Journal of Pure and Applied Algebra 225 (2021) 106754

Contents lists available at ScienceDirect

Journal of Pure and Applied Algebra

www.elsevier.com/locate/jpaa

Brauer groups of 1-motives

Cristiana Bertolin *, Federica Galluzzi

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto 10, Italy

A R T I C L E I N F O

Article history:
Received 14 October 2020
Received in revised form 16 March 2021
Available online 8 April 2021
Communicated by C.A. Weibel

MSC:
14F22; 16H05

Keywords:
Gerbes on a stack
Azumaya algebras over a stack
Brauer group of a stack
1-motives

A B S T R A C T

Over a normal base scheme, we prove the generalized Theorem of the Cube for 1-motives and that a torsion class of the group $\mathrm{H}^2_{ét}(M, \mathbb{G}_{m,M})$ of a 1-motive M, whose pull-back via the unit section $\epsilon : S \to M$ is zero, comes from an Azumaya algebra. In particular, we deduce that over an algebraically closed field of characteristic zero, all classes of $\mathrm{H}^2_{ét}(M, \mathbb{G}_{m,M})$ come from Azumaya algebras.

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* Corresponding author.
E-mail addresses: cristiana.bertolin@unito.it (C. Bertolin), federica.galluzzi@unito.it (F. Galluzzi).

https://doi.org/10.1016/j.jpaa.2021.106754

0. Introduction

Grothendieck has defined the Brauer group $\mathrm{Br}(X)$ of a scheme X as the group of similarity classes of Azumaya algebras over X. In [25, I, §1] he constructed an injective group homomorphism

$$\delta : \mathrm{Br}(X) \longrightarrow \mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m) \tag{0.1}$$

from the Brauer group of X to the étale cohomology group $\mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$. This homomorphism is not in general bijective, as pointed out by Grothendieck in [25, II, §2], where he found a scheme X whose Brauer group is a torsion group but whose étale cohomology group $\mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$ is not torsion. However, if X is quasi-compact the elements of $\delta(\mathrm{Br}(X))$ are torsion elements of $\mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$, and so Grothendieck asked in [25] the following question:

Question. *For a quasi-compact scheme X, is the image of $\mathrm{Br}(X)$ via the homomorphism δ (0.1) the torsion subgroup $\mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)_{\mathrm{Tors}}$ of $\mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$?*

Grothendieck showed that if X is regular, the étale cohomology group $\mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$ is a torsion group, and so under this hypothesis the question is whether the Brauer group of X is all of $\mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)$.

The following well-known results are related to this question: If X has dimension ≤ 1 or if X is regular and of dimension ≤ 2, then the Brauer group of X is all of $\mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_{m,X})$ ([25, II, Cor 2.2]). Gabber (unpublished theorem) showed that the Brauer group of a quasi-compact and separated scheme X endowed with an ample invertible sheaf is isomorphic to $\mathrm{H}^2_{\acute{e}t}(X, \mathbb{G}_m)_{\mathrm{Tors}}$. A different proof of this result was found by de Jong (see [18]).

In [20] Giraud introduced gerbes in the general setting of non abelian cohomology following Grothendieck's ideas: in particular he proved that gerbes give a geometrical description of classes of the group $\mathrm{H}^2(X, \mathbb{G}_m)$.

The aim of this paper is to extend Grothendieck's theory of Brauer groups to 1-motives, using gerbes as fundamental tools.

In particular,

- we study gerbes on stacks which are not separated;
- we study Azumaya algebras and Brauer groups for stacks which are not separated;
- we apply the above results to 1-motives using the dictionary between length two complexes of abelian sheaves and Picard stacks developed by Deligne in [17, Exposé XVIII, §1.4]. Remark that the Picard stacks associated to 1-motives are not algebraic in the sense of [30] since they are not quasi-separated.

We proceed in the following way:

Let $\mathbf{S}$ be a site. In Section 1 we associate to a stack in groupoids $\mathcal{X}$ over $\mathbf{S}$ the site $\mathbf{S}(\mathcal{X})$, which allows us to study the notion of sheaf and gerbe on a stack.

In Section 2 we prove the following homological interpretation of F-gerbes, with F an abelian sheaf on a site $\mathbf{S}$: the Picard 2-stack $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F)$ of F-gerbes is equivalent (as Picard 2-stack) to the Picard 2-stack associated to the complex $F[2]$, where $F[2] = [F \to 0 \to 0]$ with F in degree -2:

$$\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F) \cong 2\mathrm{st}(F[2]) \tag{0.2}$$

(Theorem 2.2). In particular, for $i = 2, 1, 0$, we have an isomorphism of abelian groups between the i-th classifying group $\mathbb{G}\mathrm{erbe}^i_{\mathbf{S}}(F)$ and the cohomological group $\mathrm{H}^i(\mathbf{S}, F)$. The equivalence of Picard 2-stacks (0.2) contains the following classical result: elements of $\mathbb{G}\mathrm{erbe}^2_{\mathbf{S}}(F)$, which are F-equivalence classes of F-gerbes, are parametrized by cohomological classes of $\mathrm{H}^2(\mathbf{S}, F)$. Always in Section 2, applying [20, Chp IV] to the site $\mathbf{S}(\mathcal{X})$ associated to a stack $\mathcal{X}$, we obtain the Picard 2-stack $\mathbb{G}\mathrm{erbe}_{\mathbf{S}(\mathcal{X})}(\mathcal{F})$ of $\mathcal{F}$-gerbes on $\mathcal{X}$, with $\mathcal{F}$ an abelian sheaf on the site $\mathbf{S}(\mathcal{X})$. We finish Section 2 proving the effectiveness of the 2-descent of $\mathbb{G}_m$-gerbes with respect to a faithfully flat morphism of schemes which is quasi-compact or locally of finite presentation (Theorem 2.7).

Let $\mathbf{S}_{ét}$ be the étale site on an arbitrary scheme S and let $\mathcal{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a locally ringed S-stack with associated étale site $\mathbf{S}_{ét}(\mathcal{X})$. In Section 3 we recall the notion of the Brauer group $\mathrm{Br}(\mathcal{X})$ of $\mathcal{X}$ and in Theorem 3.4 we establish an injective group homomorphism

$$\delta : \mathrm{Br}(\mathcal{X}) \longrightarrow \mathrm{H}^2_{ét}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}}), \tag{0.3}$$

which extends Grothendieck's group homomorphism (0.1) to locally ringed S-stacks.

Let $M = [u : X \to G]$ be a 1-motive defined over a scheme S, with X an S-group scheme which is, locally for the étale topology, a constant group scheme defined by a finitely generated free $\mathbb{Z}$-module, G an extension of an abelian S-scheme by an S-torus, and finally $u : X \to G$ a morphism of S-group schemes. Since in [17, Exposé XVIII, §1.4] Deligne associates to any length two complex of abelian sheaves a Picard stack, in Section 4 we can define the Brauer group of the 1-motive M as the Brauer group $\mathrm{Br}(\mathcal{M})$ of the associated Picard stack $\mathcal{M}$ and by Theorem 3.4 we have an injective group homomorphism $\delta : \mathrm{Br}(\mathcal{M}) \to \mathrm{H}^2_{ét}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}})$. At the end of Section 4 we prove the effectiveness of the descent of Azumaya algebras and of $\mathbb{G}_m$-gerbes with respect to the quotient map $\iota : G \to [G/X] \cong \mathcal{M}$ (Lemma 4.2 and Lemma 4.3).

Denote by $s_{ij} := \mathcal{M} \times_{\mathbf{S}} \mathcal{M} \to \mathcal{M} \times_{\mathbf{S}} \mathcal{M} \times_{\mathbf{S}} \mathcal{M}$ the map which inserts the unit section $\epsilon : \mathbf{S} \to \mathcal{M}$ of $\mathcal{M}$ into the k-th factor for $k \in \{1, 2, 3\} - \{i, j\}$. If ℓ is a prime number distinct from the residue characteristics of S, we say that the 1-motive M satisfies **the generalized Theorem of the Cube** for the prime ℓ if the homomorphism

$$\begin{array}{rcl} \prod_{(i,j)\in\{1,2,3\}} s^*_{ij} : \mathrm{H}^2_{ét}(\mathcal{M}^3, \mathbb{G}_{m,\mathcal{M}^3})(\ell) & \longrightarrow & \left(\mathrm{H}^2_{ét}(\mathcal{M}^2, \mathbb{G}_{m,\mathcal{M}^2})(\ell)\right)^3 \\ x & \longmapsto & (s^*_{12}(x), s^*_{13}(x), s^*_{23}(x)) \end{array}$$

is injective, where (ℓ) denotes the ℓ-primary component (Definition 5.1). We start Section 5 studying the consequences of the generalized Theorem of the Cube for 1-motives. In Corollary 5.6 we show that if the base scheme is connected, reduced, normal and noetherian, extensions of abelian schemes by split tori satisfy the generalized Theorem of the Cube for any prime ℓ distinct from the residue characteristics of S (Corollary 5.6). Then, as a consequence of the effectiveness of the 2-descent of $\mathbb{G}_m$-gerbes with respect to the quotient map $\iota : G \to [G/X] \cong \mathcal{M}$ (Lemma 4.3), we get Theorem 5.7: 1-motives, which are defined over a connected, reduced, normal and noetherian scheme S, and whose underlying tori are split, satisfy the generalized Theorem of the Cube for any prime ℓ distinct from the residue characteristics of S. Note that in [8, Thm 5.1] S. Brochard and the first author prove the Theorem of the Cube (involving the $\mathrm{H}^1(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}})$ instead of $\mathrm{H}^2(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}})$) for 1-motives, and in [9] the authors show that the sheaf of divisorial correspondences of extensions of abelian schemes by tori is representable.

In Section 6 we investigate Grothendieck's Question for 1-motives and our answer is contained in Theorem 6.2 which states that if $M = [u : X \to G]$ is 1-motive defined over a normal and noetherian scheme S and if the extension G underlying M satisfies the generalized Theorem of the Cube for a prime number ℓ distinct from the residue characteristics of S, then the ℓ-primary component of the kernel of the homomorphism $\mathrm{H}^2_{ét}(\epsilon) : \mathrm{H}^2_{ét}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}}) \to \mathrm{H}^2_{ét}(S, \mathbb{G}_{m,S})$ induced by the unit section $\epsilon : S \to M$ of M, is contained in the Brauer group of M:

$$\ker\left[\mathrm{H}^2_{ét}(\epsilon) : \mathrm{H}^2_{ét}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}}) \longrightarrow \mathrm{H}^2_{ét}(S, \mathbb{G}_{m,S})\right](\ell) \subseteq \mathrm{Br}(\mathcal{M}).$$

We prove this result as follows: first we show this theorem for an extension of an abelian scheme by a torus using Hoobler's Theorem [27, Thm 3.3] (Proposition 6.1). Then, thanks to the effectiveness of the descent of Azumaya algebras and of $\mathbb{G}_m$-gerbes with respect to the quotient map $\iota : G \to [G/X] \cong \mathcal{M}$, we get the required statement for M. We finish Section 6 giving a positive answer to Grothendieck's Question for 1-motives (and so in particular for semi-abelian varieties) over an algebraically closed field of characteristic zero (Corollary 6.3).

In the last years, several authors have worked with the Brauer group of stacks (see for example [1], [19], [31]) but most of them focus on algebraic or separated stacks. Moreover the techniques used in this paper are rather different from the ones used in [1], [19], [31]. Since the Picard stack associated to a 1-motive is not quasi-separated, we recall the theory of Brauer group of stacks.

An important role in this paper is played by the 2-descent theory of gerbes for which we add an Appendix.

Acknowledgment

We are very grateful to Pierre Deligne for his comments on the first version of this paper and for his communication on 2-descent theory for stacks (see Appendix). We would like to thank also the referee for the very useful comments.

Notation

Stack language

Here we refer mainly to [20]. Let $\mathbf{S}$ be a site. A **stack** over $\mathbf{S}$ is a fibered category $\mathcal{X}$ over $\mathbf{S}$ such that

- (*Gluing condition on objects*) descent is effective for objects in $\mathcal{X}$, and
- (*Gluing condition on arrows*) for any object U of $\mathbf{S}$ and for every pair of objects X, Y of the category $\mathcal{X}(U)$, the presheaf of arrows $\mathrm{Arr}_{\mathcal{X}(U)}(X, Y)$ of $\mathcal{X}(U)$ is a sheaf over U.

For the notions of morphisms of stacks (i.e. catesian functors) and morphism of cartesian functors we refer to [20, Chp. II 1.2]. An **equivalence** (resp. **isomorphism**) **of stacks** $F : \mathcal{X} \to \mathcal{Y}$ is a morphism of stacks which is an equivalence (resp. isomorphism) of fibered categories over $\mathbf{S}$, that is $F(U) : \mathcal{X}(U) \to \mathcal{Y}(U)$ is an equivalence (resp. isomorphism) of categories for any object U of $\mathbf{S}$. A **stack in groupoids** over $\mathbf{S}$ is a stack $\mathcal{X}$ over $\mathbf{S}$ such that for any object U of $\mathbf{S}$ the category $\mathcal{X}(U)$ is a groupoid, i.e. a category in which all arrows are invertible. Recall that 2-morphisms of stacks in groupoids are automatically invertible. *From now on, all stacks will be stacks in groupoids.*

A **gerbe** over the site $\mathbf{S}$ is a stack $\mathcal{G}$ over $\mathbf{S}$ such that

- $\mathcal{G}$ is locally not empty: for any object U of $\mathbf{S}$, there exists a covering $\{\phi_i : U_i \to U\}_{i \in I}$ for which the set of objects of the category $\mathcal{G}(U_i)$ is not empty for all $i \in I$;
- $\mathcal{G}$ is locally connected: for any object U of $\mathbf{S}$ and for each pair of objects g_1 and g_2 of $\mathcal{G}(U)$, there exists a covering $\{\phi_i : U_i \to U\}_{i \in I}$ of U such that the set of arrows from $g_{1|U_i}$ to $g_{2|U_i}$ in $\mathcal{G}(U_i)$ is not empty for all $i \in I$.

A **morphism** (resp. **isomorphism**) **of gerbes** is just a morphism (resp. isomorphism) of stacks whose source and target are gerbes, and a 2-morphism of gerbes is a morphism of cartesian functors. An **equivalence of gerbes** is an equivalence of stacks.

