Exact G2-structures on unimodular Lie algebras

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EXACT G_2-STRUCTURES ON UNIMODULAR LIE ALGEBRAS

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ABSTRACT. We consider seven-dimensional unimodular Lie algebras $\mathfrak{g}$ admitting exact G_2-structures, focusing our attention on those with vanishing third Betti number $b_3(\mathfrak{g})$. We discuss some examples, both in the case when $b_2(\mathfrak{g}) \neq 0$ and in the case when the Lie algebra $\mathfrak{g}$ is (2,3)-trivial, i.e., when both $b_2(\mathfrak{g})$ and $b_3(\mathfrak{g})$ vanish. These examples are solvable, as $b_3(\mathfrak{g}) = 0$, but they are not strongly unimodular, a necessary condition for the existence of lattices on the simply connected Lie group corresponding to $\mathfrak{g}$. More generally, we prove that any seven-dimensional (2,3)-trivial strongly unimodular Lie algebra does not admit any exact G_2-structure. From this, it follows that there are no compact examples of the form $(\Gamma\backslash G, \varphi)$, where G is a seven-dimensional simply connected Lie group with (2,3)-trivial Lie algebra, $\Gamma \subset G$ is a co-compact discrete subgroup, and φ is an exact G_2-structure on $\Gamma\backslash G$ induced by a left-invariant one on G.

1. Introduction

Let M be a seven-dimensional smooth manifold. A G_2-*structure* on M is a reduction of the structure group of its frame bundle from $GL(7, \mathbb{R})$ to the compact exceptional Lie group G_2. In [12], Gray proved that a smooth 7-manifold carries G_2-structures if and only if it is orientable and spin.

The existence of a G_2-structure on M is characterized by the existence of a globally defined 3-form $\varphi \in \Omega^3(M)$ satisfying a certain nondegeneracy condition. The G_2-*form* φ gives rise to a Riemannian metric g_φ with volume form dV_φ via the identity

$$g_\varphi(X, Y)\, dV_\varphi = \frac{1}{6}\, \iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi, \tag{1.1}$$

for any pair of vector fields X, Y on M, where ι_X denotes the contraction by X. Moreover, at each point $x \in M$ there exists a basis $\{e^1, \ldots, e^7\}$ of the cotangent space T_x^*M for which

$$\varphi|_x = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}.$$

Here, e^{127} stands for $e^1 \wedge e^2 \wedge e^7$, and so on. Such a basis is called an *adapted* G_2-*basis*.

A G_2-structure is said to be *closed* (or *calibrated*) if the defining 3-form φ satisfies the equation $d\varphi = 0$, while it is called *co-closed* (or *co-calibrated*) if $d *_\varphi \varphi = 0$, $*_\varphi$ being the Hodge operator defined by g_φ and dV_φ. When both of these conditions hold, the intrinsic torsion of the G_2-structure vanishes identically, the Riemannian metric g_φ is Ricci-flat, and $\mathrm{Hol}(g_\varphi) \subseteq G_2$ (cf. [2, 9]). In this case, the G_2-structure is said to be *torsion-free*.

The existence of a torsion-free G_2-structure on a compact 7-manifold M imposes various constraints on the topology. For instance, the third Betti number must satisfy $b_3(M) \geq 1$

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[1], and the first Betti number $b_1(M) \in \{0,1,3,7\}$ [14]. Moreover, if $\mathrm{Hol}(g_\varphi) = \mathrm{G}_2$, then the fundamental group $\pi_1(M)$ is finite [14], and so $b_1(M) = 0$. However, a non-compact manifold with a torsion-free G_2-structure may have vanishing third Betti number. An example is given by the metric cone of the nearly Kähler flag manifold $\mathbb{F}_{1,2}$ [2].

In the literature, all known examples of compact 7-manifolds M admitting a closed G_2-structure, but not torsion-free G_2-structures, have $b_1(M) > 0$ and $b_3(M) > 0$ (see [4, 6, 7, 8]). A longstanding open question concerns the existence of closed G_2-structures on compact 7-manifolds with $b_3(M) = 0$, such as the 7-sphere. Notice that, in this case, any closed G_2-structure would be defined by an exact 3-form.

Motivated by this problem, one may investigate the existence of such type of examples when $M = \Gamma\backslash\mathrm{G}$, where G is a seven-dimensional simply connected Lie group, $\Gamma \subset \mathrm{G}$ is a co-compact discrete subgroup (lattice), and M is endowed with an *exact* G_2-structure induced by a left-invariant one on G.

Recall that the de Rham cohomology groups of $\Gamma\backslash\mathrm{G}$ are isomorphic to those of the Chevalley-Eilenberg complex of the Lie algebra $\mathfrak{g}$ of G when, for instance, the latter is nilpotent or completely solvable (cf. [18, 13]). Thus, a possible strategy to study the problem consists in looking for suitable Lie algebras $\mathfrak{g}$ with third Betti number $b_3(\mathfrak{g}) = 0$ and admitting closed G_2-structures, and see whether the corresponding simply connected Lie group admits a lattice. Since seven-dimensional nilpotent Lie algebras cannot admit exact G_2-structures (cf. [4]), and since any Lie algebra satisfying $b_3(\mathfrak{g}) = 0$ is solvable and not nilpotent (see e.g. [15]), one may firstly focus on the latter case. By a result of Garland [11], every simply connected solvable Lie group containing a lattice must be strongly unimodular (see Definition 2.2 for details). This provides a further restriction on $\mathfrak{g}$.

In Section 2, we focus on Lie algebras $\mathfrak{g}$ satisfying the conditions $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$. They are known as (2,3)-*trivial* in the literature, and they are always solvable with codimension one derived algebra $[\mathfrak{g},\mathfrak{g}]$ [15].

In Example 2.1, we show the existence of a unimodular (2,3)-trivial Lie algebra $\mathfrak{s}$ admitting closed G_2-structures. Since $\mathfrak{s}$ is unimodular, by Hodge duality it also satisfies $b_4(\mathfrak{s}) = 0 = b_5(\mathfrak{s})$. However, we prove that it is not strongly unimodular, thus the corresponding simply connected solvable Lie group does not admit lattices. Notice that this example reveals an interesting difference between closed G_2-structures and symplectic structures (see [10] for further results in this direction). Indeed, both structures are defined by closed differential forms satisfying the same non-degeneracy condition, namely their orbit under the action of the general linear group is open, but a unimodular Lie algebra cannot admit exact symplectic forms (cf. [5]).

