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The quaternionic Calabi Conjecture on abelian Hypercomplex Nilmanifolds Viewed as Tori Fibrations

Giovanni Gentili and Luigi Vezzoni*

Dipartimento di Matematica G. Peano, Università di Torino, Via Carlo Alberto 10, 10123 Torino, Italy giovanni.gentili@unito.it; luigi.vezzoni@unito.it

**Correspondence to be sent to: e-mail: luigi.vezzoni@unito.it*

We study the quaternionic Calabi–Yau problem in HyperKähler manifolds with torsion geometry, introduced by Alesker and Verbitsky in [5], on eight-dimensional two-step nilmanifolds M with an Abelian hypercomplex structure. We show that on these manifolds the quaternionic Monge–Ampère equation can always be solved for any data that are invariant under the action of a three-torus.

1 Introduction

Since Yau proved the Calabi–Yau conjecture in [28], other Calabi–Yau-type problems have been introduced in various geometric contexts.

In the present paper, we focus on a generalization of the Calabi–Yau problem to HyperKähler manifolds with torsion (HKT) geometry, which was introduced by Alesker and Verbitsky in [5].

HKTs were introduced by Howe and Papadopoulos in [18] in the framework known as ''geometries with torsion." In a nutshell, they can be thought of as hypercomplex manifolds admitting a special compatible Riemannian metric.

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A *hypercomplex manifold* is a $4n$-dimensional real manifold M equipped with a triple of complex structures J_1, J_2, J_3 satisfying the quaternionic relations

$$J_1 J_2 = -J_2 J_1 = J_3 \,. \tag{1}$$

If g is a Riemannian metric on M that is compatible with J_1, J_2, J_3, then (M, J_1, J_2, J_3, g) is usually called a *hyperHermitian* manifold. According to the classical definition, a hyperHermitian manifold (M, J_1, J_2, J_3, g) is said HKT, if there exists an affine connection ∇ on M, which preserves the hyperHermitian structure and has totally skew-symmetric torsion. If such ∇ exists, it is necessarily unique. The existence of ∇ can be characterized in terms of the differential equation

$$\partial\Omega = 0 \,,$$

where ∂ is taken with respect to J_1 ,

$$\Omega = \omega_{J_2} + \mathrm{i}\omega_{J_3} \,,$$

and

$$\omega_{J_r}(\cdot,\cdot) = g(J_r\cdot,\cdot) \,.$$

In this context Ω is called the *HKT form* of the HKT structure, and one may think of it as the analogue of the fundamental form in Kähler geometry. The hypercomplex condition (1) implies that Ω is of type $(2, 0)$ with respect to J_1, and it satisfies

$$\Omega\,(J_2\cdot, J_2\cdot) = \bar{\Omega}$$

and

$$\Omega\,(X, J_2 X) > 0 \quad \text{for every nowhere vanishing real vector field } X \text{ on } M.$$

Moreover, Ω determines the metric g via the relation

$$g(X, Y) = \mathrm{Re}\,\Omega\,(X, J_2 Y) \,, \quad \text{for any real vector fields } X, Y \text{ on } M.$$

An HKT structure can then be defined alternatively, as a hypercomplex structure together with an HKT form.

In [5] the authors introduced the following Calabi–Yau-type problem in HKT geometry. Let $(M, J_1, J_2, J_3, \Omega)$ be a compact $4n$-dimensional HKT manifold for which the canonical bundle of (M, J_1) is holomorphically trivial, and suppose $F \in C^\infty(M)$ is a function satisfying

$$\int_M (\mathrm{e}^F - 1)\, \Omega^n \wedge \bar{\Theta} = 0\,, \tag{2}$$

where Θ is a nonvanishing holomorphic $(2n, 0)$-form on (M, J_1). The *quaternionic Calabi–Yau problem* consists in finding an HKT form $\tilde{\Omega}$ on (M, J_1, J_2, J_3) such that

$$\tilde{\Omega}^n = \mathrm{e}^F \Omega^n\,. \tag{3}$$

Just like the classical version, the quaternionic Calabi–Yau problem, too, can be rewritten in the form of a Monge–Ampère equation. Indeed, results in [6] guarantee the unknown HKT form $\tilde{\Omega}$ can be written in terms of an HKT potential $\varphi \in C^\infty(M)$ as follows

$$\tilde{\Omega} = \Omega + \partial\partial_{J_2}\varphi\,.$$

Here ∂_{J_2} is the so-called *twisted Dolbeault operator*

$$\partial_{J_2} = -J_2^{-1}\bar{\partial}J_2$$

and the complex structure J_2 acts on k-forms α by

$$J_2\alpha(X_1, \ldots, X_k) = (-1)^k \alpha(J_2X_1, \ldots, J_2X_k)\,.$$

Equation (3) reads, in terms of φ and F,

$$(\Omega + \partial\partial_{J_2}\varphi)^n = \mathrm{e}^F \Omega^n\,. \tag{4}$$

It has been conjectured in [5] that the above equation can always be solved under assumption (2). The authors of the same paper propose the continuity method as a natural approach to attack the problem, much in the same spirit of Yau's proof of the Calabi conjecture [28]. The hard part in this line of thought is to establish a priori estimates. Alesker and Verbitsky [5] showed the solution is unique up to an additive constant and proved a C^0-estimate. The latter was later generalized by Alesker and Shelukhin [3] and then by Sroka [19] in a more general setting. Alesker gave evidence for believing the conjecture in [2], where he proved that the quaternionic Monge–Ampère equation has solutions if the manifold admits a flat hyperKähler metric compatible with the underlying hypercomplex structure.

The research of the present paper moves from [10, 11, 15, 23, 24, 26], where it is studied the symplectic Calabi-Yau conjecture [12, 27] on torus fibrations under some symmetries on the data. In the same spirit, we study the quaternionic Monge–Ampère equation on compact quotients of eight-dimensional nilpotent Lie groups endowed with an *Abelian* HKT structure.

By a result of Dotti and Fino [13] the only non-Abelian eight-dimensional two-step nilpotent Lie groups admitting an Abelian hypercomplex structure are

$$N_1 = H_1(2) \times \mathbb{R}^3\,, \qquad N_2 = H_2(1) \times \mathbb{R}^2\,, \qquad N_3 = H_3(1) \times \mathbb{R}\,,$$

where $H_i(n)$ denotes the real ($i = 1$), complex ($i = 2$), and quaternionic ($i = 3$) Heisenberg group. Each N_i contains a canonical co-compact lattice Γ_i, and the nilmanifold $M_i = \Gamma_i\backslash N_i$, that is, the quotient of N_i by Γ_i, inherits the structure of a principal T^3-bundle over a five-dimensional torus T^5 and also an HKT structure (J_1, J_2, J_3, g) (see section 2 for details). In view of [8] the nilmanifolds M_i are not Kählerian, since a compact nilmanifold admits a Kähler metric if and only if it is a torus.

Moreover, the canonical bundle of (M_i, J_1) is holomorphically trivial [7,Theorem 2.7] and M_i carries a left-invariant holomorphic volume form Θ. Hence, it is quite natural to wonder whether the Alesker–Verbitsky conjecture might hold on these spaces.

Our main result is the following

Theorem 1.1 The quaternionic Monge–Ampère equation (4) on (M_i, J_1, J_2, J_3, g) can be solved for every T^3-invariant map $F \in C^\infty(M_i)$ satisfying (2).

Since we are assuming F is invariant under the action of the fibre T^3, it can be regarded as a smooth function on the base T^5. Furthermore, condition (2) can be written as

$$\int_{T^5} (\mathrm{e}^F - 1)\,dx^1 \cdots dx^5 = 0\,. \tag{5}$$

By imposing the same invariance property on the HKT potential φ, we reduce the quaternionic Monge–Ampère equation on (M_i, J_1, J_2, J_3, g) to

$$(\varphi_{11} + \varphi_{22} + \varphi_{33} + \varphi_{44} + 1)(\varphi_{55} + 1) - \varphi_{15}^2 - \varphi_{25}^2 - \varphi_{35}^2 - \varphi_{45}^2 = \mathrm{e}^F\,, \tag{6}$$

where φ_{rs} denotes the second derivative of φ in the real coordinates $x^r, x^s \in \{x^1, \dots, x^5\}$ on T^5. Then, we prove that equation (6) has a solution $\varphi \in C^\infty(T^5)$ whenever F satisfies (5).

The strategy for proving Theorem 1 goes as follows: in section 3 we prove the C^0-estimate for our equation. Then in section 4 we deduce an a priori C^0-estimate for the Laplacian of a solution to our equation, and in section 5 we achieve the $C^{2,\alpha}$-estimate by applying a general result of Alesker [2]. Eventually, we complete the proof in section 6 by applying the continuity method.

2 Preliminaries

Let G be an eight-dimensional Lie group with a left-invariant hypercomplex structure (J_1, J_2, J_3) (every complex structure J_i is left-invariant). Assume that J_1 is *Abelian*, meaning

$$[J_1 X, J_1 Y] = [X, Y], \qquad \text{for every } X, Y \in \mathfrak{g}\,,$$

where $\mathfrak{g}$ is the Lie algebra of G. Recall that this is equivalent to requiring that the Lie algebra $\mathfrak{g}^{1,0}$ of left-invariant vector fields of type $(1,0)$ on (G, J_1) is Abelian. It also implies that any left-invariant $(p,0)$-form on (G, J_1) is ∂-closed. If g is a left-invariant Riemannian metric on G compatible with (J_1, J_2, J_3), the hyperHermitian structure (J_1, J_2, J_3, g) is HKT because the corresponding form Ω is ∂-closed.

