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(Article begins on next page)

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NUCLEARITY OF RAPIDLY DECREASING ULTRADIFFERENTIABLE FUNCTIONS AND TIME-FREQUENCY ANALYSIS

CHIARA BOITI, DAVID JORNET, ALESSANDRO OLIARO, AND GERHARD SCHINDL

ABSTRACT. We use techniques from time-frequency analysis to show that the space $\mathcal{S}_\omega$ of rapidly decreasing ω-ultradifferentiable functions is nuclear for every weight function $\omega(t) = o(t)$ as t tends to infinity. Moreover, we prove that, for a sequence $(M_p)_p$ satisfying the classical condition $(M1)$ of Komatsu, the space of Beurling type $\mathcal{S}_{(M_p)}$ when defined with L^2 norms is nuclear exactly when condition $(M2)'$ of Komatsu holds.

1. Introduction

One of the main properties of a nuclear space is that the Schwartz kernel theorem holds, which gives, for instance, a different representation of a continuous and linear pseudodifferential operator as an integral operator in terms of its kernel. This is very useful for the study of the propagation of singularities or the behaviour of wave front sets of pseudodifferential operators. See, for example, [1, 5, 10, 11, 23, 24] and the references therein.

In fact, in [5] the first three authors of the present work imposed the following condition on the weight function: there is $H > 1$ such that for every $t > 0$, $2\omega(t) \leq \omega(Ht) + H$ (see [8, Corollary 16 (3)]), to have that the space $\mathcal{S}_\omega(\mathbb{R}^d)$ is nuclear (see [6]). Hence they could analyse the kernel of some pseudodifferential operators [5, Section 4]. In the present paper, we complete the study begun in [6] and prove that $\mathcal{S}_\omega(\mathbb{R}^d)$ is nuclear for every weight function $\omega(t) = o(t)$ as t tends to infinity (see Definition 2.1). Hence, now the powers of the logarithm $\omega(t) = \log^\beta(1+t)$, $\beta > 0$, are allowed as weight functions and, in particular, we recover a known result for the weight $\omega(t) = \log(1+t)$, namely, that the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ is a nuclear space.

To see that $\mathcal{S}_\omega(\mathbb{R}^d)$ is nuclear we establish an isomorphism, which is new in the literature, with some Fréchet sequence space. We use expansions in terms of Gabor frames, that are a fundamental tool in time-frequency analysis. This is motivated by the rapid decay of the Gabor coefficients of a function in $\mathcal{S}_\omega(\mathbb{R}^d)$ when ω is a subadditive function, as we showed in [5]. More precisely, we proved that $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ if and only if

$$\sup_{\sigma \in \alpha_0 \mathbb{Z}^d \times \beta_0 \mathbb{Z}^d} e^{\lambda \omega(\sigma)} |V_\varphi u(\sigma)| < +\infty, \qquad \text{for all } \lambda > 0,$$

where $\alpha_0, \beta_0 > 0$ are sufficiently small so that $\{\Pi(\sigma)\varphi\}_{\sigma \in \alpha_0 \mathbb{Z}^d \times \beta_0 \mathbb{Z}^d}$ is a Gabor frame in $L^2(\mathbb{R}^d)$ for a fixed window function $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$, $V_\varphi u$ is the short-time Fourier transform of u and $\Pi(\sigma)$ is the time-frequency shift defined as $\Pi(\sigma)\varphi(y) = e^{i\langle y, \beta_0 n \rangle} \varphi(y - \alpha_0 k)$, for $\sigma = (\alpha_0 k, \beta_0 n)$. The usual properties of modulation spaces in [5] hold only when the weight function ω is

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subadditive. However, the expansion in terms of Gabor frames is still possible when the weight is non subadditive and satisfies $\omega(t) = o(t)$ as t tends to infinity. In fact, we prove here that $\mathcal{S}_\omega(\mathbb{R}^d)$ is isomorphic to a topological subspace of the sequence space

$$(1.1) \qquad \tilde{\Lambda}_\omega := \{c = (c_\sigma)_{\sigma \in \alpha_0 \mathbb{Z}^d \times \beta_0 \mathbb{Z}^d} : \|c\|_k := \sup_{\sigma \in \alpha_0 \mathbb{Z}^d \times \beta_0 \mathbb{Z}^d} |c_\sigma| e^{k\omega(\sigma)} < +\infty, \ \forall\, k \in \mathbb{N}\}.$$

The isomorphism is defined in (2.6) by the restriction on its image of the *analysis* operator, that maps $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ to its Gabor coefficients $\{V_\varphi u(\sigma)\}_{\sigma \in \alpha_0 \mathbb{Z}^d \times \beta_0 \mathbb{Z}^d}$. As a consequence, $\mathcal{S}_\omega(\mathbb{R}^d)$ is nuclear by an application of Grothendieck-Pietsch criterion to the space $\tilde{\Lambda}_\omega$ (Proposition 3.2). This isomorphism is not the only one existing in the literature, and it should be compared with the one given by Aubry [2], *only* for the one-variable case, obtaining that $\mathcal{S}_\omega(\mathbb{R})$ is isomorphic to the different sequence space

$$\Lambda_\omega := \left\{ (c_k)_{k \in \mathbb{N}_0} : \ \sup_{k \in \mathbb{N}_0} |c_k| e^{j\omega(k^{1/2})} < +\infty, \ \forall\, j \in \mathbb{N}_0 \right\}.$$

Aubry uses expansions in terms of the Hermite functions, as Langenbruch [18] did previously for spaces defined by sequences in the sense of Komatsu.

Finally, in the last section of the paper, and without using techniques from time frequency-analysis, we characterize when the Beurling space of ultradifferentiable functions $\mathcal{S}_{(M_p)}(\mathbb{R}^d)$ (see formula (4.1) for the definition) in the sense of Komatsu is nuclear. We can give such a characterization when the space is defined by L^2 norms. We explain and motivate a little bit this result. Pilipović, Prangoski and Vindas [22] showed that

$$(1.2) \qquad S_{(M_p)}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \ \sup_{\alpha,\beta \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} \frac{j^{|\alpha+\beta|}}{M_{|\alpha+\beta|}} \|x^\alpha \partial^\beta f(x)\|_\infty < +\infty, \ \forall\, j \in \mathbb{N} \right\},$$

is nuclear when $(M_p)_p$ satisfies the standard conditions $(M1)$ (defined below in formula (1.5)) and $(M2)$ (that we do not define here because it is not used), which is stronger than $(M2)'$, defined below in formula (1.4). On the other hand, using the isomorphism of [18], we proved in [6] that the space $S_{(M_p)}(\mathbb{R}^d)$ is nuclear if $(M_p)_p$ satisfies that there is $H > 0$ such that for any $C > 0$ there is $B > 0$ with

$$(1.3) \qquad s^{s/2} M_p \le BC^s H^{s+p} M_{s+p}, \qquad \forall s, p \in \mathbb{N}_0$$

and $(M2)'$ *(stability under differential operators)*:

$$(1.4) \qquad \exists A, H > 0 : \quad M_{p+1} \le A H^p M_p, \qquad \forall p \in \mathbb{N}_0.$$

The condition (1.3) is quite natural and not restrictive at all and it is used by Langenbruch [18] to show that the Hermite functions are elements of $S_{(M_p)}(\mathbb{R}^d)$. Moreover, Langenbruch also proves in [18, Remark 2.1] that under these two conditions (1.3) and (1.4), $S_{(M_p)}(\mathbb{R}^d) = \mathcal{S}_{(M_p)}(\mathbb{R}^d)$. If we do not assume (1.4) and consider only $\mathcal{S}_{(M_p)}(\mathbb{R}^d)$ (the space defined with L^2 norms), after a careful reading of the proofs of some results of [18] in the Beurling case and the use of techniques of Petzsche [20], we are able to prove here that under the additional conditions $(M1)$ *(logarithmic convexity)*:

$$(1.5) \qquad M_p^2 \le M_{p-1} M_{p+1}, \qquad \forall p \in \mathbb{N},$$

and that $M_p^{1/p} \to +\infty$ as $p \to +\infty$, $\mathcal{S}_{(M_p)}(\mathbb{R}^d)$ is nuclear if and only if $(M_p)_p$ satisfies $(M2)'$.

The paper is organized as follows. In Section 2, we show that Gabor frames have a stable behaviour with the only assumption $\omega(t) = o(t)$ as t tends to infinity on the weight function. Indeed, we see that the *analysis* and *synthesis* operators are well defined and continuous in the suitable spaces (Propositions 2.9 and 2.10), defining an isomorphism between $\mathcal{S}_\omega(\mathbb{R}^d)$ and a subspace of $\tilde{\Lambda}_\omega$. In Section 3 we recover for this setting some known properties of Köthe echelon spaces to see that the sequence space $\tilde{\Lambda}_\omega$ is nuclear. And, finally, in Section 4 we characterize the nuclearity of $\mathcal{S}_{(M_p)}(\mathbb{R}^d)$.

2. Gabor frame operators in $\mathcal{S}_\omega(\mathbb{R}^d)$

Let us condider weight functions of the form:

Definition 2.1. *A* weight function *is a continuous increasing function $\omega : [0, +\infty) \to [0, +\infty)$ satisfying the following properties:*

(α) *there is $L \geq 1$ such that $\omega(2t) \leq L(\omega(t) + 1)$, for each $t \geq 0$;*

(β) $\omega(t) = o(t)$ *as $t \to +\infty$;*

(γ) *there are $a \in \mathbb{R}, b > 0$ such that $\omega(t) \geq a + b\log(1+t), \; t \geq 0$;*

(δ) *the map $t \mapsto \varphi_\omega(t) := \omega(e^t)$ is convex.*

For $\zeta \in \mathbb{C}^d$, we put $\omega(\zeta) := \omega(|\zeta|)$, where $|\zeta|$ denotes the Euclidean norm of ζ.