More generally, we prove that it is not possible to obtain compact examples of the form $M = \Gamma\backslash\mathrm{G}$, when the Lie algebra of the simply connected Lie group G is (2,3)-trivial and strongly unimodular. In detail, using a characterisation of (2,3)-trivial Lie algebras by Madsen and Swann [15], we first obtain the classification of all seven-dimensional (2,3)-trivial strongly unimodular Lie algebras up to isomorphism (Theorem 2.3). Then, we show that none of them admits exact G_2-structures (Theorem 2.4).

In Section 3, we discuss an example of a seven-dimensional unimodular Lie algebra $\mathfrak{h}$ satisfying $b_3(\mathfrak{h}) = 0$, $b_2(\mathfrak{h}) \neq 0$ and admitting closed G_2-structures. Also in this case, it turns out that the solvable Lie algebra $\mathfrak{h}$ is not strongly unimodular. This leads to the following open problem: *is there any seven-dimensional Lie algebra* $\mathfrak{g}$ *satisfying* $b_3(\mathfrak{g}) = 0$ *which is strongly unimodular and admits closed* G_2*-structures?*

2. (2,3)-trivial strongly unimodular Lie algebras

In this section, we focus on seven-dimensional (2,3)-trivial Lie algebras. First of all, we recall that being (2,3)-trivial imposes strong constraints on the structure of a Lie algebra. Indeed, as shown in [15], any finite-dimensional Lie algebra $\mathfrak{g}$ different from $\mathbb{R}, \mathbb{R}^2$ and satisfying $b_3(\mathfrak{g}) = 0$ is solvable and not nilpotent. Moreover, if also $b_2(\mathfrak{g}) = 0$, then the derived algebra $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ is a codimension one ideal, and $\mathfrak{g}$ is the semidirect product $\mathfrak{g} = \mathfrak{n} \rtimes \mathbb{R}$. We emphasise that $\mathfrak{n}$ is nilpotent, as $\mathfrak{g}$ is solvable.

In the next example, we describe a seven-dimensional $(2,3)$-trivial unimodular Lie algebra admitting closed G_2-structures. This shows in particular that exact G_2-structures behave differently from exact symplectic structures, as the latter do not exist on unimodular Lie algebras (cf. [5]).

Example 2.1. Let $\mathfrak{s} = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle$ be the seven-dimensional solvable Lie algebra with the following nonzero Lie brackets

$$\begin{array}{llll} [e_1, e_3] = e_4 - 3e_6, & [e_1, e_4] = e_5, & [e_1, e_5] = -e_6, & [e_2, e_3] = e_5, \\ [e_2, e_4] = -e_6, & [e_1, e_7] = 2e_1, & [e_2, e_7] = 4e_2, & [e_3, e_7] = -\frac{9}{2}e_3 + 6e_5, \\ [e_4, e_7] = -\frac{5}{2}e_4 + 6e_6, & [e_5, e_7] = -\frac{1}{2}e_5, & [e_6, e_7] = \frac{3}{2}e_6. & \end{array}$$

Notice that $\mathfrak{s}$ is the semidirect product $\mathfrak{s} = \mathfrak{n} \rtimes \mathbb{R}$, where $\mathbb{R} = \langle e_7 \rangle$ and $\mathfrak{n} = \langle e_1, \ldots, e_6 \rangle$ is isomorphic to the unique six-dimensional 4-step nilpotent Lie algebra with $b_1(\mathfrak{n}) = 3$ and $b_2(\mathfrak{n}) = 4$ (cf. [19, Table A.1]). Moreover, $\mathfrak{s}$ is completely solvable, i.e., for each $X \in \mathfrak{s}$ the map $\mathrm{ad}_X \in \mathrm{End}(\mathfrak{s})$ has only real eigenvalues.

Let $(e^1, \ldots, e^7)$ be the dual basis of $(e_1, \ldots, e_7)$. Then, the structure equations of $\mathfrak{s}$ can also be written by means of the Chevalley-Eilenberg differentials of the covectors e^i. In detail:

$$de^1 = -2e^{17}, \quad de^2 = -4e^{27}, \quad de^3 = \frac{9}{2}e^{37}, \quad de^4 = \frac{5}{2}e^{47} - e^{13},$$
$$de^5 = \frac{1}{2}e^{57} - 6e^{37} - e^{14} - e^{23}, \quad de^6 = -\frac{3}{2}e^{67} - 6e^{47} + 3e^{13} + e^{15} + e^{24}, \quad de^7 = 0.$$

Using these equations, it is possible to compute the cohomology groups of the Chevalley-Eilenberg complex $(\Lambda^\bullet(\mathfrak{s}^*), d)$ of $\mathfrak{s}$, obtaining

$$H^1(\mathfrak{s}) = \langle [e^7] \rangle, \quad H^6(\mathfrak{s}) = \langle [e^{123456}] \rangle, \quad H^7(\mathfrak{s}) = \langle [e^{1234567}] \rangle, \quad H^i(\mathfrak{s}) = 0, \ i = 2, 3, 4, 5.$$

We immediately see that $b_2(\mathfrak{s}) = 0 = b_3(\mathfrak{s})$, hence $\mathfrak{s}$ is (2,3)-trivial. Moreover, since $H^7(\mathfrak{s}) \cong \mathbb{R}$, the Lie algebra $\mathfrak{s}$ is unimodular. Recall that this is equivalent to having $\mathrm{tr}(\mathrm{ad}_X) = 0$ for each $X \in \mathfrak{s}$.

The Lie algebra $\mathfrak{s}$ admits closed (hence exact) G_2-structures. An example is given by the following 3-form

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245} = d\left(\frac{1}{6}e^{12} + \frac{23}{7}e^{34} + 2e^{36} - 2e^{45} + e^{56}\right).$$

Consequently, the simply connected solvable Lie group S with Lie algebra $\mathfrak{s}$ is endowed with a *left-invariant* exact G_2-structure obtained from φ via left multiplication.

It is well-known that a Lie group admits a lattice only if it is unimodular (cf. [17]). When the Lie group is solvable, there is a stronger necessary condition for the existence of lattices, namely the group must be *strongly unimodular* (cf. [11, Prop. 3.3]). We recall the definition here.

