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DOI:10.1007/s10649-019-09926-2

Swidan, O., Sabena, C. & Arzarello, F. (online first 2019; print version 2020). Disclosure of mathematical relationships with a digital tool: a three layer-model of meaning. *Educational Studies in Mathematics*, 103, 83–101.

https://doi.org/10.1007/s10649-019-09926-2

Disclosure of mathematical relationships with a digital tool: a three layer-model of meaning

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Abstract This paper examines mathematical meaning-making from a phenomenological perspective and considers how a specific dynamic digital tool can prompt students to disclose the relationships between a function and its antiderivatives. Drawing on case study methodology, we focus on a pair of grade 11 students and analyze how the tool's affordances and the students' engagement in the interrogative processes of sequential questioning and answering allow them to make sense of the mathematical objects and their relationships and, lastly, of the mathematical activity in which they are engaged. A three-layer model of meaning of the students' disclosure process emerges, namely a) disclosing objects, b) disclosing relationships, and c) disclosing functional relationships. The model sheds light on how the students' *interrogative processes* help them make sense of mathematical concepts as they work on tasks with a digital tool, an issue that has rarely been explored. The study's implications and limitations are discussed.

Keywords antiderivative; dynamic digital tools; indefinite integral; function; phenomenology; disclosure.

1. Introduction

Digital tools are intensively used across a vast range of mathematics education settings and in a wide variety of schools. According to the socio-cultural approach to knowledge, these tools are central to cognition, and collectively, they play a fundamental role in the development of robust cognitive skills (Bartolini Bussi & Mariotti, 2008) and in shaping our ways of thinking (Radford, 2008). Indeed, research on the design of mathematics learning software has shown that the use of different designs causes students to develop correspondingly varied ways of thinking about and tackling mathematical tasks (Drijvers, 2015; Yerushalmy & Chazan, 2008).

In the context of learning calculus, research has focused mainly on how student interaction with digital tools helps them conceptualize calculus ideas (Arzarello & Robutti, 2010; Swidan & Yerushalmy, 2016). In particular, studies have emphasized two aspects: the use of digital tools as facilitative resources for solving problems; and the use of symbolic, graphic, and numeric digital tools to illustrate the concepts of co-variation, derivation and integration (Tall, 2010; Thompson & Carlson, 2017). However, studies that examine how students learn the indefinite integral graphically by using digital tools are relatively rare (in this paper, the indefinite integral of a function $f(x)$ means the set of all its antiderivatives $F(x)+C$, where $F'(x)= f(x)$).

In this article, we examine a dynamic digital tool whose display shows two graphs, one of a function and the other of its antiderivatives, in two linked Cartesian systems. Swidan and Yerushalmy (2014) used this kind of tool to identify the learning processes involved in understanding the concept of an indefinite integral and to elucidate how students form an association between the two graphs. Following the work of Swidan and Yerushalmy (2014), we conjecture that such a dynamic digital tool can foster student engagement in processes of questioning and answering sequentially and that these interrogative processes play important roles in learning complex mathematics concepts. To shed light on how such interrogative processes help the students make sense of mathematical concepts when they are taught with digital tools — an issue that has rarely been explored — we follow Rota (1991) and take a phenomenological perspective. In the next section we will elaborate on the Rota's phenomenological perspective. Following that, we will discuss the interrogative processes which we assume to play an important role in disclosing mathematical meanings.

2. Theoretical framework

2.1. A phenomenological perspective

As is well known, the learning of mathematical concepts and ways of thinking requires significant time and effort investments. Crucial roles are played by the problems and questions students confront (and ask), by the tools they use, and by the teacher's interventions. As emphasized by Radford (2010, p. 4), students must be taught to "see and recognize things according to 'efficient' cultural means" and to convert their "eye (and other human senses) into a sophisticated intellectual organ". Namely, it is necessary to promote a "lengthy process of domestication" (*ibid.*) of how they look at things while learning mathematics. An essential part of this process is "seeing and being able to see in mathematics", according to the title of a wonderful book by de Finetti (1967). We delve into the matter more rigorously in line with the elaboration on Husserl phenomenology provided by the mathematician and philosopher G. C. Rota (1991). The starting point is the key phenomenological assumption, pointed out by Rota, that there is "no such thing as true seeing", but "there is only *seeing as*" (*ibid.*, p. 239). For example, an increasing and decreasing continuous graph in a Cartesian plane may be "seen as" the picture of a mountain by young students. The realization of this goal of school curricula, namely, that the students eventually "see" this drawing "as" the graph of a mathematical function (but also "as" an "increasing function", "as" a "continuous function", etc.), is a long, delicate process. According to Rota, this process is referred to as disclosure.

A Husserlian concept, disclosure refers to the process by which people make sense of and interpret the world around them and various situations in the world in the contexts in which they are exposed. The first encounter with the world is with and through *perceptual senses*:

> "The world is primarily a world of sense […]. Our primary concern is with sense itself, how it originates in the world, how it functions in the world. In short how it relevates […]. The basic relationship to the world is […] our senses" (*ibid.*, p. 61).

Disclosure happens when one is able to grasp an object's *functionality* in a given context, for example, in a learning situation:

> "Sense-making depends ultimately on our own being-in-the-world, on the situation of our interacting, our dealing with the contextual situation in the world […]. If you deconstruct the notion of an object, what you find is pure functionality, the pure 'being good for' of that object or something. So that the world, instead of being a world of objects, will become a world of functions, of tools. […] The world is disclosed to us not just as a system of functions, but as a network of related functions" (*ibid.*, p. 159).

This delicate process is far from natural: on the contrary, students must be educated to make sense of what they disclose when they are exposed to a mathematical situation. Generally speaking and depending on the ages and the backgrounds of the students, between different students, a given mathematical situation may evoke different contexts and lead to different sense-making (see the example of the function graph made above). According to Rota, rather than being isolated, these different contexts are instead *layered* upon one another, and the layers can generate different meanings over time:

> Side-by-side with our realization that sense is purely contextual goes the realization that contexts are not units. Contexts themselves are layered upon each other in various ways, and to be in a context is not to be in just one context. … Be-ing in a context does not in any way presume that such be-ing is be-ing in one context at a time. (*ibid.*, p. 126)

In a previous study, Arzarello, Ascari, Baldovino, and Sabena (2011) showed how the practice of suitable didactic techniques by the teacher may promote different layers of meaning in students' disclosures of calculus concepts. They refer to two related didactic techniques: making present things that are absent to students, and prompting the students by asking them suitable questions to direct their attention to some specific feature or element of the concept being addressed by the teacher (Mason, 2008). These tasks can be accomplished by the teachers through a variety of multimodal channels, for example, gestures and gazes are very effective (Arzarello, Paola, Robutti, & Sabena, 2009). Likewise, appropriate tools can play mediating roles in how the students make sense of novel concepts. Tools can not only trigger the disclosure process, they can also sustain it through different layers of meaning. In particular, tools may allow one to foster suitable contexts in which the students can engage in interrogative processes that focus their attention on the functionalities of the objects to which they are exposed in teaching situations.

