On the Reidemeister spectrum of an Abelian group

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DE GRUYTER Forum Math. 2019; 31(1): 199–214

Research Article

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https://doi.org/10.1515/forum-2017-0184
Received September 3, 2017; revised May 3, 2018

Abstract: The Reidemeister number of an automorphism ϕ of an Abelian group G is calculated by determining the cardinality of the quotient group $G/(\phi - 1_G)(G)$, and the Reidemeister spectrum of G is precisely the set of Reidemeister numbers of the automorphisms of G. In this work we determine the full spectrum of several types of group, paying particular attention to groups of torsion-free rank 1 and to direct sums and products. We show how to make use of strong realization results for Abelian groups to exhibit many groups where the Reidemeister number is infinite for all automorphisms; such groups then possess the so-called R_∞-property. We also answer a query of Dekimpe and Gonçalves by exhibiting an Abelian 2-group which has the R_∞-property.

Keywords: Reidemeister number, Reidemeister spectrum, R_∞-property, rank one group, type

MSC 2010: 20K01, 20K10, 20K15, 20K25, 20K30

Communicated by: Christopher D. Sogge

1 Introduction

If G is a multiplicatively written group, possibly non-commutative, and ϕ is an automorphism of G, then the ϕ-conjugacy class (or twisted conjugacy class) of an element $x \in G$ is the set $\{gx\phi(g^{-1})\}$, and the *Reidemeister number* $R(\phi)$ of ϕ is the cardinality of the set of ϕ-conjugacy classes of G; the *Reidemeister spectrum* $\mathrm{RSpec}(G)$ of G is defined as the set of cardinals $\{R(\phi) : \phi \in \mathrm{Aut}(G)\}$. Note that some authors make no distinction between different infinite cardinals, and for an infinite group, they simply take the spectrum to be those finite cardinals which occur as Reidemeister numbers along with the symbol ∞. Reidemeister numbers are studied in several areas of mathematics: for example, if $f : X \to X$ is a continuous map on the connected compact polyhedron X with universal cover $\bar{X}$, then the Reidemeister number of f is related to the number of lifting classes of f under conjugacy via $\pi_1(X)$; Reidemeister numbers are also crucial invariants in Nielsen fixed point theory. For more details of these connections, we refer to [4] and the references contained therein.

When the group G is an additively written Abelian group, the notion of a Reidemeister number is considerably simplified: if $\phi \in \mathrm{Aut}(G)$, then the elements x, y are in the same ϕ-conjugacy class if and only if $x \equiv y \bmod (\phi - 1_G)(G)$. Consequently, the Reidemeister number of ϕ coincides with the index of the subgroup $(\phi - 1_G)(G)$ in G.

For the rest of this work we will assume that, with the exception of references to automorphism groups, the word group means an additively written Abelian group.

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The inspiration for this paper comes from the work of Dekimpe and Gonçalves [4], where Reidemeister numbers in the context of infinite Abelian groups were studied for the first time. A related concept plays a central role in the discussions in [4]: a group G has the R_∞-property if for every automorphism ϕ of G, the Reidemeister number $R(\phi)$ is infinite. (We remark that this definition also makes sense in the non-commutative setting.)

The present work is divided into a further 4 sections. In Section 2 we introduce a generalisation of the R_∞-property that takes account of our approach utilising the different cardinalities of infinite sets, and we derive some elementary properties that are useful in the subsequent sections.

In Section 3 we focus on the question of determining the Reidemeister spectrum of an arbitrary torsion-free group of rank 1. Surprisingly, these subgroups of the rational group $\mathbb{Q}$ can have very complicated spectra. We supplement our general discussion with some illustrative examples.

In Section 4 direct sums and direct products are considered and although it is difficult to give a complete description of the spectrum of a sum or a product, there are some natural cases where such a complete description may be obtained. In our discussion of the spectrum of an infinite direct product of copies of the group of integers $\mathbb{Z}$, we point out set-theoretic issues which arise and restrict our discussions to situations where we assume the validity of the Generalised Continuum Hypothesis (GCH).

Section 5 is largely devoted to applications of the so-called realization approach, i.e., certain rings are 'realized' as endomorphism rings of infinite Abelian groups. This is an area that has played a central role in Abelian group theory since the 1960s. It was pointed out in [4] that prior to their work no example had appeared in the literature of an Abelian group having the R_∞-property. They exhibited reduced torsion-free groups of cardinality at most $2^{\aleph_0}$ which had the R_∞-property. We extend this, *inter alia*, by exhibiting reduced groups of arbitrarily large cardinality in which the spectrum consists of a single cardinal, the cardinality of the group itself.

Section 5.1 answers a comment made in [4, Remark 3.4]: we exhibit an Abelian 2-group which has the R_∞-property.

Our terminology is largely standard and may be found in the classical volumes of Fuchs [5, 6]; additional material on rank 1 groups pertinent to Section 3 may be found in [1] or [8]. Our notation is in accord with [5, 6], and we reserve the notation $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{P}$ for the group of rational integers, the group of rational numbers and the set of all rational primes, respectively. The notation $\mathrm{End}(G)$, $\mathrm{Aut}(G)$ shall denote the endomorphism ring and the automorphism group, respectively, of the Abelian group G. When we are assuming the validity of the set-theoretical hypothesis (GCH) in a theorem, we insert (GCH) prior to the statement of the theorem.

2 Elementary facts

If G is an additively written Abelian group and ϕ is an automorphism of G, then the *Reidemeister number* of ϕ is just the cardinality of the quotient group $G/(\phi - 1_G)(G)$. It is denoted by $R(\phi)$. Our first result shows that certain automorphisms always produce the same Reidemeister number.

Proposition 2.1. *If G is an arbitrary Abelian group and ϕ is any automorphism of G, then*
(i) $R(\phi) = R(\phi^{-1})$;
(ii) *if ψ is conjugate to ϕ, say $\psi = \alpha^{-1}\phi\alpha$ for some $\alpha \in \mathrm{Aut}(G)$, then $R(\psi) = R(\phi)$;*
(iii) *if α is an automorphism of G and β is a surjective endomorphism of G, then $R(\phi) = |G/\alpha(\phi - 1)\beta(G)|$.*

Proof. Since $\phi(G) = G$, we have $\phi^{-1}(G) = G$, hence $(\phi^{-1} - 1_G)(G) = (\phi^{-1} - 1_G)(\phi(G)) = (\phi - 1_G)(G)$, and so $R(\phi^{-1}) = |G/(\phi^{-1} - 1_G)| = |G/(\phi - 1_G)(G)| = R(\phi)$, as required.

For the second part, observe that $\psi - 1_G = \alpha^{-1}(\phi - 1_G)\alpha$ and so $(\psi - 1_G)(G) = \alpha^{-1}(\phi - 1_G)(G)$. However, since α^{-1} is monic, we have the isomorphism $G/(\phi - 1_G)(G) \cong \alpha^{-1}(G)/\alpha^{-1}(\phi - 1_G)(G)$, thus

$$|G/(\psi - 1_G)(G)| = |G/\alpha^{-1}(\phi - 1_G)(G)| = |\alpha^{-1}(G)/\alpha^{-1}(\phi - 1_G)(G)| = |G/(\phi - 1_G)(G)|,$$

and so $R(\psi) = R(\phi)$.

For the final part, observe that $\alpha(G) = G$ and $\alpha(\phi - 1)\beta(G) = \alpha(\phi - 1)(G)$, since both α, β are surjective. Since α is also monic, we have $G/(\phi - 1)(G) \cong \alpha(G)/\alpha(\phi - 1)(G)$. Hence,

$$R(\phi) = |G/(\phi - 1)(G)| = |\alpha(G)/\alpha(\phi - 1)(G)| = |G/\alpha(\phi - 1)\beta(G)|,$$

as required. □

We define the *Reidemeister spectrum* of an Abelian group G as the set of cardinals $\{R(\phi) : \phi \in \mathrm{Aut}(G)\}$ and denote this spectrum by $\mathrm{RSpec}(G)$. Then a group G is said to have the *R_λ-property* for a cardinal λ if $\lambda \leq \inf \mathrm{RSpec}(G)$; the property corresponding to $\lambda = \aleph_0$ is traditionally called the *R_∞- property* and we shall continue to use this terminology. Notice that the cardinality, $|G|$, of a group G is always a member of $\mathrm{RSpec}(G)$, since $R(1_G) = |G|$. Hence, the Reidemeister spectrum of an infinite group must always contain at least one infinite cardinal. Our first example shows that this minimum requirement may actually be obtained even when the group in question is uncountable; clearly a countable group can have only one infinite cardinal in its Reidemeister spectrum.

Example 2.2. If $G = J_p$, the additive group of p-adic integers, then
(i) $\mathrm{RSpec}(G) = \{p^n\ (n \geq 0); 2^{\aleph_0}\}$ if $p \neq 2$,
(ii) $\mathrm{RSpec}(G) = \{2^n\ (n \geq 1); 2^{\aleph_0}\}$ if $p = 2$.

Proof. In either case the automorphism group of G is just the set of p-adic units acting as scalar multiplication and this group is known to consist of those p-adic integers α having a non-zero entry r_0 in the standard representation $\alpha = r_0 + r_1 p + \cdots + r_n p^n + \cdots$, where for each n, $0 \leq r_n \leq p - 1$. There are three possibilities for α:
(a) $\alpha = 1$,
(b) $\alpha = 1 + r_s p^s + \cdots$, where $r_s \neq 0$,
(c) $\alpha = r_0 + r_s p^s + \cdots$, where $r_s \neq 0$ and $1 < r_0 \leq p - 1$.
Note that case (c) cannot occur if $p = 2$.

In case (a) it is immediate that $R(\alpha) = R(1) = |G|$. In case (b) we have that $\alpha - 1 = p^s u$ for some p-adic unit u and so $(\alpha - 1)(G) = p^s G$, whence $R(\alpha) = |G/p^s G| = |\mathbb{Z}(p^s)| = p^s$.

In the final case (c) we have $\alpha - 1 = (r_0 - 1) + r_s p^s + \cdots$ and $r_0 - 1 \neq 0$ so that $\alpha - 1$ is itself a p-adic unit. Thus, $(\alpha - 1)(G) = G$ and so $R(\alpha) = 1$.

