A SEQUENCE RELATED TO THE STERN SEQUENCE

BY

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DISSERTATION

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Abstract

In this dissertation we define and study a two-parameter family of recursive sequences which we call the *bow sequences*. The general bow sequence is defined similarly to the Stern sequence, and many of the properties of the bow sequences are related to known properties of the Stern sequence. In particular, we derive the generating function for the general bow sequence, and give interpretations of the generating function for two basic cases. We also determine properties of the bow sequences modulo 2 and 3, and give conjectures for the behavior of the bow sequences modulo d for $d \geq 4$. Finally, we discuss ideas for future research.

For my parents, for always believing in me.

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Chapter 1

Introduction

In this dissertation we define and study a family of recursive sequences which we call the *bow sequences*. The general bow sequence is defined similarly to the Stern sequence, and many of the properties of the bow sequences are related to known properties of the Stern sequence. We begin by discussing the main properties of the Stern sequence, and follow with the corresponding properties for the bow sequences as well as properties for some special cases.

1.1 History

In 1858, the mathematician M. A. Stern [17] studied the so-called "diatomic" array of integers, which was motivated by a function studied by Eisenstein. The diatomic array can be completely understood by consideration of the Stern sequence. The Stern sequence is defined as follows:

$$s(0) = 0, \quad s(1) = 1; \tag{1.1}$$

$$s(2n) = s(n), \quad \text{for } n \geq 1; \tag{1.2}$$

$$s(2n+1) = s(n) + s(n+1), \quad \text{for } n \geq 1. \tag{1.3}$$

The terms in the Stern sequence can be written in an array, where the r^{th} row consists of $s(n)$ for $2^r \leq n \leq 2^{r+1}$ for $r \geq 0$. The even entries in each row are copied from the previous row, and the odd entries are found by adding adjacent entries in the row above; it is a Pascal triangle with memory. The first five rows of the array are given in Table 1.1.

De Rham [5] was the first to consider the Stern sequence as defined above. He attributed the name to Bachmann [2], who considered only the diatomic array. The related *Stern-Brocot array* [7] was used in defining Minkowski's ?-function [11], and the Stern sequence has recently been used to understand 2-regular sequences [1] and the Tower of Hanoi graph [9]. The following are

1																1
1								2								1
1				3				2				3				1
1		4		3		5		2		5		3		4		1
1	5	4	7	3	8	5	7	2	7	5	8	3	7	4	5	1

Table 1.1: The first five rows of the diatomic array for the Stern sequence

a few of the important properties of the Stern sequence:

Proposition 1.1.1. [4], [10], [14], [17] *For* $n \geq 0$, *every pair of consecutive terms of the Stern sequence is relatively prime, or* $\gcd(s(n), s(n+1)) = 1$. *Moreover, given* $a,\ b \geq 1$ *such that* $\gcd(a,\ b) = 1$, *there is exactly one* n *such that* $s(n) = a$ *and* $s(n+1) = b$.

Proposition 1.1.1 implies the following proposition.

Proposition 1.1.2. [10], [14] *There are exactly* $\phi(m)$ *odd integers* $n \geq 1$ *with the property that* $s(n) = m$.

The reason is that, since

$$\big(s(2k), s(2k+1)\big) = \big(s(k), s(k) + s(k+1)\big), \tag{1.4}$$

it follows that in every pair $(a, b) = (s(2k), s(2k+1))$ we have $a < b$ and $\gcd(a, b) = 1$. In particular, for a fixed m, there are exactly $\phi(m)$ possible values for $s(2k)$ when $s(2k+1) = m$.

Next we will show how we can quickly determine all such odd n. As shown in [10], continued fractions can be used to determine a formula for the Stern sequence. Let $t(n)$ be defined as follows:

$$t(n) := \frac{s(n)}{s(n+1)}. \tag{1.5}$$

Then we can see by Proposition 1.1.1 that $(t(n))$ provides an enumeration of the positive rationals. Moreover, one can recover $s(n)$ and $s(n+1)$ from $t(n)$. By considering binary representations, we can find a simple closed formula for $t(n)$. This formula gives rise to a direct formula for $s(n)$.

Suppose n is odd. Then the binary representation of n consists of a_1 1's, followed by a_2 0's, a_3 1's, and so on, ending with a_{2v} 0's and a_{2v+1} 1's. We shall write $n \sim [a_1, \ldots, a_{2v+1}]$. Then we have the following proposition.

Proposition 1.1.3. [10] *If n is odd and $n \sim [a_1, \ldots, a_{2v+1}]$, then*

$$t(n) = \frac{s(n)}{s(n+1)} = a_{2v+1} + \frac{1}{a_{2v} + \frac{1}{\cdots + \frac{1}{a_1}}}. \tag{1.6}$$

This formula can be used to determine specific solutions to $s(n) = m$ for a fixed m.

Example 1. Suppose n is odd and $s(n) = 10$. Since $\gcd(10, s(n+1)) = 1$ and $s(n+1) < 10$, we must have $s(n+1) \in \{1, 3, 7, 9\}$. We can compute the finite continued fraction in each case as follows:

$$\begin{aligned} \frac{10}{1} &= 10, \\ \frac{10}{3} &= 3 + \frac{1}{3}, \\ \frac{10}{7} &= 1 + \frac{1}{2 + \frac{1}{3}}, \\ \frac{10}{9} &= 1 + \frac{1}{9}. \end{aligned}$$

Since the second and fourth continued fractions have an even number of denominators, we adjust the last denominator as follows:

$$\begin{aligned} \frac{10}{3} &= 3 + \frac{1}{2 + \frac{1}{1}}, \\ \frac{10}{9} &= 1 + \frac{1}{8 + \frac{1}{1}}. \end{aligned}$$

Now, reading these denominators from right to left, we see that $s(n) = 10$ when $[n]_2 \sim [10], [1, 2, 3], [3, 2, 1]$, or $[1, 8, 1]$. By converting back from binary, we find that $s(n) = 10$ for n odd exactly when $n = 1023,\ 39,\ 57$, or 513. For example, $39 = [1\ 0\ 0\ 1\ 1\ 1]_2$.

Next, we consider the terms of the Stern sequence modulo d. While the patterns are more complicated for higher moduli, the terms of the Stern sequence exhibit a very regular property modulo 2.

Proposition 1.1.4. [13], [14], [15] *For $n \geq 0$, $s(n)$ is even precisely when $3 \mid n$.*

The formulas for the terms of the Stern sequence which are congruent to k modulo d are more complex. Formulas for the cases $d = 3$ and 4 can be found in [14], and [15]. We will, however, state the following result for the distribution of pairs modulo d.

Proposition 1.1.5. [14] *The pair* $(s(n) \pmod{d}, s(n+1) \pmod{d})$ *is uniformly distributed amongst all possible pairs* (i, j) *with* $\gcd(i, j, d) = 1$.

Remark. This proposition implies that the density of terms in the Stern sequence which are divisible by a prime p is $\frac{1}{p+1}$.

1.2 Questions of Interest

In this dissertation we define a new two-parameter family of recursive sequences called the *bow sequences*, which have the flipped recursion from the Stern recursion. The general bow sequence is defined as follows:

$$b_{\alpha,\beta}(0) = 0, \quad b_{\alpha,\beta}(1) = \alpha, \quad b_{\alpha,\beta}(2) = \beta; \tag{1.7}$$

$$b_{\alpha,\beta}(2n) = b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1), \quad \text{for } n \geq 2; \tag{1.8}$$

$$b_{\alpha,\beta}(2n+1) = b_{\alpha,\beta}(n), \quad \text{for } n \geq 1. \tag{1.9}$$

We shall discuss properties of the bow sequences and how these properties relate to known properties of the Stern sequence. In particular, we would like to answer the following questions:

Question 1. What is the greatest common divisor of consecutive terms in the various bow sequences?

It is not true, in general, that the greatest common divisor of two consecutive terms in the bow sequences is 1. In fact, $b_{1,0}(13) = b_{1,0}(14) = 2$, and $b_{0,1}(21) = b_{0,1}(22) = 2$, for example. We can, however, state a property similar to Proposition 1.1.1 for three consecutive terms in the general bow sequence. In Chapter 2 we will show that

$$\gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) = \gcd(\alpha, \beta).$$

Question 2. Can we find a formula for $b_{\alpha,\beta}(n)$ that does not depend on the recursion?

Although we have not discovered a binary or continued fraction expression for the general bow sequence which is similar to the formula for the Stern sequence found in Proposition 1.1.3, we have determined a generating function for the general bow sequence and the values of the bow sequence for special values of α, β, and n. These formulas and generating functions can be found in Chapters 2 and 3. We also have work towards a reduced form involving matrices in Chapter 7.

Question 3. What is the distribution of terms $b_{\alpha,\beta}(n)$ modulo d for a given pair (α, β) and an integer d?

For the case $d = 2$ we find that $\frac{3}{7}$ of the terms are even for a given pair $\{\alpha, \beta\}$ where $\gcd(\alpha, \beta)$ is odd. Clearly, in the case where $\gcd(\alpha, \beta)$ is even, all of the terms in the bow sequence are even. Theorems detailing exactly where even terms occur for each pair $\{\alpha, \beta\}$ can be found in Chapter 4. For higher moduli, we give a conjecture for the fraction of terms which are congruent to k modulo d in Chapter 5. We have proved the conjecture to be true in the case $d = 3$, and we present numerical evidence for $d \leq 10$ in Appendix C.

Question 4. What is the distribution of triples

$$(b_{\alpha,\beta}(n) \pmod d,\ b_{\alpha,\beta}(n+1) \pmod d,\ b_{\alpha,\beta}(n+2) \pmod d)$$

for a given pair $\{\alpha, \beta\}$ and integer d?

Although pairs of terms are considered in the Stern sequence, we will mainly consider triples for the general bow sequence. Similar to Proposition 1.1.5, in Chapter 5 we show for $d = 2$ and 3 that if $\gcd(\alpha, \beta) = 1$, the triple

$$(b_{\alpha,\beta}(n) \pmod d,\ b_{\alpha,\beta}(n+1) \pmod d,\ b_{\alpha,\beta}(n+2) \pmod d)$$

is uniformly distributed amongst all triples $(i \pmod d,\ j \pmod d,\ k \pmod d)$ such that

$$\gcd(i, j, k, d) = 1.$$

Remark. This implies that if $\gcd(\alpha, \beta) = 1$, then the density of terms in the bow sequence which are divisible by a prime p is $\frac{p^2-1}{p^3-1}$. We have proved the formula to be true for $p = 2$ and 3, and we give numerical evidence to support this conjecture for larger values in Appendix B.

We have a more complicated conjecture detailing the density of terms which would be divisible by an integer d in Chapter 5.

Question 5. For a given m, how many even integers $n \geq 2$ are there such that $b_{\alpha,\beta}(n) = m$?

In Proposition 1.1.2 we considered the number of odd integers $n \geq 1$ with the property that $s(n) = m$. For the general bow sequence, we shall consider the even terms since (1.9) tells us that if $b_{\alpha,\beta}(n) = m$, then

$$b_{\alpha,\beta}(2^r(n+1) - 1) = m, \text{ for } r \geq 0.$$

Accordingly, for specific pairs $\{\alpha, \beta\}$, we would like to consider even integers $n \geq 2$ with the property that $b_{\alpha,\beta}(n) = m$. A preliminary discussion of these properties can be found in Chapter 7.

Chapter 2

The General Bow Sequence

We begin by defining the bow sequences recursively as in Chapter 1. The general *bow sequence*, $b_{\alpha,\beta}(n)$, for α, $\beta \in \mathbb{Z}$ is defined by:

$$b_{\alpha,\beta}(0) = 0, \quad b_{\alpha,\beta}(1) = \alpha, \quad b_{\alpha,\beta}(2) = \beta; \tag{2.1}$$

$$b_{\alpha,\beta}(2n) = b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1), \quad \text{for } n \geq 2; \tag{2.2}$$

$$b_{\alpha,\beta}(2n+1) = b_{\alpha,\beta}(n), \quad \text{for } n \geq 1. \tag{2.3}$$

We define $b_{\alpha,\beta}(0) = 0$ to simplify results, although the designation does not affect later terms in the sequence, as $b_{\alpha,\beta}(0)$ does not enter into the recurrence.

Here is a table listing the first 17 values of the general bow sequence:

n	$b_{\alpha,\beta}(n)$
0	0
1	α
2	β
3	α
4	$\alpha + \beta$
5	β
6	$2\alpha + \beta$
7	α
8	$\alpha + 2\beta$
9	$\alpha + \beta$
10	$2\alpha + 2\beta$
11	β
12	$3\alpha + \beta$
13	$2\alpha + \beta$
14	$2\alpha + 2\beta$
15	α
16	$2\alpha + 3\beta$

Table 2.1: The general bow sequence

Note that the terms of the general bow sequence are not necessarily distinct. In particular, we can see from Table 2.1 that $b_{\alpha,\beta}(10) = b_{\alpha,\beta}(14)$.

It is also worth noting that (2.2) fails in the case where $n = 1$, since if $\alpha \neq 0$, then $b_{\alpha,\beta}(2) \neq b_{\alpha,\beta}(1) + b_{\alpha,\beta}(2)$.

2.1 Preliminaries

By considering the recursion, we see that $b_{\alpha,\beta}(n)$ is linear in α and β. Accordingly, we can write

$$b_{\alpha,\beta}(n) = \alpha b_{1,0}(n) + \beta b_{0,1}(n). \tag{2.4}$$

For simplicity, we shall define

$$b_0(n) := b_{0,1}(n),$$

and

$$b_1(n) := b_{1,0}(n).$$

In this dissertation we mainly consider the sequences $b_0(n)$ and $b_1(n)$, as all other cases of the general bow sequence can be determined from these two cases by applying equation (2.4).

Remark. Three applications of the recursion to $2n$, $2n+1$, and $2n+3$ give the following. For $n \geq 2$,

$$b_{\alpha,\beta}(2n) = b_{\alpha,\beta}(2n+1) + b_{\alpha,\beta}(2n+3). \tag{2.5}$$

For $\alpha, \beta \geq 0$, the following inequalities follow directly:

$$b_{\alpha,\beta}(2n) \geq b_{\alpha,\beta}(2n+1),$$
$$b_{\alpha,\beta}(2n) \geq b_{\alpha,\beta}(2n+3).$$

Notice that equation (2.5) can be rewritten as

$$b_{\alpha,\beta}(2n+1) = b_{\alpha,\beta}(2n-2) - b_{\alpha,\beta}(2n-1). \tag{2.6}$$

By applying (1.2) and (1.3) to the Stern sequence, we can construct a property similar to (2.5) for the Stern sequence: for $n \geq 1$,

$$s(2n+1) = s(2n) + s(2n+2). \tag{2.7}$$

Remark. Note that iterating (2.2) and (2.3) also gives the following identities:

$$\begin{aligned} b_{\alpha,\beta}(4n) &= 2b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1), \qquad (2.8)\\ b_{\alpha,\beta}(4n+1) &= b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1),\\ b_{\alpha,\beta}(4n+2) &= b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2),\\ b_{\alpha,\beta}(4n+3) &= b_{\alpha,\beta}(n). \end{aligned}$$

This means that for $\alpha,\ \beta \geq 0$ we have

$$\begin{aligned} b_{\alpha,\beta}(4n+3) \leq b_{\alpha,\beta}(4n+1) &\leq b_{\alpha,\beta}(4n+2), \text{ and}\\ b_{\alpha,\beta}(4n) &= b_{\alpha,\beta}(2n) + b_{\alpha,\beta}(n). \qquad (2.9) \end{aligned}$$

2.2 Triatomic Arrays

We can present the values of the general bow sequence as a "triatomic" array where the first row is $b_{\alpha,\beta}(1)$, $b_{\alpha,\beta}(2)$, $b_{\alpha,\beta}(3)$, and the second row is $b_{\alpha,\beta}(3)$, $b_{\alpha,\beta}(4)$, $b_{\alpha,\beta}(5)$, $b_{\alpha,\beta}(6)$, $b_{\alpha,\beta}(7)$. In general the r-th row contains $b_{\alpha,\beta}(2^r-1)$, ..., $b_{\alpha,\beta}(2^{r+1}-1)$. The odd entries in each row are copied from the previous row in Table 2.2. The even entries are found by adding the two numbers in the previous row that occur to the right of the entry. For example, the second entry in the third row, $b_{\alpha,\beta}(8) = \alpha + 2\beta$, is obtained by adding the second and third entries in the second row of the array, namely, $\alpha+\beta$ and β. The last even entry in each row is obtained by summing the first two entries in that same row. Here are the first three rows of the array:

α				β				α
α		$\alpha+\beta$		β		$2\alpha+\beta$		α
α	$\alpha+2\beta$	$\alpha+\beta$	$2\alpha+2\beta$	β	$3\alpha+\beta$	$2\alpha+\beta$	$2\alpha+2\beta$	α

Table 2.2: First three rows of the triatomic array for the general bow sequence

The triatomic arrays for $b_0(n)$ and $b_1(n)$ are quite different from each other. As we will see in Chapter 5, $b_0(n)$ is much more regular than $b_1(n)$, but the two sequences share many properties. Consider the tables below:

0								1								0
0				1				1				1				0
0		2		1		2		1		1		1		2		0
0	3	2	3	1	3	2	2	1	2	1	3	1	2	2	3	0

Table 2.3: First four rows of the triatomic array for $b_0(n)$

1								0								1
1				1				0				2				1
1		1		1		2		0		3		2		2		1
1	2	1	3	1	2	2	3	0	5	3	4	2	3	2	3	1

Table 2.4: First four rows of the triatomic array for $b_1(n)$

2.3 Greatest Common Divisors and Maxima

One interesting property of the bow sequences is that although two consecutive terms may share a nontrivial factor, three consecutive terms can only share factors which divide $\gcd(\alpha, \beta)$.

Theorem 2.3.1. *For all $\{\alpha, \beta\}$ and $n > 0$,*

$$\gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) = \gcd(\alpha, \beta). \tag{2.10}$$

Proof. Observe that $\gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) = \gcd(\alpha, \beta)$ for $n = 1$. Now assume that this holds true for $n < k$.