2.2. Interrogative processes

Processes of questioning and answering sequentially are central to teaching-learning mathematics. Questions asked by the teacher may *direct the students' attention* to some mathematical features that characterize the phenomena on which the teacher wants the students to focus (Mason, 2008). Also, having the *students themselves engage in the interrogative process* is generally considered to be "a useful process in their pursuit of learning, in that questioning is one of the most important ways students can support their own learning to become literate, well-educated people" (Boaler & Humphreys, 2005, p. 72). Indeed, it is "every competent teacher's dream that students will ask 'good' mathematical questions" (Mason, 2014, p. 518). Research in mathematics education has focused mostly on the *teacher's questioning* as an instructional strategy (e.g., Weiland, Hudson, & Amador, 2014) and has focused rarely on *students' questioning*.

In our study, we rely on Hintikka's *logic of inquiry approach* to frame the role of students' interrogative processes within the mathematics learning process. Hintikka introduced the logic of inquiry as the logic of questioning and answering sequentially (in an *interrogative process*). The idea is similar to the notion advanced by Socrates, who claimed that rational knowledge should be sought through questioning (Hintikka, 1999). More precisely, Hintikka conceives of the quest for new knowledge as an interrogative process between two players. The first player, called the 'inquirer', actively seeks knowledge and asks questions, while the second player is called 'nature' or 'oracle' and responds to the questions of the first player. While the inquirer seeks knowledge and tries to prove a conclusion that has been reached based on prior experience or even on theoretical premises, the oracle, a source of knowledge, is questioned by the inquirer in his/her search for the truth. The questioning strategy used by the inquirer in the logic of inquiry approach is central to the quest for new knowledge: within a phenomenological perspective, we can say that it is through the process of questioning that the sought knowledge is disclosed.

Summing up, we use the phenomenological perspective set out by Rota to study students meaning-making processes, paying special attention to the role of the interrogative processes involved in them.

3. The calculus integral sketcher and its features

The digital tool used in this study is the CIS (Shternberg, Yerushalmy & Zilber, 2004), which displays two Cartesian coordinate systems (stacked one above the other) that are dynamically linked according to the indefinite integral of a function (Figs. 1 and 2a). The basic actions the students can perform in the CIS comprise *construction*, which allows one to generate a graph of a function on the upper Cartesian system of the CIS using the icon buttons; and *dragging*, which allows one to manipulate (move, stretch, etc.) the graphs. The graphical user interface of the CIS contains seven icons  in line with the different situations one can encounter in one-variable function graphs (Schwartz & Yerushalmy, 1995). Users can sketch a graph of a function on the upper Cartesian system by using one or more of the icons. In response, the CIS displays a function graph consisting of segments whose shapes resemble the respective icons that were chosen. In all cases, the CIS generates the antiderivative function graph in the lower Cartesian system (Fig. 2a). When the upper graph is manipulated (Fig. 2c and d), the lower graph (antiderivative function graph) changes in response to satisfy the mathematical relationship between the graph of the function and that of its antiderivatives.

The educational intentionality of the CIS digital tool is to enable students – via their exploration, observation and manipulation of the graphs – to disclose the mathematical relationships that link the graphs. For example, dragging a constant function graph vertically upward on the upper system causes the inclination of the antiderivative function graph on the lower system to increase (Fig. 1a and b). This action may prompt the students to direct their attention to the relationship between the increases in the *y*-values of the function graph and the increase in the inclination of the antiderivative. Likewise, when the constant function graph is stretched horizontally, its *y*-value remains fixed. By causing the antiderivative linear function graph to stretch while its inclination does not change (Fig. 1c and d), this action may promote the students' understanding of the relationship between the two graphs – that is, between the fixed *y*-value of the function graph and the fixed inclination of the antiderivative function graph. Furthermore, it can also help them grasp that the *y*-value of the function graph is also the slope of the linear antiderivative function graph.

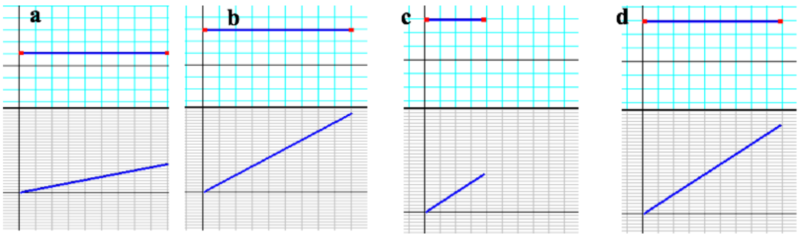

Fig. 1 Basic functionality of the CIS. (**a,b**) Dragging the function graph upward increases the inclination of the antiderivative function graph. (**c,d**) Stretching the function graph horizontally has no effect on the inclination of the antiderivative function graph.

Generally, in CIS students can (a) *drag the function graph by selecting either the entire segment or its end points* (in the former case, the user can only drag the segments vertically, whereas in the latter case the user can drag and stretch the segment in all directions), and (b) *drag the antiderivative function graph vertically* (this does not have any effect on the function graph) (Fig. 2d).

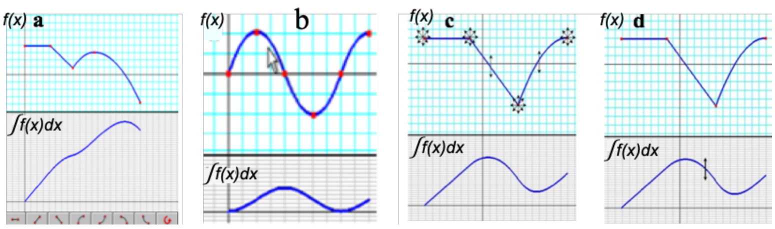

Fig. 2 General view of (**a**) basic CIS interface, (**b**) connected function graph (upper Cartesian system) and antiderivative function graph (lower Cartesian system), (**c**) dragging options of the function graph and (**d**) dragging option of the antiderivatives graph

4. Research questions

In this paper, we describe how working with the CIS digital tool can promote students' disclosure of the relationship between a function graph and its corresponding antiderivatives function graph. To that end, we were guided by the following questions:

1. How do the features of the CIS contribute to the students' disclosure of the mathematical relationships embedded in it?
2. How do the students' interrogative processes support the disclosure process?