Since case (c) does not occur when $p = 2$, we get the desired result. □

The following elementary result is surprising useful.

Proposition 2.3. *If G is a 2-divisible group, i.e., $G = 2G$, then $1 \in \mathrm{RSpec}(G)$. In particular, if G is a p-group with $p \neq 2$, then $1 \in \mathrm{RSpec}(G)$.*

Proof. The mapping $\phi\colon G \to G$ with $\phi(g) = -g$ is an automorphism of any Abelian group. Furthermore, $(\phi - 1) = -2$ and so $(\phi - 1)(G) = (-2)(G) = G$, since G is 2-divisible. Hence, $R(\phi) = 1$, as required.

The particular case follows since a p-group with $p \neq 2$ is always 2-divisible. □

In the next example we calculate the spectra of divisible groups of the form $\bigoplus_\lambda \mathbb{Q}$ or $\bigoplus_\lambda \mathbb{Z}(p^\infty)$ (with p an arbitrary prime) for any cardinal λ.

Example 2.4. (i) Let $D = D_1 \oplus D_2 \oplus \cdots \oplus D_n$, where all the D_i are isomorphic and D_i is either $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$ for a fixed, but arbitrary prime p. Then $\mathrm{RSpec}(D) = \{1, \aleph_0\}$.
(ii) If $D = \bigoplus_{i<\lambda} D_i$, where λ is an infinite cardinal, all the D_i are isomorphic and D_i is either $\mathbb{Q}$ or $\mathbb{Z}(p^\infty)$ for a fixed, but arbitrary prime p, then $\mathrm{RSpec}(D) = \{1, \mu\aleph_0 : 1 \leq \mu \leq \lambda\}$.

Proof. (i) Let ϕ be an arbitrary automorphism of D, then $(\phi - 1)(D)$ is a divisible subgroup of D and hence there is a direct decomposition $D = (\phi - 1)(D) \oplus X$, where X is either 0 or has rank k $(1 \leq k < n)$. Then $R(\phi) = |X|$ and so $\mathrm{RSpec}(D) \subseteq \{1, \aleph_0\}$. However, as $R(1) = \aleph_0$ and $R(-1) = |D/2D| = 1$, we have equality: $\mathrm{RSpec}(D) = \{1, \aleph_0\}$.

(ii) The first step of the proof is identical to that in part (i) and yields $\mathrm{RSpec}(D) \subseteq \{1, \mu\aleph_0\}$, since the cardinality of a direct complement of $(\phi - 1)(D)$, for an arbitrary automorphism ϕ of D, is precisely $\mu\aleph_0$ with $1 \leq \mu \leq \lambda$. Thus, $\mathrm{RSpec}(D) \subseteq \{1, \mu\aleph_0 : 1 \leq \mu \leq \lambda\}$.

Conversely, as in part (i), $R(-1) = 1$, $R(1) = |D| = \lambda$. Suppose then that κ is an arbitrary cardinal with $\aleph_0 \le \kappa < \lambda$. Write $D = C \oplus Y$, where $|Y| = \kappa$. Now define automorphisms α, β of C and Y, respectively, such that $\beta = 1_Y$ and α is an automorphism of C such that $\alpha - 1$ is a surjective endomorphism of C; this is possible since $\mathbb{Q}$ and each $\mathbb{Z}(p^\infty)$, for any prime p, have this property. Set $\theta = \alpha \oplus \beta$ so that θ is an automorphism of D, and a straightforward calculation gives that $R(\theta) = \kappa$. Hence, we have the desired equality $\mathrm{RSpec}(D) = \{1, \mu\aleph_0 : 1 \le \mu \le \lambda\}$. □

We say the spectrum of G is *trivial* if it consists of just the single cardinal $|G|$; it is said to be $\mathbb{Q}$-*like* if contains just the two cardinals 1 and $|G|$. Thus, homogeneous divisible groups have, by Example 2.4, spectra which are $\mathbb{Q}$-like if and only if they are of rank $\le \aleph_0$. We shall see in Section 5 that there are arbitrary large reduced groups with trivial Reidemeister spectra and also arbitrary large reduced groups with $\mathbb{Q}$-like spectra.

The calculation of the Reidemeister spectrum of a direct sum $A \oplus B$ seems, in general, to be difficult. Certainly $\mathrm{RSpec}(A \oplus B) \supseteq \{\lambda_i\mu_j : i \in I, j \in J\}$, where $\mathrm{RSpec}(A) = \{\lambda_i : i \in I\}$ and $\mathrm{RSpec}(B) = \{\mu_j : j \in J\}$, since if $\alpha \in \mathrm{Aut}(A)$ and $\beta \in \mathrm{Aut}(B)$, then $\alpha \oplus \beta \in \mathrm{Aut}(A \oplus B)$ and $R(\alpha \oplus \beta) = R(\alpha) \cdot R(\beta)$. It follows easily that we get the equality $\mathrm{RSpec}(A \oplus B) = \{\lambda_i\mu_j : i \in I, j \in J\}$ if $\mathrm{Hom}(A, B) = 0 = \mathrm{Hom}(B, A)$. The spectrum may, however, be bigger, as shown in our next example.

Example 2.5. If $G = J_2$, the additive group of 2-adic integers, then $\mathrm{RSpec}(G \oplus G) = \{2^n\ (n \ge 0); 2^{\aleph_0}\}$. In particular, $\mathrm{RSpec}(G \oplus G)$ strictly contains the set of products taken from the set $\{2^n\ (n \ge 1); 2^{\aleph_0}\}$.

Proof. Let $H = G \oplus G$ and let ϕ be an arbitrary automorphism of H. By Proposition 2.1 (iii), the Reidemeister number of ϕ may be calculated as $R(\phi) = |G/\alpha(\phi - 1)\beta(G)|$ for any automorphism α and surjection β of H.

Let $\phi - 1_H$ have matrix representation $\Delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a, b, c, d \in \mathrm{End}(G) = J_2$.

Since $\mathrm{End}(G) = J_2$ is a principal ideal domain, $\phi - 1_H$ can be reduced to Smith normal form, i.e., there exist automorphisms α, β of H such that $\alpha(\phi - 1_h)\beta$ has matrix representations a diagonal matrix of the form $\left(\begin{smallmatrix} \delta & 0 \\ 0 & \epsilon \end{smallmatrix}\right)$, where $\delta, \epsilon \in \{0, 2^n\ (n \ge 0)\}$. Hence, $\alpha(\phi - 1)\beta(H) = \delta(G) \oplus \epsilon(G)$ and so $R(\phi) = |G/\delta(G) \oplus G/\epsilon(G)|$. A straightforward check of the various possibilities gives that $R(\phi)$ belongs to the set $\{2^n\ (n \ge 0); 2^{\aleph_0}\}$, so that $\mathrm{RSpec}(H) \subseteq \{2^n\ (n \ge 0); 2^{\aleph_0}\}$. If $n \ge 2$, then the value 2^n certainly belongs to $\mathrm{RSpec}(H)$, since it can be obtained as a product of the form $\lambda\mu$, where $\lambda, \mu \in \mathrm{RSpec}(G)$.

It suffices to show that $1, 2 \in \mathrm{RSpec}(H)$. Let ϕ be the endomorphism of H having a matrix representation $\left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$; note that ϕ is actually an automorphism of H since its determinant is a unit in J_2. However, $\phi - 1_{G \oplus G}$ then has a matrix representation $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and this too is an automorphism of H. Hence, $R(\phi) = 1$.

Finally, consider the endomorphism ψ of H with matrix representation $\left(\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right)$. This too is an automorphism of H with $(\psi - 1_H)(H) = G \oplus 2G$ and hence $R(\psi) = 2$. □

We can, however, obtain a useful result about the R_λ-property when we have the property that $\mathrm{Hom}(A, B) = 0$.

Proposition 2.6. *If* $\mathrm{Hom}(A, B) = 0$ *and B has the R_λ-property, then $G = A \oplus B$ has the R_λ-property. In particular, if B has the R_∞-property, then so does G.*

Proof. Every endomorphism of G has a matrix representation of the form $\left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \beta \end{smallmatrix}\right)$ for suitable endomorphisms α, β, γ and hence an automorphism of G must be of the form $\left(\begin{smallmatrix} \alpha & \gamma \\ 0 & \beta \end{smallmatrix}\right)$, where $\alpha \in \mathrm{Aut}(A)$, $\beta \in \mathrm{Aut}(B)$.

But then if $\phi \in \mathrm{Aut}(G)$, $\phi - 1 = \left(\begin{smallmatrix} \alpha - 1_A & \gamma \\ 0 & \beta - 1_B \end{smallmatrix}\right)$, so that $(\phi - 1)(G) \subseteq ((\phi - 1_A)(A) + \gamma(B)) \oplus (\beta - 1_B)(B))$.

So, for the quotient, we have

$$\left|\frac{G}{(\phi - 1)(G)}\right| \geqslant \left|\frac{A}{A_1}\right| \cdot \left|\frac{B}{(\beta - 1_B)(B)}\right|,$$

where $A_1 = (\alpha - 1_A)(A) + (\gamma(B))$. Since $|\frac{A}{A_1}| \geqslant 1$, it follows that

$$\left|\frac{G}{(\phi - 1)(G)}\right| \geqslant \left|\frac{B}{(\beta - 1_B)(B)}\right| \geqslant \lambda.$$

The R_∞-property statement is then immediate. □

Proposition 2.7. *Let $G = A \oplus B$ and suppose G has the R_λ-property for some infinite cardinal λ, then one of the summands A or B must have the R_λ-property. In particular, if G has the R_∞-property, then one of A, B must have the R_∞-property.*

Proof. If not, A has an automorphism α with $R(\alpha) < \lambda$ and similarly B has an automorphism β with $R(\beta) < \lambda$. But then the direct sum $\alpha \oplus \beta$ is an automorphism with $R(\alpha \oplus \beta) = R(\alpha) \cdot R(\beta) < \lambda$, a contradiction. The statement for the R_∞-property is then immediate. □

Our next result indicates that the maximal divisible subgroup of a group is of no importance in relation to whether or not a non-divisible group has the R_λ-property.