Case 1: $k = 2n+1$, with $n < k$. Then,

$$\begin{aligned}
&\gcd(b_{\alpha,\beta}(k), b_{\alpha,\beta}(k+1), b_{\alpha,\beta}(k+2)) \\
&= \gcd(b_{\alpha,\beta}(2n+1), b_{\alpha,\beta}(2n+2), b_{\alpha,\beta}(2n+3)) \\
&= \gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2), b_{\alpha,\beta}(n+2)) \\
&= \gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) \\
&= \gcd(\alpha, \beta).
\end{aligned}$$

Case 2: $k = 2n$, with $n < k$. Then,

$$\begin{aligned}
&\gcd(b_{\alpha,\beta}(k), b_{\alpha,\beta}(k+1), b_{\alpha,\beta}(k+2)) \\
&= \gcd(b_{\alpha,\beta}(2n), b_{\alpha,\beta}(2n+1), b_{\alpha,\beta}(2n+2)) \\
&= \gcd(b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2)) \\
&= \gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) \\
&= \gcd(\alpha, \beta).
\end{aligned}$$

Thus $\gcd(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) = \gcd(\alpha, \beta)$ for all $\{\alpha, \beta\}$ and $n > 0$. ■

Surprisingly, Theorem 2.3.1 has implications for pairs. Although we have seen that pairs frequently share common factors, when the pairs

$$(b_0(n), b_0(n+1)), \text{ and } (b_1(n), b_1(n+1))$$

are taken together, the resulting quadruple does not have a common factor.

Theorem 2.3.2. *For $n \geq 0$,* $\gcd(b_0(n), b_0(n+1), b_1(n), b_1(n+1)) = 1$.

Proof. Suppose p is prime, and let

$$p \mid \gcd(b_0(n), b_0(n+1), b_1(n), b_1(n+1)).$$

Then, p must necessarily divide $b_0(n)$ and $b_0(n+1)$, so we know by Theorem 2.3.1 that $p \nmid b_0(n+2)$. Similarly, $p \nmid b_1(n+2)$. So let

$$b_0(n+2) \equiv r \pmod{p} \qquad \text{and} \qquad b_1(n+2) \equiv s \pmod{p},$$

where $1 \leq r, s \leq p-1$.

Consider $b_{s,-r}(n) = sb_1(n) - rb_0(n)$. Thus $p \mid b_{s,-r}(n)$, and $p \mid b_{s,-r}(n+1)$. Then, since

$$b_{s,-r}(n+2) \equiv sr + (-r)s \equiv 0 \pmod{p},$$

we know that $p \mid b_{s,-r}(n+2)$. But this implies that $p \mid \gcd(r, s)$, which is a contradiction. Thus $\gcd(b_0(n), b_0(n+1), b_1(n), b_1(n+1)) = 1$. ■

Remark. Note that Theorem 2.3.2 fails if we consider only three of the four terms. For example, consider $n = 2149948$. Then we have

$$(b_0(n), b_0(n+1), b_1(n), b_1(n+1)) = (2070, 1815, 1430, 1254).$$

When we consider the triples, we find that

$$\begin{aligned} \gcd(b_0(n), b_0(n+1), b_1(n)) &= 5, \\ \gcd(b_0(n), b_0(n+1), b_1(n+1)) &= 3, \\ \gcd(b_0(n), b_1(n), b_1(n+1)) &= 2, \\ \gcd(b_0(n+1), b_1(n), b_1(n+1)) &= 11. \end{aligned}$$

However, clearly $\gcd(b_0(n), b_0(n+1), b_1(n), b_1(n+1)) = 1$.

Similarly, we have the following theorem.

Theorem 2.3.3. *For $n \geq 0$, $\gcd(b_0(n), b_0(n+2), b_1(n), b_1(n+2)) = 1$.*

Proof. Suppose p is prime, and let

$$p \mid \gcd(b_0(n), b_0(n+2), b_1(n), b_1(n+2)).$$

Then, p must necessarily divide $b_0(n)$ and $b_0(n+2)$, so we know by Theorem 2.3.1 that $p \nmid b_0(n+1)$. Similarly, $p \nmid b_1(n+1)$. So let

$$b_0(n+1) \equiv r \pmod{p} \qquad \text{and} \qquad b_1(n+1) \equiv s \pmod{p},$$

where $1 \leq r, s \leq p-1$.

Consider $b_{s,-r}(n) = sb_1(n) - rb_0(n)$. Thus $p \mid b_{s,-r}(n)$, and $p \mid b_{s,-r}(n+2)$. Then, since

$$b_{s,-r}(n+1) \equiv sr + (-r)s \equiv 0 \pmod{p},$$

we know that $p \mid b_{s,-r}(n+1)$. But this implies that $p \mid \gcd(r,s)$, which is a contradiction. Thus $\gcd(b_0(n), b_0(n+2), b_1(n), b_1(n+2)) = 1$. ■

For the next theorem, we will need the Fibonacci numbers, defined as usual by

$$F_0 = 0, \quad F_1 = 1;$$
$$F_n = F_{n-1} + F_{n-2}, \quad \text{for } n \geq 2.$$

Theorem 2.3.4. *For $r \geq 1$,*

$$b_{\alpha,\beta}(2^r) = \alpha F_{r-1} + \beta F_r. \tag{2.11}$$

In particular, for $r \geq 1$,

$$b_0(2^r) = F_r, \quad b_1(2^r) = F_{r-1}, \quad \text{and} \quad b_{1,1}(2^r) = F_{r+1}.$$

Proof. First, note that by applying (2.9) we find

$$b_{\alpha,\beta}(2^r) = b_{\alpha,\beta}(2^{r-1}) + b_{\alpha,\beta}(2^{r-2}).$$

Secondly, recall by (2.4) that

$$b_{\alpha,\beta}(2^r) = \alpha b_1(2^r) + \beta b_0(2^r).$$

Thus we consider only these two cases. All that remains is to check the initial conditions. We find that

$$(b_0(2), b_0(4)) = (0,1) = (F_0, F_1),$$

and

$$(b_1(2), b_1(4)) = (1,1) = (F_1, F_2).$$

Thus the initial conditions are satisfied. ■

Next, we estimate the size of $b_{\alpha,\beta}(n)$. Let $I_r = (2^{r-1}, 2^r] \cap \mathbb{Z}$. To be specific,

$$I_r = \{2^{r-1} + 1, \cdots, 2^r\}.$$

Note that

$$I_r = (2I_{r-1} - 1) \cup 2I_{r-1}.$$

We have the following upper bound.

Theorem 2.3.5. *For $r \geq 1$,*

$$\max_{n \in I_r} |b_{\alpha,\beta}(n)| \leq \max\{|\alpha|, |\beta|\} F_{r+1}. \tag{2.12}$$

Moreover, for $\beta \geq \alpha \geq 0$,

$$\max_{n \in I_r} |b_{\alpha,\beta}(n)| = b_{\alpha,\beta}(2^r) = \alpha F_{r-1} + \beta F_r. \tag{2.13}$$

Proof. Since $b_{\alpha,\beta}(n) \geq 0$ for α, $\beta \geq 0$, $|b_{\alpha,\beta}(n)| \leq b_{|\alpha|,|\beta|}(n)$, and we may assume that α, $\beta \geq 0$. Observe that for α, $\beta \geq 0$ and $\gamma = \max\{\alpha, \beta\}$,

$$\begin{aligned} b_{\alpha,\beta}(n) \leq b_{\gamma,\gamma}(n) &= \gamma b_{1,1}(n) \\ &= \gamma \left(b_0(n) + b_1(n)\right). \end{aligned}$$

For $r = 1, 2$, it is easy to see that $\max_{n \in I_r} b_{1,1}(n)$ is F_{r+1}. Now assume that this holds for all $r < k$. Let $r = k$. Then

$$\begin{aligned} \max_{n \in I_k} b_{1,1}(n) &= \max_{n \in I_{k-1}} \{b_{1,1}(2n), b_{1,1}(2n-1)\} \\ &= \max_{n \in I_{k-1}} \{b_{1,1}(n) + b_{1,1}(n+1), b_{1,1}(n-1)\}. \end{aligned}$$

However, one of n or $n+1$ is odd. Thus, if $n, n+1 \in I_{k-1}$, then

$$\begin{aligned}\max_{n \in I_{k-1}} \{b_{1,1}(n) + b_{1,1}(n+1), b_{1,1}(n-1)\} &\leq \max\{F_k + F_{k-1}, F_k\} \\ &= \max\{F_{k+1}, F_k\} \\ &= F_{k+1}.\end{aligned}$$

The previous argument fails if $n+1$ is not in I_{k-1}. However, then

$$n + 1 = 2^{k-1} + 1,$$

and

$$b_{1,1}(2^{k-1} + 1) = b_{1,1}(2^{k-2}) = F_{k-1}$$

by Theorem 2.3.4.

Thus we have shown that

$$\max_{n \in I_k} b_{\gamma,\gamma}(n) \leq \gamma F_{k+1}.$$

Now we will show that $\max_{n \in I_r} b_0(n) \leq F_r$. For $r = 1, 2$, it is easy to see that $\max_{n \in I_r} b_0(n) = F_r$. Now assume that this statement also holds for all $r < k$. Let $r = k$. Then

$$\begin{aligned}\max_{n \in I_k} b_0(n) &= \max_{n \in I_{k-1}} \{b_0(2n), b_0(2n-1)\} \\ &= \max_{n \in I_{k-1}} \{b_0(n) + b_0(n+1), b_0(n-1)\}.\end{aligned}$$

Similarly, one of n or $n+1$ is odd, with the same result if $n+1 = 2^{k-1}+1$. Thus, if $n, n+1 \in I_{k-1}$, then

$$\begin{aligned}\max_{n \in I_{k-1}} \{b_0(n) + b_0(n+1), b_0(n-1)\} &\leq \max\{F_{k-1} + F_{k-2}, F_{k-1}\} \\ &= \max\{F_k, F_{k-1}\} \\ &= F_k.\end{aligned}$$

With $\delta > 0$ and $b_{0,\delta}(n) = \delta b_0(n)$, we have $\max_{n \in I_r} b_{0,\delta}(n) \leq \delta F_r$.

Now we shall show that the maximum occurs at $n = 2^r$. Let $\beta \geq \alpha \geq 0$. Then, by setting $\gamma = \alpha$ and $\delta = \beta - \alpha$ in the previous results above, we find

$$\begin{aligned} b_{\alpha,\beta}(2^r) &= \alpha b_{1,1}(2^r) + (\beta - \alpha) b_0(2^r) \\ &\leq \alpha F_{r+1} + (\beta - \alpha) F_r \\ &= \alpha F_{r-1} + \beta F_r. \end{aligned}$$

However, by Theorem 2.3.4 we know that $b_{\alpha,\beta}(2^r) = \alpha F_{r-1} + \beta F_r$, thus the maximum occurs at $n = 2^r$. ■

2.4 Formulas for $b_{\alpha,\beta}(2^r n + k)$

Here are some results on $b_{\alpha,\beta}(2^r n + k)$ for fixed values of k.

Theorem 2.4.1. *The following formulas hold for the general bow sequence.*

(1). $b_{\alpha,\beta}(2^r n - 4) = F_{r-1} b_{\alpha,\beta}(n) + F_{r-2} b_{\alpha,\beta}(n+1) + 2 b_{\alpha,\beta}(n-1)$, *for* $r \geq 2$, $n \geq 2$.

(2). $b_{\alpha,\beta}(2^r n - 3) = F_{r-1} b_{\alpha,\beta}(n) + F_{r-2} b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n-1)$, *for* $r \geq 2$, $n \geq 2$.

(3). $b_{\alpha,\beta}(2^r n - 2) = F_r b_{\alpha,\beta}(n) + F_{r-1} b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n-1)$, *for* $r \geq 1$, $n \geq 2$.

(4). $b_{\alpha,\beta}(2^r n - 1) = b_{\alpha,\beta}(n-1)$, *for* $r \geq 0$, $n \geq 1$.

(5). $b_{\alpha,\beta}(2^r n) = F_{r+1} b_{\alpha,\beta}(n) + F_r b_{\alpha,\beta}(n+1)$, *for* $r \geq 0$, $n \geq 2$.

(6). $b_{\alpha,\beta}(2^r n + 1) = F_r b_{\alpha,\beta}(n) + F_{r-1} b_{\alpha,\beta}(n+1)$, *for* $r \geq 1$, $n \geq 2$.

(7). $b_{\alpha,\beta}(2^r n + 2) = (F_{r+1} - 1) b_{\alpha,\beta}(n) + F_r b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2)$, *for* $r \geq 1$, $n \geq 2$.

(8). $b_{\alpha,\beta}(2^r n + 3) = F_{r-1} b_{\alpha,\beta}(n) + F_{r-2} b_{\alpha,\beta}(n+1)$, *for* $r \geq 2$, $n \geq 2$.

(9). $b_{\alpha,\beta}(2^r n + 4) = (F_r + F_{r-2} - 1) b_{\alpha,\beta}(n) + (F_{r-1} + F_{r-3}) b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2)$, *for* $r \geq 3$, $n \geq 2$.

Proof. To prove (4) we apply the recurrence repeatedly and obtain

$$b_{\alpha,\beta}(2^r n-1) = b_{\alpha,\beta}(2^{r-1}n-1) = \cdots = b_{\alpha,\beta}(n-1).$$

We show (5) by induction on r. First note by (2.9) that

$$b_{\alpha,\beta}(2^{r+2}n) = b_{\alpha,\beta}(2^{r+1}n) + b_{\alpha,\beta}(2^r n).$$

So all we need to do is to verify the base cases $r = 0$, and 1:

$$\begin{aligned} b_{\alpha,\beta}(n) &= F_1 b_{\alpha,\beta}(n) + F_0 b_{\alpha,\beta}(n+1), \\ b_{\alpha,\beta}(2n) &= F_2 b_{\alpha,\beta}(n) + F_1 b_{\alpha,\beta}(n+1). \end{aligned}$$

Since $F_0 = 0$, and $F_1 = F_2 = 1$, these are immediate.

Property (6) is due to property (5) along with the recursion. We show (7) by induction on r. First note that for $r = 1, 2$ the assertion is that

$$\begin{aligned} b_{\alpha,\beta}(2n+2) &= (F_2 - 1)b_{\alpha,\beta}(n) + F_1 b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2), \\ b_{\alpha,\beta}(4n+2) &= (F_3 - 1)b_{\alpha,\beta}(n) + F_2 b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2). \end{aligned}$$

Since $F_1 = F_2 = 1$, and $F_3 = 2$, these are immediate. Assuming that this holds for $r < k$, we now let $r = k$:

$$\begin{aligned} b_{\alpha,\beta}(2^k n + 2) &= b_{\alpha,\beta}(2^{k-1}n + 1) + b_{\alpha,\beta}(2^{k-1}n + 2) \\ &= F_{k-1} b_{\alpha,\beta}(n) + F_{k-2} b_{\alpha,\beta}(n+1) + (F_k - 1) b_{\alpha,\beta}(n) \\ &\quad + F_{k-1} b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2) \\ &= (F_{k+1} - 1) b_{\alpha,\beta}(n) + F_k b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n+2). \end{aligned}$$

Property (8) is due to property (6). Property (9) is due to (7) and (8). Property (3) is due to (4) and (5). Property (2) is due to property (3). Lastly, property (1) is due to properties (3) and (4). ■

We get the following corollary for $b_0(n)$ and $b_1(n)$.

Corollary 2.4.2. *For $r \geq 1$, the following are true for $b_0(n)$ and $b_1(n)$:*

(1). $b_0(3 \cdot 2^r) = F_r$, $b_0(5 \cdot 2^r) = F_{r+2}$, $b_0(7 \cdot 2^r) = 2F_r$, $b_0(9 \cdot 2^r) = 2F_r + F_{r+1}$, $b_0(11 \cdot 2^r) = 2F_{r+2}$, $b_0(13 \cdot 2^r) = 2F_r + F_{r+1}$, and $b_0(15 \cdot 2^r) = 3F_r$.

(2). $b_1(3{\cdot}2^r) = F_{r+2}$, $b_1(5{\cdot}2^r) = 2F_r$, $b_1(7{\cdot}2^r) = F_{r+2}$, $b_1(9{\cdot}2^r) = 2F_r{+}F_{r+1}$, $b_1(11 \cdot 2^r) = 3F_r$, $b_1(13 \cdot 2^r) = 2F_{r+2}$, and $b_1(15 \cdot 2^r) = 2F_r + F_{r+1}$.

Proof. These properties can be verified by letting $n = 3, 5, 7, 9, 11, 13$, and 15 in Theorem 2.4.1(5). ■

Remark. It is interesting to note that the second property in (2) implies that $b_1(5 \cdot 2^r)$ is always even. Then, by the recurrence, we also know that $b_1(5 \cdot 2^r + 1)$ is also always even. This gives an infinite number of even pairs $\{b_1(5 \cdot 2^r), b_1(5 \cdot 2^r + 1)\}$. Also, the third property in (1) implies that $b_0(7 \cdot 2^r)$ is always even. Hence, we have an infinite number of even pairs $\{b_0(7 \cdot 2^r), b_0(7 \cdot 2^r + 1)\}$. Similarly, we also have an infinite number of pairs $\{b_0(15 \cdot 2^r), b_0(15 \cdot 2^r + 1)\}$ and $\{b_1(11 \cdot 2^r), b_1(11 \cdot 2^r + 1)\}$ which are multiples of three.

Remark. Note that

$$b_0(5 \cdot 2^r) = F_{r+2} = b_1(7 \cdot 2^r), \quad \text{and} \quad b_1(5 \cdot 2^r) = 2F_r = b_0(7 \cdot 2^r).$$

Additionally, we note that

$$b_0(9 \cdot 2^r) = b_1(9 \cdot 2^r),$$

which we will use in Chapter 7.

Lemma 2.4.3. *$b_0(n) = 0 \iff n = 2^r - 1$, for $r \geq 0$.*

Proof. Theorem 2.4.1 shows that if $n = 2^r - 1$, then $b_0(n) = 0$. Suppose $b_0(n) = 0$. If n is even, then $b_0(2k) = 0$ means $b_0(k) + b_0(k + 1) = 0$, and the only time this happens is when $k = 0$, as we can see from Table 2.3. Otherwise, n must be odd, which means that the other zeroes come from the recursion, and thus $n = 2^r - 1$. ■

Lemma 2.4.4. *$b_1(n) = 0 \iff n = 0$ or $n = 3 \cdot 2^r - 1$, for $r \geq 0$.*

Proof. First, we know by Theorem 2.4.1(4) that for $r \geq 0$,

$$b_1(3 \cdot 2^r - 1) = b_1(2) = 0.$$

Now, suppose $b_1(n) = 0$. If n was even, then $b_1(2k) = 0$ means $n = 2$ or $b_1(k) + b_1(k+1) = 0$, which never happens, as we can see from Table 2.4. So n must be either 2 or odd. Thus all the other zeroes come from the recursion, and $n = 3 \cdot 2^r - 1$. ■

For $r \geq 1$, there exist infinitely many n and infinitely many m such that

$$F_r \mid \gcd(b_0(n), b_0(n+1)), \text{ and } F_r \mid \gcd(b_1(m), b_1(m+1)).$$

In fact, if we let

$$P_r = \{n \mid n = (2^r - 1)2^j, j \geq 1\} \quad \text{and} \quad Q_r = \{n \mid n = (3 \cdot 2^r - 1)2^j, j \geq 1\},$$

then we have the following theorem.