5. Method

5.1. Context, setting, participants, data collection

The study involved 11 pairs of 17-year-old students from two different science and technology oriented schools, so-called because mathematics and physics at these schools are compulsory subjects, in Israel. In addition, students are encouraged to enroll in technology courses such as computer science, electronics, and robotics. The procedure was conducted in the computer labs at the students' respective schools. The students volunteered to participate in after-school meetings. At the time when the experiments were performed, the students had already learned the concepts of function and derivative but not that of the integral as an antiderivative.[1] The students were familiar with using the derivative symbolically. They were familiar with the conventional function graph software, which was part of their previous study of functions. Each pair of students shared one computer. The paper's first author introduced the students briefly to the program interface and showed them how to operate it via demonstration, for example, how to create single and multi-segment graphs and how to drag the graphs. The classroom teacher was asked not to intervene during the experiment to ensure that the students' interaction with the CIS was captured. The teacher's role was restricted to co-designing (with the first author) a task – to explore the possible connections between the two graphs on the screen – that the students performed during the experiment. The written task also directed the students to explore different graphs. The teacher informed the students that their exploration activity would be complete when they felt that they could sketch an antiderivative function graph for any given function without the help of the digital tool. The written task and the verbal instructions were given to the students as detailed in Table 1.

Table 1 Written task and verbal instructions given to the students by the first author

[1] They had already encountered the integral symbol and its computational uses in their physics and electronics lessons, however. The integral in such cases had the meaning of an area (definite integral). Nonetheless, they had not been exposed to the fundamental theorem of calculus, which links the two meanings.

Written task	Verbal instructions
The graphs in the upper and lower Cartesian systems are linked. Your task is to explore and explain the possible connections between the two graphs. You should be able to construct such pairs of graphs yourself without the software. While working on the task, you may use the CIS tool to generate graphs using the icon buttons and to change or drag the graphs as needed.	Before the students began their exploration activity, the first author gave them the following instructions: • Generate a function graph in the upper system by clicking the icons, and conjecture about the mathematical relationship between the displayed graphs. • Throughout your exploration activity, each student pair works together. When one of you finds a relationship, you should try to describe that relationship to your partner. • The activity will be complete when you are able to sketch an antiderivative function graph for any given function.

5.2. Data Analysis

The data presented below were generated from two rounds of analysis. The first round consisted of repeatedly watching the videos to examine the students' process of disclosure of the relationships between each function and antiderivative graph pair. We identified four phases in this process: in each phase, students disclose a specific mathematical component involved in the relationship between a function and its antiderivative graphs. For a given student, a phase starts when the student discloses a specific mathematical component and ends when the student transitions to another component. The four phases are detailed in the following sections.

In the second round of analysis, a case-study methodology was adopted to more closely analyze the roles of the interrogative processes and of the digital tools in the students' disclosure of the relationship between the graphs. We chose one pair of students (Muner and Amir – pseudonyms) because i) they successfully completed the task and ii) during their activity with the digital tool, their behavior reflected all four of the detected phases of the disclosure process. In considering the video and the transcript diachronically, each multimodal utterance by the pair of students was micro analyzed in terms of the actions that the students performed with the tool, their questions, and their gestures and gazes to monitor how and where their attention was focused. For example, we distinguished between the different kinds of *actions* the students performed with the tool: graph construction, dragging the entire upper graph (U-MoveFig), dragging portions of the upper graph from points on it (U-MovePoin), tracing the upper graph (U-Trace), and tracing the lower graph (L-trace). We also distinguished between the different kinds of questions asked by the students (as *interrogative* aspects of the students' behavior) – rhetorical, what, why, how, and how much questions. Finally, we distinguished between the different ways the students approached and thought about the graph, which we place under the category of *phenomenology*. These include, approaching the graph locally (e.g., by looking at discreet points on the graph), examining graph properties (e.g., by looking at increasing/decreasing segments on the graph), considering the relationships between the graphs (e.g., by associating points or segments in one graph with points or segments in the other graph), or distinguishing the kinds of graphs (e.g., linear graph). The timeline analysis (Figure 3) illustrates how we analyzed the data in the second round of analysis. Each square in the timeline is corresponds to 10 seconds.

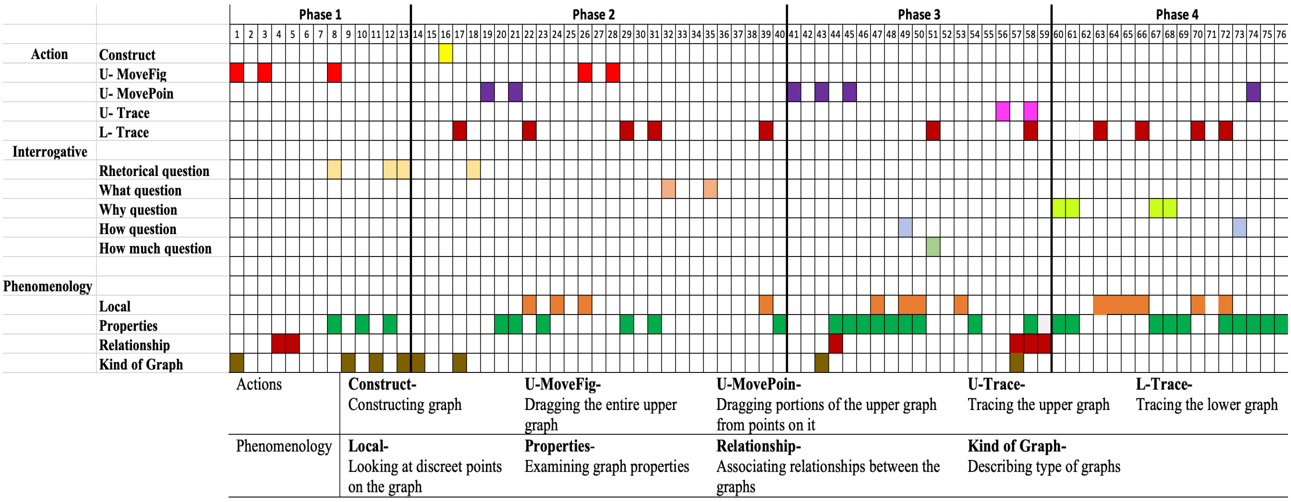

Fig. 3 Timeline analysis

To ensure the reliability of the data analysis, the paper's three authors independently coded the data in each round and then reached an agreement regarding the data after discussing the analytical process that each performed. The results of the data analysis are presented in the following section.

6. Results

6.1. Phases of the students' disclosure of the relationship between the two graphs

In the first round of data analysis, which considered the entire set of data, the following four phases were identified: (I) Disclosure of the function-derivative relationship, (II) Disclosure of the function graph as the slope function, (III) Disclosure of the function graph as the derivative, and (IV) Disclosure of the function-integral relationship. In the next section, we describe each phase in greater detail.

