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FINITE GROUPS WITH LARGE CHEBOTAREV INVARIANT*

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ABSTRACT

A subset $\{g_1, \dots, g_d\}$ of a finite group G is said to invariably generate G if the set $\{g_1^{x_1}, \dots, g_d^{x_d}\}$ generates G for every choice of $x_i \in G$. The Chebotarev invariant $C(G)$ of G is the expected value of the random variable n that is minimal subject to the requirement that n randomly chosen elements of G invariably generate G. The authors recently showed that for each $\epsilon > 0$, there exists a constant c_ϵ such that $C(G) \leq (1+\epsilon)\sqrt{|G|} + c_\epsilon$. This bound is asymptotically best possible. In this paper we prove a partial converse: namely, for each $\alpha > 0$ there exists an absolute constant δ_α such that if G is a finite group and $C(G) > \alpha\sqrt{|G|}$, then G has a section X/Y such that $|X/Y| \geq \delta_\alpha\sqrt{|G|}$, and $X/Y \cong \mathbb{F}_q \rtimes H$ for some prime power q, with $H \leq \mathbb{F}_q^\times$.

1. Introduction

Following [10] and [5], we say that a subset $\{g_1, g_2, \dots, g_d\}$ of a group G *invariably generates* G if $\{g_1^{x_1}, g_2^{x_2}, \dots, g_d^{x_d}\}$ generates G for each d-tuple $(x_1, x_2 \dots, x_d) \in G^d$. The *Chebotarev invariant* $C(G)$ of G is the expected value of the random

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variable n which is minimal subject to the requirement that n randomly chosen elements of G invariably generate G.

Motivated by the problem of finding field extensions K/F such that a fixed finite group G occurs as the Galois group of K/F, E. Kowalski and D. Zywina carried out a detailed investigation of the invariant $C(G)$ in [12]. Amongst many interesting results, they show that $C(G)$ can be quite large in comparison to $|G|$. More precisely, it is shown that if $G \cong G_q := \mathbb{F}_q \rtimes \mathbb{F}_q^\times$, then

$$C(G) = q - \sum_{1 \neq d | q-1} \frac{\mu(d)}{q(1-d^{-1})(1-d^{-1}+q^{-1})}.$$

In particular, $C(G_q) \sim \sqrt{|G_q|}$ as $q \to \infty$. It was also conjectured in [12] that these are the "worst" cases: that is, that $C(G) = O(\sqrt{|G|})$ as $|G| \to \infty$. The conjecture was proved by the first author in [15], and was later improved in [17] where it is shown that for each $\epsilon > 0$, there exists a constant c_ϵ such that $C(G) \leq (1+\epsilon)\sqrt{|G|} + c_\epsilon$. Furthermore, one has $C(G) \leq \frac{5}{3}\sqrt{|G|}$ when G is soluble.

In this paper, we prove a partial converse. Informally, we prove that the the only examples where $C(G)$ is a constant times $\sqrt{|G|}$ are those groups with a "large" section isomorphic to a subgroup of G_q, for some prime power q. Our main result reads as follows.

THEOREM 1: *Fix a constant $\alpha > 0$. There exists absolute constants β_α, γ_α, δ_α and k_α, depending only on α, such that whenever G is a finite group with the property that $C(G) > \alpha\sqrt{|G|}$, then G has a factor group $\overline{G}$ such that*

(i) $\overline{G} \cong V \rtimes H$, *with* $V \cong \mathbb{F}_q^k$, *and* $H \leq \Gamma L_1(q) \wr \mathrm{Sym}(k)$, *with q a prime power and $k \leq k_\alpha$;*
(ii) $|\overline{G}| \geq \delta_\alpha \sqrt{|G|}$; *and*
(iii) $\beta_\alpha |V| \leq |H| \leq \gamma_\alpha |V|$.

Our approach utilises the theory of crowns in finite groups, which we describe in Section 2. We also require a characterisation of those irreducible linear groups $H \leq GL(V)$ such that the set $H^*(V) := \{h \in H : v^h = v \text{ for some } v \in V \backslash \{0\}\}$ is bounded above by an absolute constant, and this is the content of Section 3. Finally, Section 4 is reserved for the proof of Theorem 1.

