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CLOSED RANGE WEIGHTED COMPOSITION OPERATORS BETWEEN L^p-SPACES

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Abstract. We characterize the closedness of ranges of weighted composition operators between L^p-spaces, where $1 \leq p \leq \infty$. When the L^p-spaces are weighted sequence spaces, several corollaries about this class of operators are also deduced.

Keywords: weighted composition operator, Lebesgue space, closed range.

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1. INTRODUCTION

Let (X, Σ, μ) and (Y, Γ, ν) be two σ-finite and complete measure spaces. Suppose $\varphi : Y \to X$ is a point mapping such that $\varphi^{-1}(E) \in \Gamma$ for all $E \in \Sigma$. Assume that φ is also non-singular, which means the measure defined by $\nu\varphi^{-1}(E) := \nu(\varphi^{-1}(E))$ for $E \in \Sigma$, is absolutely continuous with respect to μ. Let $u : Y \to \mathbb{C}$ be a Γ-measurable function. The functions u and φ induce a *weighted composition operator* uC_φ from $L^p(\mu)$ $(1 \leq p \leq \infty)$ into the linear space of all Γ-measurable functions on Y by

$$(uC_\varphi f)(y) := u(y)f(\varphi(y)) \quad \text{for every } f \in L^p(\mu) \text{ and } y \in Y.$$

The non-singularity of φ guarantees that uC_φ is a well-defined mapping of equivalence classes of functions. When $u \equiv 1$ (resp. $(X, \Sigma, \mu) = (Y, \Gamma, \nu)$ and $\varphi(x) = x$ for all $x \in X$), the corresponding operator, denoted by C_φ (resp. by M_u), is called a *composition operator* (resp. a *multiplication operator*). Observe that $uC_\varphi = M_u \circ C_\varphi$. Suppose uC_φ maps $L^p(\mu)$ into $L^q(\nu)$, where $1 \leq p, q \leq \infty$. Since norm convergence in the L^p-norm implies pointwise convergence a.e. of a subsequence, it follows from the closed graph theorem that uC_φ is also bounded.

Operator-theoretic properties of weighted composition maps from $L^p(\mu)$ into itself were obtained in [1, 10, 12] and such operators acting between two distinct

L^p-spaces in [4–7, 9]. In this paper, we investigate closed range weighted composition operators between L^p-spaces. Singh and Kumar [11] characterized multiplication and composition operators on $L^2(\mu)$ with closed ranges. Takagi and Yokouchi [13, 14] not only extended the results of Singh and Kumar to a general L^p-space, but also obtained characterizations for the closedness of ranges of such operators between distinct L^p-spaces. Our results generalize their findings to the weighted case. While this question has been considered in [6], the results and proofs therein are sketchy and incomplete. We shall provide proofs that are somewhat different from those in [6], deduce interesting consequences when the underlying spaces are weighted sequence spaces and illustrate the results with examples.

2. PRELIMINARIES

Let f be a complex-valued Σ-measurable function on X. Its support, written as $\operatorname{supp} f$, is defined by

$$\operatorname{supp} f := \{x \in X : f(x) \neq 0\}.$$

For $1 \leq p \leq \infty$, we define

$$\|f\|_p := \begin{cases} \left(\int_X |f|^p \, d\mu \right)^{1/p} & \text{if } 1 \leq p < \infty, \\ \inf\{M > 0 : \mu(\{x \in X : |f(x)| > M\}) = 0\} & \text{if } p = \infty. \end{cases}$$

The Lebesgue space, denoted by $L^p(\mu)$, consists of all (equivalence classes of) Σ-measurable functions f on X for which $\|f\|_p < \infty$. It is a Banach space under the norm $\|\cdot\|_p$ and is written as $\|\cdot\|_{L^p(\mu)}$. The functions in $L^\infty(\mu)$ are said to be essentially bounded.

Let $w := \{w_n\}_{n\in\mathbb{N}}$ be a sequence of positive real numbers. If we take $X = \mathbb{N}$, $\Sigma = \mathcal{P}(\mathbb{N})$ (the power set of $\mathbb{N}$) and $\mu(E) = \sum_{n\in E} w_n$ for every $E \in \mathcal{P}(\mathbb{N})$, then $L^p(\mu)$ is the weighted sequence space $l^p(w)$ for $1 \leq p < \infty$. If $w_n = 1$ for all $n \in \mathbb{N}$, $l^p(w)$ is just the classical sequence space l^p. When $p = \infty$, we define $l^\infty(w)$ (or simply l^∞) as the space of all bounded sequences of complex numbers.

Analogously, we may define $L^p(\nu)$ and $\|\cdot\|_{L^p(\nu)}$ for $1 \leq p \leq \infty$. Moreover, we define $\operatorname{supp} g$ similarly for a complex-valued Γ-measurable function g on Y.

In the sequel, we adopt the following decomposition of (X, Σ, μ):

$$X = \left(\bigcup_{i=1}^{\infty} A_i \right) \cup B,$$

where $\{A_i\}_{i=1}^\infty$ is a countable collection of pairwise disjoint atoms and B, being disjoint from each A_i, is nonatomic. This decomposition is unique in the sense that equality of two Σ-measurable sets interpreted as their symmetric difference is of zero μ-measure. The σ-finiteness of (X, Σ, μ) ensures that $\mu(A_i) < \infty$ for every $i \in \mathbb{N}$. Moreover, if $X = \bigcup_{i=1}^\infty A_i$ (resp. $X = B$), then (X, Σ, μ) is said to be atomic (resp. nonatomic).

The two facts below will be useful.

(a) Let E be a nonatomic set in Σ with $\mu(E) > 0$. For every real number α satisfying $0 < \alpha < \mu(E)$, there is a set $E_\alpha \in \Sigma$ with $E_\alpha \subset E$ and $\mu(E_\alpha) = \alpha$.

(b) A Σ-measurable function $f : X \to \mathbb{C}$ is constant μ-a.e. on an atom M in (X, Σ, μ). Consequently, we view an atom M as a 'point' at which f takes a constant value denoted by $f(M)$.

For $1 \le q < \infty$, the measure μ_q defined by

$$\mu_q(E) = \int_{\varphi^{-1}(E)} |u|^q \, d\nu \quad \text{for every } E \in \Sigma$$

is absolutely continuous with respect to μ. The corresponding Radon–Nikodym derivative, denoted by $[d\mu_q/d\mu]$, satisfies an important property.