Note that condition (α) implies

$$(2.1) \qquad \omega(t_1 + t_2) \leq L(\omega(t_1) + \omega(t_2) + 1), \qquad t_1, t_2 \geq 0.$$

We denote by φ_ω^* the *Young conjugate* of φ_ω, defined by

$$\varphi_\omega^*(s) := \sup_{t \geq 0}\{ts - \varphi_\omega(t)\}.$$

We recall that φ_ω^* is an increasing and convex function satisfying $(\varphi_\omega^*)^* = \varphi_\omega$ (see [15]). Moreover $\varphi_\omega^*(s)/s$ is increasing. For a collection of further well-known properties of φ_ω^* we refer, for instance, to [7, Lemma 2.3].

We consider the following notation for the Fourier transform of $u \in L^1(\mathbb{R}^d)$:

$$\mathcal{F}(u)(\xi) = \hat{u}(\xi) := \int_{\mathbb{R}^d} u(x)e^{-i\langle x, \xi \rangle}dx, \qquad \xi \in \mathbb{R}^d,$$

with standard extensions to more general spaces of functions or distributions. We recover from [3] the following

Definition 2.2. *The space $\mathcal{S}_\omega(\mathbb{R}^d)$ is the set of all $u \in L^1(\mathbb{R}^d)$ such that $u, \hat{u} \in C^\infty(\mathbb{R}^d)$ and for each $\lambda > 0$ and each $\alpha \in \mathbb{N}_0^d$ we have*

$$\sup_{x \in \mathbb{R}^d} e^{\lambda\omega(x)}|\partial^\alpha u(x)| < +\infty \quad and \quad \sup_{\xi \in \mathbb{R}^d} e^{\lambda\omega(\xi)}|\partial^\alpha \hat{u}(\xi)| < +\infty.$$

The corresponding strong dual of ultradistributions will be denoted by $\mathcal{S}_\omega'(\mathbb{R}^d)$.

We denote by T_x, M_ξ and $\Pi(z)$, respectively, the *translation*, the *modulation* and the *phase-space shift* operators, defined by

$$T_x f(y) = f(y - x), \quad M_\xi f(y) = e^{i\langle y, \xi\rangle} f(y)$$
$$\Pi(z) f(y) = M_\xi T_x f(y) = e^{i\langle y, \xi\rangle} f(y - x)$$

for $x, y, \xi \in \mathbb{R}^d$ and $z = (x, \xi)$.

For a window function $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ the *short-time Fourier transform* (briefly STFT) of $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$ is defined, for $z = (x, \xi) \in \mathbb{R}^{2d}$, by

$$(2.2) \qquad\qquad V_\varphi f(z) := \langle f, \Pi(z)\varphi\rangle$$

$$(2.3) \qquad\qquad\qquad = \int_{\mathbb{R}^d} f(y)\overline{\varphi(y - x)} e^{-i\langle y, \xi\rangle} dy,$$

where the brackets $\langle\cdot, \cdot\rangle$ in (2.2) and the (formal) integral in (2.3) denote the conjugate linear action of $\mathcal{S}'_\omega$ on $\mathcal{S}_\omega$, consistent with the inner product $\langle\cdot, \cdot\rangle_{L^2}$.

By condition (γ) of Definition 2.1 it is easy to deduce that $\mathcal{S}_\omega(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$. Hence, $\mathcal{S}_\omega(\mathbb{R}^d)$ can be equivalently defined as the set of all $u \in \mathcal{S}(\mathbb{R}^d)$ that satisfy the conditions of Definition 2.2. The Fourier transform $\mathcal{F} : \mathcal{S}_\omega(\mathbb{R}^d) \to \mathcal{S}_\omega(\mathbb{R}^d)$ is a continuous automorphism, that can be extended in the usual way to $\mathcal{S}'_\omega(\mathbb{R}^d)$ and, moreover, the space $\mathcal{S}_\omega(\mathbb{R}^d)$ is an algebra under multiplication and convolution. On the other hand, for $u, \psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ we have $V_\psi u \in \mathcal{S}_\omega(\mathbb{R}^{2d})$. Moreover, for $u \in \mathcal{S}'_\omega(\mathbb{R}^d)$ the short-time Fourier transform is well defined and belongs to $\mathcal{S}'_\omega(\mathbb{R}^{2d})$. We refer to [3, 14, 5] for subadditive weights, and to [4, 7] for non-subadditive weights; in particular, all results of [5, Section 2] are valid in the non-subadditive case also.

We shall need the following theorem from [7]:

Theorem 2.3. *Given a function $u \in \mathcal{S}(\mathbb{R}^d)$ and $1 \leq p, q \leq +\infty$, we have that $u \in \mathcal{S}_\omega(\mathbb{R}^d)$ if and only if one of the following equivalent conditions is satisfied:*

(a)   i) $\forall \lambda > 0,\ \alpha \in \mathbb{N}_0^d\ \exists C_{\alpha,\lambda} > 0\ s.t.\ \|e^{\lambda\omega(x)}\partial^\alpha u(x)\|_{L^p} \leq C_{\alpha,\lambda}$,

     ii) $\forall \lambda > 0,\ \alpha \in \mathbb{N}_0^d\ \exists C_{\alpha,\lambda} > 0\ s.t.\ \|e^{\lambda\omega(\xi)}\partial^\alpha \hat{u}(\xi)\|_{L^q} \leq C_{\alpha,\lambda}$;

(b)   i) $\forall \lambda > 0,\ \alpha \in \mathbb{N}_0^d\ \exists C_{\alpha,\lambda} > 0\ s.t.\ \|e^{\lambda\omega(x)}x^\alpha u(x)\|_{L^p} \leq C_{\alpha,\lambda}$,

     ii) $\forall \lambda > 0,\ \alpha \in \mathbb{N}_0^d\ \exists C_{\alpha,\lambda} > 0\ s.t.\ \|e^{\lambda\omega(\xi)}\xi^\alpha \hat{u}(\xi)\|_{L^q} \leq C_{\alpha,\lambda}$;

(c)   i) $\forall \lambda > 0\ \exists C_\lambda > 0\ s.t.\ \|e^{\lambda\omega(x)}u(x)\|_{L^p} \leq C_\lambda$,

     ii) $\forall \lambda > 0\ \exists C_\lambda > 0\ s.t.\ \|e^{\lambda\omega(\xi)}\hat{u}(\xi)\|_{L^q} \leq C_\lambda$;

(d)   i) $\forall \lambda > 0, \beta \in \mathbb{N}_0^d\ \exists C_{\beta,\lambda} > 0\ s.t.\ \sup_{\alpha\in\mathbb{N}_0^d} \|x^\beta\partial^\alpha u(x)\|_{L^p} e^{-\lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)} \leq C_{\beta,\lambda}$,

     ii) $\forall \mu > 0, \alpha \in \mathbb{N}_0^d\ \exists C_{\alpha,\mu} > 0\ s.t.\ \sup_{\beta\in\mathbb{N}_0^d} \|x^\beta\partial^\alpha u(x)\|_{L^q} e^{-\mu\varphi_\omega^*\left(\frac{|\beta|}{\mu}\right)} \leq C_{\alpha,\mu}$;

(e) $\forall \mu, \lambda > 0\ \exists C_{\mu,\lambda} > 0\ s.t.\ \sup_{\alpha,\beta\in\mathbb{N}_0^d} \|x^\beta\partial^\alpha u(x)\|_{L^p} e^{-\lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)} e^{-\mu\varphi_\omega^*\left(\frac{|\beta|}{\mu}\right)} \leq C_{\mu,\lambda}$;

(f) $\forall \lambda > 0\ \exists C_\lambda > 0\ s.t.\ \sup_{\alpha,\beta\in\mathbb{N}_0^d} \|x^\beta\partial^\alpha u(x)\|_{L^p} e^{-\lambda\varphi_\omega^*\left(\frac{|\alpha+\beta|}{\lambda}\right)} \leq C_\lambda$;

(g) $\forall \mu, \lambda > 0\ \exists C_{\mu,\lambda} > 0\ s.t.\ \sup_{\alpha\in\mathbb{N}_0^d} \|e^{\mu\omega(x)}\partial^\alpha u(x)\|_{L^p} e^{-\lambda\varphi_\omega^*\left(\frac{|\alpha|}{\lambda}\right)} \leq C_{\mu,\lambda}$;

(h) *Given $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$, $\forall \lambda > 0\ \exists C_\lambda > 0\ s.t.\ \|V_\psi u(z)e^{\lambda\omega(z)}\|_{L^{p,q}} \leq C_\lambda$.*

Let us set, for $\lambda \in \mathbb{R} \setminus \{0\}$,

$$m_\lambda(z) = e^{\lambda \omega(z)}, \qquad z \in \mathbb{R}^{2d},$$

and consider the weighted $L^{p,q}$ spaces

$$L_{m_\lambda}^{p,q}(\mathbb{R}^{2d}) := \Big\{ F \text{ measurable on } \mathbb{R}^{2d} \text{ such that}$$

$$\|F\|_{L_{m_\lambda}^{p,q}} := \Big(\int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |F(x,\xi)|^p m_\lambda(x,\xi)^p dx \Big)^{q/p} d\xi \Big)^{1/q} < +\infty \Big\},$$

for $1 \leq p, q < +\infty$, and

$$L_{m_\lambda}^{\infty,q}(\mathbb{R}^{2d}) := \Big\{ F \text{ measurable on } \mathbb{R}^{2d} \text{ such that}$$

$$\|F\|_{L_{m_\lambda}^{\infty,q}} := \Big(\int_{\mathbb{R}^d} \big(\operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |F(x,\xi)| m_\lambda(x,\xi) \big)^q d\xi \Big)^{1/q} < +\infty \Big\},$$

$$L_{m_\lambda}^{p,\infty}(\mathbb{R}^{2d}) := \Big\{ F \text{ measurable on } \mathbb{R}^{2d} \text{ such that}$$

$$\|F\|_{L_{m_\lambda}^{p,\infty}} := \operatorname*{ess\,sup}_{\xi \in \mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |F(x,\xi)|^p m_\lambda(x,\xi)^p dx \Big)^{1/p} < +\infty \Big\},$$

for $1 \leq p, q \leq +\infty$ with $p = +\infty$ or $q = +\infty$ respectively. If $p = q$ we write $L_{m_\lambda}^p(\mathbb{R}^d) = L_{m_\lambda}^{p,q}(\mathbb{R}^d)$.