2. Crowns in finite groups

Before defining the notion of a crown in a finite group, we require some terminology. First, let L be a monolithic primitive group. That is, L is a finite group with a unique minimal normal subgroup $V \not\leq \mathrm{Frat}(L)$. For each positive integer k, write L^k for the k-fold direct product of L. The *crown-based power of L of size k* is the subgroup L_k of L^k defined by

$$L_k = \{(l_1, \dots, l_k) \in L^k \mid l_1 \equiv \cdots \equiv l_k \bmod V\}.$$

Equivalently, $L_k = V^k \operatorname{Diag} L^k$.

Next, let G be a finite group. We say that a group V is a *G-group* if G acts on V via automorphisms. Following [9], we say that two irreducible G-groups V_1 and V_2 are *G-equivalent* and we put $V_1 \sim_G V_2$, if there are isomorphisms $\phi : V_1 \to V_2$ and $\Phi : V_1 \rtimes G \to V_2 \rtimes G$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & V_1 & \longrightarrow & V_1 \rtimes G & \longrightarrow & G & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle\phi} & & \big\downarrow{\scriptstyle\Phi} & & \big\| & & \\
1 & \longrightarrow & V_2 & \longrightarrow & V_2 \rtimes G & \longrightarrow & G & \longrightarrow & 1.
\end{array}$$

Note that two G-isomorphic G-groups are G-equivalent. In the abelian case, the converse is true: if V_1 and V_2 are abelian and G-equivalent, then V_1 and V_2 are also G-isomorphic. It is proved (see for example [9, Proposition 1.4]) that two chief factors V_1 and V_2 of G are G-equivalent if and only if either they are G-isomorphic, or there exists a maximal subgroup M of G such that $G/\operatorname{Core}_G(M)$ has two minimal normal subgroups N_1 and N_2 G-isomorphic to V_1 and V_2 respectively. For example, the minimal normal subgroups of a crown-based power L_k are all L_k-equivalent.

Let $V = X/Y$ be a chief factor of G. A complement U to V in G is a subgroup U of G such that $UV = G$ and $U \cap X = Y$. We say that $V = X/Y$ is a Frattini chief factor if X/Y is contained in the Frattini subgroup of G/Y; this is equivalent to saying that V is abelian and there is no complement to V in G. The number of non-Frattini chief factors G-equivalent to V in any chief series of G does not depend on the series, and so this number is well-defined: we will write it as $\delta_V(G)$. We now define L_V, the monolithic primitive group associated to V, by

$$L_V := \begin{cases} V \rtimes (G/C_G(V)) & \text{if } V \text{ is abelian,} \\ G/C_G(V) & \text{otherwise.} \end{cases}$$

If V is a non-Frattini chief factor of G, then L_V is a homomorphic image of G. More precisely, there exists a normal subgroup N of G such that $G/N \cong L_V$ and $\mathrm{soc}(G/N) \sim_G V$. Consider now all the normal subgroups N of G with the property that $G/N \cong L_V$ and $\mathrm{soc}(G/N) \sim_G V$: the intersection $R_G(V)$ of all these subgroups has the property that $G/R_G(V)$ is isomorphic to the crown-based power $(L_V)_{\delta_V(G)}$. The socle $I_G(V)/R_G(V)$ of $G/R_G(V)$ is called the V-crown of G and it is a direct product of $\delta_V(G)$ minimal normal subgroups G-equivalent to V.

We now record a lemma and two propositions which will be crucial in our proof of Theorem 1. The lemma reads as follows.

Lemma 2: [1, Lemma 1.3.6] *Let G be a finite group with trivial Frattini subgroup. There exists a chief factor V of G and a non trivial normal subgroup U of G such that $I_G(V) = R_G(V) \times U$.*

To state the propositions, we need some additional notation. For a finite group G, and an abelian chief factor V of G, set $H_V = H_V(G) := G/C_G(V)$, $m = m_V = m_V(G) := \dim_{\mathrm{End}_G(V)} \mathrm{H}^1(H_V, V)$, and write $H^* = H^*(V) = H^*_G(V)$ for the set of elements h of H_V which fix a non-zero vector in V. Also, let $\delta_V = \delta_V(G)$, and set $\theta_V = \theta_V(G) = 0$ if $\delta_V = 1$, and $\theta_V = 1$ otherwise. Finally, let $q_V = q_V(G) := |\,\mathrm{End}_G(V)|$ and $n_V = n_V(G) := \dim_{\mathrm{End}_G(V)} V$. Note that $\mathrm{End}_G(V)$ is a finite field, since V is finite and irreducible.