Lemma 2.1 ([7, Lemma 2])**.** *If uC_φ is a weighted composition operator from $L^p(\mu)$ into $L^q(\nu)$, where $1 \le p, q < \infty$, then*

$$\|uC_\varphi f\|^q_{L^q(\nu)} = \int_X \left[\frac{d\mu_q}{d\mu}\right] |f|^q \, d\mu \quad \textit{for every } f \in L^p(\mu).$$

Let $(B_1, \|\cdot\|_{B_1})$ and $(B_2, \|\cdot\|_{B_2})$ be two Banach spaces. For a bounded linear operator T from B_1 into B_2, we denote its kernel and range by $\ker T$ and $T(B_1)$ respectively. The following result, which can be found in [2], will be used frequently in the subsequent sections. We state it for quick reference:

There exists a constant $c > 0$ such that $\|Tx\|_{B_2} \ge c\,\|x\|_{B_1}$ for all $x \in B_1$ if and only if $\ker T = \{0\}$ and $T(B_1)$ is closed in B_2. (2.1)

The condition "$\|Tx\|_{B_2} \ge c\,\|x\|_{B_1}$ for all $x \in B_1$" is usually referred to as "T is bounded below".

3. MAIN RESULTS: $1 \le p, q < \infty$

In this section, we characterize the closedness of ranges of weighted composition operators from $L^p(\mu)$ into $L^q(\nu)$, where $1 \le p, q < \infty$. For the sake of completeness, we state without proof the following result which characterizes weighted composition maps from $L^p(\mu)$ to $L^p(\nu)$ that have closed ranges. To this end, let $\mathfrak{C}$ be the collection of all Σ-measurable sets E such that

(a) $\mu(E) < \infty$, and

(b) whenever $G \in \Sigma$ satisfies $G \subset E$ and $\int_{\varphi^{-1}(G)} |u|^p \, d\nu = 0$, $\mu(G) = 0$.

Theorem 3.1 ([6, Theorem 3.4])**.** *Let uC_φ be a weighted composition operator from $L^p(\mu)$ into $L^p(\nu)$, where $1 \le p < \infty$. The following statements are equivalent.*

(i) *The operator uC_φ has closed range.*

(ii) *There is a constant $c > 0$ such that $\left[\frac{d\mu_p}{d\mu}\right] \ge c$ μ-a.e. on $\operatorname{supp}\left[\frac{d\mu_p}{d\mu}\right]$.*

(iii) *There is a constant $\alpha > 0$ such that $\int_{\varphi^{-1}(E)} |u|^p \, d\nu \ge \alpha\mu(E)$ for all $E \in \mathfrak{C}$.*

Here is an interesting corollary of Theorem 3.1.

Corollary 3.2. *Suppose that $1 \le p < \infty$ and $0 < \inf_{n\in\mathbb{N}} w_n \le \sup_{n\in\mathbb{N}} w_n < \infty$.*

(a) *If uC_φ is a weighted composition operator from $l^p(w)$ to $l^p(w)$ such that $\inf_{n\in\operatorname{supp} u} |u(n)| > 0$, then uC_φ has closed range.*

(b) *Every composition operator from $l^p(w)$ to $l^p(w)$ has closed range. In particular, all composition operators from l^p to l^p have closed ranges.*

Proof. It suffices to prove (a) only, since (b) follows from (a) directly.

Put $\alpha := \inf_{n\in\mathbb{N}} w_n$, $\beta := \sup_{n\in\mathbb{N}} w_n$ and $\gamma := \inf_{n\in\operatorname{supp} u} |u(n)|$ $(0 < \alpha, \beta, \gamma < \infty)$. In view of Theorem 3.1, we are to show that there is a constant $c > 0$ such that $[d\mu_p/d\mu](n) \ge c$ for every $n \in \operatorname{supp}[d\mu_p/d\mu]$.

Note that we have

$$\begin{aligned}\left[\frac{d\mu_p}{d\mu}\right](n)\mu(\{n\}) &= \int\limits_{\{n\}} \left[\frac{d\mu_p}{d\mu}\right] d\mu \\ &= \int\limits_{\varphi^{-1}(\{n\})\cap\operatorname{supp} u} |u|^p \, d\mu \\ &= \sum_{i\in\varphi^{-1}(\{n\})\cap\operatorname{supp} u} |u(i)|^p \mu(\{i\})\end{aligned} \tag{3.1}$$

for all $n \in \mathbb{N}$. From (3.1), we see that for each $n \in \operatorname{supp}[d\mu_p/d\mu]$, there is some $j \in \operatorname{supp} u$ with $\varphi(j) = n$. Then

$$\left[\frac{d\mu_p}{d\mu}\right](n) = \frac{\sum_{i\in\varphi^{-1}(\{n\})\cap\operatorname{supp} u} |u(i)|^p \mu(\{i\})}{\mu(\{n\})} \ge \frac{|u(j)|^p\mu(\{j\})}{\mu(\{n\})} \ge \frac{\gamma^p\alpha}{\beta}.$$

By taking $c = (\gamma^p\alpha)/\beta$, we obtain the desired result. □

Example 3.3. Let $X = [0, \infty)$ be equipped with the Lebesgue measure μ on the σ-algebra Σ of Borel sets in $[0, \infty)$. If

$$\varphi(x) := 2x\chi_{[0,1)}(x) + \left(\frac{8}{3}x - \frac{8}{3}\right)\chi_{[1,\frac{7}{2})}(x) + \left(\frac{3}{8}x + \frac{11}{16}\right)\chi_{[\frac{7}{2},\infty)}(x) \quad \text{for } x \in X,$$

it follows from [3, Proposition 2.1] that C_φ is a composition operator from $L^p(\mu)$ $(1 \le p < \infty)$ into itself with

$$\left[\frac{d\mu_p}{d\mu}\right](x) = \frac{7}{8}\chi_{[0,2)}(x) + \frac{73}{24}\chi_{[2,\frac{20}{3})}(x) + \frac{8}{3}\chi_{[\frac{20}{3},\infty)}(x) \quad \mu\text{-a.e. on } X.$$

Using Theorem 3.1, we see that the range of C_φ is closed.