Definition 2.2. Let G be a simply connected solvable Lie group with Lie algebra $\mathfrak{g}$. Denote by $\mathfrak{n}^0 := \mathfrak{n}$ the nilradical of $\mathfrak{g}$ and, for each positive integer $i \geq 1$, let $\mathfrak{n}^i := [\mathfrak{n}, \mathfrak{n}^{i-1}]$ denote the i^{th} term in the descending central series of $\mathfrak{n}$. The Lie algebra $\mathfrak{g}$ is *strongly unimodular* if for all $X \in \mathfrak{g}$ the restriction of ad_X to each space $\mathfrak{n}^i/\mathfrak{n}^{i+1}$ is traceless. In this case, the Lie group G is said to be *strongly unimodular*.

Notice that any solvable strongly unimodular Lie algebra $\mathfrak{g}$ is unimodular. Indeed, if the nilradical $\mathfrak{n}$ of $\mathfrak{g}$ is s-step nilpotent, then we have the vector space decomposition

$$\mathfrak{g} = \mathfrak{g}/\mathfrak{n} \oplus \mathfrak{n}/\mathfrak{n}^1 \oplus \mathfrak{n}^1/\mathfrak{n}^2 \oplus \cdots \oplus \mathfrak{n}^{s-2}/\mathfrak{n}^{s-1} \oplus \mathfrak{n}^{s-1}, \tag{2.1}$$

where $\mathfrak{g}/\mathfrak{n}$ is abelian since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{n}$. Now, for every $X \in \mathfrak{g}$, the endomorphism $\mathrm{ad}_X \in \mathrm{End}(\mathfrak{g})$ preserves every summand of the splitting (2.1). From this and the Definition 2.2, we immediately get that ad_X is traceless, i.e., $\mathfrak{g}$ is unimodular. On the other hand, a unimodular solvable Lie algebra may not be strongly unimodular. This is the case, for instance, of the Lie algebra $\mathfrak{s}$ considered in Example 2.1. Indeed, its nilradical $\mathfrak{n}$ has the following descending central series

$$\mathfrak{n}^0 = \mathfrak{n}, \quad \mathfrak{n}^1 = \langle e_4 - 3e_6, e_5, e_6 \rangle, \quad \mathfrak{n}^2 = \langle e_5, e_6 \rangle, \quad \mathfrak{n}^3 = \langle e_6 \rangle, \quad \mathfrak{n}^4 = \{0\},$$

and the restriction of ad_{e_7} to $\mathfrak{n}^3$ is not traceless. This shows in particular that the simply connected solvable Lie group S does not contain any lattice.

It is then natural to ask the following.

Question 1. Is there any solvable strongly unimodular Lie algebra admitting exact G_2-structures?

As we are interested in examples having as many zero Betti numbers as possible, we investigate this problem under the assumption that the Lie algebra is $(2,3)$-trivial. Our strategy is the following: we first determine all possible (2,3)-trivial, strongly unimodular Lie algebras of dimension seven up to isomorphism, and then we investigate whether any of them admits exact G_2-structures. In the next theorem, we deal with the first problem.

Theorem 2.3. *A (2,3)-trivial, strongly unimodular seven-dimensional Lie algebra is solvable and its six-dimensional nilradical is either abelian or isomorphic to one of the following nilpotent Lie algebras*

$$\left(0,0,0,0,e^{12},e^{34}\right), \quad \left(0,0,0,0,e^{13}-e^{24},e^{14}+e^{23}\right).$$

Proof. By [15, Thm. 4.3], a Lie algebra $\mathfrak{g}$ with derived algebra $\mathfrak{n} := [\mathfrak{g}, \mathfrak{g}]$ is (2,3)-trivial if and only if $\mathfrak{g}$ is solvable, $\mathfrak{n}$ is nilpotent of codimension one in $\mathfrak{g}$ and $H^1(\mathfrak{n})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{n})^{\mathfrak{g}} = H^3(\mathfrak{n})^{\mathfrak{g}}$. Here, $H^k(\mathfrak{n})$ denotes the k^{th} cohomology group of the Chevalley-Eilenberg complex

$\left(\Lambda^\bullet(\mathfrak{n}^*), \hat{d}\right)$ of $\mathfrak{n}$, and $H^k(\mathfrak{n})^{\mathfrak{g}}$ denotes the fixed point set of the $\mathfrak{g}$-action on $H^k(\mathfrak{n})$ for which $\mathfrak{n} \subset \mathfrak{g}$ acts trivially and every $X \in \mathfrak{g}/\mathfrak{n} \cong \mathbb{R}$ acts as follows:

$$X \cdot [\alpha] = [\iota_X d\alpha], \quad \forall\ [\alpha] \in H^k(\mathfrak{n}),$$

where d is the Chevalley-Eilenberg differential on $\mathfrak{g}$.

This characterisation leads us to considering seven-dimensional solvable Lie algebras of the form $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$, where $\mathfrak{n}$ is a six-dimensional nilpotent Lie algebra and $\mathbb{R} = \langle X \rangle$ with $\mathrm{ad}_X = D \in \mathrm{Der}(\mathfrak{n})$. To determine all possible Lie algebras of this type, up to isomorphism, we proceed as follows. It is known that there exist 34 six-dimensional nilpotent Lie algebras up to isomorphism. For each of them, we consider the basis $\{e_1, \ldots, e_6\}$ of $\mathfrak{n}$ for which its structure equations are those given in [19, Table A.1]. We can then compute the generic derivation $D \in \mathrm{Der}(\mathfrak{n})$ and consider the 6×6 matrix associated to it with respect to the basis $\{e_1, \ldots, e_6\}$. This matrix may have up to 36 unknown entries, which we denote by a_i. Finally, we let $\mathbb{R} = \langle e_7 \rangle$ with $\mathrm{ad}_{e_7} = D$.

Now, for any six-dimensional nilpotent Lie algebra $\mathfrak{n}$, we have to check whether the Lie algebra $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R} = \langle e_1, \ldots, e_7 \rangle$ can both be strongly unimodular and satisfy $H^1(\mathfrak{n})^{\mathfrak{g}} = \{0\} = H^2(\mathfrak{n})^{\mathfrak{g}} = H^3(\mathfrak{n})^{\mathfrak{g}}$. Clearly, $\mathfrak{g}$ is strongly unimodular if and only if the restriction of $\mathrm{ad}_{e_7} = D$ to each space $\mathfrak{n}^i/\mathfrak{n}^{i+1}$ is traceless. This gives a set of linear equations in the real variables a_i, say $\mathcal{S}$. As for the second condition, we first determine a basis of the cohomology groups $H^k(\mathfrak{n})$, $k = 1, 2, 3$, and then we compute the square matrix A_k associated to the linear endomorphism $H^k(\mathfrak{n}) \ni [\alpha] \mapsto e_7 \cdot [\alpha]$ with respect to that basis. The entries of these matrices are polynomials in the variables a_i, and the condition $H^k(\mathfrak{n})^{\mathfrak{g}} = \{0\}$ is equivalent to A_k being non-singular.