Proposition 3: [17, Proposition 8 and the Proof of Theorem 1] *Let G be a finite group with trivial Frattini subgroup, and let U, V and $R = R_G(V)$ be as in Lemma 2. If U is non-abelian, then there exists absolute constants b_1, b_2 and b_3 such that*

$$C(G) \leq C(G/U) + \lceil b_3(\log|G|)^2\rceil + \frac{b_1}{b_2}\sqrt{|G|^3}\log|G|(1 - b_2/\log|G|)^{\lceil b_3(\log|G|)^2\rceil}.$$

Proposition 4: [17, Proposition 8 and the Proof of Theorem 1] *Let G be a finite group with trivial Frattini subgroup, and let U, V and $R = R_G(V)$ be as in Lemma 2. Suppose that V is abelian, and write $q = q_V$, $n = n_V$ and $H = H_V$, $H^* = H^*(V)$ and $m = m_V$. Also, set $\delta = \delta_V$ and $\theta = \theta_V$. Set*

$$\alpha_U := \begin{cases} \sum_{0\le i\le \delta-1} \frac{q^\delta}{q^\delta - q^i} \le \delta + \frac{q}{(q-1)^2} & \textit{if } H = 1, \\ \min\left\{ \left(\delta\cdot\theta + m + \frac{q}{q-1}\right)\frac{|H|}{|H^*|}, \left(\lceil\frac{\delta\cdot\theta}{n}\rceil + \frac{q^n}{q^n-1}\right)|H| \right\} & \textit{otherwise.} \end{cases}$$

Then

$$C(G) \le C(G/U) + \alpha_U.$$

We conclude this section with the theorem of the first author mentioned in the introduction.

Theorem 5: [15, Main Theorem] *There exists an absolute constant C such that $C(G) \le C\sqrt{|G|}$ for any finite group G.*

3. Irreducible linear groups with few elements fixing a non-zero vector

Let V be a finite dimensional vector space over an arbitrary field. In this section, our aim is to characterise the groups $H \le GL(V)$, such that the set of elements which fix at least one non-zero vector in V has cardinality bounded above by an absolute constant. For ease of notation, we will write

$$H^* = H^*(V) := \{h \in H : v^h = v \text{ for some } v \in V\backslash\{0\}\}$$

for such a subgroup H. Our main result reads as follows.

Proposition 6: *Let V be a vector space of dimension n over a field F, and fix a constant $c > 0$. Suppose that H is an irreducible subgroup of $GL(V)$ with the property that $|H^*| \le c$. Then there exists positive integers m and k such that $n = mk$, and $H \le R \wr \mathrm{Sym}(k)$, where either $|R|$ has order bounded above by a function of $|H^*|$, or $R \cong \Gamma_1(F_m)$ for some extension field F_m of F of degree m.*

Proposition 6 will follows almost immediately from our next result. Recall that if F is a field, then an irreducible subgroup H of a linear group $GL_n(F)$ is called *weakly quasiprimitive* if every characteristic subgroup of G is homogeneous.

Proposition 7: *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that if F is a field, n is a positive integer, and $H \le GL_n(F)$ is finite and weakly quasiprimitive, then either $|H| \le f(|H^*|)$, or H is a subgroup of $\Gamma L_1(F_n)$, for some extension field F_n of F of degree n.*

Proof. If $n = 1$, then $\Gamma L_n(F) = GL_n(F)$. Thus, we may assume that $n > 1$. Fix a subgroup H of $GL_n(F)$. We want to prove that if H is not a subgroup of $\Gamma L_1(F_n)$ for some extension field F_n of F of degree n, then $|H|$ is bounded in terms of $|H^*|$.

Suppose first that every characteristic abelian subgroup of H is contained in $Z(GL_n(F))$. Let L be the generalised Fitting subgroup of H. Our aim is to prove that $|L|$ is bounded above in terms of $|H^*|$. Since L is self-centralising, this will show that $|H|$ is bounded above in terms of $|H^*|$, which will give us what we need.

