PROBLEMS IN EXTREMAL GRAPH THEORY

BY

LALE OZKAHYA

DISSERTATION

Submitted in partial fulfillment of the requirements
for the degree of Doctor of Philosophy in Mathematics
in the Graduate College of the
University of Illinois at Urbana-Champaign, 2010

Urbana, Illinois

Doctoral Committee:

Professor Douglas B. West, Chair
Professor Zoltán Füredi, Director of Research
Professor Alexandr V. Kostochka, Co-Chair
Assistant Professor Sujith Vijay

Abstract

We consider a variety of problems in extremal graph and set theory.

The *chromatic number* of G, $\chi(G)$, is the smallest integer k such that G is k-colorable. The *square* of G, written G^2, is the supergraph of G in which also vertices within distance 2 of each other in G are adjacent. A graph H is a *minor* of G if H can be obtained from a subgraph of G by contracting edges. We show that the upper bound for $\chi(G^2)$ conjectured by Wegner (1977) for planar graphs holds when G is a K_4-minor-free graph. We also show that $\chi(G^2)$ is equal to the bound only when G^2 contains a complete graph of that order.

One of the central problems of extremal hypergraph theory is finding the maximum number of edges in a hypergraph that does not contain a specific forbidden structure. We consider as a forbidden structure a fixed number of members that have empty common intersection as well as small union. We obtain a sharp upper bound on the size of uniform hypergraphs that do not contain this structure, when the number of vertices is sufficiently large. Our result is strong enough to imply the same sharp upper bound for several other interesting forbidden structures such as the so-called strong simplices and clusters.

The *n-dimensional hypercube*, Q_n, is the graph whose vertex set is $\{0,1\}^n$ and whose edge set consists of the vertex pairs differing in exactly one coordinate. The generalized Turán problem asks for the maximum number of edges in a subgraph of a graph G that does not contain a forbidden subgraph H. We consider the Turán problem where G is Q_n and H is a cycle of length $4k+2$ with $k \geq 3$. Confirming a conjecture of Erdős (1984), we show that the ratio of the size of such a subgraph of Q_n over the number of edges of Q_n is $o(1)$, i.e. in the limit this ratio approaches 0 as n approaches infinity.

Acknowledgments

I thank Sujith Vijay for serving in my thesis committee and for his valuable comments on my thesis. I thank Douglas West for many great courses on combinatorics. I am also grateful to him for the seminars and summer REGS meetings he organized. I thank Alexandr Kostochka for many valuable discussions on the problems we worked together. I am grateful to my advisor Zoltán Füredi for willing to share his insights on many problems. I learned a lot from him.

I am grateful to my friends for making my years spent as a graduate student at Champaign more enjoyable, especially Nejan Huvaj and Çiğdem Şengül. I also thank Yelda Aydın, Jeong-Ok Choi and Nil Şırıkçı for our fruitful discussions on mathematics. Most of all I thank my parents for their support and love.

Table of Contents

Chapter 1

Introduction

In the following, we briefly mention the main results. In the last section of this chapter, some terminology is provided for the reader who is unfamiliar with the concepts in graph theory.

1.1 Coloring of the squares of planar graphs

Coloring the vertices of a graph such that vertices near each other receive distinct colors has been an interesting graph theory problem that arose from an application, the so-called *frequency assignment problem*. The general version of this problem in graph theory is called $L(p,q)$-labeling. An $L(p,q)$*-labeling*, a coloring is a coloring of the vertices such that the neighboring vertices and the vertices separated by distance 2 receive labels differing by at least p and q, respectively, where p and q are integers.

In Chapter 2, we present our main result, which concerns the special type of $L(p,q)$-labeling, where p=q=1. We can formulate this labeling problem also as a proper coloring of the *square* of a graph. The *square* of a graph G, written G^2, is the supergraph of G in which also vertices separated by distance 2 in G are adjacent. If G is a graph with maximum degree Δ, then the chromatic number $\chi(G^2)$, and even the clique number of G^2, may be of the order of Δ^2. A trivial lower bound on $\chi(G^2)$ is $\Delta(G)+1$, since G^2 contains a clique of size at least $\Delta(G)+1$. Since $\Delta(G^2) \leq \Delta^2(G)$, it follows that $\chi(G^2) \leq \Delta^2(G)+1$. The graphs C_5 and the Petersen graph have the property that $\chi(G^2) = \Delta^2(G)+1$.

This problem is also motivated by a conjecture of Wegner [95] in 1977, and it is still open except for a few partial results. He made the following conjecture.

Conjecture 1.1.1 (Wegner [95]). *If G be a planar graph with maximum degree $\Delta(G)$, then*

$$\chi(G^2) \leqslant \begin{cases} \Delta(G)+5 & \text{if } 4 \leqslant \Delta(G) \leqslant 7, \\ \lfloor \frac{3}{2}\Delta(G) \rfloor + 1 & \text{if } \Delta(G) \geqslant 8. \end{cases}$$

A graph G has a graph H as a *minor* if H can be obtained from a subgraph of G by contracting edges. We not only verify this conjecture for K_4-minor-free graphs, a subfamily of planar graphs, but we also show that it is sharp for only two special examples of graphs. It would be interesting to look at whether our result also holds for the list coloring version of this problem. Any counterexample to that would also disprove the conjecture of Kostochka and Woodall [65], which states that $\mathrm{ch}(G^2) = \chi(G^2)$ for every graph G.

1.2 Families without clusters

A *set system* (or a *family* or a *hypergraph*) is a collection of sets that are called *members* of the family. One of the central problems of extremal hypergraph (or set) theory is finding the maximum number of edges of a hypergraph that does not contain a forbidden configuration, say H. Hypergraphs with this property are called *extremal H-free hypergraphs*. These problems have been studied intensively during the last half century in particular in the excellent book by Babai and Frankl [9].

A family is *k-uniform* if all its members have size k. The k-uniform family containing all subsets of S is denoted by $\binom{S}{k}$. The most well-known result in this area is the Erdős–Ko–Rado (EKR) Theorem [37]. It states that if $\mathcal{F} \subset \binom{[n]}{k}$ and $\mathcal{F}$ has no two disjoint members are forbidden, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. This is often called the *EKR bound*.

We are interested in finding the size of extremal hypergraphs that do not contain a configuration of sets called an $\mathbf{a}$-cluster. It is defined as follows. Given a p-tuple $\mathbf{a}$ of positive integers, say $\mathbf{a} = (a_1, \ldots, a_p)$, such that $k = a_1 + \cdots + a_p$, an $\mathbf{a}$-*cluster* $\mathcal{A}$ in a k-uniform family $\mathcal{F}$ is a subfamily $\{F_0, \ldots, F_p\}$ such that the sets $F_i \setminus F_0$ and $F_0 \setminus F_i$ for $1 \leq i \leq p$ are pairwise disjoint, and $|F_i \setminus F_0| = |F_0 \setminus F_i| = a_i$. The sets $F_0 \setminus F_i$ for $1 \leq i \leq p$ are said to form an $\mathbf{a}$-*partition* of F_0, and F_0 is called the *host* of the $\mathbf{a}$-cluster. In Chapter 3, we show

that the size of an $\mathbf{a}$-cluster-free k-uniform family $\mathcal{F}$ (excluding the special case $\mathbf{a} = \mathbf{1}$) is at most the EKR bound, and equality holds only if the overall intersection of the members of $\mathcal{F}$ is not empty, i.e. $\mathcal{F}$ is a star. Our result not only proves a conjecture of Mubayi in [80], but also it generalizes the conjecture by considering a larger set of forbidden configurations with small unions.

There is also a hypergraph version of Turán's problem, which asks for the size of the extremal k-uniform hypergraph not containing a complete hypergraph on $d+1$ elements. This problem is open whenever $d+1 > k > 2$. The case $d = k = 2$ is a special case of Turán's theorem proved by Mantel [77] in 1907. In Chapter 3, we discuss additional famous results and open problems in this area.

1.3 Extremal cycle-free subgraphs of the hypercube

In this thesis, an *H-free* graph is a graph not containing H as a subgraph. The *Turán number*, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an H-free graph with n vertices. An H-free graph with with n vertices and $\mathrm{ex}(n, H)$ edges is an *extremal H-free graph*. The *Turán graph*, $T_{n,r}$, is the extremal K_{r+1}-free graph with n vertices; we write $t(n, r)$ for its number of edges. This graph is a complete r-partite graph with parts of size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$. Mantel provided in 1907 the first Turán-type result by determining the extremal K_3-free graphs. He proved that $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$. Much later, Turán (1941) generalized this result to K_r-free graphs for all r, showing that if G is a graph on n vertices and at least $t(n, r-1)$ edges containing no K_r, then G must be the Turán graph $T_{n,r-1}$.

One may ask a similar question for any graph other than the complete graph. In fact, Turán graphs are the extremal graphs for various class of graphs. Indeed, for a graph H with $\chi(H) = r \geq 3$, the size of the extremal H-free graph on n vertices is the same as $t(n, r-1)$, i.e. $\mathrm{ex}(K_n, H) = e(T_{n,r-1})$. On the other hand, the Turán problem for bipartite graphs is much more complicated and still unsolved in general. The graphs with chromatic number 2 are called *degenerate* graphs due to Simonovits [90]. All of the H-free extremal graphs have size $\theta(n^2)$ (asymptotically of the same order with n^2) in the case of non-degenerate H. However, in the case of degenerate H, the extremal H-free graphs have size $o(n^2)$.

Another longstanding Turán problem was motivated by a conjecture of Erdős [35]. The graph Q_n is defined by the vertex set $V(Q_n) = \{0,1\}^n$ and $E(Q_n)$ is the set of pairs differing in exactly one coordinate. The *Turán number* for the hypercube, denoted by $\mathrm{ex}(Q_n, H)$, is the maximum number of edges in a subgraph of Q_n that does not contain any copy of H. Erdős [35] conjectured in 1984 that $\mathrm{ex}(Q_n, C_4) = (0.5 + o(1))e(Q_n)$ and $\mathrm{ex}(Q_n, C_{2k}) = o(1)e(Q_n)$ for $k \geq 3$. The conjecture is disproved for C_6 and settled for C_{4k}, when $k \geq 2$, by Chung [22]. We show that this conjecture is true for C_{4k+2}, when $k \geq 3$. The problem of finding the order of magnitude of $\mathrm{ex}(Q_n, C_{2k})$ in terms of $e(Q_n)$ is still open for $k = 2, 3, 5$.

1.4 Some preliminaries

In this section, we summarize elementary definitions in graph theory that are used in the following chapters. Most of our definitions and notation follows West [97].

1.4.1 Structure and families of graphs

In this work, we consider only finite and simple graphs, i.e. with no multiple edges or loops. The *vertex set* of a graph G is denoted by $V(G)$. An *edge* is a pair of vertices and the *edge set* of G is denoted by $E(G)$. The *order* of G is the size of $V(G)$, denoted by $n(G)$. The *size* of G is $|E(G)|$, denoted by $e(G)$. For an edge e with endpoints u and v, we say that u and v are *incident* to e and e is *induced* by them *in* G. The *deletion* of a vertex v in G is to obtain a graph with vertex set $V(G) \setminus \{v\}$ and edge set induced by $V(G) \setminus \{v\}$ in G. The endpoints u and v of an edge are *adjacent* to each other or *neighbors* of each other. The set of neighbors of a vertex v in G is denoted by $N_G(v)$ or just $N(v)$, and the size of $N(v)$ is the *degree* of v, denoted by $d_G(v)$. The *maximum degree* of a graph G is denoted by $\Delta(G)$. A graph G' is a *subgraph* of G if $V(G')$ is a subset of $V(G)$ and all edges of G' are also present in $E(G)$. For a subset X of $V(G)$, the subgraph of G *induced* by X is the subgraph , denoted by $G[X]$, whose vertex set is X and whose edge set consists of all edges of G having both endpoints in X.

A *complete* graph is a graph, in which the vertices are pairwise adjacent. The vertex

set of a complete graph is called a *clique* and a clique of size r is called an *r-clique*. The *clique number* of G, $\omega(G)$, is the size of largest clique in G. A graph G is *r-partite* (*bipartite* for r=2) if there is a partition of its vertex set into r parts such that each edge in G has endpoints in different parts.

A *matching* is a set of edges whose sets of endpoints are pairwise disjoint. A *perfect matching* M of a graph G is a matching such that each vertex in $V(G)$ is incident to some edge of M. A vertex set is an *independent* set if it does not contain any pair of adjacent vertices. The size of a largest independent set of G and a largest matching of G are denoted by $\alpha(G)$ and $\alpha'(G)$, respectively. A *vertex cover* of G is a set of vertices that contains at least one endpoint of each edge in $E(G)$. Similarly, an *edge cover* of G is a set S of edges such that for each vertex of G, there is at least one edge in S incident to it. The size of a smallest vertex cover of G and a smallest edge cover of G are denoted by $\beta(G)$ and $\beta'(G)$, respectively.

A *path* of *length* k is a graph with $k+1$ vertices, call them $v_0, \ldots, v_k$, whose edge set consists of the pairs $v_i v_{i+1}$ with $0 \leq i \leq k-1$. A *u, v-path* is a path whose endpoints are u and v. Similarly, a *cycle* of *length* k is a graph with k vertices $v_0, \ldots, v_{k-1}$, whose edge set consists of the pairs $v_i v_{i+1}$ for $0 \leq i \leq k-1$ with subscript addition modulo k. The *girth* of G is the length of a shortest cycle in G if G contains a cycle. A cycle of length 3 is also called a *triangle*. The *n-dimensional hypercube*, denoted by Q_n, is the graph whose vertex set is $\{0,1\}^n$, the set of n-tuple with entries either 0 or 1, and whose edge set is the set of pairs that differ in exactly one coordinate.

Two graphs G and H are *isomorphic* if there is a bijection $f : V(G) \to V(H)$ such that $f(u)f(v)$ is an edge of H if and only if uv is an edge in G; then we write $G \cong H$ or $G = H$. Graphs can be partitioned into equivalence classes under the isomorphism relation; and each equivalence class is called an *isomorphism class*. We use K_n, P_n and C_n for the isomorphism classes of complete graphs, cycles and paths with n vertices.

We call a graph G *connected* if there is a path in G between any pair of vertices. A *component* of G is a maximal connected subgraph of G. A graph G is *k-connected* if either G is a complete graph with $k+1$ vertices or G has at least $k+2$ vertices and there is no set of $k-1$ vertices whose deletion makes G disconnected. The *connectivity* of G is the largest

k such that G is k-connected. A vertex of G is called a *cut-vertex* if its deletion increases the number of components of G.

A *curve* is the image of a continuous map from [0,1] to $\mathbb{R}^2$. A *polygonal curve* is a curve composed of finitely many line segments. A *drawing* of a graph G is a representation of G such that vertices correspond to distinct points and edges correspond to polygonal curves connecting its endpoints. A graph G is *planar* if it can be drawn in a plane without any pair of crossing edges; such a drawing is called a *planar embedding* of G. A *region* is an open set in the plane such that any two of its points have a polygonal curve connecting them. The *faces* of a planar graph are the maximal regions of the plane that does not contain any point used in the embedding. A finite graph has an unbounded face, also called the *outer face*. An *outerplanar* graph G is a graph with a planar embedding where all vertices are on the outer face.

We *subdivide* an edge by replacing it with a path of length 2. We *contract* an edge with endpoints u and v by identifying u with v after deleting the edge(s) with endpoints u and v. To *duplicate* an edge e in G is to add an edge sharing the same endpoints with e to G. A graph G has a graph H as a *minor* if H can be obtained from a subgraph of G by contracting edges and deleting isolated vertices; the vertices of G that are vertices of this copy of H are called *branch vertices*. The *square* of G, written G^2, is the supergraph of G in which also vertices separated by distance 2 in G are adjacent.

1.4.2 Coloring

A *coloring* f of $V(G)$ using the set S is a function or labeling $f : V(G) \to S$, where $f(v)$ is called the "color" of vertex v. A *proper coloring* of the vertices of a graph G is a coloring where neighboring vertices have distinct colors. A *proper k-coloring* of the vertex set of a graph G is an assignment of colors from a set of k colors to $V(G)$ such that adjacent vertices have different colors. A graph is *k-colorable* if there is a proper k-coloring of G. The *chromatic number* of G, written $\chi(G)$, is the smallest integer k such that G is k-colorable. A *list assignment* on a graph G is a function L that assigns a set $L(v)$ of colors to each vertex v; and it is *k-uniform* if $|L(v)| = k$ for all v. The *list chromatic number* or *choosability*

of G, denoted by $\mathrm{ch}(G)$, is the minimum k such that for any k-uniform list assignment L on G, there exists a proper coloring in which each vertex is assigned a color from its list. Obviously, $\mathrm{ch}(G)$ is at least as large as $\chi(G)$.

1.4.3 Functions

We denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the largest and smallest integers with value at most and at least x, respectively. Given a positive integer d, we define $\binom{x}{d}$ as $x(x-1)\dots(x-d+1)/d!$. For comparison between the limits of the order of magnitudes of functions, we use the "Oh" notation. If $\limsup_{n\to\infty} |\frac{f(n)}{g(n)}| < \infty$, then $f = O(g)$ or $g = \Omega(f)$. If the functions f and g are asymptotically of the same order of magnitude, i.e. $f = O(g)$ and $f = \Omega(g)$, then we write $f = \Theta(g)$. If $\limsup_{n\to\infty} |\frac{f(n)}{g(n)}| = 0$, then $f = o(g)$ or $g = \omega(f)$. The functions f and g are asymptotically equal if $\limsup_{n\to\infty} |\frac{f(n)}{g(n)}| = 1$.

1.4.4 Turán numbers

A *copy* of H in G is a subgraph of G isomorphic to H. The *Turán number*, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an H-free graph with n vertices. The *generalized Turán number*, denoted by $\mathrm{ex}(G, H)$, is the maximum size of a subgraph of G that does not contain any copy of H. We call an H-free graph G of size $\mathrm{ex}(G, H)$ (or asymptotically of the same order of magnitude with $\mathrm{ex}(G, H)$) an *extremal graph*. The *Turán graph*, $T_{n,r}$, is the extremal K_{r+1}-free graph with n vertices and $t(n, r)$ edges; it is an r-partite graph with parts of size $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$.

1.4.5 Set systems

A *set system* or a *family* is a collection of sets called *members* of the family. A family is *k-uniform* if all members are of size k. We define $[n]$ to be $\{1, \dots, n\}$. The family of all subsets of S is denoted by 2^S. The k-uniform family containing all k-element subsets of S is denoted by $\binom{S}{k}$. A family $\mathcal{F} \subset \binom{S}{k}$ is called *k-partite* if there is a partition of into sets $X_1, \dots, X_k$ such that each member of $\mathcal{F}$ shares at most one element with X_i, for $1 \le i \le k$.

In this case, $X_1, \ldots, X_k$ are *partite sets* of $\mathcal{F}$. If $\mathcal{F} \subset \binom{[n]}{k}$ is k-partite with partite sets $X_1, \ldots, X_k$, then for any $S \subset [n]$, the *projection* $\Pi(S)$ *of* S is defined to be $\{i : S \cap X_i \neq \emptyset\}$. The *trace* of a family $\mathcal{F}$ on a set A, denoted by $\mathcal{F}|A$, is defined to be $\{F \cap A : F \in \mathcal{F}\}$.

A k-uniform family $\mathcal{F} \subset \binom{S}{k}$ is a *star* if the common intersection of its members is a vertex v and $|\mathcal{F}| = \binom{|S|-1}{k-1}$, i.e. $\mathcal{F}$ contains all possible edges containing v. A family $\{F_1, \ldots, F_s\}$ of distict sets is called a *delta-system* or a *sunflower* of *size* s with *center* C if $F_i \cap F_j = C$ for $1 \leq i < j \leq s$. A family $\mathcal{F}$ is *d-wise intersecting* (or *intersecting* if d=2) if each d-tuple of its members has nonempty intersection.

Chapter 2

A Brooks-type bound for squares of K_4-minor-free graphs

2.1 Distance colorings in graphs

A generalization of vertex coloring is $L(p,q)$-labeling. We define $d_G(u,v)$ as the distance between the vertices u and v in a graph G. For integers p and q, an $L(p,q)$-*labeling* of a graph G is a mapping $L : V(G) \to [k] \cup \{0\}$ such that

- $|L(u) - L(v)| \geq p$ if $d_G(u,v)$=1, and
- $|L(u) - L(v)| \geq q$ if $d_G(u,v)$=2.

The p,q-*span* of G, denoted by $\lambda_q^p(G)$, is the minimum k for which an $L(p,q)$-labeling into $[k] \cup \{0\}$ exists.

The applications of this type of coloring arise from the channel assignment problem in radio and cellular phone systems, where the vertices and labels represent the transmitter locations and the frequency channels, respectively. The natural condition that non-neighboring transmitters in close proximity are assigned different frequency channels motivates the $L(p,q)$-labeling problem. This problem is also known as the *frequency assignment problem.*

Griggs and Yeh [52] studied the $L(2,1)$-labeling problem on graphs. They conjectured that for any graph G with $\Delta(G) \geq 2$, $\lambda_2^1(G) \leq \Delta^2(G)$. They confirmed this conjecture for various graph classes such as paths, cycles, trees, and graphs with diameter 2. Chang and Kuo [19] showed that $\lambda_2^1(G) \leq \Delta^2(G) + \Delta(G)$. A more general result given by Král [69] yields that $\lambda_2^1(G) \leq \Delta^2(G)+\Delta(G)-1$ and the present best result $\lambda_2^1(G) \leq \Delta^2(G)+\Delta(G)-2$ is due to Gonçalves [51].

Because of the natural setup of the frequency assignment problem, it also has been an attractive problem to consider for planar graphs. One of the strongest results on coloring of

planar graphs states that if G is a planar graph, then $\chi(G) \leq 4$. This is known as the *Four Color Theorem*, proved in 1977 by Appel and Haken [6] (see also Appel et al. [7]).