Summing up, for each six-dimensional nilpotent Lie algebra $\mathfrak{n}$, we have to solve the set of linear equations $\mathcal{S}$ under the constraints $\det(A_k) \neq 0$, $k = 1, 2, 3$. For 31 out of 34 non-isomorphic six-dimensional nilpotent Lie algebras this is not possible.

As an example, let us consider the first nilpotent Lie algebra appearing in [19, Table A.1]. Its structure equations with respect to the dual basis $\{e^1, \ldots, e^6\}$ of $\{e_1, \ldots, e_6\}$ are the following

$$\mathfrak{n} = \left(0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{34} - e^{25}\right).$$

The descending central series of $\mathfrak{n}$ is

$$\mathfrak{n}^1 = \langle e_3, e_4, e_5, e_6 \rangle, \quad \mathfrak{n}^2 = \langle e_4, e_5, e_6 \rangle, \quad \mathfrak{n}^3 = \langle e_5, e_6 \rangle, \quad \mathfrak{n}^4 = \langle e_6 \rangle, \quad \mathfrak{n}^5 = \{0\},$$

and the matrix associated to a generic derivation $D \in \mathrm{Der}(\mathfrak{n})$ with respect to the basis $\{e_1, \ldots, e_6\}$ is the following

$$D = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2a_1 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & 3a_1 & 0 & 0 & 0 \\ a_4 & 0 & a_3 & 4a_1 & 0 & 0 \\ a_5 & a_6 & -a_2 & a_3 & 5a_1 & 0 \\ a_7 & a_8 & a_5 & -a_4 & a_2 & 7a_1 \end{pmatrix}.$$

Now, we see that $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$ is strongly unimodular if and only if $a_1 = 0$. On the other hand, we have $H^1(\mathfrak{n}) = \langle [e^1], [e^2] \rangle$, and we see that

$$e_7 \cdot [e^1] = [\iota_{e_7} de^1] = -a_1[e^1], \quad e_7 \cdot [e^2] = [\iota_{e_7} de^2] = -2a_1[e^2].$$

Thus, $\det(A_1) = 2(a_1)^2$, whence it follows that $\mathfrak{g}$ cannot be both (2,3)-trivial and strongly unimodular. In the remaining 30 cases, we proceed similarly. We obtain that any solution of $\mathcal{S}$ implies $\det(A_1) = 0$ when $\mathfrak{n}$ is one of the following Lie algebras

$$\begin{array}{cc}
\left(0,0,e^{12},e^{13},e^{14},e^{34}-e^{25}\right), & \left(0,0,e^{12},e^{13},e^{14},e^{15}\right), \\
\left(0,0,e^{12},e^{13},e^{14}+e^{23},e^{15}+e^{24}\right), & \left(0,0,e^{12},e^{13},e^{14},e^{15}+e^{23}\right), \\
\left(0,0,e^{12},e^{13},e^{23},e^{14}\right), & \left(0,0,0,e^{12},e^{14}-e^{23},e^{15}+e^{34}\right), \\
\left(0,0,0,e^{12},e^{14},e^{15}+e^{23}\right), & \left(0,0,0,e^{12},e^{14},e^{15}+e^{23}+e^{24}\right), \\
\left(0,0,0,e^{12},e^{14},e^{15}+e^{24}\right), & \left(0,0,0,e^{12},e^{14},e^{15}\right), \\
\left(0,0,0,e^{12},e^{13},e^{14}+e^{35}\right), & \left(0,0,0,e^{12},e^{23},e^{14}+e^{35}\right), \\
\left(0,0,0,e^{12},e^{23},e^{14}-e^{35}\right), & \left(0,0,0,e^{12},e^{14},e^{24}\right), \\
\left(0,0,0,e^{12},e^{13}-e^{24},e^{14}+e^{23}\right), & \left(0,0,0,e^{12},e^{14},e^{13}-e^{24}\right), \\
\left(0,0,0,e^{12},e^{13}+e^{14},e^{24}\right), & \left(0,0,0,e^{12},e^{13},e^{14}+e^{23}\right), \\
\left(0,0,0,e^{12},e^{13},e^{24}\right), & \left(0,0,0,e^{12},e^{13},e^{14}\right), \\
\left(0,0,0,0,e^{12},e^{15}+e^{34}\right), & \left(0,0,0,0,e^{12},e^{15}\right), \\
\left(0,0,0,0,e^{12},e^{14}+e^{25}\right), & \left(0,0,0,0,0,e^{12}+e^{34}\right).
\end{array}$$

Any solution of $\mathcal{S}$ implies $\det(A_2) = 0$ in the following cases

$$\left(0,0,0,e^{12},e^{13},e^{23}\right), \quad \left(0,0,0,0,e^{12},e^{14}+e^{23}\right).$$

Finally, any solution of $\mathcal{S}$ implies $\det(A_3) = 0$ when $\mathfrak{n}$ is one of the following

$$\begin{array}{cc}
\left(0,0,e^{12},e^{13},e^{23},e^{14}-e^{25}\right), & \left(0,0,e^{12},e^{13},e^{23},e^{14}+e^{25}\right), \\
\left(0,0,0,0,e^{12},e^{13}\right), & \left(0,0,0,0,0,e^{12}\right),
\end{array}$$

We now claim that there exist (2,3)-trivial and strongly unimodular Lie algebras $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$ when $\mathfrak{n}$ is one of the Lie algebras of [19, Table A.1] not appearing above, namely

$$\mathfrak{a} = (0,0,0,0,0,0), \quad \mathfrak{n}_1 = \left(0,0,0,0,e^{12},e^{34}\right), \quad \mathfrak{n}_2 = \left(0,0,0,0,e^{13}-e^{24},e^{14}+e^{23}\right).$$

Let us examine each case separately.