Corollary 2.8. *An Abelian group G has the R_λ-property if and only if its "reduced part" has the R_λ-property. In particular, a group has the R_∞-property precisely when its "reduced part" has the R_∞-property.*

Proof. ($\Leftarrow$) If $G = d(G) \oplus H$, where $d(G)$ is the divisible part of G and H is reduced, then $\mathrm{Hom}(d(G), H) = 0$, so G has the R_λ-property by Proposition 2.6.

($\Rightarrow$) Conversely, since $d(G)$ does not have the R_λ-property by Proposition 2.3, it follows from Proposition 2.7 that the "reduced part" must have the R_λ-property. □

Our final result in this section is a simple generalisation of the fact that we have used several times relating $R(\alpha \oplus \beta)$ to $R(\alpha) \cdot R(\beta)$.

Proposition 2.9. *If $G = \bigoplus_{i \in I} A_i$ and if $\psi \in \mathrm{Aut}(G)$ leaves each A_i invariant, then $R(\psi) = \prod_{i \in I} R(\psi_i)$ where $\psi_i = \psi \restriction A_i$ is an automorphism of A_i.*

Proof. If $\psi \in \mathrm{Aut} G$, then $\psi = \prod_{i \in I} \psi_i$, where $\psi_i = \psi \restriction A_i$, and so $\psi - 1_G = \bigoplus_{i \in I} (\psi_i - 1_{A_i})$, hence

$$R(\psi) = \left| \frac{G}{(\psi - 1_G)(G)} \right| = \prod_{i \in I} \left| \frac{A_i}{(\psi_i - 1_{A_i})(A_i)} \right| = \prod_{i \in I} R(\psi_i).$$ □

3 Spectra of rational groups

We begin the section by recalling some concepts and notations. Let G be a torsion-free abelian group. Given a prime p, the p-height of $x \in G$, denoted by $h_p^G(x)$, is the largest integer k such that p^k divides x in G; if no such maximal integer exists, we set $h_p^G(x) = \infty$. Now let $p_1, p_2, \ldots$ be an increasing sequence of all primes. Then the sequence

$$\chi_G(x) = (h_{p_1}^G(x), h_{p_2}^G(x), \ldots, h_{p_n}^G(x), \ldots)$$

is said to be the height-sequence of x. We omit the subscript G if no ambiguity arises. For any two height-sequences $\chi = (k_1, k_2, \ldots, k_n, \ldots)$ and $\mu = (l_1, l_2, \ldots, l_n, \ldots)$, we set $\chi \geq \mu$ if $k_n \geq l_n$ for all n. Moreover, χ and μ will be considered equivalent if $\sum_n |k_n - l_n|$ is finite (we set $\infty - \infty = 0$). An equivalence class of height-sequences is called a type. If $\chi(x)$ belongs to the type $\mathbf{t}$, then we say that x is of type $\mathbf{t}$. For more details, see [5, 6]. Now suppose that σ is a type, let $\mathcal{I}(\sigma)$ denote the set of primes corresponding to an entry ∞ in σ and let $\mathcal{F}(\sigma)$ denote the primes corresponding to finite entries in σ; clearly, $\mathbb{P} = \mathcal{I}(\sigma) \,\dot\cup\, \mathcal{F}(\sigma)$ and if there is no danger of ambiguity, we shall refer to these sets of primes simply as $\mathcal{I}$, $\mathcal{F}$. If e is a positive integer, let e_σ denote the factor of e obtained by deleting all prime factors p with $p \in \mathcal{I}(\sigma)$; thus the only primes occurring in the primary decomposition of e_σ correspond to entries in σ which are finite. We shall refer to e_σ as the σ-factor of e. The significant feature of the σ-factor that we will utilise is contained in the following proposition.

Proposition 3.1. *If G is of rank 1 and type σ, then for each integer e, the quotient $G/eG = G/e_\sigma G$ is cyclic of order e_σ.*

Proof. See, for example, [1, Theorem 1.4] or [8, Corollary 2.5.1]. □

Suppose that G is of rank 1 and type $\sigma = (\ell_1, \ell_2, \ldots)$, and $\mathbb{P} = \mathcal{I} \cup \mathcal{F}$. Suppose further G is not divisible so that $\mathcal{F} \neq \emptyset$. Then $\mathrm{End}(G)$ is a rank 1 ring with type equal to the reduced type of G, i.e., $t(\mathrm{End}(G)) = (m_1, m_2, \ldots)$, where $m_i = \infty$ if m_i corresponds to a prime in $\mathcal{I}$ but is 0 otherwise.

Hence, an element of $\mathrm{Aut}(G)$ is either ± 1 or it is $\pm \prod_{p_i \in \mathcal{I}_1} p_i^{\alpha_i} / \prod_{p_j \in \mathcal{I}_2} p_j^{\beta_j}$, where α_i, β_j are non-zero integers and I_1, I_2 are disjoint (possibly empty) finite subsets of $\mathcal{I}$.

In the case where the automorphism ϕ is ± 1, we see that $|\phi - 1| = 0$ or 2 and so $R(\phi) = \aleph_0$ or $|G/2G|$. The σ-factor of 2 is equal to 1 if $2 \in \mathfrak{I}$ and is equal to 2 otherwise; hence it follows from Proposition 3.1 that $R(\pm 1) = \{1, \aleph_0\}$ or $\{2, \aleph_0\}$.

In the remaining case we set $\psi = \prod_{p_i \in \mathfrak{I}_1} p_i^{\alpha_i}$ and $\theta = \prod_{p_j \in \mathfrak{I}_2} p_j^{\beta_j}$, so that $\phi = \pm\psi/\theta$. Then we have $|\phi - 1| = |\pm\psi - \theta|/\theta$. Notice that θ is actually an automorphism of G, so $G = \theta(G)$ and hence we get the simplification that $(\phi - 1)(G) = (\psi \pm \theta)(G)$. Unfortunately, the integers $\psi \pm \theta$ may be divisible by primes which belong to $\mathfrak{I}$ and so each of these terms must be replaced by its corresponding σ-factor. It follows from Proposition 3.1 that the entries in the Reidemeister spectrum arising in this second case are precisely the integers $(\psi + \theta)_\sigma$ and $(\psi - \theta)_\sigma$, where $\psi = \prod_{\mathfrak{I}_1} p_i^{\alpha_i}$, $\theta = \prod_{\mathfrak{I}_2} p_j^{\beta_j}$ and $\mathfrak{I}_1, \mathfrak{I}_2$ are disjoint subsets of $\mathfrak{I}$.

We record this as follows.

Theorem 3.2. *If G is a rank 1 torsion-free group of type σ, then*

$$\operatorname{RSpec}(G) = \begin{cases} \{(\psi + \theta)_\sigma, (\psi - \theta)_\sigma\} \cup \{1, \aleph_0\} & \text{if } 2 \in \mathfrak{I}(\sigma), \\ \{(\psi + \theta)_\sigma, (\psi - \theta)_\sigma\} \cup \{2, \aleph_0\} & \text{if } 2 \notin \mathfrak{I}(\sigma), \end{cases}$$

where $\psi = \prod_{\mathfrak{I}_1} p_i^{\alpha_i}$, $\theta = \prod_{\mathfrak{I}_2} p_j^{\beta_j}$ and $\mathfrak{I}_1, \mathfrak{I}_2$ are disjoint (possibly empty) finite subsets of $\mathfrak{I}$.

The above description is, unfortunately, rather complicated, since the number-theoretic consequences are not easily calculated. Note also that $\mathfrak{I}_1$ or $\mathfrak{I}_2$ empty means that the corresponding term ψ or θ is simply equal to 1. We present now specific examples when the set $\mathfrak{I}(\sigma)$ is simple.

Example 3.3. Let G be a rank 1 group of type σ and suppose that $\mathfrak{I}(\sigma) = \{p\}$. Then

$$\operatorname{RSpec}(G) = \begin{cases} \{p^m + 1, p^m - 1 \text{ where } m \geq 1\} \cup \{1, \aleph_0\} & \text{if } 2 \in \mathfrak{I}(\sigma), \\ \{p^m + 1, p^m - 1 \text{ where } m \geq 1\} \cup \{2, \aleph_0\} & \text{if } 2 \notin \mathfrak{I}(\sigma). \end{cases}$$

Proof. As $\mathfrak{I}(\sigma)$ consists of a single prime, the automorphisms of G are simply of the form $\pm p^m$, where m may be positive or negative. However, as we have seen in Proposition 2.1, the Reidemeister number of an automorphism is equal to that of its inverse, so it suffices to consider the situation where m is positive. The result then follows immediately from the general discussion above. □

We note that a similar result was already obtained in [4].

To simplify notation, for our next result for a positive integer n, we shall write n_p, n_q for the factors of n obtained by dividing by the highest power of p, respectively q, that occurs in the decomposition of n.

Example 3.4. Let G be a rank 1 group of type σ and suppose that $\mathfrak{I}(\sigma) = \{p, q\}$. Then

$$\operatorname{RSpec}(G) = \begin{cases} \{(p^m \pm 1)_q, (q^r \pm 1)_p, p^m q^r \pm 1, |p^m \pm q^r|, \text{ where } m, r \geq 1\} \cup \{1, \aleph_0\} & \text{if } 2 \in \{p, q\}, \\ \{(p^m \pm 1)_q, (q^r \pm 1)_p, p^m q^r \pm 1, |p^m \pm q^r|, \text{ where } m, r \geq 1\} \cup \{2, \aleph_0\} & \text{if } 2 \notin \{p, q\}. \end{cases}$$

Proof. We give a detailed proof in the hope of making the general situation clearer. Given that G has exactly two primes corresponding to an entry ∞ in the type σ, it follows that every automorphism ϕ of G is of one of the following forms:

(i) $\pm p^{\pm n}$, where $n \geq 0$,
(ii) $\pm q^{\pm m}$, where $m \geq 1$,
(iii) $\pm p^n q^m$ or $\pm 1/p^n q^m$, where $n, m \geq 1$,
(iv) $\pm p^n/q^m$ or $\pm q^m/p^n$, where $n, m \geq 1$.