Table 2 Distribution of the phases among the student pairs

	Phase I	Phase II	Phase III	Phase IV
Slem-Maka	√	√		
Saja-Salm	√	√	√	
Amir-Muner	√	√	√	√
Rem-Shai	√	√	√	
Noga-Neya	√	√	√	√
Moha-Moat	√	√		
Mary-Zohar	√	√	√	

Narm-Mira	√	√	
Hamz-Moad	√	√	√
Aro-Nimat	√	√	√
Ans-Mis	√	√	√

The analysis revealed that three pairs of students showed only the first two phases of the disclosure process and another six pairs of students linked the graphs using the first three phases. The remaining two pairs of students, who used all four phases of disclosure to link the graphs, were the only ones who disclosed the function-integral relationship.

6.2. Interrogative processes and student interaction with the digital tool

In this subsection, we report our findings from the second round of our analysis. To that end, we include here selected excerpts from the students' conversations during the experiment to illustrate the four phases of disclosure and to highlight the interrogative processes and the actions with the digital tool (which were also observed in most of the students who participated in the learning experiment).

6.2.1. Phase 1: Disclosure of the function-derivative relationship

The first phase of the disclosure process concerns the function-derivative relationship. Initially, the students Muner and Amir generate a constant function by using the first icon  and dragging the entire function graph vertically up and down.

1	Muner	[*Drags the entire function graph up and down (Fig. 4a, 4b)*] Linear function which is parallel to the x-axis.
2	Amir	Change it.
3	Muner	[*Continues to drag the function graph up and down (Fig. 4c)*] On the x-axis
4	Amir	[*Points to the upper system*] This is the function and that is the derivative [*points to the lower system]*.
5	Muner	This is the function [*points to the lower system*] and that is its derivative [*points to the upper system*]. This is not the derivative [*points to the upper system*]. We could consider this as a new function [*points to the lower system*] and that its derivative. [*Silent for 5 seconds*] But why? Why? Because if we drag it upward [*drags the entire function graph upward (Fig. 4d)*], the slope changes.

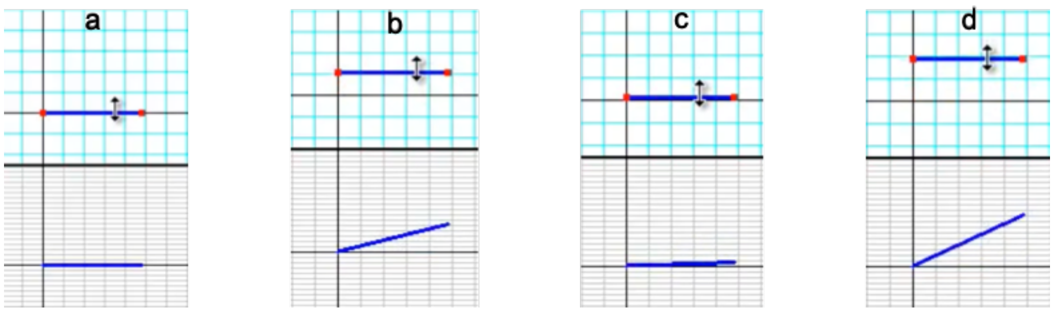

Fig. 4 Muner's exploration of the constant-linear case

As the students generate the constant function and drag it vertically, their attention is attracted by the type of the function ("linear", see line 1). But while Muner continues to drag the function graph (Fig. 4c), Amir shifts his attention to the relationship between the two graphs (line 4), which he interprets in terms of his knowledge about functions and derivatives. Muner (line 5) soon builds on Amir's interpretation, but he challenges his classmate by inverting the way the relationship should be considered, namely, by considering the lower graph as the function and the upper graph as its derivative, "we could consider this as a new function and that its derivative". The *why* question signals that he is looking for a deeper justification, and to this end, he uses the dragging tool (Fig. 4d) and observes what happens to the relationship between the graphs "If we drag it upward the slope changes" (line 5). Muner's observation of what happens when he performs the dragging action prompts him to attend to the *covariation between the function graph and the slope of the graph of the antiderivative*. In this covariation, he considers the two function graphs globally, specifically noticing how changes in one graph are linked to the changes in the slope of the other graph (a covariation of a covariation), which is cognitively demanding. In our interpretation, this covariation of covariation is disclosed by the student in response to i) the type of the chosen graphs, which are well-known to the students (constant-linear), and ii) the features of the CIS, specifically, the displays of the two graphs that are dynamically linked according to the mathematical relationships.

6.2.2. Phase 2: Disclosure of the function graph as the slope function

To continue to explore the relationships between the two graphs, the students use the second icon  to construct a new function graph. While the upper Cartesian system displays a linear function graph, the lower Cartesian system displays a quadratic function graph (Fig. 5a). The students begin to look at and discuss specific features of the graphs, as shown in the next transcript. This shift in attention from a global perspective to a perspective focused on specific details of the graphs is prompted by a *how* question from Amir:

31	Amir	How does the slope increase?
32	Muner	You should consider, from here to here. [*He points to the first segment on the antiderivative function graph as in Fig. 5a. He then moves the mouse in a "staircase shape" as in Fig. 5e.*] What is the slope value from here to here?
33	Amir	It is about one. [*Muner moves the curser and points to the function graph (Fig. 5b).*] And here?
34	Muner	It is about one. [*Muner moves the cursor on the antiderivative function graph, as if tracing a short segment*] and from here to here it is two (Fig. 5c) [*again moving the cursor in a "staircase shape" as in Fig. 5e*].
35	Amir	What is the value here? [*Muner points to the function graph (Fig. 5d*).] It is two.

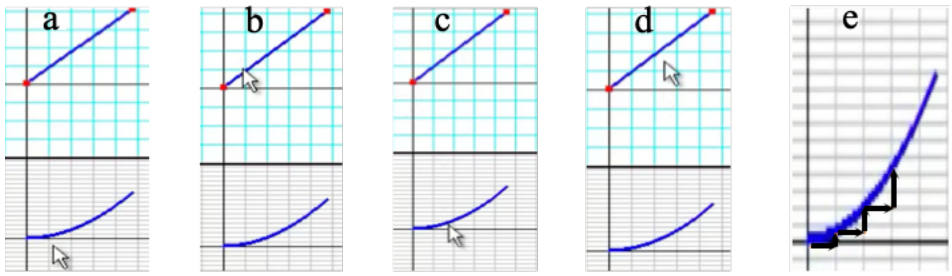

Fig. 5 Exploration by the students of the linear-quadratic case

To answer Amir's *how* question, Muner mixes three different kinds of actions on the CIS tool: pointing to the upper graph (lines 32 and 35, Fig. 5b and d), making a virtual segment on the lower graph (line 34, Fig. 5a and c), and making a staircase shape while pointing at the lower graph (lines 32 and 34, Fig. 5e). By examining the local properties of the function, Muner attempts to clarify the relationship between the two function graphs. We emphasize that in this process, in addition to considering what is *present* on the screen, i.e., the two graphs, and pointing at it, Muner *makes present what is absent*, that is, he introduces some *objects* into the scene that are not actually present but that become apparent as a result of the movements he carries out with the cursor. The first movement – his portrayal of staircase-shaped movement along the antiderivative graph – may be interpreted as evoking the classical triangle based on which the notion of slope was introduced to students in their regular mathematics lessons. The second movement is his drawing of a short segment whose slope is approximately one, as he clarifies in his words (line 34).