For the abelian Lie algebra $\mathfrak{a}$, we have $\mathrm{Der}(\mathfrak{a}) \cong \mathrm{End}(\mathfrak{a})$ and $H^k(\mathfrak{a}) \cong \Lambda^k(\mathfrak{a}^*)$. To obtain a (2,3)-trivial Lie algebra $\mathfrak{g} = \mathfrak{a} \rtimes_D \mathbb{R}$, it is sufficient to consider a diagonal derivation

$$D = \mathrm{diag}\left(a_1, a_2, a_3, a_4, a_5, a_6\right),$$

with $a_i \neq 0$, and $a_i \neq -a_j, -a_j - a_k$, whenever i, j, k are distinct (see also [16, Ex. 5.1] for a similar case). Moreover, the Lie algebra $\mathfrak{g}$ is strongly unimodular if and only if $\mathrm{tr}(D) = \sum_{i=1}^{6} a_i = 0$. This last equation clearly admits solutions under the above contraints.

The Lie algebra $\mathfrak{n}_1$ has the following derived series

$$(\mathfrak{n}_1)^0 = \mathfrak{n}_1, \quad (\mathfrak{n}_1)^1 = \langle e_5, e_6 \rangle, \quad (\mathfrak{n}_1)^i = \{0\},\ i \geq 2,$$

and the generic derivation $D_1 \in \mathrm{Der}(\mathfrak{n}_1)$ has the following expression

$$D_1 = \begin{pmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_5 & a_6 & 0 & 0 \\ 0 & 0 & a_7 & a_8 & 0 & 0 \\ a_9 & a_{10} & a_{11} & a_{12} & a_1 + a_4 & 0 \\ a_{13} & a_{14} & a_{15} & a_{16} & 0 & a_5 + a_8 \end{pmatrix}. \tag{2.2}$$

Thus, the seven-dimensional Lie algebra $\mathfrak{g} = \mathfrak{n}_1 \rtimes_{D_1} \mathbb{R}$ is strongly unimodular if and only if

$$a_1 + a_4 + a_5 + a_8 = 0.$$

We can now compute the determinant of the matrices A_1, A_2, A_3 (see the Appendix for their expression), and notice that there exist solutions of the equation above for which $\det(A_k) \neq 0$, for $k = 1, 2, 3$. An example is given by

$$a_1 = 1, \quad a_4 = 3, \quad a_5 = 2, \quad a_8 = -6, \quad a_i = 0 \text{ otherwise}.$$

Finally, let us consider the Lie algebra $\mathfrak{n}_2$. Its derived series is

$$(\mathfrak{n}_2)^0 = \mathfrak{n}_2, \quad (\mathfrak{n}_2)^1 = \langle e_5, e_6 \rangle, \quad (\mathfrak{n}_2)^i = \{0\},\ i \geq 2,$$

and the generic derivation $D_2 \in \mathrm{Der}(\mathfrak{n}_2)$ is

$$D_2 = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 \\ -a_2 & a_1 & -a_4 & a_3 & 0 & 0 \\ a_5 & a_6 & a_7 & a_8 & 0 & 0 \\ -a_6 & a_5 & -a_8 & a_7 & 0 & 0 \\ a_9 & a_{10} & a_{11} & a_{12} & a_1 + a_7 & a_2 + a_8 \\ a_{13} & a_{14} & a_{15} & a_{16} & -a_2 - a_8 & a_1 + a_7 \end{pmatrix}. \tag{2.3}$$

From this, we see that $\mathfrak{g} = \mathfrak{n}_2 \rtimes_{D_2} \mathbb{R}$ is strongly unimodular if and only if

$$a_1 + a_7 = 0.$$

There exist solutions of this equation such that $\det(A_k) \neq 0$, for $k = 1, 2, 3$. For instance

$$a_1 = 1, \quad a_4 = 2, \quad a_5 = 2, \quad a_7 = -1, \quad a_8 = 1, \quad a_i = 0 \text{ otherwise}.$$

Also in this case, the explicit expression of the matrices A_1, A_2, A_3 can be found in the Appendix. □

We now show that there are no exact G_2-structures on any seven-dimensional solvable Lie algebra of the form $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$, when $\mathfrak{n} = \mathfrak{a}, \mathfrak{n}_1$, and when $\mathfrak{n} = \mathfrak{n}_2$ and $D \in \mathrm{Der}(\mathfrak{n}_2)$ is such that $\mathfrak{g}$ is strongly unimodular. Combining this result with Theorem 2.3, we get that there are no (2,3)-trivial, strongly unimodular seven-dimensional Lie algebras admitting exact G_2-structures.

Theorem 2.4. *Any seven-dimensional solvable Lie algebra* $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$*, with* $\mathfrak{n} = \mathfrak{a}, \mathfrak{n}_1$*, does not admit exact* G_2*-structures. Moreover, the same conclusion holds true if* $\mathfrak{n} = \mathfrak{n}_2$ *and the derivation* $D \in \mathrm{Der}(\mathfrak{n}_2)$ *is such that* $\mathfrak{g}$ *is strongly unimodular.*

Proof. Recall that a 3-form ϕ on a seven-dimensional Lie algebra $\mathfrak{g}$ defines a G_2-structure if and only if

$$\iota_v\phi \wedge \iota_v\phi \wedge \phi \neq 0 \in \Lambda^7(\mathfrak{g}^*), \quad \forall\, v \in \mathfrak{g} \smallsetminus \{0\}. \tag{2.4}$$

We have to consider the seven-dimensional solvable Lie algebra $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$ in the case when the nilradical $\mathfrak{n} = \langle e_1, \ldots, e_6\rangle$ is one of $\mathfrak{a}, \mathfrak{n}_1, \mathfrak{n}_2$, with the structure equations given in the proof of Theorem 2.3, and $\mathbb{R}$ is generated by a vector e_7 such that ad_{e_7} is a generic derivation $D \in \mathrm{Der}(\mathfrak{n})$.

Under these assumptions, we need to show that there are no exact 3-forms satisfying the condition (2.4) when $\mathfrak{n} = \mathfrak{a}, \mathfrak{n}_1$, and when $\mathfrak{n} = \mathfrak{n}_2$ with $D \in \mathrm{Der}(\mathfrak{n}_2)$ such that $\mathfrak{g}$ is strongly unimodular. In what follows, we denote the Chevalley-Eilenberg differential on $\mathfrak{g}$ and on $\mathfrak{n}$ by d and $\hat{d}$, respectively. Recall that for any 2-form $\gamma \in \Lambda^k(\mathfrak{n}^*)$ the following identity holds

$$d\gamma = \hat{d}\gamma + (-1)^{k+1}\, D^*\gamma \wedge e^7,$$

where $D^*\gamma(v_1, \ldots, v_k) = \gamma(Dv_1, \ldots, v_k) + \cdots + \gamma(v_1, \ldots, Dv_k)$, for all $v_1, \ldots, v_k \in \mathfrak{n}$.