We now consider weighted maps between distinct L^p-spaces that have closed ranges. A few notations are in order. Let $E \in \Sigma$ with $\mu(X \setminus E) > 0$, we define

$$L^p|_E(\mu) = \{f \in L^p(\mu) : f = 0 \ \mu\text{-a.e. on } X \setminus E\}.$$

Then, $L^p|_E(\mu)$ (equipped with the $\|\cdot\|_{L^p(\mu)}$ norm) is a closed subspace of $L^p(\mu)$ and is thus a Banach space. The restriction of uC_φ to $L^p|_E(\mu)$ is denoted by $uC_\varphi|_E$.

Theorem 3.4. *Let uC_φ be a weighted composition operator from $L^p(\mu)$ into $L^q(\nu)$, where $1 \leq q < p < \infty$. The following statements are equivalent.*

(i) *The operator uC_φ has closed range.*
(ii) *The operator uC_φ has finite rank.*
(iii) $\left[\frac{d\mu_q}{d\mu}\right] = 0$ *μ-a.e. on B and the set* $\left\{i \in \mathbb{N} : \left[\frac{d\mu_q}{d\mu}\right](A_i) > 0\right\}$ *is finite.*
(iv) $\nu(\varphi^{-1}(B) \cap \operatorname{supp} u) = 0$ *and the set* $\{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \operatorname{supp} u) > 0\}$ *is finite.*

Proof. (ii)⇒(i) This follows from the fact that every finite-dimensional normed space is closed.

(iii)⇔(iv) Note that

$$\int_E \left[\frac{d\mu_q}{d\mu}\right] d\mu = \int_{\varphi^{-1}(E)} |u|^q \, d\nu = \int_{\varphi^{-1}(E) \cap \operatorname{supp} u} |u|^q \, d\nu$$

for every $E \in \Sigma$. Thus, $[d\mu_q/d\mu]$ vanishes μ-a.e. on B if and only if

$$\int_{\varphi^{-1}(B) \cap \operatorname{supp} u} |u|^q \, d\nu = 0,$$

which in turn is equivalent to $\nu(\varphi^{-1}(B) \cap \operatorname{supp} u) = 0$. For each $i \in \mathbb{N}$, we have

$$\left[\frac{d\mu_q}{d\mu}\right](A_i)\mu(A_i) = \int_{A_i} \left[\frac{d\mu_q}{d\mu}\right] d\mu = \int_{\varphi^{-1}(A_i) \cap \operatorname{supp} u} |u|^q \, d\nu.$$

Therefore, $[d\mu_q/d\mu](A_i) > 0$ if and only if $\nu(\varphi^{-1}(A_i) \cap \operatorname{supp} u) > 0$. The equivalence of (iii) and (iv) has been established.

(iii)⇒(ii) Suppose (iii) holds. To show that uC_φ has finite rank, it suffices to prove the set

$$S := \{g \in uC_\varphi(L^p(\mu)) : \|g\|_{L^q(\nu)} \leq 1\}$$

is compact in $L^q(\nu)$.

If the set $\{i \in \mathbb{N} : [d\mu_q/d\mu](A_i) > 0\}$ is empty, then uC_φ is just the zero operator and (ii) trivially holds. Otherwise, we may assume (upon a suitable re-indexing of atoms if necessary) there exists some $k \in \mathbb{N}$ with

$$\left[\frac{d\mu_q}{d\mu}\right](A_i) \begin{cases} > 0 & \text{if } i = 1, 2, \ldots, k, \\ = 0 & \text{if } i = k+1, k+2, \ldots. \end{cases}$$

Take an arbitrary sequence $\{uC_\varphi f_n\}_{n\in\mathbb{N}}$ in S. Since $[d\mu_q/d\mu] = 0$ μ-a.e. on B, it follows that

$$\begin{aligned}\|uC_\varphi f_n\|^q_{L^q(\nu)} &= \int_{\bigcup_{i\in\mathbb{N}} A_i} \left[\frac{d\mu_q}{d\mu}\right] |f_n|^q \, d\mu + \int_B \left[\frac{d\mu_q}{d\mu}\right] |f_n|^q \, d\mu \\ &= \sum_{i=1}^k \left[\frac{d\mu_q}{d\mu}\right](A_i)|f_n(A_i)|^q \mu(A_i) \\ &= \sum_{i=1}^k \alpha_i |f_n(A_i)|^q,\end{aligned}$$

where

$$\alpha_i := [d\mu_q/d\mu](A_i)\mu(A_i) > 0.$$

With $\|uC_\varphi f_n\|_{L^q(\nu)} \le 1$ for all $n \in \mathbb{N}$, we have $|f_n(A_i)| \le (1/\alpha_i)^{1/q}$ for $i = 1, 2, \dots, k$ and $n \in \mathbb{N}$. By the Bolzano–Weierstrass theorem, there is a subsequence of natural numbers $\{n_j\}_{j\in\mathbb{N}}$ such that for each fixed $i = 1, 2, \dots, k$, the sequence $\{f_{n_j}(A_i)\}_{j\in\mathbb{N}}$ converges. Suppose $\lim_{j\to\infty} f_{n_j}(A_i) = \zeta_i$, say, and define $f = \sum_{i=1}^k \zeta_i \chi_{A_i}$. Then, $f \in L^p(\mu)$ and

$$\|uC_\varphi f\|^q_{L^q(\nu)} = \sum_{i=1}^k \alpha_i |\zeta_i|^q \le 1,$$

i.e. $uC_\varphi f \in S$. Moreover,

$$\begin{aligned}\|uC_\varphi f_{n_j} - uC_\varphi f\|^q_{L^q(\nu)} &= \sum_{i=1}^k \alpha_i |f_{n_j}(A_i) - f(A_i)|^q \\ &= \sum_{i=1}^k \alpha_i |f_{n_j}(A_i) - \zeta_i|^q \to 0 \quad \text{as } j \to \infty.\end{aligned}$$

Hence the set S is compact in $L^q(\nu)$.

(i)⇒(iii) We first prove that if uC_φ has closed range, then $[d\mu_q/d\mu] = 0$ μ-a.e. on B. Suppose, on the contrary,

$$\mu(\{x \in B : [d\mu_q/d\mu](x) > 0\}) > 0.$$

Then there is a constant $\delta > 0$ for which the set

$$G := \{x \in B : [d\mu_q/d\mu](x) \ge \delta\}$$

has positive μ-measure. We may assume $\mu(G) < \infty$. As G is nonatomic, we can further assume that $\mu(X \setminus G) > 0$.

Consider the operator $uC_\varphi|_G$ defined on $L^p|_G(\mu)$. We claim that:

(a) $uC_\varphi|_G(L^p|_G(\mu))$ is closed in $L^q(\nu)$ (under the assumption uC_φ has closed range),
(b) $\ker uC_\varphi|_G = \{0\}$.