Here we consider generic weight functions ω satisfying (α) of Definition 2.1 (weaker than subadditivity). In this case modulation spaces lack several properties. Hence we prove directly some results on Gabor frames in $\mathcal{S}_\omega(\mathbb{R}^d)$ without using modulation spaces. If ω is subadditive we know, by Theorem 4.2 of [14], that for any fixed $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$ its dual window ψ, in the sense of the theory of Gabor frames (see Gröchenig [12]), belongs to $\mathcal{S}_\omega(\mathbb{R}^d)$ (see also [13, Thm. 4.2] and [16]). In our case we will fix $\varphi_0(x) := e^{-|x|^2}$ the Gaussian function, $\alpha_0, \beta_0 > 0$ such that $\{\Pi(\sigma)\varphi_0\}_{\sigma \in \alpha_0 \mathbb{Z}^d \times \beta_0 \mathbb{Z}^d}$ is a Gabor frame for $L^2(\mathbb{R}^d)$, and then prove that the canonical dual window ψ_0 of φ_0 is in $\mathcal{S}_\omega(\mathbb{R}^d)$. To this aim we start by the following

Lemma 2.4. *Let ω be a weight function. There exists then a subadditive weight function σ such that $\omega(t) = o(\sigma(t))$ as $t \to +\infty$.*

Proof. Let us consider $\omega_0(t) = \max\{0, t-1\}$. This is a continuous increasing function $\omega_0 : [0, +\infty) \to [0, +\infty)$ that satisfies (α), (γ) and (δ) of Definition 2.1 and moreover $\omega_0|_{[0,1]} \equiv 0$ and ω_0 is concave on $[1 + \infty)$.

Then, by Lemma 1.7 and Remark 1.8(1) of [9], there exists a weight function λ satisfying (α), (γ) and (δ) and such that $\lambda|_{[0,1]} \equiv 0$, λ concave on $[1, +\infty)$ and

$$\omega(t) = o(\lambda(t)), \qquad \text{as } t \to +\infty,$$
$$\lambda(t) = o(\omega_0(t)) = o(t), \qquad \text{as } t \to +\infty.$$

Since $\lambda(t+1)$ is concave on $[0, +\infty)$ with $\lambda(1) = 0$, we have that $\sigma(t) := \lambda(t) + \lambda(2)$ is the required weight function. $\qquad \square$

Proposition 2.5. *Let $\varphi_0(x) = e^{-|x|^2}$ be the Gaussian function and let ψ_0 be a dual window of φ_0. Then $\varphi_0, \psi_0 \in \mathcal{S}_\omega(\mathbb{R}^d)$ for every weight function ω.*

Proof. Let ω be a weight function as in Definition 2.1. By Lemma 2.4 there exists a subadditive weight function σ such that $\omega(t) = o(\sigma(t))$ as $t \to +\infty$. Then $\mathcal{S}_\sigma(\mathbb{R}^d) \subseteq \mathcal{S}_\omega(\mathbb{R}^d)$.

Clearly $\varphi_0 \in \mathcal{S}_\sigma(\mathbb{R}^d) \subseteq \mathcal{S}_\omega(\mathbb{R}^d)$ by condition (β). Since σ is subadditive, by [14, Thm. 4.2], its dual window $\psi_0 \in \mathcal{S}_\sigma(\mathbb{R}^d) \subseteq \mathcal{S}_\omega(\mathbb{R}^d)$ and the proof is complete. $\square$

We fix, once and for all, $\varphi_0(x) = e^{-|x|^2}$, $\alpha_0, \beta_0 > 0$ such that $\{\Pi(\sigma)\varphi_0\}_{\sigma \in \alpha_0 \mathbb{Z}^d \times \beta_0 \mathbb{Z}^d}$ is a Gabor frame for L^2 and ψ_0 the canonical dual window of φ_0 (see [12, Section 7.3]). For the lattice $\Lambda := \alpha_0 \mathbb{Z}^d \times \beta_0 \mathbb{Z}^d$, we consider the *analysis operator* C_{φ_0} acting on a function $f \in L^2(\mathbb{R}^d)$

$$C_{\varphi_0} f := \langle f, \Pi(\sigma)\varphi_0 \rangle, \qquad \sigma \in \Lambda,$$

and the *synthesis operator* D_{ψ_0} acting on a sequence $c = (c_{k,n})_{k,n \in \mathbb{Z}^d}$

$$D_{\psi_0} c = \sum_{k,n \in \mathbb{Z}^d} c_{k,n} \Pi(\alpha_0 k, \beta_0 n)\psi_0.$$

It is well known (see, for instance, [12]) that

$$D_{\psi_0} C_{\varphi_0} = \mathrm{Id}, \qquad \text{the identity on } L^2(\mathbb{R}^d),$$

since ψ_0 is the canonical dual window of φ_0, and then

$$(2.4) \qquad D_{\psi_0} C_{\varphi_0} = \mathrm{Id}, \qquad \text{on } \mathcal{S}_\omega(\mathbb{R}^d) \subset L^2(\mathbb{R}^d).$$

Later on we shall explain more precisely this identity on $\mathcal{S}_\omega(\mathbb{R}^d)$.

We denote by $\ell_{m_\lambda}^{p,q}$, for $1 \le p,q \le +\infty$ and $\lambda \in \mathbb{R} \setminus \{0\}$, the space of all sequences $a = (a_{kn})_{k,n \in \mathbb{Z}^d}$, with $a_{kn} \in \mathbb{C}$ for every $k,n \in \mathbb{Z}^d$, such that

$$\|a\|_{\ell_{m_\lambda}^{p,q}} := \left(\sum_{n \in \mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d} |a_{kn}|^p m_\lambda(k,n)^p \right)^{q/p} \right)^{1/q} < +\infty,$$

if $1 \le p,q < +\infty$,

$$\|a\|_{\ell_{m_\lambda}^{\infty,q}} := \left(\sum_{n \in \mathbb{Z}^d} \left(\sup_{k \in \mathbb{Z}^d} |a_{kn}| m_\lambda(k,n) \right)^q \right)^{1/q} < +\infty,$$

$$\|a\|_{\ell_{m_\lambda}^{p,\infty}} := \sup_{n \in \mathbb{Z}^d} \left(\sum_{k \in \mathbb{Z}^d} |a_{kn}|^p m_\lambda(k,n)^p \right)^{1/p} < +\infty,$$

for $1 \le p,q \le +\infty$ with $p = +\infty$ or $q = +\infty$ respectively.

Then we say that a measurable function F on $\mathbb{R}^{2d}$ belongs to the *amalgam space* $W(L_{m_\lambda}^{p,q})$ for the sequence

$$a_{kn} := \operatorname*{ess\,sup}_{(x,\xi) \in [0,1]^{2d}} |F(k+x, n+\xi)| = \|F \cdot T_{(k,n)} \chi_Q\|_{L^\infty},$$

where χ_Q is the characteristic function of the cube $Q = [0,1]^{2d}$, when $a = (a_{kn})_{k,n \in \mathbb{Z}^d} \in \ell_{m_\lambda}^{p,q}$. Equivalently, $F \in W(L_{m_\lambda}^{p,q})$ if and only if

$$(2.5) \qquad |F| \le \sum_{k,n \in \mathbb{Z}^d} b_{kn} T_{(k,n)} \chi_Q$$

for some $b = (b_{kn})_{k,n \in \mathbb{Z}^d} \in \ell_{m_\lambda}^{p,q}$ (cf. [12, pg. 222]). The amalgam space $W(L_{m_\lambda}^{p,q})$ is endowed with the norm

$$\|F\|_{W(L_{m_\lambda}^{p,q})} = \|a\|_{\ell_{m_\lambda}^{p,q}}.$$

In what follows we shall need the Young estimate for $L_{m_\lambda}^{p,q}$:

Proposition 2.6. *Let ω be a weight function and L as in (2.1). Set, for every $\lambda \in \mathbb{R}$,*

$$\mu(\lambda) := \begin{cases} \lambda L, & \lambda \geq 0 \\ \lambda/L, & \lambda < 0, \end{cases} \qquad \nu(\lambda) := \begin{cases} \lambda L, & \lambda \geq 0 \\ |\lambda|, & \lambda < 0. \end{cases}$$

Then, for $F \in L_{m_{\mu(\lambda)}}^{p,q}$ and $G \in L_{m_{\nu(\lambda)}}^1$, with $1 \leq p, q \leq +\infty$, we have that $F * G \in L_{m_\lambda}^{p,q}$ and

$$\|F * G\|_{L_{m_\lambda}^{p,q}} \leq C_\lambda \|F\|_{L_{m_{\mu(\lambda)}}^{p,q}} \|G\|_{L_{m_{\nu(\lambda)}}^1}$$

for a constant $C_\lambda > 0$ depending on λ.

Proof. Let us first assume $1 \leq p, q < +\infty$. From the definition of convolution

$$\|F * G\|_{L_{m_\lambda}^{p,q}} \leq \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \left(e^{\lambda \omega(x,\xi)} \int_{\mathbb{R}^{2d}} |F(x-y, \xi-\eta)G(y,\eta)| dy d\eta \right)^p dx \right)^{q/p} d\xi \right)^{1/q}.$$

Now, for $\lambda \geq 0$ we have, by (2.1),

$$\lambda \omega(x,\xi) \leq \lambda L(\omega(x-y, \xi-\eta) + \omega(y,\eta) + 1),$$

so that

$$\|F * G\|_{L_{m_\lambda}^{p,q}} \leq e^{\lambda L} \left\| \left(|F| e^{\lambda L \omega(\cdot)} \right) * \left(|G| e^{\lambda L \omega(\cdot)} \right) \right\|_{L^{p,q}}.$$

By the standard Young's inequality for (non weighted) $L^{p,q}$ spaces we obtain

$$\begin{aligned} \|F * G\|_{L_{m_\lambda}^{p,q}} &\leq C_\lambda \||F| e^{\lambda L \omega(\cdot)}\|_{L^{p,q}} \||G| e^{\lambda L \omega(\cdot)}\|_{L^1} \\ &= C_\lambda \|F\|_{L_{m_{\lambda L}}^{p,q}} \|G\|_{L_{m_{\lambda L}}^1} = C_\lambda \|F\|_{L_{m_{\mu(\lambda)}}^{p,q}} \|G\|_{L_{m_{\nu(\lambda)}}^1}. \end{aligned}$$

For $\lambda < 0$ we have, by (2.1),

$$\lambda \omega(x,\xi) \leq \frac{\lambda}{L} \omega(x-y, \xi-\eta) - \lambda \omega(y,\eta) - \lambda = \frac{\lambda}{L} \omega(x-y, \xi-\eta) + |\lambda| \omega(y,\eta) - \lambda,$$

and then, as before,

$$\|F * G\|_{L_{m_\lambda}^{p,q}} \leq C_\lambda \|F\|_{L_{m_{\lambda/L}}^{p,q}} \|G\|_{L_{m_{|\lambda|}}^1} = C_\lambda \|F\|_{L_{m_{\mu(\lambda)}}^{p,q}} \|F\|_{L_{m_{\nu(\lambda)}}^1}$$

for some $C_\lambda > 0$. The proof for $p = +\infty$ and/or $q = +\infty$ is similar. $\square$

We have the following proposition, analogous to [12, Prop. 11.1.4]. We give the proof for the convenience of the reader.