As we mentioned in the introduction, by [13] the only eight-dimensional nilpotent, non-Abelian, Lie groups carrying a left-invariant HKT structure (J_1, J_2, J_3, g) such that every J_i is Abelian are

$$N_1 = H_1(2) \times \mathbb{R}^3\,, \qquad N_2 = H_2(1) \times \mathbb{R}^2\,, \qquad N_3 = H_3(1) \times \mathbb{R}\,,$$

where

$$H_1(2) = \left\{ \begin{pmatrix} 1 & x^1 & x^4 & y^1 \\ 0 & 1 & 0 & x^3 \\ 0 & 0 & 1 & x^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \quad H_2(1) = \left\{ \begin{pmatrix} 1 & x^1 + \mathrm{i}x^2 & y^3 + \mathrm{i}y^2 \\ 0 & 1 & x^4 + \mathrm{i}x^3 \\ 0 & 0 & 1 \end{pmatrix} \right\},$$

$$H_3(1) = \left\{ \begin{pmatrix} 1 & q & h - \frac{1}{2}q\bar{q} \\ 0 & 1 & -\bar{q} \\ 0 & 0 & 1 \end{pmatrix} \mid q = x^1 + \mathrm{i}x^4 + \mathrm{j}x^3 + \mathrm{k}x^2, h = \mathrm{i}y^3 + \mathrm{j}y^2 + \mathrm{k}y^1 \right\}.$$

Above, $x^1, \dots, x^4, y^1, y^2, y^3 \in \mathbb{R}$ and $\mathrm{i}, \mathrm{j}, \mathrm{k}$ are the familiar units of the skew field of quaternions, which are known to obey the relations

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = -1\,, \qquad \mathrm{ij} = -\mathrm{ji} = \mathrm{k}.$$

Note that each group N_i is diffeomorphic to $\mathbb{R}^8$, and there are global coordinates

$$N_1 = H_1(2)_{x^1,\dots,x^4,y^1} \times \mathbb{R}^3_{y^2,y^3,x^5}\,, \qquad N_2 = H_2(1)_{x^1,\dots,x^4,y^2,y^3} \times \mathbb{R}^2_{y^1,x^5}\,,$$
$$N_3 = H_3(1)_{x^1,\dots,x^4,y^1,y^2,y^3} \times \mathbb{R}_{x^5}\,.$$

The Lie algebras of the N_i can be characterized in terms of left-invariant frames $\{e_1, \dots, e_8\}$ satisfying the following structure equations:

N_1: $[e_1, e_2] = -[e_3, e_4] = e_5$, and all other brackets vanish;
N_2: $[e_1, e_3] = [e_2, e_4] = e_6$, $[e_1, e_4] = -[e_2, e_3] = e_7$, and all other brackets vanish;
N_3: $[e_1, e_2] = -[e_3, e_4] = e_5$, $[e_1, e_3] = [e_2, e_4] = e_6$, $[e_1, e_4] = -[e_2, e_3] = e_7$, and all other brackets vanish.

In each case, using the frame $\{e_1, \dots, e_8\}$ we can define the left-invariant HKT structure as consisting of the standard metric

$$g = \sum_{r=1}^{8} e^r \otimes e^r$$

and the three complex structures (J_1, J_2, J_3) defined by

$$J_r(e_1) = e_{r+1}\,, \quad J_r(e_5) = e_{r+5}\,, \quad r = 1, 2, 3\,.$$

Let us fix co-compact lattices

$$\Gamma_1 = \mathbb{Z}^3 \times \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b^t \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{Z}^2,\ c \in \mathbb{Z} \right\} \subset N_1\,;$$
$$\Gamma_2 = \mathbb{Z}^2 \times \left\{ \begin{pmatrix} 1 & z & u \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} \mid u, z, w \in \mathbb{Z} + \mathrm{i}\mathbb{Z} \right\} \subset N_2\,;$$

$$\Gamma_3 = \mathbb{Z} \times \left\{ \begin{pmatrix} 1 & q & h - \frac{1}{2}q\bar{q} \\ 0 & 1 & -\bar{q} \\ 0 & 0 & 1 \end{pmatrix} \mid q \in \mathbb{Z} + \mathrm{i}\mathbb{Z} + \mathrm{j}\mathbb{Z} + \mathrm{k}\mathbb{Z}\,, \quad h \in \mathrm{i}\mathbb{Z} + \mathrm{j}\mathbb{Z} + \mathrm{k}\mathbb{Z} \right\} \subset N_3\,.$$

For $r = 1, 2, 3$ we denote by $M_r = \Gamma_r \backslash N_r$ the compact nilmanifold obtained by quotienting N_r by Γ_r. The left-invariant quadruple (J_1, J_2, J_3, g) on N_r induces an HKT structure on M_r. Let $\{Z_1, \ldots, Z_4\}$ indicate the left-invariant (1,0)-frame $Z_r = e_{2r-1} - \mathrm{i}J_1(e_{2r-1})$, $r = 1, \ldots, 4$, and denote by $\{\zeta^1, \ldots, \zeta^4\}$ the dual (1,0)-coframe. Taking in account

$$\partial_{J_2} = -J_2^{-1}\bar{\partial}J_2\,,$$

we deduce the following identity, holding for every smooth real map φ on M_r:

$$\begin{aligned}
\partial\partial_{J_2}\varphi =& \partial J_2\bar{\partial}\varphi = \partial J_2\left(\bar{Z}_1(\varphi)\bar{\zeta}^1 + \bar{Z}_2(\varphi)\bar{\zeta}^2 + \bar{Z}_3(\varphi)\bar{\zeta}^3 + \bar{Z}_4(\varphi)\bar{\zeta}^4\right) \\
=& \partial\left(\bar{Z}_1(\varphi)\zeta^2 - \bar{Z}_2(\varphi)\zeta^1 + \bar{Z}_3(\varphi)\zeta^4 - \bar{Z}_4(\varphi)\zeta^3\right) \\
=& \left(Z_1\bar{Z}_1(\varphi) + Z_2\bar{Z}_2(\varphi)\right)\zeta^{12} + \left(Z_3\bar{Z}_2(\varphi) - Z_1\bar{Z}_4(\varphi)\right)\zeta^{13} + \left(Z_4\bar{Z}_2(\varphi) + Z_1\bar{Z}_3(\varphi)\right)\zeta^{14} \\
& - \left(Z_3\bar{Z}_1(\varphi) + Z_2\bar{Z}_4(\varphi)\right)\zeta^{23} + \left(Z_2\bar{Z}_3(\varphi) - Z_4\bar{Z}_1(\varphi)\right)\zeta^{24} + \left(Z_3\bar{Z}_3(\varphi) + Z_4\bar{Z}_4(\varphi)\right)\zeta^{34}\,.
\end{aligned}$$

Since

$$\Omega = 2(\zeta^{12} + \zeta^{34})\,,$$

it follows that

$$\begin{aligned}
(\Omega + \partial\partial_{J_2}\varphi)^2 = & \, 2\left(Z_1\bar{Z}_1(\varphi) + Z_2\bar{Z}_2(\varphi) + 2\right)\left(Z_3\bar{Z}_3(\varphi) + Z_4\bar{Z}_4(\varphi) + 2\right)\zeta^{1234} \\
& - 2\left(Z_3\bar{Z}_2(\varphi) - Z_1\bar{Z}_4(\varphi)\right)\left(Z_2\bar{Z}_3(\varphi) - Z_4\bar{Z}_1(\varphi)\right)\zeta^{1234} \\
& - 2\left(Z_4\bar{Z}_2(\varphi) + Z_1\bar{Z}_3(\varphi)\right)\left(Z_3\bar{Z}_1(\varphi) + Z_2\bar{Z}_4(\varphi)\right)\zeta^{1234}\,,
\end{aligned}$$

in other words

$$(\Omega + \partial\partial_{J_2}\varphi)^2 = 2\Big(\big(Z_1\bar{Z}_1(\varphi) + Z_2\bar{Z}_2(\varphi) + 2\big)\big(Z_3\bar{Z}_3(\varphi) + Z_4\bar{Z}_4(\varphi) + 2\big) \\ - \big(Z_3\bar{Z}_2(\varphi) - Z_1\bar{Z}_4(\varphi)\big)\big(Z_2\bar{Z}_3(\varphi) - Z_4\bar{Z}_1(\varphi)\big) \\ - \big(Z_4\bar{Z}_2(\varphi) + Z_1\bar{Z}_3(\varphi)\big)\big(Z_3\bar{Z}_1(\varphi) + Z_2\bar{Z}_4(\varphi)\big)\Big)\zeta^{1234}\,. \quad (7)$$

Furthermore, every manifold M_i is naturally a principal T^3-bundle over T^5 with projection

$$\pi : M_i \to T^5_{x^1\ldots x^5}\,.$$

A smooth function on M_i is invariant under the action of the principal fibre T^3 if and only if it depends only on the five coordinates $\{x^1, \ldots, x^5\}$. What is more, T^3-invariant functions on M_i are naturally identified with functions on T^5. As mentioned in the introduction, for a T^3-invariant real map F condition (2) becomes (5). Further assuming that the HKT potential φ is T^3-invariant, equation (4) can be written as (6) on T^5.