Observe that in each of the cases (i)–(iv) we have automorphisms which are inverse to each other, so, since inverse automorphisms have the same Reidemeister number, we can reduce to consideration of the following cases:

$$\text{(a) } \phi = \pm 1, \quad \text{(b) } \phi = \pm p^n, \quad \text{(c) } \phi = \pm q^m, \quad \text{(d) } \phi = \pm p^n q^m \quad \text{and} \quad \text{(e) } \phi = \pm p^n/q^m,$$

where in all cases $n, m \geq 1$.

In case (a) we again get $R(\phi) = |G| = \aleph_0$ or $R(\phi) = 1$ if $2 \in \{p, q\}$ and $R(\phi) = 2$ if $2 \notin \{p, q\}$; in case (b) $|\phi - 1| = p^n \pm 1$ and so we need to determine the order of the quotients $G/(p^n \pm 1)(G)$. Note that $(p^n \pm 1, p) = 1$

but it is possible that q may divide $p^n \pm 1$. Consequently, $(p^n \pm 1)(G) = (p^n \pm 1)_q(G)$, since $qG = G$. By Proposition 3.1, we deduce that $(p^n \pm 1)_q(G)$ has index in G, precisely, $(p^n \pm 1)_q$ and so $(p^n \pm 1)_q \in \mathrm{RSpec}(G)$ for all $n \geq 1$. Case (c) follows in an identical way to case (b) yielding $(q^m \pm 1)_p \in \mathrm{RSpec}(G)$.

In case (d) $|\phi \pm 1|$ is either $p^n q^m - 1$ or $p^n q^m + 1$ and since these integers are relatively prime to both p and q, an easy argument using Proposition 3.1 gives that $p^n q^m \pm 1 \in \mathrm{RSpec}(G)$.

In the final case (e) we note that $|\phi - 1| = |(\pm p^n - q^m)/q^m|$. However, as G is q-divisible, the expression $(|\phi - 1|)(G)$ reduces to $(|p^n \pm q^m|)(G)$. Furthermore, neither p nor q can divide $|p^n \pm q^m|$ and so the index of $(|p^n \pm q^m|)(G)$ in G is precisely $|p^n \pm q^m|$.

Combining the outcomes of cases (a)–(e), we obtain the desired result. □

The calculation of the actual numbers arising in an example of the type illustrated by Example 3.4 is straightforward in theory but numbers become large very quickly; for example, if G is the rank 1 group where $\mathcal{I}(\sigma) = \{2, 3\}$, then the first twenty elements of $\mathrm{RSpec}(G)$ are 1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, 35, 37, 41, 43, 47, 49, 53, 55, 59, but by the time one reaches the 130th entry in the spectrum, the value has increased to 1015. Furthermore, it is not easy to determine whether or not a given number belongs to the spectrum. We illustrate this further in our next example.

Example 3.5. If G is the rank 1 group of type σ where $\mathcal{I}(\sigma) = \{3, 5\}$, then the first twenty elements of $\mathrm{RSpec}(G)$ are $\{2, 4, 8, 14, 16, 22, 26, 28, 32, 34, 44, 46, 52, 56, 74, 76, 82, 86, 98, 104\}$, but the number 2^{2k} does not belong to the spectrum if k is an odd integer and $k \geq 3$.

Proof. The numerical calculation is straightforward but the argument to show that the appropriate powers of 2 cannot occur is more number-theoretic in nature; note that the requirement that the power of 2 be odd is necessary since $2^7 = 128 = 5^3 + 3$. Furthermore, the requirement that k be odd follows from the fact that $2^4 = (3 \times 5) + 1$. Exactly as in Example 3.4, we need to show that if k is odd and $k \geq 3$, then there are no integer solutions to the following equations:

(1) the 5-factor $(3^m \pm 1)_5 = 2^{2k}$,
(2) the 3-factor $(5^m \pm 1)_3 = 2^{2k}$,
(3) $3^m 5^n \pm 1 = 2^{2k}$,
(4) $5^r - 3^s = \pm 2^{2k}$,
(5) $5^r + 3^s = \pm 2^{2k}$.

We remark at the outset that the conditions imposed on k are not needed in every case; we will use the appropriate conditions in each case and draw the final conclusion by incorporating all the component parts.

Case (1): The equation $(3^m \pm 1)_5 = 2^{2k}$ has no integer solutions if $k \geq 3$. Suppose, for a contradiction that $3^m \pm 1 = 5^r 2^{2k}$, where $k \geq 3$. Then reducing modulo 16, we get that $3^m \pm 1 \equiv 0 \bmod 16$. Looking at the residues of the powers of 3 modulo 16, we see that the only possible solutions occur if $m = 4n$ for some integer n and we choose the negative value in the expression ± 1; note also that $n = 0$ is not possible since then $m = 0$ and so $1 \pm 1 = 5^r 2^{2k} \geq 64$, a contradiction.

Thus, we have $3^{4n} - 1 = 5^r 2^{2k}$, so that $(3^{2n} - 1)(3^{2n} + 1) = 5^r 2^{2k}$, and we see that $(3^{2n} - 1) = 5^s 2^x$, $(3^{2n} + 1) = 5^t 2^y$, where $s + t = r$, $x + y = 2k$. However, if both $s, t \geq 1$, subtracting would lead to the contradiction that $5 | 2$. So at least one of s, t equals zero.

Suppose that $t = 0$; then $3^{2n} + 1 = 2^y$. If $y = 2, 3, \ldots$, then the left-hand side of the last equation is congruent to $2 \bmod 4$, while the right-hand side is congruent to $0 \bmod 4$, a contradiction; if $y = 0$ or 1, we are also led to the contradiction that $3^{2n} + 1 = 1$ or 2, where $n > 0$.

Finally, suppose that $s = 0$ so that $3^{2n} - 1 = 2^x$. Factorizing, we get that $3^n - 1 = 2^\alpha$, $3^n + 1 = 2^\beta$, where $\alpha + \beta = x$ and $\alpha < \beta$. Hence, $2^\beta = 2^\alpha + 2$, which is impossible if $\alpha = 0$ or if $\alpha > 1$; hence $\alpha = 1$, $\beta = 2$ and so $x = 3$. Thus, we must have $3^{2n} - 1 = 8$ so that $n = 1$, whence $m = 4$. But $3^4 - 1 = 80 = 5 \cdot 2^4$, which implies $k = 2$, a contradiction. Thus, case (1) is complete.

Case (2): The equation $(5^m \pm 1)_3 = 2^{2k}$ has no integer solutions if $k \geq 2$. The proof for case (2) is essentially identical to, and indeed a little simpler than that for case (1) and is left to the reader.

Case (3): The equation $3^m 5^n \pm 1 = 2^{2k}$ has no integer solutions if k is odd and $k \geq 3$. We separate the proof into two cases corresponding to the values 1 and -1.

(a) Suppose $3^m 5^n + 1 = 2^{2k}$. Then $3^m 5^n = (2^k - 1)(2^k + 1)$ so that $2^k - 1 = 3^x 5^u$, $2^k + 1 = 3^y 5^v$, where $x + y = m$, $u + v = n$. Now if $1 \leq x < m$, then $y \geq 1$, leading to the contradiction that $3|2$; if $x = 0$, then $2^k - 1 = 5^u$, $2^k + 1 = 3^m 5^v$, which forces $v = 0$ for otherwise $5|2$. If $x = m$, then $2^k - 1 = 3^m 5^u$, $2^k + 1 = 5^v$, and if $v \neq n$, then we have the contradiction that $5|2$. So we are left in the situation where either

$$2^k - 1 = 5^n, \quad 2^k + 1 = 3^m, \tag{1}$$

or

$$2^k - 1 = 3^m, \quad 2^k + 1 = 5^n. \tag{2}$$

Looking at equation (1), we see that $2^k \equiv 1 \bmod 5$, but this is only possible if k is even, a contradiction. Similarly from equation (2), we have that $2^k \equiv 4 \bmod 5$, which can happen only if k is of the form $2 + 4z$ for some integer z; in particular, k is again even, a contradiction. Thus, in subcase (a) there are no solutions.

(b) Now suppose that $3^m 5^n - 1 = 2^{2k}$; notice that we may assume that $m, n > 0$ for otherwise the situation is handled by cases 1 and 2 above, since we are assuming $k \geq 3$. Reducing modulo 8, we have that $3^m 5^n \equiv 1 \bmod 8$ and this can only happen if both m, n are even, say $m = 2x$, $n = 2y$. Set $z = 3^x 5^y$ and note that $x, y > 0$. Thus, we need to find solutions of the equation $z^2 - 1 = 2^{2k}$ and we deduce that $z - 1 = 2^u$, $z + 1 = 2^v$, where $u + v = 2k$ and $u < v$. If $u \geq 2$, we immediately get the contradiction that $4|(z - 1) + (z + 1) = 2$. If $u = 0$, then $3^x 5^y - 1 = 2^u = 1$ and this is clearly impossible since $x, y > 0$. Finally, if $u = 1$, we have $3^x 5^y - 1 = 2$ from which it follows that $x = 1$, $y = 0$, a contradiction, since $y > 0$. So in subcase (b) there are also no solutions and this completes case (3).

Case (4): The equation $5^r - 3^s = \pm 2^{2k}$ has no integer solutions if k is odd and $k \geq 3$. Reducing modulo 8, we see that $5^r \bmod 8 = 3^s \bmod 8$, which implies that both r, s are even, say $r = 2t$, $s = 2u$. For convenience, we consider the positive and negative possibilities separately.

(c) Suppose that $5^{2t} - 3^{2u} = 2^{2k}$ and $k \geq 3$ with k odd. By rearranging the equation and factorizing, we get $(5^t - 2^k)(5^t + 2^k) = 3^{2u}$, so that $5^t - 2^k = 3^\alpha$, $5^t + 2^k = 3^\beta$ and $\alpha + \beta = 2u$, $\alpha < \beta$. As before, $\alpha \geq 1$ is impossible and so $\alpha = 0$. Thus, we have $5^t = 1 + 2^k$ and so $t \neq 0$. However, since k is odd, $2^k \equiv 2 \bmod 5$ or $2^k \equiv 3 \bmod 5$, which leads to the contradiction that either 3 or 4 is congruent to 0 mod 5. So there are no solutions corresponding to subcase (c).