A **strictly commutative Picard stack** over the site $\mathbf{S}$ (just called a Picard stack) is a stack $\mathcal{P}$ over $\mathbf{S}$ endowed with a morphism of stacks $\otimes : \mathcal{P} \times_{\mathbf{S}} \mathcal{P} \to \mathcal{P}$, called the group law of $\mathcal{P}$, and two natural isomorphisms a and c, expressing the associativity and the commutativity constraints of the group law of $\mathcal{P}$, such that $\mathcal{P}(U)$ is a strictly commutative Picard category for any object U of $\mathbf{S}$ (see [17] 1.4.2 for more details). An **additive functor** $(F, \sum) : \mathcal{P}_1 \to \mathcal{P}_2$ between two Picard stacks is a morphism of stacks $F : \mathcal{P}_1 \to \mathcal{P}_2$ endowed with a natural isomorphism $\sum : F(a \otimes_{\mathcal{P}_1} b) \cong F(a) \otimes_{\mathcal{P}_2} F(b)$ (for all $a, b \in \mathcal{P}_1$) which is compatible with the natural isomorphisms a and c underlying $\mathcal{P}_1$ and $\mathcal{P}_2$.

A **strict 2-category** (just called 2-category) $\mathbb{A} = (A, C(a,b), K_{a,b,c}, U_a)_{a,b,c \in A}$ is given by the following data: a set A of objects $a, b, c, ...$; for each ordered pair (a,b) of objects of A, a category $C(a,b)$; for each ordered triple (a,b,c) of objects A, a composition functor $K_{a,b,c} : C(b,c) \times C(a,b) \to C(a,c)$, that satisfies the associativity law; for each object a, a unit functor $U_a : \mathbf{1} \to C(a,a)$ where $\mathbf{1}$ is the terminal category, that provides a left and right identity for the composition functor.

A **2-stack** over the site $\mathbf{S}$ is a fibered 2-category $\mathbb{X}$ over $\mathbf{S}$ (i.e. a family of 2-categories indexed by objects of $\mathbf{S}$, see [15, 1.10 p.29] for more details) such that

- 2-descent is effective for objects in $\mathbb{X}$ (see [15, 1.10 p.31]), and
- for any object U of $\mathbf{S}$ and for every pair of objects X, Y of the 2-category $\mathbb{X}(U)$, the fibered category of arrows $\mathrm{Arr}_{\mathbb{X}(U)}(X,Y)$ of $\mathbb{X}(U)$ is a stack over $\mathbf{S}_{|U}$.

For the notions of morphisms of 2-stacks (i.e. cartesian 2-functors), morphisms of cartesian 2-functors, modifications of 2-stacks and equivalences of 2-stacks, we refer to [26, Chp I]. A **2-stack in 2-groupoids** over $\mathbf{S}$ is a 2-stack $\mathbb{X}$ over $\mathbf{S}$ such that for any object U of $\mathbf{S}$ the 2-category $\mathbb{X}(U)$ is a 2-groupoid. *From now on, all 2-stacks will be 2-stacks in 2-groupoids.*

Let S be an arbitrary scheme and denote by $\mathbf{S}$ the site of S for a Grothendieck topology that we will fix later. We will call a stack, a Picard stack, a 2-stack over $\mathbf{S}$ respectively an S-stack, a Picard S-stack, an S-2-stack.

1. Recall on sheaves, gerbes and Picard stacks on a stack

Let $\mathbf{S}$ be a site. Let $\mathcal{X}$ be a stack over $\mathbf{S}$. We always assume that fibered (2-)categories come with a fixed cleavage (see [16, §2, §6]). Deligne furnished us the following definition of site associated to a stack.

Definition 1.1. The **site $\mathbf{S}(\mathcal{X})$ associated to $\mathcal{X}$ over $\mathbf{S}$** is the site defined in the following way:

- the category underlying $\mathbf{S}(\mathcal{X})$ consists of the objects (U,u) with U an object of $\mathbf{S}$ and u an object of $\mathcal{X}(U)$, and of the arrows $(\phi, \Phi) : (U,u) \to (V,v)$ with $\phi : U \to V$ a morphism of $\mathbf{S}$ and $\Phi : \phi^* v \to u$ an isomorphism in $\mathcal{X}(U)$. We call the pair (U,u) an **open of** $\mathcal{X}$ with respect to the chosen topology.
- the topology on $\mathbf{S}(\mathcal{X})$ is the one generated by the pre-topology for which a covering of (U,u) is a family $\{(\phi_i, \Phi_i) : (U_i, u_i) \to (U,u)\}_i$ such that the morphism of $\mathbf{S}$ $\coprod \phi_i : \coprod U_i \to U$ is a covering of U.

Definition 1.2. A **sheaf (of sets)** $\mathcal{F}$ **on** $\mathcal{X}$ is a system $(\mathcal{F}_{U,u}, \theta_{\phi,\Phi})$, where for any object (U,u) of $\mathbf{S}(\mathcal{X})$, $\mathcal{F}_{U,u}$ is a sheaf on $\mathbf{S}_{|U}$, and for any arrow $(\phi, \Phi) : (U,u) \to (V,v)$ of $\mathbf{S}(\mathcal{X})$, $\theta_{\phi,\Phi} : \mathcal{F}_{V,v} \to \phi_* \mathcal{F}_{U,u}$ is a morphism of sheaves on $\mathbf{S}_{|V}$, such that

(*i*) if $(\phi, \Phi) : (U,u) \to (V,v)$ and $(\gamma, \Gamma) : (V,v) \to (W,w)$ are two arrows of $\mathbf{S}(\mathcal{X})$, then $\gamma_* \theta_{\phi,\Phi} \circ \theta_{\gamma,\Gamma} = \theta_{\gamma \circ \phi, \phi^* \Gamma \circ \Phi}$;

(*ii*) if $(\phi, \Phi) : (U,u) \to (V,v)$ is an arrow of $\mathbf{S}(\mathcal{X})$, the morphism of sheaves $\phi^{-1} \mathcal{F}_{V,v} \to \mathcal{F}_{U,u}$, obtained by adjunction from $\theta_{\phi,\Phi}$, is an isomorphism.

To simplify notations, we denote just $(\mathcal{F}_{U,u})$ the sheaf $\mathcal{F} = (\mathcal{F}_{U,u}, \theta_{\phi,\Phi})$. The set of **global sections** $\Gamma(\mathcal{X}, \mathcal{F})$ of a sheaf $\mathcal{F}$ on $\mathcal{X}$ is the set of families $(s_{U,u})$ of sections of $\mathcal{F}$ on the objects (U,u) of $\mathbf{S}(\mathcal{X})$ such that for any arrow $(\phi, \Phi) : (U,u) \to (V,v)$ of $\mathbf{S}(\mathcal{X})$, $\mathrm{res}_\phi s_{V,v} = s_{U,u}$.

An **abelian sheaf** $\mathcal{F}$ **on** $\mathcal{X}$ is a system $(\mathcal{F}_{U,u})$ verifying the conditions (*i*) and (*ii*) of Definition 1.2, where the $\mathcal{F}_{U,u}$ are abelian sheaves on $\mathbf{S}_{|U}$. We denote by $\mathrm{Ab}(\mathcal{X})$ the category of abelian sheaves on $\mathcal{X}$. According to [23, Exp II, Prop. 6.7] and [21, Thm 1.10.1], the category $\mathrm{Ab}(\mathcal{X})$ is an abelian category with enough injectives. Let $\mathrm{R}\Gamma(\mathcal{X}, -)$ be the right derived functor of the functor $\Gamma(\mathcal{X}, -) : \mathrm{Ab}(\mathcal{X}) \to \mathrm{Ab}$ of global sections (here

Ab is the category of abelian groups). The i-th cohomology group $\mathrm{H}^i\big(\mathrm{R}\Gamma(\mathcal{X},-)\big)$ of $\mathrm{R}\Gamma(\mathcal{X},-)$ is denoted by $\mathrm{H}^i(\mathcal{X},-)$.

A **stack on** $\mathcal{X}$ is a stack $\mathcal{Y}$ over $\mathbf{S}$ endowed with a morphism of stacks $P:\mathcal{Y}\to\mathcal{X}$ (called the structural morphism) such that for any object (U,x) of $\mathbf{S}(\mathcal{X})$ the fibered product $U\times_{x,\mathcal{X},P}\mathcal{Y}$ is a stack over $\mathbf{S}_{|U}$.

A **gerbe on** $\mathcal{X}$ is stack $\mathcal{G}$ over $\mathbf{S}$ endowed with a morphism of stacks $P:\mathcal{G}\to\mathcal{X}$ (called the structural morphism) such that for any object (U,x) of $\mathbf{S}(\mathcal{X})$ the fibered product $U\times_{x,\mathcal{X},P}\mathcal{G}$ is a gerbe over $\mathbf{S}_{|U}$. A **morphism of gerbes on** $\mathcal{X}$ is a morphism of gerbes which is compatible with the underlying structural morphisms.

Let $F:\mathcal{X}\to\mathcal{Y}$ be a morphism of S-stacks and let $\mathcal{G}$ be a gerbe on $\mathcal{Y}$. The **pull-back of** $\mathcal{G}$ **via** F is the fibered product

$$F^*\mathcal{G}:=\mathcal{X}\times_{F,\mathcal{Y},P}\mathcal{G} \tag{1.1}$$

of $\mathcal{X}$ and $\mathcal{G}$ via the morphism $F:\mathcal{X}\to\mathcal{Y}$ and the structural morphism $P:\mathcal{G}\to\mathcal{Y}$ underlying $\mathcal{G}$ (see [7, Def 2.14] for the definition of fibered product of S-stacks).

A **Picard stack on** $\mathcal{X}$ is a stack $\mathcal{P}$ over $\mathbf{S}$ endowed with a morphism of stacks $P:\mathcal{P}\to\mathcal{X}$ (called the structural morphism), with a morphism of stacks $\otimes:\mathcal{P}\times_{P,\mathcal{X},P}\mathcal{P}\to\mathcal{P}$, and with two natural isomorphisms a and c, such that $U\times_{x,\mathcal{X},P}\mathcal{P}$ is a Picard stack over $\mathbf{S}_{|U}$ for any object (U,x) of $\mathbf{S}(\mathcal{X})$.

A **Picard 2-stack on** $\mathcal{X}$ is a 2-stack $\mathbb{P}$ over $\mathbf{S}$ endowed with a morphism of 2-stacks $P:\mathbb{P}\to\mathcal{X}$ (called the structural morphism - here we see $\mathcal{X}$ as a 2-stack), with a morphism of 2-stacks $\otimes:\mathbb{P}\times_{P,\mathcal{X},P}\mathbb{P}\to\mathbb{P}$, and with two natural 2-transformations a and c, such that $U\times_{x,\mathcal{X},P}\mathbb{P}$ is a Picard 2-stack over $\mathbf{S}_{|U}$ for any object (U,x) of $\mathbf{S}(\mathcal{X})$ (for more details see [7, §1] or [6]). An **additive 2-functor** $(F,\lambda_F):\mathbb{P}_1\to\mathbb{P}_2$ between two Picard 2-stacks on $\mathcal{X}$ is given by a morphism of 2-stacks $F:\mathbb{P}_1\to\mathbb{P}_2$ and a natural 2-transformation $\lambda_F\colon \otimes_{\mathbb{P}_2}\circ F^2\to F\circ\otimes_{\mathbb{P}_1}$, which are compatible with the structural morphisms of 2-stacks $P_1:\mathbb{P}_1\to\mathcal{X}$ and $P_2:\mathbb{P}_2\to\mathcal{X}$ and with the natural 2-transformations a and c underlying $\mathbb{P}_1$ and $\mathbb{P}_2$. An **equivalence of Picard 2-stacks on** $\mathcal{X}$ is an additive 2-functor whose underlying morphism of 2-stacks is an equivalence of 2-stacks.

Denote by $2\mathbb{P}\mathrm{icard}(\mathcal{X},\mathbf{S})$ the category whose objects are Picard 2-stacks on $\mathcal{X}$ and whose arrows are isomorphism classes of additive 2-functors. Applying [36, Cor 6.5] to the site $\mathbf{S}(\mathcal{X})$, we have the following equivalence of categories

$$2\mathrm{st}:\mathcal{D}^{[-2,0]}(\mathbf{S}(\mathcal{X}))\longrightarrow 2\mathbb{P}\mathrm{icard}(\mathcal{X},\mathbf{S}), \tag{1.2}$$

where $\mathcal{D}^{[-2,0]}(\mathbf{S}(\mathcal{X}))$ is the derived category of length three complexes of abelian sheaves on $\mathcal{X}$. Via this equivalence, Picard 2-stacks (resp. Picard stacks) on $\mathcal{X}$ correspond to length three (resp. two) complexes of abelian sheaves on $\mathcal{X}$. Therefore, the theory of Picard stacks is included in the theory of Picard 2-stacks. We denote by $[\ ]$ the inverse equivalence of 2st.

If $\mathcal{P}$ is a Picard stack over a site $\mathbf{S}$ we define its **classifying groups** $\mathcal{P}^i$ for $i=1,0$ in the following way: $\mathcal{P}^1$ is the group of isomorphism classes of objects of $\mathcal{P}$ and $\mathcal{P}^0$ is the group of automorphisms of the neutral object e of $\mathcal{P}$. We define the classifying groups $\mathbb{P}^i$ for $i=2,1,0$ of a Picard 2-stack $\mathbb{P}$ over a site $\mathbf{S}$ recursively: $\mathbb{P}^2$ is the group of equivalence classes of objects of $\mathbb{P}$, $\mathbb{P}^1=\mathcal{A}\mathrm{ut}^1(e)$ and $\mathbb{P}^0=\mathcal{A}\mathrm{ut}^0(e)$ where $\mathcal{A}\mathrm{ut}(e)$ is the Picard stack of automorphisms of the neutral object e of $\mathbb{P}$. Explicitly, $\mathbb{P}^1$ is the group of isomorphism classes of objects of $\mathcal{A}\mathrm{ut}(e)$ and $\mathbb{P}^0$ is the group of automorphisms of the neutral object of $\mathcal{A}\mathrm{ut}(e)$. We have the following link between the classifying groups $\mathbb{P}^i$ and the cohomology groups $\mathrm{H}^i(\mathbf{S},[\mathbb{P}])$ of the complex $[\mathbb{P}]$ associated to $\mathbb{P}$ via (1.2): $\mathbb{P}^i\cong\mathrm{H}^{i-2}(\mathbf{S},[\mathbb{P}])$ for $i=0,1,2$.