Theorem 2.4.5. *For $r \geq 1$, if $n \in P_r$, then $F_r \mid \gcd(b_0(n), b_0(n+1))$, and if $r \geq 2$ and $m \in Q_{r-2}$, then $F_r \mid \gcd(b_1(m), b_1(m+1))$.*

Proof. By Theorem 2.4.1(5) we know that

$$\begin{aligned} b_{\alpha,\beta}(2^r(2k+1)) &= F_{r+1} b_{\alpha,\beta}(2k+1) + F_r b_{\alpha,\beta}(2k+2) \\ &= F_{r+1} b_{\alpha,\beta}(k) + F_r(b_{\alpha,\beta}(k+1) + b_{\alpha,\beta}(k+2)). \end{aligned}$$

By Lemmas 2.4.3 and 2.4.4,

$$k = 2^r - 1 \implies b_0(k) = 0, \text{ and } k = 3 \cdot 2^r - 1 \implies b_1(k) = 0.$$

So if we choose $n = (2^r - 1)2^j$ for $r \geq 2$, $j \geq 1$, and $m = (3 \cdot 2^r - 1)2^j$ for $r, j \geq 1$, then $F_r \mid \gcd(b_0(n), b_0(n+1))$, and $F_{r+2} \mid \gcd(b_1(m), b_1(m+1))$. ■

Lemma 2.4.6. *For any integer r, there exists m such that $r | F_m$.*

Proof. For $d \geq 2$, let S_d denote the set $\{\overline{u}, \overline{v}\}$ of pairs of residue classes modulo d. Then the (invertible) map h on S_d defined by $h(\overline{u}, \overline{v}) = (\overline{v}, \overline{u+v})$ has the property that $h(\overline{F}_n, \overline{F}_{n+1}) = (\overline{F}_{n+1}, \overline{F}_{n+2})$. Since h induces a permutation on S_d, $(\overline{0}, \overline{1})$ belongs to a cycle of length r, and it follows that $h^r(\overline{0}, \overline{1}) = (\overline{0}, \overline{1})$. In particular, for every integer k, $h^{kr}(\overline{0}, \overline{1}) = (\overline{0}, \overline{1})$, so that $\overline{F}_{kr} = 0$; that is $F_{kr} \equiv 0 \pmod{d}$ or $d \mid F_{kr}$. ■

Theorem 2.4.5 and Lemma 2.4.6 imply the following theorem.

Theorem 2.4.7. *For all $d > 0$, there exist infinitely many n and infinitely many m such that*

$$d \mid \gcd(b_0(n), b_0(n+1)) \text{ and } d \mid \gcd(b_1(m), b_1(m+1)).$$

Proof. Given d, there exists r so that $d \mid F_{kr}$ for each k, and it follows that $b_0(n)$ and $b_0(n+1)$ are both multiples of d for $n \in P_{kr}$. Similarly, $b_1(m)$ and $b_1(m+1)$ are both multiples of d for $m \in Q_{kr}$. ■

2.5 Properties of the Summatory Function

Next, we consider the function which is the sum of the r^{th} row of the triatomic array,

$$F_{\alpha,\beta}(r) := \sum_{n \in I_{r+1}} b_{\alpha,\beta}(n) = \sum_{n=2^r+1}^{2^{r+1}} b_{\alpha,\beta}(n).$$

Lemma 2.5.1. *For $r \geq 5$,*

$$F_{\alpha,\beta}(r) = 3F_{\alpha,\beta}(r-1) - \alpha F_{r-5} - \beta F_{r-4}. \tag{2.14}$$

Proof. Starting with the definition of $F_{\alpha,\beta}(r)$ and applying the recurrence, we find

$$\begin{aligned}
F_{\alpha,\beta}(r) &= \sum_{n\in I_{r+1}} b_{\alpha,\beta}(n) \\
&= \sum_{n\in I_r} b_{\alpha,\beta}(2n) + b_{\alpha,\beta}(2n-1) \\
&= \sum_{n\in I_r} b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1) + b_{\alpha,\beta}(n-1) \\
&= \left(\sum_{n\in I_r} 3b_{\alpha,\beta}(n)\right) + b_{\alpha,\beta}(2^r+1) - b_{\alpha,\beta}(2^{r-1}+1) + b_{\alpha,\beta}(2^{r-1}) - b_{\alpha,\beta}(2^r) \\
&= 3F_{\alpha,\beta}(r-1) + b_{\alpha,\beta}(2^{r-1}) - b_{\alpha,\beta}(2^{r-2}) + b_{\alpha,\beta}(2^{r-1}) - b_{\alpha,\beta}(2^r).
\end{aligned}$$

Then, by applying Theorem 2.3.4 and simplifying we get

$$\begin{aligned}
F_{\alpha,\beta}(r) &= 3F_{\alpha,\beta}(r-1) + 2(\alpha F_{r-2} + \beta F_{r-1}) - \alpha F_{r-3} - \beta F_{r-2} - \alpha F_{r-1} - \beta F_r \\
&= 3F_{\alpha,\beta}(r-1) + \alpha\left(F_{r-1} - 3F_{r-3}\right) + \beta\left(F_r - 3F_{r-2}\right) \\
&= 3F_{\alpha,\beta}(r-1) + \alpha\left(F_{r-2} - 2F_{r-3}\right) + \beta\left(F_{r-1} - 2F_{r-2}\right) \\
&= 3F_{\alpha,\beta}(r-1) + \alpha\left(F_{r-4} - F_{r-3}\right) + \beta\left(F_{r-3} - F_{r-2}\right) \\
&= 3F_{\alpha,\beta}(r-1) - \alpha F_{r-5} - \beta F_{r-4}.
\end{aligned}$$

■

This lemma suggests a relationship of the following form.

Theorem 2.5.2. *For $r \geq 1$,*

$$F_{\alpha,\beta}(r) = \alpha\left(\frac{7}{5}\cdot 3^{r-1} + \frac{3}{5}\cdot F_r - \frac{4}{5}\cdot F_{r-1}\right) + \beta\left(\frac{2}{5}\cdot 3^r - \frac{1}{5}\cdot F_r + \frac{3}{5}\cdot F_{r-1}\right).$$

Proof. By considering Table 2.1, we check that this holds for small r. Suppose that the formula holds for $r < k$. Then by Lemma 2.5.1,

$$
\begin{aligned}
F_{\alpha,\beta}(k) = 3\alpha\left(\frac{7}{5}\cdot 3^{k-2} + \frac{3}{5}\cdot F_{k-1} - \frac{4}{5}\cdot F_{k-2}\right) + 3\beta\left(\frac{2}{5}\cdot 3^{k-1} - \frac{1}{5}\cdot F_{k-1} + \frac{3}{5}\cdot F_{k-2}\right) \\
+ \alpha F_{k-1} - 3\alpha F_{k-3} + \beta F_k - 3\beta F_{k-2} \\
= \alpha\left(\frac{7}{5}\cdot 3^{k-1} + \frac{9}{5}\cdot F_{k-1} - \frac{12}{5}\cdot F_{k-2} + F_{k-1} - 3F_{k-3}\right) \\
+ \beta\left(\frac{2}{5}\cdot 3^{k} - \frac{3}{5}\cdot F_{k-1} + \frac{9}{5}\cdot F_{k-2} + F_k - 3F_{k-2}\right) \\
= \alpha\left(\frac{7}{5}\cdot 3^{k-1} + \frac{3}{5}\cdot F_{k} - \frac{4}{5}\cdot F_{k-1}\right) + \beta\left(\frac{2}{5}\cdot 3^{k} - \frac{1}{5}\cdot F_{k} + \frac{3}{5}\cdot F_{k-1}\right).
\end{aligned}
$$

Thus the theorem holds for $r \geq 1$. ■

Remark. Theorem 2.5.2 implies that

$$\frac{F_{\alpha,\beta}(r)}{3^r} = \frac{7\alpha + 6\beta}{15} + \mathcal{O}\left(\left(\frac{\phi}{3}\right)^r\right) = \frac{7\alpha + 6\beta}{15} + o(1).$$

Remark. Interestingly, the sum of the r^{th} row of the diatomic array for the Stern sequence is $3^r + 1$.

Remark. Note that $F_{-6,7}(r) = -5F_r + 9F_{r-1}$, which means that the average value of $b_{-6,7}(n) \to 0$ as $n \to \infty$.

Now suppose we sum all the terms of the general bow sequence up to the N^{th} term. Define a new function as follows:

Definition. $E_{\alpha,\beta}(N)$ is defined as the finite sum of the first N terms of the bow sequence

$$E_{\alpha,\beta}(N) := \sum_{k=1}^{N} b_{\alpha,\beta}(k). \tag{2.15}$$

Theorem 2.5.3. *For $N \geq 1$ we have the following*

$$E_{\alpha,\beta}(2N) = 3E_{\alpha,\beta}(N) - \alpha - b_{\alpha,\beta}(N) + b_{\alpha,\beta}(N+1). \tag{2.16}$$

Proof. One can calculate quickly that this formula holds for $N < 4$. Let $N \geq 4$. We start with the definition, separate terms and apply the recursion to find

$$
\begin{aligned}
E_{\alpha,\beta}(2N) &= \sum_{k=1}^{2N} b_{\alpha,\beta}(k) \\
&= b_{\alpha,\beta}(1) + b_{\alpha,\beta}(2) + b_{\alpha,\beta}(3) + \sum_{k=4}^{2N} b_{\alpha,\beta}(k) \\
&= b_{\alpha,\beta}(1) + b_{\alpha,\beta}(2) + b_{\alpha,\beta}(3) + \sum_{k=2}^{N} (b_{\alpha,\beta}(2k) + b_{\alpha,\beta}(2k+1)) - b_{\alpha,\beta}(2N+1) \\
&= b_{\alpha,\beta}(1) + b_{\alpha,\beta}(2) + b_{\alpha,\beta}(3) + \sum_{k=2}^{N} (2b_{\alpha,\beta}(k) + b_{\alpha,\beta}(k+1)) - b_{\alpha,\beta}(2N+1).
\end{aligned}
$$

Using Table 2.1 we see that

$$
\begin{aligned}
E_{\alpha,\beta}(2N) &= 2\alpha + \beta + \sum_{k=2}^{N} (2b_{\alpha,\beta}(k) + b_{\alpha,\beta}(k+1)) - b_{\alpha,\beta}(2N+1) \\
&= \sum_{k=1}^{N} 2b_{\alpha,\beta}(k) + \sum_{k=1}^{N} b_{\alpha,\beta}(k+1) - b_{\alpha,\beta}(2N+1).
\end{aligned}
$$

We reindex the second sum and apply (2.15) to obtain

$$
\begin{aligned}
E_{\alpha,\beta}(2N) &= 2E_{\alpha,\beta}(N) + \sum_{k=2}^{N+1} b_{\alpha,\beta}(k) - b_{\alpha,\beta}(2N+1) \\
&= 2E_{\alpha,\beta}(N) + E_{\alpha,\beta}(N) - \alpha + b_{\alpha,\beta}(N+1) - b_{\alpha,\beta}(2N+1).
\end{aligned}
$$

Then by applying the recursion we get the desired result

$$
E_{\alpha,\beta}(2N) = 3E_{\alpha,\beta}(N) - \alpha - b_{\alpha,\beta}(N) + b_{\alpha,\beta}(N+1).
$$

■

Corollary 2.5.4. *For $r \geq 3$ and $N = 2^r$ we have*

$$
E_{\alpha,\beta}(2N) = 3E_{\alpha,\beta}(N) - \alpha(F_{r-3} + 1) - \beta F_{r-2}. \tag{2.17}
$$

Proof. By Theorem 2.5.3 we have

$$E_{\alpha,\beta}(2N) = 3E_{\alpha,\beta}(N) - \alpha - b_{\alpha,\beta}(2^r) + b_{\alpha,\beta}(2^r + 1).$$

By applying Theorem 2.4.1 and simplifying we obtain

$$\begin{aligned} E_{\alpha,\beta}(2N) &= 3E_{\alpha,\beta}(N) - \alpha - (\alpha F_{r-1} + \beta F_r) + (\alpha F_{r-2} + \beta F_{r-1}) \\ &= 3E_{\alpha,\beta}(N) - \alpha - \alpha F_{r-3} - \beta F_{r-2} \\ &= 3E_{\alpha,\beta}(N) - \alpha(F_{r-3} + 1) - \beta F_{r-2}. \end{aligned}$$

■

Corollary 2.5.4 suggests the following relationship.

Theorem 2.5.5. *For $r \geq 2$ we have*

$$E_{\alpha,\beta}(2^{r+1}) = \alpha\left(\frac{7}{10}\cdot 3^{r-1} - \frac{1}{5}\cdot F_r + \frac{3}{5}\cdot F_{r-1} + \frac{1}{2}\right) + \beta\left(\frac{1}{5}\cdot 3^r + \frac{2}{5}\cdot F_r - \frac{1}{5}\cdot F_{r-1}\right). \tag{2.18}$$

Proof. By considering Table 2.1, we check that this holds for small $r \geq 2$. Suppose that the formula holds for $r < k$. Then, by Corollary 2.5.4,

$$\begin{aligned} E_{\alpha,\beta}(2^r) &= 3\alpha\left(\frac{7}{10}\cdot 3^{r-2} - \frac{1}{5}\cdot F_{r-1} + \frac{3}{5}\cdot F_{r-2} + \frac{1}{2}\right) \\ &\qquad + 3\beta\left(\frac{1}{5}\cdot 3^{r-1} + \frac{2}{5}\cdot F_{r-1} - \frac{1}{5}\cdot F_{r-2}\right) - \alpha(F_{r-4} + 1) - \beta F_{r-3} \\ &= \alpha\left(\frac{7}{10}\cdot 3^{r-1} - \frac{3}{5}\cdot F_{r-1} + \frac{9}{5}\cdot F_{r-2} + \frac{3}{2} - F_{r-4} - 1\right) \\ &\qquad + \beta\left(\frac{1}{5}\cdot 3^r + \frac{6}{5}\cdot F_{r-1} - \frac{3}{5}\cdot F_{r-2} - F_{r-3}\right). \end{aligned}$$

Then, by simplifying the Fibonacci terms, we find

$$\begin{aligned} E_{\alpha,\beta}(2^r) &= \alpha\left(\frac{7}{10}\cdot 3^{r-1} - \frac{3}{5}\cdot F_{r-1} + \frac{4}{5}\cdot F_{r-2} + F_{r-3} + \frac{1}{2}\right) \\ &\qquad + \beta\left(\frac{1}{5}\cdot 3^r + \frac{1}{5}\cdot F_{r-1} + \frac{2}{5}\cdot F_{r-2}\right) \\ &= \alpha\left(\frac{7}{10}\cdot 3^{r-1} - \frac{1}{5}\cdot F_r + \frac{3}{5}\cdot F_{r-1} + \frac{1}{2}\right) + \beta\left(\frac{1}{5}\cdot 3^r + \frac{2}{5}\cdot F_r - \frac{1}{5}\cdot F_{r-1}\right). \end{aligned}$$

Thus the theorem holds for $r \geq 2$. ■

Remark. Theorem 2.5.5 implies that

$$\frac{E_{\alpha,\beta}(2^r)}{3^r} = \frac{7\alpha + 6\beta}{30} + \mathcal{O}\left(\left(\frac{\phi}{3}\right)^r\right) = \frac{7\alpha + 6\beta}{30} + o(1).$$

Remark. By comparison, the sum of the first 2^r terms of the Stern sequence is

$$\frac{1}{2}\left(3^r + 1\right).$$

Chapter 3

Generating Functions

We use the following notation for generating functions:

$$G^{a_1,a_2,a_3,\ldots,a_m}(x) := x\prod_{j=0}^{\infty}(1+x^{a_1\cdot 2^j}+x^{a_2\cdot 2^j}+\cdots+x^{a_m\cdot 2^j}) \tag{3.1}$$

$$:= \sum_{n=1}^{\infty} c^{a_1,a_2,a_3,\ldots,a_m}(n)x^n. \tag{3.2}$$

Combinatorially, this means that $c^{a_1,a_2,a_3,\ldots,a_m}(n)$ is the number of ways of writing $n-1$ as the sum $\sum_{i\geq 0}\gamma_i 2^i$ where $\gamma_i \in \{0, a_1, a_2, a_3, \ldots, a_m\}$.

First consider

$$\frac{1}{x}G^1(x) = \prod_{n=0}^{\infty}(1+x^{2^j}) = \frac{1}{1-x} = \sum_{k=0}^{\infty} x^k. \tag{3.3}$$

Thus we recover the fact that every integer $n > 0$ has a unique representation

$$n = \sum_{i=0}^{\infty}\gamma_i 2^i, \tag{3.4}$$

where $\gamma_i \in \{0, 1\}$. In this chapter we examine $G^{1,2}(x)$, $G^{1,3}(x)$, and $G^{2,3}(x)$.

3.1 The Generating Function for the Stern Sequence

The generating function for the Stern sequence, $s(n)$, is defined by

$$S(x) := \sum_{n=0}^{\infty} s(n)x^n. \tag{3.5}$$

Separating into even and odd terms as in [14] and [15] we find

$$\begin{aligned} S(x) &= \sum_{n=0}^{\infty} s(2n)x^{2n} + \sum_{n=0}^{\infty} s(2n+1)x^{2n+1} \\ &= \sum_{n=0}^{\infty} s(n)x^{2n} + \sum_{n=0}^{\infty} s(n)x^{2n+1} + \sum_{n=0}^{\infty} s(n+1)x^{2n+1} \\ &= \left(1 + x + \frac{1}{x}\right) S(x^2). \end{aligned}$$

Since $s(0) = 0$ we can write $S(x) = xT(x)$.

$$T(x) = \sum_{n=0}^{\infty} s(n+1)x^n \tag{3.6}$$

and

$$xT(x) = \left(1 + x + \frac{1}{x}\right) x^2 T(x^2),$$

thus

$$T(x) = (1 + x + x^2)T(x^2). \tag{3.7}$$

Since $1 \leq s(n) \leq n$, we know that $\lim_{n\to\infty}(s(n))^{1/n} = 1$. Thus $S(x)$ has radius of convergence 1, is analytic on the open unit disk, and $S(x^r)$ is defined for $|x| < 1$.

Iterating (3.7) N times we obtain

$$T(x) = \left(\prod_{j=0}^{N-1} (1 + x^{2^j} + x^{2^{j+1}}) \right) T\left(x^{2^N}\right).$$

We find that for $|x| < 1$,

$$\lim_{N\to\infty} T\left(x^{2^N}\right) = T(0) = s(1) = 1.$$

Thus we find, as in [15], that

$$S(x) = xT(x) = x\prod_{j=0}^{\infty}(1 + x^{2^j} + x^{2\cdot 2^j}). \tag{3.8}$$

By equation (3.1) we see that $S(x) = G^{1,2}(x)$.

Remark. Combinatorially, the generating function tells us that $s(n) = c^{1,2}(n)$.