To prove (a), take any convergent sequence $\{uC_\varphi|_G f_n\}_{n\in\mathbb{N}}$ in $uC_\varphi|_G(L^p|_G(\mu))$ and let $g \in L^q(\nu)$ satisfy $\|uC_\varphi|_G f_n - g\|_{L^q(\nu)} \to 0$ as $n \to \infty$. Note that $\{uC_\varphi|_G f_n\}_{n\in\mathbb{N}}$ can be regarded as a sequence in $uC_\varphi(L^p(\mu))$. The closedness of range of uC_φ yields a function $f \in L^p(\mu)$ with $g = uC_\varphi f$ ν-a.e. on Y. Since

$$\begin{aligned}\|uC_\varphi|_G f_n - g\|^q_{L^q(\nu)} &= \|uC_\varphi|_G f_n - uC_\varphi f\|^q_{L^q(\nu)} \\ &= \int_G \left[\frac{d\mu_q}{d\mu}\right] |f_n - f|^q \, d\mu + \int_{X\setminus G} \left[\frac{d\mu_q}{d\mu}\right] |f_n - f|^q \, d\mu \\ &= \int_G \left[\frac{d\mu_q}{d\mu}\right] |f_n - f|^q \, d\mu + \int_{X\setminus G} \left[\frac{d\mu_q}{d\mu}\right] |f|^q \, d\mu\end{aligned}$$

for every $n \in \mathbb{N}$, it follows that $\int_{X\setminus G} [d\mu_q/d\mu]|f|^q \, d\mu = 0$. Now,

$$\|uC_\varphi|_G f\chi_G - uC_\varphi f\|^q_{L^q(\nu)} = \int_{X\setminus G} \left[\frac{d\mu_q}{d\mu}\right] |f|^q \, d\mu = 0.$$

Thus, $g = uC_\varphi|_G f\chi_G$ ν-a.e. on Y, where $f\chi_G \in L^p|_G(\mu)$.

To prove (b), we take any $f \in \ker uC_\varphi|_G$. Then $uC_\varphi|_G f = 0$ ν-a.e. on Y, or

$$\int_G [d\mu_q/d\mu]|f|^q \, d\mu = \|uC_\varphi|_G f\|^q_{L^q(\nu)} = 0.$$

This, together with the inequalities

$$0 \le \delta \int_G |f|^q \, d\mu \le \int_G \left[\frac{d\mu_q}{d\mu}\right] |f|^q \, d\mu,$$

implies that $f = 0$ μ-a.e. on G. With $f \in L^p|_G(\mu)$, (b) follows.

By (2.1), there exists a constant $c > 0$ such that

$$\|uC_\varphi|_G f\|_{L^q(\nu)} \ge c\|f\|_{L^p(\mu)}$$

for all $f \in L^p|_G(\mu)$. We claim that this is impossible by showing that for each $\alpha > 0$, there is some $f_\alpha \in L^p|_G(\mu)$ satisfying

$$\|uC_\varphi|_G f_\alpha\|_{L^q(\nu)} < \alpha\|f_\alpha\|_{L^p(\mu)}.$$

For every $n \in \mathbb{N}$, define

$$G_n = \{x \in G : n - 1 \le [d\mu_q/d\mu](x) < n\}.$$

Then

$$G = \left(\bigcup_{n=1}^{\infty} G_n \right) \cup \left\{ x \in G : \left[\frac{d\mu_q}{d\mu} \right] (x) = \infty \right\}.$$

Since uC_φ is a bounded operator from $L^p(\mu)$ into $L^q(\nu)$, it follows from [5, Theorem 2.3] that $[d\mu_q/d\mu] \in L^{p/(p-q)}(\mu)$. In particular, $[d\mu_q/d\mu]$ is finite-valued μ-a.e. on X and

$$\mu \left(\left\{ x \in G : \left[\frac{d\mu_q}{d\mu} \right] (x) = \infty \right\} \right) = 0.$$

With $\mu(G) > 0$, we see that $\mu(G_N) > 0$ for some $N \in \mathbb{N}$. Fix any $\alpha > 0$. By the nonatomicity of G_N, we may choose a set $E_\alpha \in \Sigma$ with $E_\alpha \subset G_N$ and $\mu(E_\alpha) = \mu(G_N)/K$, where $K \in \mathbb{N}$ and

$$K > (N^{\frac{p}{p-q}} \mu(G_N))/\alpha^{\frac{pq}{p-q}}.$$

Take $f_\alpha := \chi_{E_\alpha}$. Then, $f_\alpha \in L^p|_G(\mu)$ and $\|f_\alpha\|_{L^p(\mu)} = (\mu(G_N)/K)^{1/p}$. Moreover,

$$\begin{aligned} \|uC_\varphi|_G f_\alpha\|_{L^q(\nu)} &= \left(\int_{E_\alpha} \left[\frac{d\mu_q}{d\mu} \right] d\mu \right)^{1/q} \\ &\leq (N\mu(E_\alpha))^{1/q} = \frac{N^{\frac{1}{q}} \mu(G_N)^{\frac{1}{p} + \frac{p-q}{pq}}}{K^{\frac{1}{p} + \frac{p-q}{pq}}} < \alpha \|f_\alpha\|_{L^p(\mu)}. \end{aligned}$$

This proves our claim. Therefore, we must have $[d\mu_q/d\mu] = 0$ μ-a.e. on B.

It remains to show that if uC_φ has closed range, then the set

$$\{i \in \mathbb{N} : [d\mu_q/d\mu](A_i) > 0\}$$

is finite.

We also argue by contradiction. Suppose, on the contrary, the set

$$\{i \in \mathbb{N} : [d\mu_q/d\mu](A_i) > 0\}$$

is infinite. Without losing generality, we assume that $[d\mu_q/d\mu](A_i) > 0$ for all $i \in \mathbb{N}$. Let $A := \bigcup_{i \in \mathbb{N}} A_i$ and assume $\mu(X \setminus A) > 0$. This time, we consider the operator $uC_\varphi|_A$ on $L^p|_A(\mu)$ (if $\mu(X \setminus A) = 0$, then (X, Σ, μ) is atomic. Upon replacing $uC_\varphi|_A$ and $L^p|_A(\mu)$ by uC_φ and $L^p(\mu)$ respectively, the proof below is still valid).