Remark. The Lie algebras of the two-step nilpotent Lie groups N_i all have four-dimensional centre $\mathfrak{z} = \{e_5, e_6, e_7, e_8\}$. Therefore, the nilmanifolds M_i can be regarded in a natural way as principal T^4-bundles over a torus T^4 if we project onto the first four coordinates $\{x^1, \ldots, x^4\}$. From this point of view, requiring all data to be invariant under the action of the fibre T^4 implies that the resulting equation can be written as the following Poisson equation on the base T^4

$$\Delta\varphi = \varphi_{11} + \varphi_{22} + \varphi_{33} + \varphi_{44} = \mathrm{e}^F - 1\,.$$

And this can be solved using standard techniques.

From this point on we shall focus on equation (6). In order to simplify the notation let us set

$$A = \varphi_{11} + \varphi_{22} + \varphi_{33} + \varphi_{44} + 1\,, \quad B = \varphi_{55} + 1\,.$$

Lemma 2.1. If $\varphi \in C^2(T^5)$ is a solution to (6), then $A > 0, B > 0$ and

$$0 < 2\mathrm{e}^{F/2} \le \Delta\varphi + 2\,. \quad (8)$$

Proof. From equation (6) we infer $AB \geq e^F > 0$. Hence, A and B have the same sign. At a point p_0 where φ attains its minimum we must have $\varphi_{55}(p_0) \geq 0$. This implies $B > 0$ and then $A > 0$. Finally, by using $A^2 + B^2 \geq 2AB$ we obtain

$$(\Delta\varphi + 2)^2 = (A + B)^2 \geq 4AB \geq 4e^F > 0\,.$$

Taking the square root produces (8). ■

Proposition 2.2. Equation (6) is elliptic. More precisely, if $\varphi \in C^2(T^5)$ denotes a solution to (6) then

$$A\xi_5^2 + B(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2) - 2\sum_{i=1}^{4} \varphi_{i5}\xi_i\xi_5 \geq \lambda(\varphi)(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2) \tag{9}$$

for every $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5) \in \mathbb{R}^5$, where

$$\lambda(\varphi) = \frac{1}{2}\left(A + B - \sqrt{(A + B)^2 - 4e^F}\right).$$

Proof. The principal symbol of the linearized equation at a solution φ equals

$$A\xi_5^2 + B(\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2) - 2\varphi_{15}\xi_1\xi_5 - 2\varphi_{25}\xi_2\xi_5 - 2\varphi_{35}\xi_3\xi_5 - 2\varphi_{45}\xi_4\xi_5$$

and the corresponding matrix is

$$P(\varphi) = \begin{pmatrix} B & 0 & 0 & 0 & -\varphi_{15} \\ 0 & B & 0 & 0 & -\varphi_{25} \\ 0 & 0 & B & 0 & -\varphi_{35} \\ 0 & 0 & 0 & B & -\varphi_{45} \\ -\varphi_{15} & -\varphi_{25} & -\varphi_{35} & -\varphi_{45} & A \end{pmatrix}.$$

Since, by (6),

$$\begin{aligned} \det(P(\varphi) - \lambda I) &= (B - \lambda)^3\left((A - \lambda)(B - \lambda) - (\varphi_{15}^2 + \varphi_{25}^2 + \varphi_{35}^2 + \varphi_{45}^2)\right) \\ &= (B - \lambda)^3\left(\lambda^2 - (A + B)\lambda + e^F\right), \end{aligned}$$

the eigenvalues are $\lambda = B$ and

$$\lambda_{\pm} = \frac{1}{2}\left(A + B \pm \sqrt{(A+B)^2 - 4e^F}\right).$$

Now $(A+B)^2 - 4e^F \geq (A-B)^2 = ((A+B) - 2A)^2 = ((A+B) - 2B)^2$, so that

$$0 < \lambda_- \leq B \leq \lambda_+ .$$

This proves the claim. ■

3 C^0-Estimate

Although the a priori C^0-estimate for equation (6) can be deduced from the C^0-estimate of the quaternionic Monge–Ampère equation, as shown in [3, 5, 19], we shall prove this fact using an argument that is specific to our setup.

Call $B_R(x_0)$ the open ball in $\mathbb{R}^N$ centred at x_0 and of radius $R > 0$. We need to recall the following results:

Theorem 3.1. Weak Harnack Estimate, Theorem 8.18 in [16]

Consider $1 \leq p < N/(N-2)$, and $q > N$, where $N > 2$ is an integer. For every $R > 0$, there exists a positive constant $C = C(N, R, p, q)$ such that

$$r^{-N/p}\|u\|_{L^p(B_{2r}(x_0))} \leq C\left(\inf_{x\in B_r(x_0)} u(x) + r^{2-2N/q}\|f\|_{L^{q/2}(B_R(x_0))}\right),$$

for any $x_0 \in \mathbb{R}^N$, $0 < r < R/4$, $f \in C^0(\mathbb{R}^N)$, and any $u \in C^2(\mathbb{R}^N)$ that is non-negative on $B_R(x_0)$ and such that $\Delta u(x) \leq f(x)$ for all $x \in B_R(x_0)$.

Theorem 3.2 Székelyhidi, [20]

Consider a map $u \in C^2(\mathbb{R}^N)$ and assume there exist a point $x_0 \in \mathbb{R}^N$ and numbers $R > 0$ and $\varepsilon > 0$, such that $\min_{|x-x_0|\leq R} u(x) = u(x_0)$, and

$$u(x_0) + 2R\varepsilon \leq \min_{|x-x_0|=R} u(x).$$

Then,

$$\varepsilon^N \leq \frac{2^N}{|B_R(0)|}\int_{\Gamma_\varepsilon} \det(D^2u),$$

where

$$\Gamma_\varepsilon = \left\{ x \in B_R(x_0) \mid u(y) \geq u(x) + \nabla u(x) \cdot (y - x),\, \forall y \in B_R(x_0),\, |\nabla u(x)| < \frac{\varepsilon}{2} \right\}.$$

Now, let us identify functions on T^5 with functions $\varphi\colon \mathbb{R}^5 \to \mathbb{R}$ that are periodic in each variable. Denote by $C^n(T^5)$ the Banach space of functions $\varphi\colon T^5 \to \mathbb{R}$ with C^n-norm

$$\|\varphi\|_{C^n} = \max_{|I| \leq n} \sup_{x \in \mathbb{R}^5} \left| \partial^I \varphi(x) \right|$$

where $I = \{i_1, \ldots, i_5\}$. We are adopting the multi-index notation $\partial^I = \partial_1^{i_1} \partial_2^{i_2} \partial_3^{i_3} \partial_4^{i_4} \partial_5^{i_5}$ with $|I| = i_1 + i_2 + i_3 + i_4 + i_5$. For $\alpha \in (0, 1)$ we also consider the Banach space $C^{n,\alpha}(T^5)$ of functions $\varphi \in C^n(T^5)$ with Hölder-continuous derivatives of order n:

$$\|\varphi\|_{C^{n,\alpha}} = \max\{\|\varphi\|_{C^n}, |\varphi|_{C^{n,\alpha}}\} < \infty\,,$$

where

$$|\varphi|_{C^{n,\alpha}} = \max_{|I|=n} \sup_{x \in \mathbb{R}^3} \sup_{0 < |h| \leq 1} \frac{\left|\partial^I \varphi(x+h) - \partial^I \varphi(x)\right|}{|h|^\alpha}\,.$$

Set

$$C_*^k(T^5) = \left\{ \varphi \in C^k(T^5) \mid \int_K \varphi = 0 \right\}$$

where

$$K = \left[-\frac{1}{2}, \frac{1}{2}\right]^5.$$

Theorem 3.4. Assume that $F \in C^0(T^5)$ satisfies (5). Let $\varphi \in C_*^2(T^5)$ be a solution to (6). Then, there is a positive constant C_0, depending on $\|F\|_{C^0}$ only, such that

$$\|\varphi\|_{C^0} \leq C_0\,. \tag{10}$$

Proof. Let $x_0 \in \mathbb{R}^5$ be a point where φ attains its minimum on K. Fix $\varepsilon > 0$ and define

$$u(x) = \varphi(x) - \max_K \varphi + 4\varepsilon \left|x - x_0\right|^2\,. \tag{11}$$

Then,

$$u(x_0)+\varepsilon=\varphi(x_0)-\max_K\varphi+\varepsilon\le\min_{|x-x_0|=1/2}\varphi(x)-\max_K\varphi+\varepsilon=\min_{|x-x_0|=1/2}u(x)$$

and by Theorem 3, with $R=1/2$, we have

$$\varepsilon^5\le\frac{2^5}{\left|B_{1/2}(0)\right|}\int_{\Gamma_\varepsilon}\det(D^2u)\,. \tag{12}$$

Differentiating (11) twice gives $D^2u=D^2\varphi+8\varepsilon I$. Hence, we may rewrite equation (6) as

$$(u_{11}+u_{22}+u_{33}+u_{44}-32\varepsilon+1)(u_{55}-8\varepsilon+1)-u_{15}^2-u_{25}^2-u_{35}^2-u_{45}^2=\mathrm{e}^F\,. \tag{13}$$