If two Picard 2-stacks $\mathbb{P}$ and $\mathbb{P}'$ are equivalent as Picard 2-stacks, then their classifying groups are isomorphic: $\mathbb{P}^i\cong\mathbb{P}'^i$ for $i=2,1,0$. The inverse affirmation is not true as explained in [3, Rem 1.3].

Let S be an arbitrary scheme and denote by $\mathbf{S}$ the site of S for a Grothendieck topology. Let $\mathcal{X}$ be an S-stack. A stack (resp. a Picard 2-stack) on $\mathcal{X}$ will be called an S-stack (resp. a Picard S-2-stack) on $\mathcal{X}$.

2. Gerbes with Abelian band on a stack

Let F be an abelian sheaf on a site $\mathbf{S}$. Denote by $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F)$ the fibered 2-category of F-gerbes over $\mathbf{S}$.

Lemma 2.1. *The fibered 2-category $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F)$ of F-gerbes is a Picard 2-stack over $\mathbf{S}$.*

Proof. By [15, §2.6] the 2-descent is effective for objects of $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F)$. Moreover, morphisms of gerbes are just morphisms of stacks and so by [15, Examples 1.11 i)], the gluing condition on arrows of $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F)$ is satisfied. Thus, the fibered 2-category $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F)$ is in fact a 2-stack over $\mathbf{S}$. In [20, Chp IV Prop 2.4.1 (i)] Giraud has defined the contracted product of gerbes (see in particular [20] Example 2.4.3 for the case of gerbes bound by abelian sheaves). He also showed that this contracted product satisfies associativity and commutativity constraints (see [20, Chp IV Cor 2.4.2 (i) and (ii)]). Hence we can conclude that the contracted product of F-gerbes endows the 2-stack of F-gerbes with a Picard structure. □

2.1. Homological interpretation of gerbes over a site

Let F be an abelian sheaf on a site $\mathbf{S}$. The **classifying groups** $\mathbb{G}\mathrm{erbe}^i_{\mathbf{S}}(F)$ for $i = 2, 1, 0$ of the Picard 2-stack $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F)$ are

- $\mathbb{G}\mathrm{erbe}^2_{\mathbf{S}}(F)$, the abelian group of F-equivalence classes of F-gerbes;
- $\mathbb{G}\mathrm{erbe}^1_{\mathbf{S}}(F)$, the abelian group of isomorphism classes of morphisms of F-gerbes from a F-gerbe to itself.
- $\mathbb{G}\mathrm{erbe}^0_{\mathbf{S}}(F)$, the abelian group of automorphisms of a morphism of F-gerbes from a F-gerbe to itself.

Theorem 2.2. *Let F be an abelian sheaf on a site $\mathbf{S}$. Then the Picard 2-stack $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F)$ of F-gerbes is equivalent (as Picard 2-stack) to the Picard 2-stack associated to the complex $F[2]$, where $F[2] = [F \to 0 \to 0]$ with F in degree -2:*

$$\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F) \cong 2\mathrm{st}(F[2]).$$

In particular, for $i = 2, 1, 0$, we have an isomorphism of abelian groups between the i-th classifying group $\mathbb{G}\mathrm{erbe}^i_{\mathbf{S}}(F)$ and the cohomological group $\mathrm{H}^i(\mathbf{S}, F)$.

Proof. It is a classical result that via the equivalence of categories stated in [17, Exposé XVIII, Prop 1.4.15], the complex $F[1]$ corresponds to the Picard stack $\mathcal{T}\mathrm{ors}(F)$ of F-torsors: $\mathcal{T}\mathrm{ors}(F) = 2\mathrm{st}(F[1])$. A higher dimensional analogue of the notion of torsor under an abelian sheaf is the notion of torsor under a Picard stack, which was introduced by Breen in [13, Def 3.1.8] and studied by the first author in [5, §2] (remark that in fact in [7] the first author introduces the notion of torsor under a Picard 2-stack, see also [4], [2] and [10]). Hence we have the notion of $\mathcal{T}\mathrm{ors}(F)$-torsors. The contracted product of torsors under a Picard 2-stack, introduced in [7, Def 2.11], endows the 2-stack $\mathbb{T}\mathrm{ors}(\mathcal{T}\mathrm{ors}(F))$ of $\mathcal{T}\mathrm{ors}(F)$-torsors with a Picard structure, and by [7, Thm 0.1] this Picard 2-stack $\mathbb{T}\mathrm{ors}(\mathcal{T}\mathrm{ors}(F))$ corresponds, via the equivalence of categories (1.2), to the complex $[\mathcal{T}\mathrm{ors}(F)][1]$:

$$\mathbb{T}\mathrm{ors}(\mathcal{T}\mathrm{ors}(F)) = 2\mathrm{st}(F[2]). \tag{2.1}$$

In [15, Prop 2.14] Breen constructs a canonical equivalence of Picard 2-stacks between the Picard 2-stack $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F)$ of F-gerbes and the Picard 2-stack $\mathbb{T}\mathrm{ors}(\mathcal{T}\mathrm{ors}(F))$ of $\mathcal{T}\mathrm{ors}(F)$-torsors:

$$\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F) \cong \mathbb{T}\mathrm{ors}(\mathcal{T}\mathrm{ors}(F)). \tag{2.2}$$

This equivalence and the equality (2.1) furnish the expected equivalence $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F) \cong 2\mathrm{st}(F[2])$. The classifying groups of the Picard 2-stack $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(F)$ are therefore

$$\mathbb{G}\mathrm{erbe}^{i}_{\mathbf{S}}(F) \cong \mathrm{H}^{i-2}(\mathbf{S}, F[2]) = \mathrm{H}^{i}(\mathbf{S}, F). \quad \square$$

Remark 2.3. Via the cohomological interpretation (2.1) of torsors under the Picard stack of F-torsors, the equivalence of Picard 2-stacks (2.2) is the geometrical counterpart of the canonical isomorphism in cohomology $\mathrm{H}^2(\mathbf{S}, F) \cong \mathrm{H}^1(\mathbf{S}, F[1])$.

2.2. Gerbes on a stack

Let $\mathcal{X}$ be a stack over a site $\mathbf{S}$ and denote by $\mathbf{S}(\mathcal{X})$ the site associated to $\mathcal{X}$. Applying [20, Chp IV] to the site $\mathbf{S}(\mathcal{X})$, we get the notion of $\mathcal{F}$-gerbes on the stack $\mathcal{X}$, with $\mathcal{F}$ an abelian sheaf on $\mathcal{X}$. We recall briefly this notion.

An $\mathcal{F}$**-gerbe** is a gerbe $\mathcal{G}$ on $\mathcal{X}$ such that for any object (U, x) of $\mathbf{S}(\mathcal{X})$ the fibered product $U \times_{x,\mathcal{X},P} \mathcal{G}$ is a $\mathcal{F}_{U,x}$-gerbe over $\mathbf{S}_{|U}$ (here $P : \mathcal{G} \to \mathcal{X}$ is the structural morphism): in particular for each i indexing a covering $\{U_i \to U\}_i$ of U, it exists an object g_i of $(U \times_{x,\mathcal{X},P} \mathcal{G})(U_i)$ and an isomorphism $\mathcal{F}_{U,x|U_i} \to \underline{\mathrm{Aut}}(g_i)$ of sheaves of groups on U_i (see [15, Def 2.3]). Consider now an $\mathcal{F}$-gerbe $\mathcal{G}$ and an $\mathcal{F}'$-gerbe $\mathcal{G}'$ on $\mathcal{X}$. Let $u : \mathcal{F} \to \mathcal{F}'$ a morphism of abelian sheaves on $\mathcal{X}$. A morphism of gerbes $m : \mathcal{G} \to \mathcal{G}'$ is an u**-morphism** if u is compatible with the morphism of bands $\mathrm{Band}(\underline{\mathrm{Aut}}(g)_{U,x}) \to \mathrm{Band}(\underline{\mathrm{Aut}}(m(g))_{U,x})$ (see [20, Chp IV 2.1.5.1]). As in [20, Chp IV Prop 2.2.6] an u-morphism $m : \mathcal{G} \to \mathcal{G}'$ is fully faithful if and only if $u : \mathcal{F} \to \mathcal{F}'$ is an isomorphism, in which case m is an equivalence of gerbes. If $\mathcal{G}$ and $\mathcal{G}'$ are two $\mathcal{F}$-gerbes on $\mathcal{X}$, instead of $\mathrm{id}_{\mathcal{F}}$-morphism $\mathcal{G} \to \mathcal{G}'$ we use the terminology $\mathcal{F}$**-equivalence** $\mathcal{G} \to \mathcal{G}'$ **of** $\mathcal{F}$**-gerbes** on $\mathcal{X}$.

$\mathcal{F}$-gerbes on $\mathcal{X}$ build a Picard 2-stack on $\mathcal{X}$, that we denote by

$$\mathbb{G}\mathrm{erbe}_{\mathbf{S}(\mathcal{X})}(\mathcal{F})$$

whose group law is given by the contracted product of $\mathcal{F}$-gerbes on $\mathcal{X}$ ([20, Chp IV 2.4.3]). Its neutral element is the stack $\mathcal{T}\mathrm{ors}(\mathcal{F})$ of $\mathcal{F}$-torsors on $\mathcal{X}$, which is called the **neutral** $\mathcal{F}$**-gerbe**. Applying Theorem 2.2 to the abelian sheaf $\mathcal{F}$ on the site $\mathbf{S}(\mathcal{X})$ (see Definition 1.2) we get

Corollary 2.4. *We have the following equivalence of Picard 2-stacks*

$$\mathbb{G}\mathrm{erbe}_{\mathbf{S}(\mathcal{X})}(\mathcal{F}) \cong 2\mathrm{st}(\mathcal{F}[2]).$$

In particular, $\mathbb{G}\mathrm{erbe}^{i}_{\mathbf{S}(\mathcal{X})}(\mathcal{F}) \cong \mathrm{H}^{i}(\mathcal{X}, \mathcal{F})$ *for* $i = 2, 1, 0$.

Hence, $\mathcal{F}$-equivalence classes of $\mathcal{F}$-gerbes on $\mathcal{X}$, which are the elements of the 0th-homotopy group $\mathbb{G}\mathrm{erbe}^{2}_{\mathbf{S}(\mathcal{X})}(\mathcal{F})$, are parametrized by cohomological classes of $\mathrm{H}^2(\mathcal{X}, \mathcal{F})$.

Remark 2.5. $\mathbb{G}\mathrm{erbe}_{\mathbf{S}(\mathcal{X})}(\mathcal{F})$ is a Picard $\mathbf{S}(\mathcal{X})$-2-stack. Via the structural morphism $F : \mathcal{X} \to \mathbf{S}$, we can view $\mathbb{G}\mathrm{erbe}_{\mathbf{S}(\mathcal{X})}(\mathcal{F})$ also as a Picard $\mathbf{S}$-2-stack $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(\mathcal{F})$. In this case we have that $\mathbb{G}\mathrm{erbe}_{\mathbf{S}}(\mathcal{F}) \cong 2\mathrm{st}(\tau_{\leq 0} RF_*(\mathcal{F}[2]))$ where $\tau_{\leq 0}$ is the good truncation in degree 0. We will not use this fact in the paper and therefore we omit the proof.

2.3. 2-descent of $\mathbb{G}_m$-gerbes

We finish this section proving the effectiveness of the 2-descent of $\mathbb{G}_m$-gerbes with respect to a faithfully flat morphism of schemes $p : S' \to S$ which is quasi-compact or locally of finite presentation. We will need the **semi-local description of gerbes** done by Breen in [16, §4] and [14, §2.3], that we recall only in the case of $\mathbb{G}_m$-gerbes. Denote by $\mathcal{T}\mathrm{ors}(\mathbb{G}_m)$ the Picard stack of $\mathbb{G}_m$-torsors. According to Breen, to have a $\mathbb{G}_m$-gerbe $\mathcal{G}$ over a site $\mathbf{S}$ is equivalent to have the data

$$\big((\mathcal{T}\mathrm{ors}(\mathbb{G}_{m,U}), \Psi_x), (\psi_x, \xi_x)\big)_{x\in\mathcal{G}(U), U\in\mathbf{S}} \tag{2.3}$$

indexed by the objects x of the $\mathbb{G}_m$-gerbe $\mathcal{G}$ (recall that $\mathcal{G}$ is locally not empty), where

- $\Psi_x : \mathcal{G}_{|U} \to \mathcal{T}\mathrm{ors}(\mathbb{G}_{m,U})$ is an equivalence of U-stacks between the restriction $\mathcal{G}_{|U}$ to U of the $\mathbb{G}_m$-gerbe $\mathcal{G}$ and the neutral gerbe $\mathcal{T}\mathrm{ors}(\mathbb{G}_{m,U})$. This equivalence is determined by the object x in $\mathcal{G}(U)$,
- $\psi_x = pr_1^*\Psi_x \circ (pr_2^*\Psi_x)^{-1} : \mathcal{T}\mathrm{ors}(pr_2^*\mathbb{G}_{m,U}) \to \mathcal{T}\mathrm{ors}(pr_1^*\mathbb{G}_{m,U})$ is an equivalence of stacks over $U \times_S U$ (here $pr_i : U \times_S U \to U$ are the projections), which restricts to the identity when pulled back via the diagonal morphism $\Delta : U \to U \times_S U$, and
- $\xi_x : pr_{23}^*\psi_x \circ pr_{12}^*\psi_x \Rightarrow pr_{13}^*\psi_x$ is a isomorphism of cartesian S-functors between morphisms of stacks on $U \times_S U \times_S U$ (here $pr_{ij} : U \times_S U \times_S U \to U \times_S U$ are the partial projections), which satisfies the compatibility condition

$$pr_{134}^*\xi_x \circ [pr_{34}^*\psi_x * pr_{123}^*\xi_x] = pr_{124}^*\xi_x \circ [pr_{234}^*\xi_x * pr_{12}^*\psi_x] \tag{2.4}$$

  when pulled back to $U \times_S U \times_S U \times_S U := U^4$ (here $pr_{ijk} : U^4 \to U \times_S U \times_S U$ and $pr_{ij} : U^4 \to U \times_S U$ are the partial projections. See [12, (6.2.7)-(6.2.8)] for more details).