Similarly, one may show that $uC_\varphi|_A(L^p|_A(\mu))$ is a closed subspace of $L^q(\nu)$ and $\ker uC_\varphi|_A = \{0\}$. By (2.1), there is a constant $d > 0$ with

$$\|uC_\varphi|_A f\|_{L^q(\nu)} \geq d\|f\|_{L^p(\mu)}$$

for all $f \in L^p|_A(\mu)$. Then

$$\left[\frac{d\mu_q}{d\mu} \right] (A_i)\mu(A_i) = \|uC_\varphi|_A \chi_{A_i}\|^q_{L^q(\nu)} \geq d^q \|\chi_{A_i}\|^q_{L^p(\mu)} = d^q \mu(A_i)^{q/p},$$

so that

$$\left[\frac{d\mu_q}{d\mu}\right](A_i)^{\frac{p}{p-q}}\mu(A_i) \geq d^{\frac{pq}{p-q}}$$

for all $i \in \mathbb{N}$. From the above inequality,

$$\sum_{i\in\mathbb{N}} d^{\frac{pq}{p-q}} \leq \sum_{i\in\mathbb{N}} \left[\frac{d\mu_q}{d\mu}\right](A_i)^{\frac{p}{p-q}}\mu(A_i) = \|uC_\varphi\|^{\frac{pq}{p-q}} < \infty.$$

This is absurd, since $\sum_{i\in\mathbb{N}} d^{\frac{pq}{p-q}} = \infty$.

Hence $[d\mu_q/d\mu](A_i) > 0$ for finitely many $i \in \mathbb{N}$ only. The proof of the implication (i)⇒(iii) is now complete. □

Theorem 3.5. *Let uC_φ be a weighted composition operator from $L^p(\mu)$ into $L^q(\nu)$, where $1 \leq p < q < \infty$. The following statements are equivalent.*

(i) *The operator uC_φ has closed range.*
(ii) *The operator uC_φ has finite rank.*
(iii) *The set $\left\{i \in \mathbb{N} : \left[\frac{d\mu_q}{d\mu}\right](A_i) > 0\right\}$ is finite.*
(iv) *The set $\{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \operatorname{supp} u) > 0\}$ is finite.*

Proof. The proofs of (iii)⇔(iv), (ii)⇒(i) and (iii)⇒(ii) are the same as those in Theorem 3.4. It suffices to prove (i)⇒(iii) only.

Suppose, on the contrary, the set $\{i \in \mathbb{N} : [d\mu_q/d\mu](A_i) > 0\}$ is infinite. We may further assume that $[d\mu_q/d\mu] > 0$ on A, where $A := \bigcup_{i\in\mathbb{N}} A_i$. By [4, Theorem 2.5], $[d\mu_q/d\mu] < \infty$ on A. Let $d > 0$ be a constant such that

$$\|uC_\varphi|_A f\|_{L^q(\nu)} \geq d\,\|f\|_{L^p(\mu)} \quad \text{for all } f \in L^p|_A(\mu).$$

Consider the operator M on $L^p|_A(\mu)$ defined by

$$Mf = \left[\frac{d\mu_q}{d\mu}\right]^{1/q} f \quad \text{for } f \in L^p|_A(\mu).$$

Since $\|Mf\|_{L^q(\mu)} = \|uC_\varphi|_A f\|_{L^q(\nu)}$, M maps $L^p|_A(\mu)$ into $L^q|_A(\mu)$ and M is bounded below. We shall show that $M(L^p|_A(\mu))$ is dense in $L^q|_A(\mu)$, for this implies that M is a surjective map between $L^p|_A(\mu)$ and $L^q|_A(\mu)$.

Take any $g \in L^q|_A(\mu)$ and let $g_n := \sum_{i=1}^n g(A_i)\chi_{A_i}$ for each $n \in \mathbb{N}$. Then

$$\|g_n - g\|^q_{L^q(\mu)} = \|g\|^q_{L^q(\mu)} - \sum_{i=1}^n |g(A_i)|^q \mu(A_i) \to 0$$

as $n \to \infty$. We also define

$$f_n = \begin{cases} [d\mu_q/d\mu]^{-1/q} g_n & \text{on } A, \\ 0 & \text{on } X \setminus A. \end{cases}$$

Note that $f_n \in L^p|_A(\mu)$ (with $\|f_n\|_{L^p(\mu)}^p = \sum_{i=1}^n \frac{|g_n(A_i)|^p \mu(A_i)}{[d\mu_q/d\mu](A_i)^{p/q}} < \infty$) and

$$\|Mf_n - g\|_{L^q(\mu)} = \|g_n - g\|_{L^q(\mu)} \to 0 \quad \text{as } n \to \infty.$$

Thus, $M(L^p|_A(\mu))$ is dense in $L^q|_A(\mu)$.

The inverse operator of M, denoted by M^{-1}, is given by $g \mapsto pg$ for $g \in L^q|_A(\mu)$, where

$$p := \begin{cases} [d\mu_q/d\mu]^{-1/q} & \text{on } A, \\ 0 & \text{on } X \setminus A. \end{cases}$$

By the bounded inverse theorem, M^{-1} is also bounded. Applying [5, Theorem 2.3], we obtain

$$\int_A \frac{1}{[d\mu_q/d\mu]^{p/(q-p)}}\, d\mu < \infty. \tag{3.2}$$

Furthermore, according to [4, Theorem 2.5], we have

$$(0 <)\ \|uC_\varphi\|^q = \sup_{i\in\mathbb{N}} \frac{[d\mu_q/d\mu](A_i)}{\mu(A_i)^{(q-p)/p}}.$$

This, together with (3.2), yields

$$\sum_{i\in\mathbb{N}} \frac{1}{\|uC_\varphi\|^{\frac{pq}{q-p}}} \le \sum_{i\in\mathbb{N}} \frac{\mu(A_i)}{[d\mu_q/d\mu](A_i)^{p/(q-p)}} = \int_A \frac{1}{[d\mu_q/d\mu]^{p/(q-p)}}\, d\mu < \infty.$$

This is absurd because

$$\sum_{i\in\mathbb{N}} 1/\|uC_\varphi\|^{\frac{pq}{q-p}} = \infty.$$

Hence the set $\{i \in \mathbb{N} : [d\mu_q/d\mu](A_i) > 0\}$ is finite if uC_φ has closed range. □

In Corollary 3.2, we showed that every composition operator from l^p to itself has closed range. This is *not* necessarily true, when the domain of the operator differs from the co-domain. The following example demonstrates this.