Now, on Γ_ε the function u is convex; therefore, the Hessian matrix $D^2u(x)$ is non-negative for all $x\in\Gamma_\varepsilon$. In particular $u_{ii}(x)\ge 0$ for all $i=1,\dots,5$ and every $x\in\Gamma_\varepsilon$. In addition,

$$u_{ii}(x)u_{55}(x)-u_{i5}^2(x)\ge 0,\qquad \text{for all } i=1,\dots,5, \text{ and every } x\in\Gamma_\varepsilon\,. \tag{14}$$

Set $\varepsilon=\varepsilon_0=1/48$, so that from (14) and (13) we obtain, for every $x\in\Gamma_{\varepsilon_0}$,

$$\begin{aligned}\frac{\Delta u(x)}{5}&\le\frac{5}{6}(u_{11}(x)+u_{22}(x)+u_{33}(x)+u_{44}(x))+\frac{1}{3}u_{55}(x)\\&\le\left(u_{11}(x)+u_{22}(x)+u_{33}(x)+u_{44}(x)+\frac{1}{3}\right)\left(u_{55}(x)+\frac{5}{6}\right)-\sum_{i=1}^{4}u_{i5}^2(x)-\frac{5}{18}\\&=\mathrm{e}^{F(x)}-\frac{5}{18}\le\mathrm{e}^{\max_K F}.\end{aligned}$$

Using again the fact that D^2u is non-negative on Γ_ε, the arithmetic–geometric mean inequality forces

$$\det(D^2u(x))\le\left(\frac{\Delta u(x)}{5}\right)^5\le\mathrm{e}^{5\max_K F},\qquad\text{for every } x\in\Gamma_{\varepsilon_0}\,. \tag{15}$$

At last, (12) and (15) imply

$$\left(\frac{1}{48}\right)^5\frac{\left|B_{1/2}(0)\right|}{2^5}\le\int_{\Gamma_{\varepsilon_0}}\det(D^2u)\le\mathrm{e}^{5\max_K F}\mathrm{meas}(\Gamma_{\varepsilon_0})\,,$$

that is,

$$\mathrm{meas}(\Gamma_{\varepsilon_0}) \geq \left|B_{1/2}(0)\right| \left(\frac{e^{-\max_K F}}{96}\right)^5 =: C\,. \tag{16}$$

Now observe that

$$u(x) \leq u(x_0) - \nabla u(x)\cdot(x_0 - x) \leq u(x_0) + \frac{\varepsilon_0}{4}\,, \qquad \text{for every } x \in \Gamma_{\varepsilon_0}\,,$$

that is,

$$\varphi(x) - \max_K \varphi + 4\varepsilon_0\,|x - x_0|^2 \leq \varphi(x_0) - \max_K \varphi + \frac{\varepsilon_0}{4} = \min_K \varphi - \max_K \varphi + \frac{\varepsilon_0}{4}\,, \qquad \text{for every } x \in \Gamma_{\varepsilon_0}\,.$$

This implies

$$\max_K \varphi - \min_K \varphi \leq \max_K \varphi - \varphi(x) + 1\,, \qquad \text{for every } x \in \Gamma_{\varepsilon_0}\,.$$

It follows that for every $p \geq 1$

$$\left(\max_K \varphi - \min_K \varphi\right)\left(\mathrm{meas}(\Gamma_{\varepsilon_0})\right)^{1/p} \leq \left(\int_{\Gamma_{\varepsilon_0}} \left(\max_K \varphi - \varphi + 1\right)^p\right)^{1/p} = \left\|\max_K \varphi - \varphi + 1\right\|_{L^p(\Gamma_{\varepsilon_0})}\,,$$

and since $\Gamma_{\varepsilon_0} \subseteq B_{1/2}(x_0) \subseteq K + x_0$, we have

$$\left\|\max_K \varphi - \varphi + 1\right\|_{L^p(\Gamma_{\varepsilon_0})} \leq \left\|\max_K \varphi - \varphi + 1\right\|_{L^p(K)}\,.$$

Therefore, since $\int_K \varphi = 0$, we have $\|\varphi\|_{C^0} \leq \max_K \varphi - \min_K \varphi$. Then, (16) implies

$$\|\varphi\|_{C^0} \leq \max_K \varphi - \min_K \varphi \leq C^{-1/p}\left(\left\|\max_K \varphi - \varphi\right\|_{L^p(K)} + 1\right), \qquad \forall p \geq 1\,. \tag{17}$$

By (8) we see that $\Delta(\max_K \varphi - \varphi) \leq 2$, and since $\max_K \varphi - \varphi \geq 0$ we can apply Theorem 3.2 with $\max_K \varphi - \varphi$ in place of u, $N = 5$, $p = 4/3$, $q = 6$, $x_0 \in K$ such that $\varphi(x_0) = \max_K \varphi$, $r = 1/2$ and $R = 3$. This eventually shows there exists a positive constant C' satisfying

$$\left\|\max_K \varphi - \varphi\right\|_{L^{4/3}(K)} \leq C'\left(\inf_K\left(\max_K \varphi - \varphi\right) + \|2\|_{L^3(K)}\right) = 2C'\,. \tag{18}$$

Estimate (10) now follows from (17) with $p = 4/3$ and (18). ■

4 C^0-estimate for the Laplacian

In this section we shall prove a C^0-estimate for the Laplacian of φ. The technique we employ is an adaptation of that found in [11].

Lemma 4.1. Let φ be a C^2 function on the n-torus T^n, fix $\mu \in \mathbb{R}$ and pick a point p_0 where $\Phi = (\Delta\varphi + 2)\mathrm{e}^{-\mu\varphi}$ attains its maximum value. Define

$$\eta_{ij} = \mu(\Delta\varphi + 2)(\varphi_{ij} + \mu\varphi_i\varphi_j) - \Delta\varphi_{ij}\,, \quad i, j = 1, \ldots, n\,.$$

Then,

$$\eta_{ii}(p_0) \geq 0\,, \ \text{ and } \sqrt{\eta_{ii}\eta_{jj}} \geq |\eta_{ij}| \text{ at } p_0\,,$$

for every $i, j = 1, \ldots, n$.

Proof. We begin by recalling the standard formulas

$$\nabla\Phi = \mathrm{e}^{-\mu\varphi}\left(\nabla\Delta\varphi - \mu(\Delta\varphi + 2)\nabla\varphi\right)$$

and

$$\begin{aligned}(\nabla\otimes\nabla)\Phi = &-\mu\mathrm{e}^{-\mu\varphi}\big(\nabla\varphi\otimes\nabla\Delta\varphi + \nabla\Delta\varphi\otimes\nabla\varphi\big) + \mu^2\mathrm{e}^{-\mu\varphi}\big((\Delta\varphi+2)\nabla\varphi\otimes\nabla\varphi\big)\\ &+ \mathrm{e}^{-\mu\varphi}\big((\nabla\otimes\nabla)\Delta\varphi - \mu(\Delta\varphi+2)(\nabla\otimes\nabla)\varphi\big)\,.\end{aligned}$$

Since

$$\nabla\Phi = 0\,, \quad (\nabla\otimes\nabla)\Phi \leq 0 \quad \text{at } p_0\,,$$

we infer

$$\nabla\Delta\varphi = \mu(\Delta\varphi + 2)\nabla\varphi \quad \text{at } p_0 \tag{19}$$

and

$$(\nabla\otimes\nabla)\Delta\varphi \leq \mu(\Delta\varphi + 2)((\nabla\otimes\nabla)\varphi + \mu\nabla\varphi\otimes\nabla\varphi) \quad \text{at } p_0\,.$$

In particular

$$\Big(\mu(\Delta\varphi+2)(\varphi_{ij}+\mu\varphi_i\varphi_j)-\Delta\varphi_{ij}\Big)^2 \leq \Big(\mu(\Delta\varphi+2)(\varphi_{ii}+\mu\varphi_i^2)-\Delta\varphi_{ii}\Big)\Big(\mu(\Delta\varphi+2)(\varphi_{jj}+\mu\varphi_j^2)-\Delta\varphi_{jj}\Big)$$

at p_0, for every $1\leq i,j\leq n$, and also

$$\mu(\Delta\varphi+2)(\varphi_{ii}+\mu\varphi_i\varphi_i)-\Delta\varphi_{ii}\geq 0 \quad \text{at } p_0\,, \quad i=1,\dots,n.$$

Hence, the claim follows. ■

Proposition 4.2. Let $F\in C^2(T^5)$ satisfy (5). There exists a positive constant C_1, depending on $\|F\|_{C^2}$ only, such that

$$\|\Delta\varphi\|_{C^0}\leq C_1(1+\|\varphi\|_{C^1}) \tag{20}$$

for any solution $\varphi\in C^4_*(T^5)$ to (6).