Since the case $q = 0$ corresponds to labeling vertices so that adjacent vertices receive labels p apart, the Four Color Theorem provides an immediate upper bound of $3p$ using a labeling of vertices with labels $\{0, p, 2p, 3p\}$ only. For the case $q \geq 1$, an obvious lower bound is $\lambda_q^p \geq q\Delta + p - q + 1$. The best known upper bound on $\lambda_q^p(G)$ for planar G is $\lambda_q^p(G) \leq (4q-2)\Delta + 10p + 38q - 24$, proved by van den Heuvel and McGuiness [92]. Molloy and Salavatipour [79] improved this upper bound asymptotically by showing $\lambda_q^p(G) \leq q\lceil \frac{5}{3}\Delta \rceil + 18p + 77q - 18$.

More results and problems regarding the relation between distances and coloring can be found in Jensen and Toft [57, Section 2.18]. This chapter is organized as follows. In Section 2.2, we provide the most recent results on the coloring of squares of graphs. We give some preliminaries in Section 2.3. In Section 2.4 we discuss the structure of the cliques of order $\lfloor \frac{3}{2}\Delta \rfloor + 1$ in the square of a K_4-minor-free graph with maximum degree Δ. In Section 2.5 we study properties of minimum counterexamples to the theorem, and in the last section we conclude the proof of Theorem 2.2.3.

2.2 Coloring of squares of graphs

The *square* of a graph G, written G^2, is the supergraph of G in which also vertices separated by distance 2 in G are adjacent. Note that $\chi(G^2)=\lambda_1^1(G) + 1$. In 1977, Wegner [95] made the following conjecture for planar graphs.

Conjecture 2.2.1 (Wegner [95])**.** *If G is a planar graph with maximum degree $\Delta(G)$, then*

$$\chi(G^2) \leqslant \begin{cases} \Delta(G) + 5 & \text{if } 4 \leqslant \Delta(G) \leqslant 7, \\ \lfloor \frac{3}{2}\Delta(G) \rfloor + 1 & \text{if } \Delta(G) \geqslant 8. \end{cases}$$

If G is a graph with maximum degree $\Delta(G)$, then the chromatic number $\chi(G^2)$, and even the clique number of G^2, may be of the order of $\Delta(G)^2$. A trivial lower bound on $\chi(G^2)$ is $\Delta(G)+1$, since G^2 contains a clique of order at least $\Delta(G)+1$. The fact that $\Delta(G^2) \leq \Delta^2(G)$

implies $\chi(G^2) \leq \Delta^2(G) + 1$. The graphs C_5 and the Petersen graph have the property that $\chi(G^2) = \Delta^2(G) + 1$.

Wegner's conjecture is still open. Wang and Lih [94] proved that if G is a planar graph with girth g and maximum degree Δ, then $\chi(G^2) \leq \Delta + 5, \Delta + 10$ and $\Delta + 16$, when $g \geq 7, 6$ and 5, respectively. Recently Havet, van den Heuvel, McDiarmid and Reed [54] proved an approximate upper bound of $\frac{3}{2}\Delta + o(\Delta)$, but the exact result has not been proved. The best upper bound so far is $5\Delta/3 + 78$, given by Molloy and Salavatipour [79].

A graph G has a graph H as a *minor* if H can be obtained from a subgraph of G by contracting edges. Kuratowski [70] proved in 1930 that a graph is planar if and only if it does not contain a K_5-minor or a $K_{3,3}$-minor, this result is also known as *Kuratowski's theorem*. A graph obtained by connecting a vertex to all vertices of an outerplanar graph is planar. Thus, Kuratowski's theorem implies a well-known result [20], that a graph is outerplanar if and only if it is K_4-minor-free and $K_{2,3}$-minor-free. A graph is called a *series-parallel* graph if it can be obtained from K_2 by applying a sequence of edge duplications and edge subdivisions. A well-known characterization of a K_4-minor-free graph is that each of its blocks is a series-parallel graph. Therefore, the class of K_4-minor-free graphs contain both outerplanar graphs and series-parallel graphs. The bound of Wegner's conjecture, if true, is sharp, as shown in Section 2.4. Moreover, for every $\Delta \geqslant 4$, there are series-parallel (hence, K_4-minor-free) graphs G with maximum degree Δ such that the chromatic number and clique number of G^2 are both equal to $\lfloor \frac{3}{2}\Delta \rfloor + 1$: see Figure 2.2, where A, B and C are independent sets of suitable orders, as explained in Section 2.4. In these examples G, the clique number of G^2 is $\lfloor \frac{3}{2}\Delta \rfloor + 1$.

Lih, Wang, and Zhu [72] proved the following theorem, which implies that Wegner's conjecture holds for K_4-minor-free graphs.

Theorem 2.2.2. [72] *If G is a K_4-minor-free graph, then*

$$\chi(G^2) \leqslant \begin{cases} \Delta(G) + 3 & \textit{if } 2 \leqslant \Delta(G) \leqslant 3, \\ \lfloor \frac{3}{2}\Delta(G) \rfloor + 1 & \textit{if } \Delta(G) \geqslant 4. \end{cases}$$

Wegner [95] showed that any for planar graph G with $\Delta(G) = 3$ the list chromatic number

$\mathrm{ch}(G)$ is 8. Cranston and Kim [25] showed that for every connected subcubic graph G, $\mathrm{ch}(G^2) \leq 8$, except for the Petersen graph. Hetherington and Woodall [55] proved that the upper bound in Theorem 2.2.2 holds not only for $\chi(G^2)$ but also for the list chromatic number $\mathrm{ch}(G^2)$. They remarked that they "strongly suspect" that the bound $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$ is attained for $\Delta \geqslant 4$ only when G^2 contains a clique of order t. We show that this suspicion is incorrect for $\Delta \in \{4, 5\}$ but correct for every $\Delta \geqslant 6$.

Theorem 2.2.3 ([66])**.** *Let G be a K_4-minor-free graph. If $\Delta(G) \geqslant 6$ and G^2 does not contain a clique of order $\lfloor \frac{3}{2}\Delta \rfloor + 1$, then $\chi(G^2) \leqslant \lfloor \frac{3}{2}\Delta \rfloor$.*

Our proof uses the approach of Hetherington and Woodall [55]. In the next section we introduce some notation and present examples for $\Delta(G) \in \{4, 5\}$. We have failed to prove the analogous result for the list chromatic number, but any counterexample to that would also disprove the conjecture of Kostochka and Woodall [65], which states that $\mathrm{ch}(G^2) = \chi(G^2)$ for every graph G.

2.3 Some preliminaries

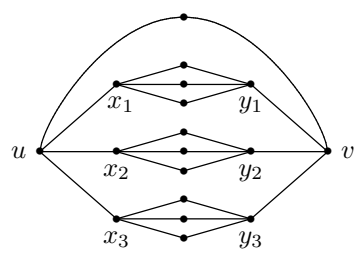

Figure 2.1: A K_4-minor-free graph G with $\Delta(G) = 4$ such that $\chi(G^2) = 7$ and $\omega(G^2)$=6.

Let G be the graph in Figure 2.1. By inspection, G is a K_4-minor-free graph, and $\omega(G^2)$=6. For $i \in \{1, 2, 3\}$, let $C_i = \{x_i, y_i\} \cup (N_G(x_i) \cap N_G(y_i))$. Let f be a proper coloring of G^2, and let $\alpha = f(u)$ and $\beta = f(v)$. Since $uv \in E(G^2)$, we have $\alpha \neq \beta$. Since x_1, x_2, and x_3 all have different colors, at most one of them has color β. Similarly, at most one of y_1, y_2, and y_3 is colored with α. Thus, for some $i \in \{1, 2, 3\}$, neither α nor β is used to color any vertex

of C_i. However, all five vertices of C_i have different colors in f; thus f uses at least seven colors, i.e., $\chi(G^2) \geqslant 7$. The example for $\Delta = 5$ is very similar, only instead of three copies of $K_{2,3}$ we take three copies of $K_{2,4}$, call that graph G'. For some $i, j \in \{1, 2, 3, 4\}$, $i \neq j$, neither α nor β is used to color any vertex of $C_i \cup C_j$ and $\chi(G'^2) \geqslant 8$. Thus for $\Delta \in \{4, 5\}$ there is a K_4-minor-free graph G with maximum degree Δ such that $\chi(G^2) = \lfloor \frac{3}{2}\Delta \rfloor + 1$ but $\omega(G^2) < \chi(G^2)$, contrary to the "strong suspicion" of Hetherington and Woodall [55]. Theorem 2.2.3 shows that this cannot happen if $\Delta \geqslant 6$.

Our proof of Theorem 2.2.3 depends heavily on the following well-known result of Dirac.

Lemma 2.3.1 (Dirac [27])**.** *Every K_4-minor-free graph has a vertex with degree at most* 2.

2.4 Structure of large cliques

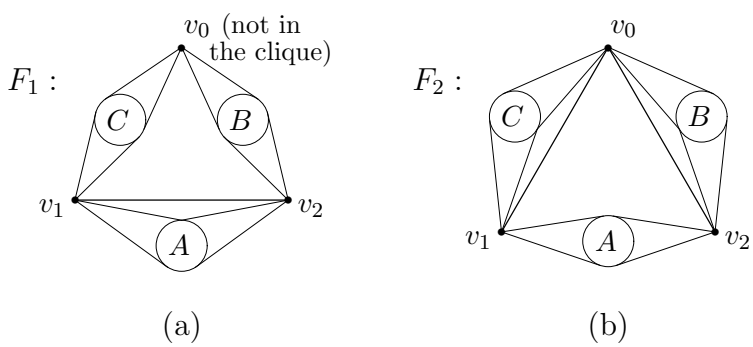

Figure 2.2: The two possible forms for $G[Q]$.

Let F denote the configuration F_1 or F_2 in Figure 2.2, where A, B, and C are sets of vertices, v_0 is adjacent to all vertices in $B \cup C$, v_1 to all vertices in $C \cup A$, and v_2 to all vertices in $A \cup B$. Let $a = |A|$, $b = |B|$ and $c = |C|$. For $F_1^2 - v_0$ or F_2^2 to be a clique of order $\lfloor \frac{3}{2}\Delta \rfloor + 1$, with $\Delta(F) \leqslant \Delta$, we require

$$a + b \leqslant \Delta - 1, \qquad a + c \leqslant \Delta - 1;$$

also, in Figure 2.2(a),

$$b + c \leqslant \Delta,$$
$$a + b + c = \lfloor \tfrac{3}{2}\Delta \rfloor - 1;$$

and, in Figure 2.2(b),

$$b + c \leqslant \Delta - 2,$$
$$a + b + c = \lfloor \tfrac{3}{2}\Delta \rfloor - 2.$$

If Δ is even, then there is a unique solution in each case. If Δ is odd, then there are three solutions in each case, depending on which one of the three inequalities is strict; but two of the three solutions are isomorphic (interchanging B and C). For the remaining, $a, b, c \geqslant \frac{1}{2}(\Delta - 3)$ in each solution, so that each of the sets A, B, C has at least two elements if $\Delta \geqslant 6$. Note also that, in F,

$$\begin{aligned} &\text{if } \Delta \text{ is even, then all of } v_0, v_1, v_2 \text{ have degree } \Delta; \\ &\text{if } \Delta \text{ is odd, then two of } v_0, v_1, v_2 \text{ have degree } \Delta \text{ and one has degree } \Delta - 1; \\ &\text{every other vertex of } F \text{ has degree } 2. \end{aligned} \tag{2.1}$$

By an *F-path* we mean a path whose endvertices are in F but whose internal vertices (if any) are not in F.

Lemma 2.4.1. *Suppose that F ($\cong F_1$ or F_2) is a subgraph of a K_4-minor-free graph G, where each of A, B and C has at least two vertices. Then $A \cup B \cup C$ is an independent set in G, and there is no F-path in G that joins two vertices in $A \cup B \cup C$, or that joins one vertex u in this set to a vertex $v \in \{v_0, v_1, v_2\}$ that is not adjacent to u in F.*

Proof. It is easy to see that if there were an edge or an F-path of the type described, then G would have a K_4-minor. For example, if there is an edge uv or an F-path from u to v, where $u \in A$ and $v \in A \cup B \cup C \cup \{v_0\}$, then there is a K_4-minor with branch vertices u, v, v_1 and v_2. (Note that, since $|A| \geqslant 2$, there is a path from v_1 to v_2 through A that does not

use u.) The remaining cases are similar. $\square$

If $Q \subseteq V(G)$ and Q induces a clique of order $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$ in G^2, then we will say that Q, its t-clique, and $G[Q]$, are all of *standard form* if there is a vertex $v \in V(G)$ such that $G[Q \cup \{v\}] \cong F_1$, or if $G[Q] \cong F_2$. We will define

$$F(G,Q) = \begin{cases} G[Q \cup \{v\}] & \text{if } G[Q \cup \{v\}] \cong F_1, \\ G[Q] & \text{if } G[Q] \cong F_2. \end{cases} \tag{2.2}$$

Lemma 2.4.2. *Let G be a 2-connected K_4-minor-free graph with maximum degree Δ, and suppose that G^2 contains a standard-form clique of order $\lfloor \frac{3}{2}\Delta \rfloor + 1$ with vertex set Q. Let $F = F(G,Q)$. Then either $G \cong F$, or Δ is odd and there is a connected subgraph H of G, and an edge uv of F, where $d_G(u) = \Delta$, $d_F(u) = \Delta - 1$ and $d_F(v) = 2$, such that $G = F \cup H$ and $F \cap H = \{u, uv, v\}$.*

Proof. It follows from Lemma 2.4.1 that F is an induced subgraph of G. Suppose that $G \ncong F$, and let $C_1, \ldots, C_k$ be the components of $G - V(F)$. Since G is 2-connected, there are at least two vertices of F that are adjacent to each component C_i. It follows from Lemma 2.4.1 again that if u and v are adjacent to the same component C_i, then $uv \in E(F)$. Since it is clearly impossible for $d_F(u)$ or $d_F(v)$ to be Δ, it follows from (2.1) that Δ is odd, and one of u and v, say u, is the unique vertex with $d_F(u) = \Delta - 1$, and $d_F(v) = 2$. Since the one edge between u and C_i raises the degree of u to its maximum possible value Δ, there is exactly one component C_1 of $G - V(F)$, and exactly two edges uu' and vv' between F and C_1. If we define H to be the union of C_1 and the path $u'uvv'$, then $G = F \cup H$ and $F \cap H = \{u, uv, v\}$ as required. $\square$

The main result of this section is the following.

Lemma 2.4.3. *Let G be a K_4-minor-free graph. If $\Delta(G) \geqslant 6$, then every clique of size $\lfloor \frac{3}{2}\Delta \rfloor + 1$ in G^2 is of standard form.*

Proof. Assume this is false, and consider a smallest K_4-minor-free graph G with maximum degree at most Δ such that G^2 contains a t-clique K with $V(K) = Q$ that is not of standard form. By the minimality of G, G has no vertices with degree 0 or 1. Therefore, by

Lemma 2.3.1, G has a vertex with degree 2. Let v be such a vertex, with neighbors u and w. We consider two cases.

Case 1: $v \notin Q$. If $u \notin Q$ or $w \notin Q$ or $uw \in E(G)$, then $(G-v)^2$ contains the t-clique K. By the minimality of G, $(G-v)[Q]$ is of standard form, which is a contradiction, since $G[Q] = (G-v)[Q]$. Therefore $u, w \in Q$ and $uw \notin E(G)$.

Let $H = G - v + uw$. Since H is a minor of G (obtained by contracting the edge uv), H is K_4-minor-free. Since $v \notin Q$, $K \subseteq H^2$. By the minimality of G, $H[Q]$ is of standard form. This implies that uw is one of the edges in Figure 2.2, and that by subdividing uw we obtain G such that $G^2[Q]$ is the t-clique K. Notice that every edge in Figure 2.2 is incident with some vertex v_i ($i \in \{0,1,2\}$). By symmetry we may assume that $u \in \{v_0, v_1\}$. If $u = v_0$ and $w \in B$ (respectively, $w \in C$), then the distance between w and C (respectively, w and B) is greater than 2, which contradicts the supposition that $G^2[Q]$ is a clique. If $u = v_1$, then we obtain a similar contradiction using A instead of B. If $uw = v_1v_2$ in F_1, then the distance between v_1 and B in G is greater than 2. Finally, if $uw = v_0v_1$ (respectively, v_0v_2) in F_2, then the distance between v_1 and B (respectively, v_2 and C) is greater than 2. In each case we have a contradiction; thus Case 1 cannot arise.

Case 2: $v \in Q$. Partition the set of vertices in Q at distance exactly two from v as $X_0 \cup X_1 \cup X_2$, where

$$X_0 := (N(u) \cap N(w)) \cap Q \setminus \{v\},$$
$$X_1 := (N(u) \setminus N(w)) \cap Q \setminus \{w\},$$
$$X_2 := (N(w) \setminus N(u)) \cap Q \setminus \{u\},$$

as shown in Figure 2.3. Let $x_i = |X_i|$ for $i = 0, 1, 2$.

Claim 2.4.4. *There is a vertex $z_0 \in V(G) \setminus \{u, v, w\}$ such that z_0 is adjacent to all vertices in $(X_1 \cup X_2) - z_0$.*

In the following is the proof of Claim 2.4.4. Since $X_1 \cup X_2 \subset Q$ by the definition of the sets X_i, and the distance between any two vertices of Q is at most 2, every vertex of X_1 is connected to every vertex of X_2 by a path of length at most 2. Let H be the subgraph of

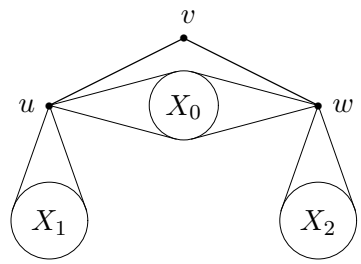

Figure 2.3: The vertex sets in Q.

G induced by the vertices of all paths of length at most 2 between X_1 and X_2. Note that $u, v, w \notin V(H)$, since there are no edges between u and X_2 or between w and X_1.

Suppose there is no vertex z_0 as in the statement of the Claim. Then there is no single vertex whose deletion disconnects all paths of H between X_1 and X_2. Thus, by Pym's version of Menger's theorem, there are two vertex-disjoint paths P_1 and let P_2 in H between X_1 and X_2. Let P_1 have endvertices $p \in X_1$ and $q \in X_2$, and P_2 have endvertices $r \in X_1$ and $s \in X_2$. Since p and s are in a clique in G^2, there is a path P_3 of length at most 2 with endvertices p and s. If P_3 is internally disjoint from P_1 and P_2, then G has a K_4-minor with branch vertices p, s, u and w. If P_3 has a central vertex t, and $t \in V(P_1)$, then G has a K_4-minor with branch vertices s, t, u and w. Similarly, if $t \in V(P_2)$, then G has a K_4-minor with branch vertices p, t, u and w. In every case we have a contradiction. This completes the proof of the claim.

The argument now splits into two subcases.

Subcase 2.1: $uw \in E(G)$. In this case $x_0 + x_1$, $x_0 + x_2 \leqslant \Delta - 2$ and, since $|Q| = t$, $x_0 + x_1 + x_2 \geqslant \lfloor \frac{3}{2}\Delta \rfloor - 2$. This implies that $x_1, x_2 \geqslant \lfloor \frac{1}{2}\Delta \rfloor \geqslant \frac{1}{2}(\Delta - 1)$.

By Claim 2.4.4, there is a vertex $z_0 \in V(G)$ such that z_0 is adjacent to every vertex in $(X_1 \cup X_2) - z_0$. Note that z_0 cannot be in X_0, because $|X_1 \cup X_2 \cup \{u, w\}| > \Delta$. If $z_0 \notin X_1 \cup X_2$, then $G[Q \cup \{z_0\}]$ has the form in Figure 2.2(a), with $A = X_0 \cup \{v\}$, $B = X_1$, $C = X_2$, and $(v_0, v_1, v_2) = (z_0, w, u)$. If $z_0 \in X_1$, then $G[Q]$ has the form in Figure 2.2(b) with $A = X_2$, $B = X_0 \cup \{v\}$, $C = X_1 - z_0$, and $(v_0, v_1, v_2) = (u, z_0, w)$. If $z_0 \in X_2$, then we handle this case similar to the previous case by interchanging X_1 with X_2 and u with w. In each case

we have a contradiction.

Subcase 2.2: $uw \notin E(G)$. In this case $x_0 + x_1$, $x_0 + x_2 \leqslant \Delta - 1$ and, since $|Q| = t$, $x_0 + x_1 + x_2 \geqslant \lfloor \frac{3}{2}\Delta \rfloor - 2$. This implies that $x_1, x_2 \geqslant \lfloor \frac{1}{2}\Delta \rfloor - 1$, so that $x_1, x_2 \geqslant 2$, since we are assuming that $\Delta \geqslant 6$.

Recall that $G[Q]$ has diameter 2. Consider the subgraph induced by the vertices of all paths of length at most 2 connecting the pairs (u, X_2), (w, X_1), and (X_1, X_2). If all these paths go through the vertex z_0 whose existence was proved in Claim 2.4.4, then $z_0 \notin X_1 \cup X_2 \cup \{u, w\}$, since u and w are not adjacent to X_2 and X_1, respectively. However, z_0 is adjacent to all vertices in $X_1 \cup X_2 \cup \{u, w\}$, so $z_0 \in Q$. Thus $z_0 \in X_0$, and $G[Q]$ has the form in Figure 2.2(b) with $A = (X_0 \cup \{v\}) - z_0$, $B = X_1$, $C = X_2$, and $(v_0, v_1, v_2) = (z_0, w, u)$.