Example 3.6. Let $T : l^1 \to l^2$ be defined by

$$T(x_1, x_2, \ldots) = (x_2, x_3, \ldots) \quad \text{for every } \{x_n\}_{n\in\mathbb{N}} \in l^1.$$

The map T is a composition operator (also known as the left shift operator) induced by $\varphi(n) = n + 1$ for $n \in \mathbb{N}$. Since

$$[d\mu_2/d\mu](n) = \begin{cases} 0 & \text{if } n = 1, \\ 1 & \text{if } n = 2, 3, \ldots, \end{cases}$$

an appeal to Theorem 3.5 shows that T does not have closed range. This fact can also be established by a direct argument: for each $n \in \mathbb{N}$, let f_n $(= \{x_j^n\}_{j\in\mathbb{N}})$ be the sequence

$$x_j^n := \begin{cases} 1/j & \text{if } j = 1, 2, \ldots, n, \\ 0 & \text{if } j = n+1, n+2, \ldots. \end{cases}$$

Then as $n \to \infty$, $\|Tf_n - g\|_{l^2} \to 0$, where $g := (1/2, 1/3, 1/4, \ldots) \in l^2$. However, $g \notin T(l^1)$.

4. MAIN RESULTS: $p = \infty$ OR $q = \infty$

In this section, we study the closedness of ranges of weighted composition maps from $L^p(\mu)$ into $L^q(\nu)$, where $p = \infty$ or $q = \infty$. When $1 \le p, q < \infty$, we rely on the Radon–Nikodym derivative $[d\mu_q/d\mu]$ in stating characterizations and constructing proofs. This is no longer viable if $q = \infty$. It is necessary to utilize properties intrinsic to uC_φ and the measure $\nu\varphi^{-1}$ instead. The following notations will be adopted in this section.

Let $S \in \Gamma$. The restriction of a Γ-measurable function $g : Y \to \mathbb{C}$ to S is denoted by $g|_S$. We also define the σ-algebra Γ_S and the measure ν_S on Γ_S by

$$\Gamma_S = \{F \cap S : F \in \Gamma\} \quad \text{and} \quad \nu_S(F) = \nu(F) \quad \text{for } F \in \Gamma_S$$

respectively. Then (S, Γ_S, ν_S) is a σ-finite and complete measure space. For $1 \le p \le \infty$ and a Γ_S-measurable function $h : S \to \mathbb{C}$, let

$$\|h\|_p := \begin{cases} \left(\int_S |h|^p \, d\nu\right)^{1/p} & \text{if } 1 \le p < \infty, \\ \inf\{M > 0 : \nu(\{y \in S : |h(y)| > M\}) = 0\} & \text{if } p = \infty. \end{cases}$$

The set $L^p(\nu_S)$ is the Banach space of all (equivalence classes of) Γ_S-measurable functions h on S such that $\|h\|_p < \infty$. We denote the norm $\|\cdot\|_p$ by $\|\cdot\|_{L^p(\nu_S)}$.

If $S \in \Sigma$, then $L^p(\mu_S)$ and $\|\cdot\|_{L^p(\mu_S)}$ are defined similarly.

When (X, Σ, μ) is atomic, we have the following sufficient condition for the closedness of range of uC_φ from $L^\infty(\mu)$ to $L^\infty(\nu)$. A general characterization, however, has not yet been obtained.

Theorem 4.1. *Suppose that (X, Σ, μ) is atomic. Let uC_φ be a weighted composition operator from $L^\infty(\mu)$ into $L^\infty(\nu)$. If there is a constant $\delta > 0$ such that $|u(y)| \ge \delta$ ν-a.e. on $\operatorname{supp} u$, then uC_φ has closed range.*

Proof. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $L^\infty(\mu)$ and g be a function in $L^\infty(\nu)$ with $\|uC_\varphi f_n - g\|_{L^\infty(\nu)} \to 0$ as $n \to \infty$. Then there is a set $F \in \Gamma$ for which $\nu(Y \setminus F) = 0$ and $uC_\varphi f_n$ converges to g uniformly on F. Put $S := \operatorname{supp} u$ and $\psi := \varphi|_S$. Consider the composition operator $C_\psi : L^\infty(\mu) \to L^\infty(\nu_S)$ defined by $C_\psi f = f \circ \psi$ for $f \in L^\infty(\mu)$. Since

$$\left| C_\psi f_n - \frac{g}{u}\Big|_S \right| = \left|\frac{1}{u}\right| |uC_\psi f_n - g| \le \frac{1}{\delta} |uC_\varphi f_n - g| \quad \text{on } S$$

and $\nu(S \setminus (F \cap S)) = 0$, it follows that $\|C_\psi f_n - (g/u)|_S\|_{L^\infty(\nu_S)} \to 0$ as $n \to \infty$, i.e. the sequence $\{C_\psi f_n\}_{n\in\mathbb{N}}$ converges to $(g/u)|_S$ $(\in L^\infty(\nu_S))$ uniformly on $F \cap S$.

We claim that $C_\psi(L^\infty(\mu))$ is closed. To prove this, let $T : L^\infty(\mu)/\ker C_\psi \to L^\infty(\nu_S)$ be defined as

$$T(f + \ker C_\psi) = C_\psi f,$$

with $f \in L^\infty(\mu)$ and $L^\infty(\mu)/\ker C_\psi$ being a Banach space equipped with the quotient norm given by

$$\|f + \ker C_\psi\| := \inf\{\|f + h\|_{L^\infty(\mu)} : h \in \ker C_\psi\}.$$

The map T is well-defined and linear. Fix any $f \in L^\infty(\mu)$. For every $h \in \ker C_\psi$,

$$\|f + h\|_{L^\infty(\mu)} \geq \|C_\psi(f + h)\|_{L^\infty(\nu_S)} = \|T(f + \ker C_\psi)\|_{L^\infty(\nu_S)}.$$

Thus,

$$\|f + \ker C_\psi\| \geq \|T(f + \ker C_\psi)\|_{L^\infty(\nu_S)}. \tag{4.1}$$

If we let

$$s := -\sum_{i \in I} f(A_i)\chi_{A_i},$$

where

$$I := \{i \in \mathbb{N} : \nu\psi^{-1}(A_i) = 0\},$$

then $s \in \ker C_\psi$ and

$$\|C_\psi f\|_{L^\infty(\nu_S)} = \|f + s\|_{L^\infty(\mu)} \ (= \sup_{i \notin I} |f(A_i)|).$$

Hence

$$\begin{aligned} \|T(f + \ker C_\psi)\|_{L^\infty(\nu_S)} &= \|C_\psi f\|_{L^\infty(\nu_S)} \\ &= \|f + s\|_{L^\infty(\mu)} \\ &\geq \|f + \ker C_\psi\|. \end{aligned} \tag{4.2}$$

From (4.1) and (4.2), T is indeed an isometry. Therefore, the range of C_ψ is closed.