To this end, extend the field F so that F is a splitting field for all subgroups of L. Then L may longer be homogeneous, but its irreducible constituents are algebraic conjugates of each other, so L acts faithfully on them. Let W be such a constituent, and let r_i, m_i, s_i, t_i, S_i and T_i be as in [8, Lemma 2.14]. In particular, the r_i are prime numbers and L is a central product of the collection of groups $O_{r_i}(G)$, T_i, where T_i is a central product of t_i copies of a quasisimple group S_i. By [8, Lemmas 2.15, 2.16 and 2.17], W decomposes as a tensor product

$$W = W_Z \otimes W_{r_1} \otimes \ldots \otimes W_{r_a} \otimes W_{s_1} \otimes \ldots \otimes W_{s_b},$$

where W_Z is a 1-dimensional module for Z; W_{r_i} is an irreducible module for $O_{r_i}(G)$ of dimension $r_i^{m_i}$; and W_{s_i} is an irreducible module for T_i of dimension $s_i^{t_i}$. In particular, $[O_{r_i}(H), W_{r_j}] = [T_i, W_{s_j}] = 1$ for $i \neq j$, and $[O_{r_i}(H), W_{s_j}] = [T_i, W_{r_j}] = 1$, for all i, j. Hence, if $a + b > 1$, then $|L|$ is bounded above in terms of $|H^*|$, as needed. So we may assume that either $L = Z(G) \circ O_r(H)$, for some prime r, or $L = Z(G) \circ T$ is a central product of t copies of a quasisimple group S. If $Z(G) \not\leq O_r(H)$ in the first case, or $Z(G) \not\leq T$ in the second case, then the same argument as above gives that $|L|$ is bounded in terms of $|H^*|$.

So we may assume that either $L = O_r(H)$, for some prime r, or $L = T$ is a central product of t copies of a quasisimple group S. Hence, W is a tensor product of m [respectively t] copies of an irreducible module for an extraspecial group of order r^3 [resp. quasisimple group]. Thus, by arguing as in the paragraph above, we can immediately reduce to the case $m = 1$ [resp. $t = 1$].

Suppose first that $L = O_r(H) = M \rtimes \langle x \rangle$ is extraspecial of order r^3, for a prime r, where M is cyclic of order r^2 if L has exponent r^2, and M is elementary abelian of order r^2 otherwise. Then, being an absolutely irreducible module for L of dimension r, W is isomorphic to $U \uparrow_M^L$, where U is a one dimensional module for M in which $Z(L)$ acts non-trivially. Hence, we may write $W = \bigoplus_{i=0}^{r-1} U \otimes x^i$. It follows that for each non-zero vector $u \in U$, x^j fixes the non-zero vector $u \otimes 1 + u \otimes x + \ldots + u \otimes x^{r-1}$. Thus, $r \leq |H^*|$, from which it follows that $|L| = r^3$ is bounded above in terms of $|H^*|$, as needed.

Finally, assume that L is quasisimple. Since L acts on L^* by conjugation, we may assume that $L^* \leq Z$ (otherwise $L \leq \mathrm{Sym}(L^*)$, which would imply that $|L|$ is bounded above in terms of $|H^*|$). However, since $Z = Z(H) \leq Z(GL_n(F))$, Z acts on V by scalar multiplication. Hence, $Z \cap H^* = 1$. It follows that $L^* = 1$, and hence that L is a Frobenius complement in the group $V \rtimes L$. Since L is perfect, it now follows from Zassenhaus' Theorem that $L \cong SL_2(5)$. Whence, $|L|$ is bounded, and this proves our claim.

Finally, assume that H has a characteristic abelian subgroup not contained in $Z(GL_n(F))$, and let $M \leq H$ be maximal with this property. Then by [16, Lemma 1.10], M is contained in $Z(GL_{\frac{n}{m}}(F_m))$ for some m dividing n, and some extension field F_m of F of degree m. Hence, $H_1 := C_H(M)$ is a subgroup of $GL_{\frac{n}{m}}(F_m)$ with the property that every characteristic abelian subgroup of H_1 is contained in $Z(GL_{\frac{n}{m}}(F_m))$. Furthermore, H_1 is weakly quasiprimitive, since it is characteristic in H. Also, the group H/H_1 is naturally embedded in $\mathrm{Gal}(F_m/F)$, its action induced by a vector space isomorphism $F_m^{\frac{n}{m}} \to F^n$. Since $H_1^*(F_m^{\frac{n}{m}}) = H_1^*(F^n)$, it follows from the arguments above that either $|H_1|$ is bounded in terms of $|H^*|$; or $n = 1$. If $|H_1|$ is bounded in terms of $|H^*|$, then so is $|H|$, since H_1 is self-centralising and normal in H. If $n = 1$, then $H_1 \leq GL_1(F_n)$, so $H \leq \Gamma L_1(F_n)$, since H/H_1 acts on $M = Z(H_1)$ via the Galois group, as described above. This completes the proof. ∎

Finally, we prove Proposition 6.

