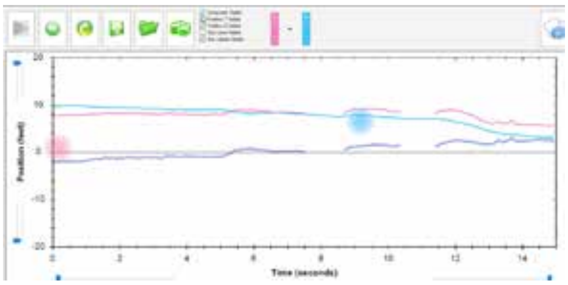

Figure 3: M's first attempt to create a difference graph

During the last seconds of the graph production (see Figure 3), he says:

M: I'm trying as hard as possible not to make the things go opposite.

Commentary

The appearance of a third graph prompted M to examine the screen seeking for additional signs that could name or account for it. There was none with a dark blue color. However, the sign at the top of the screen "pink minus blue," which had been displayed from the beginning of this session but had remained unused, offered him a compelling interpretation ("it's obvious"): the dark blue line "it's called minus." Besides the two remotes with clear referents, the minus was a third component, which was immediately clasped by the third graph. The dark blue graph seemed to announce its name. M was eager to be the first to use WiiGraph to obtain a dark blue graph that remained on the horizontal axis. M generated the graph shown in Figure 3 slowly moving the pink and blue Wiimotes back and forth in opposite directions. He seemed to move his arms exerting an effort, as if he had to push them back and forth. Perhaps RN's prior example, in which he had moved his arms in that way, had tacitly suggested to M that this is the kind of motion to perform. However, upon seeing that the dark blue graph refused to stay on zero, except for an interval around the 7th second, his arms tensed as if trying to push the dark blue graph to the center. M reflected on this sense of effort ("trying as hard as possible") as striving "not to make the things go opposite". Among other possibilities, this "going opposite" might have been the dark blue graph shifting in a direction opposite to the desired one. This episode suggests how the kinesthetic interpretation of a symbolic expression that we would consider algebraic (i.e. pink minus blue) is not "given" on its own, but demands novel interpretive acts involving matters of a qualitatively different nature, such as the fastness and slowness, or the proximity and remoteness, of his hands. A pattern of motion is not there to be seized, but needs to be created. The use of WiiGraph incorporates body motion — a complex realm of sensibility and performance — to generate graphical shapes, dramatically broadening the sensory qualities at play in obtaining intended and unintended graphical shapes. Furthermore, kinesthesia inherently awakens bodily feelings. How is a minus responsive to kinesthetic actions? In our interpretation, the graph called minus was not just a visual display out there, but also a curve that resisted physical efforts seeming to possess a will of its own.

Episode 2: D works with the difference graph

After M obtained several additional graphs, he gives the Wiimotes to D, who starts a new graph. He stands still in the same position for all the session, keeping steadily the remotes at the same distance from the sensor (See Figure 4).

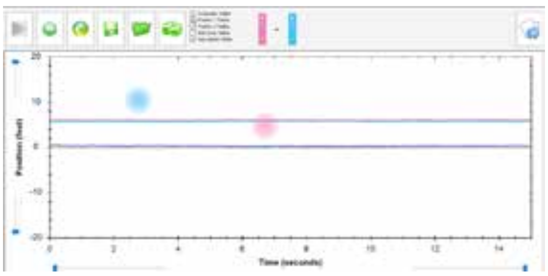

Figure 4: D generates a graph staying still with the Wiimotes next to each other

RN: So that, that's a perfect zero [around the 8th second, everyone laughs] [ending his graph, D relaxes his position, shrugs his shoulder and smiles].

NA: This is one way to get it.

RN: (…) try to do it while you walk [D generates the graph shown in Figure 5].

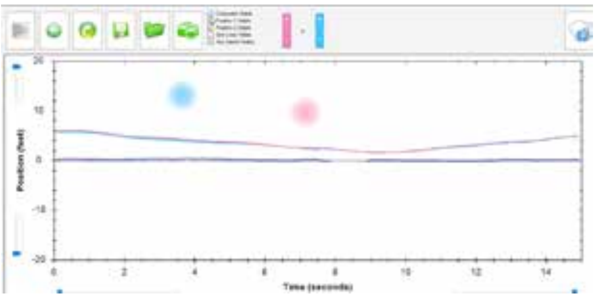

Figure 5: D keeps the difference graph on zero while walking

D: (…) You don't have to keep the remotes in (…) one position.

RN: Like, keeping [them] together?

D: keep them at the same level.