Proposition 2.7. *Let ω be a weight function, L as in (2.1) and $\lambda > 0$. If $F \in W(L_{m_{\lambda L}}^{p,q})$ is continuous, then for every $\alpha, \beta > 0$ there exists a constant $C_{\alpha,\beta,\lambda} > 0$ such that*

$$\left\| F|_{\alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d} \right\|_{\ell_{\tilde{m}_\lambda}^{p,q}} \leq C_{\alpha,\beta,\lambda} \|F\|_{W(L_{m_{\lambda L}}^{p,q})},$$

for $\tilde{m}_\lambda(k,n) := m_\lambda(\alpha k, \beta n)$.

Proof. The continuity of F is necessary in order that $F(\alpha k, \beta n)$ is well defined. For $(\alpha k, \beta n) \in (r, s) + [0,1]^{2d}$ with $(r, s) \in \mathbb{Z}^d \times \mathbb{Z}^d$ we have

$$\tilde{m}_\lambda(k, n) = e^{\lambda \omega(\alpha k, \beta n)} \leq \sup_{(x, \xi) \in [0,1]^{2d}} e^{\lambda L(\omega(r,s) + \omega(x,\xi) + 1)}$$

$$= e^{\lambda L} e^{\lambda L \omega(r,s)} \sup_{(x, \xi) \in [0,1]^{2d}} e^{\lambda L \omega(x,\xi)} = C_\lambda m_{\lambda L}(r, s)$$

for $C_\lambda = e^{\lambda L} \sup_{(x,\xi) \in [0,1]^{2d}} e^{\lambda L \omega(x,\xi)}$. Then

$$|F(\alpha k, \beta n)| m_\lambda(\alpha k, \beta n) \leq \operatorname*{ess\,sup}_{(x, \xi) \in [0,1]^{2d}} |F(r + x, s + \xi)| \cdot C_\lambda m_{\lambda L}(r, s)$$

$$\leq C_\lambda \|F \cdot T_{(r,s)} \chi_Q\|_{L^\infty} \cdot m_{\lambda L}(r, s).$$

Since there are at most $\tilde{C}_\alpha := \left(\left[\frac{1}{\alpha} \right] + 1 \right)^d$ points $\alpha k \in r + [0,1]^d$ we obtain

$$\left(\sum_{k \in \mathbb{Z}^d} |F(\alpha k, \beta n)|^p m_\lambda(\alpha k, \beta n)^p \right)^{1/p} \leq \left(\tilde{C}_\alpha C_\lambda^p \sum_{r \in \mathbb{Z}^d} \|F \cdot T_{(r,s)} \chi_Q\|_{L^\infty}^p m_{\lambda L}(r, s)^p \right)^{1/p}.$$

Analogously, there are at most $\tilde{C}_\beta := \left(\left[\frac{1}{\beta} \right] + 1 \right)^d$ points $\beta n \in s + [0,1]^d$ and therefore

$$\left\| F|_{\alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d} \right\|_{\ell_{\tilde{m}_\lambda}^{p,q}} \leq \left(\sum_{s \in \mathbb{Z}^d} \tilde{C}_\beta \left(\tilde{C}_\alpha C_\lambda^p \sum_{r \in \mathbb{Z}^d} \|F \cdot T_{(r,s)} \chi_Q\|_{L^\infty}^p m_{\lambda L}(r, s)^p \right)^{q/p} \right)^{1/q}$$

$$\leq \tilde{C}_\beta^{1/q} \tilde{C}_\alpha^{1/p} C_\lambda \|F\|_{W(L_{m_{\lambda L}}^{p,q})}.$$

$\square$

Proposition 2.8. *Let ω be a weight function, L as in (2.1) and $\lambda > 0$. If $F \in L_{m_{\lambda L}}^{p,q}$ and $G \in L_{m_{\lambda L^2}}^1$, then $F * G \in W(L_{m_\lambda}^\infty)$ and*

$$\|F * G\|_{W(L_{m_\lambda}^\infty)} \leq C_\lambda \|F\|_{L_{m_{\lambda L}}^\infty} \|G\|_{L_{m_{\lambda L^2}}^1}.$$

Proof. From the definition of the norm in $W(L_{m_\lambda}^\infty)$ we have

$$\|F * G\|_{W(L_{m_\lambda}^\infty)}$$

$$= \sup_{k, n \in \mathbb{Z}^d} \left\{ \left[\operatorname*{ess\,sup}_{(x, \xi) \in [0,1]^{2d}} \left| \int_{\mathbb{R}^{2d}} F(x + k - y, \xi + n - \eta) G(y, \eta) \, dy \, d\eta \right| \right] e^{\lambda \omega(k,n)} \right\}.$$

By (2.1), it is easy to see that

$$\omega(k, n) \leq L \omega(x + k - y, \xi + n - \eta) + L^2 \omega(x, \xi) + L^2 \omega(y, \eta) + L^2 + L.$$

Therefore, we obtain

$$\|F * G\|_{W(L_{m_\lambda}^\infty)} \leq e^{\lambda(L^2 + L)} \sup_{k, n \in \mathbb{Z}^d} \left\{ \operatorname*{ess\,sup}_{(x, \xi) \in [0,1]^{2d}} \left| \int_{\mathbb{R}^{2d}} e^{\lambda L \omega(x + k - y, \xi + n - \eta)} |F(x + k - y, \xi + n - \eta)| \right. \right.$$

$$\left. \left. \cdot e^{\lambda L^2 \omega(y, \eta)} |G(y, \eta)| \, dy \, d\eta \right| e^{\lambda L^2 \omega(x, \xi)} \right\}.$$

Since $(x, \xi) \in [0,1]^{2d}$ we have that $e^{\lambda L^2 \omega(x,\xi)}$ is bounded by a constant depending on λ (and L), so we obtain

$$\|F * G\|_{W(L^\infty_{m_\lambda})} \leq C_\lambda \sup_{k,n \in \mathbb{Z}^d} \left\{ \operatorname*{ess\,sup}_{(x,\xi) \in [0,1]^{2d}} \left| \left(e^{\lambda L \omega(\cdot,\cdot)} |F(\cdot,\cdot)| \right) * \left(e^{\lambda L^2 \omega(\cdot,\cdot)} |G(\cdot,\cdot)| \right)(x+k, \xi+n) \right| \right\}$$

$$= C_\lambda \left\| \left(e^{\lambda L \omega} |F| \right) * \left(e^{\lambda L^2 \omega} |G| \right) \right\|_{L^\infty(\mathbb{R}^{2d})},$$

for some $C_\lambda > 0$.

By Young's inequality we finally deduce

$$\|F * G\|_{W(L^\infty_{m_\lambda})} \leq C_\lambda \|e^{\lambda L \omega} F\|_{L^\infty} \|e^{\lambda L^2 \omega} G\|_{L^1} = C_\lambda \|F\|_{L^\infty_{m_{\lambda L}}} \|G\|_{L^1_{m_{\lambda L^2}}}.$$

$\square$

Now, our aim is to show that there is an isomorphism between $\mathcal{S}_\omega(\mathbb{R}^d)$ and its image through the analysis operator C_{φ_0}:

$$(2.6) \qquad\qquad C_{\varphi_0} : \ \mathcal{S}_\omega(\mathbb{R}^d) \longrightarrow \operatorname{Im} C_{\varphi_0} \subseteq \tilde{\Lambda}_\omega,$$

where $\tilde{\Lambda}_\omega$ is defined in (1.1).

The following proposition holds for every window function $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$ and in particular for our fixed window $\varphi_0 \in \mathcal{S}_\omega(\mathbb{R}^d)$:

Proposition 2.9. *Let ω be a weight function and $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$. The analysis operator*

$$C_\varphi : \ \mathcal{S}_\omega(\mathbb{R}^d) \longrightarrow \tilde{\Lambda}_\omega$$

is continuous.

Proof. It is known that if $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ then for every $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$|V_\varphi f(z)| \leq C_\lambda e^{-\lambda \omega(z)}, \qquad z \in \mathbb{R}^{2d}.$$

In fact, this property is proved in [14] when ω is subadditive, but it is still true in the general case (Theorem 2.3). Since $C_\varphi f = (V_\varphi f(\sigma))_{\sigma \in \Lambda}$ we have $C_\varphi f \in \tilde{\Lambda}_\omega$.

Now, we prove that the operator C_φ is continuous. By [12, Lemma 11.3.3]

$$|V_\varphi f(z)| \leq \frac{1}{(2\pi)^d \|\varphi\|_{L^2}^2} (|V_\varphi f| * |V_\varphi \varphi|)(z), \qquad \forall z \in \mathbb{R}^{2d}.$$

By Propositions 2.7 and 2.8, for every fixed $\lambda > 0$ we obtain

$$\sup_{\sigma \in \Lambda} |V_\varphi f(\sigma)| e^{\lambda \omega(\sigma)} = \||V_\varphi f|_{\alpha_0 \mathbb{Z}^d \times \beta_0 \mathbb{Z}^d}\|_{\ell^\infty_{\tilde{m}_\lambda}} \leq C_\lambda \|V_\varphi f\|_{W(L^\infty_{m_{\lambda L}})}$$

$$\leq C'_\lambda \|V_\varphi f\|_{L^\infty_{m_{\lambda L^2}}} \|V_\varphi \varphi\|_{L^1_{m_{\lambda L^3}}}$$

for $\tilde{m}_\lambda(k,n) = m_\lambda(\alpha_0 k, \beta_0 n)$ and for some $C_\lambda, C'_\lambda > 0$ (α_0 and β_0 are fixed). Observe that, since $f, \varphi \in \mathcal{S}_\omega(\mathbb{R}^d)$, then $V_\varphi f \in L^\infty_{m_{\lambda L^2}}$ and $V_\varphi \varphi \in L^1_{m_{\lambda L^3}}$ for every $\lambda > 0$ by Theorem 2.3(h).