Proof of Proposition 6. If H is primitive, then the result follows immediately from Proposition 7. Thus, we may assume that H is not primitive. Then V may be decomposed into a system $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$ of imprimitivity for H. Let $\Gamma := \{W_1, \ldots, W_k\}$, let $S := H^\Gamma$ denote the induced (transitive) action of H on Γ, and let $R := \mathrm{Stab}_H(W_1)^{W_1}$ denote the induced action of $\mathrm{Stab}_H(W_1)$ on W_1. Then H is isomorphic to a subgroup of the wreath product $R \wr S$.

Finally, since $\mathrm{Stab}_H(W)$ induces R on W, we have $|R^*(W_1)| \leq |H^*(V)|$. Hence, Proposition 7 implies that either $R \leq \Gamma L_1(F_m)$, for some extension F_m of F of degree m, or $|R|$ is bounded above by a function of $|H^*|$. This completes the proof. ∎

4. The proof of Theorem 1

We begin our preparations towards the proof of Theorem 1 with a lemma concerning the cohomology of an irreducible linear group which has a bounded number of elements fixing a non-zero vector.

Lemma 8: *There exists an absolute constant c such that if V is a vector space of dimension n over a field F of characteristic $p > 0$, and H is an irreducible subgroup of $GL(V)$ with the property that $|H| > \sqrt{|V|}$, then $2^m \leq c|H^*|^4$, where $m := \dim_F \mathrm{H}^1(H,V)$ and $F := \mathrm{End}_H V$.*

Proof. Clearly we may assume that $m > 0$. Then, it is proven in [15, Lemma 9] that

(1) H has a unique minimal normal subgroup N, which is non-abelian.
(2) If S is a component of H, then $C_H(S) \subseteq H^*$.
(3) If W is an irreducible N-submodule of V not centralised by S, then $m \leq \dim_F \mathrm{H}^1(S,W)$.

Write $N = S_1 \times \ldots \times S_t \cong S^t$, and view H as a subgroup in the wreath product $\mathrm{Aut}(N) = \mathrm{Aut}(S) \wr K$, where K denote the induced action of H on the components in N. Suppose first that $t > 1$. Then (2) implies that $S_i \subseteq H^*$ for all i. Hence, $|H^*| \geq 1 + t(|S|-1)$. Also, $|H^*| \geq C_H(S_1) \geq |H \cap B||\,\mathrm{Stab}_K(1)| = |H \cap B|\frac{|K|}{t}$, where $B := \mathrm{Aut}(S_2) \times \ldots \times \mathrm{Aut}(S_t)$. Note that $|H| \leq |H \cap B||\,\mathrm{Aut}(S)||K|$. It follows that $|H| \leq |H^*|t|\,\mathrm{Aut}(S)| \leq |H^*|t(|S|-1)^2 \leq |H^*|^3$.

Next, it is shown by Guralnick and Hoffman in [7, Theorem 1] that $m \leq \frac{n}{2}$. Since we also have $|H| > \sqrt{|V|}$, it follows that

$$m \leq \frac{n}{2} \leq \log \sqrt{|V|} < \log |H| \leq \log |H^*|^3.$$

Thus, we may assume that $H \leq \mathrm{Aut}(S)$ is almost simple. Before distinguishing cases, we make some remarks. First, $p = \mathrm{char}\, F$ divides $|H|$, since $\mathrm{H}^1(H,V) \neq 0$. Furthermore, $|H^*| \geq |H|_p$, since every element of a Sylow p-subgroup of H fixes a non-zero vector in V. Finally, note that we may assume that S is not sporadic, since there are a bounded number of such groups having an irreducible module with non-zero cohomology.

Thus, we have two cases.

(a) $S \cong \mathrm{Alt}(k)$. In this case, we have $\frac{n}{2} \leq \log \sqrt{|V|} \leq \log |H| \leq k \log k$, as long as $k > 6$. Hence, by [15, Proof of Proposition 10], we have $m \leq 4 \log k$ and $|H|_p > \frac{k}{2}$, if k is large enough. Hence $2^m \leq k^4 \leq 16|H^*|^4$ in this case. If

k is bounded, then m is also bounded, since $m \leq \frac{n}{2} \leq \log |H|$. Hence, the result also follows in this case.