During this clarification, other questions emerge; this time they are *what* questions about the values of the slope at different points (lines 32, 33, 35); they prompt Amir to focus on specific points in the function graph. In line 33, Amir answers Muner's question and poses a new one, "and here?" The sequence of questions prompts Muner to focus on the correspondence between the y-values of points on the function graph and the tangent slope values of the antiderivative graph: this is visible in the vertical correspondence made through pointing gestures on the screen, which shift between points with the same x-value in the two graphs. Here again we observe that the students *make present what is absent*: in this case, what is made present is not an object, but rather, *key features of the relationship between the two graphs*, i.e., the correspondence between the y-values of points on the function graph and the tangent slope values of the antiderivative graph. In both cases, the action of *making present what is absent* is introduced through gestures or actions with the pointer, which in this sense becomes an extension of the hand (its function is that of a finger that points at an object on the screen).

6.2.3. Phase 3: Disclosure of the function graph as the derivative

Soon thereafter, the students click the fourth icon (), which constructs an increasing and concave down branch of a quadratic function graph. In the lower Cartesian system, an increasing and concave up antiderivative function graph is displayed (Fig. 6a). This display prompts a process of inquiry from the students that is guided by a *why* question that Amir directs at his classmate:

67.	Amir	But why was it concave down [*pointing to the upper graph*] and it becomes concave up [*pointing to the lower graph*]?
68	Muner	[*Drags the function graph from its right endpoint (Fig. 6c-d) and remains silent for 5 seconds*] Because the rate of change in the slope is decreasing. Because here in two steps on the x-axis it rises one unit [*pointing while tracing a "staircase shape" on the graph, as indicated in Fig. 6d*], but here in one step on the x-axis it rises one unit.

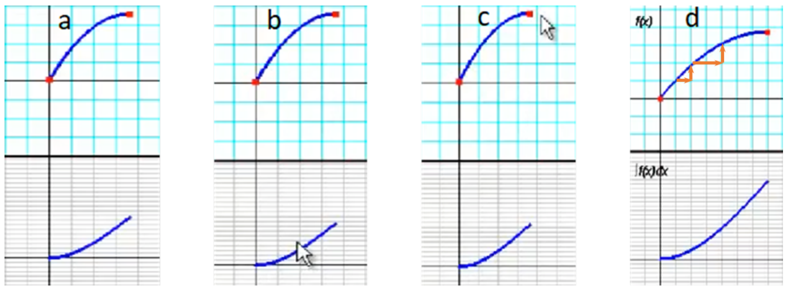

Fig. 6 Exploration by the students of the increasing and concave down function graph

Amir's question focuses on the difference between the concavities of the graphs (line 67); namely, he asks why the upper function is concave *down* while the antiderivative is concave *up*. Muner drags the function graph (Fig. 6c and d) to get a hint as to the answer to Amir's question. Looking at the function graph globally, Muner explains the differences between the concavity of the two graphs by remarking that "the rate of change in the slope is decreasing" (line 68). This remark is soon clarified by looking at the function graph locally by using the staircase gesture to justify his claim about the decrement of the function's rate of change.

The explanations given by Muner do not answer Amir's question. In the following excerpt, we see that Amir refines his question by asking new *how* and *why* questions.

80	Amir	Okay the slope is rising, but how is it rising, like this [*traces the concave down graph with the cursor, as illustrated in Fig. 7a*] or like that [*traces the graph with the cursor, as illustrated Fig. 7b*]? Now it is rising like this [*traces again the function graph as illustrated in Fig. 7a*]. Why? If it was like this [*as illustrated in Fig. 7b*] it is also increasing. It is a new situation we didn't explore.
81	Muner	*Adds a new segment using the sixth icon*  *to obtain the graph shown in Fig. 7c.*
82	Amir	It is the derivative [*points to the function graph*]. Here the derivative is zero [*pointing to the leftmost point in the function graph in Fig. 7c*], the slope here is

zero [*pointing to the leftmost point in the antiderivative function graph in Fig. 7c*].

83 Muner How much here [*pointing to maximum point in the function graph in Fig. 7c*]? It is two, okay? If the value of the derivative is two, what does it mean in the integral? It means the slope of the function in this neighbor is two.

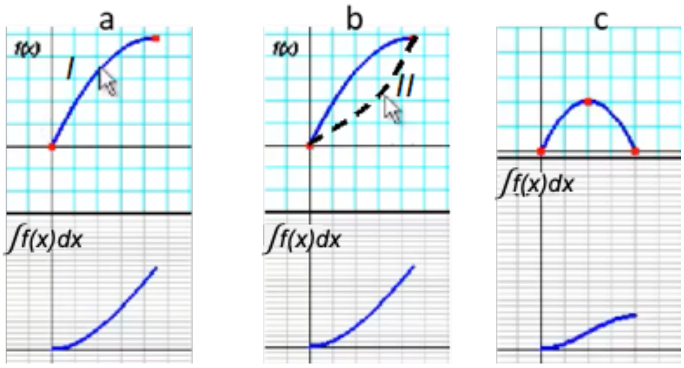

Fig. 7 Exploration by the students of the concave up and concave down function graphs

Apparently unsatisfied with Muner's explanation, Amir describes his confusion regarding the two possible graphic representations of an increasing function – increasing and concave up or increasing and concave down – and especially how they are distinguished from one another (line 80).

To resolve Amir's confusion, the two students construct a new graph by pressing the  icon and they obtain Fig. 7c. Amir clarifies that the graph in the upper Cartesian system is "the derivative" (line 82). He also describes the relationship between the two graphs at a specific point – the origin. His utterances in line 82 suggest that he disclosed that the function graph is the derivative (global property) and clarifies this disclosure by referring with words and pointing gestures to the relationship between specific points, namely, correlating the value of the upper function ("the derivative is zero") with the slope of the lower function ("the slope here is zero").