Therefore, for every fixed $\lambda > 0$ there exists a constant $C''_\lambda = C'_\lambda \|V_\varphi \varphi\|_{L^1_{m_{\lambda L^3}}} > 0$ such that

$$\sup_{\sigma \in \Lambda} |V_\varphi f(\sigma)| e^{\lambda \omega(\sigma)} \leq C''_\lambda \|V_\varphi f\|_{L^\infty_{m_{\lambda L^2}}}.$$

This gives the continuity by Theorem 2.3(h). $\square$

The following proposition is valid for any $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$.

Proposition 2.10. *Let ω be a weight function and $\psi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$. Then the synthesis operator*

$$D_\psi : \ \tilde{\Lambda}_\omega \longrightarrow \mathcal{S}_\omega(\mathbb{R}^d)$$

is continuous.

Proof. Let $c = (c_\sigma)_{\sigma \in \Lambda} \in \tilde{\Lambda}_\omega$. For simplicity, we denote c_σ by c_{kn} for $\sigma = (\alpha_0 k, \beta_0 n)$. We start proving that $D_\psi c \in \mathcal{S}_\omega(\mathbb{R}^d)$. We shall apply Theorem 2.3(c) with $p = +\infty$. So, first, we have to see that $D_\psi c \in \mathcal{S}(\mathbb{R}^d)$.

By definition

$$(2.7) \qquad (D_\psi c)(t) = \sum_{k,n \in \mathbb{Z}^d} c_{kn} M_{\beta_0 n} T_{\alpha_0 k} \psi(t) = \sum_{k,n \in \mathbb{Z}^d} c_{kn} e^{i\langle \beta_0 n, t \rangle} \psi(t - \alpha_0 k).$$

Now, we see that $D_\psi c \in C^\infty(\mathbb{R}^d)$. To that aim we show that for each $\gamma \in \mathbb{N}_0^d$, the series

$$(2.8) \qquad \sum_{k,n \in \mathbb{Z}^d} \partial_t^\gamma \left[c_{kn} e^{i\langle \beta_0 n, t \rangle} \psi(t - \alpha_0 k) \right]$$

is uniformly convergent on $t \in \mathbb{R}^d$. Let us compute

$$\partial_t^\gamma \left[c_{kn} e^{i\langle \beta_0 n, t \rangle} \psi(t - \alpha_0 k) \right] = \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} c_{kn} \partial_t^\mu \left(e^{i\langle \beta_0 n, t \rangle} \right) \partial_t^{\gamma - \mu} \psi(t - \alpha_0 k)$$

$$(2.9) \hspace{4cm} = c_{kn} \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} (i\beta_0 n)^\mu e^{i\langle \beta_0 n, t \rangle} \partial_t^{\gamma - \mu} \psi(t - \alpha_0 k).$$

Since $(c_{kn})_{k,n \in \mathbb{Z}^d} \in \tilde{\Lambda}_\omega$, for every $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$|c_{kn}| \leq C_\lambda e^{-\lambda \omega(\alpha_0 k, \beta_0 n)}, \qquad k, n \in \mathbb{Z}^d.$$

Now, since ω is increasing it is obvious that $\omega(t, s) \geq \frac{1}{2}(\omega(t) + \omega(s))$. Therefore

$$(2.10) \qquad |c_{kn}| \leq C_\lambda e^{-\lambda \omega(\alpha_0 k, \beta_0 n)} \leq C_\lambda e^{-\frac{\lambda}{2}\omega(\alpha_0 k)} e^{-\frac{\lambda}{2}\omega(\beta_0 n)}.$$

Since

$$\omega(t) \leq L(\omega(\alpha_0 k - t) + \omega(\alpha_0 k) + 1),$$

we obtain

$$(2.11) \qquad \begin{aligned} |c_{kn}| & \leq C_\lambda e^{-\frac{\lambda}{2}\omega(\beta_0 n)} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} \\ & \leq C_\lambda e^{-\frac{\lambda}{2}\omega(\beta_0 n)} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} e^{-\frac{\lambda}{4}\left[\frac{1}{L}\omega(t) - \omega(\alpha_0 k - t) - 1\right]} \\ & \leq C_\lambda e^{-\frac{\lambda}{4L}\omega(0)} e^{\frac{\lambda}{4}} e^{-\frac{\lambda}{2}\omega(\beta_0 n)} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} e^{\frac{\lambda}{4}\omega(\alpha_0 k - t)} \\ & = C_\lambda' e^{-\frac{\lambda}{2}\omega(\beta_0 n)} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} e^{\frac{\lambda}{4}\omega(\alpha_0 k - t)}. \end{aligned}$$

Then we have, by (2.9), for $C_{\lambda,\gamma} = C'_\lambda \max_{\mu \leq \gamma} |\beta_0|^{|\mu|-|\gamma|}$, since $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ (see Definition 2.2),

$$\left| \partial_t^\gamma \left[c_{kn} e^{i\langle \beta_0 n, t \rangle} \psi(t - \alpha_0 k) \right] \right| \leq |c_{kn}| \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} |\beta_0 n|^{|\mu|} |\partial_t^{\gamma-\mu} \psi(t - \alpha_0 k)|$$

$$= |c_{kn}| \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} |\beta_0|^{|\mu|-|\gamma|} |\beta_0|^{|\gamma|} n^{|\mu|} |\partial_t^{\gamma-\mu} \psi(t - \alpha_0 k)|$$

(2.12)
$$\leq \sum_{\mu \leq \gamma} \binom{\gamma}{\mu} C_{\lambda,\gamma} e^{-\frac{\lambda}{2}\omega(\beta_0 n)} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} |\beta_0 n|^{|\gamma|} \left| \partial_t^{\gamma-\mu} \psi(t - \alpha_0 k) \right| e^{\frac{\lambda}{4}\omega(t-\alpha_0 k)}$$

$$\leq C'_{\lambda,\gamma} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} |\beta_0 n|^{|\gamma|} e^{-\frac{\lambda}{2}\omega(\beta_0 n)}$$

for some $C'_{\lambda,\gamma} > 0$. Hence, for $\lambda > 0$ sufficiently large the series

$$\sum_{k,n \in \mathbb{Z}^d} \partial_t^\gamma \left[c_{kn} e^{i\langle \beta_0 n, t \rangle} \psi(t - \alpha_0 k) \right]$$

is uniformly convergent on $t \in \mathbb{R}^d$. This implies that $D_\psi c \in C^\infty(\mathbb{R}^d)$ for every $c \in \tilde{\Lambda}_\omega$.

In particular we can differentiate $D_\psi c$ in (2.7) term by term, so that, to prove that $D_\psi c \in \mathcal{S}(\mathbb{R}^d)$, we can estimate, for every $\gamma, \mu \in \mathbb{N}_0^d$,

$$|t^\mu \partial_t^\gamma (D_\psi c)| = \left| t^\mu \sum_{k,n \in \mathbb{Z}^d} \partial_t^\gamma \left[c_{kn} e^{i\langle \beta_0 n, t \rangle} \psi(t - \alpha_0 k) \right] \right|$$

$$\leq |t|^{|\mu|} \sum_{k,n \in \mathbb{Z}^d} C_{\lambda,\gamma} \sum_{\tilde{\mu} \leq \gamma} \binom{\gamma}{\tilde{\mu}} e^{-\frac{\lambda}{2}\omega(\beta_0 n)} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} |\beta_0 n|^{|\gamma|} \left| \partial_t^{\gamma-\tilde{\mu}} \psi(t - \alpha_0 k) \right| e^{\frac{\lambda}{4}\omega(t-\alpha_0 k)}.$$

Since

$$|t|^{|\mu|} \leq (|t - \alpha_0 k| + |\alpha_0 k|)^{|\mu|} \leq 2^{|\mu|}(1 + |t - \alpha_0 k|^{|\mu|})(1 + |\alpha_0 k|^{|\mu|}),$$

we obtain

$$|t^\mu \partial_t^\gamma (D_\psi c)| \leq \sum_{k,n \in \mathbb{Z}^d} 2^{|\mu|} C_{\lambda,\gamma}(1 + |\alpha_0 k|^{|\mu|}) e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} |\beta_0 n|^{|\gamma|} e^{-\frac{\lambda}{2}\omega(\beta_0 n)}$$

$$\cdot \sum_{\tilde{\mu} \leq \gamma} \binom{\gamma}{\tilde{\mu}} (1 + |t - \alpha_0 k|^{|\mu|}) |\partial_t^{\gamma-\tilde{\mu}} \psi(t - \alpha_0 k)| e^{\frac{\lambda}{4}\omega(t-\alpha_0 k)}$$

(2.13)
$$\leq C_{\lambda,\gamma,\mu} \sum_{k,n \in \mathbb{Z}^d} (1 + |\alpha_0 k|^{|\mu|}) e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} |\beta_0 n|^{|\gamma|} e^{-\frac{\lambda}{2}\omega(\beta_0 n)},$$

for some $C_{\lambda,\gamma,\mu} > 0$ because $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$, by Theorem 2.3(b). Since the series in (2.13) converges for $\lambda > 0$ sufficiently large, we have $D_\psi c \in \mathcal{S}(\mathbb{R}^d)$.