This contradiction shows that not all of the paths mentioned go through z_0. By symmetry, interchanging X_1 and X_2 if necessary, we may assume that there is a vertex $q \in X_2$ such that there is a shortest path (of length at most 2) from u to q that does not contain z_0 and clearly does not contain w. Now G has a K_4-minor with branch vertices u, w, q and z_0. (This uses the fact that $|X_1| \geqslant 2$ and $|X_2| \geqslant 2$.) This contradiction completes the proof of Lemma 2.4.3. □

2.5 Structure of minimum counterexamples

Definition 2.5.1. Let $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$. If Theorem 2.2.3 fails for Δ, then among the K_4-minor-free graphs with maximum degree at most Δ whose square has chromatic number at least t but does not contain K_t, choose one having the fewest vertices, and within that the fewest edges. We will call such a graph a (Δ, t)-*graph*.

In this section, we derive a number of properties of (Δ, t)-graphs. We also introduce some terminology that will be used in the proof of Theorem 2.2.3 in the final section. Note that

$$t - 1 = \lfloor \tfrac{3}{2}\Delta \rfloor \geqslant \Delta + 3. \tag{2.3}$$

Lemma 2.5.2. *If G is a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$, then G is 2-connected.*

Proof. Clearly G is connected. Suppose that G has a cut-vertex v. Let $G = G' \cup G''$, where $G' \cap G'' = \{v\}$, $|V(G')| > 1$, and $|V(G'')| > 1$. By the minimality of G, there are proper colorings f' and f'' of G'^2 and G''^2 respectively, using colors in $\{1, 2, \ldots, \lfloor \frac{3}{2}\Delta \rfloor\}$. Permute colors in f'', if necessary, so that v has color $f'(v)$ and no G''-neighbor of v has the same color as any G'-neighbor of v; this is possible, since $|N_G(v) \cup \{v\}| \leqslant \Delta + 1 < \lfloor \frac{3}{2}\Delta \rfloor$. Now the union of the two colorings is a proper $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G, and this contradicts the definition of G. □

For a graph G with $\Delta(G) \geqslant 3$, we follow [55] in denoting by G_1 the graph whose vertices are the vertices of degree at least 3 in G, defined by making two vertices adjacent in G_1 if and only if they are either adjacent in G or connected in G by a path whose internal vertices all have degree 2. Note that, G_1 is a minor of G.

Lemma 2.5.3. *Let G be a graph that does not contain a vertex with degree 0 or 1 or two adjacent vertices with degree 2. Then G_1 has no isolated vertices, and if G is 2-connected, then either G_1 is 2-connected or $G_1 \cong K_2$.*

Proof. Suppose that there is an isolated vertex v in G_1. Then all neighbors of v in G have degree 2. But only two of these vertices can neighbor each other, so there is a vertex in $N_G(v)$, that is adjacent to a vertex of degree 3 other than v. Therefore, v cannot be an isolated vertex of G_1. Note that G_1 can be obtained from G by contracting some edges, each of which has an endvertex of degree 2 at the time of its contraction, and deleting multiple edges. Neither of these operations can create a cut-vertex, and so if G is 2-connected, then G_1 is nonseparable, i.e., it is 2-connected or K_2. □

Lemma 2.5.4. *If G is a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$, then*
(a) *G does not contain a vertex with degree 0 or 1 or two adjacent vertices with degree 2;*
(b) *G_1 exists and is 2-connected.*

Proof. Suppose first that G contains two adjacent vertices u and w of degree 2. Then $(G - \{u, w\})^2 = G^2 - \{u, w\}$. By the minimality of G, $(G - \{u, w\})^2$ is $\lfloor \frac{3}{2}\Delta \rfloor$-colorable. Since $d_{G^2}(u), d_{G^2}(w) \leqslant \Delta + 2 < \lfloor \frac{3}{2}\Delta \rfloor$, we can extend a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G - \{u, w\})^2$ to

G^2, by coloring u and w with available colors not used on $N_{G^2}(u)$ and $N_{G^2}(w)$, respectively. This contradicts the fact that $\chi(G^2) > \lfloor \frac{3}{2}\Delta \rfloor$. Thus G does not contain two adjacent vertices of degree 2. Also, by the minimality of G, it has no vertex with degree 0 or 1. This proves (a).

Since G is 2-connected by Lemma 2.5.2, it follows immediately from (a) and Lemma 2.5.3 that G_1 exists and is either 2-connected or K_2. But if $G_1 \cong K_2$, with vertices u, v, say, then every vertex of G is adjacent to u, and so G^2 is a complete graph; thus G cannot be a (Δ, t)-graph, and this contradiction proves (b). □

Definition 2.5.5. For $u, v \in V(G)$, define

$$M_{uv} := \{x \in N_G(u) \cap N_G(v) : d_G(x) = 2\},$$

$$\epsilon_{uv} := \begin{cases} 1 & \text{if } uv \in E(G), \\ 0 & \text{otherwise,} \end{cases}$$

and

$$d_{uv} := |M_{uv}| + \epsilon_{uv}.$$

Lemma 2.5.6. *Let G be a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$. If $v \in V(G)$ and $N_{G_1}(v) = \{u, w\}$, then $d_{uv} \geqslant \lfloor \frac{1}{2}\Delta \rfloor$ and $d_{vw} \geqslant \lfloor \frac{1}{2}\Delta \rfloor$.*

Proof. Since $v \in V(G_1)$, $d_{uv} + d_{vw} = d_G(v) \geqslant 3$. W.l.o.g. we may assume that $d_{uv} \geqslant 2$, so that $M_{uv} \neq \emptyset$. Let $x \in M_{uv}$; then $(G - x)^2 = G^2 - x$. By the minimality of G, $(G - x)^2$ has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f. Let

$$N_2(x) := (N(u) \setminus \{x\}) \cup (N(v) \setminus N(u)) \cup \{u, v\},$$

which is the set of G^2-neighbors of x. We may assume that $|N_2(x)| \geqslant \lfloor \frac{3}{2}\Delta \rfloor$, since otherwise we can extend f to G^2 by giving x a color that is not used on any vertex in $N_2(x)$. Since $|N(u)| \leqslant \Delta$ and $|N(v) \setminus (N(u) \cup \{u\})| \leqslant d_{vw}$, it follows that $\Delta - 1 + d_{vw} + 2 \geqslant \lfloor \frac{3}{2}\Delta \rfloor$, so that

$$d_{vw} \geqslant \lfloor \tfrac{1}{2}\Delta \rfloor - 1 \geqslant 2. \tag{2.4}$$

By symmetry we may assume also that

$$d_{uv} \geqslant \lfloor \tfrac{1}{2}\Delta \rfloor - 1. \tag{2.5}$$

Suppose now that the lemma is false, say $d_{vw} < \lfloor \frac{1}{2}\Delta \rfloor$. Now (2.4) and its derivation imply

$$d_{vw} = \lfloor \tfrac{1}{2}\Delta \rfloor - 1, \qquad |N_2(x)| = \lfloor \tfrac{3}{2}\Delta \rfloor, \quad \text{and} \quad d_G(u) = \Delta. \tag{2.6}$$

If $uv \in E(G)$, then $v \in N(u) \setminus \{x\}$, and so we have counted v twice in our estimate for $|N_2(x)|$. Hence we may assume that $uv \notin E(G)$. If $vw \notin E(G)$, then the degree of v in G^2 is at most $\Delta + 2 \leqslant \lfloor \frac{3}{2}\Delta \rfloor - 1$, and so we can uncolor v, color x, and then recolor v. Thus we may assume that $vw \in E(G)$. If $uw \in E(G)$ then, since $vw \in E(G)$, we obtain $|N(v) \setminus (N(u) \cup \{u\})| = d_{vw} - 1$, and so $|N_2(x)| < \lfloor \frac{3}{2}\Delta \rfloor$. Thus we may assume that $uw \notin E(G)$. Let y be a vertex in M_{vw}. The picture now is as in Figure 2.4.

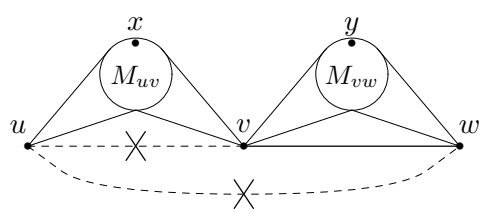

Figure 2.4: The neighborhood of a vertex v contradicting Lemma 2.5.6.

If $d_{uv} < \lfloor \frac{1}{2}\Delta \rfloor$, then by the same argument we can deduce that $uv \in E(G)$ and $vw \notin E(G)$. Since this is not so, we can strengthen (2.5) to

$$d_{uv} \geqslant \lfloor \tfrac{1}{2}\Delta \rfloor \geqslant 3. \tag{2.7}$$

Let G' be the graph obtained from G by deleting all vertices in $M_{uv} \cup M_{vw} \cup \{v\}$ and adding an edge joining u and w. Now G' is a minor of G, and so G' is K_4-minor-free and connected, since G is.

Suppose that G' has a cut-vertex y. If $y \in \{u, w\}$, then y is also a cut-vertex in G.

Similarly, if $y \notin \{u, w\}$, then since $uw \in E(G')$, vertices u and w are in the same component of $G' - y$, and hence y is a cut-vertex in G. Since Lemma 2.5.2 implies that G is 2-connected, G has no cut-vertex. Since no vertex of G has become a cut-vertex of G', G' also has no cut-vertex, and so G' is 2-connected. (Clearly $G' \ncong K_2$, otherwise v is a cut-vertex of G.)

Suppose now that G'^2 contains a t-clique Q. By Lemma 2.4.3, Q is of standard form, and so $F(G', Q)$, defined by (2.2), is one of the graphs shown in Figure 2.2. Let $F = F(G', Q)$. Since $G' - uw \subset G$, and G^2 has no t-clique, it follows that $uw \in E(F)$. Now, $d_F(u) \leqslant d_{G'}(u) = \Delta + 1 - d_{uv} < \Delta - 1$ by (2.6) and(2.7). By (2.1), therefore, $d_F(u) = 2$ and $d_F(w) \geqslant \Delta - 1$, with strict inequality if Δ is even. However, $d_F(w) \leqslant d_{G'}(w) \leqslant \Delta + 1 - d_{vw} = \Delta + 2 - \lfloor \frac{1}{2}\Delta \rfloor$, by (2.6). The only possibility is that $\Delta = 7$, $d_{vw} = 2$, and $d_F(w) = d_{G'}(w) = 6$. It now follows from Lemma 2.4.2 that $F = G'$, so $d_{G'}(u) = d_F(u) = 2$ and, since $d_G(u) = \Delta = 7$ by (2.6), $d_{uv} = 6$ and $d_G(v) = d_{uv} + d_{vw} = 8 > \Delta$. This contradiction shows that G'^2 has no t-clique.

By the minimality of G, there is a proper $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f of G'^2. We will use f to give a proper $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2. Since $uw \in E(G')$, color $f(u)$ is not used on vertices in $N_{G'}(w) - u$. Therefore, we can use $f(u)$ to color y. We consecutively color vertices in M_{vw}, then v, and then vertices in M_{uv}. We can do this, since at the moment of coloring, each vertex in $M_{vw} \cup \{v\}$ has at most $d_G(w)$ colored G^2-neighbors, and (because $f(y) = f(u)$) each vertex in M_{uv} has at most $|N_2(x)| - 1 = \lfloor \frac{3}{2}\Delta \rfloor - 1$ colored neighbors.

This contradiction shows that $d_{vw} \geqslant \lfloor \frac{1}{2}\Delta \rfloor$, and it follows by symmetry that $d_{uv} \geqslant \lfloor \frac{1}{2}\Delta \rfloor$. This completes the proof of Lemma 2.5.6. □

Lemma 2.5.7. *If G is a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$, then the graph G_1 cannot have two adjacent vertices with degree 2.*

Proof. Suppose that there are two adjacent vertices $x, y \in V(G_1)$ with degree 2. Let w and z, respectively, be the other neighbors of x and y in G_1.

Suppose first that $w = z$. Note that z cannot be a cut-vertex of G_1, since G_1 is 2-connected by Lemma 2.5.4. Thus z also has degree 2 in G_1, which is a triangle. Let V_0 consist of the vertices in $\{x, y, z\}$ that are not adjacent in G to another vertex of this set, and let

$V_1 = \{x, y, z\} \setminus V_0$. Now $M_{xy} \cup M_{xz} \cup M_{yz} \cup V_1$ is a clique in G^2, with order at most $\lfloor \frac{3}{2}\Delta \rfloor$, since G is a (Δ, t)-graph. Thus these vertices can be colored with at most $\lfloor \frac{3}{2}\Delta \rfloor$ colors, and the vertices in V_0 are now easily colored, since each has degree at most $\Delta + 2$ in G^2.

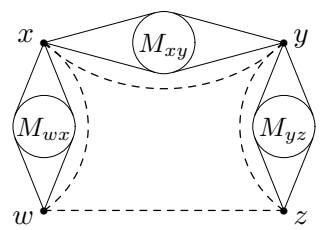

Figure 2.5: The neighborhood of vertices x and y contradicting Lemma 2.5.7.

Thus we may assume that $w \neq z$. (See Figure 2.5, where the broken edges may or may not be present.) By Lemma 2.5.6,

$$\lfloor \tfrac{1}{2}\Delta \rfloor \leqslant d_{wx} \leqslant \lceil \tfrac{1}{2}\Delta \rceil \quad \text{and} \quad \lfloor \tfrac{1}{2}\Delta \rfloor \leqslant d_{yz} \leqslant \lceil \tfrac{1}{2}\Delta \rceil, \tag{2.8}$$

since $d_{wx} = d_G(x) - d_{xy} \leqslant \Delta - d_{xy}$, and similarly for d_{yz}. Without loss of generality, we may assume that $d_{wx} \leqslant d_{yz}$. Let $s = d_{wx} - 1$; note that $s \geqslant 2$ by Lemma 2.5.6. Also

$$d_{wx} = s + 1 \quad \text{and} \quad d_{yz} \leqslant s + 2 \tag{2.9}$$

by (2.8). Let G' be the graph obtained from G by deleting all vertices in $M_{wx} \cup M_{xy} \cup M_{yz} \cup \{x, y\}$, and adding s vertices, $v_1, \ldots, v_s$, each of which is adjacent to w and z. By the definition of s,

$$d_{G'}(w) \leqslant \Delta - 1 \quad \text{and} \quad d_{G'}(z) \leqslant \Delta - 1; \tag{2.10}$$

in particular, the maximum degree of G' is at most Δ.

Since G is 2-connected, G' also is 2-connected. Since G' is a minor of G, G' does not have a K_4-minor. If G'^2 contains a t-clique Q, then Q is of standard form by Lemma 2.4.3, and Q clearly contains at least one of the vertices v_i. Thus at least one of w and z has degree Δ in G' by (2.1), but this contradicts (2.10). Thus G'^2 has no t-clique. By the minimality of

G, G'^2 has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f. We will extend f to a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G. Color s vertices of M_{wx} and s vertices of M_{yz} with the colors $f(v_i)$ $(1 \leqslant i \leqslant s)$. Now consecutively color the remaining vertices in $M_{wx} \cup M_{yz}$, which is possible, since each of these vertices has at most Δ colored G^2-neighbors at the moment of its coloring.

We now color x. If $wx \notin E(G)$, then the number of colored G^2-neighbors of x does not exceed $d_{wx} + d_{yz} \leqslant \Delta + 1$ by (2.8), so there are at least two spare colors for x by (2.3). If $wx \in E(G)$, then the number of colored G^2-neighbors of x does not exceed

$$|\{w\} \cup N_G(w) \setminus \{x\}| + d_{yz} \leqslant \Delta + (s+2) \tag{2.11}$$

by (2.9). Since s colors are used on both M_{wx} and M_{yz}, at most $\Delta + 2 < \lfloor \frac{3}{2}\Delta \rfloor$ colors are forbidden for x, and x can be colored.

We now color y in the same way as x. We have an extra restriction, that $f(y) \neq f(x)$. If $yz \notin E(G)$, then there is no problem, as we have at least one spare color for y. If $yz \in E(G)$ then, since $d_{wx} = s + 1$, we can replace the term $(s + 2)$ by $(s + 1)$ on the RHS of (2.11), which exactly compensates for the extra color $f(x)$ that is forbidden for y. Thus y can be colored.

Finally, note that if $v \in M_{xy}$, then

$$d_{G^2}(v) = (d_G(x) - \epsilon_{xy}) + (d_G(y) - \epsilon_{xy}) - (d_{xy} - \epsilon_{xy} - 1) \tag{2.12}$$

$$\leqslant d_G(x) + d_G(y) - d_{xy} + 1, \tag{2.13}$$

where the first term in (2.12) counts x and all its neighbors except v and y, the second term counts y and all its neighbors except v and x, and the third term subtracts the $|M_{xy}| - 1$ vertices of $M_{xy} \setminus \{v\}$ that have been counted twice in the first two terms. The number of distinct colors that cannot be used on v is at most $d_{G^2}(v) - s$. Thus if $d_{G^2}(v) \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$, then we can color v, since $s \geqslant 2$. If $d_{G^2}(v) > \lfloor \frac{3}{2}\Delta \rfloor + 1$, then, by (2.13) and Lemma 2.5.6, Δ is odd, $d_G(x) = d_G(y) = \Delta$, $d_{xy} = \lfloor \frac{1}{2}\Delta \rfloor$, and $d_{G^2}(v) = \lfloor \frac{3}{2}\Delta \rfloor + 2$. But then $d_{wx} = d_{yz} = \Delta - d_{xy} = \lceil \frac{1}{2}\Delta \rceil \geqslant 4$, and so $s \geqslant 3$ and $d_{G^2}(v) - s \leqslant \lfloor \frac{3}{2}\Delta \rfloor - 1$. In every case, $d_{G^2}(v) - s < \lfloor \frac{3}{2}\Delta \rfloor$, and so we can consecutively color all the vertices of M_{xy} to obtain a

$\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2. This contradicts the definition of G, and this contradiction completes the proof of Lemma 2.5.7. $\square$

Lemma 2.5.8. *If G is a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$, then the graph G_1 cannot contain a 4-cycle $wxyz$ such that x and z both have degree 2 in G_1.*

Proof. Suppose there is such a 4-cycle $wxyzw$ in G_1; call it C. By Lemma 2.5.7, G_1 does not contain two adjacent vertices with degree 2, and so w and y both have degree at least 3 in G_1. By Lemma 2.5.6, Δ is odd and

$$d_{wx} = d_{xy} = d_{yz} = d_{zw} = \lfloor \tfrac{1}{2}\Delta \rfloor, \tag{2.14}$$

and w and y each have exactly one edge in G that is not counted in (2.14). Let these edges join w and y to w' and y' respectively. Note that $|M_{wx}| = \lfloor \frac{1}{2}\Delta \rfloor$ if $wx \notin E(G)$ and $|M_{wx}| = \lfloor \frac{1}{2}\Delta \rfloor - 1$ if $wx \in E(G)$, and similarly for the other edges of C.

Suppose first that $wy \in E(G)$, so that $w' = y$, $y' = w$, and

$$V(G) = M_{wx} \cup M_{xy} \cup M_{yz} \cup M_{zw} \cup \{w, x, y, z\}.$$

Now we can color the vertices of G^2 with $\Delta + 3 \leqslant \lfloor \frac{3}{2}\Delta \rfloor$ colors, by coloring the vertices of M_{wx} and those of M_{yz} from the same set of $\lfloor \frac{1}{2}\Delta \rfloor$ colors, coloring the vertices of M_{xy} and M_{zw} from another set of $\lfloor \frac{1}{2}\Delta \rfloor$ colors, and giving the remaining four colors to w, x, y, z.

We may therefore suppose that $wy \notin E(G)$. Form G' from G by deleting x, z, and all their neighbors except w and y. By the minimality of G, there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f of G'^2. We will extend this coloring to G^2. We may assume that $f(y) \neq f(w)$, since y has at most $\Delta + 1$ colored neighbors in G^2 and so can be recolored if necessary. Choose disjoint sets A and B of $\lfloor \frac{1}{2}\Delta \rfloor$ colors each, which do not include any of the colors of w, w', y, y'. If there is a color not in $A \cup B \cup f(\{w, w', y, y'\})$, then let γ be such a color and let $\alpha = \gamma$ and $\beta = \gamma$; otherwise, the colors of w, w', y and y' are all distinct (and $\Delta = 7$), and we let $\alpha = f(w')$ and $\beta = f(y')$.

Color all vertices of M_{wx} and M_{yz} with colors from A, and color all vertices of M_{xy} and M_{zw} with colors from B, ensuring that if $|M_{wx}| = |M_{yz}| = |A| - 1$, then one color from A is

not used at all, and similarly with B. If G contains all four edges of C, then there is a color in A and one in B that we have not used, and we can use these on x and z. If G omits only one edge of C, say the edge wx, then we can color x with α and use a color from B to color z. If G contains edges wx and wz (only) of C, then we can color x with the color from A that is not used on M_{wx}, and z with the color from B that is not used on M_{wz}. If G contains edges wx and xy (only) of C, then we can color x with color γ if it exists. If γ does not exist, then let v be the unique vertex in M_{yz} whose color is not used on M_{wx}, color x with $f(v)$, and recolor v with $f(w)$. Now z can be colored, since it has only $\Delta + 1$ neighbors in G^2. Finally, if G does not contain two adjacent edges of C, without loss of generality, assume $wx, yz \notin E(G)$. Now we can color x with α and y with β. Every other case is similar to one of these, leading to a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G, and this contradiction proves Lemma 2.5.8. $\square$

Let a 2-*thread* in G_1 be a path of length 2 whose internal vertex has degree 2 in G_1.

Lemma 2.5.9. *If G is a (Δ, t)-graph, where $\Delta \geqslant 6$ and $t = \lfloor \frac{3}{2}\Delta \rfloor + 1$, then the graph G_1 has a triangle $xywx$ such that $d_{G_1}(w) = 2$ and $d_{G_1}(y) = 3$.*

Proof. By Lemma 2.5.4, G_1 is 2-connected and so does not contain a vertex with degree 0 or 1. By Lemma 2.5.7, G_1 does not contain two adjacent vertices with degree 2. Let G_2 be the graph obtained from G_1 by suppressing each vertex v of degree 2 (i.e., contracting one edge incident with v) and removing multiple edges; in other words, $G_2 = (G_1)_1$. It follows from Lemma 2.5.3 that G_2 exists and is 2-connected or is K_2. If $G_2 \cong K_2$, with vertices w, y, then, since $d_{G_1}(w) \geqslant 3$, G_1 contains at least two 2-threads wxy and wzy between w and y, and so contains a 4-cycle $wxyzw$ of the sort that was proved impossible in Lemma 2.5.8. Thus G_2 is 2-connected and has minimum degree at least 2.