(d) Suppose that $5^{2t} - 3^{2u} = -2^{2k}$ with $k \geq 3$ and k odd. Analogously to the situation in subcase (c), we get $(3^u - 2^k)(3^u + 2^k) = 5^{2t}$, so that $3^u - 2^k = 5^\alpha$, $3^u + 2^k = 5^\beta$, where $\alpha + \beta = 2t$, $\alpha < \beta$. As in subcase (c), this immediately reduces to $\alpha = 0$, so that $3^u = 1 + 2^k$ and so $5^\beta = 5^{2t} = 1 + 2^k + 2^k = 1 + 2^{k+1}$. Since $k \geq 3$ with k odd, we have that $2^{2k} = 2^{2r}$ for some integer $r \geq 2$. Thus, $5^{2t} = 1 + 2^{2r}$, where $r \geq 2$ and this is impossible by a special case of case (2) above. Hence, there are no solutions to subcase (d) and so there are no solutions in case (4).

Case (5): The equation $5^r + 3^s = 2^{2k}$ has no integer solutions if $k \geq 3$ and k is odd. Note that we may assume that both $r, s > 0$; if either is zero, the equation has no integer solutions by cases 1 and 2, respectively, while $r = s = 0$ is impossible since $k \geq 3$. Reducing modulo 3, we have that $5^r \bmod 3 = 2^{2k} \bmod 3$, and since the latter term is congruent to 1 mod 3, we must have that r is even.

On the other hand, if we reduce modulo 8, we get that $5^r \bmod 8 + 3^s \bmod 8 \equiv 0 \bmod 8$. Since 5^r is congruent to either 5 or 1 modulo 8 and 3^s is congruent to either 3 or 1 modulo 8, the previous equation can only be satisfied by choosing the residues to 5 and 3, respectively. But then r must be odd, a contradiction. Hence, there are no integer solutions in case (5).

Taking all the cases together, we get the desired result that for the group G of type σ, where $\mathcal{I}(\sigma) = \{3, 5\}$, the integers 2^{2k}, $k \geq 3$ and k odd, do not belong to the Reidemeister spectrum $\mathrm{RSpec}(G)$. □

We consider one final example which is the exact opposite of the situation considered in Example 3.3. Recall that $\mathbb{Q}^{(q)_{q \in \mathbb{P} \setminus \{p\}}}$ is a rank 1 group in which every prime different from p is a unit, i.e., $\mathbb{Q}^{(q)_{q \in \mathbb{P} \setminus \{p\}}}$ is of type σ with $\mathcal{I}(\sigma) = \mathbb{P} \setminus \{p\}$.

Example 3.6. Let $G = \mathbb{Q}^{(q)_{q\in\mathbb{P}\setminus\{p\}}}$. Then

$$\mathrm{RSpec}(G) = \begin{cases} \{1, p, p^2, \dots; \aleph_0\} & \text{if } p \neq 2, \\ \{2, 2^2, \dots; \aleph_0\} & \text{if } p = 2. \end{cases}$$

Proof. Apply Theorem 3.2 to this situation and observe that the Reidemeister numbers that occur arise from terms of the form ± 1 or $\psi \pm \theta$, where ψ, θ are products of primes different from p.

Observe firstly that if $\phi = 1 + p^n$, where $n \geq 1$, then ϕ is an automorphism of G and it follows easily from Proposition 3.1 that $R(\phi) = p^n$. Hence, $\{p^n : n \geq 1\} \subseteq \mathrm{RSpec}(G)$. Furthermore, the choice $\phi = 1_G$ shows that $\aleph_0 \in \mathrm{RSpec}(G)$. If $p \neq 2$ and $\phi = -1$, then $\phi - 1$ is an automorphism of G and hence $1 \in \mathrm{RSpec}(G)$ in this case. In summary, we have

$$\mathrm{RSpec}(G) \supseteq \begin{cases} \{1, p, p^2, \dots; \aleph_0\} & \text{if } p \neq 2, \\ \{2, 2^2, \dots; \aleph_0\} & \text{if } p = 2. \end{cases}$$

We now derive the reverse inclusion. When $p \neq 2$, the expressions $|\psi \pm \theta|_\sigma$ are then of the form p^n, where $n \geq 0$ and so the possible values of the Reidemeister numbers of G are contained in this set along with the Reidemeister numbers arising from the automorphisms ± 1, which as we have seen above yield the values $\aleph_0$ and 1, respectively. So, for $p \neq 2$, we have the desired equality $\mathrm{RSpec}(G) = \{1, p, p^2, \dots; \aleph_0\}$.

When $p = 2$ however, the expressions $|\psi \pm \theta|$ are necessarily even and so $|\psi \pm \theta|_\sigma$ can only take the form 2^n, where $n \geq 1$. Furthermore, since the automorphisms ± 1 yield Reidemeister numbers $\aleph_0$, 2, in this case, we have $\mathrm{RSpec}(G) \subseteq \{2, 2^2, \dots; \aleph_0\}$, which gives us the desired result. □

4 Sums and products

In this section we consider direct sums and products of groups focusing in the first section on the homogeneous situation, i.e., where we consider sums and products of a single fixed group and here we are able in some situations to determine the complete spectrum of the sum or product; we remark that infinite products produce set-theoretic difficulties which are somewhat outside the scope of the main interest of this paper. In the second part our results are similar but somewhat more general than those in the corresponding section of [4].

4.1 Homogeneous sums and products

We make the following ad hoc definition.

Definition 4.1. A unit u in a ring R is said to be *strong* if $u - 1$ is also a unit in R.

Note that the identity is never strong. We can characterize strong units as follows.

Proposition 4.2. *The following are equivalent for any unital ring R:*
(i) *R has a strong unit,*
(ii) *R has a unit which is the sum of two units,*
(iii) *every unit of R is a sum of two units.*

Proof. The implication (i) ⇒ (ii) is trivial. Suppose then that (ii) holds for some unit $u = v + w$, where u, v, w are all units of R. If x is an arbitrary unit of R, then $x = (xu)u^{-1} = (xv + xw)u^{-1} = xvu^{-1} + xwu^{-1}$ and each of the terms in the sum is a unit of R, so (iii) holds.

Finally, suppose that (iii) holds and $x = a + b$, with a, b units of R. Then it follows immediately that each of xa^{-1}, xb^{-1}, $a^{-1}x$ and $b^{-1}x$ is a strong unit, so that (i) holds. □

In the context of Reidemeister numbers of an Abelian group G, the significance of a strong unit in the endomorphism ring $\mathrm{End}(G)$ is clear: the existence of a strong unit in $\mathrm{End}(G)$ suffices to show that $1 \in \mathrm{RSpec}(G)$.

We remark that such existence is not, however, necessary: if $D = \mathbb{Z}(2^\infty)$ is the Prüfer quasi-cyclic 2-group, then $\operatorname{End}(D) = J_2$ and if α is an automorphism of D, then $\alpha - 1$ is necessarily of the form 2θ for some endomorphism θ of D, hence α is not strong. However, if we choose $\alpha = 1 + 2 + \cdots \in J_2$, then $\alpha - 1$ is a surjective endomorphism of D and so $R(\alpha) = 1$.

We can deduce immediately the following, which we have previously established in Proposition 2.3.

Proposition 4.3. *If G is an Abelian p-group with $p > 2$, then G has an automorphism ϕ with $R(\phi) = 1$ and consequently G does not have the R_∞-property.*

Proof. Since $p > 2$, multiplication by $p - 1$ is a strong unit in $\operatorname{End}(G)$. □

The presence of strong units in matrix rings is well known and easily established, see, for example, [4, Proposition 3.5]. We sketch the proof for completeness.

Proposition 4.4. *If R is an arbitrary unital ring, then the matrix ring $M_n(R)$ of all $n \times n$ matrices over R has a strong unit for all integers $n \geq 2$.*

Proof. If $n = 2$, let $B = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and note that B is invertible with inverse $\left[\begin{smallmatrix} 1 & -1 \\ -1 & 2 \end{smallmatrix}\right]$. Furthermore, $B - I = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ is also invertible with inverse $\left[\begin{smallmatrix} 0 & 1 \\ 1 & -1 \end{smallmatrix}\right]$. So B is a strong unit in $M_2(R)$.

If $n = 3$, the matrix

$$D = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \text{ is invertible with inverse } \begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

and

$$D - I = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \text{ is invertible with inverse } \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Hence, D is strong unit in $M_3(R)$.

Now suppose that $n = 3r$. Then the block diagonal matrix $\operatorname{diag}(D_1, D_2, \ldots D_r)$, where each $D_i = D$, is a strong unit in $M_{3r}(R)$.

If $n = 3r + 2$, then the block diadonal matrix $\operatorname{diag}(B, D_1, D_2, \ldots D_r)$, where each $D_i = D$, is a strong unit in $M_{3r+2}(R)$.

Finally, if $n = 3r + 1$, then the block diadonal matrix $\operatorname{diag}(B, B, D_1, D_2, \ldots D_{r-1})$, where each $D_i = D$, is a strong unit in $M_{3r+1}(R)$. □

Corollary 4.5. *If A is a fixed Abelian group and if $G = A^{(\kappa)}$, $H = A^\kappa$ are, respectively, the direct sum and direct product of κ copies of A, for any (finite or infinite) cardinal $\kappa \geq 2$, then G and H both have automorphisms which are strong units in their respective endomorphism rings. In particular, $1 \in \operatorname{RSpec}(G) \cap \operatorname{RSpec}(H)$.*

Proof. If κ is finite, then $G = H$ and $\operatorname{End}(G) = M_\kappa(R)$, where $R = \operatorname{End}(A)$ and so the result follows from the previous proposition. If κ is infinite, then $\operatorname{End}(G) \cong M_2(R)$ and $\operatorname{End}(H) \cong M_2(R_1)$, where $R = \operatorname{End}(G)$ and $R_1 = \operatorname{End}(H)$, and hence the result again follows from the previous proposition. □

We can, however, say somewhat more in certain situations. For convenience, let us write $[1, \kappa]$ for the set of all cardinals μ such that $1 \leq \mu \leq \kappa$.

Theorem 4.6. *If S_κ is the free group of rank κ, for some infinite cardinal κ, then* $\operatorname{RSpec}(S_\kappa) = [1, \kappa]$.

Proof. Observe that $\operatorname{RSpec}(S_\kappa) \subseteq [1, \kappa]$. We show the reverse inclusion.