Therefore, the $\mathbb{G}_m$-gerbe $\mathcal{G}$ may be reconstructed from the local data $(\mathcal{T}\mathrm{ors}(\mathbb{G}_m), \Psi_x)_x$ using the transition data (ψ_x, ξ_x). We call the equivalences of stacks Ψ_x the **local neutralizations** of the $\mathbb{G}_m$-gerbe $\mathcal{G}$ defined by the local objects $x \in \mathcal{G}(U)$. The transition data (ψ_x, ξ_x) are in fact 2-descent data. See Appendix for this reconstruction of a $\mathbb{G}_m$-gerbe via local neutralizations and 2-descent data.

In §5 we will need the semi-local description of a $\mathbb{G}_m$-equivalence class of a $\mathbb{G}_m$-gerbe which consists in the following data: a family $(\mathcal{T}\mathrm{ors}^1(\mathbb{G}_{m,U}))_{U\in\mathbf{S}}$ of groups of isomorphism classes of $\mathbb{G}_m$-torsors, bijections $\mathcal{T}\mathrm{ors}^1(pr_2^*\mathbb{G}_{m,U}) \to \mathcal{T}\mathrm{ors}^1(pr_1^*\mathbb{G}_{m,U})$ of their pull-backs on $U \times_S U$ via the projections $pr_i : U \times_S U \to U$, and compatibility conditions on the pull-back on $U \times_S U \times_S U$ of these bijections (here we use the above notations).

Remark 2.6. In this paper, Breen's semi-local description of gerbes allows us to reduce of one the degree of the cohomology groups involved: instead of working with gerbes, which are cohomology classes of $\mathrm{H}^2(\mathbf{S}, \mathbb{G}_m)$, we can work with torsors, which are cohomology classes of $\mathrm{H}^1(\mathbf{S}, \mathbb{G}_m)$.

Theorem 2.7. *Let $p : S' \to S$ be a faithfully flat morphism of schemes which is quasi-compact or locally of finite presentation. To have a $\mathbb{G}_{m,S}$-gerbe over $\mathbf{S}$ is equivalent to have a triple*

$$(\mathcal{G}', \varphi, \gamma)$$

where $\mathcal{G}'$ is a $\mathbb{G}_{m,S'}$-gerbe over $\mathbf{S}'$ and (φ, γ) are 2-descent data on $\mathcal{G}'$ with respect to $p : S' \to S$. More precisely,

- *$\mathcal{G}'$ is a $\mathbb{G}_{m,S'}$-gerbe over $\mathbf{S}'$,*
- *$\varphi : p_1^*\mathcal{G}' \to p_2^*\mathcal{G}'$ is an equivalence of gerbes over $S' \times_S S'$, where $p_i : S' \times_S S' \to S'$ are the natural projections,*
- *$\gamma : p_{23}^*\varphi \circ p_{12}^*\varphi \Rightarrow p_{13}^*\varphi$ is a natural isomorphism over $S' \times_S S' \times_S S'$, where $p_{ij} : S' \times_S S' \times_S S' \to S' \times_S S'$ are the partial projections,*

such that over $S' \times_S S' \times_S S' \times_S S'$ the compatibility condition

$$p_{134}^* \gamma \circ [p_{34}^* \varphi * p_{123}^* \gamma] = p_{124}^* \gamma \circ [p_{234}^* \gamma * p_{12}^* \varphi] \tag{2.5}$$

is satisfied, where $p_{ijk} : S' \times_S S' \times_S S' \times_S S' \times_S S' \to S' \times_S S' \times_S S'$ and $p_{ij} : S' \times_S S' \times_S S' \times_S S' \to S' \times_S S'$ are the partial projections.

Under this equivalence, the pull-back $p^* : \mathbb{G}\mathrm{erbe}_S(\mathbb{G}_{m,S}) \to \mathbb{G}\mathrm{erbe}_{S'}(\mathbb{G}_{m,S'})$ is the additive 2-functor which forgets the 2-descent data: $p^*(\mathcal{G}', \varphi, \gamma) = \mathcal{G}'$.

Proof. Let $(\mathcal{G}', \varphi, \gamma)$ be a triplet as in the statement. According to Appendix, the data (φ, γ) satisfying the equality (2.5) are 2-descent data for the gerbe $\mathcal{G}'$. As observed in Lemma 2.1, the fibered 2-category of $\mathbb{G}_m$-gerbes builds a 2-stack (that is, in particular, the 2-descent is effective for objects), and so $\mathcal{G}'$ with its 2-descent data corresponds to a $\mathbb{G}_{m,S}$-gerbe $\mathcal{G}$ over $\mathbf{S}$. □

3. The Brauer group of a locally ringed stack

Let $\mathcal{X}$ be a stack over a site $\mathbf{S}$ and let $\mathbf{S}(\mathcal{X})$ be its associated site.

A **sheaf of rings $\mathcal{A}$ on** $\mathcal{X}$ is a system $(\mathcal{A}_{U,u})$ verifying the conditions (i) and (ii) of Definition 1.2, where the $\mathcal{A}_{U,u}$ are sheaves of rings on $\mathbf{S}_{|U}$. Consider the sheaf of rings $\mathcal{O}_\mathcal{X}$ on $\mathcal{X}$ given by the system $(\mathcal{O}_{\mathcal{X}\ U,u})$ with $\mathcal{O}_{\mathcal{X}\ U,u}$ the structural sheaf of U. The sheaf of rings $\mathcal{O}_\mathcal{X}$ is **the structural sheaf of the stack** $\mathcal{X}$ and the pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a **ringed stack**. An $\mathcal{O}_\mathcal{X}$**-module** $\mathcal{M}$ is a system $(\mathcal{M}_{U,u})$ verifying the conditions (i) and (ii) of Definition 1.2, where the $\mathcal{M}_{U,u}$ are sheaves of $\mathcal{O}_U$-modules on $\mathbf{S}_{|U}$. An $\mathcal{O}_\mathcal{X}$**-algebra** $\mathcal{A}$ is a system $(\mathcal{A}_{U,u})$ verifying the conditions (i) and (ii) of Definition 1.2, where the $\mathcal{A}_{U,u}$ are sheaves of $\mathcal{O}_U$-algebras on $\mathbf{S}_{|U}$. An $\mathcal{O}_\mathcal{X}$-module $\mathcal{M}$ is **of finite presentation** if the $\mathcal{M}_{U,u}$ are sheaves of $\mathcal{O}_U$-modules of finite presentation.

Now let S be an arbitrary scheme and let $\mathbf{S}_{\acute{e}t}$ be the étale site on S. Let $\mathcal{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a **locally ringed** S**-stack**, i.e. a ringed stack such that, for any object (U,u) of the associated étale site $\mathbf{S}_{\acute{e}t}(\mathcal{X})$, and for any section $f \in \mathcal{O}_{\mathcal{X}\ U,u}(U)$, there exists a covering $\{(U_i, u_i) \to (U,u)\}_{i \in I}$ of (U,u) such that for any $i \in I$ either $f|_{(U_i,u_i)}$ or $(1-f)|_{(U_i,u_i)}$ is invertible in $\Gamma(U_i, \mathcal{O}_{\mathcal{X}\ U_i,u_i})$

An **Azumaya algebra** over $\mathcal{X}$ is an $\mathcal{O}_\mathcal{X}$-algebra $\mathcal{A} = (\mathcal{A}_{U,u})$ of finite presentation as $\mathcal{O}_\mathcal{X}$-module which is, locally for the topology of $\mathbf{S}_{\acute{e}t}(\mathcal{X})$, isomorphic to a matrix algebra, i.e. for any open (U,u) of $\mathcal{X}$ there exists a covering $\{(\phi_i, \Phi_i) : (U_i, u_i) \to (U,u)\}_i$ in $\mathbf{S}_{\acute{e}t}(\mathcal{X})$ such that $\mathcal{A}_{U,u} \otimes_{\mathcal{O}_{U,u}} \mathcal{O}_{U_i} \cong M_{r_i}(\mathcal{O}_{U_i,u_i})$ for any i. Azumaya algebras over $\mathcal{X}$, that we denote by

$$\mathrm{Az}(\mathcal{X}),$$

build an S-stack on $\mathcal{X}$ by [22, Exposé VIII 1.1, 1.2] (see also [30, (3.4.4)]). Two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ over $\mathcal{X}$ are **Brauer-equivalent** if there exist two locally free $\mathcal{O}_\mathcal{X}$-modules $\mathcal{E}$ and $\mathcal{E}'$ of finite rank such that

$$\mathcal{A} \otimes_{\mathcal{O}_\mathcal{X}} \underline{\mathrm{End}}_{\mathcal{O}_\mathcal{X}}(\mathcal{E}) \cong \mathcal{A}' \otimes_{\mathcal{O}_\mathcal{X}} \underline{\mathrm{End}}_{\mathcal{O}_\mathcal{X}}(\mathcal{E}').$$

The above isomorphism defines an equivalence relation because of the isomorphism of $\mathcal{O}_\mathcal{X}$-algebras $\underline{\mathrm{End}}_{\mathcal{O}_\mathcal{X}}(\mathcal{E}) \otimes_{\mathcal{O}_\mathcal{X}} \underline{\mathrm{End}}_{\mathcal{O}_\mathcal{X}}(\mathcal{E}') \cong \underline{\mathrm{End}}_{\mathcal{O}_\mathcal{X}}(\mathcal{E} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{E}')$. We denote by $[\mathcal{A}]$ the equivalence class of an Azumaya algebra $\mathcal{A}$ over $\mathcal{X}$. The set of equivalence classes of Azumaya algebra is a group under the group law given by $[\mathcal{A}][\mathcal{A}'] = [\mathcal{A} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{A}']$. A **trivialization** of an Azumaya algebra $\mathcal{A}$ over $\mathcal{X}$ is a couple $(\mathcal{L}, a)$ with $\mathcal{L}$ a locally free $\mathcal{O}_\mathcal{X}$-module and $a : \underline{\mathrm{End}}_{\mathcal{O}_\mathcal{X}}(\mathcal{L}) \to \mathcal{A}$ an isomorphism of sheaves of $\mathcal{O}_\mathcal{X}$-algebras. An Azumaya algebra $\mathcal{A}$ is **trivial** if it exists a trivialization of $\mathcal{A}$. The class of any trivial Azumaya algebra is the neutral element of the above group law. The inverse of a class $[\mathcal{A}]$ is the class $\left[\mathcal{A}^0\right]$ with $\mathcal{A}^0$ the opposite $\mathcal{O}_\mathcal{X}$-algebra of $\mathcal{A}$.

Definition 3.1. Let $\mathcal{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a locally ringed S-stack. The **Brauer group** of $\mathcal{X}$, denoted by $\mathrm{Br}(\mathcal{X})$, is the group of equivalence classes of Azumaya algebras over $\mathcal{X}$.

Br(−) is a functor from the category of locally ringed S-stacks (objects are locally ringed S-stacks and arrows are isomorphism classes of morphisms of locally ringed S-stacks) to the category Ab of abelian groups. Remark that the above definition generalizes to stack the classical notion of Brauer group of a scheme: in fact if $\mathcal{X}$ is a locally ringed S-stack associated to an S-scheme X, then $\mathrm{Br}(\mathcal{X}) = \mathrm{Br}(X)$.

Consider the following sheaves of groups on $\mathcal{X}$: the multiplicative group $\mathbb{G}_{m,\mathcal{X}}$, the linear general group $\mathrm{GL}(n,\mathcal{X})$, and the sheaf of groups $\mathrm{PGL}(n,\mathcal{X})$ on $\mathcal{X}$ defined by the system $(\mathrm{PGL}(n,\mathcal{X})_{U,u})$ where $\mathrm{PGL}(n,\mathcal{X})_{U,u} = \underline{\mathrm{Aut}}\big(M_n(\mathcal{O}_{\mathcal{X}\,U,u})\big)$ (automorphisms of $M_n(\mathcal{O}_{\mathcal{X}\,U,u})$ as a sheaf of $\mathcal{O}_{\mathcal{X}\,U,u}$-algebras). We have the following

Lemma 3.2. *Assume $n > 0$. The sequence of sheaves of groups on $\mathcal{X}$*

$$1 \longrightarrow \mathbb{G}_{m,\mathcal{X}} \longrightarrow \mathrm{GL}(n,\mathcal{X}) \longrightarrow \mathrm{PGL}(n,\mathcal{X}) \longrightarrow 1 \tag{3.1}$$

is exact.

Proof. It is enough to show that for any étale open (U,u) of $\mathcal{X}$, the restriction to the étale site of U of the sequence $1 \to \mathbb{G}_{m_{U,u}} \to \mathrm{GL}(n)_{U,u} \to \mathrm{PGL}(n)_{U,u} \to 1$ is exact and this follows by [32, IV, Prop. 2.3. and Cor 2.4.]. □

Let $\mathrm{Lf}(\mathcal{X})$ be the S-stack on $\mathcal{X}$ of locally free $\mathcal{O}_{\mathcal{X}}$-modules. Let $\mathcal{A}$ be an Azumaya algebra over $\mathcal{X}$. Consider the morphism of S-stacks on $\mathcal{X}$

$$\mathrm{End} : \mathrm{Lf}(\mathcal{X}) \longrightarrow \mathrm{Az}(\mathcal{X}), \qquad \mathcal{L} \longmapsto \underline{\mathrm{End}}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{L}) \tag{3.2}$$

Following [20, Chp IV 2.5], let $\delta(\mathcal{A})$ be the fibered category over $\mathbf{S}_{ét}$ of trivializations of $\mathcal{A}$ defined in the following way:

- for any $U \in \mathrm{Ob}(\mathbf{S}_{ét})$, the objects of $\delta(\mathcal{A})(U)$ are trivalizations of $\mathcal{A}_{|U}$, i.e. pairs $(\mathcal{L}, a)$ with $\mathcal{L} \in \mathrm{Ob}(\mathrm{Lf}(\mathcal{X})(U))$ and $a \in \underline{\mathrm{Isom}}_U\big(\underline{\mathrm{End}}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{L}), \mathcal{A}_{|U}\big)$,
- for any arrow $f : V \to U$ of $\mathbf{S}_{ét}$, the arrows of $\delta(\mathcal{A})$ over f with source $(\mathcal{L}', a')$ and target $(\mathcal{L}, a)$ are arrows $\varphi : \mathcal{L}' \to \mathcal{L}$ of $\mathrm{Lf}(\mathcal{X})$ over f such that $\mathrm{Az}(\mathcal{X})(f) \circ a' = a \circ \mathrm{End}(\varphi)$, with $\mathrm{Az}(\mathcal{X})(f) : \mathcal{A}_{|V} \to \mathcal{A}_{|U}$.