Since G_2 is a minor of G_1, G_2 is K_4-minor-free. So, by Lemma 2.3.1, G_2 has a vertex y with degree 2; let its neighbors in G_2 be x and z. By Lemma 2.5.8, there cannot be two or more 2-threads in G_1 connecting x and y or connecting y and z, and so y is connected to each of x and z by an edge, or a 2-thread, or both. By the definition of G_2, $d_{G_1}(y) > 2$, and so there is no loss of generality in assuming that y is connected to x in G_1 by an edge and a 2-thread ywx, forming a triangle $xywx$. If y is connected to z by a 2-thread but not by

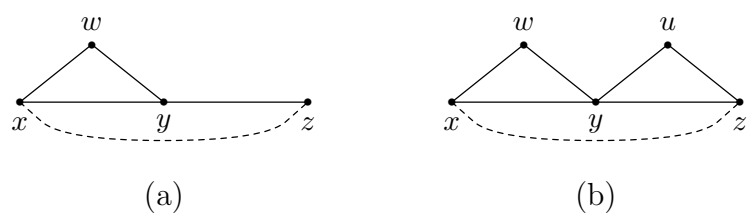

Figure 2.6: The subgraphs induced by $N_{G_1}(y) \cup \{y\}$ in G_1.

an edge, then redefine z to be the middle vertex of this 2-thread. Now y and its neighbors in G_1 induce one of the graphs in Figure 2.6 (where the broken edges may or may not be present). However, the graph in Figure 2.6(b) is impossible because, $d_G(y)$ would be at least $d_{uy} + d_{wy} + 2 \geqslant \Delta + 1$, by Lemma 2.5.6. Therefore, y and its neighbors in G_1 induce the subgraph in Figure 2.6(a). $\square$

2.6 Proof of the main theorem

Let $\Delta \geqslant 6$ and $t := \lfloor \frac{3}{2}\Delta \rfloor + 1$. If the theorem fails for Δ, then there exists a (Δ, t)-graph G (defined at the start of Section 2.5). By Lemma 2.5.9, G_1 contains a subgraph of the form depicted in Figure 2.6(a). In G, this corresponds to the subgraph depicted in Figure 2.7, where the broken edges may or may not be present. Among all possible subgraphs of this form in G, choose one such that d_{wy} is as small as possible. By Lemma 2.5.6,

$$d_{wx} \geqslant \lfloor \tfrac{1}{2}\Delta \rfloor \quad \text{and} \quad d_{wy} \geqslant \lfloor \tfrac{1}{2}\Delta \rfloor. \tag{2.15}$$

Since $d_{wx} + d_{wy} = d_G(w) \leqslant \Delta$, it follows that equality holds in both parts of (2.15) if Δ is even, and in at least one part if Δ is odd.

If $v \in M_{wx}$, then

$$d_{G^2}(v) = (d_G(w) - \epsilon_{wx}) + (d_G(x) - \epsilon_{wx}) - (d_{wx} - \epsilon_{wx} - 1) - \epsilon_{wy}\epsilon_{xy}, \tag{2.16}$$

where the first term in (2.16) counts w and all its neighbors except v and x, the second

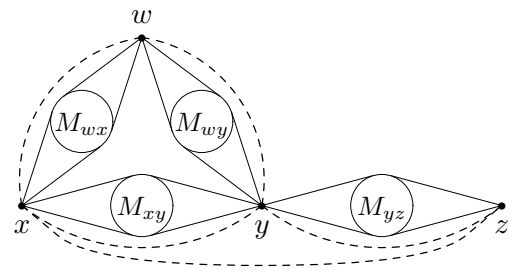

Figure 2.7: The induced subgraph of G.

term counts x and all its neighbors except v and w, the third term subtracts the $|M_{wx}| - 1$ vertices of $M_{wx} \setminus \{v\}$ that have been counted twice in the first two terms, and the last term accounts for y, which is also counted twice if $wy, xy \in E(G)$. Let $p = d_{G^2}(v)$, which is the same for all $v \in M_{wx}$. It follows from (2.16), using (2.15) in the third line, that

$$\begin{aligned} p &= d_G(w) + d_G(x) + 1 - d_{wx} - \epsilon_{wx} - \epsilon_{wy}\epsilon_{xy} \\ &\leqslant 2\Delta + 1 - d_{wx} - \epsilon_{wx} - \epsilon_{wy}\epsilon_{xy} \\ &\leqslant \lceil \tfrac{3}{2}\Delta \rceil + 1 \\ &= \begin{cases} \lfloor \frac{3}{2}\Delta \rfloor + 2 \text{ if } \Delta \text{ is odd,} \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 \text{ if } \Delta \text{ is even.} \end{cases} \end{aligned} \tag{2.17}$$

Similarly, let $q = d_{G^2}(v)$ for all $v \in M_{wy}$. Then

$$\begin{aligned} q &= d_G(w) + d_G(y) + 1 - d_{wy} - \epsilon_{wy} - \epsilon_{wx}\epsilon_{xy} \\ &\leqslant 2\Delta + 1 - d_{wy} - \epsilon_{wy} - \epsilon_{wx}\epsilon_{xy} \\ &\leqslant \begin{cases} \lfloor \frac{3}{2}\Delta \rfloor + 2 \text{ if } \Delta \text{ is odd,} \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 \text{ if } \Delta \text{ is even.} \end{cases} \end{aligned} \tag{2.18}$$

Let G_w denote the graph obtained from G by deleting w and all its neighbors except x and y. Let G^- be obtained from G_w by deleting y and all its neighbors except x and z. Let G^+ be obtained from G^- by adding the edge xz if it is not already present. Let $N'(x) := N_{G^-}(x)$ and $N'(z) := N_{G^-}(z)$. Since, by (2.15), $d_{wx} + d_{xy}$ and $d_{wy} + d_{xy}$ are both at least $\lfloor \frac{1}{2}\Delta \rfloor + 1$,

it follows that

$$|N'(x)| \leqslant \lceil \tfrac{1}{2}\Delta \rceil - 1 \quad \text{and} \quad d_{yz} \leqslant \lceil \tfrac{1}{2}\Delta \rceil - 1. \tag{2.19}$$

Let $S = N'(x) \cup M_{xy} \cup M_{yz} \cup \{x, y\}$ and $S^+ = S \cup \{z\}$. Note that

$$|S| \leqslant (\lceil \tfrac{1}{2}\Delta \rceil - 1) + (\Delta - d_{wy}) + 2 \leqslant \Delta + 2, \tag{2.20}$$

by (2.15) and (2.19). Recall that $t - 1 = \lfloor \frac{3}{2}\Delta \rfloor \geqslant \Delta + 3$ by (2.3).

Lemma 2.6.1. *If $z \notin N'(x)$, and either* (i) *or* (ii) *holds, and at least one of* (iii) *and* (iv) *holds*:

(i) $|N'(x)| = 1$;

(ii) $|N'(x)| = 2$ *and* $M_{yz} = \emptyset$;

(iii) *there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f of $(G^-)^2$ such that all vertices in $N'(x) \cup \{x, z\}$ have different colors*;

(iv) $(G^+)^2$ *has no t-cliques.*

Then there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$ such that all vertices in S^+ have different colors.

Proof. We start by proving a claim, which is needed only in one special case, but which cannot be avoided.

Claim 2.6.2. *If $d_G(z) = \Delta \in \{6, 7\}$, and* (ii) *and* (iii) *hold ($N'(x) = \{u_1, u_2\}$), then f can be chosen so that some vertex in $N'(z)$ has the same color as one of u_1, u_2, x.*

Let us assume that this is not true for the given f, so the $\lfloor \frac{3}{2}\Delta \rfloor = \Delta + 3$ distinct colors are those of u_1, u_2, x, z and the $\Delta - 1$ vertices in $N'(z)$. Note that x has degree 2 in G^-. Choose a vertex $z_1 \in N'(z)$. There are three cases.

Case 1: *x is a cut-vertex in G^-.* Here u_1 and u_2 are in different components of $G^- - x$. Let u_1 be in the component not containing z_1, and interchange the colors $f(u_1)$ and $f(z_1)$ throughout this component.

Case 2: *there are two internally disjoint paths between x and z in G^-.* Here there is no path between u_1 and u_2 in $G^- - \{x, z\}$, otherwise G contains a K_4-minor. Thus u_1 and u_2

are in different components of $G^- - \{x, z\}$. Let u_1 be in the component not containing z_1, and interchange the colors $f(u_1)$ and $f(z_1)$ throughout this component.

Case 3: *neither of the previous cases applies.* Now there is a cut-vertex $v \in V(G^-)$ such that x and z are in different components of $G^- - v$. Let $C(x)$ be the component that contains x, and let α be a color not in $f(N(v) \cup \{v, z\})$. If $\alpha \in f(N'(z))$, then interchange colors $f(x)$ and α throughout $C(x)$. Otherwise, $\alpha \in f(\{u_1, u_2, x\})$, by the first sentence of the proof, so choose $z_1 \in N'(z)$ such that $f(z_1) \neq f(v)$, and interchange colors $f(z_1)$ and α throughout $C(x)$. This concludes the proof of Claim 2.6.2.

We can now prove Lemma 2.6.1. Suppose first that (iii) holds. Transfer the given coloring f to $(G_w)^2$, and extend it to all uncolored vertices in $N_G(z)$ by consecutively coloring each of them differently from all colored vertices in the set $N'(x) \cup N_G(z) \cup \{x, z\}$; call this set T. This is possible, because if we are coloring a vertex in T, then there are at most $|T| - 1$ vertices in T that are colored already. Thus, at each stage, the number of colored vertices in T is at most $\lfloor \frac{3}{2}\Delta \rfloor - 1$ unless $|N'(x)| = 2$ (in which case (ii) holds), and $|N_G(z)| = \Delta$, and $\lfloor \frac{3}{2}\Delta \rfloor = \Delta + 3$. We have shown in Claim 2.6.2 that in this case we can choose f so that the colors of the vertices in T are not all distinct.

We can now consecutively color all vertices in M_{xy}, and y if $yz \notin E(G)$, by coloring each of them differently from all colored vertices in S^+, of which there are at most $|S^+| - 1 \leqslant \Delta + 2$ by (2.20). This gives the required $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$.

This proves the result when (iii) holds. Suppose now that (iv) holds. Since G^+ is a minor of G, G^+ is K_4-minor-free, and its maximum degree is clearly at most Δ. By hypothesis (iv), $(G^+)^2$ has no t-cliques, and so, by the minimality of G, $(G^+)^2$ has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f, in which all vertices in $N'(x) \cup \{x, z\}$ necessarily have different colors; thus (iii) holds, and the result follows. □

Lemma 2.6.3. *If there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f of $(G_w)^2$ in which all vertices of S have different colors and $f(x) \neq f(z)$, then f can be chosen so that there exists a vertex $u \in N'(x)$ with $f(u) \neq f(z)$.*

Proof. If the conclusion fails for the given f, then $|N'(x)| \leqslant 1$. Since G is 2-connected (by Lemma 2.5.2), z is not a cut-vertex, and so $|N'(x)| = 1$ and $N'(x) \neq \{z\}$. Let $N'(x) = \{u\}$.

Since $f(u) = f(z)$, $d_G(u, z) \geqslant 3$ and $xz \notin E(G)$.

Suppose that $(G^+)^2$ has a t-clique, with vertex set Q, say. By Lemma 2.4.3, Q is of standard form in G^+, and so $F(G^+, Q)$, defined by (2.2), is one of the graphs shown in Figure 1. Since x has degree 2 in G^+, and G^2 has no t-cliques, it follows that $x \in Q$ and u is connected to z by more than one path of length 2 in G^+. This is impossible, since $d_G(u, z) \geqslant 3$. Thus $(G^+)^2$ has no t-cliques. Thus hypotheses (i) and (iv) of Lemma 2.6.1 hold, and the result follows from Lemma 2.6.1. □

Lemma 2.6.4. *If there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f of $(G_w)^2$ in which all vertices of S have different colors and $f(x) \neq f(z)$, then there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2.*

Proof. By Lemma 2.6.3, we may assume that there is a vertex $u \in N'(x)$ such that $f(u) \neq f(z)$. (We use u in Case 1.)

We first color w differently from all the colored vertices in $(N_G(x) \setminus M_{wx}) \cup M_{yz} \cup \{x, z\}$, of which there are at most $\Delta - d_{wx} + d_{yz} + 2 \leqslant \Delta + 2$ by (2.15) and (2.19).

Case 1: *either Δ is even, or $d_G(w) < \Delta$, or $d_{wx} = \lfloor \frac{1}{2}\Delta \rfloor + 1$.* In this case we will first color consecutively all vertices in M_{wy}, each of them differently from the (at most $\Delta - 1$) colored neighbors of y and from w, x, y, a total of at most $\Delta + 2$ colors; note that $\Delta + 2 \leqslant \lfloor \frac{3}{2}\Delta \rfloor - 1$ by (2.3). In doing this, we take care to use the color $f(u)$ on one vertex of M_{wy}. We now consecutively color the vertices of M_{wx}, in four subcases.

Subcase 1.1: *$wx \in E(G)$.* Here $p \leqslant \lfloor \frac{3}{2}\Delta \rfloor$ by the hypothesis of Case 1 and (2.17) (since $\epsilon_{wx} = 1$). Since every vertex in M_{wx} has two G^2-neighbors with the same color $f(u)$, the vertices in M_{wx} can all be colored.

Subcase 1.2: *$wx, wy \notin E(G)$.* Here $p \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$, so if we try to color the vertices of M_{wx} as in Subcase 1.1 it is only with the last vertex that we may fail. If this happens, then uncolor w, color the last vertex in M_{wx}, and recolor w, which is possible, since w has at most $\Delta + 2$ neighbors in G^2.

Subcase 1.3: *$wx \notin E(G)$ and $wy, xy \in E(G)$.* Here $p \leqslant \lfloor \frac{3}{2}\Delta \rfloor$ by (2.17), and we color as in Subcase 1.1.

Subcase 1.4: *$wx, xy \notin E(G)$ and $wy \in E(G)$.* Here $p \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$. Now x is not adjacent to the vertices of M_{wy} in G^2, and so when we color M_{wy}, as well as using $f(u)$ on one vertex of M_{wy}, we also use $f(x)$ on another vertex. Then, when we color the vertices of M_{wx} as in Subcase 1.1, each has two pairs of G^2-neighbors with the same color, and the coloring succeeds.

Case 2: $d_{wy} \neq \lfloor \frac{1}{2}\Delta \rfloor$. Here Δ is odd, $d_{wx} = \frac{1}{2}(\Delta - 1)$, and $d_{wy} = \frac{1}{2}(\Delta + 1)$, by (2.15). Note that Cases 1 and 2 are exhaustive, since if $d_{wy} = \lfloor \frac{1}{2}\Delta \rfloor$, then the hypotheses of Case 1 are satisfied.

We will first color consecutively all vertices in M_{wx}, each of them differently from the (at most $\Delta - 1$) colored neighbors of x and from w, x, y, a total of at most $\Delta + 2 \leqslant \lfloor \frac{3}{2}\Delta \rfloor - 1$. Case 2 now divides into three.

Case 2a: *either $M_{yz} \neq \emptyset$, or $M_{yz} = \emptyset$* (here $yz \in E(G)$) and $f(z)$ is not used on any vertex of $N'(x)$. Choose $v \in M_{yz}$ in the first case and let $v = z$ in the second. When we color M_{wx}, we make sure to use $f(v)$ on one vertex of M_{wx}. We can now color the vertices in M_{wy} exactly as in Case 1, interchanging x with y and p with q, and using v instead of u and (2.18) instead of (2.17). In each subcase, q satisfies the same upper bound as was given for p in the corresponding subcase of Case 1.

Case 2b: *$M_{yz} = \emptyset$ and $f(z) \in f(N'(x))$ and $d_G(y) < \Delta$.* Here there is no vertex v as in Case 2a, but in each subcase the upper bound for q is one less than in Case 2a, and so the argument works with no need for v.

Case 2c: *$M_{yz} = \emptyset$ and $f(z) \in f(N'(x))$ and $d_G(y) = \Delta$.* Here $d_{xy} = \frac{1}{2}(\Delta - 3)$, and so $|N'(x)| \leqslant 2$. Let $N'(x) = \{u_1, u_2\}$, where for the moment we allow the possibility that $u_1 = u_2$. We may assume that

$$f(z) \in f(N'(x)) \tag{2.21}$$

for every $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f of $(G_w)^2$ satisfying the hypotheses of the lemma, since otherwise the result follows by Case 2a.

Subcase 2c.1: *$z \in \{u_1, u_2\}$, i.e., $xz \in E(G)$.* Here we have a t-clique in G^2, a contradiction, unless either all of the edges wx, wy, xy are in G, or none of these edges are in G. If all of

these edges are in G, then $q \leqslant \lfloor \frac{3}{2}\Delta \rfloor - 1$ by (2.18), and so we can color all the vertices in M_{wy}. If none of the edges wx, wy, xy are in G, then $q \leqslant \lfloor \frac{3}{2}\Delta \rfloor + 1$, but we can use $f(y)$ on some vertex of M_{wx} and also uncolor w before coloring the last vertex of M_{wy}, after which it easy to recolor w.

Subcase 2c.2: $z \notin \{u_1, u_2\}$. Assume $f(u_1) = f(z)$. This implies that $d_G(u_1, z) \geqslant 3$.

If $(G^+)^2$ has no t-cliques, then hypotheses (i) or (ii), and (iv), of Lemma 2.6.1 hold, and the $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$ whose existence was proved in Lemma 2.6.1 contradicts (2.21). So we may assume that $(G^+)^2$ has a t-clique Q. By Lemma 2.4.3, Q is of standard form in G^+, i.e., $F(G^+, Q)$, defined by (2.2), is isomorphic to one of the graphs F_1 and F_2 in Figure 2.2. Let $F = F(G^+, Q)$.

Since G^2 has no t-cliques, and x has degree at most $3 < \Delta - 1$ in G^+, it follows from (2.1) that x has degree 2 in F and the three vertices of degree at least $\Delta - 1$ in F are z, another neighbor u_i of x, and a third vertex w'. Now u_i and z have common neighbors other than x in F, and hence in G. Since $d_G(u_1, z) \geqslant 3$, it follows that $i \neq 1$, so $u_1 \neq u_2$ and the 'big' vertices in F are z, u_2 and w'. It follows from this that Q is the only t-clique in $(G^+)^2$.

Since x has a G^+-neighbor u_1 that is not in F, and G is 2-connected (by Lemma 2.5.2), it follows from Lemma 2.4.2 that either $\{u_2, x\}$ or $\{x, z\}$ is a cutset of G, and there is a subgraph H of G^+ such that $G^+ = F \cup H$ where $F \cap H = \{u_2, u_2x, x\}$ or $\{x, xz, z\}$. There are two cases to consider.

Case 1: $F \cap H = \{u_2, u_2x, x\}$. Here u_2 is a cut-vertex of G^-. The given coloring f of $(G_w)^2$ induces a coloring of $(G^-)^2$, and we can easily permute colors in this induced coloring so that z has a different color from both u_1 and x (and, necessarily, from u_2). Now hypotheses (ii) and (iii) of Lemma 2.6.1 hold, and the $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$ whose existence was proved in Lemma 2.6.1 contradicts (2.21).

Case 2: $F \cap H = \{x, xz, z\}$. In this case z is the vertex of degree $\Delta - 1$ in F, and both x and z have degree 2 in H. Now, H^2 has no t-cliques, since we have already seen that Q is the only t-clique in $(G^+)^2$. Since H is a minor of G, it is K_4-minor-free. By the minimality of G, H^2 has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f'. Let $z_1 \neq x$ be the other neighbor of z in H. Note that $f'(u_1) \neq f'(z)$ and $f'(x) \neq f'(z_1)$.

Let $C' = F - xz$. Note that C' is a configuration of the same type as the configuration C in Figure 2.7 that we have been working with, with the vertices w', z, u_2, x playing the roles of w, x, y, z respectively; and this configuration exists in G^- with z having exactly one neighbor, z_1, outside C'. Let us emphasize this by writing $x' = z$, $y' = u_2$, $z' = x$, and $u' = z_1$. Since $d_{wy} > \lfloor \frac{1}{2}\Delta \rfloor$ by the hypothesis of Case 2, we may assume that $d_{w'y'} > \lfloor \frac{1}{2}\Delta \rfloor$ also, since otherwise we would have chosen to work with C' rather than C at the start of Section 2.6.

Let $H^- = H - xz$. Note that H^- is obtained from G^- by deleting w', y' and all their neighbors other than x' and z'. In other words, C', H^-, and G^- are related to each other in exactly the same way that C, G^-, and G are. Also, f' is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(H^-)^2$ in which the vertices u', x', z' (i.e., z_1, z, x) all have different colors. With respect to C', H^-, and G^-, therefore, hypotheses (i) and (iii) of Lemma 2.6.1 hold, and the proof of that Lemma, and Case 2a of this Lemma, show that f' can be extended to a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G^-)^2$, in which necessarily $f'(u_2) \neq f'(z)$, since $d_{G^-}(u_2, z) \leqslant 2$.