Proof. For starters,

$$\Delta e^F=\Delta AB+A\Delta B+2\nabla A\cdot\nabla B-2\sum_{i=1}^{4}\Big(\big|\nabla\varphi_{i5}\big|^2+\varphi_{i5}\Delta\varphi_{i5}\Big)\,. \tag{21}$$

Let p_0 and η_{ij} be as in Lemma 2 with

$$\mu=\frac{\varepsilon}{\max(\Delta\varphi+2)}$$

and $\varepsilon\in(0,1)$ to be determined later. Then, by using (9) with

$$\xi_i=\mathrm{sgn}(\varphi_{i5})\sqrt{\eta_{ii}},\; i=1,\dots,4\,,\quad \xi_5=\sqrt{\eta_{55}}\,,$$

we find

$$\mu(\Delta\varphi+2)\left(A(\varphi_{55}+\mu\varphi_5^2)+B\sum_{i=1}^{4}(\varphi_{ii}+\mu\varphi_i^2)\right)-A\underbrace{\Delta\varphi_{55}}_{\Delta B}-B\underbrace{\sum_{i=1}^{4}\Delta\varphi_{ii}}_{\Delta A}-2\sum_{i=1}^{4}\varphi_{i5}\xi_i\xi_5\geq 0\,.$$

at p_0. Lemma 2 now implies

$$\varphi_{i5}\xi_i\xi_5=\big|\varphi_{i5}\big|\sqrt{\eta_{ii}}\sqrt{\eta_{55}}\geq\varphi_{i5}\eta_{i5}\,,\quad \text{at } p_0\,,$$

that is,

$$\varphi_{i5}\xi_i\xi_5 \geq \varphi_{i5}\left(\mu(\Delta\varphi+2)(\varphi_{i5}+\mu\varphi_i\varphi_5)-\Delta\varphi_{i5}\right) \text{ at } p_0\,.$$

Therefore, we obtain

$$\mu(\Delta\varphi+2)\left(A(\varphi_{55}+\mu\varphi_5^2)+B\sum_{i=1}^{4}(\varphi_{ii}+\mu\varphi_i^2)\right)-2\sum_{i=1}^{4}\varphi_{i5}\left(\mu(\Delta\varphi+2)(\varphi_{i5}+\mu\varphi_i\varphi_5)\right)$$
$$\geq A\Delta B+B\Delta A-2\sum_{i=1}^{4}\varphi_{i5}\Delta\varphi_{i5}\,,\quad \text{at } p_0\,.$$

By (21), and the definition of A, B, at the point p_0 we have

$$\begin{aligned}\Delta e^F \leq&\mu(\Delta\varphi+2)\left(A(B-1)+B(A-1)\right)+2\nabla A\cdot\nabla B\\ &+\mu^2(\Delta\varphi+2)\left(A\varphi_5^2+B\sum_{i=1}^{4}\varphi_i^2\right)-2\mu(\Delta\varphi+2)\sum_{i=1}^{4}\left(\varphi_{i5}^2+\mu\varphi_{i5}\varphi_i\varphi_5\right)\\ =&2\mu(\Delta\varphi+2)\left(AB-\sum_{i=1}^{4}\varphi_{i5}^2\right)-\mu(\Delta\varphi+2)(A+B)+2\nabla A\cdot\nabla B\\ &+\mu^2(\Delta\varphi+2)\left(A\varphi_5^2+B(\varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2)-2\sum_{i=1}^{4}\varphi_{i5}\varphi_i\varphi_5\right)\\ \leq&2\mu(\Delta\varphi+2)e^F-\mu(\Delta\varphi+2)^2+2\nabla A\cdot\nabla B+2\mu^2(\Delta\varphi+2)\left(A\varphi_5^2+B(\varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2)\right).\end{aligned}$$

Observe that in the last inequality we used (9) with $\xi_i = \varphi_i(p_0)$ for $i = 1,\dots,4$ and $\xi_5 = -\varphi_5(p_0)$.

By (19) we then have

$$\mu^2(\Delta\varphi+2)^2\,|\nabla\varphi|^2=|\nabla\Delta\varphi|^2=|\nabla(A+B)|^2=|\nabla A|^2+|\nabla B|^2+2\nabla A\cdot\nabla B\geq 2\nabla A\cdot\nabla B\,,\quad \text{at } p_0\,,$$

and with the help of

$$A\varphi_5^2+B(\varphi_1^2+\varphi_2^2+\varphi_3^2+\varphi_4^2)\leq A\,|\nabla\varphi|^2+B\,|\nabla\varphi|^2=(\Delta\varphi+2)\,|\nabla\varphi|^2$$

we deduce

$$\mu(\Delta\varphi(p_0)+2)^2\leq-\Delta e^F(p_0)+2\mu(\Delta\varphi(p_0)+2)e^{F(p_0)}+3\mu^2(\Delta\varphi(p_0)+2)^2\,|\nabla\varphi(p_0)|^2\,. \tag{22}$$

Let us set

$$m=\Delta\varphi(p_0)+2\,,\quad \varphi_0=\varphi(p_0)\,.$$

Since p_0 is a maximum point for Φ, clearly

$$\max\Phi=me^{-\mu\varphi_0}\,.$$

From (22) we obtain

$$\mu m^2 \leq \left\|\Delta \mathrm{e}^F\right\|_{C^0} + 2\mu m \left\|\mathrm{e}^F\right\|_{C^0} + 3\mu^2 m^2 \|\nabla\varphi\|^2_{C^0} \,. \tag{23}$$

Now fix a point p_1 where $\Delta\varphi + 2$ reaches its maximum, and call $\varphi_1 = \varphi(p_1)$. Then

$$m \leq \max(\Delta\varphi + 2) = \mathrm{e}^{\mu\varphi_1}\Phi \leq m\mathrm{e}^{\mu(\varphi_1-\varphi_0)} \leq m\mathrm{e}^{2\mu\|\varphi\|_{C^0}}. \tag{24}$$

By the definition of μ and inequality (8), we have

$$2\mu = \frac{2}{\max(\Delta\varphi+2)}\,\varepsilon \leq \frac{1}{\mathrm{e}^{\min(F/2)}}\,\varepsilon \leq \mathrm{e}^{-\min(F/2)}\,,$$

hence by (24)

$$\varepsilon \exp\left(-\mathrm{e}^{-\min(F/2)}\|\varphi\|_{C^0}\right) \leq \varepsilon \mathrm{e}^{-2\mu\|\varphi\|_{C^0}} = \mu \max(\Delta\varphi+2)\mathrm{e}^{-2\mu\|\varphi\|_{C^0}} \leq \mu m$$

and also

$$\exp\left(-\mathrm{e}^{-\min(F/2)}\|\varphi\|_{C^0}\right)\max(\Delta\varphi+2) \leq \mathrm{e}^{-2\mu\|\varphi\|_{C^0}}\max(\Delta\varphi+2) \leq m\,.$$

Next, we multiply the last two inequalities and use (23), recalling that $\mu m \leq \varepsilon$, to the effect that

$$\varepsilon \exp\left(-2\mathrm{e}^{-\min(F/2)}\|\varphi\|_{C^0}\right)\max(\Delta\varphi+2) \leq \left\|\Delta\mathrm{e}^F\right\|_{C^0} + 2\varepsilon\left\|\mathrm{e}^F\right\|_{C^0} + 3\varepsilon^2\|\nabla\varphi\|^2_{C^0}.$$

Put otherwise,

$$\|\Delta\varphi\|_{C^0} \leq \exp\left(2\mathrm{e}^{-\min(F/2)}\|\varphi\|_{C^0}\right)\left(\frac{1}{\varepsilon}\left\|\Delta\mathrm{e}^F\right\|_{C^0} + 2\left\|\mathrm{e}^F\right\|_{C^0} + 3\varepsilon\|\nabla\varphi\|^2_{C^0}\right),$$

and by choosing

$$\varepsilon = \frac{1}{1+\|\nabla\varphi\|_{C^0}}$$

the claim is straighforward. ■

The next theorem will provide us with an a priori C^1-estimate for φ. Together with Proposition 2, it will give an a priori C^0-bound for $\Delta\varphi$.

Theorem 4.3. For all solutions $\varphi \in C^4_*(T^5)$ of equation (6) with $F \in C^2(T^5)$ satisfying (5) there exists a positive constant C_2, depending on $\|F\|_{C^2}$ only, such that

$$\|\varphi\|_{C^1} \leq C_2. \tag{25}$$

Proof. Fix $0 < \alpha < 1$ and $p = \frac{3}{1-\alpha} > 3$. Morrey's inequality says

$$\|\varphi\|_{C^{1,\alpha}} \leq C\,\|\varphi\|_{W^{2,p}}$$

for some positive constant C depending only on α. Elliptic L^p-estimates for the Laplacian also generate another constant C', still depending on α only, such that

$$\|\varphi\|_{W^{2,p}} \leq C'\left(\|\varphi\|_{L^p} + \|\Delta u\|_{L^p}\right)\,.$$

If $\varphi \in C^2(T^5)$, the C^0-estimate (10) for φ and bound (20) for $\Delta\varphi$ imply

$$\|\varphi\|_{L^p} + \|\Delta\varphi\|_{L^p} \leq \|\varphi\|_{C_0} + \|\Delta\varphi\|_{C^0} \leq C_0 + C_1(1 + \|\varphi\|_{C^1})\,.$$

Using standard interpolation theory (see [16,section 6.8]), for any $\varepsilon > 0$ there is a constant $P_\varepsilon > 0$, such that

$$\|\varphi\|_{C^1} \leq P_\varepsilon\,\|\varphi\|_{C^0} + \varepsilon\,\|\varphi\|_{C^{1,\alpha}}\,, \qquad \text{for every } \varphi \in C^{1,\alpha}(T^5)\,.$$

Putting all this together, we obtain

$$\|\varphi\|_{C^1} \leq P_\varepsilon C_0 + \varepsilon K_0\left(C_0 + C_1(1 + \|\varphi\|_{C^1})\right) = P_\varepsilon C_0 + \varepsilon K_0(C_0 + C_1) + \varepsilon K_0 C_1\,\|\varphi\|_{C^1}\,,$$

for some positive constant K_0, again depending on α only. This produces (25) once we choose

$$\varepsilon < \frac{1}{K_0 C_1}\,.$$

■

Corollary 4.4. Assume that $F \in C^2(T^5)$ satisfies (5) and let $\varphi \in C^4_*(T^5)$ be a solution to (6). Then, there exists a positive constant C_3, depending on $\|F\|_{C^2}$ only, such that

$$\|\Delta\varphi\|_{C^0} \leq C_3\,.$$

5 $C^{2,\alpha}$-estimate

The $C^{2,\alpha}$-estimate for our equation (6) can be deduced directly from the general result of Alesker, which we state next. It holds for compact hypercomplex manifolds that are locally flat, in the sense that they are locally isomorphic to $\mathbb{H}^n$.