Commentary

D came to create a difference graph with a clear plan: stay still with the two remotes next to each other. He had a well-defined sense that a dark blue graph on the horizontal axis "converted" into the two Wiimotes being equally distant from the sensor. Moreover, D easily showed in Figure 5 that that condition was indifferent to his walking distance from the sensor ("You don't have to keep the remotes in one position"). The point, he said, was to "keep them at the same level." His choice of words ("same level") reflects a phenomenon, we think, of great significance: "level" is customarily a term for height, which was relevant to the light blue and pink graphs being at the same height, but not necessarily to the Wiimotes that could be at different heights while keeping equal distances to the sensor. D articulated an instance of a type of semiotic sliding between qualities of the graph and qualities of the remotes, such that they could apply indistinctly to one or the other.

Episode 3: Z works with the difference graph above and below the *x*-axis

In between Episodes 2 and 3 the children generated graphs to either keep the dark blue graph above or below the horizontal axis. Along that sequence, Z generated the graph shown in Figure 6.

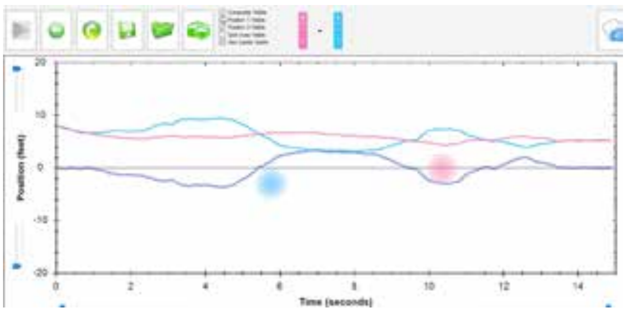

Figure 6: Z generates a graph in which the difference graphs goes above and below zero

RN: So, how did you change [the dark blue line] from below to above?

Z: Em, by changing which controller was in front.

RN: So which one was in front here? [points to the dark blue graph around the 4th second]

Z: Em, [light] blue.

RN: (…) And do you have a sense for why for the blue, for the dark blue line, to be below [the *x*-axis] then the pink has to be below [the light blue graph]?

Z: Em, yep. Em, it's something to do with like maths and, like, because on there, it says the [likely pointing at the light blue one remote on the screen] has been taken away and then it's hard to tell because it's not actual numbers but if you have more on one side, that will be a negative number… then, then, if you have them on the other side, it'll be a positive number, which is that [points to the screen with the remote].

Commentary

In this exchange Z describes qualitative differences of one kind (i.e. graphical configurations on the computer screen) and qualitative differences of another kind (i.e. bodily motions and postures) percolating onto each other. Sometimes these qualitative differences mutually communicate along critical points, such as the blue and pink Wiimotes being next to each other, as the condition that tips the cases of the minus graph being positive or negative. In other cases, the qualitative differences of each kind adopt corresponding ordinal arrangements along more or less, such as a Wiimote being closer or farther to the sensor bar matching its graph being higher or lower on the screen. Some of these qualitative differences can, in principle, be located on a metric scale, such as the distance between each remote and the sensor or the height of each graph at a given time. However, in this episode Z does not operate with metric scales. He said, for instance, "if you have *more on one side*, that will be a negative number" or "it's not actual numbers." Or he distinguished which remote is on front, rather than estimating distances between them. The focus of this commentary is to foreground ordinal arrangements of qualitative differences of unlike kinds, via critical points or corresponding alignments of more and less.

Discussion

In the introduction we distinguished the attainment of abstraction along paths of white or black light. Whereas the mode of white light calls for navigating a *surplus* or overabundance of sensible qualities, the case of dark light is one of deficit of sensible qualities enabling zones of vagueness. In the episodes described above, the radical expansion of relevant sensible qualities encompasses the infinite nuances of kinesthesia. In our commentary for Episode 1, we described such kinesthetic expansion as a vivid broadening of the sensory qualities at play.

The introduction also stated the question of this study: What kind of navigation arrives at abstraction across a *surplus* of sensible qualities, of the white light type (in terms of generals)? Our commentary to Episode 2 suggested a type of semiotic sliding between qualities of the graph and qualities of the remotes, such that they could apply indistinctly to one or the other. Additionally, in the commentary to Episode 3 we considered the notion of qualitative differences of dissimilar kinds percolating onto each other, either via critical points cutting across them or mutual exchanges between corresponding ordinal arrangements of differences of degrees (i.e. more/less). All these processes amount to opening channels of flow and exchange among sensible qualities, such that these cease to be self-enclosed and start to configure a plane of unity, which, far from denying their differences, brings them into mutual circulation and, therefore, speaks directly to navigating generals.

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