Since $\mathrm{Lf}(\mathcal{X})$ and $\mathrm{Az}(\mathcal{X})$ are S-stacks on $\mathcal{X}$, $\delta(\mathcal{A})$ is also an S-stack on $\mathcal{X}$ (see [20, Chp IV Prop 2.5.4 (i)]). Observe that the morphism of S-stacks $\mathrm{End} : \mathrm{Lf}(\mathcal{X}) \to \mathrm{Az}(\mathcal{X})$ is locally surjective on objects by definition of Azumaya algebra. Moreover, it is locally surjective on arrows by exactness of the sequence (3.1). Therefore as in [20, Chp IV Prop 2.5.4 (ii)], $\delta(A)$ is a gerbe over $\mathcal{X}$, called the **gerbe of trivializations of** $\mathcal{A}$. For any object (U,u) of $\mathbf{S}_{ét}(\mathcal{X})$ the morphism of sheaves of groups on U

$$(\mathbb{G}_{m,\mathcal{X}})_{U,u} = (\mathcal{O}^*_{\mathcal{X}})_{U,u} \longrightarrow \big(\underline{\mathrm{Aut}}(\mathcal{L}, a)\big)_{U,u},$$

that sends a section g of $(\mathcal{O}^*_{\mathcal{X}})_{U,u}$ to the multiplication $g \cdot - : (\mathcal{L}, a)_{U,u} \to (\mathcal{L}, a)_{U,u}$ by this section, is an isomorphism. This means that the gerbe $\delta(\mathcal{A})$ is in fact a $\mathbb{G}_{m,\mathcal{X}}$-gerbe. By Corollary 2.4 we can then associate to any Azumaya algebra $\mathcal{A}$ over $\mathcal{X}$ a cohomological class in $\mathrm{H}^2_{ét}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}})$, denoted by $\overline{\delta(\mathcal{A})}$, which is given by the $\mathbb{G}_{m,\mathcal{X}}$-equivalence class of $\delta(\mathcal{A})$ in $\mathbb{G}\mathrm{erbe}^2_S(\mathbb{G}_{m,\mathcal{X}})$.

Proposition 3.3. *An Azumaya algebra $\mathcal{A}$ over $\mathcal{X}$ is trivial if and only if its cohomological class $\overline{\delta(\mathcal{A})}$ in $\mathrm{H}^2_{ét}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}})$ is zero.*

Proof. The Azumaya algebra $\mathcal{A}$ is trivial if and only if the gerbe $\delta(\mathcal{A})$ admits a global section if and only if its corresponding class $\overline{\delta(\mathcal{A})}$ is zero in $\mathrm{H}^2_{ét}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}})$. □

Theorem 3.4. *The morphism*

$$\begin{aligned}\delta : \mathrm{Br}(\mathcal{X}) &\longrightarrow \mathrm{H}^2_{\acute{e}t}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}}) \\ [\mathcal{A}] &\longmapsto \overline{\delta(\mathcal{A})}\end{aligned}$$

is an injective group homomorphism.

Proof. Let $\mathcal{A}, \mathcal{B}$ be two Azumaya algebras over $\mathcal{X}$. For any $U \in \mathrm{Ob}(\mathbf{S}_{\acute{e}t})$, the morphism of gerbes

$$\begin{aligned}\delta(\mathcal{A})(U) \times \delta(\mathcal{B})(U) &\longrightarrow \delta(\mathcal{A} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{B})(U) \\ ((\mathcal{L}, a), (\mathcal{M}, b)) &\longmapsto (\mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{M}, a \otimes_{\mathcal{O}_\mathcal{X}} b)\end{aligned}$$

is a +-morphism, where $+ : \mathbb{G}_{m,\mathcal{X}} \times \mathbb{G}_{m,\mathcal{X}} \to \mathbb{G}_{m,\mathcal{X}}$ is the group law underlying the sheaf $\mathbb{G}_{m,\mathcal{X}}$. Therefore

$$\overline{\delta(\mathcal{A})} + \overline{\delta(\mathcal{B})} = \overline{\delta(\mathcal{A} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{B})} \tag{3.3}$$

in $\mathrm{H}^2_{\acute{e}t}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}})$. This equality shows first that $\overline{\delta(\mathcal{A})} = -\overline{\delta(\mathcal{A}^0)}$ and also that

$$[\mathcal{A}] = [\mathcal{B}] \Leftrightarrow [\mathcal{A} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{B}^0] = 0 \overset{\text{Prop 3.3}}{\Leftrightarrow} \overline{\delta(\mathcal{A} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{B}^0)} = 0 \overset{(3.3)}{\Leftrightarrow} \overline{\delta(\mathcal{A})} + \overline{\delta(\mathcal{B}^0)} = 0 \Leftrightarrow \overline{\delta(\mathcal{A})} = \overline{\delta(\mathcal{B})}$$

These equivalences prove that the morphism $\delta : \mathrm{Br}(\mathcal{X}) \to \mathrm{H}^2_{\acute{e}t}(\mathcal{X}, \mathbb{G}_{m,\mathcal{X}})$ is well-defined and injective. Finally always from the equality (3.3) we get that δ is a group homomorphism. □

4. Gerbes and Azumaya algebras over 1-motives

Let $M = [u : X \to G]$ be a 1-motive defined over a scheme S, denote by $\mathcal{M}$ its associated Picard S-stack under the equivalence constructed in [17, Exposé XVIII, Prop 1.4.15] and denote by $\mathbf{S}(\mathcal{M})$ the site associated to the stack $\mathcal{M}$ as in Definition 1.1.

Definition 4.1.

(1) The S-stack of Azumaya algebras over the 1-motive M is the S-stack of Azumaya algebras $\mathrm{Az}(\mathcal{M})$ over $\mathcal{M}$.
(2) The Brauer group of the 1-motive M is the Brauer group $\mathrm{Br}(\mathcal{M})$ of $\mathcal{M}$.
(3) A $\mathbb{G}_m$-gerbe on the 1-motive M is a $\mathbb{G}_{m,\mathcal{M}}$-gerbe on $\mathcal{M}$ (i.e. a $\mathbb{G}_{m,\mathcal{M}}$-gerbe on the site $\mathbf{S}(\mathcal{M})$).

By [30, (3.4.3)] the associated Picard S-stack $\mathcal{M}$ is isomorphic to the quotient stack $[G/X]$ (where X acts on G via the given morphism $u : X \to G$). Note that in general it is not algebraic in the sense of [30] because it is not quasi-separated. However the quotient map

$$\iota : G \longrightarrow [G/X] \cong \mathcal{M}$$

is representable, étale and surjective. The fiber product $G \times_{[G/X]} G$ is isomorphic to $X \times_S G$. Via this identification, the projections $q_i : G \times_{[G/X]} G \to G$ (for $i = 1, 2$) correspond respectively to the second projection $p_2 : X \times_S G \to G$ and to the map $\mu : X \times_S G \to G$ given by the action $(x, g) \mapsto u(x)g$. We can further identify the fiber product $G \times_{[G/X]} G \times_{[G/X]} G$ with $X \times_S X \times_S G$ and the partial projections $q_{13}, q_{23}, q_{12} : G \times_{[G/X]} G \times_{[G/X]} G \to G \times_{[G/X]} G$ respectively with the map $m_X \times \mathrm{id}_G : X \times_S X \times_S G \to X \times_S G$ where m_X denotes the group law of X, the map $\mathrm{id}_X \times \mu : X \times_S X \times_S G \to X \times_S G$, and the partial projection

$p_{23} : X \times_S X \times_S G \to X \times_S G$. The effectiveness of the descent of Azumaya algebras with respect to the quotient map $\iota : G \to [G/X]$ is proved in the following Lemma (see [35, (9.3.4)] for the definition of pull-back of $\mathcal{O}_{\mathcal{M}}$-algebras):

Lemma 4.2. *The pull-back $\iota^* : \mathrm{Az}(\mathcal{M}) \to \mathrm{Az}(G)$ is an equivalence of S-stacks between the S-stack of Azumaya algebras on $\mathcal{M}$ and the S-stack of X-equivariant Azumaya algebras on G. More precisely, to have an Azumaya algebra $\mathcal{A}$ on $\mathcal{M}$ is equivalent to have a pair*

$$(A, \varphi)$$

where A is an Azumaya algebra on G and $\varphi : p_2^* A \to \mu^* A$ is an isomorphism of Azumaya algebras on $X \times_S G$ that satisfies (up to canonical isomorphisms) the cocycle condition

$$(m_X \times id_G)^* \varphi = \big((\mathrm{id}_X \times \mu)^* \varphi\big) \circ \big((p_{23})^* \varphi\big). \tag{4.1}$$

Under this equivalence, the pull-back $\iota^* : \mathrm{Az}(\mathcal{M}) \to \mathrm{Az}(G)$ is the morphism of stacks which forgets the descent datum: $\iota^*(A, \varphi) = A$.

Proof. Since the assertion is local for the topology on $\mathbf{S}_{\acute{e}t}(\mathcal{M})$, it suffices to prove it for any open (U, u) of $\mathcal{M}$, where U is an object of $\mathbf{S}_{\acute{e}t}$ and x is an object of $\mathcal{M}(U)$. The descent of quasi-coherent modules is known for the morphism $\iota_U : G \times_{\iota, \mathcal{M}, x} U \to U$ obtained by base change (see [30, Thm (13.5.5)]). The additional algebra structure descends by [29, II Thm 3.4]. Finally the Azumaya algebra structure descends by [28, III, Prop 2.8]. Since an Azumaya algebra on $\mathcal{M}$ is by definition a collection of Azumaya algebras on the various schemes U, the general case follows. □

Now we prove also the effectiveness of the 2-descent of $\mathbb{G}_m$-gerbes under the quotient map $\iota : G \to \mathcal{M}$, using the result of Section 2.3.

Lemma 4.3. *To have a $\mathbb{G}_{m,\mathcal{M}}$-gerbe $\mathcal{G}$ on $\mathcal{M}$ is equivalent to have a triplet*

$$(\mathcal{G}', \varphi, \gamma)$$

where $\mathcal{G}'$ is a $\mathbb{G}_{m,G}$-gerbe on G and (φ, γ) are 2-descent data on $\mathcal{G}'$ with respect to $\iota : G \to [G/X]$, that is

- *$\varphi : p_2^* \mathcal{G}' \to \mu^* \mathcal{G}'$ is an equivalence of gerbes on $X \times_S G$,*
- *$\gamma : \big((\mathrm{id}_X \times \mu)^* \varphi\big) \circ \big((p_{23})^* \varphi\big) \Rightarrow (m_X \times id_G)^* \varphi$ is a natural isomorphism on $X \times_S X \times_S G \cong G \times_{[G/X]} G \times_{[G/X]} G$,*

which satisfies the compatibility condition

$$p_{134}^* \gamma \circ [p_{34}^* \varphi * p_{123}^* \gamma] = p_{124}^* \gamma \circ [p_{234}^* \gamma * p_{12}^* \varphi] \tag{4.2}$$

when pulled back to $X \times_S X \times_S X \times_S G \cong G \times_{[G/X]} G \times_{[G/X]} G \times_{[G/X]} G := G^4$ (here $pr_{ijk} : G^4 \to G \times_{[G/X]} G \times_{[G/X]} G$ and $pr_{ij} : G^4 \to G \times_{[G/X]} G$ are the partial projections).

Proof. A $\mathbb{G}_{m,\mathcal{M}}$-gerbe on $\mathcal{M}$ is by definition a collection of $\mathbb{G}_{m,U}$-gerbes over the various objects U of $\mathbf{S}$. Hence it is enough to prove that for any object U of $\mathbf{S}$ and any object x of $\mathcal{M}(U)$, the 2-descent of $\mathbb{G}_m$-gerbes with respect to the morphism $\iota_U : G \times_{\iota, \mathcal{M}, x} U \to U$ obtained by base change is effective. But this is a consequence of Theorem 2.7. □

5. The generalized theorem of the Cube for 1-motives and its consequences

We use the same notation of the previous Section. We denote by $\mathcal{M}^3 = \mathcal{M}\times_{\mathbf{S}}\mathcal{M}\times_{\mathbf{S}}\mathcal{M}$ (resp. $\mathcal{M}^2 = \mathcal{M}\times_{\mathbf{S}}\mathcal{M}$) the fibered product of 3 (resp. 2) copies of $\mathcal{M}$. Since any Picard stack admits a global neutral object, it exists a unit section denoted by $\epsilon : \mathbf{S} \to \mathcal{M}$. Consider the map

$$s_{ij} := \mathcal{M} \times_{\mathbf{S}} \mathcal{M} \to \mathcal{M} \times_{\mathbf{S}} \mathcal{M} \times_{\mathbf{S}} \mathcal{M}$$

which inserts the unit section $\epsilon : \mathbf{S} \to \mathcal{M}$ into the k-th factor for $k \in \{1, 2, 3\} - \{i, j\}$. If ℓ is a prime number and H is an abelian group, $\mathrm{H}(\ell)$ denotes the ℓ-primary component of H.

Definition 5.1. Let M be a 1-motive defined over a scheme S. Let ℓ be a prime number distinct from the residue characteristics of S. The 1-motive M **satisfies the generalized Theorem of the Cube for the prime** ℓ if the natural homomorphism

$$\begin{array}{rcl} \prod s_{ij}^* : \mathrm{H}^2_{\acute{e}t}(\mathcal{M}^3, \mathbb{G}_{m,\mathcal{M}^3})(\ell) & \longrightarrow & \big(\mathrm{H}^2_{\acute{e}t}(\mathcal{M}^2, \mathbb{G}_{m,\mathcal{M}^2})(\ell)\big)^3 \\ x & \longmapsto & (s_{12}^*(x), s_{13}^*(x), s_{23}^*(x)) \end{array} \tag{5.1}$$

is injective.