Suppose firstly that r is an integer with $r \geq 1$. Write $S_\kappa = S_1 \oplus S_2$, where $S_1 = \mathbb{Z} \oplus \mathbb{Z}$ and $S_2 \cong S_\kappa$. Now define a mapping $\phi\colon S_\kappa \to S_\kappa$ by $\phi = \phi_1 \oplus \phi_2$, where ϕ_2 is an automorphism which is a strong unit in $\operatorname{End}(S_2)$ and ϕ_1 has a matrix representation $\left[\begin{smallmatrix} r+1 & 1 \\ r & 1 \end{smallmatrix}\right]$. Since ϕ_1 is an automorphism of S_1, the mapping ϕ is certainly an automorphism of S_κ. Furthermore, the subgroup $(\phi - 1_{S_\kappa})(S_\kappa)$ has index in S_κ equal to that of $(\phi_1 - 1_{S_1})(S_1)$ in S_1. A direct calculation shows that $(\phi_1 - 1_{S_1})(S_1) = \{(rx + y, rx) \mid x, y \in \mathbb{Z}\}$ and this is easily seen to be precisely $\mathbb{Z} \oplus r\mathbb{Z}$, so that $R(\phi) = r$.

Now suppose that μ is an arbitrary infinite cardinal in $[1, \kappa]$. Again we write $S_\kappa = S_1 \oplus S_2$, where $S_1 = \bigoplus_\mu \mathbb{Z}$ and $S_2 \cong S_\kappa$; this is easily arranged. Now consider the map $\psi = 1_{S_1} \oplus \theta$, where θ is a strong unit in $\mathrm{End}(S_2)$. Clearly, $|S_\kappa/(\psi - 1_{S_\kappa})(S_\kappa)| = |S_1| = \mu$, whence $R(\psi) = \mu$. Thus, we get the desired result that $\mathrm{RSpec}(S_\kappa) = [1, \kappa]$. □

The situation for direct products is vastly more complicated due to the fact that cardinal exponentiation occurs in calculating the cardinality of a direct product, which means that many set-theoretical issues arise. Since these would represent a major diversion from the main thrust of this work, we content ourselves with the simplest type of case and assume the validity of the Generalised Continuum Hypothesis (GCH).

Theorem 4.7 (GCH). *If $n \geq 0$ is an integer and $P = \prod_{\aleph_n} \mathbb{Z}$, then* $\mathrm{RSpec}(P) = \{1, 2, \dots; \aleph_0, \aleph_1, \dots, \aleph_{n+1}\}$.

Proof. Since we are assuming (GCH), $2^{\aleph_k} = \aleph_{k+1} = (\aleph_k)^+$ for each $k \geq 0$ and it is clear that $\mathrm{RSpec}(P) \subseteq \{1, 2, \dots; \aleph_0, \aleph_1, \dots \aleph_{n+1}\}$. To show that every integer $r \geq 1$ occurs in the spectrum, we follow an identical argument to that used in the proof of Theorem 4.6 above. Clearly, $R(1_P) = \aleph_{n+1}$, so it suffices to show the existence of an automorphism having Reidemeister number equal to $\aleph_k$ for $k \geq 0$.

Set $P = \mathbb{Z} \oplus P_1$, where $P_1 \cong P$, and choose a strong unit $\phi_1 \in \mathrm{End}(P_1)$. Then if $\phi = 1 \oplus \phi_1$, it is clear that ϕ is an automorphism of P and $(\phi - 1_P)(P) = P_1$, whence $R(\phi) = |\mathbb{Z}| = \aleph_0$, as required. If $\kappa \in \{\aleph_1, \dots \aleph_n\}$, set $\kappa^- = \aleph_{k-1}$ for $\kappa = \aleph_k$. Now decompose $P = \prod_{\kappa^-} \mathbb{Z} \oplus P_1$, where $P_1 \cong P$. Arguing exactly as in the case where $P = \mathbb{Z} \oplus P_1$ above, we find an automorphism ϕ of P with $R(\phi) = |P/P_1| = |\prod_{\kappa^-} \mathbb{Z}| = \kappa$. □

We now consider a somewhat more complex situation and for convenience we restrict our attention to groups of at most countable rank.

Let $G = A^{(\alpha)}$, where $2 \leq \alpha \leq \aleph_0$, be a homogeneous completely decomposable group of type σ which is not divisible, and as in Section 3, let $\mathcal{I} = \mathcal{I}(\sigma)$ denote the set of primes corresponding to an entry ∞ in σ and let $\mathcal{F} = \mathcal{F}(\sigma)$ denote the primes corresponding to finite entries in σ; by assumption, $\mathcal{F} \neq \emptyset$.

Clearly, the spectrum of G is contained in the set $[1, \aleph_0]$. If an integer d lies in the spectrum and d is divisible by some prime $p \in \mathcal{I}$, then G has a homomorphic image, C say, which is a cyclic group of order p. This is impossible since G is p-divisible for all $p \in \mathcal{I}$ and hence any homomorphic image is also p-divisible. Since C is q-divisible for all primes $q \neq p$, it would be a cyclic divisible p-group, a contradiction. Thus, $\mathrm{RSpec}(G) \subseteq \{1, d, \aleph_0 \mid (d, p) = 1 \text{ for all} p \in \mathcal{I}\}$.

Conversely, we must show that every such value occurs as the Reidemeister number of some automorphism of G; since $R(1_G) = \aleph_0$, we may restrict our attention to the finite values. So let d be an integer ≥ 1 with the property that $(d, p_i) = 1$ for all primes $p_i \in I$. We consider three cases: (i) $\alpha = 2$, (ii) $\alpha = 3$, (iii) $\alpha \geq 4$.

Case (i): Let $\Delta = \left[\begin{smallmatrix} d+1 & 1 \\ d & 1 \end{smallmatrix}\right]$, so that Δ represents an automorphism of G. Exactly as in the proof of Theorem 4.6, $\Delta - 1_G$ has index in G equal to $|A/dA| = d$, since $(d, p_i) = 1$ for all primes p_i, $(i \in I)$. Hence, $R(\Delta) = d$.

Case (ii): Let

$$\Delta = \begin{bmatrix} d+1 & 1 & 1 \\ d & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}.$$

Then Δ is invertible and

$$\Delta - 1_G = \begin{bmatrix} d & 1 & 1 \\ d & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

and a straightforward check shows that the image $(\Delta - 1_G)(G)$ has index in G equal to $|A/dA| = d$, so that $R(\Delta) = d$.

Case (iii): If $\alpha \geq 4$, write $G = G_1 \oplus G_2$, where G_1 is the direct sum of two copies of A and G_2 is the direct sum of $\alpha - 2 \geq 2$ copies of A (where $\aleph_0 - 2 = \aleph_0$). Consider the automorphism $\phi = \phi_1 \oplus \phi_2$, where ϕ_2 is a strong unit in $\mathrm{End}(G_2)$ and ϕ_1 has a matrix representation equal to that of Δ in case (i). A straightforward calculation gives that $R(\phi) = d$.

Hence, we have established the following theorem.

Theorem 4.8. *If $G = A^{(\alpha)}$, where $2 \le \alpha \le \aleph_0$, is a homogeneous completely decomposable group of type σ which is not divisible and $\mathcal{I}$ denotes the set of primes corresponding to an entry ∞ in σ, then*

$$\mathrm{RSpec}(G) = \{1, d, \aleph_0 \mid d \text{ is a positive integer and } (d, p_i) = 1 \text{ for all } p_i \in \mathcal{I}\}.$$

4.2 Non-homogeneous sums and products

We now switch our attention to situations that are not homogeneous, focusing primarily on the existence of automorphisms with Reidemeister number 1; such groups clearly fail to have the R_∞-property. By assuming the existence of surjections between the various groups in the direct sum or product we are enabled to make progress.

Proposition 4.9. *Suppose $\{A_i\}$, $(i \ge 1)$ is a family of groups such that for each $i > 1$, there is a surjection $\pi_i : A_i \twoheadrightarrow A_{i-1}$. Then the group $G = \bigoplus_{i=1}^{\infty} A_i$ has an automorphism φ with $R(\varphi) = 1$.*

Proof. Define a homomorphism $\pi : G \to G$ by setting $\pi(a_1) = 0$ for all $a_1 \in A_1$ and $\pi(a_i) = \pi_i(a_i)$ for all $a_i \in A_i$. Note that π is a surjection $G \to G$ (but not an isomorphism). We claim that $\varphi = 1 - \pi$ is an automorphism of G. To see this note that π is locally nilpotent: if $x \in A_i$, then $\pi^i(x) = 0$, and so the formal series $\psi = 1 + \pi + \pi^2 + \cdots$ is a well-defined endomorphism of G. Clearly, $\varphi\psi = 1_G = \psi\varphi$ and so φ is invertible as claimed. Furthermore, $\varphi - 1 = -\pi$ is surjective and hence $R(\varphi) = 1$. □

Corollary 4.10. *If $G = \bigoplus_{i=1}^{N} A_i$ for a finite integer N and there is a surjection $\pi_i : A_i \to A_{i-1}$ for each i, then G has an automorphism φ with $R(\varphi) = |A_N|$.*

Proof. Taking π as in Proposition 4.9, we have $\pi(G) = A_1 \oplus \cdots \oplus A_{N-1}$ and π is nilpotent since $\pi^N(A_i) = 0$ for all i. So $\varphi = 1 - \pi$ has an inverse $1 + \pi + \cdots + \pi^{N-1}$ and $\mathrm{Im}(\varphi - 1) = \mathrm{Im}(-\pi) = \mathrm{Im}(\pi) = A_1 \oplus \cdots \oplus A_{N-1}$. So $|\frac{G}{(\varphi - 1_G)(G)}| = |A_N|$, as claimed and so $R(\varphi) = |A_N|$. □

We have immediately from Proposition 4.9 that certain unbounded direct sums of cyclic p-groups (for arbitrary primes p) fail to have the R_∞-property.