Remark. Consider the generating function modulo 2. Proceeding as in [13] we find the following

$$\begin{aligned} S(x) &= x\prod_{j=0}^{\infty}(1 + x^{2^j} + x^{2\cdot 2^j}) \\ &= x\prod_{j=0}^{\infty}\frac{1 - x^{3\cdot 2^j}}{1 - x^{2^j}} \\ &\equiv x\prod_{j=0}^{\infty}\frac{1 + x^{3\cdot 2^j}}{1 + x^{2^j}} \pmod 2 \\ &\equiv x\frac{1}{1 - x^3}(1 - x) \pmod 2 \\ &\equiv \frac{x + x^2}{1 - x^3} \pmod 2. \end{aligned} \tag{3.9}$$

Thus we can clearly see that $s(n)$ is even precisely when $3 \mid n$. We obtain similar results for the bow sequences.

3.2 The Generating Function for the General Bow Sequence

We can now consider the generating function for the general bow sequence. First, let

$$B_{\alpha,\beta}(x) = \sum_{n\geq 0} b_{\alpha,\beta}(n)x^n = \alpha B_{1,0}(x) + \beta B_{0,1}(x). \tag{3.10}$$

It follows from (2.12) that $B_{1,0}(x)$ and $B_{0,1}(x)$ have radius of convergence 1.

By repeatedly applying the recursion and tweaking the limits, we have

$$\begin{aligned}
B_{\alpha,\beta}(x) &= \alpha x + \beta x^2 + \sum_{n\geq 3} b_{\alpha,\beta}(n)x^n \\
&= \alpha x + \beta x^2 + \sum_{n\geq 1} b_{\alpha,\beta}(2n+1)x^{2n+1} + \sum_{n\geq 2} b_{\alpha,\beta}(2n)x^{2n} \\
&= \alpha x + \beta x^2 + \sum_{n\geq 1} b_{\alpha,\beta}(n)x^{2n+1} + \sum_{n\geq 2}(b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1))x^{2n} \\
&= \alpha x + \beta x^2 + \sum_{n\geq 1} b_{\alpha,\beta}(n)x^{2n+1} + \sum_{n\geq 2} b_{\alpha,\beta}(n)x^{2n} + \sum_{n\geq 2} b_{\alpha,\beta}(n+1)x^{2n} \\
&= \alpha x + \beta x^2 + \sum_{n\geq 1} b_{\alpha,\beta}(n)x^{2n+1} + \sum_{n\geq 2} b_{\alpha,\beta}(n)x^{2n} + \sum_{n\geq 3} b_{\alpha,\beta}(n)x^{2n-2} \\
&= \alpha x + \beta x^2 + x\sum_{n\geq 1} b_{\alpha,\beta}(n)x^{2n} + \sum_{n\geq 2} b_{\alpha,\beta}(n)x^{2n} + \frac{1}{x^2}\sum_{n\geq 3} b_{\alpha,\beta}(n)x^{2n} \\
&= \alpha x + \beta x^2 + x\sum_{n\geq 1} b_{\alpha,\beta}(n)x^{2n} + \sum_{n\geq 1} b_{\alpha,\beta}(n)x^{2n} - \alpha x^2 \\
&\qquad + \frac{1}{x^2}\sum_{n\geq 1} b_{\alpha,\beta}(n)x^{2n} - \alpha - \beta x^2 \\
&= -\alpha + \alpha x - \alpha x^2 + \left(\frac{1}{x^2} + 1 + x\right) B_{\alpha,\beta}(x^2) \\
&= -\alpha(1 - x + x^2) + \frac{1}{x^2}(1 + x^2 + x^3)B_{\alpha,\beta}(x^2).
\end{aligned}$$

Since $b_{\alpha,\beta}(0) = 0$ and $b_{\alpha,\beta}(1) = \alpha$, we can write $B_{\alpha,\beta}(x) = \alpha x + x^2 C_{\alpha,\beta}(x)$, with $C_{\alpha,\beta}(0) = \beta$, so $B_{\alpha,\beta}(x^2) = \alpha x^2 + x^4 C_{\alpha,\beta}(x^2)$. Thus we have

$$\begin{aligned}B_{\alpha,\beta}(x) &= \alpha x + x^2 C_{\alpha,\beta}(x)\\ &= \frac{1}{x^2}(1+x^2+x^3)(\alpha x^2 + x^4 C_{\alpha,\beta}(x^2)) - \alpha(1-x+x^2).\end{aligned}$$

Solving for $C_{\alpha,\beta}(x)$, we find

$$\begin{aligned}C_{\alpha,\beta}(x) &= \frac{1}{x^2}\Big(\alpha(1-x+x^2+x^3) + x^2 C_{\alpha,\beta}(x^2)(1+x^2+x^3) - \alpha(1-x+x^2)\Big)\\ &= \alpha x + C_{\alpha,\beta}(x^2)(1+x^2+x^3).\end{aligned}$$

We can feed this identity back into the equation to get

$$\begin{aligned}C_{\alpha,\beta}(x) &= \alpha x + (1+x^2+x^3)(\alpha x^2 + C_{\alpha,\beta}(x^4)(1+x^4+x^6))\\ &= \alpha x + (1+x^2+x^3)\alpha x^2 + (1+x^2+x^3)(1+x^4+x^6)C_{\alpha,\beta}(x^4).\end{aligned}$$

After N steps,

$$C_{\alpha,\beta}(x) = \alpha x + \alpha \sum_{k=0}^{N} x^{2\cdot 2^k} \prod_{j=0}^{k}(1 + x^{2\cdot 2^j} + x^{3\cdot 2^j}) + \prod_{k=0}^{N+1}(1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}) C_{\alpha,\beta}(x^{2^{N+2}}).$$

Since, $C_{\alpha,\beta}(x) = \beta + x\cdot P(x)$ for some $P(x)$, $C_{\alpha,\beta}(x^{2^{N+2}}) - \beta = x^{2^{N+2}} P(x^{2^{N+2}})$. Thus for $|x| < 1$, $C_{\alpha,\beta}(x^{2^{N+2}}) - \beta \to 0$ as $N \to \infty$; hence we have

$$C_{\alpha,\beta}(x) = \alpha x + \alpha \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k}(1 + x^{2\cdot 2^j} + x^{3\cdot 2^j}) + \beta \prod_{k=0}^{\infty}(1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}).$$

Thus we have the following theorem.

Theorem 3.2.1. *The generating function for the general bow sequence is:*

$$B_{\alpha,\beta}(x) = \alpha x + \alpha x^3 + \alpha x^2 \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k} (1 + x^{2\cdot 2^j} + x^{3\cdot 2^j}) \qquad (3.11)$$
$$+ \beta x^2 \prod_{k=0}^{\infty} (1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}).$$

Since $B_{\alpha,\beta}(x) = \alpha B_{1,0}(x) + \beta B_{0,1}(x)$, we can state two combinatorial implications. First, we have

$$B_{0,1}(x) := \sum_{n=0}^{\infty} b_0(n) x^n = x^2 \prod_{k=0}^{\infty} (1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}) = x G^{2,3}(x) \qquad (3.12)$$

for the case when $\alpha = 0$, $\beta = 1$. We interpret this statement as:

Corollary 3.2.2. *Combinatorially, $b_0(n) = c^{2,3}(n-1)$ is the number of ways of writing $n-2$ as the sum $\sum_i c_i 2^i$ where $c_i \in \{0, 2, 3\}$.*

We also have

$$B_{1,0}(x) := \sum_{n=0}^{\infty} b_1(n) x^n = x + x^3 + x^2 \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k} (1 + x^{2\cdot 2^j} + x^{3\cdot 2^j}) \quad (3.13)$$

for the case when $\alpha = 1$, $\beta = 0$. We interpret this statement as:

Corollary 3.2.3. *Combinatorially, $b_1(n)$ is the number of ways of writing $n - 2 - 2^{k+1}$ as the sum $\sum_{i=0}^{k} c_i 2^i$ where $c_i \in \{0, 2, 3\}$ and $k \in \mathbb{N}$.*

Remark. By rearranging, we find that $b_1(n)$ is also the number of ways of writing $n-2$ as the sum $\sum_{i=0}^{k} c_i 2^i$ where $k \in \mathbb{N}$, $c_i \in \{0, 2, 3\}$ for $i \le k-1$ and $c_k \in \{2, 4, 5\}$.

Alternatively, since $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$, we get the following formula

$$B_{1,0}(x) = x + x^3 + x^3 \sum_{k=0}^{\infty} \prod_{j=0}^{k} (x^{1\cdot 2^j} + x^{3\cdot 2^j} + x^{4\cdot 2^j}), \qquad (3.14)$$

which means that $b_1(n)$ is the number of ways of writing $n-3$ as a finite sum $\sum_{i=0}^{k} c_i 2^i$ where $c_i \in \{1, 3, 4\}$.

3.3 A Relative of the Bow Sequences

After considering the generating functions $G^{1,2}(x)$ and $G^{2,3}(x)$, one might be curious about the sequence that has the following generating function

$$\begin{aligned} Y(x) &:= G^{1,3}(x) && (3.15)\\ &= x\prod_{k=0}^{\infty}(1+x^{2^k}+x^{3\cdot 2^k}) = \sum_{n=0}^{\infty} y(n)x^n. && (3.16)\end{aligned}$$

We see that $y(n) = c^{1,3}(n)$ is the number of ways of writing $n-1$ as the sum $\sum_{i\geq 0} c_i 2^i$ where $c_i \in \{0,1,3\}$. We can define $y(n)$ recursively as follows:

$$\begin{aligned} y(0) = 0, \quad y(1) = 1; && (3.17)\\ y(2n+1) = y(n+1), \quad \text{for } n \geq 1; && (3.18)\\ y(2n) = y(n) + y(n-1), \quad \text{for } n \geq 1. && (3.19)\end{aligned}$$

By analogy with $b_{\alpha,\beta}(n)$, we can define the general sequence by:

$$\begin{aligned} y_{\alpha,\beta}(0) = \alpha, \quad y_{\alpha,\beta}(1) = \beta; && (3.20)\\ y_{\alpha,\beta}(2n+1) = y_{\alpha,\beta}(n+1), \quad \text{for } n \geq 1; && (3.21)\\ y_{\alpha,\beta}(2n) = y_{\alpha,\beta}(n) + y_{\alpha,\beta}(n-1), \quad \text{for } n \geq 1. && (3.22)\end{aligned}$$

In particular, $y(n) = y_{0,1}(n)$. We can find the generating function of $y_{\alpha,\beta}(n)$ using the techniques of this chapter. Let the generating function be defined as

$$Y_{\alpha,\beta}(x) := \sum_{n\geq 0} y_{\alpha,\beta}(n)x^n.$$

Then

$$\begin{aligned}
Y_{\alpha,\beta}(x) &= \alpha + \beta x + \sum_{n\geq 2} y_{\alpha,\beta}(n)x^n \\
&= \alpha + \beta x + \sum_{n\geq 1} y_{\alpha,\beta}(2n)x^{2n} + \sum_{n\geq 1} y_{\alpha,\beta}(2n+1)x^{2n+1} \\
&= \alpha + \beta x + \sum_{n\geq 1}(y_{\alpha,\beta}(n) + y_{\alpha,\beta}(n-1))x^{2n} + \sum_{n\geq 1} y_{\alpha,\beta}(n+1)x^{2n+1} \\
&= \alpha + \beta x + \sum_{n\geq 0} y_{\alpha,\beta}(n)x^{2n} - \alpha + \sum_{n\geq 0} y_{\alpha,\beta}(n)x^{2n+2} + \sum_{n\geq 2} y_{\alpha,\beta}(n)x^{2n-1} \\
&= \beta x + Y_{\alpha,\beta}(x^2) + x^2 Y_{\alpha,\beta}(x^2) + \sum_{n\geq 0} y_{\alpha,\beta}(n)x^{2n-1} - \frac{\alpha}{x} - \beta x \\
&= -\frac{\alpha}{x} + \frac{1}{x}Y_{\alpha,\beta}(x^2) + Y_{\alpha,\beta}(x^2) + x^2 Y_{\alpha,\beta}(x^2) \\
&= -\frac{\alpha}{x} + \frac{1}{x}Y_{\alpha,\beta}(x^2)(1 + x + x^3).
\end{aligned}$$

Now, let $Y_{\alpha,\beta}(x) = \alpha + xD_{\alpha,\beta}(x)$. So $Y_{\alpha,\beta}(x^2) = \alpha + x^2 D_{\alpha,\beta}(x^2)$. Substituting this into our equation for $Y_{\alpha,\beta}(x)$ we get

$$\begin{aligned}
Y_{\alpha,\beta}(x) &= -\frac{\alpha}{x} + \frac{1}{x}Y_{\alpha,\beta}(x^2)(1 + x + x^3) \\
&= -\frac{\alpha}{x} + \frac{1}{x}(\alpha + x^2 D_{\alpha,\beta}(x^2))(1 + x + x^3).
\end{aligned}$$

Then we solve for $D_{\alpha,\beta}(x)$ as follows

$$\begin{aligned}
xD_{\alpha,\beta}(x) &= -\frac{\alpha}{x} + \frac{1}{x}(\alpha + x^2 D_{\alpha,\beta}(x^2))(1 + x + x^3) \\
&= -\frac{\alpha}{x} - \alpha + (\frac{\alpha}{x} + xD_{\alpha,\beta}(x^2))(1 + x + x^3) \\
&= -\frac{\alpha}{x} - \alpha + \frac{\alpha}{x} + \alpha + \alpha x^2 = xD_{\alpha,\beta}(x^2) + x^2 D_{\alpha,\beta}(x^2) + x^4 D_{\alpha,\beta}(x^2) \\
&= \alpha x^2 + D_{\alpha,\beta}(x^2)(x + x^2 + x^4).
\end{aligned}$$

Dividing each side by x we find

$$D_{\alpha,\beta}(x) = \alpha x + D_{\alpha,\beta}(x^2)(1 + x + x^3).$$

Next, we shall substitute this equation into itself repeatedly to obtain the following equation after N iterations

$$D_{\alpha,\beta}(x) = \alpha x + \alpha \sum_{k=0}^{N} x^{2\cdot 2^k} \prod_{j=0}^{k} (1 + x^{2^j} + x^{3\cdot 2^j}) + D_{\alpha,\beta}(x^{2^{N+1}}) \prod_{j=0}^{N} (1 + x^{2^j} + x^{3\cdot 2^j}).$$

Let $N \to \infty$. By the definition of $D_{\alpha,\beta}(x)$ we see that for $|x| < 1$, $D_{\alpha,\beta}(x^{2^{N+1}}) \to \beta$ as $N \to \infty$. Now we get the following formula

$$D_{\alpha,\beta}(x) = \alpha x + \alpha \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k} (1 + x^{2^j} + x^{3\cdot 2^j}) + \beta \prod_{j=0}^{\infty} (1 + x^{2^j} + x^{3\cdot 2^j}).$$

Thus we obtain the following theorem.

Theorem 3.3.1. *For $y_{\alpha,\beta}(n)$,*

$$\begin{aligned} Y_{\alpha,\beta}(x) = \alpha + \alpha x^2 + \alpha x \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k} (1 + x^{2^j} + x^{3\cdot 2^j}) \\ + \beta x \prod_{j=0}^{\infty} (1 + x^{2^j} + x^{3\cdot 2^j}). \end{aligned} \tag{3.23}$$

Since $Y_{\alpha,\beta}(x) = \alpha Y_{1,0}(x) + \beta Y_{0,1}(x)$, we can state another combinatorial implication. First, we have

$$Y_{1,0}(x) = 1 + x^2 + x \sum_{k=0}^{\infty} x^{2\cdot 2^k} \prod_{j=0}^{k} (1 + x^{2^j} + x^{3\cdot 2^j}) \tag{3.24}$$

for the case where $\alpha = 1$, $\beta = 0$. We interpret this statement as:

Corollary 3.3.2. *Combinatorially, $y_{1,0}(n)$ is the number of ways of writing $n - 1 - 2^{k+1}$ as the sum $\sum_{i=0}^{k} c_i 2^i$ where $k \in \mathbb{N}$, $c_i \in \{0, 1, 3\}$.*

Remark. By rearranging, we find that $y_{1,0}(n)$ is also the number of ways of writing $n - 1$ as the sum $\sum_{i=0}^{k} c_i 2^i$, where $k \in \mathbb{N}$, $c_i \in \{0, 1, 3\}$ for $i \leq k - 1$, and $c_k \in \{2, 3, 5\}$.

Alternatively, since $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$, we get the following formula

$$Y_{1,0}(x) = 1 + x^2 + x^2 \sum_{k=0}^{\infty} \prod_{j=0}^{k} (x^{1\cdot 2^j} + x^{2\cdot 2^j} + x^{4\cdot 2^j}), \tag{3.25}$$

which means that $y_{1,0}(n)$ is the number of ways of writing $n-2$ as a finite sum $\sum_{i=0}^{k} c_i 2^i$, where $c_i \in \{1,2,4\}$.

Let

$$Y_n(x) := x \prod_{k=0}^{n} \left(1 + x^{2^k} + x^{3\cdot 2^k}\right), \tag{3.26}$$

and define

$$B_n(x) := x^2 \prod_{k=0}^{n} \left(1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}\right). \tag{3.27}$$

Then we know that $Y_n(x) = Y(x)$ up to $x^{2^{n+1}}$, and $B_n(x) = B_{0,1}(x)$ up to $x^{2^{n+2}}$. Consider

$$Y_n\left(\frac{1}{x}\right) = \frac{1}{x} \prod_{k=0}^{n} \left(1 + \frac{1}{x^{2^k}} + \frac{1}{x^{3\cdot 2^k}}\right).$$

If we multiply by

$$x^3 \prod_{k=0}^{n} x^{3\cdot 2^k},$$

we find

$$\begin{aligned} x^{3\cdot(2^{n+1}-1)+3} Y_n\left(\frac{1}{x}\right) &= x^2 \prod_{k=0}^{n} \left(x^{3\cdot 2^k} + x^{2\cdot 2^k} + 1\right) \\ &= B_n(x). \end{aligned}$$

Thus,

Theorem 3.3.3.

$$x^{3\cdot 2^{n+1}} Y_n\left(\frac{1}{x}\right) = B_n(x).$$

Theorem 3.3.3 can be used to find a "weak" relationship between $y(n)$ and $b_0(m)$, but first we need the following definition.

Definition. For an integer $N \geq 1$, we say that

$$\sum_{n=0}^{\infty} a(n)x^n \equiv \sum_{n=0}^{\infty} b(n)x^n \pmod{x^N}$$

if $a(n) = b(n)$ for $n < N$.

Remark. No information is assumed about $a(n)$ and $b(n)$ for $n \geq N$.