Theorem 5.1. Theorem 4.1 in [2]

Let M be a $4n$-dimensional compact HKT manifold whose underlying hypercomplex structure is locally flat. Suppose $\varphi \in C^2(M)$ is a solution to the quaternionic Monge–Ampère equation (4). Then,

$$\|\varphi\|_{C^{2,\alpha}} \le C$$

for some $\alpha \in (0,1)$ and a positive constant C, both depending on M, Ω, $\|F\|_{C^2}$, $\|\varphi\|_{C_0}$ and $\|\tilde{\Delta}\varphi\|_{C^0}$, where

$$\tilde{\Delta}\varphi = \frac{\partial\partial_{J_2}\varphi \wedge \Omega^{n-1}}{\Omega^n}$$

and Ω is the HKT form.

The HKT structures we are considering on M_r are flat for the Obata connection [13,Proposition 6.1]. Hence the underlying hypercomplex structure is locally flat. Moreover, for T^3-invariant functions the operator $\tilde{\Delta}$ acts as a multiple of the Laplace operator, hence Theorem 6 and Corollary 1 imply

Proposition 5.2. Assume $F \in C^2(T^5)$ satisfies (5). For every solution $\varphi \in C^4_*(T^5)$ to equation (6), there exist $\alpha \in (0,1)$ and a positive constant C_4, depending on $\|F\|_{C^2}$, $\|\varphi\|_{C^0}$ only, such that

$$\|\varphi\|_{C^{2,\alpha}} \le C_4\,.$$

6 Proof of Theorem 1

In this section we shall use the previously established a priori estimates in order to prove the following result. This will then imply Theorem 1.

Theorem 6.1. Let $F \in C^\infty(T^5)$ satisfy (5). Then, equation (6) admits a solution $\varphi \in C^\infty_*(T^5)$.

Proof. For $t \in [0,1]$, we define

$$F_t = \log(1 - t + te^F)$$

and set

$$S_t = \left\{\varphi \in C^\infty_*(T^5) \mid (\varphi_{11} + \varphi_{22} + \varphi_{33} + \varphi_{44} + 1)(\varphi_{55} + 1) - \varphi_{15}^2 - \varphi_{25}^2 - \varphi_{35}^2 - \varphi_{45}^2 = e^{F_t}\right\},$$

and $S = \bigcup_{t\in[0,1]} S_t$. Clearly $0 \in S_0$, and S_1 is the set of smooth solutions of (6). We thus need to show that $S_1 \neq \emptyset$. For any $t \in [0,1]$ the map F_t satisfies (5) and

$$\max_{t\in[0,1]} \|F_t\|_{C^2} < \infty .$$

Proposition 3. therefore implies there exists $\alpha \in (0,1)$ such that

$$\sup_{\varphi\in S} \|\varphi\|_{C^{2,\alpha}} < \infty . \tag{26}$$

Let

$$\tau = \sup\{t \in [0,1] \mid S_t \neq \emptyset\} .$$

We claim that $S_\tau \neq \emptyset$ and $\tau = 1$.

$S_\tau \neq \emptyset$. Let $\{t_k\} \subseteq [0,1]$ be an increasing sequence converging to τ, and for any $k \in \mathbb{N}$ we fix $\varphi_k \in S_{t_k}$. Condition (26) implies that $\{\varphi_k\}$ is a sequence in $C^{2,\alpha}_*(T^5)$, so by the Ascoli–Arzelà theorem there exists a subsequence $\{\varphi_{k_j}\}$ converging to some ψ in $C^{2,\alpha/2}_*(T^5)$. The function ψ satisfies

$$(\psi_{11} + \psi_{22} + \psi_{33} + \psi_{44} + 1)(\psi_{55} + 1) - \psi_{15}^2 - \psi_{25}^2 - \psi_{35}^2 - \psi_{45}^2 = e^{F_\tau} .$$

In view of Proposition 1, equation (6) is elliptic, and elliptic regularity (see e.g., [21,Theorem 4.8, Chapter 14]) implies that ψ is in fact C^∞. Therefore, $S_\tau \neq \emptyset$, as required.

$\tau = 1$. Assume that, by contradiction, $\tau < 1$, and consider the nonlinear operator

$$T\colon C^{2,\alpha}_*(T^5) \times [0,1] \to C^{0,\alpha}_*(T^5)$$

defined by

$$T(\varphi,t)=(\varphi_{11}+\varphi_{22}+\varphi_{33}+\varphi_{44}+1)(\varphi_{55}+1)-\varphi_{15}^2-\varphi_{25}^2-\varphi_{35}^2-\varphi_{45}^2-\mathrm{e}^{F_t}\,.$$

Since $S_\tau\neq\emptyset$, there exists $\psi\in C^\infty_*(T^5)$ such that $T(\psi,\tau)=0$. Let $L\colon C^{2,\alpha}_*(T^5)\to C^{0,\alpha}_*(T^5)$ be the first variation of T with respect to the first variable. Then,

$$Lu=Au_{55}+B(u_{11}+u_{22}+u_{33}+u_{44})-2C_1u_{15}-2C_2u_{25}-2C_3u_{35}-2C_4u_{45}$$

where

$$A=(\psi_{11}+\psi_{22}+\psi_{33}+\psi_{44}+1)\,,\qquad B=(\psi_{55}+1)\,,\qquad C_i=\psi_{i5}\,,$$

which implies that L is elliptic since $\psi\in S_\tau$. The strong maximum principle guarantees L is injective because $L\varphi=0$ forces φ to be constant. Furthermore, ellipticity implies that L has closed range, and Schauder theory together with the method of continuity (see [16,Theorem 5.2]) ensures L is surjective. Hence, by the Implicit Function theorem there exists $\varepsilon>0$, such that for every fixed $t\in(\tau-\varepsilon,\tau+\varepsilon)$, equation

$$T(\varphi,t)=0$$

has a solution φ, which is additionally smooth by elliptic regularity. Therefore, $S_t\neq\emptyset$ for every $t\in(\tau,\tau+\varepsilon)$, which contradicts the maximality of τ. ■

7 Further Developments

As a follow-up to the present work, we plan to study the quaternionic Monge–Ampère equation on other homogeneous spaces.

The manifold M_2, for instance, can be regarded as a T^2-bundle over T^6, so it is quite natural to wonder whether Theorem 1 might extend to T^2-invariant functions (instead of T^3-invariant). We shall next describe this setup for M_2 and point out the differences from the T^3-invariant setting considered in Theorem 1.

From (7) the quaternionic Monge–Ampère equation (4) on (M_2,J_1,J_2,J_3,g) reduces to the following partial differential equation (PDE) on the six-dimensional base T^6 when the map F is T^2-invariant

$$(\varphi_{11}+\varphi_{22}+\varphi_{33}+\varphi_{44}+1)(\varphi_{55}+\varphi_{66}+1) - (\varphi_{35}-\varphi_{26})^2-(\varphi_{45}-\varphi_{16})^2-(\varphi_{46}+\varphi_{15})^2-(\varphi_{36}+\varphi_{25})^2 = \mathrm{e}^F\,, \quad (27)$$

where φ is an unknown function in $C^\infty(T^6)$. By calling

$$A=\varphi_{11}+\varphi_{22}+\varphi_{33}+\varphi_{44}+1\,, \qquad B=\varphi_{55}+\varphi_{66}+1$$

and

$$a_1=\varphi_{35}-\varphi_{26}\,,\qquad a_2=\varphi_{45}-\varphi_{16}\,,\qquad a_3=\varphi_{46}+\varphi_{15}\,,\qquad a_4=\varphi_{36}+\varphi_{25}\,,$$

we may rewrite (27) as

$$AB-\sum_{i=1}^4 a_i^2=\mathrm{e}^F\,. \quad (28)$$

The above is elliptic and

$$B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)+A(\xi_5^2+\xi_6^2)-2a_1(\xi_3\xi_5-\xi_2\xi_6)-2a_2(\xi_4\xi_5-\xi_1\xi_6) - 2a_3(\xi_4\xi_6+\xi_1\xi_5)-2a_4(\xi_3\xi_6+\xi_2\xi_5)>0, \quad (29)$$

for every $\xi\in\mathbb{R}^6$, $\xi\neq 0$.