By Theorem 2.3(c), to see that $D_\psi c \in \mathcal{S}_\omega(\mathbb{R}^d)$ it is now enough to prove that, for every $\tilde{\lambda} > 0$, the following two conditions hold:

(2.14)
$$\sup_{t \in \mathbb{R}^d} e^{\tilde{\lambda}\omega(t)} |D_\psi c(t)| < +\infty,$$

(2.15)
$$\sup_{\xi \in \mathbb{R}^d} e^{\tilde{\lambda}\omega(\xi)} |\widehat{D_\psi c}(\xi)| < +\infty.$$

To prove (2.14) we use the calculations in (2.11) and obtain, for every $\lambda \geq 4L\tilde{\lambda}$,

$$e^{\tilde{\lambda}\omega(t)}|D_\psi c(t)| \leq e^{\tilde{\lambda}\omega(t)} \sum_{k,n\in\mathbb{Z}^d} |c_{kn}||\psi(t-\alpha_0 k)|$$

$$(2.16) \qquad \leq \sum_{k,n\in\mathbb{Z}^d} C_\lambda e^{-\frac{\lambda}{2}\omega(\beta_0 n)} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} e^{\tilde{\lambda}\omega(t)} e^{-\frac{\lambda}{4L}\omega(t)} e^{\frac{\lambda}{4}} e^{\frac{\lambda}{4}\omega(\alpha_0 k - t)} |\psi(t-\alpha_0 k)|$$

$$(2.17) \qquad \leq \tilde{C}_\lambda e^{-\left(\frac{\lambda}{4L}-\tilde{\lambda}\right)\omega(0)} \sum_{k,n\in\mathbb{Z}^d} e^{-\frac{\lambda}{2}\omega(\beta_0 n)} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)},$$

for some $\tilde{C}_\lambda > 0$, since $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$. For λ sufficiently large the series in (2.17) converges and hence (2.14) is proved.

To prove (2.15) let us now consider

$$\widehat{D_\psi c}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle t,\xi\rangle} \sum_{k,n\in\mathbb{Z}^d} c_{kn} e^{i\langle\beta_0 n,t\rangle} \psi(t-\alpha_0 k) dt.$$

Since the series

$$e^{-i\langle t,\xi\rangle} \sum_{k,n\in\mathbb{Z}^d} c_{kn} e^{i\langle\beta_0 n,t\rangle} \psi(t-\alpha_0 k)$$

converges uniformly and moreover, by (2.16) with $\tilde{\lambda} = 0$ and λ large enough,

$$\left| e^{-i\langle t,\xi\rangle} \sum_{k,n\in[-N,N]^d} c_{kn} e^{i\langle\beta_0 n,t\rangle} \psi(t-\alpha_0 k) \right| \leq \sum_{k,n\in[-N,N]^d} |c_{kn}||\psi(t-\alpha_0 k)|$$

$$(2.18) \qquad \leq \sum_{k,n\in\mathbb{Z}^d} \tilde{C}_\lambda e^{\frac{\lambda}{4}} e^{-\frac{\lambda}{2}\omega(\beta_0 n)} e^{-\frac{\lambda}{4}\omega(\alpha_0 k)} e^{-\frac{\lambda}{4L}\omega(t)}$$

$$\leq \tilde{C}'_\lambda e^{-\frac{\lambda}{4L}\omega(t)} \in L^1(\mathbb{R}^d),$$

by the Dominated Convergence Theorem

$$\widehat{D_\psi c}(\xi) = \sum_{k,n\in\mathbb{Z}^d} c_{kn} \int_{\mathbb{R}^d} e^{-i\langle t,\xi\rangle} e^{i\langle\beta_0 n,t\rangle} \psi(t-\alpha_0 k) dt$$

$$= \sum_{k,n\in\mathbb{Z}^d} c_{kn} \int_{\mathbb{R}^d} e^{-i\langle t+\alpha_0 k,\xi-\beta_0 n\rangle} \psi(t) dt$$

$$= \sum_{k,n\in\mathbb{Z}^d} c_{kn} e^{-i\langle\alpha_0 k,\xi-\beta_0 n\rangle} \hat{\psi}(\xi-\beta_0 n).$$

Then

$$(2.19) \qquad \left| e^{\tilde{\lambda}\omega(\xi)} \widehat{D_\psi c}(\xi) \right| \leq e^{\tilde{\lambda}\omega(\xi)} \sum_{k,n\in\mathbb{Z}^d} |c_{kn}||\hat{\psi}(\xi-\beta_0 n)|$$

and since $\hat{\psi} \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfies the same estimates as ψ the proof of (2.15) is similar to that of (2.14) and so $D_\psi c \in \mathcal{S}_\omega(\mathbb{R}^d)$.

Now, we see that D_ψ is continuous. To this aim we have to estimate (2.14) and (2.15), for every $\tilde\lambda > 0$, by some seminorm of $c = (c_{kn})_{k,n\in\mathbb{Z}^d}$ in $\tilde\Lambda_\omega$. Writing, for every $\lambda > 0$,

$$|c_{kn}| \leq \sup_{k,n\in\mathbb{Z}^d} \left(|c_{kn}|e^{\lambda\omega(\alpha_0 k,\beta_0 n)}\right) \cdot e^{-\lambda\omega(\alpha_0 k,\beta_0 n)},$$

and proceeding as to obtain (2.17), with $\sup_{k,n\in\mathbb{Z}^d}\left(|c_{kn}|e^{\lambda\omega(\alpha_0 k,\beta_0 n)}\right)$ instead of C_λ in (2.10), we obtain that for every $\tilde\lambda > 0$ there exist $\lambda > 0$ and $C_{\tilde\lambda} > 0$ such that

$$\sup_{t\in\mathbb{R}^d} e^{\tilde\lambda\omega(t)}|D_\psi c(t)| \leq C_{\tilde\lambda} \sup_{k,n\in\mathbb{Z}^d} (|c_{kn}|e^{\lambda\omega(\alpha_0 k,\beta_0 n)}).$$

Similarly, from (2.19),

$$\sup_{\xi\in\mathbb{R}^d} e^{\tilde\lambda\omega(\xi)}|\widehat{D_\psi c}(\xi)| \leq C'_{\tilde\lambda} \sup_{k,n\in\mathbb{Z}^d} (|c_{kn}|e^{\lambda\omega(\alpha_0 k,\beta_0 n)}),$$

for some $C'_{\tilde\lambda} > 0$. Therefore D_ψ is continuous and the proof is complete. $\square$

We already know from the general theory of Gabor frames that $D_{\psi_0}C_{\varphi_0} = \mathrm{Id}$ on $\mathcal{S}_\omega(\mathbb{R}^d)$, as already observed in (2.4). Hence the operator in (2.6) is injective, surjective, continuous and its inverse $D_{\psi_0}|_{\mathrm{Im}\,C_{\varphi_0}}$ is continuous. Since we consider on $\mathrm{Im}\,C_{\varphi_0}$ the topology induced by $\tilde\Lambda_\omega$, to see that $\mathcal{S}_\omega(\mathbb{R}^d)$ is nuclear it is enough to check that $\tilde\Lambda_\omega$ is nuclear [19, Prop. 28.6].

3. Nuclearity of $\mathcal{S}_\omega(\mathbb{R}^d)$

In this section we show that $\tilde\Lambda_\omega$ is nuclear by an application of Grothendieck-Pietsch criterion. For a countable lattice Λ, we consider a matrix

$$(3.1) \qquad\qquad A = (a_{\sigma,k})_{\substack{\sigma\in\Lambda,\\ k\in\mathbb{N}}}$$

of Köthe type with positive entries, in the sense that A satisfies

$$(3.2) \qquad\qquad a_{\sigma,k} > 0 \qquad \forall\sigma\in\Lambda, k\in\mathbb{N},$$
$$(3.3) \qquad\qquad a_{\sigma,k} \leq a_{\sigma,k+1} \qquad \forall\sigma\in\Lambda, k\in\mathbb{N}.$$

We denote

$$\tilde\lambda^p(A) := \left\{ c = (c_\sigma)_{\sigma\in\Lambda} : \ \|c\|_k := \left(\sum_{\sigma\in\Lambda}|c_\sigma|^p a_{\sigma,k}^p\right)^{1/p} < +\infty, \ \forall k\in\mathbb{N}\right\}, \quad 1\leq p < +\infty,$$

$$\tilde\lambda^\infty(A) := \left\{ c = (c_\sigma)_{\sigma\in\Lambda} : \ \|c\|_k := \sup_{\sigma\in\Lambda}|c_\sigma|a_{\sigma,k} < +\infty, \ \forall k\in\mathbb{N}\right\}$$

$$\tilde c_0(A) := \left\{ c \in \tilde\lambda^\infty(A) : \ \lim_{|\sigma|\to+\infty}|c_\sigma|a_{\sigma,k} = 0, \ \forall k\in\mathbb{N}\right\}.$$

We put

$$\tilde\ell^p := \left\{ c = (c_\sigma)_{\sigma\in\Lambda} : \ \left(\sum_{\sigma\in\Lambda}|c_\sigma|^p\right)^{1/p} < +\infty, \ \forall k\in\mathbb{N}\right\}, \qquad 1\leq p < +\infty.$$

Analogously, we define $\tilde{\ell}^\infty$ and $\tilde{c}_0$. The spaces $\tilde{\ell}^p$, for $1 \le p \le +\infty$, and $\tilde{c}_0$ are Banach spaces, while $\tilde{\lambda}^p(A)$, for $1 \le p \le +\infty$, and $\tilde{c}_0(A)$ are Fréchet spaces. We consider the canonical basis $(e_\eta)_{\eta \in \Lambda}$:

$$e_\eta = (\delta_{\eta\sigma})_{\sigma \in \Lambda} = \begin{cases} 1, & \sigma = \eta \\ 0, & \sigma \ne \eta. \end{cases}$$

Since Λ is countable, it is obvious that $(e_\eta)_{\eta \in \Lambda}$ is a Schauder basis for $\tilde{c}_0(A)$ and $\tilde{\lambda}^p(A)$, for $1 \le p < +\infty$.

The following result is analogous to [19, Prop. 28.16]. We give the proof in the case of lattices for the sake of completeness.