Let $\mathfrak{n} = \mathfrak{a}$ be the abelian Lie algebra, and consider the generic 2-form $\alpha = \beta + \eta \wedge e^7$ on $\mathfrak{g}$, where $\beta \in \Lambda^2(\mathfrak{a}^*)$ and $\eta \in \Lambda^1(\mathfrak{a}^*)$. Then, the generic exact 3-form on $\mathfrak{g}$ is given by

$$\phi = d\alpha = d\beta + d\eta \wedge e^7 = \hat{d}\beta - D^*\beta \wedge e^7 + \hat{d}\eta \wedge e^7 = -D^*\beta \wedge e^7,$$

as $\hat{d}\gamma = 0$ for any $\gamma \in \Lambda^k(\mathfrak{a}^*)$. It is clear that this 3-form cannot define a G_2-structure, since there exists some nonzero $v \in \mathfrak{a}$ such that $\iota_v\varphi \wedge \iota_v\varphi \wedge \varphi = 0$.

Let us now consider the Lie algebra $\mathfrak{g} = \mathfrak{n}_1 \rtimes_{D_1} \mathbb{R}$, where $D_1 \in \mathrm{Der}(\mathfrak{n}_1)$ is given by (2.2). Let $\alpha \in \Lambda^2(\mathfrak{g}^*)$, we write $\alpha = \sum_{1\leq i<j\leq 7} c_{ij}e^i \wedge e^j$, with $c_{ij} \in \mathbb{R}$, and see that a generic exact 3-form on $\mathfrak{g}$ has the following expression

$$\begin{aligned}
d\alpha = {} & (-a_1c_{12} - a_4c_{12} + a_9c_{25} - a_{10}c_{15} + a_{13}c_{26} - a_{14}c_{16} + c_{57})e^{127} - c_{16}e^{134} \\
& - (a_1c_{16} + a_3c_{26} + a_5c_{16} + a_8c_{16} + a_9c_{56})e^{167} - c_{26}e^{234} - c_{35}e^{123} + c_{56}e^{126} \\
& - (a_2c_{16} + a_4c_{26} + a_5c_{26} + a_8c_{26} + a_{10}c_{56})e^{267} - c_{45}e^{124} - c_{56}e^{345} \\
& - (a_1c_{13} + a_3c_{23} + a_5c_{13} + a_7c_{14} - a_9c_{35} + a_{11}c_{15} - a_{13}c_{36} + a_{15}c_{16})e^{137} \\
& - (a_2c_{13} + a_4c_{23} + a_5c_{23} + a_7c_{24} - a_{10}c_{35} + a_{11}c_{25} - a_{14}c_{36} + a_{15}c_{26})e^{237} \\
& - (2a_5c_{36} + a_7c_{46} + a_8c_{36} + a_{11}c_{56})e^{367} - (a_1c_{25} + a_2c_{15} + 2a_4c_{25} - a_{14}c_{56})e^{257} \\
& - (a_1c_{14} + a_3c_{24} + a_6c_{13} + a_8c_{14} - a_9c_{45} + a_{12}c_{15} - a_{13}c_{46} + a_{16}c_{16})e^{147} \\
& - (a_2c_{14} + a_4c_{24} + a_6c_{23} + a_8c_{24} - a_{10}c_{45} + a_{12}c_{25} - a_{14}c_{46} + a_{16}c_{26})e^{247} \\
& - (a_5c_{34} + a_8c_{34} - a_{11}c_{45} + a_{12}c_{35} - a_{15}c_{46} + a_{16}c_{36} - c_{67})e^{347} \\
& - (a_5c_{46} + a_6c_{36} + 2a_8c_{46} + a_{12}c_{56})e^{467} - (2a_1c_{15} + a_3c_{25} + a_4c_{15} - a_{13}c_{56})e^{157} \\
& - (a_1c_{35} + a_4c_{35} + a_5c_{35} + a_7c_{45} - a_{15}c_{56})e^{357} - c_{56}(a_4 + a_1 + a_8 + a_5)e^{567} \\
& - (a_1c_{45} + a_4c_{45} + a_6c_{35} + a_8c_{45} - a_{16}c_{56})e^{457}.
\end{aligned}$$

This exact 3-form does not define a G_2-structure, as we always have $\iota_{e_6}d\alpha \wedge \iota_{e_6}d\alpha \wedge d\alpha = 0$, for all a_i and c_{ij}.

We are left with the case $\mathfrak{g} = \mathfrak{n}_2 \rtimes_{D_2} \mathbb{R}$. Recall from the proof of Theorem 2.3 that $\mathfrak{g}$ is strongly unimodular if and only if the entries of the derivation $D_2 \in \mathrm{Der}(\mathfrak{n}_2)$ given in (2.3) satisfy the condition $a_1 + a_7 = 0$. Now, similarly as in the previous case, we can compute the expression of the generic exact 3-form $d\alpha \in \Lambda^3(\mathfrak{g}^*)$ and see that

$$\iota_{e_6} d\alpha \wedge \iota_{e_6} d\alpha \wedge d\alpha = -12\,(c_{56})^3\,(a_1 + a_7)\,e^{1234567}.$$

Thus, if $\mathfrak{g}$ is strongly unimodular, then $d\alpha$ cannot define a G_2-structure. □

3. A unimodular example with $b_2 \neq 0$ and $b_3 = 0$

Let $\mathfrak{h} = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7 \rangle$ be the seven-dimensional solvable Lie algebra with the following nonzero Lie brackets

$$\begin{aligned} &[e_1, e_3] = 4\,e_3, \quad [e_1, e_4] = -e_4, \quad [e_1, e_5] = -5\,e_5, \quad [e_1, e_6] = -e_6, \quad [e_1, e_7] = 3\,e_7, \\ &[e_2, e_3] = 3\,e_3, \quad [e_2, e_5] = -4\,e_5, \quad [e_2, e_6] = -e_6, \quad [e_2, e_7] = 2\,e_7, \\ &[e_3, e_5] = -e_6, \quad [e_3, e_6] = -e_7. \end{aligned} \tag{3.1}$$

This Lie algebra is the semidirect product $\mathfrak{h} = \mathbb{R}^2 \ltimes \mathfrak{m}$, where the abelian Lie algebra $\mathbb{R}^2 = \langle e_1, e_2 \rangle$ acts on the nilradical $\mathfrak{m} = \langle e_3, e_4, e_5, e_6, e_7 \rangle$ of $\mathfrak{h}$ via the representation

$$\rho : \mathbb{R}^2 \to \mathrm{Der}(\mathfrak{m}), \quad \rho(e_1) = \mathrm{diag}(4, -1, -5, -1, 3), \quad \rho(e_2) = \mathrm{diag}(3, 0, -4, -1, 2).$$

Moreover, $\mathfrak{h}$ is easily seen to be unimodular and completely solvable.