In view of the claim, we obtain a function f in $L^\infty(\mu)$ such that $C_\psi f = (g/u)|_S$ ν-a.e. on S. With $g = 0$ ν-a.e. on $Y \setminus S$, we conclude that $uC_\varphi f = g$ ν-a.e. on Y. This completes the proof. □

We digress momentarily to consider invertible weighted composition operators from $L^p(\mu)$ onto $L^p(\nu)$. When $1 \leq p < \infty$ and (X, Σ, μ) is nonatomic, characterizations for this class of operators were obtained by the authors in [8, Theorem 1.2]. We now completely characterize these maps for $p = \infty$, without assuming the nonatomicity of (X, Σ, μ).

Theorem 4.2. *Let uC_φ be a weighted composition operator from $L^\infty(\mu)$ into $L^\infty(\nu)$. Then it is invertible if and only if the following conditions are all satisfied:*

(i) *There exists a constant $\delta > 0$ such that $|u| \geq \delta$ ν-a.e. on Y.*
(ii) *μ is absolutely continuous with respect to $\nu\varphi^{-1}$.*
(iii) *For each set $F \in \Gamma$, there is a set $E \in \Sigma$ such that $\varphi^{-1}(E) = F$.*

Proof. Assume uC_φ is invertible. By (2.1), there exists a constant $c > 0$ with

$$\|uC_\varphi f\|_{L^\infty(\nu)} \geq c\|f\|_{L^\infty(\mu)} \quad \text{for all } f \in L^\infty(\mu).$$

We claim that $|u| \geq c/2$ ν-a.e. on Y. Otherwise, there is a set $G \in \Gamma$ such that $\nu(G) > 0$ and $|u| < c/2$ on G. Let $g \in L^\infty(\mu)$ be the function with $uC_\varphi g = \chi_G$. Then

$$|g| \leq \|g\|_{L^\infty(\mu)} \leq \frac{1}{c}\|uC_\varphi g\|_{L^\infty(\nu)} = \frac{1}{c}\|\chi_G\|_{L^\infty(\nu)} = \frac{1}{c} \quad \mu\text{-a.e. on } X.$$

It follows that

$$1 = |\chi_G| = |uC_\varphi g| = |u||g \circ \varphi| < \frac{c}{2} \cdot \frac{1}{c} = \frac{1}{2} \quad \nu\text{-a.e. on } G.$$

This absurdity justifies our claim. To prove (ii), suppose $\nu\varphi^{-1}(E) = 0$ for $E \in \Sigma$. Since $uC_\varphi \chi_E = 0$, the injectivity of uC_φ implies that $\chi_E = 0$, i.e. $\mu(E) = 0$.

It remains to prove (iii). Choose any set $F \in \Gamma$ with $\nu(F) < \infty$. Let $g \in L^\infty(\mu)$ be the function with $uC_\varphi g = \chi_F$, or $C_\varphi g = (1/u)\chi_F$. Let $\mathcal{E} := \{\varphi^{-1}(E) : E \in \Sigma\}$. Since $C_\varphi g$ is $\mathcal{E}$-measurable, so is $(1/u)\chi_F$. By writing $Y = \bigcup_{i=1}^\infty F_i$, where $\{F_i\}_{i=1}^\infty$ is an increasing sequence of Γ-measurable sets with finite ν-measures, we have $1/u = \lim_{i\to\infty}(1/u)\chi_{F_i}$ on Y. It follows that $1/u$ is $\mathcal{E}$-measurable. Hence χ_F is also $\mathcal{E}$-measurable for all $F \in \Gamma$ satisfying $\nu(F) < \infty$.

Conversely, assume all the three conditions hold. The equality $\varphi^{-1}(E) = F$ in (iii) may be expressed as $C_\varphi \chi_E = \chi_F$. In view of this and the fact that simple functions (not necessarily with supports of finite measures) are dense in $L^\infty(\mu)$ (or $L^\infty(\nu)$), it suffices to show that

$$\|C_\varphi f\|_{L^\infty(\nu)} = \|f\|_{L^\infty(\mu)} \tag{4.3}$$

for every simple function $f \in L^\infty(\mu)$. This will establish the injectivity of C_φ. Together with the inclusion $(1/u)L^\infty(\nu) \subset L^\infty(\nu)$, this shows that uC_φ is invertible as well. To verify (4.3), write $f = \sum_{i=1}^n c_i\chi_{E_i}$, where the E_i's are pairwise disjoint and $\mu(E_i) > 0$ for all i. By (ii), we have $\nu\varphi^{-1}(E_i) > 0$ and so

$$\|C_\varphi f\|_{L^\infty(\nu)} = \left\|\sum_{i=1}^n c_i\chi_{\varphi^{-1}(E_i)}\right\|_{L^\infty(\nu)} = \max_{i=1,2,\dots,n} |c_i| = \|f\|_{L^\infty(\mu)}.$$

□

The proof of the following result is analogous to that of Theorem 3.4 and is thus omitted.

Theorem 4.3. *Let uC_φ be a weighted composition operator from $L^\infty(\mu)$ into $L^q(\nu)$, where $1 \leq q < \infty$. The following statements are equivalent.*

(i) *The operator uC_φ has closed range.*
(ii) *The operator uC_φ has finite rank.*
(iii) $\nu(\varphi^{-1}(B) \cap \operatorname{supp} u) = 0$ *and the set* $\{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \operatorname{supp} u) > 0\}$ *is finite.*

Corollary 4.4. *Suppose that $1 \leq q < p \leq \infty$ and (X, Σ, μ) is nonatomic. If uC_φ is a weighted composition operator from $L^p(\mu)$ into $L^q(\nu)$ with closed range, then it is the zero operator.*

Theorem 4.5. *Let uC_φ be a weighted composition operator from $L^p(\mu)$ into $L^\infty(\nu)$, where $1 \leq p < \infty$. The following statements are equivalent.*

(i) *The operator uC_φ has closed range.*
(ii) *The operator uC_φ has finite rank.*
(iii) *The set $\{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \operatorname{supp} u) > 0\}$ is finite.*

Proof. (ii)⇒(i) Since $uC_\varphi(L^p(\mu))$ is finite-dimensional, it is also closed in $L^\infty(\nu)$.