Corollary 4.11. *If $G = \bigoplus_{i \ge 1} B_i$, where each B_i is a homocyclic p-group of finite rank and of exponent n_i with $n_1 < n_2 < \cdots$, then G has an automorphism φ with $R(\varphi) = 1$.*

Proof. Let $B_i = \bigoplus_{k=1}^{m_i} \langle e_{ik} \rangle$, where each m_i is a finite integer. Clearly, for each r with $2 \le r \le m_i$, there is a surjection from $\langle e_{ir} \rangle \twoheadrightarrow \langle e_{i(r-1)} \rangle$. Furthermore, for each $i > 1$, there is a 'connecting' surjection from $\langle e_{i1} \rangle \twoheadrightarrow \langle e_{i-1(m_{i-1})} \rangle$. Listing the groups $\langle e_{ik} \rangle$ as a sequence, it is easy to see that the conditions of Proposition 4.9 hold and hence G has an automorphism ϕ with $R(\phi) = 1$. □

Note that in light of Proposition 4.3, the previous result is really only significant when $p = 2$.

In light of Corollary 4.5 and Proposition 2.9 we can show the following corollary.

Corollary 4.12. *If $G = \bigoplus_{i \ge 1} B_i$, where each B_i is a non-trivial homocyclic p-group of exponent n_i, with $n_1 < n_2 < \cdots$, then G has an automorphism φ with $R(\varphi) = 1$.*

Proof. Split $G = C \oplus D$ with $C = \bigoplus B_i$ when $rk(B_i)$ is infinite and $D = \bigoplus B_i$ for those B_i where $rk(B_i)$ is finite. We consider two cases: (i) D is unbounded and (ii) D is bounded.

In the first case, as D is unbounded, the conditions of Corollary 4.11 hold and so D has an automorphism ψ with $R(\psi) = 1$. Moreover, by Corollary 4.5 each homocyclic component of C has an automorphism φ_i with $R(\varphi_i) = 1$. If $\varphi : C \to C$ is the automorphism $\bigoplus \varphi_i$, then the automorphism α of G, given by $\alpha = \varphi \oplus \psi$, has $R(\alpha) = 1$ by Proposition 2.9.

In case (ii), as D is bounded there is an integer N such that for all $i > N$, B_i has infinite rank. We further split D as follows: for each $i > N$, let $B_i = \langle e_i \rangle \oplus B_i'$, where $\langle e_i \rangle$ is a cyclic group of exponent n_i. Now set $D' = D \oplus \bigoplus_{i > N} \langle e_i \rangle$ and observe, by Corollary 4.11, that now D' has an automorphism ψ with $R(\psi) = 1$. Furthermore, D' is a summand of G with a complement consisting entirely of homocyclic groups of infinite rank. Now the final part of the argument in case (i) can be applied to give an automorphism α of G with $R(\alpha) = 1$. □

It is also possible to derive a result similar to Corollary 4.12 by replacing the group G with its torsion completion $\bar{G}$. (Note that $\bar{G}$ can be thought of as the torsion subgroup of the direct product of the groups B_n in Corollary 4.12.)

The key idea required is a generalization of Proposition 4.9.

Proposition 4.13. *Let $B = \bigoplus_{i=1}^{\infty} B_{n_i}$, where each B_{n_i} is a nonzero cyclic p-group of exponent p^{n_i}, with $n_1 < n_2 < \cdots$, and let $\bar{B}$ denote the torsion completion of B. Then $\bar{B}$ has an automorphism $\bar{\varphi}$ with $R(\bar{\varphi}) = 1$.*

Proof. Fix generators e_i for each cyclic subgroup B_{n_1} and let π denote the left Bernouilli shift on B, i.e., the mapping which sends the generator e_1 to zero and the generator e_i to the generator e_{i-1} for each $i \geq 2$. Define $\varphi\colon B \to B$ by $\varphi = 1_B - \pi$ and note that φ is then an automorphism of B.

Let $\bar{\pi}$ denote the unique extension of π to an endomorphism of $\bar{B}$. Then $\bar{\varphi} = 1_{\bar{B}} - \bar{\pi}$ is an extension of φ and is an automorphism since φ is an automorphism of B, see Leptin's result [6, Theorem 69.1]. Then $\bar{\varphi} - 1_{\bar{B}} = \bar{\pi}$ and, by a continuity argument, $\bar{\pi}(\bar{B}) = \overline{\pi(B)} = \bar{B}$, so $\bar{\pi}$ is surjective. Hence, $R(\bar{\varphi}) = 1$. □

An application of the idea used in the proof of Corollary 4.12 yields the following useful theorem.

Theorem 4.14. *If B is an unbounded direct sum of cyclic p-groups and $\bar{B}$ is its torsion completion, then $\bar{B}$ has an automorphism φ with $R(\varphi) = 1$.*

Our final result in this section is an extension to certain products of a result given in [4, Proposition 3.7] for products of cyclic 2-groups; the argument used in the proof is similar to that in [4].

Proposition 4.15. *Let $\{A_i : 1 \leq i < \infty\}$ be a family of groups such that for each $i \geq 2$, there exists a surjection $\rho_i\colon A_i \to A_{i-1}$. Then the group $G = \prod_{i=1}^{\infty} A_i$ has an automorphism φ such that $R(\varphi) = 1$.*

Proof. Define $\pi\colon G \to G$ by $\pi(a_1, \dots, a_n, \dots) = (\rho_2(a_2), \dots, \rho_n(a_n), \rho_{n+1}(a_{n+1}), \dots)$. There is no ambiguity then in letting $\pi^2 : G \to G$ be the composition $\pi \circ \pi$.

Now define $\chi\colon G \to G$ by

$$\chi(x_1, x_2, x_3, \dots) = (\pi(x_2) + \pi^2(x_3), \pi(x_3), \pi(x_4) + \pi^2(x_5), \pi(x_5), \dots).$$

We claim χ is an onto mapping. To see this, note that we can recursively solve the necessary equations arising from $\tilde{y} = (y_1, y_2, \dots, y_n, \dots) = \chi(x_1, x_2, \dots)$, where $\tilde{y}$ is an arbitrary element of G. For example, x_3 is easily obtained since $\pi_3(x_3) = y_2$, and from this, x_2 can be obtained, and then repeating this pattern, one can find all the required x_i $(i \geq 2)$ and the choice of x_1 is arbitrary.

Now set $\varphi = 1 + \chi$ so that $\varphi(a_1, a_2, \dots, a_n, \dots) = (a_1 + \pi(a_2) + \pi^2(a_3), a_2 + \pi(a_3), \dots)$. We claim that φ is an automorphism of G; clearly, if we show this, then the desired result $R(\varphi) = 1$ follows immediately from the fact that χ is surjective.

Suppose that $\tilde{x} = (x_1, x_2, \dots) \in \operatorname{Ker}\varphi$. Then

$$(0, 0, 0, \dots) = (x_1 + \pi(x_2) + \pi^2(x_3), x_2 + \pi(x_3), x_3 + \pi(x_4) + \pi^2(x_5), x_4 + \pi(x_5), \dots).$$

Taking the first 2 co-ordinates, we get the equations

$$0 = x_1 + \pi(x_2) + \pi^2(x_3), \quad 0 = x_2 + \pi(x_3).$$

The second equation implies $\pi(x_2) + \pi^2(x_3) = 0$ whence $x_1 = 0$. Similarly, from the second pair of equations we get $x_3 = 0$ and this immediately implies $x_2 = 0$, as $\pi(x_3) = 0$ also. Repeating in this way, we deduce $\operatorname{Ker}\varphi = \{0\}$.

Finally, suppose $(y_1, y_2, \dots)$ is an arbitrary element of G. We shall show that φ is surjective by solving $(y_1, y_2, \dots, y_n, \dots) = \varphi((x_1, \dots, x_n), \dots)$.

Again, working in pairs, we must solve

$$y_1 = x_1 + \pi(x_2) + \pi^2(x_3), \quad y_2 = x_2 + \pi(x_3).$$

Taking the image of y_2 under π, we deduce $y_1 - \pi(y_2) = x_1$. Similarly, working with the second pair of co-ordinates, we get $x_3 = y_3 - \pi(y_4)$ and so $x_2 = y_2 - \pi(y_3) + \pi^2(y_4)$. Repeating in this way, we can find a solution $(x_1, \dots, x_n, \dots)$ of the equation $\varphi((x_1, \dots, x_n, \dots)) = (y_1, \dots, y_n, \dots)$ and so φ is onto as required. □

5 Use of realization theorems

In this final section we utilize some results that realize rings as endomorphism rings of Abelian groups to calculate Reidemeister spectra. We consider firstly the situation in relation to torsion-free groups. The fundamental result in this area is the celebrated theorem of Corner [2] for countable rings.

Theorem 5.1. *Every countable reduced torsion-free ring is the endomorphism ring of a countable reduced torsion-free group.*

Applying this result of Corner to the ring $A = \mathbb{Z}[X]$ of polynomials over the integers, we find a countable reduced torsion-free group G with $\mathrm{End}(G) = A$. Furthermore, it follows from the proof of Corner's theorem that the group G is a pure subgroup of the $\mathbb{Z}$-adic completion of A and it contains A as a pure subgroup: $A \leq_* G \leq_* \hat{A}$. Since $\mathrm{Aut}(G) = \pm 1$, we have trivially that $R(1) = \aleph_0$ while $|G/-2(G)| \geq |(A + 2G)/2G|$. However, it follows from the purity of A in G that $A/(A \cap 2G) \cong A/2A \cong \mathbb{Z}_2[X]$ and so $R(-1) = \aleph_0$ also; hence $\mathrm{RSpec}(G) = \{\aleph_0\}$ in this case. In fact, Corner's theorem is easily modified to show the existence of $2^{\aleph_0}$ groups G_i with $\mathrm{End}(G_i) = A$ but $\mathrm{Hom}(G_i, G_j) = 0$ for all pairs $i \neq j$. We record this as follows.

Theorem 5.2. *There exists a family of $2^{\aleph_0}$ countable reduced torsion-free groups $\{G_i\}$ such that for each i, $\mathrm{RSpec}(G_i) = \{\aleph_0\}$ and for each pair $i \neq j$, $\mathrm{Hom}(G_i, G_j) = 0$. In particular, each group G_i has a trivial spectrum and has the R_∞-property.*

Corner's theorem can be generalized to a much wider class of rings (see, for example, [7, Chapter 12]) but the same fundamentals occur in the construction. We shall make use of the following simplified version of [7, Theorem 12.3.12].