We now show that the simply connected solvable Lie group H corresponding to $\mathfrak{h}$ does not contain any lattice. Indeed, the only nonzero terms in the descending central series of the nilradical $\mathfrak{m}$ of $\mathfrak{h}$ are

$$\mathfrak{m}^0 = \mathfrak{m}, \quad \mathfrak{m}^1 = \langle e_6, e_7 \rangle, \quad \mathfrak{m}^2 = \langle e_7 \rangle,$$

and from (3.1) it follows that $\mathrm{ad}_{e_1}|_{\mathfrak{m}^2} = 3\,\mathrm{Id}_{\mathfrak{m}^2}$. Hence, $\mathfrak{h}$ is not strongly unimodular.

Let $(e^1, \ldots, e^7)$ be the dual basis of $(e_1, \ldots, e_7)$. Then, the Chevalley-Eilenberg differentials of the covectors e^i are the following:

$$\begin{aligned} &de^1 = 0 = de^2, \qquad de^3 = -4\,e^{13} - 3\,e^{23}, \quad de^4 = e^{14}, \\ &de^5 = 5\,e^{15} + 4\,e^{25}, \quad de^6 = e^{16} + e^{26} + e^{35}, \quad de^7 = -3\,e^{17} - 2\,e^{27} + e^{36}. \end{aligned}$$

Using these equations, we compute the cohomology groups of the Chevalley-Eilenberg complex $(\Lambda^\bullet(\mathfrak{h}^*), d)$ of $\mathfrak{h}$, obtaining:

$$\begin{aligned} &H^1(\mathfrak{g}^*) = \langle [e^1], [e^2] \rangle, \quad H^2(\mathfrak{g}^*) = \langle [e^1 \wedge e^2] \rangle, \quad H^3(\mathfrak{g}^*) = H^4(\mathfrak{g}^*) = \{0\}, \\ &H^5(\mathfrak{g}^*) = \langle [e^{34567}] \rangle, \quad H^6(\mathfrak{g}^*) = \langle [e^{234567}], [e^{134567}] \rangle, \quad H^7(\mathfrak{g}^*) = \langle [e^{1234567}] \rangle. \end{aligned}$$

Therefore, the Betti numbers of the Lie algebra $\mathfrak{h}$ are the following

$$b_1(\mathfrak{h}) = b_6(\mathfrak{h}) = 2, \qquad b_2(\mathfrak{h}) = b_5(\mathfrak{h}) = b_7(\mathfrak{h}) = 1, \qquad b_3(\mathfrak{h}) = b_4(\mathfrak{h}) = 0.$$

We now describe an explicit example of a left-invariant exact G_2-structure φ on H.

Let us consider the following basis of $\mathfrak{h}^*$:

$$\begin{aligned} &E^1 = e^3, \quad E^2 = \frac{1}{4\sqrt{3}}\,e^2, \quad E^3 = -\sqrt{3}\,e^4 + 2\sqrt{3}\,e^6, \\ &E^4 = e^5, \quad E^5 = e^4 + 2\,e^6, \quad E^6 = 8\sqrt{3}\,e^7, \quad E^7 = e^1 + \frac{3}{4}\,e^2. \end{aligned}$$

The structure equations of $\mathfrak{h}$ with respect to this new basis are:

$$\begin{cases} dE^1 = 4\,E^{17}, \\ dE^2 = 0, \\ dE^3 = -E^{37} + 2\sqrt{3}\,E^{14} - \sqrt{3}\,E^{23} + 6\,E^{25}, \\ dE^4 = -5\,E^{47} + \sqrt{3}E^{24}, \\ dE^5 = -E^{57} + 2\,E^{14} + 2\,E^{23} - \sqrt{3}\,E^{25}, \\ dE^6 = 3\,E^{67} + 2\,E^{13} + 2\sqrt{3}\,E^{15} + \sqrt{3}\,E^{26}, \\ dE^7 = 0. \end{cases}$$

We consider the G_2-structure φ on $\mathfrak{h}$ with adapted G_2-basis $(E^1, \ldots, E^7)$, that is,

$$\varphi = E^{127} + E^{347} + E^{567} + E^{135} - E^{146} - E^{236} - E^{245}.$$

This is easily seen to be a closed, hence exact, G_2-structure. In particular, we have

$$\varphi = d\left(-\frac{1}{4}\,E^{12} + \frac{1}{6}\,E^{34} - \frac{1}{2}\,E^{56}\right).$$

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Appendix A.

In this appendix, we explicitly write the matrices associated to the endomorphisms A_1, A_2, A_3, introduced in the proof of Theorem 2.3, in the case when $\mathfrak{n} = \mathfrak{n}_1$ and $\mathfrak{n} = \mathfrak{n}_2$. For each of these Lie algebras, we write an ordered basis of the cohomology group $H^k(\mathfrak{n}^*)$, $k = 1, 2, 3$, and the matrix associated to A_k with respect to that basis. For the sake of convenience, we first impose the conditions on $D \in \mathrm{Der}(\mathfrak{n})$ for which $\mathfrak{g} = \mathfrak{n} \rtimes_D \mathbb{R}$ is strongly unimodular.