Now, consider the following generating functions:

$$\begin{aligned} V(x) &:= \sum_{n=0}^{\infty} v(n)x^n = \prod_{k=0}^{\infty} \left(1 + x^{2^k} + x^{3\cdot 2^k}\right), \\ W(x) &:= \sum_{n=0}^{\infty} w(n)x^n = \prod_{k=0}^{\infty} \left(1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}\right). \end{aligned}$$

For an integer $r \geq 0$, let

$$\begin{aligned} V_r(x) &= \sum_{n=0}^{\infty} v_r(n)x^n = \prod_{k=0}^{r} \left(1 + x^{2^k} + x^{3\cdot 2^k}\right), \\ W_r(x) &= \sum_{n=0}^{\infty} w_r(n)x^n = \prod_{k=0}^{r} \left(1 + x^{2\cdot 2^k} + x^{3\cdot 2^k}\right). \end{aligned}$$

As in our work towards Theorem 3.3.3, observe that

$$\begin{aligned} W_r\left(\frac{1}{x}\right) &= \prod_{k=0}^{r} \left(1 + \frac{1}{x^{2\cdot 2^k}} + \frac{1}{x^{3\cdot 2^k}}\right) \\ &= \left(\prod_{k=0}^{r} \frac{1}{x^{3\cdot 2^k}}\right) \prod_{k=0}^{r} \left(x^{3\cdot 2^k} + x^{2^k} + 1\right), \end{aligned}$$

so that $V_r(x) = x^{3(2^{r+1}-1)} W_r(\frac{1}{x})$. We can see that $V_r(x)$ is a polynomial of degree $M_r := 3(2^{r+1} - 1)$. So we have

$$\sum_{n=0}^{M_r} v_r(n)x^n = \sum_{n=0}^{M_r} w_r(n)x^{M_r - n},$$

hence

$$v_r(n) = w_r(M_r - n). \tag{3.28}$$

Now we consider $V(x)$ up to x^{M_r}. We see that

$$\begin{aligned}
V(x) &= V_r(x)\left(1 + x^{2^{r+1}} + x^{3\cdot 2^{r+1}}\right)\left(1 + x^{2^{r+2}} + x^{3\cdot 2^{r+2}}\right)\cdots \\
&\equiv V_r(x)\left(1 + x^{2^{r+1}}\right)\left(1 + x^{2^{r+2}}\right) \pmod{x^{M_r}} \\
&\equiv V_r(x)\left(1 + x^{2^{r+1}} + x^{2^{r+2}}\right) \pmod{x^{M_r}}.
\end{aligned}$$

Since $3 \cdot 2^r > M_r$, it is also true that

$$V(x) \equiv V_r(x)\left(1 + x^{2^{r+1}} + x^{2^{r+2}} + x^{3\cdot 2^{r+1}} + \cdots\right) \pmod{x^{M_r}}.$$

Thus,

$$V(x) \equiv \frac{V_r(x)}{1 - x^{2^{r+1}}} \pmod{x^{M_r}},$$

or equivalently,

$$\left(1 - x^{2^{r+1}}\right) V(x) \equiv V_r(x) \pmod{x^{M_r}}.$$

Thus, with the restriction that if $n < 0$, $v(n) = 0$, for $m < M_r$, we have

$$v_r(m) = v(m) - v(m - 2^{r+1}). \tag{3.29}$$

Similarly, we have

$$\begin{aligned}
W(x) &= W_r(x)\left(1 + x^{2\cdot 2^{r+1}} + x^{3\cdot 2^{r+1}}\right)\left(1 + x^{2\cdot 2^{r+2}} + x^{3\cdot 2^{r+2}}\right)\cdots \\
&\equiv W_r(x)\left(1 + x^{2\cdot 2^{r+1}}\right) \pmod{x^{M_r}}.
\end{aligned}$$

Since $4 \cdot 2^{r+1} > M_r$,

$$W(x) \equiv W_r(x)\frac{1}{1 - x^{2\cdot 2^{r+1}}} \pmod{x^{M_r}},$$

or equivalently,

$$\left(1 - x^{2\cdot 2^{r+1}}\right) W(x) \equiv W_r(x) \pmod{x^{M_r}}.$$

Thus, with the restriction that if $n < 0$, $w(n) = 0$, for $m < M_r$, we have

$$w_r(m) = w(m) - w(m - 2 \cdot 2^{r+1}). \tag{3.30}$$

If we now limit ourselves to $0 \leq n \leq M_r$, by (3.28), (3.29), and (3.30) we have:

$$v(n) - v(n - 2^{r+1}) = w(3 \cdot 2^{r+1} - 3 - n) - w(2^{r+1} - 3 - n). \tag{3.31}$$

Recalling that $v(n) = y(n+1)$ and $w(n) = b_0(n+2)$, we substitute these into (3.31) to find

$$y(n+1) - y(n+1-2^{r+1}) = b_0(3 \cdot 2^{r+1} - 1 - n) - b_0(2^{r+1} - 1 - n).$$

Setting $m = n + 1$ we have the following theorem.

Theorem 3.3.4. *For $m \leq 3 \cdot 2^{r+1} - 2$,*

$$y(m) - y(m - 2^{r+1}) = b_0(3 \cdot 2^{r+1} - m) - b_0(2^{r+1} - m), \tag{3.32}$$

where $y(k) = b_0(k) = 0$ for $k < 0$.

Remark. There is no m such that both $m - 2^{r+1} > 0$ and $2^{r+1} - m > 0$.

Next, consider the generating function

$$G^{2,3}(x) = x \prod_{j=0}^{\infty} (1 + x^{2 \cdot 2^j} + x^{3 \cdot 2^j}). \tag{3.33}$$

We may examine this generating function mod 2.

Theorem 3.3.5. $G^{2,3}(x) \equiv \frac{x+x^3+x^4+x^5}{1-x^7} \pmod 2$.

Proof. We shall show that

$$\begin{aligned} &x \prod_{j=0}^{k} (1 + x^{2 \cdot 2^j} + x^{3 \cdot 2^j}) - \frac{x + x^3 + x^4 + x^5}{1 - x^7} \\ &\equiv \frac{(1 + x^2 + x^3 + x^4) x^{2^{k+2}+1} (1 + x^{2^{k+1}})}{1 - x^7} \pmod 2. \end{aligned} \tag{3.34}$$

For the case $k=0$, (3.34) becomes

$$x(1+x^{2\cdot 2^0}+x^{3\cdot 2^0})-\frac{x+x^3+x^4+x^5}{1-x^7}=\frac{x^5+x^8+x^{10}+x^{11}}{1-x^7}$$
$$=\frac{(1+x^2+x^3+x^4)x^5(1+x^2)}{1-x^7},$$

which is what we want.

Now, assuming that (3.34) holds for k, we see that for $k+1$ we have

$$\begin{aligned}
&x\prod_{j=0}^{k+1}(1+x^{2\cdot 2^j}+x^{3\cdot 2^j})-\frac{x+x^3+x^4+x^5}{1-x^7}\\
&=\left(\frac{(1+x^2+x^3+x^4)x^{2^{k+2}+1}(1+x^{2^{k+1}})}{1-x^7}+\frac{x+x^3+x^4+x^5}{1-x^7}\right)(1+x^{2\cdot 2^{k+1}}+x^{3\cdot 2^{k+1}})\\
&\quad-\frac{x+x^3+x^4+x^5}{1-x^7}\\
&=\frac{1+x^2+x^3+x^4}{1-x^7}\cdot\left[\left(x^{2^{k+2}+1}\left(1+x^{2^{k+1}}\right)+x\right)\left(1+x^{2\cdot 2^{k+1}}+x^{3\cdot 2^{k+1}}\right)-x\right]\\
&=\frac{1+x^2+x^3+x^4}{1-x^7}\cdot\left[x^{2\cdot 2^{k+1}+1}+x^{3\cdot 2^{k+1}+1}+x^{4\cdot 2^{k+1}+1}+x^{5\cdot 2^{k+1}+1}+x^{2\cdot 2^{k+1}+1}\right.\\
&\quad\left.+x^{5\cdot 2^{k+1}+1}+x^{6\cdot 2^{k+1}+1}+x^{3\cdot 2^{k+1}+1}+x-x\right].
\end{aligned}$$

Then, considering this equation modulo 2, we remove terms with coefficient 2 to obtain

$$\begin{aligned}
&x\prod_{j=0}^{k+1}(1+x^{2\cdot 2^j}+x^{3\cdot 2^j})-\frac{x+x^3+x^4+x^5}{1-x^7}\\
&\equiv\frac{1+x^2+x^3+x^4}{1-x^7}\cdot\left(x^{4\cdot 2^{k+1}+1}+x^{6\cdot 2^{k+1}+1}\right)\pmod 2\\
&\equiv\frac{(1+x^2+x^3+x^4)(x^{2^{k+3}+1})(1+x^{2^{k+2}})}{1-x^7}\pmod 2.
\end{aligned}$$

Thus as $k\to\infty$ we have

$$x\prod_{j=0}^{k}(1+x^{2\cdot 2^j}+x^{3\cdot 2^j})\to\frac{x+x^3+x^4+x^5}{1-x^7}. \tag{3.35}$$

■

From Theorem 3.3.5 it is easy to see that if

$$G^{2,3}(x) = \sum_{n=0}^{\infty} c^{2,3}(n)x^n,$$

then $c^{2,3}(n)$ is even exactly when $n \equiv 0, 2, 6 \pmod 7$. Note that

$$c^{2,3}(n) = b_0(n+1).$$

Thus we have the following corollary.

Corollary 3.3.6. *$b_0(n)$ is even precisely when $n \equiv 0, 1, 3 \pmod 7$.*

Next, consider $Y(x) = G^{1,3}(x)$ modulo 2.

Theorem 3.3.7. $G^{1,3}(x) \equiv \frac{x+x^2+x^3+x^5}{1-x^7} \pmod 2$.

Proof. We shall show, to be precise, that

$$\begin{aligned} &x\prod_{j=0}^{k}(1 + x^{2^j} + x^{3\cdot 2^j}) - \frac{x + x^2 + x^3 + x^5}{1 - x^7} \\ &\equiv \frac{(1 + x + x^2 + x^4)x^{2^{k+1}+1}(1 + x^{2^{k+2}})}{1 - x^7} \pmod 2. \end{aligned} \tag{3.36}$$

For the case $k = 0$, (3.36) becomes

$$\begin{aligned} x(1 + x^{2^0} + x^{3\cdot 2^0}) - \frac{x + x^2 + x^3 + x^5}{1 - x^7} &= \frac{x^3 + x^4 + x^5 + x^8 + x^9 + x^{11}}{1 - x^7} \\ &= \frac{(1 + x + x^2 + x^4)x^3(1 + x^4)}{1 - x^7}, \end{aligned}$$

which is what we want.

Now, assuming that (3.36) holds for k, we see that for $k+1$ we have

$$
\begin{aligned}
&x\prod_{j=0}^{k+1}(1+x^{2^j}+x^{3\cdot 2^j})-\frac{x+x^2+x^3+x^5}{1-x^7}\\
&=\left(\frac{(1+x+x^2+x^4)x^{2^{k+1}+1}(1+x^{2^{k+2}})}{1-x^7}+\frac{x+x^2+x^3+x^5}{1-x^7}\right)(1+x^{2^{k+1}}+x^{3\cdot 2^{k+1}})\\
&\quad-\frac{x+x^2+x^3+x^5}{1-x^7}\\
&=\frac{1+x+x^2+x^4}{1-x^7}\cdot\left[\left(x^{2^{k+1}+1}\left(1+x^{2^{k+2}}\right)+x\right)\left(1+x^{2^{k+1}}+x^{3\cdot 2^{k+1}}\right)-x\right]\\
&=\frac{1+x+x^2+x^4}{1-x^7}\cdot\left[x^{2^{k+1}+1}+x^{3\cdot 2^{k+1}+1}+x^{2\cdot 2^{k+1}+1}+x^{4\cdot 2^{k+1}+1}+x^{2\cdot 2^{k+1}+1}\right.\\
&\quad\left.+x^{4\cdot 2^{k+1}+1}+x^{6\cdot 2^{k+1}+1}+x^{3\cdot 2^{k+1}+1}+x-x\right].
\end{aligned}
$$

Then, considering this equation modulo 2, we remove terms with coefficient 2 to obtain

$$
\begin{aligned}
&x\prod_{j=0}^{k+1}(1+x^{2^j}+x^{3\cdot 2^j})-\frac{x+x^2+x^3+x^5}{1-x^7}\\
&\equiv\frac{1+x+x^2+x^4}{1-x^7}\cdot\left(x^{2\cdot 2^{k+1}+1}+x^{6\cdot 2^{k+1}+1}\right)\pmod 2\\
&\equiv\frac{(1+x+x^2+x^4)(x^{2^{k+2}+1})(1+x^{2^{k+3}})}{1-x^7}\pmod 2.
\end{aligned}
$$

Thus as $k\to\infty$ we have

$$
x\prod_{j=0}^{k}(1+x^{2^j}+x^{3\cdot 2^j})\to\frac{x+x^2+x^3+x^5}{1-x^7}. \tag{3.37}
$$

■

From Theorem 3.3.7 it is easy to see that if $G^{1,3}(x)=\sum_{n=0}^{\infty}c^{1,3}(n)x^n$, then $c^{1,3}(n)$ is even exactly when $n\equiv 0,4,6\pmod 7$. Recall that $y(n)=c^{1,3}(n)$. We have the following corollary.

Corollary 3.3.8. *$y(n)$ is even precisely when $n\equiv 0,4,6\pmod 7$.*

We also observe the following theorem.

Theorem 3.3.9. $G^{1,k-1,k}(x)=\frac{x}{(1-x)(1-x^{k-1})}$.

Proof. Starting with the definition, we know that

$$G^{1,k-1,k}(x) = x \prod_{j=0}^{\infty} \left(1 + x^{2^j} + x^{(k-1)\cdot 2^j} + x^{k\cdot 2^j}\right).$$

Then, by factoring and simplifying, we find that

$$\begin{aligned} G^{1,k-1,k}(x) &= x \prod_{j=0}^{\infty} \left(1 + x^{2^j}\right) \prod_{i=0}^{\infty} \left(1 + x^{(k-1)\cdot 2^i}\right) \\ &= x \cdot \frac{1}{1-x} \cdot \frac{1}{1-x^{k-1}} \\ &= \frac{x}{(1-x)(1-x^{k-1})}. \end{aligned}$$

■

Remark. We can write $G^{1,k-1,k}(x)$ as

$$\begin{aligned} G^{1,k-1,k} &= x(1 + x + x^2 + \cdots)(1 + x^{k-1} + x^{2(k-1)} + \cdots) \\ &= x(1 + x + \cdots + x^{k-2} + 2x^{k-1} + \cdots + 2x^{2k-3} + 3x^{2k-2} + \cdots). \end{aligned}$$

Thus $a^{1,k-1,k} = \left\lfloor \frac{n}{k-1} \right\rfloor + 1$.

Corollary 3.3.10. *Combinatorially, $a^{1,k-1,k}(n)$ is the number of ways of writing $n-1$ as the sum $\sum_{i\geq 0} c_i 2^i$ where $c_i \in \{0, 1, k-1, k\}$.*

Remark. Alternatively, if we let $c_i = a_i + (k-1)b_i$, with $a_i, b_i \in \{0, 1\}$, then $a^{1,k-1,k}(n)$ is the number of ways of writing $n - 1 = r + (k-1)s$, where $r = \sum_{i\geq 0} a_i 2^i$, and $s = \sum_{i\geq 0} b_i 2^i$.

Chapter 4

The General Bow Sequence Modulo 2

We begin with a few statements about the general bow sequence modulo 2 before proceeding to consider $b_0(n)$ and $b_1(n)$ in greater detail.

4.1 The Behavior of $b_{\alpha,\beta}(n)$ Modulo 2

Let $A_{\alpha,\beta}(m)$ denote the following expression:

$$A_{\alpha,\beta}(m) = b_{\alpha,\beta}(m-1) + b_{\alpha,\beta}(m) + b_{\alpha,\beta}(m+2). \tag{4.1}$$

Then we have the following theorem.

Theorem 4.1.1. *For $m \geq 2$, $A_{\alpha,\beta}(m)$ is always even, except that for $k \geq 0$, we have $A_{\alpha,\beta}(3 \cdot 2^k) \equiv \alpha \pmod 2$.*

Proof. For small m and k, we use Table 2.1 to note the following:

$$\begin{aligned} A_{\alpha,\beta}(2) &= b_{\alpha,\beta}(1) + b_{\alpha,\beta}(2) + b_{\alpha,\beta}(4) = 2\alpha + 2\beta \equiv 0 \pmod 2, \\ A_{\alpha,\beta}(3) &= b_{\alpha,\beta}(2) + b_{\alpha,\beta}(3) + b_{\alpha,\beta}(5) = \alpha + 2\beta \equiv \alpha \pmod 2. \end{aligned}$$

Then, by applying (2.5) for $m \geq 2$, we see that

$$\begin{aligned} A_{\alpha,\beta}(2m+1) &= b_{\alpha,\beta}(2m) + b_{\alpha,\beta}(2m+1) + b_{\alpha,\beta}(2m+3) \\ &\equiv 0 \pmod 2. \end{aligned}$$

Thus we know that $A_{\alpha,\beta}(3) \equiv \alpha \pmod 2$, and $A_{\alpha,\beta}(2m+1) \equiv 0 \pmod 2$ for $m \geq 2$.

For even terms, by applying the recurrence, we find

$$\begin{aligned}A_{\alpha,\beta}(2m) &= b_{\alpha,\beta}(2m-1) + b_{\alpha,\beta}(2m) + b_{\alpha,\beta}(2m+2)\\ &= b_{\alpha,\beta}(m-1) + b_{\alpha,\beta}(m) + 2b_{\alpha,\beta}(m+1) + b_{\alpha,\beta}(m+2)\\ &\equiv A_{\alpha,\beta}(m) \pmod 2.\end{aligned}$$

Putting these together, we find that for $k \geq 0$ and $m \geq 2$, $A_{\alpha,\beta}(3\cdot 2^k) \equiv \alpha \pmod 2$, and otherwise $A_{\alpha,\beta}(m) \equiv 0 \pmod 2$. ■

Remark. Note that this implies that for $m \geq 1$,

$$b_0(m-1) + b_0(m) + b_0(m+2) \equiv 0 \pmod 2. \tag{4.2}$$

We use the following theorem to prove several results about $b_0(n)$ and $b_1(n)$.

Theorem 4.1.2. *If $(x_n) \subseteq \mathbb{Z}$ is a sequence of integers, and for all $n \geq 0$*

$$x_{n+3} + x_{n+1} + x_n \equiv 0 \pmod 2,$$

then for all n, $x_{n+7} \equiv x_n \pmod 2$.