Note that the color modifications required by Claim 2.6.2 and Lemma 2.6.3 have not been needed here, and the colors of vertices in H have not changed. Thus all vertices in $N'(x) \cup \{x, z\}$ now have different colors. This shows that, with respect to C, hypotheses (ii) and (iii) of Lemma 2.6.1 hold, and the $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$, whose existence was proved in Lemma 2.6.1, contradicts (2.21). This completes the proof of Lemma 2.6.4. $\square$

Now let $s = d_{yz}$ and let $\hat{G}$ be the graph obtained from G^- by adding s vertices $y_1, \ldots, y_s$ of degree 2, each with neighbors x and z. By (2.19), $d_{\hat{G}}(x) \leqslant \Delta - 1$, and so the maximum degree of $\hat{G}$ is at most Δ.

Lemma 2.6.5. *The graph $\hat{G}$ has no K_4-minor, and $(\hat{G})^2$ has no t-cliques.*

Proof. The graph obtained from G^- by adding just one vertex y_1 adjacent to x and z is a minor of G, and so is K_4-minor-free. Adding $s - 1$ further vertices of degree 2 in parallel with y_1 cannot create a K_4-minor, and so $\hat{G}$ is K_4-minor-free.

Now suppose that $(\hat{G})^2$ has a t-clique, with vertex set Q, say. Then Q must contain at least one new vertex y_i. By Lemma 2.4.3, Q is of standard form. By (2.1), in $F(\hat{G}, Q)$, x

and z both have degree Δ if Δ is even, and if Δ is odd, then one of them has degree Δ and the other has degree at least $\Delta - 1$. This implies the statements (*1)–(*3) below, where (*1) and (*2) hold because of (2.19) and the argument that gave rise to (2.19), and (*3) holds because otherwise a vertex $v \in \{x, z\}$ that has degree Δ in $F(\hat{G}, Q)$ would have degree greater than Δ in $\hat{G}$, which we have already seen to be impossible.

(*1) Δ is odd and $|N'(x)| = d_{yz} = s = \frac{1}{2}(\Delta - 1)$;

(*2) $d_{wx} = d_{wy} = \frac{1}{2}(\Delta - 1)$, $d_{xy} = 1$ and $d_G(w) = \Delta - 1$;

(*3) all vertices $y_1, \ldots, y_s$ are in Q.

Let $G^* = \hat{G} - y_s$; the graph G^* is a K_4-minor-free graph with maximum degree at most Δ whose square by (*3) has no t-clique. By the minimality of G, $(G^*)^2$ has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f. We will use f to construct a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2. First we use colors $f(y_1), \ldots, f(y_{s-1})$ to color all but one vertex, say v_{yz}, in M_{yz}, and to color y if $yz \in E(G)$. Then we choose a vertex $u \in N'(x)$ such that $f(u) \neq f(z)$ and we color v_{yz} with a color not used on any vertex in $N_G(z) \cup \{u, x, z\}$. There remain at most two uncolored vertices in G_w: possibly y, and, by (*2), at most one vertex in M_{xy}. Each of these vertices (if they exist) can be colored differently from all the colored vertices in $N'(x) \cup M_{xy} \cup \{x, y, z\}$, of which by (*1) and (*2) there are at most $\frac{1}{2}(\Delta - 1) + 4 \leqslant \Delta$, since here $\Delta \geqslant 7$.

At this point we have a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$. It may fail to satisfy the hypotheses of Lemma 2.6.4, but only because it is possible that $v_{yz} \in M_{yz}$ may have the same color as some vertex in $N'(x)$. However, we have ensured that $u \in N'(x)$ does not have the same color as any vertex in $M_{yz} \cup \{z\}$, and this is enough to ensure that Case 1 in the proof of Lemma 2.6.4 works and gives a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2. This contradicts the choice of G as a (Δ, t)-graph, and this contradiction shows that $(\hat{G})^2$ has no t-cliques. $\square$

Now we will prove Theorem 2.2.3. By Lemma 2.6.5 and the minimality of G, $\hat{G}^2$ has a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring f; clearly $f(x) \neq f(z)$. We will use f to construct a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$. First, we use $f(y_1), \ldots, f(y_s)$ to color all vertices in M_{yz}, and to color y if $yz \in E(G)$. Then we consecutively color all vertices in M_{xy}, and y if $yz \notin E(G)$, differently from all colored vertices in S (see (2.20)). The result is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of $(G_w)^2$ such that all vertices in S have different colors. It now follows from Lemma 2.6.4 that there is a $\lfloor \frac{3}{2}\Delta \rfloor$-coloring of G^2,

and this contradiction completes the proof of Theorem 2.2.3.

Chapter 3

Unavoidable subhypergraphs: $\mathbf{a}$-clusters

3.1 Forbidden families with small unions

One of the central problems of extremal hypergraph theory is maximizing the size of a hypergraph that does not contain a forbidden configuration. The hypergraphs, that maximize the number of edges under that condition are called *extremal* hypergraphs.

We are interested in finding the size of the extremal hypergraphs that do not contain a configuration of sets called an **a**-cluster.

Definition 3.1.1. Given a p-tuple $\mathbf{a}$ of positive integers, say $\mathbf{a} = (a_1, \ldots, a_p)$, such that $k = a_1 + \cdots + a_p$, an $\mathbf{a}$-*cluster* $\mathcal{A}$ in a k-uniform family $\mathcal{F}$ is a subfamily $\{F_0, \ldots, F_p\}$ such that the sets $F_i \backslash F_0$ and $F_0 \backslash F_i$ for $1 \leq i \leq p$ are pairwise disjoint, and $|F_i \backslash F_0| = |F_0 \backslash F_i| = a_i$. The sets $F_0 \setminus F_i$ for $1 \leq i \leq p$ are said to form an $\mathbf{a}$-*partition* of F_0, and F_0 is called the *host* of the **a**-cluster.

Various cases of the problem of finding the largest k-uniform hypergraph not containing an **a**-cluster are inspired by some of the oldest theorems and conjectures of extremal combinatorics. The first result in this area was obtained by Erdős, Ko and Rado. A family having no two disjoint members is an *intersecting family.*

Erdős-Ko-Rado Theorem (EKR theorem [37])**.** *If $\mathcal{F} \subset \binom{[n]}{k}$ with $n \geq 2k$ is an intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Moreover, equality holds only if $\mathcal{F}$ is a star.*

A *d-dimensional simplex* (or *d-simplex*) is a collection of sets $F_1, \ldots, F_{d+1}$ such that every d of them intersect, but $\cap_i F_i = \emptyset$. In that sense, a 1-simplex is a family consisting of two disjoint members; the EKR Theorem gives the maximum size of a family without a 1-simplex.

There is also hypergraph version of Turán's problem, which asks for the size of the extremal k-uniform hypergraph without a complete hypergraph on $d+1$ elements. This problem is open whenever $d+1 > k > 2$. The case $d = k = 2$ is a special case of Turán's theorem, proved by Mantel [77] in 1907.

Erdős posed the more general form of this Turán problem. A *triangle* is a family of three sets F_1, F_2, F_3 such that each pair has nonempty intersection, but the overall intersection is empty. Erdős asked for the size of an extremal triangle-free hypergraph. He conjectured in [34] that extremal k-uniform hypergraph on n elements has size $\binom{n-1}{k-1}$ for $n \geq 3k/2$. Partial results for this conjecture were solved in [11, 26, 42, 43]. Finally, Mubayi and Verstraëte [83] settled it, where equality holds only for a star. Chvátal [23] proved Erdős' conjecture for k=3 in a more general form. He proved that if $n \geq k+2 \geq 5$, $\mathcal{F} \subset \binom{[n]}{k}$ and $\mathcal{F}$ does not contain a $k-1$-simplex, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. He generalized the question of Erdős by setting the forbidden configuration to be d-simplex. Note that, if d and k have values close to each other, then a d-simplex in a k-uniform family is a configuration with small union which is the main feature of the forbidden families we discuss in this chapter.

Chvátal's Simplex Conjecture ([23]). *Let $k \geq d+1 \geq 3$, $n \geq k(d+1)/d$ and $\mathcal{F} \subset \binom{[n]}{k}$. If $\mathcal{F}$ contains no d-simplex, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$. Equality holds only when $\mathcal{F}$ is a star.*

This conjecture is already proved for $d = 1$ and $d = 2$ as mentioned earlier and it is still open for larger values of d. Frankl and Füredi [45] proved Chvátal's Simplex Conjecture for sufficiently large n.

Another generalization of the EKR Theorem was proposed by Katona in [60]. He asked for the size of the extremal family $\mathcal{F}$ assuming that $|F_1 \cup F_2 \cup F_3| \leq s$ implies $F_1 \cap F_2 \cap F_3 \neq \emptyset$ for any $F_1, F_2, F_3 \in \mathcal{F}$. It is shown that the EKR bound holds for the cases $s \geq 2k$ and sufficiently large n in [44] and later for $3k/2 \leq n < 2k$ in [83]. Mubayi generalized the result for $n \geq 3k/2$ in [80]. He also showed in [81] that the same bound follows, if $|F_1 \cup F_2 \cup F_3 \cup F_4| \leq 2k$ implies $F_1 \cap F_2 \cap F_3 \cap F_4 \neq \emptyset$ for sufficiently large n, which led him to make the following conjecture.

Conjecture 3.1.2. (Mubayi, [80]) *Let $k \geq d \geq 2$, $n \geq dk/(d-1)$ and $\mathcal{F} \subset \binom{[n]}{k}$ containing*

no (k,d)-cluster. Call a family of k-sets $\{F_1,\dots,F_d\}$ a (k,d)-cluster if

$$|F_1 \cup F_2 \cup \cdots \cup F_d| \le 2k \quad \text{and} \quad F_1 \cap F_2 \cdots \cap F_d = \emptyset.$$

Then $|\mathcal{F}| \le \binom{n-1}{k-1}$, with equality only if $\mathcal{F}$ is a star.

The case $d = k$ follows from a theorem of Chvatal [23], as was observed by Chen, Liu, and Wang [21]. Keevash and Mubayi [63] proved both Conjecture 3.1 and 3.1.2 when k/n and $n/2 - k$ are bounded away from zero. Recently, Jiang, Pikhurko, and Yilma [58] proved a more general result concerning the so-called "strong simplices".

Mubayi and Ramadurai [82] showed that Conjecture 3.1.2 is true when $2 \le d \le k$ and sufficiently large n. Their proof uses the stability method pioneered by Erdős and Simonovits [89]. The stability method has been recently used to solve classical problems in extremal combinatorics. The idea of the method is to show that given a forbidden configuration F if the F-free (hyper)graph has size close the size of the extremal (hyper)graph, then also its structure is nearly the same as the extremal (hyper)graph's. This method is mostly not applicable to hypergraph problems because the structure of the extremal hypergraph is not known. Recently, however, there has been more progress in extremal hypergraph theory with examples such as [50], [61], [62], [64].

In this chapter, we give a stronger generalization that proves Conjecture 3.1.2 for sufficiently large $n > n_0(k)$, where we use the *delta-system method* developed by Frankl and Füredi in [45]. We also use a complicated version of the stability method described in Section 3.6. In our result, we additionally give an explicit structure of the unavoidable subhypergraphs, the so-called $\mathbf{a}$-clusters.

Theorem 3.1.3. *Fix integers k and p with $k > p > 1$. If $\mathcal{F} \subset \binom{[n]}{k}$ with $|\mathcal{F}| > \binom{n-1}{k-1}$ and n is sufficiently large, i.e. $n > n_0(k)$, then $\mathcal{F}$ contains an $\mathbf{a}$-cluster, where $\mathbf{a} = (a_1,\dots,a_p)$. Moreover, if $|\mathcal{F}| = \binom{n-1}{k-1}$ and $\mathcal{F}$ is $\mathbf{a}$-cluster-free, then it is a star.*

Our $n_0(k)$ is very large, it is double exponential in k. The case $k = p$, i.e., $\mathbf{a} = \mathbf{1} = (1,1,\dots,1)$, is special and is discussed in Section 3.9.

3.2 Definitions and the delta-system method

Definition 3.2.1. For $F \in \mathcal{F}$, we denote the *intersection structure* of F in $\mathcal{F}$ by $\mathcal{I}(F, \mathcal{F})$, defined by

$$\mathcal{I}(F, \mathcal{F}) = \{F \cap F' : F' \in \mathcal{F}, F \neq F'\}$$

Definition 3.2.2. A family $\mathcal{F} \subset \binom{S}{k}$ is called *k-partite* if there is a partition of into sets $X_1, \ldots, X_k$ such that each member of $\mathcal{F}$ shares at most one element with X_i, for $1 \leq i \leq k$. In this case, $X_1, \ldots, X_k$ are *partite sets* of $\mathcal{F}$. If $\mathcal{F} \subset \binom{[n]}{k}$ is k-partite with partite sets $X_1, \ldots, X_k$, then for any $S \subset [n]$, the *projection* $\Pi(S)$ *of* S is defined to be $\{i : S \cap X_i \neq \emptyset\}$; and the *projection* $\Pi(\mathcal{F})$ *of* $\mathcal{F}$ is $\{\Pi(F) : F \in \mathcal{F}\}$.

A k-partite family $\mathcal{F}$ is *homogeneous* (or has *homogeneous* intersection structure) if $\Pi(\mathcal{I}(F, \mathcal{F}))$ is the same for each $F \in \mathcal{F}$.

Definition 3.2.3. For a homogeneous k-partite family $\mathcal{F}$ with $\mathcal{J} := \Pi(\mathcal{I}(F, \mathcal{F}))$ for each $F \in \mathcal{F}$, we let $r(\mathcal{J})$ be the *rank* of $\mathcal{J}$, defined by

$$r(\mathcal{J}) = \min\{|A| : A \subset [k], \nexists B \in \mathcal{J}, A \subset B\}.$$

The rank is k only if $\mathcal{J} = 2^{[k]} \setminus \{[k]\}$; otherwise, it is at most $k - 1$.

The *delta-system method* is described in the following theorem due to Füredi.

Theorem 3.2.4 (Füredi [47])**.** *For any two positive integers k and s there exists a positive constant $c(k, s)$ such that every family $\mathcal{F} \subset \binom{[n]}{k}$ contains a subfamily $\mathcal{F}^* \subset \mathcal{F}$ satisfying*

(3.2.4.1) $|\mathcal{F}^*| \geqslant c(k, s)|\mathcal{F}|$,

(3.2.4.2) *$\mathcal{F}^*$ is k-partite,*

(3.2.4.3) *there is a family $\mathcal{J} \subset 2^{[k]} \setminus \{[k]\}$ such that $\Pi(\mathcal{I}(F, \mathcal{F}^*)) = \mathcal{J}$ holds for all $F \in \mathcal{F}^*$,*

(3.2.4.4) *$\mathcal{J}$ is closed under intersection, (i.e., $A, B \in \mathcal{J}$ imply $A \cap B \in \mathcal{J}$),*

(3.2.4.5) *every member of $\mathcal{I}(F, \mathcal{F}^*)$ is the center of a delta-system of size s formed by members of $\mathcal{F}^*$ for all $F \in \mathcal{F}^*$.*

We frequently use (3.2.4.5) in the following forms:

(3.2.4.5a) if $F_1, F_2 \in \mathcal{F}^*$, $M \in \mathcal{I}(F_1, \mathcal{F}^*)$ and $M \subset F_2$, then $M \in \mathcal{I}(F_2, \mathcal{F}^*)$,

(3.2.4.5b) if $F_1 \in \mathcal{F}^*$, $F_2 \in \mathcal{F}$, $M \in \mathcal{I}(F_1, \mathcal{F}^*)$ and $M \subset F_2$, then $M \in \mathcal{I}(F_2, \mathcal{F})$,

(3.2.4.5c) if $F_1 \in \mathcal{F}^*$, $F_2 \in \mathcal{F}$, $M \in \mathcal{I}(F_1, \mathcal{F}^*)$, $M \subset F_2$, $|S| < s$, and $M \cap S = \emptyset$, then there exists an $F_3 \in \mathcal{F}^*$ such that $M = F_2 \cap F_3$ and $S \cap F_3 = \emptyset$.

Definition 3.2.5. For a subset $S \subset F \in \mathcal{F}$, denote the *degree* of S in $\mathcal{F}$ by $\deg_{\mathcal{F}}(S)$, defined by

$$\deg_{\mathcal{F}}(S) = |\{F : F \in \mathcal{F},\ S \subset F\}|.$$

A subset of $F \in \mathcal{F}$ is called a *private subset of* F *in* $\mathcal{F}$ if its degree in $\mathcal{F}$ is one.

We assume throughout this chapter that the family $\mathcal{F} \subset \binom{[n]}{k}$ is an $\mathbf{a}$-cluster-free family, where $\mathbf{a} \neq \mathbf{1}$, and $|\mathcal{F}| \geq \binom{n-1}{k-1}$. Also the constant s in Theorem 3.2.4 is fixed as $s = 2k$, and c_1 denotes the constant $c(k, s)$ in (3.2.4.1).

In Section 3.3, the subfamily $\mathcal{F}^* \subset \mathcal{F}$ is a k-partite subfamily satisfying (3.2.4.2)–(3.2.4.5). In that section, we obtain a significant relation between the rank and the intersection structure of $\mathcal{F}^*$. In Section 3.4 and 3.5, we partition the family $\mathcal{F}$ such that one part has very rich intersection structure, called $\mathcal{F}_1$. It will be shown that the remaining parts of $\mathcal{F}$ have negligible size and therefore $|\mathcal{F}_1|$ dominates $|\mathcal{F}|$. In Section 3.6, we obtain a partition of $\mathcal{F}_1$ using a partition of $\Delta_{k-2}(\mathcal{F}_1)$. By using this partition and the version of Kruskal-Katona theorem due to Lovász, we obtain a stability-type result. Section 3.7 concludes the proof of Theorem 3.1.3 . In Section 3.8, we show that a special case of Conjecture 3.1.2, when $d = k + 1$, is implied by a well-known lemma of Bollobás. Finally, in Section 3.9 we discuss the special case when $\mathbf{a}$=$\mathbf{1}$, and we provide some open problems.

3.3 Rank and shadow

Let the subfamily $\mathcal{F}^* \subset \mathcal{F}$ be a k-partite subfamily satisfying (3.2.4.2)–(3.2.4.5). We define $\mathcal{J}$ as $\Pi(\mathcal{I}(F, \mathcal{F}^*))$ for all $F \in \mathcal{F}^*$. Note that $\mathcal{J}$ is closed under intersection, and every member of $\mathcal{I}(F, \mathcal{F}^*)$ is the center of a delta-system of size $2k$ formed by the members of $\mathcal{F}^*$.

Definition 3.3.1. For a family $\mathcal{F}$ and $A \subset F \in \mathcal{F}$ with $(F \setminus A) \in \mathcal{I}(F, \mathcal{F})$, we denote by $F(A)$ an arbitrary member of $\mathcal{F}$ such that $F(A) \cap F = F \setminus A$.

Lemma 3.3.2. *If $r(\mathcal{J}) \geq k-1$, then $r(\mathcal{J}) = k-1$, i.e., it is impossible that $(F \setminus \{x_i\}) \in \mathcal{I}(F, \mathcal{F}^*)$ for all $1 \leq i \leq k$.*

Proof. We assume, on the contrary, that $r(\mathcal{J}) = k$. Because $\mathcal{J}$ is closed under intersection, $\mathcal{J} = 2^{[k]} \setminus \{[k]\}$. Let $F = A_1 \cup \cdots \cup A_p$ be a **a**-partition of F with $|A_i| = a_i$ for $1 \leq i \leq p$. Since each $F \setminus A_i \in \mathcal{I}(F, \mathcal{F}^*)$, there are delta-systems in $\mathcal{F}^*$ with these centers. By applying (3.2.4.5c), one can choose $F(A_1), \ldots, F(A_p)$ that do not intersect outside F. The subfamily $\{F, F(A_1), \ldots, F(A_p)\}$ is an **a**-cluster in $\mathcal{F}^*$, a contradiction. □

Definition 3.3.3. We use the notation $\Delta_\ell(\mathcal{H})$ for the *ℓ-shadow* of the family $\mathcal{H}$, i.e.,

$$\Delta_\ell(\mathcal{H}) = \{L : |L| = \ell, \exists H \in \mathcal{H} \text{ with } \ L \subset H\}.$$

Corollary 3.3.4. *$\mathcal{F}$ is not too dense, i.e. $|\Delta_{k-1}(\mathcal{G})| \geqslant c_1|\mathcal{G}|$ for all $\mathcal{G} \subset \mathcal{F}$.*

Proof. We apply Theorem 3.2.4 to $\mathcal{G}$ to obtain $\mathcal{G}^*$ such that $\Pi(\mathcal{I}(G, \mathcal{G})) = \mathcal{J}_{\mathcal{G}}$. Lemma 3.3.2 implies that $r(\mathcal{J}_{\mathcal{G}}) \leq k-1$. Therefore, each $G \in \mathcal{G}^*$ has a private $(k-1)$-subset in $\mathcal{G}^*$ and $|\Delta_{k-1}(\mathcal{G}^*)| \geq |\mathcal{G}^*|$. This implies

$$|\Delta_{k-1}(\mathcal{G})| \geq |\Delta_{k-1}(\mathcal{G}^*)| \geq |\mathcal{G}^*| \geq c_1|\mathcal{G}|. \tag{3.1}$$

□

3.4 The intersection structure of $\mathcal{F}_1$ and $\mathcal{F}_2$

Before proving Lemma 3.4.1 and Lemma 3.4.3, we define the following subfamilies of $\mathcal{F}$. We apply Theorem 3.2.4 to $\mathcal{F}$ with $s = 2k$ to obtain a homogeneous k-partite family and we define $\mathcal{G}_1$ to be $\mathcal{F}^*$. We also define $\mathcal{J}_1$ to be $\Pi(\mathcal{I}(G, \mathcal{G}_1))$ for all $G \in \mathcal{G}_1$. We apply Theorem 3.2.4 again to $\mathcal{F} \setminus \mathcal{G}_1$ and we define $\mathcal{G}_2$ to be $(\mathcal{F} \setminus \mathcal{G}_1)^*$ and $\mathcal{J}_2$ to be $\Pi(\mathcal{I}(G, \mathcal{G}_2))$ for

all $G \in \mathcal{G}_2$. We repeat this procedure until either $\mathcal{F} \setminus (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m) = \emptyset$ or $r(\mathcal{J}_{m+1}) \leq k-2$ for some m.