5.1. Its consequences

Proposition 5.2. *Let M be a 1-motive satisfying the generalized Theorem of the Cube for a prime ℓ distinct from the residue characteristics of S. Let $N : \mathcal{M} \to \mathcal{M}$ be the multiplication by N on the Picard S-stack $\mathcal{M}$. Then for any $y \in \mathrm{H}^2_{\acute{e}t}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}})(\ell)$ we have that*

$$N^*(y) = N^2 y + \Big(\frac{N^2 - N}{2}\Big)\big((-id_{\mathcal{M}})^*(y) - y\big). \tag{5.2}$$

Proof. First we prove that given three contravariant functors $F, G, H : \mathcal{P} \to \mathcal{M}$, we have the following equality for any y in $\mathrm{H}^2_{\acute{e}t}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}})(\ell)$

$$(F + G + H)^*(y) - (F + G)^*(y) - (F + H)^*(y) - (G + H)^*(y) + F^*(y) + G^*(y) + H^*(y) = 0. \tag{5.3}$$

Let $pr_i : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ the projection onto the i^{th} factor. Put $m_{i,j} = pr_i \otimes pr_j : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ and $m = pr_1 \otimes pr_2 \otimes pr_3 : \mathcal{M} \times \mathcal{M} \times \mathcal{M} \to \mathcal{M}$, where $\otimes$ is the law group of the Picard S-stack $\mathcal{M}$. The element

$$z = m^*(y) - m_{1,2}^*(y) - m_{1,3}^*(y) - m_{2,3}^*(y) + pr_1^*(y) + pr_2^*(y) + pr_3^*(y)$$

of $\mathrm{H}^2_{\acute{e}t}(\mathcal{M}^3, \mathbb{G}_{m,\mathcal{M}^3})(\ell)$ is zero when restricted to $S \times \mathcal{M} \times \mathcal{M}$, $\mathcal{M} \times S \times \mathcal{M}$ and $\mathcal{M} \times \mathcal{M} \times S$ (this restriction is obtained inserting the unit section $\epsilon : S \to \mathcal{M}$). Thus it is zero in $\mathrm{H}^2_{\acute{e}t}(\mathcal{M}^3, \mathbb{G}_{m,\mathcal{M}^3})(\ell)$ by the generalized Theorem of the Cube for ℓ. Finally, pulling back z by $(F, G, H) : \mathcal{P} \to \mathcal{M} \times \mathcal{M} \times \mathcal{M}$ we get (5.3).

Now, setting $F = N$, $G = id_{\mathcal{M}}$, $h = (-id_{\mathcal{M}})$ we get

$$N^*(y) = (N + id_{\mathcal{M}})^*(y) + (N - id_{\mathcal{M}})^*(y) + 0^*(y) - N^*(y) - (id_{\mathcal{M}})^*(y) - (-id_{\mathcal{M}})^*(y).$$

We rewrite this as

$$(N + id_{\mathcal{M}})^*(y) - N^*(y) = N^*(y) - (N - id_{\mathcal{M}})^*(y) + (id_{\mathcal{M}})^*(y) + (-id_{\mathcal{M}})^*(y).$$

If we denote $z_1 = y$ and $z_N = N^*(y) - (N - id_{\mathcal{M}})^*(y)$, we obtain $z_{N+1} = z_N + y + (-id_{\mathcal{M}})^*(y)$. By induction, we get $z_N = y + (N - id_{\mathcal{M}})(y + (-id_{\mathcal{M}})^*(y))$. From the equality $N^*(y) = z_N + (N - id_{\mathcal{M}})^*(y)$ we have

$$N^*(y) = z_N + z_{N-1} + \cdots + z_1,$$

and therefore we are done. □

Corollary 5.3. *Let M be a 1-motive satisfying the generalized Theorem of the Cube for a prime ℓ. Then, if $\ell \neq 2$, the ℓ^n-torsion elements of $\mathrm{H}^2_{\acute{e}t}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}})$ are contained in*

$$\ker \left[(\ell^n_{\mathcal{M}})^* : \mathrm{H}^2_{\acute{e}t}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}}) \longrightarrow \mathrm{H}^2_{\acute{e}t}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}}) \right]$$

and if $\ell = 2$, they are contained in

$$\ker \left[(2^{n+1}_{\mathcal{M}})^* : \mathrm{H}^2_{\acute{e}t}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}}) \longrightarrow \mathrm{H}^2_{\acute{e}t}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}}) \right].$$

Proof. The result follows by (5.2). □

5.2. Its proof

We finish this section by searching the hypothesis we should put on the base scheme S in order to have that the 1-motive $M = [X \xrightarrow{u} G]$ satisfies the generalized Theorem of the Cube. From now on we will switch freely between the two equivalent notion of invertible sheaf $\mathcal{L}$ on the extension G and $\mathbb{G}_m$-torsor $\underline{\mathrm{Isom}}(\mathcal{O}_G, \mathcal{L})$ on G. The extension G fits into the following short exact sequence

$$0 \longrightarrow T \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0$$

The pull-back of gerbes defined in (1.1) allows us to define an homomorphism of abelian groups $\pi^* : \mathbb{G}\mathrm{erbe}^2_S(\mathbb{G}_{m,A}) \to \mathbb{G}\mathrm{erbe}^2_S(\mathbb{G}_{m,G})$.

Proposition 5.4. *Let S be a normal scheme. Let G be an extension of an abelian S-scheme A by $\mathbb{G}^r_m$. The pull-back $\pi^* : \mathbb{G}\mathrm{erbe}^2_S(\mathbb{G}_{m,A}) \to \mathbb{G}\mathrm{erbe}^2_S(\mathbb{G}_{m,G})$ is surjective.*

Proof. Denote by $\mathcal{T}\mathrm{orsRig}(\mathbb{G}_{m,G})$ the Picard S-stack of $\mathbb{G}_m$-torsors on G with rigidification along the unit section $\epsilon_G : S \to G$. Because of this unit section ϵ_G, the group of isomorphism classes of $\mathbb{G}_m$-torsors over G with rigidification is canonically isomorphic to the quotient of the group of isomorphism classes of $\mathbb{G}_m$-torsors over G by the group of isomorphism classes of $\mathbb{G}_m$-torsors over S:

$$\mathcal{T}\mathrm{orsRig}^1(\mathbb{G}_{m,G}) \cong \mathcal{T}\mathrm{ors}^1(\mathbb{G}_{m,G}) / \mathcal{T}\mathrm{ors}^1(\mathbb{G}_{m,S}). \tag{5.4}$$

Denote by $\mathcal{C}\mathrm{ub}(G, \mathbb{G}_m)$ the Picard S-stack of $\mathbb{G}_m$-torsors on G with cubical structure and by $\mathcal{C}\mathrm{ub}^i(G, \mathbb{G}_m)$ for $i = 1, 0$ its classifying groups. Roughly speaking a $\mathbb{G}_m$-torsor on G is cubical if it satisfies the Theorem of the Cube, for details see [11, Def 2.2]. In [11, Prop 2.4], Breen proves the Theorem of the Cube for extensions of abelian schemes by tori which are defined over a normal scheme, that is the forgetful additive functor $\mathcal{C}\mathrm{ub}(G, \mathbb{G}_m) \to \mathcal{T}\mathrm{orsRig}(\mathbb{G}_{m,G})$ is an equivalence of Picard S-stacks. In particular

$$\mathcal{C}\mathrm{ub}^1(G, \mathbb{G}_m) \cong \mathcal{T}\mathrm{orsRig}^1(\mathbb{G}_{m,G}). \tag{5.5}$$

With our hypothesis, by [33, Chp I, Rem 7.2.4], any $\mathbb{G}_m$-torsors on G with cubical structure comes from a $\mathbb{G}_m$-torsors on A with cubical structure. Using the isomorphisms (5.4) and (5.5), we get the surjection

$$\pi^* : \mathcal{T}\mathrm{ors}^1(\mathbb{G}_{m,A}) \longrightarrow \mathcal{T}\mathrm{ors}^1(\mathbb{G}_{m,G}). \tag{5.6}$$

Now let $\mathcal{G}$ be a $\mathbb{G}_{m,G}$-gerbe on G. Breen's semi-local description of gerbes (2.3) asserts that to have $\mathcal{G}$ is equivalent to have the local data $(\mathcal{T}\mathrm{ors}(\mathbb{G}_{m,G,U}), \Psi_x)_{x\in\mathcal{G}(U),U\in\mathbf{S}_{fppf}}$ endowed with the transition data (ψ_x, ξ_x). Let $y = \pi(P(x)) \in A(U)$, where $P : \mathcal{G} \to G$ is the structural morphism of $\mathcal{G}$. The equivalence of $U \times_S U$-stacks $\psi_x : \mathcal{T}\mathrm{ors}(pr_2^*\mathbb{G}_{m,G,U}) \to \mathcal{T}\mathrm{ors}(pr_1^*\mathbb{G}_{m,G,U})$ induces a bijection between the classifying groups $\psi_x^1 : \mathcal{T}\mathrm{ors}^1(pr_2^*\mathbb{G}_{m,G,U}) \to \mathcal{T}\mathrm{ors}^1(pr_1^*\mathbb{G}_{m,G,U})$. Because of the surjection (5.6), all torsors over G come from torsors over A up to isomorphisms, and so we have a bijection

$$\psi_y^1 : \mathcal{T}\mathrm{ors}^1(pr_2^*\mathbb{G}_{m,A,U}) \to \mathcal{T}\mathrm{ors}^1(pr_1^*\mathbb{G}_{m,A,U})$$

such that $\psi_y^1 = (\pi^*)^*\psi_x^1$. By pull-back via (5.6), the isomorphism of cartesian S-functors $\xi_x : pr_{23}^*\psi_x \circ pr_{12}^*\psi_x \Rightarrow pr_{13}^*\psi_x$ gives $\xi_y^1 = (\pi^*)^*\xi_x^1 : pr_{23}^*\psi_y^1 \circ pr_{12}^*\psi_y^1 \Rightarrow pr_{13}^*\psi_y^1$ over $U \times_S U \times_S U$, which satisfies an analogue of the equality (2.4). The local data $(\mathcal{T}\mathrm{ors}^1(\mathbb{G}_{m,A,U}))_{y=\pi(P(x)),x\in\mathcal{G}(U)}$ endowed with the transition data (ψ_y^1, ξ_y^1), which are defined on isomorphism classes of $\mathbb{G}_m$-torsors, furnish a $\mathbb{G}_{m,A}$-equivalence class $\overline{\mathcal{G}}'$ of a $\mathbb{G}_{m,A}$-gerbe on A such that $\pi^*(\overline{\mathcal{G}}') = \overline{\mathcal{G}}$. □

We give an immediate application of this result in the case of extensions over a field k.

Corollary 5.5. *Let G be an extension of an abelian variety by a torus over a field k. Then* $\mathrm{Br}(G) \cong \mathrm{H}^2_{\acute{e}t}(G, \mathbb{G}_{m,G})$.

Proof. By Gabber's unpublished result [18], if A is an abelian variety defined over a field k, then $\mathrm{Br}(A) \cong \mathrm{H}^2_{\acute{e}t}(A, \mathbb{G}_{m,A})$. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{Br}(A) & \xrightarrow{\cong} & \mathrm{H}^2_{\acute{e}t}(A, \mathbb{G}_m) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \pi^*} \\ \mathrm{Br}(G) & \xhookrightarrow[\delta]{} & \mathrm{H}^2_{\acute{e}t}(G, \mathbb{G}_m) \end{array} \tag{5.7}$$

where $\pi : G \to A$ is the surjective morphism of varieties underlying G and π^* denotes the pull-back maps of Azumaya algebras and cohomological classes. By [25, II, Prop 1.4] the cohomological groups $\mathrm{H}^2_{\acute{e}t}(G, \mathbb{G}_m)$ and $\mathrm{H}^2_{\acute{e}t}(A, \mathbb{G}_m)$ are torsion groups and so Theorem 2.2 and Proposition 5.4 imply that $\pi^* : \mathrm{H}^2_{\acute{e}t}(A, \mathbb{G}_m) \to \mathrm{H}^2_{\acute{e}t}(G, \mathbb{G}_m)$ is surjective. Hence the injective homomorphism δ on the bottom row is surjective too. □

Let G_i be an extension of an abelian S-scheme A_i by an S-torus for $i = 1, 2, 3$, and denote by $\epsilon_i^G : S \to G_i$ its unit section. Let $s_{ij}^G := G_i \times_S G_j \to G_1 \times_S G_2 \times_S G_3$ be the map obtained from the unit section $\epsilon_k^G : S \to G_k$ after the base change $G_i \times_S G_j \to S$ (i.e. the map which inserts $\epsilon_k^G : S \to G_k$ into the k-th factor for $k \in \{1, 2, 3\} - \{i, j\}$).

Corollary 5.6. *Let S be a connected, reduced, normal and noetherian scheme and let G_i be an extension of an abelian S-scheme A_i by $\mathbb{G}_m^{r_i}$ for $i = 1, 2, 3$. Let ℓ be a prime distinct from the residue characteristics of S. Then, with the above notation, the natural homomorphism*

$$\begin{array}{rcl} \prod s_{ij}^{G*} : \mathrm{H}^2_{\acute{e}t}(G_1 \times_S G_2 \times_S G_3, \mathbb{G}_m)(\ell) & \longrightarrow & \prod_{(i,j)\in\{1,2,3\}} \mathrm{H}^2_{\acute{e}t}(G_i \times_S G_j, \mathbb{G}_m)(\ell) \\ x & \longmapsto & (s_{12}^{G*}(x), s_{13}^{G*}(x), s_{23}^{G*}(x)) \end{array} \tag{5.8}$$

is injective.

In particular, if the base scheme is connected, reduced, normal and noetherian, an extension of an abelian S-scheme by $\mathbb{G}_m^r$ satisfies the generalized Theorem of the Cube for any prime ℓ distinct from the residue characteristics of S.