Theorem 5.3. *Let λ be an infinite cardinal, then there exists a family $\{G_i\}$ of $2^{\lambda^{\aleph_0}}$ reduced torsion-free groups G of cardinality $\lambda^{\aleph_0}$ such that*
(i) *$B \leq_* G_i \leq_* \hat{B}$, where B is free of rank λ and $\hat{B}$ is the $\mathbb{Z}$-adic completion of B,*
(ii) $\mathrm{End}(G_i) = \mathbb{Z}$,
(iii) *for each pair $i \neq j$,* $\mathrm{Hom}(G_i, G_j) = 0$.

If we choose a group G coming from Theorem 5.3, then it is again immediate that G has but two automorphisms, 1 and -1. Trivially, $R(1) = |G| = \lambda^{\aleph_0} \geq \lambda$. An identical argument to that used before Theorem 5.2 shows that $|G/2G| \geq |B/2B| = \lambda$. Thus, we have the following theorem.

Theorem 5.4. *There exist, for each infinite cardinal λ, a family of $2^{\lambda^{\aleph_0}}$ reduced torsion-free groups $\{G_i\}$ each having the R_λ-property and* $\mathrm{Hom}(G_i, G_j) = 0$ *for every pair $i \neq j$.*

The groups arising in Theorem 5.3 have just two automorphisms but clearly they are not examples of groups with $\mathbb{Q}$-like spectra since they satisfy the R_∞-property. Nor is it clear that they have trivial spectra. However, if we impose the restriction that we work with cardinals λ satisfying the equality $\lambda^{\aleph_0} = \lambda$, then we can achieve the desired result. We mention that the collection of cardinals satisfying this condition actually forms a proper class, for example, if $\lambda = \mu^{\aleph_0}$ for an arbitrary cardinal μ, then $\lambda^{\aleph_0} = \lambda$.

We record this as follows.

Theorem 5.5. *There exists, for each infinite cardinal λ satisfying $\lambda^{\aleph_0} = \lambda$, a family of 2^λ reduced torsion-free groups $\{G_i\}$ each having trivial Reidemeister spectrum and* $\mathrm{Hom}(G_i, G_j) = 0$ *for every pair $i \neq j$.*

Our next example provides reduced groups of arbitrary large cardinality which have $\mathbb{Q}$-like spectra. Again we restrict to cardinals λ satisfying $\lambda^{\aleph_0} = \lambda$.

Theorem 5.6. *If λ is an infinite cardinal with the property that $\lambda^{\aleph_0} = \lambda$, then there is a reduced torsion-free group G of cardinality λ with $\mathbb{Q}$-like Reidemeister spectrum,* $\mathrm{RSpec}(G) = \{1, \lambda\}$.

Proof. We use a simple variant of Theorem 5.3 above, realizing the ring $\mathbb{Q}^{(2)}$ of rationals having denominators a power of 2 as endomorphism ring rather than $\mathbb{Z}$; note that this immediately gives us the fact that the group is reduced. Exactly as before, the construction of such a group G is obtained in such a way that G contains a pure subgroup B of the form $\bigoplus_\lambda \mathbb{Q}^{(2)}$. Note that since $|\mathbb{Q}^{(2)}/k\mathbb{Q}^{(2)}| \neq 1$ if the integer k is not, up to sign, a power of 2, it follows that $|B/kB| = \lambda$ if $k = 0$ or $|k| \neq 2^n$ for some integer $n > 0$. Furthermore, as $\lambda \geq |G/kG| \geq |(B + kG)/kG| = |B/B \cap kG| = |B/kB|$, we have that $|G/kG| = \lambda$ if $k = 0$ or $|k| \neq 2^n$ for some integer $n > 0$.

Let ϕ be an arbitrary automorphism of G. Then it is well known that the units of $\mathbb{Q}^{(2)}$ are precisely the rationals of the form $\pm 2^n$ for $n \in \mathbb{Z}$. Since, as observed in Proposition 2.1, $R(\phi) = R(\phi^{-1})$, we may restrict our discussion to the cases $\phi = \pm 2^n$, $n \geq 0$.

If ϕ is of the form $\phi = \pm 2^n$ with $n > 1$, then $|\phi - 1|$ is not a power of 2 and so, as observed above, $R(\phi) = \lambda$. The remaining cases consist of the four possibilities that $\phi \in \{1, -1, 2, -2\}$, or equivalently that $\phi - 1 \in \{0, -2, 1, -3\}$. A straightforward calculation then gives that $R(\phi) = \lambda, 1, 1, \lambda$, respectively, since $|G/2G| = 1$ and $|G/3G| = \lambda$.

Thus, $\mathrm{RSpec}(G) = \{1, \lambda\}$ and G has a $\mathbb{Q}$-like Reidemeister spectrum, as required. □

5.1 Torsion groups

We now turn our attention to torsion groups and specifically to 2-groups. In [4, Remark 3.4], Dekimpe and Gonçalves remarked that they did not know of an Abelian 2-group which has the R_∞-property. We now show how to produce a large family of such groups. We will make use of an old result of Pierce [9] and some generalizations due to Corner [3].

Recall that a p-group G is said to be a *standard Pierce p-group* if

(i) G has a standard basic subgroup $B = \bigoplus_{i=1}^\infty \langle b_i \rangle$, where $o(b_i) = p^i$ and $|G| = 2^{\aleph_0}$,
(ii) $\mathrm{End}(G)$ is the split extension of the ring of p-adic integers J_p by the ideal of small endomorphisms $E_s(G)$.

The terminology arises from the fact that Pierce first established the existence of such a group in his seminal paper [9]; Corner in [3] showed that there exists a family of $2^{2^{\aleph_0}}$ such groups and the only homomorphisms between different members of the family are small homomorphisms.

Assume now that G is a standard Pierce 2-group with basic subgroup $B = \bigoplus_{i=1}^\infty \langle b_i \rangle$, where $o(b_i) = 2^i$ and $|G| = 2^{\aleph_0}$. Let ϕ be an arbitrary automorphism of G. Since all endomorphisms of G are of the form $r + \theta$, it follows that $\phi = r + \theta$, where θ is small and r is a unit of J_2, and so $r = 1 + r_1 2 + r_2 2^2 + \cdots$ for integers $r_i \in \{0, 1\}$. In particular, $r - 1$ is either 0 or of the form $2^k u$ for some $k \geq 1$, with u being a unit in J_2. So $\psi = \phi - 1$ is either θ or $2^k u + \theta$ for some $k \geq 1$ and θ small. If ψ is small, then the quotient $G/\psi(G)$ is of cardinality $2^{\aleph_0}$, since small endomorphisms have countable images. So we may restrict our consideration to the case where $\psi = \phi - 1 = 2^k u + \theta$ for some $k \geq 1$ and θ small.

We claim that $R(\phi)$ is infinite. Suppose, for a contradiction, that $\psi(G)$ is of finite index in G. Then, by a well-known property of p-groups (see, for example, [9, Lemma 16.5]), there is a subgroup X of G with $X \leq \psi(G)$, $|G/X|$ finite and X is a direct summand of G (hence of course of X also); say $G = X \oplus Y$.

Let π denote the projection of G onto X along Y, so that $\pi\psi(G) = X$ is of finite index in G. Since π is an idempotent in $\mathrm{End}(G)$, it must have the form $t + \theta'$, where $t = 0$ or 1.

The first case cannot occur, for then π is small and so $\pi\psi$ is small, whence by standard properties of small homomorphisms (see, for example, [9, §3]), $X = \pi\psi(G) = \pi\psi(B)$ is countable and so X is of uncountable index in G, a contradiction.

So $\pi = 1 + \theta'$ and $\pi\psi = 2^k u + \theta_0$ for some small endomorphism θ_0 of G. It follows from [9, Lemma 16.1 (d)] that $\mathrm{Ker}\, \pi\psi \leq G[2^m]$ for some integer $m \geq k$.

The crucial part of the argument is based on [9, Theorem 16.8 (3)]. Since the basic subgroup B of G is standard and θ_0 is small, there is an $x \in G$ with $h(x) = 0$, $h(\theta_0(x)) > 0$ and $x \in \langle b_{m+1} \rangle \oplus \langle b_{m+2} \rangle \oplus \langle b_{m+3} \rangle \oplus \cdots$. Hence, $\pi\psi(x) = (2^k u + \theta_0)(x)$ has height > 0 but, as $x \in \langle b_{m+1} \rangle \oplus \langle b_{m+2} \rangle \oplus \langle b_{m+3} \rangle \oplus \cdots$, $h(2^m x) = m$. Suppose that $\pi\psi(x) = 2\pi\psi(y)$ for some $y \in G$. Then $x = 2y + z$, where $z \in \mathrm{Ker}\, \pi\psi \leq G[2^m]$, whence $2^m x = 2^{m+1} y$, a contradiction. Since ϕ was arbitrary we have established the following result.

Theorem 5.7. *If G is a standard Pierce 2-group, then G has the R_∞-property.*

Remark 5.8. The Pierce 2-group G above exhibits the interesting property that it is an extension $0 \to B \to G \to D \to 0$ of a group B with $1 \in \mathrm{RSpec}(B)$ by a group D with $1 \in \mathrm{RSpec}(D)$, but G itself has the R_∞-property. To see this, choose B basic in G and then $1 \in \mathrm{RSpec}(B)$ follows by Corollary 4.11; note that the quotient $D = G/B$ is divisible and so $1 \in \mathrm{RSpec}(D)$ follows from Proposition 2.3.

Using Corner's results in [3] and the fact that the image of a small homomorphism from a group with countable basic subgroup is itself countable, we have the following corollary.

Corollary 5.9. *There is a family of $2^{2^{\aleph_0}}$ reduced 2-groups each of which has the R_∞-property and every homomorphic image from one member of the family to a different member of the family is countable.*

The situation for mixed groups is easy to handle when the group splits.

Theorem 5.10. *If X is a torsion-free group with the R_λ-property for an infinite cardinal λ, then $G = X \oplus T$ has the R_λ-property for any torsion group T.*

Proof. If T is torsion and X is torsion-free, then $\mathrm{Hom}(T, X) = 0$ and the result follows immediately from Proposition 2.6. □

Acknowledgment: We would like to thank the referee for numerous suggestions which have greatly improved the presentation of this work.

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