Let

$$\mathcal{F}_1 = \cup_i \{\mathcal{G}_i : r(\mathcal{J}_i) = k-1 \text{ and } |\Delta_{k-1}(\mathcal{J}_i)| = k-1\}.$$

We define $\mathcal{F}_2 = \cup_i \{\mathcal{G}_i : \mathcal{G}_i \notin \mathcal{F}_1\}$ and $\mathcal{F}_3 = \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$.

Lemma 3.4.1. *If $F \in \mathcal{G}_j \subset \mathcal{F}_1$ for some j and $F \setminus \{x_i\} \in \mathcal{I}(F, \mathcal{G}_j)$ for $2 \leq i \leq k$, then $F \setminus \{x_1\}$ is a private subset of F in $\mathcal{F}$. Moreover, in this case*

$$F_1 \in \mathcal{F} \text{ and } |F_1 \cap F| \geq k-2 \ \text{ imply } \ x_1 \in F_1. \tag{3.2}$$

Proof. Assume that $F \cap F_1 = \{x_2, \ldots, x_k\}$ for some $F_1 \in \mathcal{F}$. Let $F = A_1 \cup \cdots \cup A_p$ be an **a**-partition of F, $|A_i| = a_i$ for all i, such that $x_1, x_2 \in A_1$. Note that $\{E : E \subset F, x_1 \in E\} \subset \mathcal{I}(F, \mathcal{G}_j)$ and there is a delta-system with center $\{x_1\} \cup A_2 \cup \cdots \cup A_p$ containing F as a member. Pick another member $F' \in \mathcal{G}_j$ of that delta-system such that $F' \cap (F_1 \setminus F) = \emptyset$ and $F' \cap F = \{x_1\} \cup A_2 \cup \cdots \cup A_p$. Let $A_1' = F' \setminus (A_2 \cup \cdots \cup A_p)$. Hence $F_1 \cap F' = F' \setminus A_1'$. Since $F' \in \mathcal{G}_j$, there exist $F_i \in \mathcal{G}_j$ for $2 \leq i \leq p$ such that $F_i \cap F' = F' \setminus A_i$ and such that $F_1, \ldots, F_p$ do not intersect outside F'. Hence the subfamily $\{F', F_1, F_2, \ldots, F_p\}$ is an **a**-cluster in $\mathcal{F}$, a contradiction.

Note that $F \setminus \{x_1\}$ is a private subset of F in $\mathcal{F}$, not only in $\mathcal{G}_j$. □

We need the following claim for the proof of Lemma 3.4.3. This lemma will be used later to obtain an upper bound on the size of $\mathcal{F}_2 \cup \mathcal{F}_3$. For the remainder of this section, we fix $F \in \mathcal{G}_j \subset \mathcal{F}_2$ for some j. Let t be fixed, $2 \leq t \leq k$, such that $F \setminus \{x_i\} \notin \mathcal{I}(F, \mathcal{G}_j)$ only for $1 \leq i \leq t$. Also we define G to be a bipartite graph with partite sets $X = \{x_1, \ldots, x_t\}$ and $Y = [n] \setminus F$ and edges $x_i y$ for $y \in Y$ if and only if $(F \setminus \{x_i\}) \cup \{y\} \in \mathcal{F}$.

Claim 3.4.2. $\alpha'(G) \leq t-2$.

Proof. We assume, on the contrary, that distinct elements $\{y_2, \ldots, y_t\}$ exist such that $(F \setminus \{x_i\} \cup \{y_i\}) \in \mathcal{F}$ for $2 \leq i \leq t$. Observe that for any $A \subset F$ and $|A| < k$

$$\text{if } |A \cap \{x_1, \ldots, x_t\}| \geq 2, \text{ then } (F \setminus A) \in \mathcal{I}(F, \mathcal{G}_j). \tag{3.3}$$

(3.3) is implied by the fact that $F \setminus A$ is the intersection of the following sets and the fact that $\mathcal{J}_j$ is closed under intersection.

$$F \setminus A = \left(\bigcap_{x_u, x_v \in A,\ u<v\leq t} (F \setminus \{x_u, x_v\}) \right) \bigcap \left(\bigcap_{x_w \in A,\ w>t} (F \setminus \{x_w\}) \right) \quad \in \mathcal{I}(F, \mathcal{G}_j).$$

Therefore, if $A_1, \ldots, A_p$ is an **a**-partition of F, and each A_i either satisfies (3.3) or $A_i = \{x_j\}$ for some j, $2 \leq j \leq t$, then one can find an **a**-cluster with host F in the following cases.

Without loss of generality, index the entries of **a** so that $a_1 \geq a_2 \geq \cdots \geq a_p$, where $a_1 \geq 2$. We define the positive integers i and ℓ by

$$\begin{aligned} a_1 + \cdots + a_{i-1} < \ \ t \ &\leq a_1 + \cdots + a_i\,; \\ \ell \ &= t - (a_1 + \cdots + a_{i-1}). \end{aligned}$$

Case 1: $\ell \geq 2$. In this case, $a_1, \ldots, a_i \geq 2$.

Use an **a**-partition of F such that $A_1, A_2, \ldots, A_{i-1} \subset \{x_1, \ldots, x_t\}$ and $|A_i \cap \{x_1, \ldots, x_t\}| = \ell$.

Case 2: $\ell = 1$ and $a_i = 1$.

Use an **a**- partition such that $A_1 \cup A_2 \cdots \cup A_i = \{x_1, \ldots, x_t\}$, $x_1 \in A_1$.

Case 3: $\ell = 1$, $a_i \geq 2$ and $a_1 \geq 3$.

Use an **a**- partition such that $A_1 \cup A_2 \cdots \cup A_i \supseteq \{x_1, \ldots, x_t, x_{t+1}\}$, $x_1, x_{t+1} \in A_1$.

Case 4: $\ell = 1$, $a_i \geq 2$, $a_1 \leq 2$ and $a_p = 1$. In this case, $a_1 = \cdots = a_i = 2$.

Use an **a**- partition such that $A_p = \{x_t\}$ and $A_1 \cup A_2 \cdots \cup A_{i-1} = \{x_1, \ldots, x_{t-1}\}$, where $x_1 \in A_1$.

Case 5: $\ell = 1$, $a_1 = \cdots = a_p = 2$.

This implies that t is odd, $t \geq 3$, and k must be even; also, $t < k$. Pick a member $F' \in \mathcal{G}_j$ such that $F' = F \setminus \{x_k\} \cup \{y_k\}$ with $y_k \neq y_2$. Let $F_2 = F \setminus \{x_2\} \cup \{y_2\}$. We have $F' \setminus F_2 = \{x_2, y_k\}$. Using again (3.3) and F_2 one can build an **a**-cluster with host F' with the partition $\{x_2, y_k\}$, $\{x_1, x_3\}$ and $\{x_4, x_5\}, \ldots, \{x_{k-2}, x_{k-1}\}$. □

Lemma 3.4.3.

$$\sum_{1 \leq i \leq t} \frac{1}{\deg_{\mathcal{F}}(F \setminus \{x_i\})} \geq 1 + \frac{1}{k-1}.$$

Proof. By the claim above and by the well-known König–Egerváry theorem, $\beta(G) \leq t-2$. Let $|X \setminus S| = \ell$, we have $\ell \geq 2$ and $|S \cap Y| = \ell - 2$. Since $N(v) \subset S \cap Y$ for each $v \in X \setminus S$, we have

$$\deg_{\mathcal{F}}(F \setminus \{v\}) = \deg_G(v) + 1 \leq |S \cap Y| + 1 = \ell - 1.$$

This yields

$$\sum_{v \in X \setminus S} \frac{1}{\deg_{\mathcal{F}}(F \setminus \{v\})} \geq \frac{\ell}{\ell - 1} \geq \frac{k}{k-1}.$$

□

3.5 $\mathcal{F}_1$ dominates $\mathcal{F}$

Lemma 3.5.1.

$$|\mathcal{F}_2| + |\mathcal{F}_3| \leq \frac{k}{c_1}\binom{n}{k-2} + (k-1)\binom{n-1}{k-2} < c_2 n^{k-2}$$

for some c_2 depending only on k.

Proof. Since the rank of $\mathcal{J}_{m+1}$ is at most $k-2$, each member of $\mathcal{G}_{m+1}$ has its private $(k-2)$-subset in $\mathcal{G}_{m+1}$. We obtain as in (3.1) that

$$c_1|\mathcal{F} \setminus (\mathcal{G}_1 \cup \cdots \cup \mathcal{G}_m)| \leq |\mathcal{G}_{m+1}| \leq |\Delta_{k-2}(\mathcal{G}_{m+1})| \leq \binom{n}{k-2}.$$

Therefore,

$$\frac{k}{k-1}|\mathcal{F}_3| \leq \frac{k}{(k-1)c_1}\binom{n}{k-2}.$$

Lemma 3.4.1 implies that every $F \in \mathcal{F}_1$ contains a private $(k-1)$-set in $\mathcal{F}$. Using this and the result of Lemma 3.4.3, we have

$$|\mathcal{F}_1| + \frac{k}{k-1}|\mathcal{F}_2| \leq \sum_{F \in \mathcal{F}} \left(\sum_{v \in F} \frac{1}{\deg_{\mathcal{F}}(F \setminus \{v\})} \right) = |\Delta_{k-1}(\mathcal{F})| \leq \binom{n}{k-1}.$$

Comparing the sum of the two inequalities above to $\binom{n-1}{k-1} \leq |\mathcal{F}|$ gives us the upper bound on $|\mathcal{F}_2| + |\mathcal{F}_3|$. □

3.6 The stability of the extremum

Each $F \in \mathcal{F}_1$ is contained in a unique $\mathcal{G}_i \subset \mathcal{F}_1$ for some $i \in [m]$ as introduced in Section 3.4. Lemma 3.4.1 implies that there exists a (unique) $\ell(F) \in [n]$ for each $F \in \mathcal{F}_1$ such that $\{E : \ell(F) \in E \subset F\} \subset \mathcal{I}(F, \mathcal{G}_i)$. In the following, we partition $\mathcal{F}_1$ based on $\ell(F)$ for each $F \in \mathcal{F}_1$.

For all $i \in [n]$, let $\mathcal{H}_i = \{F \in \mathcal{F}_1 : \ell(F) = i\}$ and $\tilde{\mathcal{H}}_i = \{H \setminus \{i\} : H \in \mathcal{H}_i\}$. By Lemma 3.4.1, the subfamilies $\Delta_{k-2}(\tilde{\mathcal{H}}_1), \ldots, \Delta_{k-2}(\tilde{\mathcal{H}}_n)$ are pairwise disjoint. Therefore, $\tilde{\mathcal{H}}_i \cap \tilde{\mathcal{H}}_j = \emptyset$ and $H_i \cap H_j = \emptyset$ for $1 \le i < j \le n$. We will need the following version of the Kruskal-Katona theorem due to Lovász.

Theorem 3.6.1 (Lovász [74])**.** *If $\mathcal{H} \subset \binom{[n]}{d}$ and $|\mathcal{H}| = \binom{x}{d}$ for some real number x such that $x \ge d$, then $|\Delta_h(\mathcal{H})| \ge \binom{x}{h}$ for all $d > h \ge 0$.*

If $\mathcal{H}_i \neq \emptyset$, then define x_i such that $|\tilde{\mathcal{H}}_i| = \binom{x_i}{k-1}$ for $i \in [n]$. Without loss of generality, we let $x_1 \ge x_i$ for all i. Then,

$$|\mathcal{H}_i| = |\tilde{\mathcal{H}}_i| \le \frac{\binom{x_i}{k-1}}{\binom{x_i}{k-2}} |\Delta_{k-2}(\tilde{\mathcal{H}}_i)| \le \frac{x_1 - k + 2}{k-1} |\Delta_{k-2}(\tilde{\mathcal{H}}_i)| \le \frac{n-k+1}{k-1} |\Delta_{k-2}(\tilde{\mathcal{H}}_i)|. \tag{3.4}$$

Lemma 3.5.1 provides a lower bound for $|\mathcal{F}_1|$.

$$\binom{n-1}{k-1} - c_2 n^{k-2} \le |\mathcal{F}_1| = \sum_{i \in [n]} |\mathcal{H}_i| \le \frac{x_1 - k + 2}{k-1} \left(\sum_{i \in [n]} |\Delta_{k-2}(\tilde{\mathcal{H}}_i)| \right) \le \frac{x_1 - k + 2}{k-1} \binom{n}{k-2}.$$

This inequality implies that $x_1 > n - c_3$ for some c_3 depending only on k. Therefore there exists a c_4, also depending only on k, such that

$$\sum_{2 \le i \le k} |\mathcal{H}_i| \le \binom{n}{k-1} - \binom{n-c_3}{k-1} < c_4 n^{k-2}.$$

This and Lemma 3.5.1 lead to

$$|\mathcal{F} \setminus \mathcal{H}_1| \le (c_2 + c_4) n^{k-2}. \tag{3.5}$$

3.7 The extremal family is unique

In this section we complete the proof of Theorem 3.1.3. Suppose that $\mathcal{F} \subset \binom{[n]}{k}$ contains no **a**-cluster and satisfies $|\mathcal{F}| \geq \binom{n-1}{k-1}$. In previous sections we have already defined $\mathcal{H}_1 \subset \mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}_3$ and showed in (3.5) that $\mathcal{H}_1$ constitutes the bulk of $\mathcal{F}$. Let a_1 be the largest among $a_1, \ldots, a_p$, we have $a_1 \geq 2$. For any $F \in \mathcal{F}$ and any $H \in \mathcal{H}_1$, if $|F \cap H| \geq k - a_1$, then $1 \in F$, otherwise there is an **a**-cluster with host H.

We partition $\mathcal{F}$ into four subfamilies:

$$\begin{aligned}
\mathcal{B} &= \{B : 1 \notin B \in \mathcal{F}\}, \\
\mathcal{C} &= \{C : 1 \in C \in \mathcal{F} \text{ and } |C \cap B| \geq k - a_1 \text{ for some } B \in \mathcal{B}\}, \\
\mathcal{D} &= \{D : 1 \in D \in \mathcal{F} \setminus \mathcal{C} \text{ and every } S \text{ with } 1 \in S \subsetneq D \\
&\qquad\qquad \text{is a center of some delta-system of } \mathcal{F} \text{ of size } 2k\}, \\
\mathcal{E} &= \{E : 1 \in E \in \mathcal{F}\} \setminus (\mathcal{C} \cup \mathcal{D}).
\end{aligned}$$

Note that $\mathcal{H}_1 \subset \mathcal{D}$ and and by (3.5), $\mathcal{C} \cup \mathcal{D}$ dominates $\mathcal{F}$.

Subfamily $\mathcal{B}$: Let $\tilde{\mathcal{D}} := \{D \setminus \{1\} : D \in \mathcal{D}\}$. By definition of $\mathcal{D}$ we have $|D \cap B| \neq k - 2$ for $D \in \tilde{\mathcal{D}}$, $B \in \mathcal{B}$. In other words, $\Delta_{k-2}(\tilde{\mathcal{D}}) \cap \Delta_{k-2}(\mathcal{B}) = \emptyset$. Hence

$$\binom{n-1}{k-2} \geq |\Delta_{k-2}(\tilde{\mathcal{D}})| + |\Delta_{k-2}(\mathcal{B})|.$$

We choose a real number $x \geq k - 1$ such that $|\Delta_{k-1}(\mathcal{B})| = \binom{x}{k-1}$. Since $\mathcal{B} \subset \mathcal{F} \setminus \mathcal{H}_1$, (3.5) gives

$$\binom{x}{k-1} = |\Delta_{k-1}(\mathcal{B})| \leq k|\mathcal{B}| < k(c_2 + c_4)n^{k-2}, \tag{3.6}$$

and $x = O(n^{\frac{k-2}{k-1}})$.

Note that

$$|\Delta_{k-2}(\tilde{\mathcal{D}})| \geq \frac{k-1}{n-k+1}|\tilde{\mathcal{D}}| \quad \text{and} \quad |\Delta_{k-2}(\mathcal{B})| \geq \frac{k-1}{x-k+2}|\Delta_{k-1}(\mathcal{B})|.$$

Multiplying both of these with $(n-k+1)/(k-1)$ and using Lemma 3.3.4 yields

$$\binom{n-1}{k-1} \geq |\tilde{\mathcal{D}}| + \frac{n-k+1}{x-k+2}|\Delta_{k-1}(\mathcal{B})| \geq |\mathcal{D}| + c_1\frac{n-k+1}{x-k+2}|\mathcal{B}|. \tag{3.7}$$

Subfamily $\mathcal{E}$: Let $\tilde{\mathcal{E}} := \{E \setminus \{1\} : E \in \mathcal{E}\}$ and apply Theorem 3.2.4 to $\tilde{\mathcal{E}}$ with $s = 2k$. Call this $(k-1)$-partite subfamily $\mathcal{E}^*$. We first show that each $E' \in \mathcal{E}^*$ has a $(k-2)$-subset that is neither in $\mathcal{I}(E', \mathcal{E}^*)$ nor in $\mathcal{I}(E', \tilde{\mathcal{D}})$. Suppose, on the contrary, that for some $E \in \mathcal{E}$, $E' := E \setminus \{1\} \in \mathcal{E}^*$, $E' = \{x_1, \ldots, x_{k-1}\}$ such that

$$E' \setminus \{x_i\} \in \begin{cases} \mathcal{I}(E', \tilde{\mathcal{D}}) & \text{for} \quad i = 1, \ldots, r \\ \mathcal{I}(E', \mathcal{E}^*) & \qquad\quad i = r+1, \ldots, k-1. \end{cases} \tag{3.8}$$

All subsets of $E' \setminus \{x_i\}$ are contained in $\mathcal{I}(E', \tilde{\mathcal{D}})$, for $1 \leq i \leq r$, and all supersets of the set $\{x_1, \ldots, x_r\}$ in E', except E' itself, are contained in $\mathcal{I}(E', \mathcal{E}^*)$. So, for all $S \subset E'$, there is a delta-system of size $2k$ with center $S \cup \{1\}$. This contradicts $E = E' \cup \{1\} \notin \mathcal{D}$ and therefore (3.8) is not true. So,

$$\binom{n-1}{k-2} \geq |\Delta_{k-2}(\tilde{\mathcal{D}})| + |\mathcal{E}^*|,$$

Note that

$$\frac{n-k+1}{k-1}|\Delta_{k-2}(\tilde{\mathcal{D}})| \geq |\tilde{\mathcal{D}}| = |\mathcal{D}|.$$

Also by Corollary 3.3.4, $|\mathcal{E}^*| \geq c_1|\tilde{\mathcal{E}}| = c_1|\mathcal{E}|$. By combining all of above, we obtain

$$\binom{n-1}{k-1} \geq |\mathcal{D}| + c_1\frac{n-k+1}{k-1}|\mathcal{E}|. \tag{3.9}$$

Subfamily $\mathcal{C}$: Define $\tilde{\mathcal{C}} := \{C \setminus \{1\} : C \in \mathcal{C}\}$. Apply Theorem 3.2.4 to $\tilde{\mathcal{C}}$ with $s = 2k$. Call this $(k-1)$-partite subfamily $\mathcal{C}^*$ and let $\mathcal{J}_C := \Pi(\mathcal{I}(C', \mathcal{C}^*))$ for any $C' \in \mathcal{C}^*$.

Claim 3.7.1. *Each $C' \in \mathcal{C}^*$ has a $(k-2)$-set that is neither in $\Delta_{k-2}(\tilde{\mathcal{D}})$ nor in $\mathcal{I}(C', \mathcal{C}^*)$.*

Proof. Suppose, on the contrary, that for some $C' = \{x_1, \ldots, x_{k-1}\} \in \mathcal{C}^*$ with $C = C' \cup \{1\} \in$

$\mathcal{C}$ and for some r, we have

$$C' \setminus \{x_i\} \in \begin{cases} \mathcal{I}(C', \tilde{\mathcal{D}}) & \text{for} \quad i = 1, \ldots, r \\ \mathcal{I}(C', \mathcal{C}^*) & \qquad\quad i = r+1, \ldots, k-1. \end{cases} \tag{3.10}$$

All subsets of $C' \setminus \{x_i\}$ are contained in $\mathcal{I}(C', \tilde{\mathcal{D}})$, for $1 \leq i \leq r$, and all supersets of the set $\{x_1, \ldots, x_r\}$ in C', except C' itself, are contained in $\mathcal{I}(C', \mathcal{C}^*)$. So, for all $S \subset C'$, there is a delta-system of size $2k$ with center $S \cup \{1\}$.

First, we claim that $\mathcal{J}_C \neq 2^{[k-1]} \setminus \{[k-1]\}$. Otherwise, there exists a member $C'' \in \mathcal{C}$ such that $C'' \setminus \{1\} \in \mathcal{C}^*$ and $|C'' \cap B| = k - a_1$ for some $B \in \mathcal{B}$. Then one can build an **a**-cluster with host C'' such that $C''(A_1) = B$. So, $r \geq 1$ in (3.10).