Amir's disclosure prompts Muner to address the same detail (line 83) but by focusing on another point, namely, the extremum point of the function graph. Similarly, Muner describes the relationship between the function graph and the antiderivative function graph. Unlike their prior attempts at understanding, here the students confidently associate the y-values of the function graph with the tangent slope of the antiderivative function graph.

6.2.4. Phase 4: Disclosure of the function-integral relationship

The students' disclosure of the relationship between the y-values of the function and the tangent slope of the antiderivative function graph allows them to explain the differences they observe in the concavities of the function and antiderivative function graphs. The excerpt below illustrates how they make present the relationship between the function and the antiderivative.

91	Muner	Look! [*cancelling the right segment of the function graph, as in Fig. 8a and dragging the function graph from its endpoint diagonally as in Fig. 8b] [silent for 15 seconds]* I get it. Oh my god, I understand it. Wait a moment! Take this point, for example, at this point where x is one the y-value is three [*Fig. 8c]*.
92	Amir	Okay! Three
93	Muner	This means its slope at one from here to here *[traces a virtual segment on the antiderivative function graph from x=0 to x=1 (Fig. 8d)]* is three. Here it is five [*Fig. 8e*], so, its slope from here to here is five [*traces a virtual segment on the antiderivative function graph from x=1 to x=2 (Fig. 8f)*].
94	Amir	Here the slope is decreasing. [*points to the function graph*]. Let's say one (*Fig.* 8g).
95	Muner	No! the slope here is not the matter. You should consider the points values.
96	Amir	Ahh, the points value
97	Muner	Here what is it value? Let us say, six [*drags the function graph to anchor its endpoint at a point whose y-value is six*]. Thus, the slope here is six [*pointing to the antiderivative function graph*]. Do not consider large intervals, take small intervals on the x-axis.

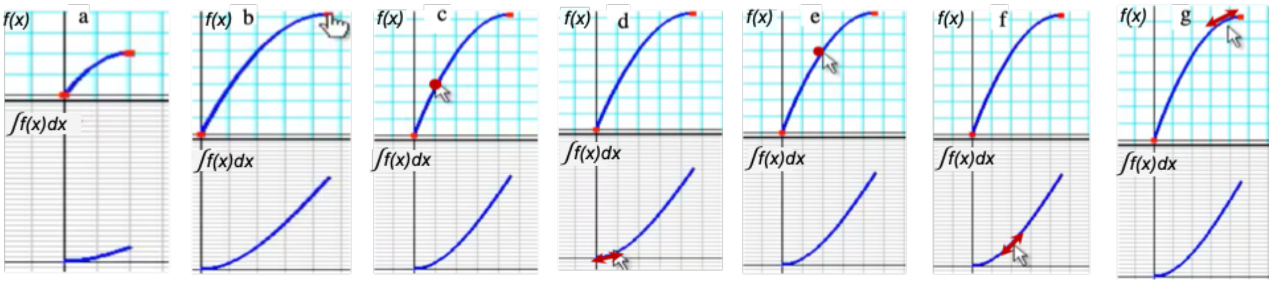

Fig. 8 Exploration by the students of (**a,b**) the relationship between the function and its antiderivative. (**c-f**) Realization by the students of the correlation between points in the function graph and the tangent slope of the antiderivative function graph. (**g**) Students draw a virtual segment on the function graph.

After observing the screen for a relatively long interval (15 sec), Muner directs his classmate's attention to the relationship between the two graphs by showing him how values of specific points in the function graph correlate with slopes of the tangents to the antiderivative graph (lines 91-93). In line 94, Amir refers to the function graph in an attempt to globally describe its tangent slope. Muner, who has just disclosed that the y-values in the function graph represent the tangent slope of the antiderivative function graph, clarifies to Amir that they must consider the y-value of the function graph but *not* the tangent slope of it. Although Amir declares that they should look at the y-value (line 96), Muner continues his explanation and poses a rhetorical question for his classmate in an effort to clarify how the two graphs are connected. Note that after working through the preceding phases of disclosure, in phase 4 the students ask fewer questions than they did in the previous phases (1 question vs. 10). Additionally, the sentences uttered by the students in phase 4 possess a distinctly explanatory modality compared with the inquiring modality of their earlier utterances. (Note that the question in line 97 is rhetorical, since Muner already knows an answer: "let us say six".)

7. Discussion

7.1 An emerging three-layer model

Based on our analysis of the activities performed by the two students and in accordance with Rota's account that disclosure is rooted in different layers (Rota, 1991), the findings show that the students' disclosure process, in which they made sense of the concept of the antiderivative in the CIS environment, occurred via *three layers of meaning*.

The first layer consists of *disclosing objects* and entails processes of making sense of each of the graphs in isolation. In the timeline analysis, this layer is connected to the components, 'local', 'properties', and 'kind of graph'. The student's attention in this layer is focused on the type of the function, specific points on the graph, and properties of a graph. The students express these notions according to their respective levels of knowledge: for instance, they 'see' the first function 'as' a "linear function parallel to the x-axis" (line 1). The sequence of icons and the dragging functions of the CIS tool facilitate this disclosure.

The second layer of meaning refers to *disclosing relationships*, i.e., making sense of the relationships between the graphs and the covariation between changes in the upper function graph and the slope of the lower function graph (the antiderivative). This layer corresponds to the 'relationship' component in the timeline analysis. The shift from the first to the second layer of meaning relied critically on

- *the tool design*, which *links and displays the function graph and its antiderivative function graph in the same interface*, thereby enabling the students to disclose any invariant phenomenon among the elements on the screen that do vary, for example, the linearity of the antiderivative function when dragging the constant function graph (lines 2-5);
- the *interrogative process* in which the students engaged and specifically, the *why, how* and *what* questions that they asked.

In asking and answering *why* questions, the students effectively justify the relationships between the function values and the corresponding tangent slope of the antiderivative function graph (lines 5 and 67). *How* questions prompt the students to shift from focusing on the graph globally to focusing on its specific features (see lines 31-32 and 80). This shift is central to the learning process, as it redirects the students' attention to the second layer of meaning, namely, the disclosure of the relationships between the mathematical objects (see lines 31-35, in which the students focus on the slope values of the antiderivative graph.) Indeed, their exploration of *how* questions also causes the students to work to explain the relationships between the two graphs (see lines 31-32 and 80). Also, their efforts to address *how* questions lead the students to ask *what* questions. The interrogative process fostered by their investigation of *what* questions, in turn, deepens the students' inquiry further still by zooming their focus to the disclosure of specific relationships between points on the function graphs and the corresponding slopes on the antiderivative function graphs. (See lines 32, 35 regarding the slope values in different points.)