Proof. We have $x_{n+3} \equiv x_n + x_{n+1} \pmod 2$, so

$$\begin{aligned}&x_3 \equiv x_0 + x_1 \pmod 2,\\ &x_4 \equiv x_1 + x_2 \pmod 2,\\ &x_5 \equiv x_2 + x_3 \equiv x_0 + x_1 + x_2 \pmod 2,\\ &x_6 \equiv x_3 + x_4 \equiv x_0 + x_2 \pmod 2,\\ &x_7 \equiv x_4 + x_5 \equiv x_0 + 2x_1 + 2x_2 \equiv x_0 \pmod 2,\\ &x_8 \equiv x_5 + x_6 \equiv 2x_0 + x_1 + 2x_2 \equiv x_1 \pmod 2,\\ &x_9 \equiv x_6 + x_7 \equiv 2x_0 + x_2 \equiv x_2 \pmod 2.\end{aligned}$$

So we can see that the lemma holds for small n. Suppose the lemma holds for $n < k$. Similarly, by the induction hypothesis we have

$$\begin{aligned} x_k &\equiv x_{k-3} + x_{k-2} \pmod 2 \\ &\equiv x_{k-10} + x_{k-9} \pmod 2 \\ &\equiv x_{k-7} \pmod 2. \end{aligned}$$

Thus for all n, $x_{n+7} \equiv x_n \pmod 2$. ■

4.2 The Behavior of $b_1(n)$ and $b_0(n)$ Modulo 2

First, define $v_{\alpha,\beta}(n)$ as follows:

$$v_{\alpha,\beta}(n) := (b_{\alpha,\beta}(7n), b_{\alpha,\beta}(7n+1), b_{\alpha,\beta}(7n+2), ..., b_{\alpha,\beta}(7n+6)) \pmod 2.$$

There is only one possibility for $v_{0,1}(n)$. We have the following theorem.

Theorem 4.2.1. *For $b_0(n)$ and all $n \geq 0$,*

$$v_{0,1}(n) \equiv v^{(0)}(n) := (0, 0, 1, 0, 1, 1, 1) \pmod 2. \tag{4.3}$$

Proof. First, we can see from Table 2.1 that the theorem holds for small n. Then, by Theorem 4.1.1, for $n \geq 1$,

$$b_0(n-1) + b_0(n) + b_0(n+2) \equiv 0 \pmod 2.$$

So by Theorem 4.1.2, for all n,

$$v_{0,1}(n) \equiv v^{(0)}(n) := (0, 0, 1, 0, 1, 1, 1) \pmod 2.$$

■

Remark. This theorem also follows immediately from Corollary 3.3.6.

Corollary 4.2.2. *For $n \geq 0$,*

$$2 | \gcd(b_0(n), b_0(n+1)) \iff 7|n. \tag{4.4}$$

Proof. The statement is an immediate consequence of Theorem 4.2.1. ■

Next, consider $b_1(n)$. By Theorem 4.1.1 we know that unless $m = 3 \cdot 2^k$,

$$b_1(m+2) \equiv b_1(m-1) + b_1(m) \pmod{2},$$

and for $m = 3 \cdot 2^k$ we have

$$b_1(m+2) \equiv b_1(m-1) + b_1(m) + 1 \pmod{2}.$$

Since $3 \cdot 2^k \equiv 3, 5, 6 \pmod{7}$, by Theorem 4.1.2 we should have three main cases for $v_{1,0}(n)$, and possibly three transition cases.

Consider the first 28 cases for $v_{1,0}(n)$ given below:

$$\begin{aligned}
v_{1,0}(0) &\equiv (0,1,0,1,1,0,0) \pmod{2},\\
v_{1,0}(1) &\equiv (1,1,1,0,0,1,0) \pmod{2},\\
v_{1,0}(2) &\equiv (0,1,0,1,1,1,0) \pmod{2},\\
v_{1,0}(3) &\equiv (0,1,0,1,1,0,0) \pmod{2},\\
v_{1,0}(4) &\equiv \cdots \equiv v_{1,0}(6) \equiv (1,0,1,1,1,0,0) \pmod{2},\\
v_{1,0}(7) &\equiv \cdots \equiv v_{1,0}(13) \equiv (1,1,1,0,0,1,0) \pmod{2},\\
v_{1,0}(14) &\equiv \cdots \equiv v_{1,0}(26) \equiv (0,1,0,1,1,1,0) \pmod{2},\\
v_{1,0}(27) &\equiv (0,1,0,1,1,0,0) \pmod{2}.
\end{aligned}$$

It appears that $v_{1,0}(n)$ cycles through 4 cases. These cases are listed below:

$$v_{1,0}(n) \equiv \begin{cases} v^{(1)}(n) := (0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 0) \\ v^{(1*)}(n) := (0,\ 1,\ 0,\ 1,\ 1,\ 0,\ 0) \\ v^{(2)}(n) := (1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0) \\ v^{(3)}(n) := (1,\ 1,\ 1,\ 0,\ 0,\ 1,\ 0). \end{cases}$$

Remark. This would imply that we have the following 4 cases for $v_{1,1}(n)$:

$$v_{1,1}(n) \equiv \begin{cases} v^{(4)}(n) := (0,\ 1,\ 1,\ 1,\ 0,\ 0,\ 1) \\ v^{(4*)}(n) := (0,\ 1,\ 1,\ 1,\ 0,\ 1,\ 1) \\ v^{(5)}(n) := (1,\ 0,\ 0,\ 1,\ 0,\ 1,\ 1) \\ v^{(6)}(n) := (1,\ 1,\ 0,\ 0,\ 1,\ 0,\ 1). \end{cases}$$

By Theorems 4.1.1 and 4.1.2, we know that $v_{1,0}(n) = v_{1,0}(n-1)$ unless $3 \cdot 2^k \in \{7n-2, 7n-1, 7n, \cdots, 7n+3\}$. We consider what happens at each of these transitions.

First, since $3 \cdot 2^k \equiv 3 \pmod 7$ precisely when $k \equiv 0 \pmod 3$, we expect a transition at $v_{1,0}(\frac{3}{7}(2^{3k}-1))$. For small k, we can see that this is exactly when case (1) transitions to case (1*), with case (1*) occurring at $v_{1,0}(\frac{3}{7}(2^{3k}-1))$. In general, suppose $v_{1,0}(n-1) = v^{(1)} = (0,1,0,1,1,1,0)$, and $3 \cdot 2^k = 7n+3$ for some k. Then, by Theorem 4.1.1 we know that

$$b_1(7n+2) + b_1(7n+3) + 1 \equiv b_1(7n+5) \pmod 2.$$

Otherwise $b_1(n-3) + b_1(n-2) \equiv b_1(n) \pmod 2$. Thus

$$\begin{aligned} v_{1,0}(n) &= (0,1,0,1,1,0,0) = v^{(1^*)} \text{ and} \\ v_{1,0}(n+1) &= (1,0,1,1,1,0,0) = v^{(2)}. \end{aligned}$$

Then, by Theorems 4.1.1 and 4.1.2, $v_{1,0}(m) = v_{1,0}(m-1) = v^{(2)}$ until

$$3 \cdot 2^k \in \{7m-2, 7m-1, 7m, \cdots, 7m+3\}.$$

Similarly, since $3 \cdot 2^k \equiv 6 \pmod 7$ precisely when $k \equiv 1 \pmod 3$, we expect a transition at $v_{1,0}(n)$, where $n = \frac{1}{7}(3 \cdot 2^{3k+1} - 6) + 1 = \frac{3}{7}(2^{3k+1} + \frac{1}{3})$. For small k, we can see that this is exactly when case (2) transitions to case (3), with case (3) occurring at $v_{1,0}(\frac{3}{7}(2^{3k+1} + \frac{1}{3}))$. In general, suppose $v_{1,0}(n) = v^{(2)} = (1,0,1,1,1,0,0)$, and $3 \cdot 2^k = 7n+6$ for some k. Then, by Theorem 4.1.1 we know that

$$b_1(7n+5) + b_1(7n+6) + 1 \equiv b_1(7n+8) \pmod 2.$$

Otherwise $b_1(n-3) + b_1(n-2) \equiv b_1(n) \pmod 2$. Thus

$$v_{1,0}(n+1) = (1,1,1,0,0,1,0) = v^{(3)}.$$

Then, by Theorems 4.1.1 and 4.1.2, $v_{1,0}(m) = v_{1,0}(m-1) = v^{(3)}$ until

$$3 \cdot 2^k \in \{7m-2, 7m-1, 7m, \cdots, 7m+3\}.$$

Thirdly, since $3 \cdot 2^k \equiv 5 \pmod 7$ precisely when $k \equiv 2 \pmod 3$, we expect a transition at $v_{1,0}(n)$, where $n = \frac{1}{7}(3 \cdot 2^{3k+2} - 5) + 1 = \frac{3}{7}(2^{3k+2} + \frac{2}{3})$. For small k, we can see that this is exactly when case (3) transitions to case (1), with case (1) occurring at $v_{1,0}(\frac{3}{7}(2^{3k+2} + \frac{2}{3}))$. In general, suppose $v_{1,0}(n) = v^{(3)} = (1,1,1,0,0,1,0)$, and $3 \cdot 2^k = 7n + 5$ for some k. Then, by Theorem 4.1.1 we know that

$$b_1(7n+4) + b_1(7n+5) + 1 \equiv b_1(7n+7) \pmod 2.$$

Otherwise $b_1(n-3) + b_1(n-2) \equiv b_1(n) \pmod 2$. Thus

$$v_{1,0}(n+1) = (0,1,0,1,1,1,0) = v^{(1)}.$$

Then, by Theorems 4.1.1 and 4.1.2, $v_{1,0}(m) = v_{1,0}(m-1) = v^{(1)}$ until

$$3 \cdot 2^k \in \{7m-2, 7m-1, 7m, \cdots, 7m+3\}.$$

We use the following definition to describe the intervals on which cases (1), (2), and (3) occur more precisely.

Define J_m as follows:

$$J_m := \left(\frac{3}{7}(2^m - 1), \frac{3}{7}(2^{m+1} - 1)\right] \cap \mathbb{Z}. \tag{4.5}$$

Then cases (1), (2), and (3) occur on the following intervals:

Theorem 4.2.3. *For $b_1(n)$ the following are true:*

(1). Case (1) occurs on $J_{3r+2} \setminus \left\{\frac{3}{7}(2^{3r+3} - 1)\right\}$, for $r \geq 0$.

(2). Case (2) occurs on J_{3r}, for $r \geq 1$.

(3). Case (3) occurs on J_{3r+1}, for $r \geq 0$.

We have also shown that case (1^*) occurs as given below:

Theorem 4.2.4. *For $b_1(n)$ and $r \geq 0$, case (1*) occurs exactly when*

$$n = \frac{3}{7}\left(2^{3r} - 1\right). \tag{4.6}$$

Remark. Thus, the case (1*) occurs for the 7-tuple

$$\left(b_1(3(2^{3r}-1)), b_1(3(2^{3r}-1)+1), \cdots, b_1(3(2^{3r}-1)+6)\right).$$

Remark. Theorems 4.2.1 and 4.2.3 imply that the following are true for all n:

(1). $b_{\alpha,\beta}(7n+6) \equiv \beta \pmod 2$.

(2). $b_{\alpha,\beta}(7n+3) + b_{\alpha,\beta}(7n+4) \equiv \beta \pmod 2$.

(3). $b_{\alpha,\beta}(7n) + b_{\alpha,\beta}(7n+2) \equiv \beta \pmod 2$.

(4). $b_{\alpha,\beta}(7n+1) + b_{\alpha,\beta}(7n+2) + b_{\alpha,\beta}(7n+3) \equiv \beta \pmod 2$.

Note that exactly three of the seven terms are even for $v^{(k)}(n)$, with $k = 0, 1, \cdots, 6$.

Theorems 4.2.1 and 4.2.3 imply the following theorem.

Theorem 4.2.5. *If at least one of $\{\alpha, \beta\}$ is odd, then the set of n for which $b_{\alpha,\beta}(n)$ is even has density $\frac{3}{7}$, and the set for which $b_{\alpha,\beta}(n)$ is odd has density $\frac{4}{7}$.*

Definition. Define m to be *purely even* if $b_0(m)$ and $b_1(m)$ are both even.

We can show that $\frac{1}{7}$ of the terms are purely even. Moreover, we have the following theorem.

Theorem 4.2.6. *Each 7-tuple $(7n,\ 7n+1,\ ...,\ 7n+6)$ contains exactly one purely even term.*

Proof. We know by Theorem 4.2.1 that for all n,

$$v_{0,1}(n) \equiv (0,\ 0,\ 1,\ 0,\ 1,\ 1,\ 1) \pmod 2.$$

This means that any purely even terms must occur at either $7n,\ 7n+1$ or $7n+3$. We also know that for $b_1(n)$ we have exactly four cases for $v_{1,0}(n)$:

$$v_{1,0}(n) \equiv \begin{cases} v^{(1)}(n) = (0,\ 1,\ 0,\ 1,\ 1,\ 1,\ 0) \\ v^{(1*)}(n) = (0,\ 1,\ 0,\ 1,\ 1,\ 0,\ 0) \\ v^{(2)}(n) = (1,\ 0,\ 1,\ 1,\ 1,\ 0,\ 0) \\ v^{(3)}(n) = (1,\ 1,\ 1,\ 0,\ 0,\ 1,\ 0). \end{cases}$$

In each of these four cases exactly one of $7n,\ 7n+1$ and $7n+3$ is even. Thus each 7-tuple $(7n,\ 7n+1,\ ...,\ 7n+6)$ contains exactly one purely even term. ■

Combining Theorem 4.2.4, Theorem 4.2.3, and Theorem 4.2.6 we state the result more explicitly in the following remark.

Remark. For $n \geq 0$, $7n + \epsilon_n$ is purely even, where

$$\epsilon_n = \begin{cases} 0, \text{ if } n \in J_{3r+2} \text{ for } r \geq 0 \\ 1, \text{ if } n \in J_{3r} \text{ for } r \geq 1 \\ 3, \text{ if } n \in J_{3r+1} \text{ for } r \geq 0. \end{cases}$$

Chapter 5

Higher Moduli

In Chapter 4 we discussed properties of the general bow sequence modulo 2. In this chapter we will use graph theory to prove several properties of the general bow sequence modulo 3, and a conjecture will be given for higher moduli.

5.1 Background

We need some definitions before we can begin. Let $G = G(V, E)$ be a directed graph with vertices $V = \{v_1, v_2, \cdots, v_m\}$ and edges $E = \{e_1, e_2, \cdots, e_t\}$, where $e_j = (x_i, y_i)$ with $x_i, y_i \in V$. Loops are allowed, but there are no repeated edges. The following definitions are taken from [18].

Definition. The *out-degree* of $v \in V$ is the number of edges e_j so that $x_j = v$. The *in-degree* of $v \in V$ is the number of edges e_j so that $y_j = v$.

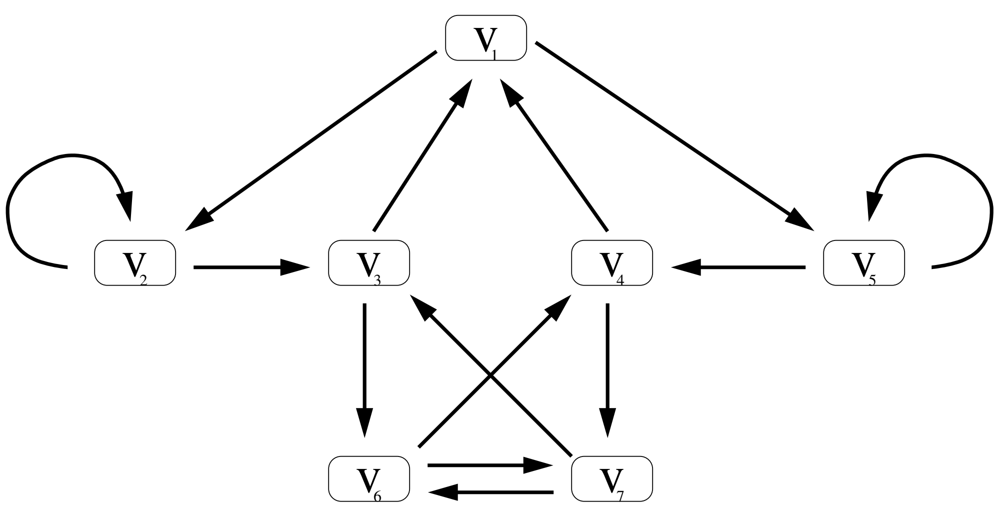

Figure 5.1: An example of a directed graph, G_0

Definition. We say that G is *k-diregular* if each vertex has in-degree and out-degree equal to k.

For example, in Figure 5.1, G_0 is a 2-diregular directed graph with vertices $V = \{v_1, v_2, \cdots, v_7\}$, and 14 edges $E = \{(v_1, v_2), (v_1, v_5), (v_2, v_3), \cdots, (v_7, v_6)\}$.

Definition. A *walk W in G of length n from v to v'* is a sequence of n edges $e_j = (x_j, y_j)$ so that $x_1 = v$, $y_j = x_{j+1}$ for $1 \leq j \leq n-1$ and $y_n = v'$.

Vertices and edges may be repeated in a walk.

Definition. The *adjacency matrix* of G is defined to be the $m \times m$ matrix $A_G = [a_{ij}]$ in which $a_{ij} = 1$ if $(v_i, v_j) \in E$ and $a_{ij} = 0$ otherwise.