For each $1 \leq i \leq r$, choose $D_i \in \mathcal{D}$ such that $C \cap D_i = C \setminus \{x_i\}$, and choose $B \in \mathcal{B}$ with $|C \cap B| \geq k - a_1$. By definition of $\mathcal{D}$, $|D_i \cap B| \leq k - a_1 - 1$. We also have

$$|D_i \cap B| + 1 \geq |C' \cap B| = |C \cap B| \geq k - a_1.$$

Therefore, all inequalities above are equality for each $1 \leq i \leq r$ and one can find an **a**-cluster with host C and $C(A_1) = B$, a contradiction. □

Since every $C' \in \mathcal{C}^*$ contains a private $(k-2)$-set not covered by any member of $\tilde{\mathcal{D}}$ we have $\binom{n-1}{k-2} \geq |\Delta_{k-2}(\tilde{\mathcal{D}})| + |\mathcal{C}^*|$. Using Corollary 3.3.4, this yields

$$\binom{n-1}{k-1} \geq |\mathcal{D}| + c_1 \frac{n-k+1}{k-1} |\mathcal{C}|. \tag{3.11}$$

In (3.7), (3.9) and (3.11) we prove that for $n > n_0(k)$, one has

$$|\mathcal{D}| + 4|\mathcal{B}| \leq \binom{n-1}{k-1}, \qquad |\mathcal{D}| + 4|\mathcal{C}| \leq \binom{n-1}{k-1}, \qquad |\mathcal{D}| + 4|\mathcal{E}| \leq \binom{n-1}{k-1}.$$

Adding these three, we obtain

$$3|\mathcal{F}| + (|\mathcal{B}| + |\mathcal{C}| + \mathcal{E}|) \leq 3\binom{n-1}{k-1}$$

implying $\mathcal{B} = \mathcal{C} = \mathcal{E} = \emptyset$. Thus $\mathcal{F} = \mathcal{D}$, $\cap\mathcal{F} \neq \emptyset$, and we are done. $\square$

3.8 Finding a $(k, k+1)$-cluster

Our first observation is that in Conjecture 3.1.2 the constraint $d \leq k$ is not necessary. We prove the case $d = k+1$. (It is not clear what is the possible maximum value of d.) We apply a classical result of Bollobás.

Definition 3.8.1. A *cross-intersecting set system* is a collection of pairs of sets $\{A_i, B_i\}$ for $i \in [m]$ such that $A_i \cap B_i = \emptyset$ and $A_i \cap B_j \neq \emptyset$ for $i \neq j$.

Theorem 3.8.2 (Bollobás [14])**.** *If $\{A_i, B_i\}$ is a cross-intersecting set system with $|A_i| \leq a$,$|B_i| \leq b$ for $i \in [m]$, then*

$$m \leq \binom{a+b}{a}.$$

Equality holds only if $\{A_1, \ldots, A_m\} = \binom{[a+b]}{a}$ and $B_i = [a+b] \setminus A_i$.

Theorem 3.8.3. *If $\mathcal{F} \subset \binom{[n]}{k}$ contains no $(k, k+1)$-cluster and $n \geq k$, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.*

Proof. Every $F \in \mathcal{F}$ has a $(k-1)$-subset $B(F) \subset F$ that is not contained by any other member of $\mathcal{F}$, otherwise there are sets $F_1, \ldots, F_k \in \mathcal{F}$ such that $F = \{x_1, \ldots, x_k\}$ and $F \cap F_i = F \setminus \{x_i\}$, a contradiction. Therefore, the sets $\{B(F), [n] - F\}$ form an intersecting set pair system as described above and Bollobás' theorem gives $|\mathcal{F}| \leq \binom{(k-1)+(n-k)}{k-1} = \binom{n-1}{k-1}$.$\square$

3.9 The case **a=1** and open problems

A k-uniform family $\{E_1, E_2, \ldots, E_q\}$ is called *k-tree* if for every i with $2 \leq i \leq q$ we have $|E_i \setminus \cup_{j<i} E_j| = 1$, and there exists an α less than i such that $|E_\alpha \cap E_i| = k-1$. The case $k = 2$ corresponds to the usual trees in graphs. Let $\mathcal{T}$ be a k-tree on v vertices, and let $\mathrm{ex}_k(n, \mathcal{T})$ denote the maximum size of a k-family on n elements not containing $\mathcal{T}$. We have

$$\mathrm{ex}_k(n, \mathcal{T}) \geq (1 - o(1)) \frac{v}{k} \binom{n}{k-1}. \tag{3.12}$$

To see this, consider a $(k-1)$-packing of maximum size $\mathcal{P} \subset \binom{[n]}{k+v-1}$, i.e. $|P \cap P'| < k-1$ for all distinct $P, P' \in \mathcal{P}$. Rödl's theorem [87] provides such a packing of size $(1-o(1))\binom{n}{k-1}/\binom{k+v-1}{k-1}$, when $n \to \infty$. By setting $\mathcal{F} = \Delta_k(\mathcal{P})$, we obtain a $\mathcal{T}$-free k-uniform hypergraph $\mathcal{F}$ with size

$$\binom{k+v-1}{k}|\mathcal{P}|, \text{ which equals } (1-o(1))\frac{v}{k}\binom{n}{k-1}.$$

Due to this observation, the following conjecture is best possible if it is true.

Conjecture 3.9.1 (Erdős and Sós for graphs, Kalai 1984 for all k, see in [45])**.**

$$\mathrm{ex}_k(n, \mathcal{T}) \leq \frac{v}{k}\binom{n}{k-1}.$$

This was proved for *star-shaped* trees by Frankl and Füredi [45], i.e., whenever $\mathcal{T}$ contains an edge wich intersects all other edges in $k-1$ vertices.

Note that a **1**-cluster is a k-tree with $v = 2k$. A Steiner system $S(n, k, t)$ is a family of k-subsets of $[n]$ such that each t-subset of $[n]$ is contained in a unique member of that family. So if an $S(n, 2k-1, k-1)$ exists, then the same type of construction used to prove (3.12) provides a **1**-cluster-free k-family of size $\binom{n}{k-1}$, slightly exceeding the EKR bound. (Such designs exist, e.g., for $k = 3$ and $n \equiv 1$ or $5 \pmod{20}$, see [12]). On the other hand, the result of Frankl and Füredi [45] that is mentioned above implies that if $\mathcal{F} \subset \binom{[n]}{k}$ is a family with more than $\binom{n}{k-1}$ members, then $\mathcal{F}$ contains every star-shaped tree with $k+1$ edges, especially it contains a **1**-cluster.

Theorem 3.1.3 is also related to the trace problem of uniform hypergraphs. Given a hypergraph H, its trace on $S \subseteq V(H)$ is defined as the set $\{E \cap S : E \in \mathcal{E}(H)\}$. Let $\mathrm{Tr}(n, r, k)$ denote the maximum number of edges in an r-uniform hypergraph of order n and not admitting the power set $2^{[k]}$ as a trace. For $k \leq r \leq n$, the bound $\mathrm{Tr}(n, r, k) \leq \binom{n}{k-1}$ was proved by Frankl and Pach [46]. Mubayi and Zhao [84] slightly reduced this upper bound by $\log_p n - k!k^k$ in the case when $k-1$ is a power of the prime p and n is large. So far, the best lower bound known was given by Ahlswede and Khachatrian [1], who showed that $\mathrm{Tr}(n, k, k) \geq \binom{n-1}{k-1} + \binom{n-4}{k-3}$ for $n \geq 2k \geq 6$. Hence, finding $\mathrm{Tr}(n, r, k)$ is still one of the

challenging open problems in extremal hypergraph theory.

Chapter 4

Extremal Cycle-free subgraphs of the hypercube

4.1 Turán problem

The *generalized Turán number* $\mathrm{ex}(G, H)$ is the maximum number of edges in an H-free subgraph of G. An H-free graph with with n vertices and $\mathrm{ex}(n, H)$ edges is an *extremal H-free graph*. The *Turán graph*, $T_{n,r}$, is defined as the extremal K_{r+1}-free graph with n vertices and $t(n, r)$ edges. This graph is an r-partite graph with parts of order $\lfloor \frac{n}{r} \rfloor$ or $\lceil \frac{n}{r} \rceil$.

Mantel [77] provided in 1907 the first Turán-type result by showing a sharp upper bound on the size of the extremal K_3-free graphs. He proved that $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$. Much later, in 1941, Turán generalized this result to all K_r-free graphs, known as Turán's theorem. It states that if G is a K_r-free graph on n vertices with $e(G) \geq t(n, r-1)$, then G is the Turán graph $T_{n,r-1}$.

One may ask a similar question for any graph other than the complete graph. In fact, Turán graphs are the extremal graphs for various classes of graphs. For a graph H with $\chi(H) = r \geq 3$, the size of the extremal H-free graph on n vertices is the same as $e(T_{n,r-1})$, i.e. $\mathrm{ex}(n, H) = t(n, r-1)$. On the other hand, the Turán problem for bipartite graphs is much more complicated and is still unsolved for all but a few of the bipartite graphs. The graphs with chromatic number 2 are also called *degenerate* graphs due to Simonovits [90]. All of the H-free extremal graphs have size $\theta(n^2)$ in the case of non-bipartite H. However, in the case of bipartite H, the extremal H-free graphs have size $o(n^2)$. Most of the known results for the Turán problem supply an error term of order $o(n^2)$, which makes it obvious in the case of bipartite graphs. In the following, we mention some of the seminal results in the Turán theory of non-bipartite and bipartite graphs.

For a family of graphs $\mathcal{F}$, let its *subchromatic number* be the number one less than the

minimum chromatic number over all of its members. The following result, known as Erdős-Stone-Simonovits theorem proved in [39, 41] states that if $\mathcal{F}$ is a family of graphs whose subchromatic number is r, then

$$\mathrm{ex}(n, \mathcal{F}) = t(n, r) + o(n^2) = \big(1 - \frac{1}{r}\big)\binom{n}{2} + o(n^2).$$

The Erdős-Stone-Simonovits theorem becomes obvious in the case of bipartite graphs. On the other hand, it gives the correct order of magnitude of the Turán number for all non-bipartite graphs. Another powerful theorem of Erdős and Simonovits is the so-called Stability Theorem. It says that the extremal graph family for any forbidden graph H with $\chi(H) = r$ is not far from the Turán graph $T_{n,r}$, both structurally and in terms of their sizes. In other words, this result gives information about the structure of the almost extremal graph families as well. These results of Erdős and Simonovits were proved in [32, 33, 89].

The Turán problem is still unsolved for all but a few of the bipartite graphs, such as even cycles and complete bipartite graphs. For extremely sparse bipartite graphs and very dense bipartite graphs, upper bounds are known and, in general, they are believed to be correct up to constant factors, but the constructions to prove that these upper bounds are tight are missing.

A simple example of a bipartite graph is a path. The Turán number for a path was first given by Erdős and Gallai [36] (1959) as $\mathrm{ex}(n, P_k) = \frac{1}{2}(k-2)n$; equality holds if $k-1$ divides n. The construction proving sharpness of the bound is a disjoint union of complete graphs on $k-1$ vertices.

The fact that $\mathrm{ex}(n, K_{1,k})$ is the same as $\mathrm{ex}(n, P_k)$ led Erdős and Sós [31] to conjecture that for every tree on $k+1$ vertices the Turán number is the same as $\mathrm{ex}(n, P_k)$. This conjecture was proved for sufficiently large trees by Ajtai, Komlós and Szemerédi [2, 3]. The same bound was proved by Sidorenko [88] for all trees on k vertices with a vertex neighboring at least $(k-2)/2$ leaves.

The rest of this section will be on the extremality results for the other two notorious classes of bipartite graphs for which the extremal graphs are not known: even cycles and complete bipartite graphs.

The first result in this area was given by Erdős [29] in 1938 as follows.

$$\mathrm{ex}(n, C_4) = \mathrm{ex}(n, K_{2,2}) = \theta(n^{\frac{3}{2}}).$$

This result was generalized by Kövári, Sós and Turán [68]. They showed that for $r \leq s$,

$$\mathrm{ex}(n, K_{r,s}) \leq \frac{1}{2}(s-1)^{1/2} n^{2-1/r} + O(n).$$

Erdős, Rényi, Sós [38] and independently Brown [17] showed that the result in the Kövari-Sós-Turán Theorem is best possible for $r = s = 2$ and infinitely many values of n. This was generalized by Füredi [48] to all $K_{2,t}$, for $t \geq 1$, as

$$\mathrm{ex}(n, K_{2,t}) = \frac{1}{2}\sqrt{t-1}\, n^{3/2} + O(n).$$

The Kövári-Sós-Turán Theorem gives the correct order of magnitude for $r = s = 3$ (Brown [17, 18]). However, the Turán number for $K_{r,s}$ in general is not known.

The best lower bound known was proved by Erdős and Spencer using a probabilistic technique as follows.

$$\mathrm{ex}(n, K_{t,t}) \geq \frac{1}{2} n^{2-2/(t+1)}$$

Another long-standing open Turán problem is the case of even cycles. The seminal result in this area was provided by Bondy and Simonovits [15] in 1974.

Theorem 4.1.1 (Bondy, Simonovits)**.** $\mathrm{ex}(n, C_{2k}) < 100kn^{1+\frac{1}{k}}$

The original form of this theorem appears in a much stronger form and says that the upper bound in Theorem 4.1.1 holds for the Turán number of each cycle of length $2t$, where $t \in [k, kn^{1/k}]$. Later Verstraëte [93] improved the upper bound in Theorem 4.1.1 to $8kn^{1+\frac{1}{k}}$.

Similar to the case of $K_{t,t}$, the first general lower bound was provided by Erdős [30], where he obtained $\mathrm{ex}(n, C_{2k}) \geq \theta(n^{1+1/(2k)})$ using probabilistic techniques. Imrich [56], Lubotzky, Phillips and Sarnak [76] and Margulis [78] have shown by using results from number theory and applying eigenvalue methods in graph theory to construct the so-called *Ramanujan graphs*, which have large chromatic number and girth. With the discovery of these graphs

the best lower bound known for the Turán number of even cycles is as below.

$$\mathrm{ex}(n, C_{2k}) \geq \theta(n^{1+\frac{3}{4k+21}})$$

Today, among all the even cycles, only the Turán numbers of C_4, C_6 and C_{10} are known asymptotically and they are of the same order of magnitude as the upper bound in Theorem 4.1.1. The constructions for the lower bound of these Turán numbers were given by Benson [10] and Erdős, Rényi, and Sós [38] using finite geometry and then simplified by Wenger [96]. Also new constructions were provided by Lazebnik, Ustimenko and Woldar [71] using applications from Lie algebras.

4.2 Turán problem on the hypercube

We denote by Q_n the *n-dimensional hypercube*, the graph whose vertex set is $\{0,1\}^n$, with vertices adjacent if they differ in exactly one coordinate. The *Turán number* for the hypercube, denoted by $\mathrm{ex}(Q_n, H)$, is the maximum number of edges in a subgraph of Q_n that does not contain any copy of H.

Definition 4.2.1. Let $c(H,n) = \frac{\mathrm{ex}(Q_n,H)}{e(Q_n)}$ and $c(H) = \lim_{n\to\infty} c(H,n)$. Note that $c(H,n)$ is a non-increasing and bounded function of n regardless of the choice of H, and therefore $c(H)$ exists.

Turán problem on the hypercube received more attention due to the following conjecture of Erdős.

Conjecture 4.2.2 (Erdős [35], 1984)**.** $c(C_{2k}) = \begin{cases} \frac{1}{2} + o(1) & k = 2 \\ o(1) & k \geq 3 \end{cases}$

The best upper bound known for $c(C_4)$ is obtained by F. Chung [22] as $0.6228 + o(1)$ and recently improved by Thomason and Wagner [91] to $0.6226 + o(1)$. Brass, Harborth and Nienborg [16] showed that the lower bound for $c(C_4, n)$ is $\frac{1}{2}(1 + 1/\sqrt{n})$, when $n = 4^r$ for integer r, and $\frac{1}{2}(1 + 0.9/\sqrt{n})$, when $n \geq 9$. Bialostocki [13] proved that in any 2-coloring of $E(Q_n)$ without a monochromatic copy of C_4, the size of each color class is at most $(n + 0.9\sqrt{n})2^{n-2}$, which provides another lower bound for $\mathrm{ex}(Q_n, C_4)$.

The problems of deciding the order of magnitude of $c(C_6)$ and $c(C_{10})$ are open as well. The best results known for $c(C_6)$ imply $1/3 \leq c(C_6) < 0.3941$; they are due to Conder [24] and Lu [75], respectively. The lower bound disproves Conjecture 4.2.2 for C_6. Conder constructed an edge-coloring of Q_n using 3 colors, where none of the color classes contain a copy of C_6. F. Chung [22] settled the conjecture of Erdős for C_{4k}, $k \geq 2$, by showing that

$$c(C_{4k}, n) \leq cn^{-\frac{1}{2}+\frac{1}{2k}}. \tag{4.1}$$

Axenovich and Martin [8] gave $c(C_{4k+2}) \leq 1/\sqrt{2}$ for $k \geq 1$. We showed in [49] that $C(C_{14})$ is 0. Here, we extend this result to $c(C_{4k+2})$ for $k \geq 3$ by using similar but simpler methods and prove Conjecture 4.2.2 for all cycles of length $4k+2$ and $k \geq 3$.

Theorem 4.2.3. *For $k \geq 3$,*

$$c(C_{4k+2}, n) \leq \begin{cases} O(n^{-\frac{1}{2k+1}}) & k \in \{3, 5, 7\}, \\ O(n^{-\frac{1}{16}+\frac{1}{16(k-1)}}) & \text{otherwise,} \end{cases}$$

i.e. $c(C_{4k+2})$ is 0.

Another variation of the definition of the Turán problem for the hypercube is to find the minimum number of edges required to intersect every copy of H in Q_n, which is equal to $e(Q_n) - \mathrm{ex}(Q_n, H)$. We define, for a subgraph H of Q_n,

$$f(H, n) = \frac{e(Q_n) - \mathrm{ex}(Q_n, H)}{e(Q_n)}, \quad f(H) = \lim_{n\to\infty} f(H, n).$$

Dejter, Emamy, and Guan [28], Harborth and Nienborg [53] and Graham, Harary, Livingston, and Stout [73] studied $f(Q_d, n)$ for small values of n. It is known that $f(Q_3) \leq 1/4$. Alon, Krech, and Szábo [4] conjectured that $f(Q_3) = 1/4$. The best lower bound known was due to a result in [73], which implies $f(Q_3) \geq 1 - (5/8)^{1/4} \approx .11086$. Offner [86] improved this lower bound to $f(Q_3) \geq .1165$. Alon, Krech, and Szábo [4] gave the following bounds:

$$\Omega(\frac{\log d}{d2^d}) = f(Q_d) \leq \begin{array}{ll} \frac{4}{(d+1)^2} & \text{if d is odd,} \\ \frac{4}{d(d+2)} & \text{if d is even.} \end{array}$$

The problem of finding minimum set of $V(Q_n)$ that intersects every copy of C_4 is studied by Kostochka [67] and by Johnson and Entringer [59]. They proved independently that such a minimum set has size $1/3|V(Q_n)|$, when n is sufficiently large.

Many results about Turán-type problems on the hypercube are motivated by the Ramsey version of these problems. In that vein, *H-polychromatic coloring* of the hypercube has attracted some attention, which is a coloring of $E(Q_n)$ such that every color class intersects every copy of H in Q_n. We define $p(H,n)$ to be the maximum number of colors with which it is possible to H-polychromatically color $E(Q_n)$. Let $p(H) = \lim_{n\to\infty} p(H,n)$. Trivially, each color class has at least $f(H,n)e(Q_n)$ edges of Q_n, which implies $p(H) \leq f^{-1}(H)$. Alon, Krech, and Szábo [4] proved for all $d \geq 1$,

$$\binom{d+1}{2} \geq p(Q_d) \geq \begin{array}{ll} \frac{(d+1)^2}{4} & \text{if d is odd,} \\ \frac{d(d+2)}{4} & \text{if d is even.} \end{array}$$

To obtain the upper bound, these authors used Ramsey-type results for hypergraphs. Subsequently, the exact order of magnitude of $p(Q_d)$ was given by Offner [85] as

$$p(Q_d) = \begin{array}{ll} \frac{(d+1)^2}{4} & \text{if d is odd,} \\ \frac{d(d+2)}{4} & \text{if d is even.} \end{array}$$

4.3 Proof of the main theorem

Remark 4.3.1. If $k = a + b - 1$ for integers a and b and G is a C_{4k+2}-free graph, then a cycle of length $4a$ and a cycle of length $4b$ in G cannot intersect at a single edge, otherwise their union contains a copy of C_{4k+2}.

For the remainder of this chapter, we assume that G is a C_{4k+2}-free subgraph of Q_n. For $k \geq 3$, we fix the value of $a \geq 2$ and let $b = k - a + 1$. Remark 4.3.1 applies to G.

We define $N(G, C_k)$ to be the number of copies of C_k in the graph G. In Section 4.4, we obtain an upper bound on $N(G, C_{4a})$ by proving

$$N(G, C_{4a}) \leq \overline{d}O(2^n n^{2a-2}) + O(2^n n^{2a-1+\frac{1}{2}+\frac{1}{2b}}), \tag{4.2}$$

where $\overline{d}$ is the average degree of G and equals $2e(G)/2^n$. In Section 4.5, we find the following lower bound for $N(G, C_{4a})$ via the lower bound on the number of copies of C_{2a} in an auxiliary graph constructed from G.

$$N(G, C_{4a}) \geq 2^n c \frac{\overline{d}^{4a}}{n^{2a}} - O(2^n n^a), \tag{4.3}$$

where $c > 0$ is a constant. By combining (4.2) with (4.3), we obtain an upper bound on $\overline{d}$ in the last section.

4.4 The upper bound on $N(G, C_{4a})$

Definition 4.4.1. We define the *direction* of an edge uv in $E(Q_n)$ as the coordinate in $[n]$, that appears in the symmetric difference of u and v, denoted by $d(uv)$. Similarly,

$$D(F) := \{d(e) : e \in E(F)\} \subset [n]\},$$

where F is any subgraph of Q_n.

A trivial upper bound on $N(G, C_{4a})$ is given by a counting argument on the edges of G. Note that, each direction on some edge of a C_{4a} appears an even number times. Let e be an edge with direction d. There are $O(n^{2a-1})$ possible ways that remaining at most $2a-1$ directions other than d could appear on a C_{4a} containing e. Therefore,

$$N(G, C_{4a}) \leq \sum_{e \in E(G)} O(n^{2a-1}) \leq e(G) O(n^{2a-1}) \leq 2^n O(n^{2a}).$$

The following lemma will help us to obtain a better bound.