(b) $S \cong^{\epsilon} X_k(r)$ is a group of Lie type. Write $R_F(S)$ for the smallest degree of a non-trivial irreducible representation of S over the field F. If char F is different to the defining characteristic for S, then we have $p^{\frac{R_F(S)}{2}} > |\operatorname{Aut}(S)|$ for $|S|$ large enough (see [13, 18, 20]). Since $\sqrt{|V|} \leq |H|$, we conclude that either $|S|$ is bounded, or char F coincides with the defining characteristic of S. In the latter case, we have $|H|_p > |S|^{\frac{1}{3}}$ by [11, Proposition 3.5]. Also, $|S| \geq |\operatorname{Aut}(S)|^{\frac{4}{5}}$ by [14, Proposition 4.4]. Hence,

$$|H^*| > |S|^{\frac{1}{3}} \geq |\operatorname{Aut}(S)|^{\frac{4}{15}} > |H|^{\frac{1}{4}} \geq 2^{\frac{m}{4}}.$$

Thus, either $|S|$ is bounded, or $2^m \leq |H^*|^4$. This gives us what we need. ∎

Next, we prove a reduction lemma.

Lemma 9: *Fix a constant $\alpha > 0$. There exists absolute constants $b = b(\alpha)$, $c = c(\alpha)$ and $c_i = c_i(\alpha)$, $1 \leq i \leq 4$, depending only on α, such that: If G is a finite group with trivial Frattini subgroup with the property that $C(G) > \alpha\sqrt{|G|}$, and U is as in Lemma 2, then one of the following holds.*

(i) U is non-abelian and $|G| \leq b$.
(ii) U is abelian and $|U| \leq c$.
(iii) U is abelian and G has a factor group $\overline{G}$ such that
  (a) $\overline{G} \cong V \rtimes H$, with $V \cong U$ an abelian chief factor of G, and $H \leq GL(V)$;
  (b) $|H^*(V)| \leq c_1$;
  (c) $\dim_{\operatorname{End}_H V} \mathrm{H}^1(H, V) \leq c_2$; and
  (d) $c_3|V| \leq |H| \leq c_4|V|$.

Proof. Adopt in its entirety the notation of Proposition 4, so that U, V and $R = R_G(V)$ are as in Lemma 2. We first consider the case where V is non-abelian. Then by Proposition 3 we have

$$\alpha\sqrt{|G|} < C(G/U) + \lceil b_3(\log |G|)^2\rceil + \frac{b_1}{b_2}\sqrt{|G|^3}\log |G|(1 - b_2/\log |G|)^{\lceil b_3(\log |G|)^2\rceil},$$

where b_1, b_2 and b_3 are the absolute constants from Proposition 3. Since $C(G/U) \leq C\sqrt{|G/U|}$, it follows that $\sqrt{|G|} \leq \alpha'\lceil b_3(\log |G|)^2\rceil + \frac{b_1}{b_2}\sqrt{|G|^3}\log |G|(1 - b_2/\log |G|)^{\lceil b_3(\log |G|)^2\rceil}$, for some constant α' depending only on α. Hence, since the square root of $|G|$ divided by the right hand side of the above equation tends to ∞ as $|G|$ tends to infinity, we must have that $|G|$ is bounded above by a constant $b = b(\alpha)$ depending only on α.

Thus, we may assume that U is abelian. Then by Proposition 4 and Theorem 5, there exists an absolute constant C such that

$$\alpha\sqrt{|G|} \leq C(G) \leq C(G/U) + \alpha_U \leq c\sqrt{\frac{|G|}{|U|}} + \alpha_U.$$

In particular, using the definition of α_U from Proposition 4, we conclude that

$$\alpha \leq \frac{c}{\sqrt{|U|}} + (\delta\cdot\theta + m + 2)\,\frac{\sqrt{|H|}}{\sqrt{|V|^{\delta}|H^*|}}, \text{ and} \tag{4.1}$$

$$\alpha \leq \frac{c}{\sqrt{|U|}} + \left(\left\lceil\frac{\delta\cdot\theta}{n}\right\rceil + 2\right)\frac{\sqrt{|H|}}{\sqrt{|V|^{\delta}}}. \tag{4.2}$$

We claim first that $\delta = 1$. Indeed, assume otherwise, and note that $\frac{|H|}{|H^*|} \leq |H|/|H_v| \leq |V|$, for any non-zero $v \in V$. Hence, since $m \leq \frac{n}{2}$, we conclude from (4.1) that

$$|V|^{\frac{\delta-1}{2}} \leq C_1(n+\delta), \tag{4.3}$$

where $C_1 = C_1(\alpha)$ depending only on α. Now, since $|U| = |V|^{\delta} = q^{n\delta}$, we conclude that there exists a constant $c = c(\alpha)$ such that if $|U| > c$ and $\delta > 1$ then $|V|^{\frac{\delta-1}{2}} > C_1(n+\delta)$.