Lemma 5.1.1. [14], [18] *Let G be a directed graph with adjacency matrix A_G, and write $A_G^n = \left[a_{ij}^{(n)}\right]$. Then $\left[a_{ij}^{(n)}\right]$ is the number of walks of length n in G from v_i to v_j.*

Consider the directed graph, G_0, with vertices v_k, $1 \leq k \leq 7$, in Figure 5.1. The adjacency matrix is as follows:

$$A_{G_0} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Since $(A_{G_0})^3$ is as given below, we know that there is exactly one walk of length 3 from v_1 to v_2. We can also see that since all entries are positive, there is at least one walk of length three between each pair of vertices in G_0.

$$(A_{G_0})^3 = \begin{pmatrix} 2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Definition. For a fixed d, let G^d be the directed graph whose vertices consist of triples in $\mathbb{Z}/d\mathbb{Z}$:

$$V^d := \{\{i \pmod d,\ j \pmod d,\ k \pmod d\} : gcd(i,j,k,d) = 1\},$$

and whose edges consist of

$$E^d := \{((i,j,k),(i+j,i,j+k)),((i,j,k),(i,j+k,j))\}.$$

That is, the edges are defined by a triple and its images by applying the recursion of the general bow sequence. In particular, if we fix α, β, and d and define

$$w(n) = (b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) \pmod d, \tag{5.1}$$

and $w(n)$ is a vertex in V^d, then we have edges

$$(w(n), w(2n)) \text{ and } (w(n), w(2n+1)).$$

Thus, if U_k is the unit vector $(0,\cdots,1,0,\cdots,0)^T$, where the 1 is in the place corresponding to $w(k) \in V^d$, then the entries of

$$\left(G^d\right)^r U_k$$

count the number of times each $v \in V^d$ appears in the set

$$\{w(2^r k), w(2^r k+1), \cdots, w(2^r k + 2^r - 1)\}.$$

Theorem 5.1.2. *G^d is 2-diregular for $d \geq 2$.*

Proof. Clearly, all vertices v have out-degree 2 by definition. Consider a vertex $\{i,j,k\}$. Then the preimages are $\{j, i-j, k-i+j\}$, and $\{i,k,j-k\}$. If $\{j,i-j,k-i+j\} \equiv \{i,k,j-k\} \pmod d$, then $i=j=k=0$. Then, since $\gcd(i,j,k,d) = 1$, we know that $\gcd(j,i-j,k-i+j,d) = 1$, and $\gcd(i,k,j-k,d) = 1$. Thus $\{i,j,k\}$ has in-degree 2 and G^d is 2-diregular. ∎

The directed graph G^2 is given in Figure 5.2. Since G^2 is a relabeling of the vertices of G_0, we can clearly see that G^2 is 2-diregular, and has adjacency

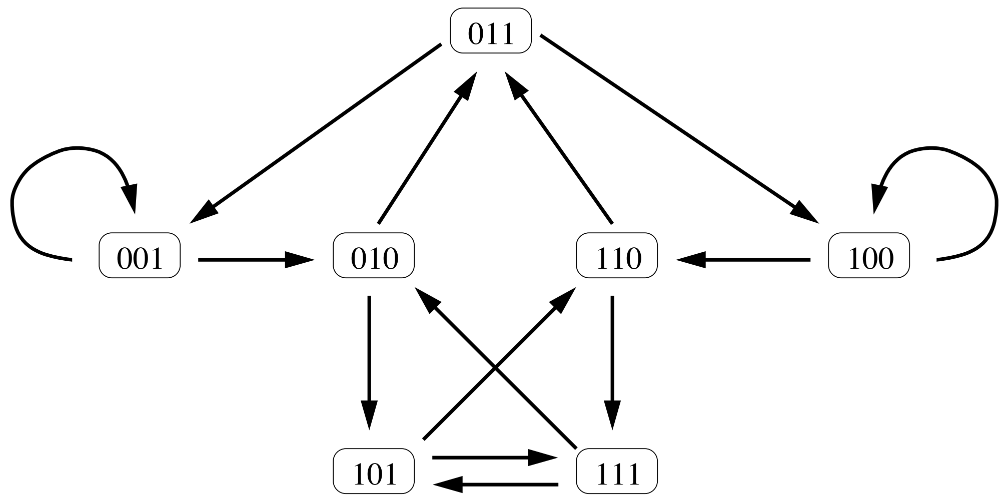

Figure 5.2: The directed graph G^2

matrix $A_{G^2} = A_{G_0}$. Since $(A_{G_0})^3$ has all positive entries, we can see that for every $v_i, v_j \in V$, there is a walk from v_i to v_j, and a walk from v_j to v_i.

Definition. [15] We say that G is *pedestrian-friendly* if there exists $r > 0$ so that for every $v,\ v' \in V$, there is a walk of length r from v to v'.

Remark. Equivalently, G is pedestrian-friendly if $a_{ij}^{(r)} > 0$ for $1 \leq i,\ j \leq m$.

Thus we can see that G^2 is pedestrian-friendly, with $r = 3$.

Theorem 5.1.3. [14], [15] *If G is a pedestrian-friendly k-diregular directed graph with m vertices, then the number of walks of length n from v_i to v_j satisfies*

$$a_{ij}^{(n)} = \frac{1}{m} \cdot k^n + \mathcal{O}\left(c^n\right) \tag{5.2}$$

for some $c < k$.

If we let $B = \frac{1}{k} A_G$, then we know that the entries of B are either 0 or $\frac{1}{k}$. Thus B is a doubly stochastic matrix: a non-negative matrix with row sums and column sums equal to 1. Since B is also pedestrian-friendly, there is some $r > 0$ such that $a_{ij}^{(r)} > 0$, and the entries of B^r are all positive. It follows from standard results in matrix theory (see [14], [15] pages 9-11, for example) that $b_{ij}^{(n)} = \frac{1}{m} + \mathcal{O}(\rho^n)$, for some $\rho < 1$.

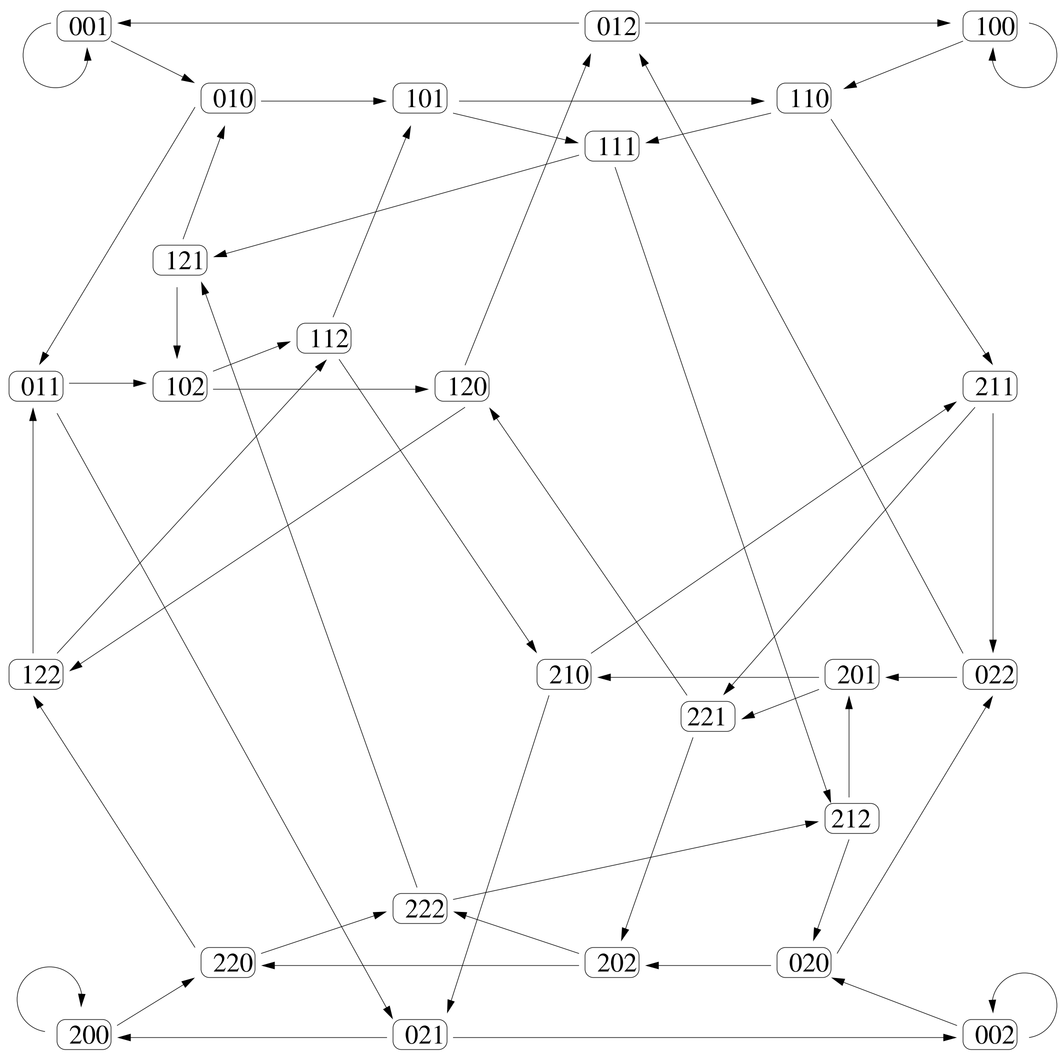

Figure 5.3: The directed graph G^3

By (5.2) and an identical argument to that in [15], it follows that

$$\lim_{N\to\infty} \frac{|\{k \leq N : w(k) = v\}|}{N} = \frac{1}{|V^d|}$$

for each $v \in V^d$. That is, each of the triples of allowed congruence classes is, asymptotically, equally likely.

We say that G is *ergodic* if G is a pedestrian-friendly k-diregular directed graph. We have the following theorem.

Theorem 5.1.4. *G^d is ergodic for $d = 2, 3$.*

Proof. We have shown that G^2 is a pedestrian-friendly 2-diregular directed graph; thus G^2 is ergodic.

The graph of G^3 is given in Figure 5.3. We know by Theorem 5.1.2 that G^3 is a 2-diregular directed graph. Also, A_{G^3} is a 26×26 square matrix. By noting that all entries in $(A_{G^3})^8$, given in Appendix A, are positive, we find that G^3 is pedestrian-friendly, and hence ergodic. ■

Conjecture 1. G^d is ergodic for $d \geq 4$.

5.2 Distributions Modulo d

We build towards a general conjecture on the density of the sets of terms equivalent to c modulo d, when $\gcd(\alpha, \beta) = 1$.

Lemma 5.2.1. *For $d \geq 2$, the number of triples $(i, j, k) \pmod d$ such that $\gcd(i, j, k, d) = 1$ is*

$$d^3 \prod_{p|d} \left(1 - \frac{1}{p^3}\right). \tag{5.3}$$

Proof. Write d in the usual prime factorization, $p_1 < p_2 < \cdots < p_n$, and $e_i \geq 1$,

$$d = {p_1}^{e_1} {p_2}^{e_2} \cdots {p_n}^{e_n}. \tag{5.4}$$

Let $A_r = \{(i, j, k) \pmod d : i \equiv j \equiv k \equiv 0 \pmod {p_r}\}$. Then we want to find the size of the set

$$S = \bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_n.$$

By the inclusion-exclusion principle, the size of S is

$$\begin{aligned}
|S| &= |\mathcal{U}| - \sum_{s=1}^{n} |A_s| + \sum_{1\le s<t\le n} |A_s \cap A_t| - \cdots + (-1)^n |A_1 \cap A_2 \cap \cdots \cap A_n| \\
&= d^3 - \sum_{s=1}^{n} \frac{d^3}{{p_s}^3} + \sum_{1\le s<t\le n} \frac{d^3}{{p_s}^3 {p_t}^3} - \cdots + (-1)^n \frac{d^3}{{p_1}^3 {p_2}^3 \cdots {p_n}^3} \\
&= d^3 \left(1 - \sum_{s=1}^{n} \frac{1}{{p_s}^3} + \sum_{1\le s<t\le n} \frac{1}{{p_s}^3 {p_t}^3} - \cdots + (-1)^n \frac{1}{{p_1}^3 {p_2}^3 \cdots {p_n}^3}\right) \\
&= d^3 \prod_{p|d} \left(1 - \frac{1}{p^3}\right).
\end{aligned}$$

■

We have the following conjectures.

Conjecture 2. For all triples $(i, j, k) \pmod d$ such that $\gcd(i, j, k) = \gcd(\alpha, \beta)$, there exists $n \ge 0$ such that

$$\{b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)\} \equiv (i, j, k) \pmod d. \tag{5.5}$$

We have numerical evidence that this conjecture holds true for at least $d \le 10$ for $b_0(n)$ and $b_1(n)$. In Appendix B, we have calculated the number of different congruence classes for triples $(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) \pmod d$ with $(a, b) = (0, 1)$ and $(1, 0)$, and their relative frequencies for $d \le 10$.

By Lemma 5.2.1 we get the following conjectured corollaries.

Conjectured Corollary. *For $gcd(\alpha, \beta) = 1$, each of the*

$$d^3 \prod_{p|d} \left(1 - \frac{1}{p^3}\right)$$

cases occur.

Conjectured Corollary. *For* $\gcd(\alpha, \beta) = 1$, *the following would be true:*

(1). The set of n for which $b_{\alpha,\beta}(n) \equiv 0 \pmod d$ has density

$$\frac{\prod_{p|d}\left(1 - \frac{1}{p^2}\right)}{d\prod_{p|d}\left(1 - \frac{1}{p^3}\right)} = \frac{1}{d} \prod_{p|d} \frac{p^2 + p}{p^2 + p + 1}.$$

(2). The set of n for which $b_{\alpha,\beta}(n) \equiv c \pmod{d}$, with $\gcd(c,d) \neq 1$ has density

$$\frac{\prod_{p|\gcd(c,d)}\left(1-\frac{1}{p^2}\right)}{d\prod_{p|d}\left(1-\frac{1}{p^3}\right)}.$$

(3). The set of n for which $b_{\alpha,\beta}(n) \equiv a \pmod{d}$, with $a \neq 1$ and $\gcd(a,d) = 1$, has density

$$\frac{1}{d\prod_{p|d}\left(1-\frac{1}{p^3}\right)}.$$

All of the results are given as a fraction of the total number of combinations.

For the first conjectured result, consider the number of combinations $(0, j, k)$ such that $\gcd(d, j, k) = 1$. Similarly to Lemma 5.2.1, there would be

$$d^2\prod_{p|d}\left(1-\frac{1}{p^2}\right)$$

combinations of this type. Thus the density of terms equivalent to 0 modulo d would be

$$\frac{d^2\prod_{p|d}\left(1-\frac{1}{p^2}\right)}{d^3\prod_{p|d}\left(1-\frac{1}{p^3}\right)}.$$

For the second conjectured result, consider the number of combinations (c, j, k) such that $\gcd(c, j, k) = 1$. Similarly to the argument above, we find that the number of combinations would be

$$d^2\prod_{p|\gcd(c,d)}\left(1-\frac{1}{p^2}\right).$$

For the third conjectured result, since all combinations (a, j, k) would be accounted for, we find that there would be d^2 such combinations, and thus the fraction of terms congruent to a would be

$$\frac{d^2}{d^3 \prod_{p|d}\left(1 - \frac{1}{p^3}\right)}.$$

Remark. We would have the following results when $\gcd(\alpha, \beta) = 1$, for example:

1. The set of n for which $b_{\alpha,\beta}(n) \equiv 0 \pmod 2$ has density $\frac{3}{7}$, and the set of n for which $b_{\alpha,\beta}(n) \equiv 1 \pmod 2$ has density $\frac{4}{7}$, which is exactly what we have shown in Theorem 4.2.5.

2. The set of n for which $b_{\alpha,\beta}(n) \equiv 0 \pmod 3$ has density $\frac{8}{26}$ and the sets of n for which $b_{\alpha,\beta}(n) \equiv 1, 2 \pmod 3$ have density $\frac{9}{26}$ in each case. We proved that this was true in Theorem 5.1.4.

3. The sets of n for which $b_{\alpha,\beta}(n) \equiv 0, 2 \pmod 4$ have density $\frac{12}{56}$ in each case, and the sets of n for which $b_{\alpha,\beta}(n) \equiv 1, 3 \pmod 4$ has density $\frac{16}{56}$.

4. The set of n for which $b_{\alpha,\beta}(n) \equiv 0 \pmod 5$ has density $\frac{24}{124}$, and the set of n for which $b_{\alpha,\beta}(n) \equiv 1, 2, 3, 4 \pmod 5$ have density $\frac{25}{124}$ in each case.

5. The set of n for which $b_{\alpha,\beta}(n) \equiv 0 \pmod 6$ has density $\frac{24}{182}$, the sets of n for which $b_{\alpha,\beta}(n) \equiv 2, 4 \pmod 6$ have density $\frac{27}{182}$, the set of n for which $b_{\alpha,\beta}(n) \equiv 3 \pmod 6$ has density $\frac{32}{182}$, and the set of n for which $b_{\alpha,\beta}(n) \equiv 1, 5 \pmod 6$ has density $\frac{36}{182}$.

These assertions are all true for $d = 2$ by the results of Chapter 4. Not only do the sets have the indicated densities, but the difference in cardinality up to N and the conjectured value is $\mathcal{O}(1)$.

Numerical evidence suggests that the remaining statements are true. In Appendix C, we have calculated the number of terms congruent to a modulo d for $d \leq 10$ and $(\alpha, \beta) = (0, 1), (1, 0)$, and $(1, 1)$. Columns two through four correspond to the size of the set

$$\{n : b_{\alpha,\beta}(n) \equiv a \pmod d, 1 \leq n \leq 10^6\},$$

while the last column gives the conjectured density in decimal form.

Chapter 6

Another Representation of $b_0(n)$

In this chapter, we discuss another representation of $b_0(n)$, found independently in unpublished work by Paul Barry [3], and referenced in Sloane's On-Line Encyclopedia of Integer Sequences [16]. The sequence was defined by Barry, but was not shown to satisfy the bow recurrence. We show that $b_0(n)$ can be determined by computing this finite sum, which is independent of the recurrence.

6.1 Preliminaries

Before we give this new equation we must first state a few formulas which will be necessary in the proof of our theorem. The first is dePolignac's formula.

DePolignac's Formula. [8], [12] *For $n \geq 1$, the exponent of the highest power of a prime p dividing $n!$ is*

$$s_p(n) := \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor . \tag{6.1}$$

We will need the following corollary for the proof of the next theorem.

Corollary 6.1.1. *For $m \geq 1$,*

$$s_2(2m+1) = s_2(2m) = s_2(m) + m. \tag{6.2}$$

Proof. We write $(2m+1)! = (2m+1)(2m)!$, thus $s_2(2m+1) = s_2(2m)$.

Then, pulling the first term out of the sum, we find

$$\begin{aligned} s_2(2m) &= \left\lfloor \frac{2m}{2} \right\rfloor + \sum_{k=2}^{\infty} \left\lfloor \frac{2m}{2^k} \right\rfloor \\ &= m + s_2(m). \end{aligned}$$

Thus $s_2(2m+1) = s_2(2m) = s_2(m) + m$. ∎

Definition. Define $\epsilon(m)$ to be the smallest non-negative residue of m modulo 2, thus $m \in \{0, 1\}$.

Let

$$c(n) := \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \epsilon\left(\binom{k}{n-2k}\right). \tag{6.3}$$

The terms of this sequence are 1, 0, 1, 1, 1, 0, 2, 1, 2, $\cdots$ We can see that this looks the same as $b_0(n+2)$. In fact, we shall prove that $c(n-2) = b_0(n)$.

Let $c(n-2) = a(n)$. We have the following lemma.

Lemma 6.1.2.

$$\binom{2a+1}{2b+1} \equiv \binom{2a+1}{2b} \equiv \binom{2a}{2b} \equiv \binom{a}{b} \pmod 2.$$

Proof. To show that the first three have the same parity, we note that

$$\frac{(2a+1)!}{(2b+1)!(2a-2b)!}, \frac{(2a+1)!}{(2b)!(2a+1-2b)!}, \text{ and } \frac{(2a)!}{(2b)!(2a-2b)!}$$

share all the same even factors in both their numerator and denominator. Thus the highest power of 2 dividing $\binom{2a+1}{2b+1}$, $\binom{2a+1}{2b}$, and $\binom{2a}{2b}$ is the same, and they are all congruent modulo 2.