Lemma 4.4.2. *Let C_1 and C_2 be cycles of length $4a$ and $4b$ in G, respectively, whose intersection contains an edge. Then $|D(C_1) \cap D(C_2)| \geq 2$.*

Proof. Let v_1 and v_2 be the endpoints of an edge in the intersection of C_1 and C_2. There must be another vertex v_3 shared by C_1 and C_2, otherwise we have a contradiction with

Remark 4.3.1. Because v_3 differs from either v_1 or v_2 in at least two coordinates, these two coordinates are also contained in the intersection of $D(C_1)$ and $D(C_2)$. $\square$

Let $\mathcal{C}$ denote the set of C_{4a}'s in G. We use a partitioning $\mathcal{C} = \mathcal{C}^1 \cup \mathcal{C}^2$ such that $\mathcal{C}^1$ is the collection of C_{4a}'s that satisfy the assumption of Lemma 4.4.2 and $\mathcal{C}^2 = \mathcal{C} \setminus \mathcal{C}^1$. Recall again that for any $C \in \mathcal{C}$, $|D(C)| \leq 2a$. However, if e is an edge contained in a C_{4a} in $\mathcal{C}^1$, then there are not "many" possible C_{4a}'s containing e (only $O(n^{2a-2})$, instead of $O(n^{2a-1})$) because of Lemma 4.4.2. Therefore, we count the C_{4a}'s as follows.

$$\begin{aligned} N(G, C_{4a}) &= \textstyle\sum_{e \in E(\mathcal{C}^1)} O(n^{2a-2}) + \sum_{e \in E(\mathcal{C}^2)} O(n^{2a-1}) \leq \\ &\leq \overline{d} 2^{n-1} O(n^{2a-2}) + \mathrm{ex}(Q_n, C_{4b}) O(n^{2a-1}) \end{aligned} \tag{4.4}$$

This and the result of Chung in (4.1) yield the following corollary.

Corollary 4.4.3. $N(G, C_{4a}) \leq \overline{d} O(2^n n^{2a-2}) + O(2^n n^{2a-1+\frac{1}{2}+\frac{1}{2b}})$.

4.5 The lower bound on $N(G, C_{4a})$

For a graph $G \subset Q_n$, we define an auxiliary graph $H(x, G)$ for each vertex $x \in Q_n$ as it was used by Chung in [22]. The vertex set of $H(x, G)$ consists of the neighbors of x in Q_n. The edge set of $H(x, G)$ is defined as follows. Consider any two vertices y and z of $H(x, G)$. There is a unique C_4 in Q_n containing x, y and z. We denote the fourth vertex of that cycle by $w = w_x(y, z)$. We let yz to be an edge of $H(x, G)$ if and only if wz and wy are edges of G.

According to the definition of H_x, we have

$$\sum_{x \in V(Q_n)} e(H_x) = \sum_{w \in V(Q_n)} \binom{\deg_G(w)}{2}.$$

By using convexity, we obtain

$$\overline{h} := \sum_{x \in V(Q_n)} e(H_x)/2^n \geq \binom{\overline{d}}{2}, \tag{4.5}$$

where $\overline{d} = 2e(G)/2^n$.

For each cycle of H_x with vertex set $\{y_1, \ldots, y_\ell\}$, $\ell \geq 3$, there exists a cycle of length 2ℓ in G with vertex set $\{y_1,\ w_x(y_1, y_2),\ \ldots,\ y_\ell,\ w_x(y_\ell, y_1)\}$. This yields

$$N(G, C_{4a}) \geq \sum_{x \in V(Q_n)} N(H_x, C_{2a}). \tag{4.6}$$

By the following theorem of Erdős and Simonovits, we have a lower bound on $N(H_x, C_{2a})$, and therefore on $N(G, C_{4a})$.

Theorem 4.5.1 ([40])**.** *Let L be a bipartite graph, where there are vertices x and y such that $L - \{x, y\}$ is a tree. Then for a graph H with n vertices, there exist constants $c_1, c_2 > 0$ such that if H contains more than $c_1 n^{3/2}$ edges, then*

$$N(H, L) \geq c_2 \frac{e^{n(L)}}{n^{2e(L)-n(L)}}.$$

We use this theorem with $L = C_{2a}$ in the following form so that the condition on the minimum number of edges is incorporated.

$$N(H_x, C_{2a}) \geq c_2 \left(\frac{e(H_x)^{2a}}{n^{2a}} - \frac{(c_1 n^{3/2})^{2a}}{n^{2a}}\right) \tag{4.7}$$

(4.6) and (4.7) imply

$$N(G, C_{4a}) \geq \sum_{x \in V(Q_n)} c_2 \left(\frac{e(H_x)^{2a}}{n^{2a}} - \frac{(c_1 n^{3/2})^{2a}}{n^{2a}}\right).$$

Finally, by (4.5) and above, we have

$$N(G, C_{4a}) \geq 2^n c_2 \frac{\overline{h}^{2a}}{n^{2a}} - O(2^n n^a) \geq 2^n c \frac{\overline{d}^{4a}}{n^{2a}} - O(2^n n^a), \tag{4.8}$$

for some constant $c > 0$.

4.6 Conclusion

Finally, (4.2) together with (4.3) yield

$$2^n c\frac{\overline{d}^{4a}}{n^{2a}} \leq \overline{d}O(2^n n^{2a-2}) + O(2^n n^{2a-1+\frac{1}{2}+\frac{1}{2b}}) + O(2^n n^a)$$

Note that the last term is negligible. Because this is an asymptotic result,

$$\text{either } \ 2^n c\frac{\overline{d}^{4a}}{n^{2a}} \leq \overline{d}O(2^n n^{2a-2}) \qquad \text{or } \ 2^n c\frac{\overline{d}^{4a}}{n^{2a}} \leq O(2^n n^{2a-1+\frac{1}{2}+\frac{1}{2b}}).$$

Therefore,

$$\overline{d} \leq \max\{O(n^{1-\frac{1}{4a-1}}), O(n^{1-\frac{1}{4a}(\frac{1}{2}-\frac{1}{2b})})\}.$$

Finally, we optimize this upper bound with respect to a. This bound is minimized when $a = 2$ (i.e. $b = k-1$) yielding that

$$\overline{d} \leq O(n^{1-\frac{1}{16}+\frac{1}{16(k-1)}}). \tag{4.9}$$

Note that another approach we could use in Section 4.4 is to consider $a = b = (k+1)/2$ for odd k. This changes the counting argument by making the set $\mathcal{C}^2$ empty. Thus, the second term in (4.4) and in Corollary 4.4.3 disappear. Following the same proof by using this upper bound, we obtain for odd k

$$\overline{d} \leq O(n^{1-\frac{1}{2k+1}}).$$

This improves (4.9) for $k = 3, 5, 7$.

Our proof also implies that $\mathrm{ex}(Q_n, \Theta_{4a-1,1,4b-1})$ is $o(e(Q_n))$ for $a, b \geq 2$, where $\Theta_{i,j,k}$ is a *Theta-graph* consisting of three paths of lengths i, j, and k having the same endpoints and distinct inner vertices.

A graph H is said to be *l-Ramsey* (or *Ramsey*) if any edge-coloring of Q_n with l colors must contain H in some color class provided that n is sufficiently large. Our result also naturally implies that C_{4k+2} is Ramsey for $k \geq 3$ which is a result of Alon, Radoičić, Sudakov, and Vondrák [5] who showed that C_{2r} is Ramsey for $r \geq 5$.

References

[1] R. Ahlswede and L.H. Khachatrian. Counterexample to the Frankl-Pach conjecture for uniform, dense families. *Combinatorica*, 17(2):299–301, 1997.

[2] M. Ajtai, J. Komlós, and E. Szemerédi. A note on Ramsey numbers. *J. Combin. Theory Ser. A*, 29(3):354–360, 1980.

[3] M. Ajtai, J. Komlós, and E. Szemerédi. A dense infinite Sidon sequence. *European J. Combin.*, 2(1):1–11, 1981.

[4] N. Alon, A. Krech, and T. Szabó. Turán's theorem in the hypercube. *SIAM J. Discrete Math.*, 21(1):66–72 (electronic), 2007.

[5] N. Alon, R. Radoičić, B. Sudakov, and J. Vondrák. A Ramsey-type result for the hypercube. *J. Graph Theory*, 53(3):196–208, 2006.

[6] K. Appel and W. Haken. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21(3):429–490, 1977.

[7] K. Appel, W. Haken, and J. Koch. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21(3):491–567, 1977.

[8] M. Axenovich and R. Martin. A note on short cycles in a hypercube. *Discrete Math.*, 306(18):2212–2218, 2006.

[9] L. Babai and P. Frankl. *Linear algebra methods in combinatorics with applications to geometry and computer science*. The University of Chicago, 1992.

[10] C. T. Benson. Minimal regular graphs of girths eight and twelve. *Canad. J. Math.*, 18:1091–1094, 1966.

[11] J.C. Bermond and P. Frankl. On a conjecture of Chvátal on m-intersecting hypergraphs. *Bull. London Math. Soc*, 9(3):310–312, 1977.

[12] T. Beth, D. Jungnickel, and H. Lenz. *Design theory*, volume 69. Cambridge University Press, 1999.

[13] A. Bialostocki. Some Ramsey type results regarding the graph of the n-cube. *Ars Combin.*, 16(A):39–48, 1983.

[14] B. Bollobas. On generalized graphs. *Acta Mathematica Hungarica*, 16(3):447–452, 1965.

[15] J. A. Bondy and M. Simonovits. Cycles of even length in graphs. *J. Combinatorial Theory Ser. B*, 16:97–105, 1974.

[16] P. Brass, H. Harborth, and H. Nienborg. On the maximum number of edges in a C_4-free subgraph of Q_n. *J. Graph Theory*, 19(1):17–23, 1995.

[17] W. G. Brown. On graphs that do not contain a Thomsen graph. *Canad. Math. Bull.*, 9:281–285, 1966.

[18] W. G. Brown. On an open problem of Paul Turán concerning 3-graphs. In *Studies in pure mathematics*, pages 91–93. Birkhäuser, Basel, 1983.

[19] G. J. Chang and D. Kuo. The $L(2,1)$-labeling problem on graphs. *SIAM J. Discrete Math.*, 9(2):309–316, 1996.

[20] G. Chartrand and F. Harary. Planar permutation graphs. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 3:433–438, 1967.

[21] W. Y. C. Chen, J. Liu, and L. X. W. Wang. Families of sets with intersecting clusters. *SIAM J. Discrete Math.*, 23(3):1249–1260, 2009.

[22] F. R. K. Chung. Subgraphs of a hypercube containing no small even cycles. *J. Graph Theory*, 16(3):273–286, 1992.

[23] V. Chvátal. An extremal set-intersection theorem. *Journal of the London Mathematical Society*, 9(2):355–359, 1974.

[24] M. Conder. Hexagon-free subgraphs of hypercubes. *J. Graph Theory*, 17(4):477–479, 1993.

[25] D. W. Cranston and S. Kim. List-coloring the square of a subcubic graph. *J. Graph Theory*, 57(1):65–87, 2008.

[26] R. Csákány and J. Kahn. A homological approach to two problems on finite sets. *Journal of Algebraic Combinatorics*, 9(2):141–149, 1999.

[27] G. A. Dirac. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.*, 27:85–92, 1952.

[28] M. R. Emamy, K. P. Guan, and I. J. Dejter. On the fault tolerance in a 5-cube. In *Proceedings of the Twentieth Manitoba Conference on Numerical Mathematics and Computing (Winnipeg, MB, 1990)*, volume 80, pages 171–176, 1991.

[29] P. Erdős. On sequences of integers no one of which divides the product of others and some related problems. *Izvestiya Naustno-Issl. Inst. Mat. i Meh. Tomsk*, 2:74–82, 1938.

[30] P. Erdős. Graph Theory and Probability, Canad. *Math. Monthly*, 11:34–38, 1959.

[31] P. Erdős. Extremal problems in graph theory. *Theory of Graphs and Appl.*, pages 29–36, 1965.

[32] P. Erdős. Some recent results on extremal problems in graph theory. Results. In *Theory of Graphs (Internat. Sympos., Rome, 1966)*, pages 117–123 (English); pp. 124–130 (French). Gordon and Breach, New York, 1967.

[33] P. Erdős. On some new inequalities concerning extremal properties of graphs. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 77–81. Academic Press, New York, 1968.

[34] P. Erdős. Topics in combinatorial analysis. *Proc. Second Louisiana Conf. on Comb., Graph Theory and Computing*, 12:2–20, 1971.

[35] P. Erdős. On some problems in graph theory, combinatorial analysis and combinatorial number theory. In *Graph theory and combinatorics (Cambridge, 1983)*, pages 1–17. Academic Press, London, 1984.

[36] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar*, 10:337–356 (unbound insert), 1959.

[37] P. Erdős, C. Ko, and R. Rado. Intersection theorems for systems of finite sets Quart. *J. Math. Oxford Ser.(2)*, 12:313–320, 1961.

[38] P. Erdős, A. Rényi, and V. T. Sós. On a problem of graph theory. *Studia Sci. Math. Hungar.*, 1:215–235, 1966.

[39] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar*, 1:51–57, 1966.

[40] P. Erdős and M. Simonovits. Cube-supersaturated graphs and related problems. In *Progress in graph theory (Waterloo, Ont., 1982)*, pages 203–218. Academic Press, Toronto, ON, 1984.

[41] P. Erdős and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.*, 52:1087–1091, 1946.

[42] P. Frankl. On Sperner families satisfying an additional condition. *J. Combin. Theory, Ser. A*, 20(1):1–11, 1976.

[43] P. Frankl. On a Problem of Chvatal and Erdős on Hypergraphs Containing No Generalized Simplex. *J. Combin. Theory, Ser. A*, 30(2):169–182, 1981.

[44] P. Frankl and Z. Füredi. A new generalization of the Erdős-Ko-Rado theorem. *Combinatorica*, 3(3):341–349, 1983.

[45] P. Frankl and Z. Füredi. Exact solution of some Turán-type problems. *Journal of Combinatorial Theory, Series A*, 45(2):226–262, 1987.

[46] P. Frankl and J. Pach. On disjointly representable sets. *Combinatorica*, 4(1):39–45, 1984.

[47] Z. Füredi. On finite set-systems whose every intersection is a kernel of a star. *Discrete Math*, 47(1):129–132, 1983.

[48] Z. Füredi. New asymptotics for bipartite Turán numbers. *J. Combin. Theory Ser. A*, 75(1):141–144, 1996.

[49] Z. Füredi and L. Özkahya. On 14-cycle-free subgraphs of the hypercube. *Combin. Probab. Comput.*, 18(5):725–729, 2009.

[50] Z. Füredi and M. Simonovits. Triple systems not containing a Fano configuration. *Combin. Probab. Comput.*, 14(4):467–484, 2005.

[51] D. Gonçalves. On the $L(p, 1)$-labelling of graphs. *Discrete Math.*, 308(8):1405–1414, 2008.

[52] J. R. Griggs and R. K. Yeh. Labelling graphs with a condition at distance 2. *SIAM J. Discrete Math.*, 5(4):586–595, 1992.

[53] H. Harborth and H. Nienborg. Maximum number of edges in a six-cube without four-cycles. *Bull. Inst. Combin. Appl.*, 12:55–60, 1994.

[54] F. Havet, J. v. d. Heuvel, C. McDiarmid, and B. Reed. List colouring squares of planar graphs. *Electronic Notes in Discrete Mathematics*, 29:515 – 519, 2007. European Conference on Combinatorics, Graph Theory and Applications.

[55] T. J. Hetherington and D. R. Woodall. List-colouring the square of a K_4-minor-free graph. *Discrete Math.*, 308(18):4037–4043, 2008.

[56] W. Imrich. Explicit construction of regular graphs without small cycles. *Combinatorica*, 4(1):53–59, 1984.

[57] T. R. Jensen and B. Toft. *Graph coloring problems*. Wiley-Interscience Series in Discrete Mathematics and Optimization. John Wiley & Sons Inc., New York, 1995. A Wiley-Interscience Publication.

[58] T. Jiang, O. Pikhurko, and Z. Yilma. Set systems without a strong simplex. *submitted*, page 17pp., June 2009.

[59] K. A. Johnson and R. Entringer. Largest induced subgraphs of the n-cube that contain no 4-cycles. *J. Combin. Theory Ser. B*, 46(3):346–355, 1989.

[60] G. Katona. A theorem of finite sets. In *Theory of graphs (Proc. Colloq., Tihany, 1966)*, pages 187–207. Academic Press, New York, 1968.

[61] P. Keevash. The Turán problem for projective geometries. *J. Combin. Theory Ser. A*, 111(2):289–309, 2005.

[62] P. Keevash and D. Mubayi. Stability theorems for cancellative hypergraphs. *J. Combin. Theory Ser. B*, 92(1):163–175, 2004.

[63] P. Keevash and D. Mubayi. Set systems without a simplex or a cluster. *Combinatorica*, to appear.

[64] P. Keevash and B. Sudakov. The Turán number of the Fano plane. *Combinatorica*, 25(5):561–574, 2005.

[65] A. V. Kostochka and D. R. Woodall. Choosability conjectures and multicircuits. *Discrete Math.*, 240(1-3):123–143, 2001.

[66] A.V. Kostochka, L. Özkahya, and D.R. Woodall. A Brooks-type bound for squares of K_4-minor-free graphs. *Discrete Mathematics*, 309:6572–6584, 2009.

[67] E.A. Kostochka. Piercing the edges of the n-dimensional unit cube. *(Russian) Diskret. Analiz Vyp. 28 Metody Diskretnogo Analiza v Teorii Grafov i Logiceskih Funkcii*, pages 55–64, 1976.

[68] T. Kövari, V. T. Sós, and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.*, 3:50–57, 1954.

[69] D. Král and R. Škrekovski. A theorem about the channel assignment problem. *SIAM J. Discrete Math.*, 16(3):426–437 (electronic), 2003.

[70] K. Kuratowski. Sur le problème des courbes gauches en topologie. *Fund. Math.*, 15:271–283, 1930.

[71] F. Lazebnik, V. A. Ustimenko, and A. J. Woldar. A new series of dense graphs of high girth. *Bull. Amer. Math. Soc. (N.S.)*, 32(1):73–79, 1995.

[72] K. Lih, W. Wang, and X. Zhu. Coloring the square of a K_4-minor free graph. *Discrete Math.*, 269(1-3):303–309, 2003.

[73] M. Livingston, Q. Stout, N. Graham, and F. Harary. Subcube fault-tolerance in hypercubes. *Univ. of Michigan Computing Research Laboratory, Tech. Rep. CRL-TR-12-87*, 1987.

[74] L. Lovász. *Combinatorial problems and exercises*. North-Holland Publishing Co., Amsterdam, 1979.

[75] L. Lu. Hexagon-free subgraphs in hypercube Q_n. *private communication*.

[76] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988.

[77] W. Mantel. Problem 28. *Wiskundige Opgaven*, 10:60–61, 1907.

[78] G. A. Margulis. Explicit constructions of graphs without short cycles and low density codes. *Combinatorica*, 2(1):71–78, 1982.

[79] M. Molloy and M. R. Salavatipour. A bound on the chromatic number of the square of a planar graph. *J. Combin. Theory Ser. B*, 94(2):189–213, 2005.

[80] D. Mubayi. Erdős-Ko-Rado for three sets. *J. Combin. Theory Ser. A*, 113(3):547–550, 2006.

[81] D. Mubayi. An intersection theorem for four sets. *Adv. Math.*, 215(2):601–615, 2007.

[82] D. Mubayi and R. Ramadurai. Set systems with union and intersection constraints. *J. Combin. Theory Ser. B*, 99(3):639–642, 2009.

[83] D. Mubayi and J. Verstraëte. Proof of a conjecture of Erdős on triangles in set-systems. *Combinatorica*, 25(5):599–614, 2005.

[84] D. Mubayi and Y. Zhao. On the VC-dimension of uniform hypergraphs. *J. Algebraic Combin.*, 25(1):101–110, 2007.

[85] D. Offner. Polychromatic colorings of subcubes of the hypercube. *SIAM J. Discrete Math.*, 22(2):450–454, 2008.

[86] D. Offner. Some Turán type results on the hypercube. *Discrete Math.*, 309(9):2905–2912, 2009.

[87] V. Rödl. On a packing and covering problem. *European J. Combin.*, 6(1):69–78, 1985.

[88] A. F. Sidorenko. Asymptotic solution for a new class of forbidden r-graphs. *Combinatorica*, 9(2):207–215, 1989.

[89] M. Simonovits. A method for solving extremal problems in graph theory, stability problems. In *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pages 279–319. Academic Press, New York, 1968.

[90] M. Simonovits. Extremal graph theory. In *Selected topics in graph theory, 2*, pages 161–200. Academic Press, London, 1983.

[91] A. Thomason and P. Wagner. Bounding the size of square-free subgraphs of the hypercube. *Discrete Math.*, 309(6):1730–1735, 2009.

[92] J. v. d. Heuvel and S. McGuinness. Coloring the square of a planar graph. *J. Graph Theory*, 42(2):110–124, 2003.

[93] J. Verstraëte. On arithmetic progressions of cycle lengths in graphs. *Combinatorics, Probability and Computing*, 9(04):369–373, 2000.

[94] W. Wang and K. Lih. Labeling planar graphs with conditions on girth and distance two. *SIAM J. Discrete Math.*, 17(2):264–275 (electronic), 2003.

[95] G. Wegner. Graphs with given diameter and a coloring problem. *preprint, University of Dortmund*, 1977.

[96] R. Wenger. Extremal graphs with no C^4's, C^6's, or C^{10}'s. *J. Combin. Theory Ser. B*, 52(1):113–116, 1991.

[97] D.B. West. *Introduction to graph theory.* Prentice Hall Upper Saddle River, NJ, 2001.