Hence, we may assume that $\delta = 1$. We will first prove that the properties (b) and (c) of Part (iii) of thge statement of the lemma hold in the factor group $\overline{G} := G/R_G(V)$. If $|H| \leq \frac{|V|}{n^2}$, then (4.1) [respectively (4.2)] implies that $|H^*|$ [resp. n] is bounded above by a constant depending only on α. Properties (b) and (c) then follow immediately.

So we may assume that $|H| > \frac{|V|}{n^2}$. We then use (4.1) and the fact that $|H|/|H_v| \leq |V|$ to deduce that $|H^*| \leq C_2(1+m^2)$, where $C_2 = C_2(\alpha)$ is a constant depending only on α. Since $|H| > \sqrt{|V|}$, if follows from Lemma 8 that $|H^*| \leq C_3(1+\log|H^*|^2)$, where $C_3 = C_3(\alpha)$ is a constant depending only on α. It follows that $|H^*|$, and hence m, are bounded above by constants depending only on α. This proves that Properties (b) and (c) hold.

Finally, the existence of c_3 follows immediately from (4.2), while the existence of c_4 follows from (4.1) and the bound $|H|/|H^*| \leq |V|$. This proves that Property (d) holds, and completes the proof. ∎

We are now ready to prove Theorem 1.

Proof of Theorem 1. Let C be the constant from Theorem 5; let f be the function from Proposition 7; let b_1, b_2 and b_3 be the constants from Proposition 3; and let $b = b(\alpha)$ and $c = c(\alpha)$ be the constants from Lemma 9. Also, let c_i, $1 \le i \le 4$, be the functions of α from Lemma 9. Note that we may assume that f, c_1, c_2 and c_4 are increasing functions, while c_3 is decreasing. Hence, we may also assume that g satisfies $g(\alpha_1\alpha_2) \ge g(\alpha_1)\alpha_2$, for $g \in \{f, c_1\}$. For ease of notation, we will sometimes write c_i in place of $c_i(\alpha)$.

Set $b_4 := \max\{b, \lceil b_3(\log b)^2\rceil + \frac{b_1}{b_2}\sqrt{b^3}\log b(1 - b_2/\log b)^{\lceil b_3(\log b)^2\rceil}\}$; $\alpha' := \max\{\alpha, C\}$; $c_5 := \max\{c, \frac{1}{c_3(\alpha')} f(\lfloor c_1(\alpha')\rfloor)^{\frac{c_1(\alpha')}{c_3(\alpha')}} \lfloor \frac{c_1(\alpha')}{c_3(\alpha')}\rfloor !\}$; and $c_6 := (2 + c_2)c_5$. Then define

$$\delta(\alpha) := \min\{f(\lfloor c_1(\beta)\rfloor) : 0 < \beta \le \alpha'\} \text{ and}$$
$$k(\alpha) := \frac{c_1(\alpha')}{c_3(\alpha')}.$$

Finally, set $\beta := c_3$ and $\gamma := c_4$. Note that by construction k is an increasing function of α, and that

$$\delta(\beta\sqrt{u}) \ge \delta(\beta)\sqrt{u} \ge \delta(\alpha)\sqrt{u}, \tag{4.4}$$

whenever $\beta \le \alpha$.

We will now prove by induction on $|G|$ that G has a factor group $\overline{G}$ such that

(i) $\overline{G} \cong V \rtimes H$, with $V \cong \mathbb{F}_q^k$, and $H \le \Gamma L_1(q) \wr \mathrm{Sym}(k)$, with q a prime power and $k \le k(\alpha)$;
(ii) $|\overline{G}| \ge \delta(\alpha)\sqrt{|G|}$; and
(iii) $\beta(\alpha)|V| \le |H| \le \gamma(\alpha)|V|$.