Proof. Let $p_{ij}^G : G_1 \times_S G_2 \times_S G_3 \to G_i \times_S G_j$ be the projection maps for $i, j = 1, 2, 3$ and let α^G be the map

$$\begin{array}{rcl} \alpha^G : \prod_{(i,j)\in\{1,2,3\}} \mathrm{H}^2_{\acute{e}t}(G_i \times_S G_j, \mathbb{G}_m)(\ell) & \longrightarrow & \mathrm{H}^2_{\acute{e}t}(G_1 \times_S G_2 \times_S G_3, \mathbb{G}_m)(\ell) \\ ((y_2, y_3), (y_1, y_3), (y_1, y_2)) & \longmapsto & \sum_{i,j=1,2,3} p_{ij}^{G*}(y_i, y_j)) \end{array} \tag{5.9}$$

Analogously we define the maps α^A and p_{ij}^A. We observe that the injectivity of the map $\prod s_{ij}^{G*}$ is equivalent to the surjectivity of the map α^G ([34, Rem page 55]). If we denote $\pi_i : G_i \to A_i$ the surjections underlying the extensions G_i (for $i = 1, 2, 3$), $(\pi_i \times \pi_j) \circ p_{ij}^G = p_{ij}^A \circ (\pi_1 \times \pi_2 \times \pi_2)$ and so we have the following commutative diagram

$$\begin{array}{ccc} \prod_{(i,j)\in\{1,2,3\}} \mathrm{H}^2_{\acute{e}t}(A_i \times_S A_j, \mathbb{G}_m)(\ell) & \xrightarrow{\alpha^A} & \mathrm{H}^2_{\acute{e}t}(A_1 \times_S A_2 \times_S A_3, \mathbb{G}_m)(\ell) \\ \downarrow & & \downarrow \\ \prod_{(i,j)\in\{1,2,3\}} \mathrm{H}^2_{\acute{e}t}(G_i \times_S G_j, \mathbb{G}_m)(\ell) & \xrightarrow{\alpha^G} & \mathrm{H}^2_{\acute{e}t}(G_1 \times_S G_2 \times_S G_3, \mathbb{G}_m)(\ell) \end{array} \tag{5.10}$$

where the vertical arrows, which are the pull-backs induced by the π_i, are surjective by Proposition 5.4 and Corollary 2.4. The top horizontal arrow is surjective since under our hypotheses abelian S-schemes satisfy the generalized Theorem of the Cube (see [27, Cor 2.6]). Hence we can conclude that also the down horizontal arrow is surjective, i.e. $\prod s_{ij}^{G*}$ is injective. □

Using the effectiveness of the 2-descent for $\mathbb{G}_m$-gerbes via the quotient map $\iota : G \to \mathcal{M}$ (Lemma 4.3), from the above corollary we get

Theorem 5.7. *1-motives, which are defined over a connected, reduced, normal and noetherian scheme S, and whose underlying tori are split, satisfy the generalized Theorem of the Cube for any prime ℓ distinct from the residue characteristics of S.*

6. Cohomological classes of 1-motives which are Azumaya algebras

Let S be a scheme. We will need the finite site on S: first recall that a morphism of schemes $f : X \to S$ is said to be **finite locally free** if it is finite and $f_*(\mathcal{O}_X)$ is a locally free $\mathcal{O}_S$-module. In particular, by [24, Prop (18.2.3)] finite étale morphisms are finite locally free. The finite site on S, denoted $\mathbf{S}_f$, is the category of finite locally free schemes over S, endowed with the topology generated from the pretopology for which the set of coverings of a finite locally free scheme T over S is the set of single morphisms $u : T' \to T$ such that u is finite locally free and $T = u(T')$ (set theoretically). There is a morphism of site $\tau : \mathbf{S}_{fppf} \to \mathbf{S}_f$. If F is a sheaf for the étale topology, then we define as in [27, §3]

$$F(T)_f := \{ y \in F(T) \mid \text{ there is a covering } u : T' \to T \text{ in } \mathbf{S}_f \text{ with } F(u)(y) = 0 \}$$

i.e. $F(T)_f$ are the elements of $F(T)$ which can be split by a finite locally free covering.

Proposition 6.1. *Let G be an extension of an abelian scheme by a torus, which is defined over a normal and noetherian scheme S, and which satisfies the generalized Theorem of the Cube for a prime number ℓ distinct from the characteristics of S. Then the ℓ-primary component of the kernel of the homomorphism*

$\mathrm{H}^2_{ét}(\epsilon) : \mathrm{H}^2_{ét}(G, \mathbb{G}_{m,G})) \to \mathrm{H}^2_{ét}(S, \mathbb{G}_{m,S})$ *induced by the unit section* $\epsilon : S \to G$ *of* G*, is contained in the Brauer group of* G*:*

$$\ker\left[\mathrm{H}^2_{ét}(\epsilon) : \mathrm{H}^2_{ét}(G, \mathbb{G}_{m,G})) \longrightarrow \mathrm{H}^2_{ét}(S, \mathbb{G}_{m,S})\right](\ell) \subseteq \mathrm{Br}(G).$$

Proof. In order to simplify notations we denote by $\ker(\mathrm{H}^2_{ét}(\epsilon))$ the kernel of the homomorphism $\mathrm{H}^2_{ét}(\epsilon) : \mathrm{H}^2_{ét}(G, \mathbb{G}_{m,G}) \to \mathrm{H}^2_{ét}(S, \mathbb{G}_{m,S})$. We identify étale and fppf cohomologies for the smooth group schemes μ_{ℓ^n} and $\mathbb{G}_m$.

(1) First we show that $\mathrm{H}^2_f(G, \tau_*\mu_{\ell^n})$ is isomorphic to $\mathrm{H}^2_{ét}(G, \mu_{\ell^n})_f$. By definition, $\mathrm{R}^1\tau_*\mu_{\ell^n}$ is the sheaf on G associated to the presheaf $U \to \mathrm{H}^1(U_{fppf}, \mu_{\ell^n})$. This latter group classifies torsors in U_{fppf} under the finite locally free group scheme μ_{ℓ^n}. Since any μ_{ℓ^n}-torsor is trivialized by using itself as an f-cover, we have that $\mathrm{R}^1\tau_*\mu_{\ell^n} = (0)$. The Leray spectral sequence for the morphism of sites $\tau : \mathbf{S}_{fppf} \to \mathbf{S}_f$ (see [32, page 309]) gives then the isomorphism

$$\mathrm{H}^2_f(G, \tau_*\mu_{\ell^n}) \cong \ker\left[\mathrm{H}^2_{ét}(G, \mu_{\ell^n}) \xrightarrow{\pi} \mathrm{H}^0_f(G, \mathrm{R}^2\tau_*\mu_{\ell^n})\right] = \mathrm{H}^2_{ét}(G, \mu_{\ell^n})_f$$

where the map π is the edge morphism which can be interpreted as the canonical morphism from the presheaf $U \to \mathrm{H}^2_{ét}(U_{fppf}, \mu_{\ell^n})$ to the associated sheaf $\mathrm{R}^2\tau_*\mu_{\ell^n}$.

(2) Now we prove that $\mathrm{H}^2_{ét}(G, \mu_{\ell^\infty})_f$ maps onto $\ker(\mathrm{H}^2_{ét}(\epsilon))(\ell)$. Let x be an element of $\ker(\mathrm{H}^2_{ét}(\epsilon))$ with $\ell^n x = 0$ for some n. The filtration on the Leray spectral sequence for $\tau : \mathbf{S}_{fppf} \to \mathbf{S}_f$ and the Kummer sequence gives the following exact commutative diagram

$$\begin{array}{ccccccc}
 & & & & \mathrm{Pic}(G) & \xrightarrow{\pi'} & \mathrm{H}^0_f(G, (\mathrm{R}^1\tau_*\mathbb{G}_m)_{\ell^n}) \\
 & & & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d'} \\
0 & \longrightarrow & \mathrm{H}^2_{ét}(G, \mu_{\ell^n})_f & \longrightarrow & \mathrm{H}^2_{ét}(G, \mu_{\ell^n}) & \xrightarrow{\pi} & \mathrm{H}^0_f(G, \mathrm{R}^2\tau_*\mu_{\ell^n}) \\
 & & \downarrow & & \downarrow{\scriptstyle i} & & \downarrow \\
0 & \longrightarrow & \mathrm{H}^2_{ét}(G, \mathbb{G}_m)_f & \longrightarrow & \mathrm{H}^2_{ét}(G, \mathbb{G}_m) & \longrightarrow & \mathrm{H}^0_f(G, \mathrm{R}^2\tau_*\mathbb{G}_m) \\
 & & & & \downarrow{\scriptstyle \ell^n} & & \\
 & & & & \mathrm{H}^2_{ét}(G, \mathbb{G}_{m,G}) & &
\end{array} \tag{6.1}$$

where $(\mathrm{R}^1\tau_*\mathbb{G}_m)_{\ell^n}$ is the cokernel of the multiplication by ℓ^n. Since $\ell^n x = 0$, we can choose an $y \in \mathrm{H}^2_{ét}(G, \mu_{\ell^n})$ such that $i(y) = x$. By Corollary 5.3, the isogeny $\ell^{2n} : G \to G$ is a finite locally free covering which splits x, that is $x \in \mathrm{H}^2_{ét}(G, \mathbb{G}_m)_f$. Moreover $i((l^{2n})^* y) = (l^{2n})^* i(y) = 0$ and so there exists an element $z \in \mathrm{Pic}(G)$ such that $d(z) = (l^{2n})^* y$. In particular $d'(\pi'(z)) = \pi((l^{2n})^* y)$ is an element of $\mathrm{H}^0_f(G, \mathrm{R}^2\tau_*\mu_{\ell^n})$. By the Theorem of the Cube for the extension G (see [11, Prop 2.4]), we have that $(l^{2n})^* z = l^{2n} z'$ for some $z' \in \mathrm{Pic}(G)$, which implies that $\pi'(z) = 0$ in $(\mathrm{R}^1\tau_*\mathbb{G}_m)_{\ell^n}(G_f)$. From the equality $\pi((l^{2n})^* y) = d'(\pi'(z)) = 0$ follows $\pi(y) = 0$, which means that y is an element of $\mathrm{H}^2_{ét}(G, \mu_{\ell^n})_f$.

(3) Here we show that $\ker(\mathrm{H}^2_{ét}(\epsilon))(\ell) \subseteq \tau^*\mathrm{H}^2_f(G, \mathbb{G}_m)$. By the first two steps, $\mathrm{H}^2_f(G, \tau_*\mu_{\ell^n})$ maps onto $\ker(\mathrm{H}^2_{ét}(\epsilon))(\ell)$. Since $\mathrm{H}^2_f(G, \tau_*\mu_{\ell^n}) \subseteq \mathrm{H}^2_f(G, \tau_*\mathbb{G}_m)$, we can then conclude that $\ker(\mathrm{H}^2_{ét}(\epsilon))(\ell) \subseteq \tau^*\mathrm{H}^2_f(G, \mathbb{G}_m)$.

(4) Let x be an element of $\ker(\mathrm{H}^2_{ét}(\epsilon))$ with $\ell^n x = 0$ for some n. Using the morphism of sites $\tau : \mathbf{S}_{fppf} \to \mathbf{S}_f$, in [27, Lem 3.2] Hoobler built a commutative diagram where upper and lower lines give rise to spectral sequences comparing Čech and sheaf cohomology for $\mathbf{S}_{fppf}$ and $\mathbf{S}_f$. Using this diagram he showed that the group $\tau^*\mathrm{H}^2_f(G, \mathbb{G}_m)$ is contained in the Čech cohomology group $\check{\mathrm{H}}^2_{ét}(G, \mathbb{G}_m)$. Thus, from step (3), x is an

element of $\check{\mathrm{H}}^2_{\acute{e}t}(G,\mathbb{G}_m)$. By Corollary 5.3, the isogeny $\ell^{2n}: G \to G$ is a finite locally free covering which splits x, that is $x \in \check{\mathrm{H}}^2_{\acute{e}t}(G,\mathbb{G}_m)_f$. In fact x is an element of $\ker\big[\check{\mathrm{H}}^2_{\acute{e}t}(\epsilon) : \check{\mathrm{H}}^2_{\acute{e}t}(G,\mathbb{G}_m) \to \check{\mathrm{H}}^2_{\acute{e}t}(S,\mathbb{G}_m)\big]_f$. Finally by [27, Prop 3.1.] we can then conclude that x is an element of $\mathrm{Br}(G)$. □

Finally we can prove

Theorem 6.2. *Let $M = [u : X \to G]$ be a 1-motive defined over a normal and noetherian scheme S. Assume that the extension G underlying M satisfies the generalized Theorem of the Cube for a prime number ℓ distinct from the residue characteristics of S. Then the ℓ-primary component of the kernel of the homomorphism $\mathrm{H}^2_{\acute{e}t}(\epsilon) : \mathrm{H}^2_{\acute{e}t}(\mathcal{M},\mathbb{G}_{m,\mathcal{M}}) \to \mathrm{H}^2_{\acute{e}t}(S,\mathbb{G}_{m,S})$ induced by the unit section $\epsilon : S \to M$ of M, is contained in the Brauer group of M:*

$$\ker\big[\mathrm{H}^2_{\acute{e}t}(\epsilon) : \mathrm{H}^2_{\acute{e}t}(\mathcal{M},\mathbb{G}_{m,\mathcal{M}}) \longrightarrow \mathrm{H}^2_{\acute{e}t}(S,\mathbb{G}_{m,S})\big](\ell) \subseteq \mathrm{Br}(\mathcal{M}).$$

Proof. We have to show that if G satisfies the generalized Theorem of the Cube for a prime ℓ, then

$$\ker\big[\mathrm{H}^2_{\acute{e}t}(\epsilon) : \mathrm{H}^2_{\acute{e}t}(\mathcal{M},\mathbb{G}_{m,\mathcal{M}})) \longrightarrow \mathrm{H}^2_{\acute{e}t}(S,\mathbb{G}_{m,S})\big](\ell) \subseteq \mathrm{Br}(\mathcal{M}).$$

In order to simplify notations we denote by $\ker(\mathrm{H}^2_{\acute{e}t}(\epsilon))$ the kernel of the homomorphism $\mathrm{H}^2_{\acute{e}t}(\epsilon)$: $\mathrm{H}^2_{\acute{e}t}(\mathcal{M},\mathbb{G}_{m,\mathcal{M}})) \to \mathrm{H}^2_{\acute{e}t}(S,\mathbb{G}_{m,S})$. Let x be an element of $\ker(\mathrm{H}^2_{\acute{e}t}(\epsilon))$ such that $\ell^n x = 0$ for some n. Let $y = \iota^* x$ the image of x via the homomorphism $\iota^* : \mathrm{H}^2_{\acute{e}t}(\mathcal{M},\mathbb{G}_{m,\mathcal{M}}) \to \mathrm{H}^2_{\acute{e}t}(G,\mathbb{G}_{m,G})$ induced by the quotient map $\iota : G \to [G/X]$. Because of the commutativity of the following diagram

$$\begin{array}{ccc} \mathrm{H}^2_{\acute{e}t}(\mathcal{M},\mathbb{G}_{m,\mathcal{M}}) & \xrightarrow{\iota^*} & \mathrm{H}^2_{\acute{e}t}(G,\mathbb{G}_{m,G}) \\ \downarrow{\scriptstyle \mathrm{H}^2_{\acute{e}t}(\epsilon)} & & \downarrow{\scriptstyle \mathrm{H}^2_{\acute{e}t}(\epsilon_G)} \\ \mathrm{H}^2_{\acute{e}t}(S,\mathbb{G}_{m,S}) & = & \mathrm{H}^2_{\acute{e}t}(S,\mathbb{G}_{m,S}) \end{array} \tag{6.2}$$

(since $\epsilon : S \to \mathcal{M}$ is the unit section of $\mathcal{M}$ and $\epsilon_G : S \to G$ is the unit section of G, $\iota \circ \epsilon_G = \epsilon$), y is in fact an element of $\ker(\mathrm{H}^2_{\acute{e}t}(\epsilon_G))(\ell)$. By Proposition 6.1, we know that $\ker(\mathrm{H}^2_{\acute{e}t}(\epsilon_G))(\ell) \subseteq \mathrm{Br}(G)$, and therefore the element y defines a class $[A]$ in $\mathrm{Br}(G)$, with A an Azumaya algebra on G.