The transition between the first and second layers of meaning is neither unidirectional nor smooth. Indeed, the two students alternately shift between them. Moreover, each new function graph that is generated prompts the students' disclosure of the characteristics of the obtained graph in a process that we consider to be part of the first layer of meaning (e.g., lines 1, 31, 80). As soon as the students disclose and familiarize themselves with the characteristics of the graph, they shift to the second layer of meaning (lines 4-5, 33-34, 82-83) and struggle to determine the relationship between the two graphs. They repeat this cycle with each pair of functions: when they generate a new graph with which they are not familiar, they typically revert to the first layer while using the *function graph generation* icons. In addition, note that the order in which the set of available icons is displayed in the CIS tool strongly influences the order of the types of function graphs that are explored by the students. This order, in turn, prompts the students to fluctuate between the first two layers of meaning.

Once the students establish each of the first two layers of meaning and cease to fluctuate between them during the disclosure process, the third layer of meaning begins to emerge. The third layer, which we did not anticipate, transcends the specificity of the graphic examples and entails the process of grasping a *general* relationship between a function and its antiderivatives, i.e., as *relationships within a set of functions*. This occurs when Muner reflects on the function and its antiderivatives and explains his insight to his classmate (lines 91-97). This third layer of disclosure allows students not only to grasp the *functionality* of the two classes of objects in the CIS context – that is, how the changes in the graph on one of the windows affects what is displayed on the other window – but also to see such mutual dependencies within a general setting as a network of mutual relationships (see quotation from Rota about sense-making in § 2.1). Accordingly, we name the third layer, *disclosing functional relationships.* In our case, they concern function – antiderivative relationships.

Here we must comment on our choice to distinguish three layers of disclosure and four phases in the students' cognitive processes. In the example of Amir and Muner, the second and third phases correspond essentially to the emergence of the students' understanding of the mathematical relationship between the upper and lower function graphs. The students' newfound understanding of this mathematical relationship developed gradually via two important realizations they have while using the digital tool. First, when using the tool's second icon, the students recognize the correspondence between the slope of the lower graph and the y-value of the upper graph . Then, when using the tool's fourth and sixth icons, which change the function's concavity in the upper screen (first the functions with a concave graph, then those with a convex graph), they become aware that the lower graph is the derivative of the upper function. The students' use of the tool's different icons helps them make sense of each upper graph vis-à-vis its matching lower graph and vice versa. Namely, they are able *to see it as* the derivative of that on the other screen and conversely. It is exactly the *mathematical sense* of this relationship between the phenomena depicted by the two screens that constitutes the essence of the third layer.

Stated generally, the layers of meaning indicate how making sense of mathematical concepts emerges from the specific interpretative processes employed by the students because of their interactions with the different situations they produce and observe on the two screens. This scenario resembles that when a child is given problems of partition and quotition to interpret (Correa, Nunes, & Bryant, 1998) and then makes sense of the division concept and algorithms while understanding quotition and partition. Also, in this case the child elaborates on the mathematical meaning possibly in two phases (e.g., in the first phase, when situations of partition are seen as related to division; in the second phase, when situations of quotition are seen to be related to division), but the two phases occur in the same layer of meaning (that of interpreting particular situations as related to division).

As the discussion shows, the roles played by the specific features of the digital tool and the interrogative process are intimately and necessarily intertwined with the students' progressive disclosure processes. For this reason, completely disentangling the roles played by each element in the students' processes of disclosure is virtually impossible. The disclosure process unifies, cognitively and epistemically, the interrogative process and the tool's affordances in the context of the problem, such that the interrogative process is triggered by the tool's affordances. Moreover, it allows students to 'navigate' within the two first layers and to arrive to the third layer at the end of learning experiment. The phenomenological perspective of disclosure differs markedly from the common 'platonic' picture of 'abstraction from the concrete' that is assumed by many other approaches: disclosing different aspects of mathematical objects and of the relationships between them is in fact a non-linear and unending process.

Finally, we note two key-processes in the students' progressive disclosure of the mathematical meanings: *focusing attention on what is present* on the screen, and *making present what is absent*. The first constitutes an almost obvious motivation for engaging students in using digital tools in mathematical explorations; however, the second, more significantly, is involved in shifting from the first layer (objects) to the second layer of disclosure (relationships). In the case of Amir and Muner, the most evident case of this second process is the "staircase shape", which is made present with the pointer and plays a role in the disclosure of the function graph as the derivative graph. The educational function of virtual figures created by the students' gestures, such as Muner's "staircase shape," has been highlighted in other studies showing how this feature contributed to exploration and argumentation processes both in geometry (Chen and Herbst, 2013) and in calculus (Yoon, Thomas, & Dreyfus, 2011). In our case, the combination of the virtual figures with those actually visible on the screen forms a new landscape in which the students can dynamically explore the mathematical concepts embedded in the function graphs.

7.2 Limitations and implications

The four main phases of disclosure were identified by analyzing 11 pairs of students. The interplay between the action with the tool and the distribution of types of questions were identified in a case study that involved only one pair of students. Clearly, then, we do not claim that our results can be generalized on a statistical basis, but rather, on an analytical basis. In fact, the layered structure by which mathematical ideas will be disclosed in stages through suitable inquires, as we have described in this paper, has both epistemic and cognitive features with didactical consequences. . In this regard, it resonates with other proposals for layered structures in mathematics learning described by other authors, albeit working within different frameworks and with their own language. To give an example, Thompson & Carlson's chapter for the *Compendium of Research in Mathematics Education* (J. Cai, 2017, p. 421 ff.), discusses a "foundational way of thinking mathematically" involving aspects of variation and covariation in the historical development of functions and in the development of students' mathematical understanding. They describe the evolution of covariational meaning and reasoning (that is according to an epistemological and cognitive lens) using a rubric elaborated in an ongoing research by the two authors and others. In their paper of 2017 they use a classification that "a researcher could use […] to describe a class of behaviors, or she could use […] as a characteristic of a person's capacity to reason variationally or covariationally. As a descriptor of a class of behaviors, individuals at various levels of sophistication could exhibit behavior that a framework level describes". For this they introduce levels of variational (Table 13.3, p. 440) and covariational (Table 13.4, p. 441) reasoning by which the understanding of functions extends and unfolds. In this way, it bears a striking resemblance to our own description of the disclosure of a layered structure. This is also true with respect to our phenomenological approach, at the center of which was the use of inquiry methods. For Thompson and Carlson underline the role of the teacher in this general inquiry method, based on suitably questioning the students and focusing their attention on appropriate features of the situations to which they are exposed.