In order to show that (27) can be solved, we need only prove an a priori C^0-estimate for the Laplacian of the solutions to (27). The natural approach consists in adapting the proof of Proposition 2 by mixing Lemma 7 with the ellipticity of the equation. In this case, however, it seems that condition (29) should be replaced with a stronger assumption, one implied by the estimate

$$2(|a_2a_3|+|a_1a_4|)<\mathrm{e}^F\,. \quad (30)$$

Applying the Laplacian operator to both sides of (28) we get

$$B\Delta A+A\Delta B+2\nabla A\cdot\nabla B-2\sum_{k=1}^4\left(|\nabla a_k|^2+a_k\Delta a_k\right)=\Delta\mathrm{e}^F,$$

which readily implies

$$\Delta \mathrm{e}^F \leq B\Delta A + A\Delta B + 2\nabla A\cdot\nabla B - 2\sum_{k=1}^{4} a_k \Delta a_k\,. \tag{31}$$

Let p_0 be a maximum point for $(\Delta\varphi + 2)\mathrm{e}^{-\mu\varphi}$, as in Lemma 7, and

$$\mu = \frac{1}{\max(\Delta\varphi+2)}\,\frac{1}{1+\|\nabla\varphi\|_{C^0}}\,.$$

Using (19), we see that the following relation holds at p_0

$$\mu^2(\Delta\varphi+2)^2\,|\nabla\varphi|^2 = |\nabla\Delta\varphi|^2 = |\nabla(A+B)|^2 = |\nabla A|^2 + |\nabla B|^2 + 2\nabla A\cdot\nabla B \geq 2\nabla A\cdot\nabla B\,,$$

that is,

$$2\nabla A\cdot\nabla B \leq \mu^2(\Delta\varphi+2)^2\,|\nabla\varphi|^2\,. \tag{32}$$

To produce an upper bound for $B\Delta A + A\Delta B - 2\sum_{k=1}^{4} a_k\Delta a_k$, we consider η_{ij} as in Lemma 2 and

$$\xi_i = \sqrt{\eta_{ii}}\,.$$

Then, at p_0 we have

$$\xi_i\xi_j \geq |\eta_{ij}|\,.$$

Moreover,

$$\begin{aligned}|a_1|(\xi_3\xi_5+\xi_2\xi_6) &\geq |a_1|\Big\{|\mu(\Delta\varphi+2)(\varphi_{35}+\mu\varphi_3\varphi_5)-\Delta\varphi_{35}| + |\mu(\Delta\varphi+2)(\varphi_{26}+\mu\varphi_2\varphi_6)-\Delta\varphi_{26}|\Big\}\\ &\geq a_1\Big\{\mu(\Delta\varphi+2)(\varphi_{35}+\mu\varphi_3\varphi_5)-\Delta\varphi_{35}-\mu(\Delta\varphi+2)(\varphi_{26}+\mu\varphi_2\varphi_6)+\Delta\varphi_{26}\Big\}\\ &= \mu(\Delta\varphi+2)(a_1^2+a_1\mu(\varphi_3\varphi_5-\varphi_2\varphi_6)) - a_1\Delta a_1\end{aligned}$$

at p_0, that is,

$$|a_1|(\xi_3\xi_5+\xi_2\xi_6) \geq \mu(\Delta\varphi+2)(a_1^2+\mu a_1(\varphi_3\varphi_5-\varphi_2\varphi_6)) - a_1\Delta a_1$$

at p_0. Similarly,

$$\left|a_2\right|(\xi_4\xi_5+\xi_1\xi_6)\geq\mu(\Delta\varphi+2)(a_2^2+\mu a_2(\varphi_4\varphi_5-\varphi_1\varphi_6))-a_2\Delta a_2\,,$$
$$\left|a_3\right|(\xi_4\xi_6+\xi_1\xi_5)\geq\mu(\Delta\varphi+2)(a_3^2+\mu a_3(\varphi_4\varphi_6+\varphi_1\varphi_5))-a_3\Delta a_3\,,$$
$$\left|a_4\right|(\xi_3\xi_6+\xi_2\xi_5)\geq\mu(\Delta\varphi+2)(a_4^2+\mu a_4(\varphi_3\varphi_6+\varphi_2\varphi_5))-a_4\Delta a_4\,,$$

at p_0. If we add up the last four inequalities and use (29) with $\xi_k=\varphi_k$ for $k=1,\dots,4$ and $\xi_5=-\varphi_5$, $\xi_6=-\varphi_6$, we end up with

$$\begin{aligned}&2|a_1|(\xi_3\xi_5+\xi_2\xi_6)+2\left|a_2\right|(\xi_4\xi_5+\xi_1\xi_6)+2\left|a_3\right|(\xi_4\xi_6+\xi_1\xi_5)+2\left|a_4\right|(\xi_3\xi_6+\xi_2\xi_5)\geq\\&\mu(\Delta\varphi+2)\left(\sum_{k=1}^4(2a_k^2-\mu B\varphi_k^2)-\mu A(\varphi_5^2+\varphi_6^2)\right)-2\sum_{k=1}^4a_k\Delta a_k\end{aligned}$$

at p_0.

To handle the last inequality, we need the following estimate

$$\begin{aligned}&B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)+A(\xi_5^2+\xi_6^2)\geq\\&\quad 2|a_1|(\xi_3\xi_5+\xi_2\xi_6)+2\left|a_2\right|(\xi_4\xi_5+\xi_1\xi_6)+2\left|a_3\right|(\xi_4\xi_6+\xi_1\xi_5)+2\left|a_4\right|(\xi_3\xi_6+\xi_2\xi_5).\end{aligned}\tag{33}$$

Notice this is stronger than (29).

In fact, if we assume (33), then

$$B\sum_{k=1}^4\xi_k^2+A(\xi_5^2+\xi_6^2)\geq\mu(\Delta\varphi+2)\left(\sum_{k=1}^4(2a_k^2-\mu B\varphi_k^2)-\mu A(\varphi_5^2+\varphi_6^2)\right)-2\sum_{k=1}^4a_k\Delta a_k$$

at p_0 and, keeping in mind the definition of ξ_k,

$$B\sum_{k=1}^4\xi_k^2+A(\xi_5^2+\xi_6^2)=\mu(\Delta\varphi+2)\left(A\sum_{k=5}^6(\varphi_{kk}+\mu\varphi_k^2)+B\sum_{k=1}^4(\varphi_{kk}+\mu\varphi_k^2)\right)-A\Delta B-B\Delta A\,,$$

at p_0.

Therefore,

$$\mu(\Delta\varphi+2)\left(A\sum_{k=5}^{6}(\varphi_{kk}+\mu\varphi_k^2)+B\sum_{k=1}^{4}(\varphi_{kk}+\mu\varphi_k^2)\right)-A\Delta B-B\Delta A\geq$$
$$\mu(\Delta\varphi+2)\left(\sum_{k=1}^{4}(2a_k^2-\mu B\varphi_k^2)-\mu A(\varphi_5^2+\varphi_6^2)\right)-2\sum_{k=1}^{4}a_k\Delta a_k\,,$$

at p_0, which implies

$$\begin{aligned}
&A\Delta B+B\Delta A-2\sum_{k=1}^{4}a_k\Delta a_k\leq\\
&\leq\mu(\Delta\varphi+2)\left(A\sum_{k=5}^{6}(\varphi_{kk}+2\mu\varphi_k^2)+B\sum_{k=1}^{4}(\varphi_{kk}+2\mu\varphi_k^2)-2\sum_{k=1}^{4}a_k^2\right)\\
&\leq\mu(\Delta\varphi+2)\left(2AB-(A+B)+2\mu(A+B)\,|\nabla\varphi|^2-2\sum_{k=1}^{4}a_k^2\right)\\
&=\mu(\Delta\varphi+2)\left(2e^F-(\Delta\varphi+2)+2\mu(\Delta\varphi+2)\,|\nabla\varphi|^2\right)\,,
\end{aligned}$$

at p_0. In other terms,

$$A\Delta B+B\Delta A-2\sum_{k=1}^{4}a_k\Delta a_k\leq\mu(\Delta\varphi+2)\left(2e^F-(\Delta\varphi+2)+2\mu(\Delta\varphi+2)\,|\nabla\varphi|^2\right)\tag{34}$$

at p_0. From (31), (32), and (34) we finally deduce

$$\mu(\Delta\varphi+2)^2\leq-\Delta e^F+2\mu(\Delta\varphi+2)e^F+3\mu^2(\Delta\varphi+2)^2\,|\nabla\varphi|^2\,,$$

at p_0. At this juncture the a priori C^0-estimate for $\Delta\varphi$ can be obtained as we did in the second part of Section 4.