(iii)⇒(ii) By the boundedness of uC_φ, we have $\nu(\varphi^{-1}(B) \cap \operatorname{supp} u) = 0$ [7, Theorem 4]. Put

$$I := \{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \operatorname{supp} u) > 0\}.$$

If (iii) holds, then it follows that

$$uC_\varphi(L^p(\mu)) = \operatorname{span}_{i \in I} uC_\varphi \chi_{A_i}.$$

Thus, $\dim(uC_\varphi(L^p(\mu))) < \infty$ and (ii) follows.

(i)⇒(iii) We prove the result by contradiction. Suppose, on the contrary, the set $\{i \in \mathbb{N} : \nu(\varphi^{-1}(A_i) \cap \operatorname{supp} u) > 0\}$ is infinite. Without losing generality, we may assume $\nu(\varphi^{-1}(A_i) \cap \operatorname{supp} u) > 0$ for all $i \in \mathbb{N}$. Let $S := \bigcup_{i \in \mathbb{N}} S_i$, where $S_i := \varphi^{-1}(A_i) \cap \operatorname{supp} u$, and put $A := \bigcup_{i \in \mathbb{N}} A_i$. Consider the weighted composition operator

$$wC_\psi : L^p(\mu_A) \to L^\infty(\nu_S)$$

induced by $\psi := \varphi|_S$ with the weight $w := u|_S$. Since wC_ψ is injective and has closed range, it follows from (2.1) that there exists some constant $c > 0$ with

$$\|wC_\psi f\|_{L^\infty(\nu_S)} \geq c\|f\|_{L^p(\mu_A)}$$

for every $f \in L^p(\mu_A)$.

For each $i \in \mathbb{N}$, we now let

$$\|w\|_{S_i,\infty} := \inf\{M > 0 : \nu(\{y \in S_i : |w(y)| > M\}) = 0\}.$$

Note that $0 < \|w\|_{S_i,\infty} < \infty$. Let

$$\Theta := \sum_{i \in \mathbb{N}} \|w\|_{S_i,\infty} \chi_{A_i}$$

and define the operator M_Θ on $L^p(\mu_A)$ by

$$M_\Theta f = \Theta f \quad \text{for } f \in L^p(\mu_A).$$

According to [7, Theorem 4] again, M_Θ is a bounded operator from $L^p(\mu_A)$ into $L^\infty(\mu_A)$ with

$$\|M_\Theta\|^p = \sup_{i \in \mathbb{N}} \|w\|^p_{S_i,\infty} / \mu(A_i) (> 0).$$

It is also plain that

$$\|M_\Theta f\|_{L^\infty(\mu_A)} = \alpha,$$

where

$$\alpha := \sup_{i\in\mathbb{N}} \|w\|_{S_i,\infty}|f(A_i)|.$$

We claim that $\|wC_\psi f\|_{L^\infty(\nu_S)} = \alpha$ as well.

If $y \in S_i$, then

$$|wC_\psi f(y)| = |w(y)f(\varphi(y))| \le \|w\|_{S_i,\infty}|f(A_i)| \le \alpha$$

ν-a.e. on S_i. Since this inequality holds for every $i \in \mathbb{N}$, we have $\|wC_\psi f\|_{L^\infty(\nu_S)} \le \alpha$. With $\|w\|_{S_i,\infty}|f(A_i)| \le \|wC_\psi f\|_{L^\infty(\nu_S)}$ for each $i \in \mathbb{N}$, we also have $\alpha \le \|wC_\psi f\|_{L^\infty(\nu_S)}$. Our claim now follows.

From the preceding paragraphs, we have $\|M_\Theta f\|_{L^\infty(\mu_A)} \ge c\|f\|_{L^p(\mu_A)}$ for $f \in L^p(\mu_A)$. Let

$$f_n := \sum_{i=1}^{n} \|w\|_{S_i,\infty}^{-1}\chi_{A_i} \quad \text{for } n \in \mathbb{N}.$$

Then,

$$\|f_n\|_{L^p(\mu_A)}^p = \sum_{i=1}^{n} \mu(A_i)/\|w\|_{S_i,\infty}^p \quad \text{and} \quad \|M_\Theta f_n\|_{L^\infty(\mu_A)} = 1.$$

Hence

$$\sum_{i=1}^{n} \frac{\mu(A_i)}{\|w\|_{S_i,\infty}^p} \le \frac{1}{c^p} \quad \text{for all } n \in \mathbb{N}.$$

Letting $n \to \infty$ yields $\sum_{i=1}^\infty \mu(A_i)/\|w\|_{S_i,\infty}^p \le 1/c^p < \infty$. On the other hand, with $\|w\|_{S_i,\infty}^p/\mu(A_i) \le \|M_\Theta\|^p$ for each $i \in \mathbb{N}$, we have $\sum_{i=1}^\infty \mu(A_i)/\|w\|_{S_i,\infty}^p = \infty$. This is a contradiction. □

From Example 3.6, not every composition operator from l^p into l^q, $p \ne q$, has closed range. Actually every uC_φ such that the image of $\operatorname{supp} u$ under φ is infinite does not have closed range.

Corollary 4.6. *Let uC_φ be a weighted composition operator from $l^p(w)$ into $l^q(w)$, where $1 \le p, q \le \infty$ and $p \ne q$. Then uC_φ has closed range if and only if $\varphi(\operatorname{supp} u)$ is finite.*

Let uC_φ be a weighted composition map from $L^p(\mu)$ to $L^q(\nu)$, where $1 \le p, q \le \infty$ and $p \ne q$. An immediate consequence of Theorems 3.4, 3.5, 4.3 and 4.5 is that if uC_φ has closed range, then uC_φ is compact. The converse, however, is *not* necessarily true. For example, let $T : l^p \to l^q$, $1 \le p < q \le \infty$, be defined by

$$T(x_1, x_2, x_3, \dots) = \left(x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots\right) \quad \text{for every } \{x_n\}_{n\in\mathbb{N}} \in l^p.$$

The operator T does not have closed range, yet it is compact by [9, Theorems 3.3 and 3.7].

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