The particular similarity between our work and Thompson and Carlson's is surely not by chance since the language of variation and covariation in investigating the relationship between functions and anti-derivative functions is almost unavoidable, as we have seen in the Muner's insights in section 6.2.1 above. On this basis, we intend to extend our research using the phenomenological framework of layers' disclosure to study the way students develop their variational and covariational reasoning while they are building mathematical models of physical phenomena. A first step towards this direction has been done by one of the authors in Arzarello (2019).

Of great relevance, in our view, is also our contribution to the discussion on how mathematical conceptual development may evolve in students when it is supported by a digital tool in combination with interrogative processes.

In addition to this theoretical contribution, the study also has pedagogical implications for the teaching-learning of mathematical concepts. The use of digital tools may promote, via a graphic approach, the students' learning of the indefinite integral concept. Based on the findings of our study, we argue that the dynamicity and the interactivity of the digital tool allowed the students not only to freely choose the functions they wanted to investigate, but also to vary them, thereby helping them focus their attention on specific aspects of the phenomenon they were exploring.

The layers of meaning model should be exploited and further studied as a tool for planning educational activities and classroom discussions in a wide range of mathematical topics, and modes of thinking about mathematical objects as in the case of covariational reasoning as we have already mentioned. Prompting students to familiarize themselves with the different layers of meaning can be a methodological tool wielded by teachers via their actions in the classroom. For example, in our case the third layer of meaning (sense of the relationships) is certainly the most challenging for the student to grasp. Thus, here the teacher can provide crucial support of the students' efforts to achieve mathematical disclosure, possibly by drawing the students' attention to their own intuitions while they interact with the digital tool. In this sense, the layers model may be complementary to existing models that frame the teacher's role during classroom discussion in artifact-based educational settings, such as the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008). In addition, our findings suggest that the teacher's intervention may also be needed during the students' exploratory activities with the CIS tool to promote the emergence of the second layer of meaning (relationships) and its dynamic interplay with the first (objects) and to foster the interplay between these two layers and the third (sense of the relationships).

References

Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, *70*(2), 97-109.

Arzarello, F., & Robutti, O. (2010). Multimodality in multi-representational environments. *ZDM-The International Journal on Mathematics Education*, *42*(7), 715-731.

Arzarello, F., Ascari, M., Baldovino, C., & Sabena, C. (2011). The teacher's activity under a phenomenological lens. In U. Behiye (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 49-56). Ankara, Turkey: PME.

Arzarello, F. (2019). La covariación instrumentada: un fenómeno de mediación semiótica y epistemológica. *Proceeding of the XV CIAEM, Conferencia Interamericana de Educación Matemática* (Plenary Lecture). Medellin, Colombia.

Bartolini Bussi, M.G. & Mariotti, M.A. (2008). Semiotic mediation in the mathematical classroom. Artefacts and signs after a Vygotskian perspective. In L. English, (Ed.), *Handbook of international research in mathematics education* (2nd revised ed., pp. 746-783). Mahwah, NJ: Lawrence Erlbaum Associates.

Boaler, J., & Humphreys, C. (2005). *Connecting mathematical ideas: Middle school video cases to support teaching and learning*. Portsmouth, NH: Heinemann.

Cai, J. (Editor) (2017). *Compendium of Research in Mathematics Education.* Reston, VA: National Council of Teachers of Mathematics.

Chen, C. L., & Herbst, P. (2013). The interplay among gestures, discourse, and diagrams in students' geometrical reasoning. *Educational Studies in Mathematics*, *83*(2), 285-307.

Correa, J., Nunes, T., & Bryant, P. (1998). Young children's understanding of division: The relationship between division terms in a noncomputational task. *Journal of Educational Psychology, 90*(2), 321-329.

de Finetti, B. (1967). *Il "saper vedere" in matematica* [The art of seeing in mathematics]. Torino, Italy: Loescher.

Drijvers, P. (2015). Digital technology in mathematics education: Why it works (or doesn't). In: S.J. Cho (Ed.), *Selected Regular Lectures from the 12th International Congress on Mathematical Education* (pp. 135-151). Cham, Switzerland: Springer.

Hintikka, J. (1999). The role of logic in argumentation. In J. Hintikka (Ed.), *Inquiry as inquiry: A logic of* *scientific discovery* (pp. 25–46). Dordrecht, Netherlands: Springer.

Mason, J. (2008). Being mathematical with & in front of learners: Attention, awareness, and attitude as sources of differences between teacher educators, teachers & learners. In T. Wood & B. Jaworski (Eds.), *International handbook of mathematics teacher education* (Vol. 4, pp. 31-56). Rotterdam, Netherlands: Sense Publishers.

Mason J. (2014). Questioning in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 513-518). Dordrecht, Netherlands: Springer.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 215–234). Rotterdam, Netherlands: Sense Publishers.

Radford, L. (2010). The eye as a theoretician: seeing structures in generalizing activities. *For the Learning of Mathematics, 30*(2), 2-7.

Rota, G. C. (1991). *The end of objectivity: a series of lectures delivered at M.I.T in October, 1973*. Oxford, England: TRUEXpress.

Schwartz, J., & Yerushalmy, M. (1995). On the need for a bridging language for mathematical modeling. *For the Learning of Mathematics*, *15*(2), 29–35.

Shternberg, B., Yerushalmy, M., & Zilber, A. (2004). The Calculus Integral Sketcher [Computer software]. Tel-Aviv, Israel: Center of Educational Technology.

Swidan, O., & Yerushalmy, M. (2014). Learning the indefinite integral in a dynamic and interactive technological environment. *ZDM-The International Journal on Mathematics Education*, *46*(4), 517–531.

Swidan, O., & Yerushalmy, M. (2016). Conceptual structure of the accumulation function in an interactive and multiple-linked representational environment. *International Journal of Research in Undergraduate Mathematics Education*, *2*(1), 30–58.

Tall, D. (2010). A sensible approach to the calculus. http://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot2010a-sensible-calculus.pdf. Accessed 12 June 2019.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. *Compendium of Research in Mathematics Education* (pp. 421-456). Reston, VA: National Council of Teachers of Mathematics.

Weiland, I. S., Hudson, R. A., & Amador, J. M. (2014). Preservice formative assessment interviews: The development of competent questioning. *International Journal of Science and Mathematics Education*, *12*(2), 329-352.

Yerushalmy, M., & Chazan, D. (2008). Technology and curriculum design: The ordering of discontinuities in school algebra. In L. English (Ed.), *Handbook of international research in mathematics education*

(2nd ed., pp. 806-837). New York, NY: Routledge.

Yoon, C., Thomas, M. O. J., & Dreyfus, T. (2011). Grounded blends and mathematical gesture spaces: developing mathematical understandings via gestures. *Educational Studies in Mathematics*, *78*(3), 371-393.