Let us point out that requiring (33) for every $\xi\in\mathbb{R}^6$ is equivalent to (30). Indeed the quadratic form

$$\begin{aligned}
Q(\xi)\;=\;&B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)+A(\xi_5^2+\xi_6^2)\\
&-2|a_1|(\xi_3\xi_5+\xi_2\xi_6)-2\left|a_2\right|(\xi_4\xi_5+\xi_1\xi_6)-2\left|a_3\right|(\xi_4\xi_6+\xi_1\xi_5)-2\left|a_4\right|(\xi_3\xi_6+\xi_2\xi_5)
\end{aligned}$$

has matrix

$$\begin{pmatrix} B & 0 & 0 & 0 & -|a_3| & -|a_2| \\ 0 & B & 0 & 0 & -|a_4| & -|a_1| \\ 0 & 0 & B & 0 & -|a_1| & -|a_4| \\ 0 & 0 & 0 & B & -|a_2| & -|a_3| \\ -|a_3| & -|a_4| & -|a_1| & -|a_2| & A & 0 \\ -|a_2| & -|a_1| & -|a_4| & -|a_3| & 0 & A \end{pmatrix},$$

which is positive definite if and only if

$$B^4\left(\left(A - B^{-1}\sum_{k=1}^{4} a_k^2\right)^2 - 4B^{-2}\left(|a_2a_3| + |a_1a_4|\right)^2\right) > 0$$

since $B > 0$. A direct computation tells that the last condition is equivalent to

$$2(|a_2a_3| + |a_1a_4|) < \mathrm{e}^F .$$

In analogy to the above discussion, the manifold M_1 arises as an S^1-bundle over a T^7-torus, and the function F may be chosen to be S^1-invariant. If so, the quaternionic Monge–Ampère equation (4) reads

$$\begin{aligned} &(\varphi_{11} + \varphi_{22} + \varphi_{33} + \varphi_{44} + 1)(\varphi_{55} + \varphi_{66} + \varphi_{77} + 1) \\ &-(\varphi_{45} - \varphi_{16} - \varphi_{27})^2 - (\varphi_{35} + \varphi_{17} - \varphi_{26})^2 \\ &-(\varphi_{36} + \varphi_{47} + \varphi_{25})^2 - (\varphi_{46} - \varphi_{37} + \varphi_{15})^2 = \mathrm{e}^F , \end{aligned}$$

where φ is an unknown function in $C^\infty(T^7)$.

Setting

$$A = \varphi_{11} + \varphi_{22} + \varphi_{33} + \varphi_{44} + 1 , \qquad B = \varphi_{55} + \varphi_{66} + \varphi_{77} + 1$$

and

$$a_1 = \varphi_{45} - \varphi_{16} - \varphi_{27} , \qquad a_2 = \varphi_{35} + \varphi_{17} - \varphi_{26} ,$$

$$a_3 = \varphi_{36} + \varphi_{47} + \varphi_{25} , \qquad a_4 = \varphi_{46} - \varphi_{37} + \varphi_{15} ,$$

the equation turns into

$$AB - \sum_{i=1}^{4} a_i^2 = \mathrm{e}^F . \tag{35}$$

The above is elliptic and

$$B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2) + A(\xi_5^2+\xi_6^2+\xi_7^2) - 2a_1(\xi_4\xi_5 - \xi_1\xi_6 - \xi_2\xi_7) \tag{36}$$

$$-2a_2(\xi_3\xi_5+\xi_1\xi_7-\xi_2\xi_6) - 2a_3(\xi_3\xi_6+\xi_4\xi_7+\xi_2\xi_5) - 2a_4(\xi_4\xi_6-\xi_3\xi_7+\xi_1\xi_5) > 0 ,$$

for every $\xi \in \mathbb{R}^7$, $\xi \neq 0$.

We proceed as in the previous case, and choose p_0 and η_{ij} as in Lemma 2 and

$$\mu = \frac{1}{\max(\Delta\varphi + 2)} \frac{1}{1 + \|\nabla\varphi\|_{C^0}} ,$$

resulting in

$$\Delta \mathrm{e}^F \leq B\Delta A + A\Delta B + \mu^2(\Delta\varphi+2)^2 |\nabla\varphi|^2 - 2\sum_{k=1}^{4} a_k \Delta a_k , \quad \text{at } p_0 .$$

Set $\xi_i = \sqrt{\eta_{ii}}$ and apply Lemma 2 to obtain

$$\begin{aligned}
&|a_1|(\xi_4\xi_5+\xi_1\xi_6+\xi_2\xi_7)\\
&\geq |a_1| \Big\{ \big|\mu(\Delta\varphi+2)(\varphi_{45}+\mu\varphi_4\varphi_5) - \Delta\varphi_{45}\big| + \big|\mu(\Delta\varphi+2)(\varphi_{16}+\mu\varphi_1\varphi_6) - \Delta\varphi_{16}\big| \\
&\quad + \big|\mu(\Delta\varphi+2)(\varphi_{27}+\mu\varphi_2\varphi_7) - \Delta\varphi_{27}\big| \Big\}\\
&\geq a_1 \Big\{ \mu(\Delta\varphi+2)(\varphi_{45}+\mu\varphi_4\varphi_5) - \Delta\varphi_{45} - \mu(\Delta\varphi+2)(\varphi_{16}+\mu\varphi_1\varphi_6) + \Delta\varphi_{16}\\
&\quad - \mu(\Delta\varphi+2)(\varphi_{27}+\mu\varphi_2\varphi_7) + \Delta\varphi_{27} \Big\}\\
&= \mu(\Delta\varphi+2)(a_1^2 + a_1\mu(\varphi_4\varphi_5 - \varphi_1\varphi_6 - \varphi_2\varphi_7)) - a_1\Delta a_1
\end{aligned}$$

at p_0, that is,

$$|a_1|(\xi_4\xi_5+\xi_1\xi_6+\xi_2\xi_7) \geq \mu(\Delta\varphi+2)(a_1^2 + \mu a_1(\varphi_4\varphi_5-\varphi_1\varphi_6-\varphi_2\varphi_7)) - a_1\Delta a_1$$

at p_0. From that we deduce

$$\left|a_2\right|(\xi_3\xi_5+\xi_1\xi_7+\xi_2\xi_6) \geq \mu(\Delta\varphi+2)(a_2^2+\mu a_2(\varphi_3\varphi_5+\varphi_1\varphi_7-\varphi_2\varphi_6)) - a_2\Delta a_2\,,$$
$$\left|a_3\right|(\xi_3\xi_6+\xi_4\xi_7+\xi_2\xi_5) \geq \mu(\Delta\varphi+2)(a_3^2+\mu a_3(\varphi_3\varphi_6+\varphi_4\varphi_7+\varphi_2\varphi_5)) - a_3\Delta a_3\,,$$
$$\left|a_4\right|(\xi_4\xi_6+\xi_3\xi_7+\xi_1\xi_5) \geq \mu(\Delta\varphi+2)(a_4^2+\mu a_4(\varphi_4\varphi_6-\varphi_3\varphi_7+\varphi_1\varphi_5)) - a_4\Delta a_4\,,$$

at p_0. The sum of the previous four inequalities, together with (36), yields

$$\begin{aligned}&2|a_1|(\xi_4\xi_5+\xi_1\xi_6+\xi_2\xi_7)+2\left|a_2\right|(\xi_3\xi_5+\xi_1\xi_7+\xi_2\xi_6)\\ &+2\left|a_3\right|(\xi_3\xi_6+\xi_4\xi_7+\xi_2\xi_5)+2\left|a_4\right|(\xi_4\xi_6+\xi_3\xi_7+\xi_1\xi_5)\geq\\ &\mu(\Delta\varphi+2)\left(\sum_{k=1}^{4}(2a_k^2-\mu B\varphi_k^2)-\mu A\sum_{k=5}^{7}\varphi_k^2\right)-2\sum_{k=1}^{4}a_k\Delta a_k\end{aligned}$$

at p_0.

We need the following estimate

$$\begin{aligned}&B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)+A(\xi_5^2+\xi_6^2+\xi_7^2)-2|a_1|(\xi_4\xi_5+\xi_1\xi_6+\xi_2\xi_7)\\ &-2|a_2|(\xi_3\xi_5+\xi_1\xi_7+\xi_2\xi_6)-2|a_3|(\xi_3\xi_6+\xi_4\xi_7+\xi_2\xi_5)-2|a_4|(\xi_4\xi_6+\xi_3\xi_7+\xi_1\xi_5)>0\,,\end{aligned}\tag{37}$$

at p_0, which is stronger than (36). Once this has been established, the result follows.

To prove (38) one has to show that the quadratic form

$$\begin{aligned}Q(\xi)&=B(\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)+A(\xi_5^2+\xi_6^2+\xi_7^2)-2|a_1|(\xi_4\xi_5+\xi_1\xi_6+\xi_2\xi_7)\\ &-2|a_2|(\xi_3\xi_5+\xi_1\xi_7+\xi_2\xi_6)-2|a_3|(\xi_3\xi_6+\xi_4\xi_7+\xi_2\xi_5)-2|a_4|(\xi_4\xi_6+\xi_3\xi_7+\xi_1\xi_5)\end{aligned}$$

on $\mathbb{R}^7$ is positive-definite. This is equivalent to demanding two things:

$$e^{2F}-4(\left|a_2a_3\right|+\left|a_1a_4\right|)^2>0\,,$$

$$\begin{aligned}&e^{3F}-4e^{F}\left((\left|a_2a_3\right|+\left|a_1a_4\right|)^2+(\left|a_1a_3\right|+\left|a_2a_4\right|)^2+(\left|a_1a_2\right|+\left|a_3a_4\right|)^2\right)\\ &-16\left(\left|a_2a_3\right|+\left|a_1a_4\right|\right)\left(\left|a_1a_3\right|+\left|a_2a_4\right|\right)\left(\left|a_1a_2\right|+\left|a_3a_4\right|\right)>0\,.\end{aligned}$$

We wrap up this overview of our future plans by observing that there exist torus fibrations whose hypercomplex structure is not locally trivial. On these spaces Alesker's theorem cannot be applied, so once the C^0-estimate of the Laplacian is at hand one needs to prove the $C^{2,\alpha}$-estimate by alternative arguments.

We expect that the study of the equation on these explicit examples will give new insight for the handling of the general case.

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