Case $\mathfrak{n} = \mathfrak{n}_1$. For the Lie algebra $\mathfrak{n}_1 = \left(0, 0, 0, 0, e^{12}, e^{34}\right)$ with generic derivation D_1 given by (2.2), we have that $\mathfrak{g} = \mathfrak{n} \rtimes_{D_1} \mathbb{R}$ is strongly unimodular if and only if $a_8 = -a_1 - a_4 - a_5$. In this case, we get

- $H^1(\mathfrak{n}_1) = \langle [e^1],\ [e^2],\ [e^3],\ [e^4] \rangle$,

$$A_1 = -\begin{pmatrix} a_1 & a_3 & 0 & 0 \\ a_2 & a_4 & 0 & 0 \\ 0 & 0 & a_5 & a_7 \\ 0 & 0 & a_6 & -a_1-a_4-a_5 \end{pmatrix};$$

- $H^2(\mathfrak{n}_1) = \langle [e^{13}], [e^{14}], [e^{15}], [e^{23}], [e^{24}], [e^{25}], [e^{36}], [e^{46}] \rangle$,

$$A_2 = \begin{pmatrix} -a_1-a_5 & -a_7 & -a_{11} & -a_3 & 0 & 0 & a_{13} & 0 \\ -a_6 & a_4+a_5. & -a_{12} & 0 & -a_3 & 0 & 0 & a_{13} \\ 0 & 0 & -2a_1-a_4 & 0 & 0 & -a_3 & 0 & 0 \\ -a_2 & 0 & 0 & -a_4-a_5 & -a_7 & -a_{11} & a_{14} & 0 \\ 0 & -a_2 & 0 & -a_6 & a_1+a_5 & -a_{12} & 0 & a_{14} \\ 0 & 0 & -a_2 & 0 & 0 & -2a_4-a_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1+a_4-a_5 & -a_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & -a_6 & 2a_4+2a_1+a_5 \end{pmatrix};$$

- $H^3(\mathfrak{n}_1) = \langle [e^{125}], [e^{135}], [e^{136}], [e^{145}], [e^{146}], [e^{235}], [e^{236}], [e^{245}], [e^{246}], [e^{346}] \rangle$,

$$A_3 = \begin{pmatrix} 2a_4+2a_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2a_1+a_5+a_4 & 0 & a_6 & 0 & a_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_5-a_4 & 0 & a_6 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_7 & 0 & a_1-a_5 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_7 & 0 & -2a_4-a_5-a_1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & a_3 & 0 & 0 & 0 & 2a_4+a_5+a_1 & 0 & a_6 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & a_5-a_1 & 0 & a_6 & 0 \\ 0 & 0 & 0 & a_3 & 0 & a_7 & 0 & a_4-a_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & 0 & a_7 & 0 & -2a_1-a_5-a_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2a_4-2a_1 \end{pmatrix}.$$

Case $\mathfrak{n} = \mathfrak{n}_2$. Consider the Lie algebra $\mathfrak{n}_2 = \left(0,0,0,0,e^{13}-e^{24}, e^{14}+e^{23}\right)$ with generic derivation D_2 given by (2.3). Then, $\mathfrak{g} = \mathfrak{n}_2 \rtimes_{D_2} \mathbb{R}$ is strongly unimodular if and only if $a_7 = -a_1$. In this case, we have

- $H^1(\mathfrak{n}_2) = \langle [e^1],\ [e^2],\ [e^3],\ [e^4] \rangle$,

$$A_1 = \begin{pmatrix} -a_1 & a_2 & -a_5 & a_6 \\ -a_2 & -a_1 & -a_6 & -a_5 \\ -a_3 & a_4 & a_1 & a_8 \\ -a_4 & -a_3 & -a_8 & a_1 \end{pmatrix};$$

- $H^2(\mathfrak{n}_2) = \langle [e^{12}], [e^{34}], [e^{13}+e^{24}], [e^{14}-e^{23}], [e^{16}+e^{25}], [-e^{15}+e^{26}], [e^{36}+e^{45}], [-e^{35}+e^{46}] \rangle$,

$$A_2 = \begin{pmatrix} -2a_1 & -2a_6 & -2a_5 & a_9-a_{14} & a_{10}+a_{13} & 0 & 0 & 0 \\ a_4 & 0 & a_8-a_2 & -\frac{1}{2}a_{15} & \frac{1}{2}a_{11}-\frac{1}{2}a_{16} & -a_6 & \frac{1}{2}a_{13}+\frac{1}{2}a_{10} & -\frac{1}{2}a_9+\frac{1}{2}a_{14} \\ -a_3 & a_2-a_8 & 0 & -\frac{1}{2}a_{16} & \frac{1}{2}a_{12}+\frac{1}{2}a_{15} & -a[_5 & -\frac{1}{2}a_{14}+\frac{1}{2}a_9 & \frac{1}{2}a_{13}+\frac{1}{2}a_{10} \\ 0 & & 0 & -a_1 & a_8+2a_2 & 0 & -a_5 & a_6 \\ 0 & 0 & 0 & -a_8-2a_2 & -a_1 & 0 & -a_6 & -a_5 \\ 0 & 2a_4 & -2a_3 & 0 & 0 & 2a_1 & a_{11}-a_{16} & a_{12}+a_{15} \\ 0 & 0 & 0 & -a_3 & a_4 & 0 & a_1 & a_2+2a_8 \\ 0 & 0 & 0 & -a_4 & -a_3 & 0 & -a_2-2a_8 & a_1 \end{pmatrix};$$

- $H^3(\mathfrak{n}_2) = \langle [e^{125}], [e^{126}], [e^{345}], [e^{346}], [-e^{135}+e^{146}], [-e^{135}+e^{236}], [e^{135}+e^{245}], [e^{136}+e^{246}], [e^{145}-e^{246}], [e^{235}-e^{246}] \rangle$,

$$A_3 = \begin{pmatrix} -2a_1 & 0 & 0 & -a_4 & -a_4 & 0 & -a_3 & 0 & -a_4 & a_2+a_8 \\ 0 & -2a_1 & -a_2-a_8 & a_5 & -a_6 & -2a_5 & -a_6 & 2a_6 & -a_5 & 0 \\ 0 & a_2+a_8 & 2a_1 & a_6 & a_5 & 0 & -a_5 & 0 & -a_6 & 0 \\ a_6 & 0 & a_4 & 0 & -a_2-2a_8 & 0 & -a_8-2a_2 & a_2+a_8 & 0 & 0 \\ a_5 & 0 & a_3 & a_2+2a_8 & 0 & -a_2-a_8 & 0 & 0 & -a_8-2a_2 & 0 \\ 0 & -a_3 & 0 & 0 & -a_2 & 0 & -a_8-2a_2 & a_2-a_8 & 0 & -a_5 \\ -a_5 & 0 & -a_3 & a_2 & 0 & a_2+a_8 & 0 & 0 & -a_8 & 0 \\ 0 & -a_4 & 0 & a_2 & 0 & -a_2+a_8 & 0 & 0 & -a_8-2a_2 & a_6 \\ -a_6 & 0 & a_4 & 0 & a_2 & 0 & a_8 & a_2+a_8 & 0 & 0 \\ -a_2-a_8 & 0 & 0 & a_3 & a_3 & -2a_3 & a_4 & -2a_4 & -a_3 & -2a_1 \end{pmatrix}.$$

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