Theorem 3.1. *Let A be as in* (3.1) *a matrix of Köthe type with positive entries. The following are equivalent:*

(a) *$\tilde{\lambda}^p(A)$ is nuclear for some $1 \le p \le +\infty$;*
(b) *$\tilde{\lambda}^p(A)$ is nuclear for all $1 \le p \le +\infty$;*
(c) *$\forall k \in \mathbb{N} \, \exists m \in \mathbb{N}, m \ge k$ s.t. $\sum_{\sigma \in \Lambda} a_{\sigma,k} a_{\sigma,m}^{-1} < +\infty$.*

Proof. If $1 \le p < +\infty$, then $\tilde{\lambda}^p(A)$ is a Fréchet space with the increasing fundamental system of seminorms $(\|\cdot\|_m)_{m \in \mathbb{N}}$ and the Schauder basis $(e_\eta)_{\eta \in \Lambda}$. We can then apply Grothendieck-Pietsch criterion (see [19, Thm. 28.15] or [21]) to $\tilde{\lambda}^p(A)$ and obtain that $\tilde{\lambda}^p(A)$ is nuclear if and only if

$$(3.4) \qquad \forall k \in \mathbb{N} \, \exists m \in \mathbb{N}, m \ge k : \sum_{\sigma \in \Lambda} \|e_\sigma\|_k \|e_\sigma\|_m^{-1} < +\infty.$$

Since

$$\|e_\sigma\|_k = \left(\sum_{\eta \in \Lambda} |\delta_{\sigma\eta}|^p a_{\eta,k}^p \right)^{1/p} = a_{\sigma,k},$$

the thesis is clear for $p < +\infty$.

Now, we treat the case $p = +\infty$. Assume that $\tilde{\lambda}^\infty(A)$ is nuclear. We prove that

$$(3.5) \qquad \forall k \in \mathbb{N} \, \exists m \in \mathbb{N}, m \ge k : \lim_{|\sigma| \to +\infty} a_{\sigma,k} a_{\sigma,m}^{-1} = 0.$$

To this aim, for every $k \in \mathbb{N}$, we denote

$$E_k := \left\{ c = (c_\sigma)_{\sigma \in \Lambda} : \|c\|_k = \sup_{\sigma \in \Lambda} |c_\sigma| a_{\sigma,k} < +\infty \right\}$$

the local space of $\tilde{\lambda}^\infty(A)$. This is a Banach space with the norm $\|\cdot\|_k$ (observe that $a_{\sigma,k} > 0$ for all $\sigma \in \Lambda, k \in \mathbb{N}$). The operator

$$\begin{aligned} A_k : \quad E_k &\longrightarrow \tilde{\ell}^\infty \\ c = (c_\sigma)_{\sigma \in \Lambda} &\longmapsto A_k(c) := (c_\sigma a_{\sigma,k})_{\sigma \in \Lambda} \end{aligned}$$

is an isometric isomorphism and $A_k(E_k) = \tilde{\ell}^\infty$. For every $k \in \mathbb{N}$, the inclusion

$$\begin{aligned} i_k : \quad \tilde{\lambda}^\infty(A) &\longrightarrow E_k \\ (c_\sigma)_{\sigma \in \Lambda} &\longmapsto (c_\sigma)_{\sigma \in \Lambda} \end{aligned}$$

is compact by [19, Lemma 24.17]. Indeed, $\tilde{\lambda}^\infty(A)$ is a locally convex space, which is nuclear (by assumption) and hence Schwartz by [19, Cor. 28.5]; moreover E_k is a Banach space and hence we can apply [19, Lemma 24.17] and obtain that there exists a neighbourhood V of 0 in $\tilde{\lambda}^\infty(A)$, that we can take of the form $\{c \in \tilde{\lambda}^\infty(A) : \ \|c\|_m < \varepsilon\}$, for some $\varepsilon > 0$ and with $m \geq k$ (the family of seminorms $(\|\cdot\|_m)_{m\in\mathbb{N}}$ is increasing), whose image through i_k is precompact, and hence compact. Moreover, for $m \geq k$ clearly $E_m \subseteq E_k$. So, for every $k \in \mathbb{N}$ there exists $m \geq k$ such that the inclusion $i_m^k = i_k|_{E_m}$

$$i_m^k : \qquad E_m \longrightarrow E_k$$
$$(c_\sigma)_{\sigma\in\Lambda} \longmapsto (c_\sigma)_{\sigma\in\Lambda}$$

is compact (and also $i_{m'}^k$ for all $m' \geq m$).

Then, we put $D := A_k \circ i_m^k \circ A_m^{-1}$:

$$D : \qquad \tilde{\ell}^\infty \longrightarrow \tilde{\ell}^\infty$$
$$(c_\sigma)_{\sigma\in\Lambda} \longmapsto (c_\sigma a_{\sigma,m}^{-1} a_{\sigma,k})_{\sigma\in\Lambda}.$$

The operator D is clearly compact. The restriction $\tilde{D} := D\big|_{\tilde{c}_0}$ satisfies $\tilde{D}(\tilde{c}_0) \subseteq \tilde{c}_0$, for $m \geq k$, since

$$|c_\sigma| a_{\sigma,m}^{-1} a_{\sigma,k} \leq |c_\sigma| a_{\sigma,m}^{-1} a_{\sigma,m} = |c_\sigma| \to 0,$$

for $c = (c_\sigma)_{\sigma\in\Lambda} \in \tilde{c}_0$. The operator $\tilde{D}$ is also compact.

For every $\varepsilon > 0$ we define, for $m \geq k$,

$$I_\varepsilon := \{\sigma \in \Lambda : \ a_{\sigma,k} a_{\sigma,m}^{-1} \geq \varepsilon\},$$

and also

$$T_\varepsilon : \quad \tilde{c}_0 \longrightarrow \tilde{c}_0$$
$$c = (c_\sigma)_{\sigma\in\Lambda} \longmapsto (T_\varepsilon(c))_{\sigma\in\Lambda} = \begin{cases} c_\sigma a_{\sigma,k}^{-1} a_{\sigma,m}, & \sigma \in I_\varepsilon \\ 0, & otherwise. \end{cases}$$

The operator $T_\varepsilon : \tilde{c}_0 \to \tilde{c}_0$ is continuous since

$$\sup_{\sigma\in\Lambda} |(T_\varepsilon(c))_\sigma| \leq \frac{1}{\varepsilon} \sup_{\sigma\in\Lambda} |c_\sigma|.$$

Now we consider

$$P_\varepsilon := \tilde{D} T_\varepsilon : \quad \tilde{c}_0 \longrightarrow \tilde{c}_0$$
$$c = (c_\sigma)_{\sigma\in\Lambda} \longmapsto \tilde{c} = (\tilde{c}_\sigma)_{\sigma\in\Lambda} = \begin{cases} c_\sigma, & \sigma \in I_\varepsilon \\ 0, & \sigma \in \Lambda \setminus I_\varepsilon. \end{cases}$$

Hence, P_ε is a compact projection on

$$S_\varepsilon := \{(c_\sigma)_{\sigma\in\Lambda} \in \tilde{c}_0 : \ c_\sigma = 0 \text{ for } \sigma \in \Lambda \setminus I_\varepsilon\} \subseteq \tilde{c}_0.$$

Since $\tilde{c}_0$ is a Banach space we can apply [19, Cor. 15.6] and obtain that the kernel $\ker(\mathrm{Id} - P_\varepsilon)$ is finite dimensional. But P_ε is a projection and hence its image $\mathrm{Im}(P_\varepsilon) = \ker(\mathrm{Id} - P_\varepsilon)$ is finite dimensional and I_ε must be finite for every $\varepsilon > 0$. Then

$$\lim_{|\sigma| \to +\infty} a_{\sigma,k} a_{\sigma,m}^{-1} = 0$$

and (3.5) is proved.

This implies that $\tilde{\lambda}^\infty(A) = \tilde{c}_0(A)$. Indeed, if $c = (c_\sigma)_{\sigma \in \Lambda} \in \tilde{\lambda}^\infty(A)$ then for every $k \in \mathbb{N}$ we find $m \in \mathbb{N}$, $m \geq k$ such that (3.5) holds and we get

$$\lim_{|\sigma| \to +\infty} |c_\sigma| a_{\sigma,k} = \lim_{|\sigma| \to +\infty} |c_\sigma| a_{\sigma,m} a_{\sigma,k} a_{\sigma,m}^{-1} = 0,$$

since $|c_\sigma| a_{\sigma,m}$ is bounded because $c \in \tilde{\lambda}^\infty(A)$ and $a_{\sigma,k} a_{\sigma,m}^{-1} \to 0$ by (3.5). Therefore $c \in \tilde{c}_0(A)$.

Now, $\tilde{c}_0(A)$ is a Fréchet space endowed with the increasing fundamental system of seminorms $(\| \cdot \|_m)_{m \in \mathbb{N}}$ and the Schauder basis $(e_\eta)_{\eta \in \Lambda}$. We can then apply Grothendieck-Pietsch criterion (3.4) to $\tilde{c}_0(A)$ for

$$\|e_\sigma\|_k = \sup_{\eta \in \Lambda} |\delta_{\sigma\eta}| a_{\eta,k} = a_{\sigma,k}.$$

Since $\tilde{c}_0(A) = \tilde{\lambda}^\infty(A)$ is nuclear by assumption, then (3.4) implies (c).