Via the isomorphisms $\mathbb{G}\mathrm{erbe}^2_{S(\mathcal{M})}(\mathbb{G}_{m,\mathcal{M}}) \cong \mathrm{H}^2_{\acute{e}t}(\mathcal{M},\mathbb{G}_{m,\mathcal{M}})$ and $\mathbb{G}\mathrm{erbe}^2_S(\mathbb{G}_{m,G}) \cong \mathrm{H}^2_{\acute{e}t}(G,\mathbb{G}_{m,G})$ obtained in Corollary 2.4, the element x corresponds to the $\mathbb{G}_{m,\mathcal{M}}$-equivalence class $\overline{\mathcal{H}}$ of a $\mathbb{G}_{m,\mathcal{M}}$-gerbe $\mathcal{H}$ on $\mathcal{M}$, and the element y corresponds to the $\mathbb{G}_{m,G}$-equivalence class $\overline{\iota^*\mathcal{H}}$ of the $\mathbb{G}_{m,G}$-gerbe $\iota^*\mathcal{H}$ on G, which is the pull-back of $\mathcal{H}$ via the quotient map $\iota : G \to [G/X]$. By the effectiveness of the 2-descent of $\mathbb{G}_m$-gerbes with respect to ι proved in Lemma 4.3, we can identify the $\mathbb{G}_{m,\mathcal{M}}$-gerbe $\mathcal{H}$ on $\mathcal{M}$ with the triplet $(\iota^*\mathcal{H},\varphi,\gamma)$, where (φ,γ) is 2-descent data on the $\mathbb{G}_{m,G}$-gerbe $\iota^*\mathcal{H}$ with respect to ι. More precisely, $\varphi : p_2^*\iota^*\mathcal{H} \to \mu^*\iota^*\mathcal{H}$ is an equivalence of gerbes on $X \times_S G$ and $\gamma : \big((\mathrm{id}_X \times \mu)^*\varphi\big) \circ \big((p_{23})^*\varphi\big) \Rightarrow (m_X \times id_G)^*\varphi$ is a natural isomorphism which satisfy the compatibility condition (4.2). Moreover via the inclusions $\ker(\mathrm{H}^2_{\acute{e}t}(\epsilon_G))(\ell) \hookrightarrow \mathrm{Br}(G) \overset{\delta}{\hookrightarrow} \mathrm{H}^2_{\acute{e}t}(G,\mathbb{G}_{m,G})$, in $\mathbb{G}\mathrm{erbe}^2_S(\mathbb{G}_{m,G})$ the class $\overline{\iota^*\mathcal{H}}$ coincides with the class $\overline{\delta(A)}$ of the gerbe of trivializations of the Azumaya algebra A.

Now we will show that the 2-descent data (φ,γ) with respect to ι on the $\mathbb{G}_{m,G}$-gerbe $\iota^*\mathcal{H}$ induces a descent datum $\varphi^A : p_2^*A \to \mu^*A$ with respect to ι on the Azumaya algebra A, which satisfies the cocycle condition (4.1). Since the statement of our main theorem involves classes of Azumaya algebras and $\mathbb{G}_m$-equivalence classes of $\mathbb{G}_m$-gerbes, we may assume that $\iota^*\mathcal{H} = \delta(A)$, and so the pair (φ,γ) are canonical 2-descent data on $\delta(A)$. The equivalence of gerbes $\varphi : p_2^*\delta(A) \to \mu^*\delta(A)$ on $X \times_S G$ implies an equivalence of categories $\varphi(U) : p_2^*\delta(A)(U) \to \mu^*\delta(A)(U)$ for any object U of $\mathbf{S}_{\acute{e}t}$ and hence we have the following diagram

$$\begin{array}{ccc} \underline{\mathrm{End}}_{\mathcal{O}_{X\times_S G}}(p_2^*\mathcal{L}_1) & \xrightarrow{a_1} & p_2^*A_{|U} \\ {\scriptstyle \mathrm{End}\varphi(U)} \downarrow & & \\ \underline{\mathrm{End}}_{\mathcal{O}_{X\times_S G}}(\mu^*\mathcal{L}_2) & \xrightarrow{a_2} & \mu^*A_{|U} \end{array} \tag{6.3}$$

with $\mathcal{L}_1$ and $\mathcal{L}_2$ objects of $\mathrm{Lf}(G)(U)$. For any object U of $\mathbf{S}_{ét}$, we define $\varphi^A_{|U} := a_2 \circ \mathrm{End}\varphi(U) \circ a_1^{-1} : p_2^*A_{|U} \to \mu^*A_{|U}$. It is an isomorphism of Azumaya algebras over U. The collection $(\varphi^A_{|U})_U$ of all these isomorphisms furnishes the expected isomorphism of Azumaya algebras $\varphi^A : p_2^*A \to \mu^*A$ on $X \times_S G$. For any object U of $\mathbf{S}_{ét}$, the natural isomorphism $\gamma : \big((\mathrm{id}_X \times \mu)^*\varphi\big) \circ \big((p_{23})^*\varphi\big) \Rightarrow (m_X \times id_G)^*\varphi$ induces $\big((\mathrm{id}_X \times \mu)^*\mathrm{End}\varphi(U)\big) \circ \big((p_{23})^*\mathrm{End}\varphi(U)\big) = (m_X \times id_G)^*\mathrm{End}\varphi(U)$ and so φ^A satisfies the cocycle condition (4.1).

By Lemma 4.2 the descent of Azumaya algebras with respect to ι is effective, and so the pair (A, φ^A) corresponds to an Azumaya algebra $\mathcal{A}$ on $\mathcal{M}$, whose equivalence class $[\mathcal{A}]$ is an element of $\mathrm{Br}(\mathcal{M})$. □

Corollary 6.3. *Let M be a 1-motive defined over an algebraically closed field of characteristic zero. Then* $\mathrm{Br}(\mathcal{M}) \cong \mathrm{H}^2_{ét}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}})$.

Proof. First we prove that $\mathrm{H}^2_{ét}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}})$ is a torsion group: in fact, as already observed, $\mathrm{H}^2_{ét}(G, \mathbb{G}_{m,G})$ is a torsion group and by Corollary 2.4 $\mathbb{G}\mathrm{erbe}^2_{S(\mathcal{M})}(\mathbb{G}_{m,\mathcal{M}}) \cong \mathrm{H}^2_{ét}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}})$ and $\mathbb{G}\mathrm{erbe}^2_S(\mathbb{G}_{m,G}) \cong \mathrm{H}^2_{ét}(G, \mathbb{G}_{m,G})$. Now by Lemma 4.3, the 2-descent of $\mathbb{G}_m$-gerbes with respect to the quotient map $\iota : G \to [G/X] \cong \mathcal{M}$ is effective and so also the group $\mathrm{H}^2_{ét}(\mathcal{M}, \mathbb{G}_{m,\mathcal{M}})$ is torsion. Finally, if S is the spectrum of an algebraically closed field of characteristic zero $\mathrm{H}^2_{ét}(S, \mathbb{G}_{m,S}) = 0$, and therefore the statement is a consequence of Theorem 6.2. □

Appendix A. A communication from P. Deligne on 2-descent theory for stacks

Let $\mathbf{S}$ be a site. For our applications, $\mathbf{S}$ will be the site of a scheme S for a Grothendieck topology, that is the category of S-schemes endowed with a Grothendieck topology.

Here we describe the 2-descent theory for stacks over $\mathbf{S}$ following Deligne's indications. We will consider two different points of view in order to describe this 2-descent.

We start with the point of view of covering sieves. Let $\mathcal{C}r$ be a covering sieve of the final object of $\mathbf{S}$. In the case of a scheme S, $\mathcal{C}r$ is a sieve generated by a covering family of arrows $(X_i \to S)_i$, that is $\mathcal{C}r$ consists of the S-schemes T such that it exists a morphism $T \to X_i$ of S-schemes for an index i. With these notation, the 2-descent data for a stack $\mathcal{X}$ with respect to the functor $\mathcal{C}r \to \mathbf{S}$ are the following:

- for any object T in $\mathcal{C}r$, a stack $\mathcal{X}_T$ over T
- for any arrow $a : T' \to T$ between objects of $\mathcal{C}r$, an equivalence of stacks $\varphi_a : a^*\mathcal{X}_T \to \mathcal{X}_{T'}$ over T',
- for any composite $T'' \overset{b}{\to} T' \overset{a}{\to} T$ in $\mathcal{C}r$, a natural isomorphism $\varphi_{ab} \Rightarrow \varphi_b \circ b^*\varphi_a$ between equivalences of stacks over T'',
- for any triple composite $T''' \overset{c}{\to} T'' \overset{b}{\to} T' \overset{a}{\to} T$ in $\mathcal{C}r$, a compatibility condition $\varphi_{bc} \circ (bc)^*\varphi_a = \varphi_c \circ c^*\varphi_{ab}$ involving the above natural isomorphisms.

Now we describe the 2-descent of stacks from the point of view of Čech-coverings. Let S' be a covering of the final object of the site $\mathbf{S}$ that we denote by S. In the case of the site of a scheme S for a Grothendieck topology, S' is a covering of S for the chosen Grothendieck topology. The 2-descent data for a stack with respect to the covering $S' \to S$ are the following:

(1) a stack $\mathcal{X}$ over S'

(2) over $S' \times_S S'$, an equivalence of stacks $\varphi : p_1^*\mathcal{X} \to p_2^*\mathcal{X}$, where $p_i : S' \times_S S' \to S'$ are the natural projections,
(3) over $S' \times_S S' \times_S S'$, a natural isomorphism $\gamma :\ p_{23}^*\varphi \circ p_{12}^*\varphi \Rightarrow p_{13}^*\varphi$, where $p_{ij} : S' \times_S S' \times_S S' \to S' \times_S S'$ are the partial projections,
(4) over $S' \times_S S' \times_S S' \times_S S'$, the compatibility condition

$$p_{134}^*\gamma \circ [p_{34}^*\varphi * p_{123}^*\gamma] = p_{124}^*\gamma \circ [p_{234}^*\gamma * p_{12}^*\varphi], \tag{A.1}$$

where $p_{ijk} : S' \times_S S' \times_S S' \times_S S' \times_S S' \to S' \times_S S' \times_S S'$ and $p_{ij} : S' \times_S S' \times_S S' \times_S S' \to S' \times_S S'$ are the partial projections.

A nice explanation of this last compatibility condition can be found in [12, page 442 diagram (6.2.8)]. The above 2-descent of stacks through Čech-coverings furnishes Breen's semi-local description of gerbes cited in Section 2.3.

Now if $\mathcal{G}$ be an $\mathcal{L}$-gerbe on $\mathbf{S}$, with $\mathcal{L}$ an abelian band, the 2-descent data for $\mathcal{G}$ with respect to the covering $S' \to S$ become in this case the following:

(1′) a neutral $\mathcal{L}$-gerbe over S', that is $\mathcal{T}\mathrm{ors}(\mathcal{L})$, or a neutralization of a $\mathcal{L}$-gerbe $\mathcal{G}'$ over S', that is an equivalence of S'-stacks $\Psi : \mathcal{G}' \to \mathcal{T}\mathrm{ors}(\mathcal{L})$,
(2′) over $S' \times_S S'$, an $\mathcal{L}$-torsor $T(1,2)$. In fact by [20, Chp IV, Prop 5.2.5 (iii)] we have an equivalence of categories between the category of equivalences between $p_1^*\mathcal{T}\mathrm{ors}(\mathcal{L})$ and $p_2^*\mathcal{T}\mathrm{ors}(\mathcal{L})$ and the category of $\mathcal{L}$-torsors.
(3′) over $S' \times_S S' \times_S S'$, an isomorphism $\gamma : T(1,2) \wedge^{\mathcal{L}} T(2,3) \to T(1,3)$ of $\mathcal{L}$-torsors,
(4′) over $S' \times_S S' \times_S S' \times_S S'$, the compatibility condition is that the two isomorphisms of $\mathcal{L}$-torsors from $T(1,2) \wedge^{\mathcal{L}} T(2,3) \wedge^{\mathcal{L}} T(3,4)$ to $T(1,4)$ should be equal.

Remark. If in (2′) the $\mathcal{L}$-torsor $T(1,2)$ is trivial, that is $T(1,2) = \mathcal{L}$, then the datum (3′) becomes a section of $\mathcal{L}$ over $S' \times_S S' \times_S S'$ and the compatibility condition (4′) becomes that this section is in fact a 2-cocycle (see also end of [14, end of 2.3]). This is the heuristic explanation of the homological interpretation of gerbes proved in Theorem 2.2.

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