Suppose first that $\mathrm{Frat}(G) = 1$, and let U, V and $R = R_V(G)$ be as in Lemma 2. We would like to reduce to the case where $|G| > b$ if V is non-abelian, and $|U| > c_5$ if V is abelian. We first deal with the non-abelian case. So assume that V is non-abelian and that $|G| \le b$. In this case, we have

$$\alpha\sqrt{|G|} < C(G/U) + b_4 \le (1 + b_4)C(G/U),$$

by Proposition 3. In particular, it follows that $C(G/U) > \alpha_1\sqrt{|G/U|}$, where

$$\alpha_1 := \frac{\alpha\sqrt{|U|}}{1 + b_4}.$$

Note that $\gamma(\alpha_1) \leq \gamma(\alpha)$, since $\alpha_1 \leq \alpha$, and γ is an increasing function. Similarly, $k(\alpha_1) \leq k(\alpha)$ and $\beta(\alpha) \leq \beta(\alpha_1)$. Furthermore, $\delta(\alpha_1) \geq \delta(\alpha)\sqrt{|U|}$ by (4.4). The inductive hypothesis now implies that G, and hence G/U, has a factor group $\overline{G}$ with the desired properties.

Next, assume that V is abelian, and that $|U| \leq c$. Then since $\alpha_U \leq c_6$, Proposition 4 yields $C(G/U) > \alpha_2\sqrt{|G/U|}$, where

$$\alpha_2 := \frac{\alpha\sqrt{|U|}}{1+c_6}.$$

As above, it now follows from the inductive hypothesis and the definitions of $\delta(\alpha)$ and $k(\alpha)$ that G has a factor group $\overline{G}$ with the desired properties.

Thus, we may assume that $|G| > b$ if U is non-abelian, and $|U| > c_5 \geq c$ otherwise. However, Lemma 9 then implies that U must be abelian, and that G has a factor group $\overline{G}$ such that

(a) $\overline{G} \cong V \rtimes H$, with $V \cong U$ an abelian chief factor of G, and $H \leq GL(V)$;
(b) $|H^*(V)| \leq c_1(\alpha)$;
(c) $\dim_{\mathrm{End}_H V} \mathrm{H}^1(H,V) \leq c_2(\alpha)$; and
(d) $c_3(\alpha)|V| \leq |H| \leq c_4(\alpha)|V|$.

Furthermore, Lemma 6 guarantees the existence of positive integers m and k, and a transitive permutation group S of degree k, such that $n = mk$ and $H \leq R \wr S$, with either $|R| \leq f(c_1)$, or $R \leq \Gamma L_1(p^m)$. Hence, we just need to prove that $k \leq k(\alpha)$. Indeed, if this is true then we must have $R \leq \Gamma L_1(p^m)$, since otherwise $|V| \leq \frac{1}{c_3(\alpha)}|H| \leq \frac{1}{c_3(\alpha)} f(c_1(\alpha))^{\frac{c_1(\alpha)}{c_3(\alpha)}} \lfloor \frac{c_1(\alpha)}{c_3(\alpha)} \rfloor!$, contradicting $|U| > c_5$.

Now, note that (b) and (d) above imply that the number of orbits of H in its action on V is bounded above by $1 + \frac{c_1}{c_3}$. Hence, the number of orbits of $X := GL_m(p) \wr \mathrm{Sym}(k)$ is bounded above by $1 + \frac{c_1}{c_3}$. Then since $GL_m(p)$ has 2 orbits in its action on the natural module $(\mathbb{F}_p)^m$, it follows that the number of orbits of X on V is precisely the number of orbits of $\mathrm{Sym}(k)$ in its action on the k-fold cartesian power $\{0,1\}^k$ by permutation of coordinates. This number is precisely $k+1$. Hence, we have $k + 1 \leq 1 + \frac{c_1}{c_3}$, and this completes the proof in the case $\mathrm{Frat}(G) = 1$.

Finally, assume that $\mathrm{Frat}(G) > 1$. Then $C(G/\mathrm{Frat}(G)) = C(G) > \beta\sqrt{|G/\mathrm{Frat}(G)|}$, where $\beta := \alpha\sqrt{|\mathrm{Frat}(G)|}$. Now, since $\alpha\sqrt{|G|} < C(G/\mathrm{Frat}(G)) \leq C\sqrt{|G/\mathrm{Frat}(G)|}$, we have $|\mathrm{Frat}(G)| \leq (\frac{C}{\alpha})^2$. Hence, $\beta \leq C$. The result now follows from the inductive hypothesis and the definitions of $\delta(\alpha)$ and $k(\alpha)$. ∎

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