On the contrary, if (c) holds then $\tilde{c}_0(A)$ is nuclear by the Grothendiech-Pietsch criterion (3.4). We see again that $\tilde{\lambda}^\infty(A) = \tilde{c}_0(A)$. If $c = (c_\sigma)_{\sigma \in \Lambda} \in \tilde{\lambda}^\infty(A)$, we have

$$|c_\sigma| a_{\sigma,k} = |c_\sigma| a_{\sigma,m} a_{\sigma,k} a_{\sigma,m}^{-1} \longrightarrow 0,$$

since $|c_\sigma| a_{\sigma,m}$ is bounded for $c \in \tilde{\lambda}^\infty(A)$ and (3.5) holds by the convergence of the series in (c). Therefore $c \in \tilde{c}_0(A)$. $\qquad \square$

Observe that, for $\Lambda = \alpha_0 \mathbb{Z}^d \times \beta_0 \mathbb{Z}^d$ as fixed in Section 2, the matrix

$$(3.6) \qquad \tilde{A} = \left(e^{k\omega(\sigma)}\right)_{\substack{\sigma \in \Lambda, \\ k \in \mathbb{N}}}$$

satisfies (3.2) and (3.3). Hence the space $\tilde{\Lambda}_\omega$ defined in (1.1) is, in fact,

$$\tilde{\lambda}^\infty(\tilde{A}) := \left\{c = (c_\sigma)_{\sigma \in \Lambda} : \|c\|_k := \sup_{\sigma \in \Lambda} |c_\sigma| e^{k\omega(\sigma)} < +\infty, \ \forall k \in \mathbb{N}\right\}.$$

Proposition 3.2. *The sequence space $\tilde{\Lambda}_\omega$ is nuclear.*

Proof. By Theorem 3.1 we have that $\tilde{\Lambda}_\omega = \tilde{\lambda}^\infty(\tilde{A})$ is nuclear if and only if

$$(3.7) \qquad \forall k \in \mathbb{N} \ \exists m \in \mathbb{N}, m \geq k, \ \text{s.t.} \ \sum_{\sigma \in \Lambda} e^{k\omega(\sigma) - m\omega(\sigma)} < +\infty.$$

Since, by condition (γ) of Definition 2.1,

$$e^{k\omega(\sigma) - m\omega(\sigma)} \leq e^{-(m-k)a} e^{-(m-k)b \log(1+|\sigma|)} = e^{-(m-k)a} \frac{1}{(1+|\sigma|)^{b(m-k)}},$$

we have, for $m > k + \frac{2d}{b}$,

$$\sum_{\sigma \in \Lambda} \frac{1}{(1+|\sigma|)^{b(m-k)}} < +\infty.$$

$\qquad \square$

As we explained at the end of Section 2, we deduce:

Theorem 3.3. *The space $\mathcal{S}_\omega(\mathbb{R}^d)$ is nuclear.*

4. Nuclearity of $\mathcal{S}_{(M_p)}(\mathbb{R}^d)$ with L^2 norms

Let $(M_p)_{p \in \mathbb{N}_0}$ be a sequence such that $M_p^{1/p} \to +\infty$ as $p \to +\infty$ and consider the locally convex space of rapidly decreasing ultradifferentiable functions

$$(4.1) \qquad \mathcal{S}_{(M_p)}(\mathbb{R}^d) := \left\{ f \in C^\infty(\mathbb{R}^d) : \sup_{\alpha,\beta \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} \frac{j^{|\alpha+\beta|}}{M_{|\alpha+\beta|}} \|x^\alpha \partial^\beta f(x)\|_2 < +\infty, \ \forall j \in \mathbb{N} \right\},$$

where $\| \cdot \|_2$ denotes the L^2 norm. We write the associated function in the usual way:

$$(4.2) \qquad\qquad\qquad M(t) = \sup_{p \in \mathbb{N}} \log \frac{t^p M_0}{M_p}.$$

Langenbruch [18] uses (1.3) to show that the Hermite functions H_γ, for $\gamma \in \mathbb{N}_0^d$, are an absolute Schauder basis in $\mathcal{S}_{(M_p)}(\mathbb{R}^d)$, where

$$H_\gamma(x) := (2^{|\gamma|}\gamma!\pi^{d/2})^{-1/2} \exp\left(-\sum_{j=0}^d \frac{x_j^2}{2} \right) h_\gamma(x),$$
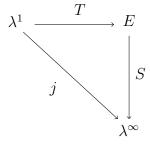
and the Hermite polynomials h_γ are given by

$$h_\gamma(x) := (-1)^{|\gamma|} \exp\left(\sum_{j=0}^d x_j^2 \right) \partial^\gamma \exp\left(-\sum_{j=0}^d x_j^2 \right), \qquad x \in \mathbb{R}^d.$$

Here we consider a matrix A^* of Köthe type with positive entries as in Section 3 for $\Lambda = \mathbb{N}_0^d$, defined by

$$(4.3) \qquad\qquad\qquad a_{\gamma,k} := e^{M(k|\gamma|^{1/2})}, \qquad \gamma \in \mathbb{N}_0^d, \, k \in \mathbb{N},$$

where $M(t)$ is the associated function defined by (4.2). We characterize when $\mathcal{S}_{(M_p)}(\mathbb{R}^d)$ is nuclear with Theorem 3.5 of [20], that we state here in our setting, for the convenience of the reader. In what follows we denote $\lambda^1 := \tilde{\lambda}^1(A^*)$ and $\lambda^\infty := \tilde{\lambda}^\infty(A^*)$.

Theorem 4.1. *Assume that the inclusion $j : \lambda^1 \to \lambda^\infty$ has dense image. Let E be a locally convex space such that we have a commutative diagram of continuous linear operators of the form*

with S injective or T with dense image. Then λ^1 is nuclear if and only if E is nuclear.

We can now prove the following:

Proposition 4.2. *Let $(M_p)_p$ be a sequence satisfying $M_p^{1/p} \to +\infty$ as $p \to +\infty$, condition (1.3) and (M1). Then $\mathcal{S}_{(M_p)}(\mathbb{R}^d)$ is nuclear if and only if the associated function $M(t)$ satisfies*

$$(4.4) \qquad \exists H > 1 \ s.t. \ M(t) + \log t \leq M(Ht) + H, \quad \forall t > 0.$$

Proof. We shall use Theorem 4.1 with $E = \mathcal{S}_{(M_p)}(\mathbb{R}^d)$. We observe that $\lambda^1 \subseteq \lambda^\infty$ and denote by j the inclusion

$$j : \ \lambda^1 \longrightarrow \lambda^\infty.$$

Let us consider the linear map

$$S : \ \mathcal{S}_{(M_p)}(\mathbb{R}^d) \ \longrightarrow \lambda^\infty$$
$$f \ \longmapsto (c_\gamma)_{\gamma \in \mathbb{N}_0^d} := (\xi_\gamma(f))_{\gamma \in \mathbb{N}_0^d},$$

where

$$\xi_\gamma(f) = \int_{\mathbb{R}^d} f(x) H_\gamma(x) dx$$

are the Hermite coefficients of f, and then the linear map

$$T : \ \lambda^1 \ \longrightarrow \mathcal{S}_{(M_p)}(\mathbb{R}^d)$$
$$(c_\gamma)_{\gamma \in \mathbb{N}_0^d} \ \longmapsto \sum_{\gamma \in \mathbb{N}_0^d} c_\gamma H_\gamma(x).$$

In Theorem 3.4 of [18] it was proved that condition (1.3) implies that S and T are continuous. Note also that the diagram in Theorem 4.1 commutes by the uniqueness of the coefficients with respect to the Schauder basis $(H_\gamma)_{\gamma \in \mathbb{N}_0^d}$.

Let us prove that j has dense image. By conditions $M_p^{1/p} \to +\infty$ and (M1), and by [20, Lemma 3.2], we have

$$\lim_{t \to +\infty} e^{M(t/h) - M(t/h')} = 0, \qquad \text{if } h > h' > 0.$$

Therefore, for every $k \in \mathbb{N}$ there exists $m \in \mathbb{N}$, $m > k$, such that

$$\lim_{|\gamma| \to +\infty} a_{\gamma,k} a_{\gamma,m}^{-1} = \lim_{|\gamma| \to +\infty} e^{M(k|\gamma|^{1/2}) - M(m|\gamma|^{1/2})} = 0,$$

and hence $\lambda^\infty = \tilde{c}_0(A^*)$, by the same arguments we used to prove that (3.5) implies $\tilde{\lambda}^\infty(A) = \tilde{c}_0(A)$ in Section 3. Then $j(\lambda^1)$ is dense in $\lambda^\infty = \tilde{c}_0(A^*)$ because

$$\tilde{c}_{00}(A^*) := \{(c_\gamma)_{\gamma \in \mathbb{N}_0^d} \in \tilde{c}_0(A^*) : \ c_\gamma = 0 \text{ except that for a finite number of indexes}\}$$

is dense in $\tilde{c}_0(A^*)$ and is contained in λ^1.

Moreover, S is injective. Hence, by Theorem 4.1, $E = \mathcal{S}_{(M_p)}(\mathbb{R}^d)$ is nuclear if and only if λ^1 is nuclear. By Theorem 3.1, the sequence space λ^1 is nuclear if and only if

$$(4.5) \qquad \forall k \in \mathbb{N} \ \exists m \in \mathbb{N}, m \geq k \ s.t. \ \sum_{\gamma \in \mathbb{N}_0^d} e^{M(k|\gamma|^{1/2}) - M(m|\gamma|^{1/2})} < +\infty.$$

The series in (4.5) converges if and only if

$$(4.6) \qquad M(t) + N \log t \leq M(H^N t) + C_{N,H}, \qquad \forall N \in \mathbb{N},$$

for some $C_{N,H} > 0$ and $N > 2d$ (see the proof of [6, Thm. 1]). This gives the conclusion since (4.6) is equivalent to (4.4) (see again the proof of [6, Thm. 1]). □

Theorem 4.3. *Let $(M_p)_p$ be a sequence satisfying $M_p^{1/p} \to +\infty$ as $p \to +\infty$, condition* (1.3) *and* (M1). *Then $\mathcal{S}_{(M_p)}(\mathbb{R}^d)$ is nuclear if and only if* $(M2)'$ *holds.*

Proof. It follows from Proposition 4.2 because, under condition (M1), condition $(M2)'$ is equivalent to condition (4.4) (see [6, Rem. 1]). □

If $(M2)'$ is satisfied then $\mathcal{S}_{(M_p)}(\mathbb{R}^d)$ can be equivalently defined with L^∞ norms as in (1.2) (see [18, Remark 2.1]) and hence $S_{(M_p)}(\mathbb{R}^d)$ is nuclear (cf. [6, Corollary 1]), but we cannot derive a characterization in terms of $(M2)'$ from the results of Langenbruch [18].

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Dipartimento di Matematica e Informatica, Università di Ferrara, Via Machiavelli n. 30, I-44121 Ferrara, Italy

   E-mail address: chiara.boiti@unife.it

Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, Camino de Vera, s/n, E-46071 Valencia, Spain

   E-mail address: djornet@mat.upv.es

Dipartimento di Matematica, Università di Torino, Via Carlo Alberto n. 10, I-10123 Torino, Italy

   E-mail address: alessandro.oliaro@unito.it

Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz n. 1, A-1090 Wien, Austria

   E-mail address: gerhard.schindl@univie.ac.at