Then, to show that $\binom{a}{b} \equiv \binom{2a}{2b} \pmod 2$, we notice that if $s_2(a) = r$, then by Corollary 6.1.1 we know that $s_2(2a) = r + a$. Similarly, if $s_2(b) = t$, then $s_2(2b) = t + b$. Lastly, if $s_2(a-b) = s$, then $s_2(2a-2b) = s + (a-b)$.

The highest power of 2 dividing $\binom{a}{b}$ is $r - s - t$, and the highest power of 2 dividing $\binom{2a}{2b}$ is $r + a - (t+b) - s - (a-b) = r - s - t$. Thus the highest power of 2 dividing $\binom{a}{b}$ and $\binom{2a}{2b}$ is the same, and they are congruent modulo 2. ∎

6.2 A New Formula for $b_0(n)$

We use dePolignac's Formula, Corollary 6.1.1, and Lemma 6.1.2 to prove that $b_0(n)$ satisfies the following formula.

Theorem 6.2.1. *For $n \geq 2$, $b_0(n)$ can be described by the following formula*

$$b_0(n) = a(n) := \sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{k}{n-2-2k}\right). \tag{6.4}$$

Proof. Clearly $b_0(2) = 1 = a(2)$. Defining $a(0) = a(1) = 0$, we shall show that the bow recursion holds for $a(n)$.

First we shall show that $a(2n+1) = a(n)$, or

$$\sum_{k=0}^{\lfloor (2n+1)/2-1 \rfloor} \epsilon\left(\binom{k}{2n-1-2k}\right) = \sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{k}{n-2-2k}\right). \tag{6.5}$$

Consider the terms in the left hand side of (6.5)

$$\sum_{k=0}^{\lfloor (2n+1)/2-1 \rfloor} \epsilon\left(\binom{k}{2n-1-2k}\right) = \sum_{k=0}^{n-1} \epsilon\left(\binom{k}{2n-1-2k}\right).$$

In general, when k is even, the term $\binom{k}{2a+1} \equiv 0 \mod 2$. Thus we need only consider the terms where k is odd. So let $k = 2a+1$ and now consider the sum

$$\sum_{a=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{2a+1}{2n-1-2(2a+1)}\right).$$

We know that

$$\sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{2k+1}{2n-1-2(2k+1)}\right) = \sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{k}{n-2-2k}\right) \tag{6.6}$$

because the terms are equal, by Lemma 6.1.2 with $a = k$, and $b = n-2-2k$. Thus, the recurrence $a(2n+1) = a(n)$ holds.

Second, we must show that $a(2n) = a(n) + a(n+1)$ holds, or

$$\sum_{k=0}^{n-1} \epsilon\left(\binom{k}{2n-2-2k}\right) = \sum_{k=0}^{\lfloor n/2-1 \rfloor} \epsilon\left(\binom{k}{n-2-2k}\right) \tag{6.7}$$

$$+\sum_{k=0}^{\lfloor (n+1)/2-1\rfloor} \epsilon\left(\binom{k}{n-1-2k}\right).$$

We shall first consider the right hand side.

By applying Lemma 6.1.2 with $a=k$ and $b=n-2-2k$, we find

$$\sum_{k=0}^{\lfloor n/2-1\rfloor} \epsilon\left(\binom{k}{n-2-2k}\right) = \sum_{k=0}^{\lfloor n/2-1\rfloor} \epsilon\left(\binom{2k+1}{2n-4-4k}\right). \tag{6.8}$$

Similarly applying Lemma 6.1.2 with $a=k$ and $b=n-1-2k$, we obtain

$$\sum_{k=0}^{\lfloor (n+1)/2-1\rfloor} \epsilon\left(\binom{k}{n-1-2k}\right) = \sum_{k=0}^{\lfloor (n+1)/2-1\rfloor} \epsilon\left(\binom{2k}{2n-2-4k}\right). \tag{6.9}$$

Summing (6.8) and (6.9) together we observe that these are the odd and even terms of the sum

$$\sum_{k=0}^{n-1} \epsilon\left(\binom{k}{2n-2-2k}\right). \tag{6.10}$$

Therefore (6.7) holds and we have shown that $a(2n)=a(n)+a(n+1)$, and hence $b_0(n)=a(n)$. ■

Chapter 7

Directions for Future Research

The discussions so far have suggested several questions that we hope to study in the future. Specifically, we plan to analyze other non-standard binary representations and their associated generating functions, as well as continuing to study the properties of the bow sequences modulo d.

7.1 Non-Standard Binary Representations

We have seen that the bow sequences have interpretations in counting the numbers of ways integers can be written in non-standard binary representations, in which the digits are sets other than $\{0,1\}$. I would like to answer the following question. Given $A \subset \mathbb{Z}$, what can we say about the function which counts the number of ways you can write

$$n = \sum_{i=0}^{k} c_i 2^i,$$

for $c_i \in A$?

In Chapter 3 we showed that the subsets $A_0 = \{0,1,2\}$, $A_1 = \{0,2,3\}$, $A_2 = \{1,3,4\}$, $A_3 = \{0,1,3\}$, and $A_4 = \{1,2,4\}$ correspond to $s(n+1)$, $b_0(n+2)$, $b_1(n+3)$, $y(n+1)$, and $y_{1,0}(n+2)$, respectively. Additionally, the sets $A_d = \{0,1,\cdots,d-1\}$ have been studied by Euler ($d=2$) [6], and Reznick [13]. I hope to prove many more such relations by considering generating functions of the form:

$$\begin{aligned} G^{a_1,a_2,a_3,\ldots,a_m}(x) &:= x\prod_{j=0}^{\infty}(1 + x^{a_1\cdot 2^j} + x^{a_2\cdot 2^j} + \cdots + x^{a_m\cdot 2^j}) \\ &:= \sum_{n=1}^{\infty} c^{a_1,a_2,a_3,\ldots,a_m}(n)x^n. \end{aligned}$$

7.2 Bow Sequences Modulo d

First, we shall consider pairs modulo d.

Definition 1. For $d \geq 2$, define $\mathcal{A}^d_{\alpha,\beta}$ as follows:

$$\mathcal{A}^d_{\alpha,\beta} := \{n : d \mid b_{\alpha,\beta}(n),\ d \mid b_{\alpha,\beta}(n+1)\}.$$

We have the following theorem.

Theorem 7.2.1. *For $n \geq 2$, $n \in \mathcal{A}^d_{\alpha,\beta} \iff 2n \in \mathcal{A}^d_{\alpha,\beta}$.*

Proof. First, suppose $n \in \mathcal{A}^d_{\alpha,\beta}$. Then by definition we know that $d \mid b_{\alpha,\beta}(n)$ and $d \mid b_{\alpha,\beta}(n+1)$, thus $d \mid b_{\alpha,\beta}(2n)$ and $d \mid b_{\alpha,\beta}(2n+1)$ by the recursion. So, by definition $2n \in \mathcal{A}^d_{\alpha,\beta}$.

Suppose $2n \in \mathcal{A}^d_{\alpha,\beta}$. Then by definition we know that $d \mid b_{\alpha,\beta}(2n)$ and $d \mid b_{\alpha,\beta}(2n+1)$. By the recursion we find that $d \mid b_{\alpha,\beta}(2n+1) \iff d \mid b_{\alpha,\beta}(n)$. Since $b_{\alpha,\beta}(2n) = b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1)$ we find that $d \mid b_{\alpha,\beta}(n)$ as well. Hence $n \in \mathcal{A}^d_{\alpha,\beta}$. ■

Question. *When is $2n + 1 \in A^d_{\alpha,\beta}$?*

We have seen that pairs frequently share common factors, and we know by Theorem 2.3.2 that when the pairs $(b_0(n), b_0(n+1))$, and $(b_1(n), b_1(n+1))$ are taken together, the resulting quadruple does not have a mutual factor.

Consider the quadruple

$$J_d(n) := (b_0(n), b_0(n+1), b_1(n), b_1(n+1)) \pmod d. \tag{7.1}$$

We have the following conjecture.

Conjecture 3. For $d \geq 2$, every congruence class for $J_d(n)$ occurs, except those where all four terms share a common factor.

We have numerical evidence supporting this conjecture for $d \leq 17$ in Table 7.1. First, we note that the number of congruence classes modulo d with no mutual factor is

$$d^4 \prod_{p \mid d} \left(1 - \frac{1}{p^4}\right). \tag{7.2}$$

Let $A(d, N)$ be the number of congruence classes taken on by $J_d(n)$ for $1 \leq n \leq N$. The third column of the table below tells us the conjectured number of congruence classes, as found by applying equation (7.2).

(d, N)	$A(d, N)$	conjectured classes
$(2, 100)$	15	15
$(3, 400)$	80	80
$(4, 2000)$	240	240
$(5, 10^4)$	624	624
$(6, 2 \cdot 10^4)$	1200	1200
$(7, 3 \cdot 10^4)$	2400	2400
$(8, 10^5)$	3840	3840
$(9, 10^5)$	6480	6480
$(10, 10^6)$	9360	9360
$(11, 10^6)$	14640	14640
$(12, 10^6)$	19200	19200
$(13, 10^6)$	28560	28560
$(14, 10^6)$	36000	36000
$(15, 10^6)$	49920	49920
$(16, 2 \cdot 10^6)$	61440	61440
$(17, 2 \cdot 10^6)$	83520	83520

Table 7.1: Congruence classes of $J_d(n)$

Also, as noted in Chapter 5, we have made some progress on other conjectures about the properties of the bow sequences modulo d, and we hope to provide proofs of those conjectures. In the future, we want to look at sequences in which the recurrence is defined differently for the even and the odd terms, and we would like to consider the properties of these sequences modulo d.

7.3 More Formulas for $b_{\alpha,\beta}(2^r + k)$

We are still working to provide a formula for $b_{\alpha,\beta}(2^r + k)$, but we have an interesting formula for $b_0(2^r + k)$.

For the next theorem, we need two new sequences. Define $C(n)$ and $D(n)$ as follows:

$$
\begin{aligned}
C(0) = 2, \quad C(1) = 1, \quad C(2) = 2; \\
C(2n+1) = D(n) - C(n), \quad \text{for } n \geq 1; \\
C(2n) = C(2n+1) + C(2n+3), \quad \text{for } n \geq 2.
\end{aligned} \tag{7.3}
$$

$$
\begin{aligned}
D(0) = 3, \quad D(1) = 2, \quad D(2) = 3; \\
D(2n+1) = C(n), \quad \text{for } n \geq 1; \\
D(2n) = D(2n+1) + D(2n+3), \quad \text{for } n \geq 2.
\end{aligned} \tag{7.4}
$$

Theorem 7.3.1. *For $r \geq 4$ and $k < 2^r$, we can compute $b_0(2^r + k)$ as follows:*

$$
b_0(2^r + k) = F_{r-4}C(k) + F_{r-3}D(k) \tag{7.5}
$$

Proof. It can be quickly verified that the theorem holds for $r = 4$, $k \leq 15$, since (7.5) reduces to

$$
b_0(2^4 + k) = D(k).
$$

Assume (7.5) holds for $r < N$, $k < 2^r$. Then for $r = N$ and $k < 2^{N-1}$,

$$
\begin{aligned}
b_0(2^r + 2k) &= b_0(2^{r-1} + k) + b_0(2^{r-1} + k + 1) \\
&= F_{r-5}C(k) + F_{r-4}D(k) + F_{r-5}C(k+1) + F_{r-4}D(k+1) \\
&= F_{r-5}D(2k+1) + F_{r-4}(C(2k+1) + D(2k+1)) \\
&\quad + F_{r-5}D(2k+3) + F_{r-4}(C(2k+3) + D(2k+3)) \\
&= F_{r-5}D(2k) + F_{r-4}(C(2k) + D(2k)).
\end{aligned}
$$

Then by noting that $F_{r-5} = F_{r-3} - F_{r-4}$, we can reduce this to

$$
b_0(2^r + 2k) = F_{r-3}D(2k) + F_{r-4}C(2k).
$$

Similarly, for $r = N$ and $k < 2^{N-1} - 1$,

$$\begin{aligned} b_0(2^r + 2k + 1) &= b_0(2^{r-1} + k) \\ &= F_{r-5}C(k) + F_{r-4}D(k) \\ &= F_{r-5}D(2k+1) + F_{r-4}(C(2k+1) + D(2k+1)). \end{aligned}$$

Then by noting that $F_{r-5} = F_{r-3} - F_{r-4}$, we can reduce this to

$$b_0(2^r + 2k) = F_{r-3}D(2k+1) + F_{r-4}C(2k+1).$$

Thus for $r \geq 4$ and $k < 2^r$,

$$b_0(2^r + k) = F_{r-4}C(k) + F_{r-3}D(k).$$

■

We have been working on a similar but more complicated formula for $b_1(2^r + k)$ which also involves $C(n)$ and $D(n)$.

We can compute values of the Stern sequence by multiplying matrices. As in [14], let $w(n)$ be defined by

$$w(n) := \begin{pmatrix} s(n) \\ s(n+1) \end{pmatrix}. \tag{7.6}$$

Then for $n \geq 1$,

$$w(2n) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} w(n), \text{ and} \tag{7.7}$$

$$w(2n+1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} w(n). \tag{7.8}$$

Also,

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}, \text{ and}$$
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

So, for example, we can find $s(3 \cdot 2^5)$, $s(3 \cdot 2^5 + 1)$ as follows:

$$w(3 \cdot 2^5) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^5 w(3), \text{ so}$$
$$w(3 \cdot 2^5) = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 11 \end{pmatrix}.$$

We can also compute values of the bow sequence by multiplying matrices. Although this method does not provide us with a closed form, we are able to get a reduced form. Let $w_{\alpha,\beta}(n)$ be defined by

$$w_{\alpha,\beta}(n) := \begin{pmatrix} b_{\alpha,\beta}(n) \\ b_{\alpha,\beta}(n+1) \\ b_{\alpha,\beta}(n+2) \end{pmatrix}. \tag{7.9}$$

Then

$$w_{\alpha,\beta}(2n) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} w_{\alpha,\beta}(n) \text{ for } n \geq 2, \text{ and} \tag{7.10}$$
$$w_{\alpha,\beta}(2n+1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} w_{\alpha,\beta}(n) \text{ for } n \geq 1. \tag{7.11}$$

We have the following theorems.

Theorem 7.3.2. *For $r \geq 1$,*

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^r = \begin{pmatrix} F_{r+1} & F_r & 0 \\ F_r & F_{r-1} & 0 \\ F_{r+1}-1 & F_r & 1 \end{pmatrix}.$$

Proof. With a quick calculation, we find that

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Assume this holds true for $r < N$. Let $r = N$. Then

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^N = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{N-1} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

By the induction hypothesis,

$$\begin{aligned} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^N &= \begin{pmatrix} F_N & F_{N-1} & 0 \\ F_{N-1} & F_{N-2} & 0 \\ F_N - 1 & F_{N-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} F_{N+1} & F_N & 0 \\ F_N & F_{N-1} & 0 \\ F_{N+1}-1 & F_N & 1 \end{pmatrix}. \end{aligned}$$

■

Example 2. We can use this formula to compute terms of the bow sequence. For example, we can compute $b_{\alpha,\beta}(3 \cdot 2^5)$ by performing the matrix multiplication

$$\begin{pmatrix} b_{\alpha,\beta}(3 \cdot 2^5) \\ b_{\alpha,\beta}(3 \cdot 2^5 + 1) \\ b_{\alpha,\beta}(3 \cdot 2^5 + 2) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^5 \begin{pmatrix} b_{\alpha,\beta}(3) \\ b_{\alpha,\beta}(4) \\ b_{\alpha,\beta}(5) \end{pmatrix}.$$

By Theorem 7.3.2 we can simplify this to

$$\begin{pmatrix} b_{\alpha,\beta}(3 \cdot 2^5) \\ b_{\alpha,\beta}(3 \cdot 2^5 + 1) \\ b_{\alpha,\beta}(3 \cdot 2^5 + 2) \end{pmatrix} = \begin{pmatrix} 8 & 5 & 0 \\ 5 & 3 & 0 \\ 7 & 5 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \alpha + \beta \\ \beta \end{pmatrix} = \begin{pmatrix} 13\alpha + 5\beta \\ 8\alpha + 3\beta \\ 12\alpha + 6\beta \end{pmatrix}.$$

We verify that this is true by Theorem 2.4.1.

Similarly, we have the following theorem.

Theorem 7.3.3. *For $r \geq 1$,*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}^r = \begin{pmatrix} 1 & 0 & 0 \\ 0 & F_{r+1} & F_r \\ 0 & F_r & F_{r-1} \end{pmatrix}.$$

Proof. With a quick calculation, we find that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Assume this holds true for $r < N$. Let $r = N$. Then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}^N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}^{N-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

By the induction hypothesis,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}^N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & F_N & F_{N-1} \\ 0 & F_{N-1} & F_{N-2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & F_{N+1} & F_N \\ 0 & F_N & F_{N-1} \end{pmatrix}.$$

■

7.4 Primitive Values

Next we shall consider for specific values of k, which values of n will give $b_{\alpha,\beta}(n) = k$. Recall that Lemmas 2.4.3 and 2.4.4 tell us that for $r \geq 0$,

$$b_0(n) = 0 \iff n = 2^r - 1, \text{ and } b_1(n) = 0 \iff n = 0 \text{ or } n = 3 \cdot 2^r - 1.$$

We can see that this means $n = 2$ is the only even value for which $b_0(n) = 0$, and likewise $n = 0, 2$ are the only even values for which $b_1(n) = 0$.

Definition. Define $b_{\alpha,\beta}(n) = k$ to be a *primitive* occurrence of k if n is even.

By the recurrence, $b_{\alpha,\beta}(2j) = k \iff b_{\alpha,\beta}((2j+1)2^r - 1) = k$, thus all other occurrences are for odd n. Define $\mathcal{P}_{\alpha,\beta}(k)$ to be the set of primitive occurrences of k.

From our argument above, $\mathcal{P}_{0,1}(0) = \{2\}$, and $\mathcal{P}_{1,0}(0) = \{0, 2\}$. We determine primitive values by applying the recurrence and examining the terms. For example, we have the following theorem.

Theorem 7.4.1. $\mathcal{P}_{1,0}(1) = \{4, 8\}$.

Proof. First, recall that $b_1(2n) = b_1(n) + b_1(n+1)$. So either $b_1(n)$ or $b_1(n+1)$ must be zero.

Case 1. $b_1(n) = 0$. Then, by Lemma 2.4.4 we know that $n = 0$ or $n = 3 \cdot 2^r - 1$. Since $b_1(0)$ does not enter into the recursion, it must be the case that $n = 3 \cdot 2^r - 1$. By Theorem 2.4.1(5) we know that for $r \geq 0$,

$$b_1(3 \cdot 2^r) = F_{r+1} b_1(3) + F_r b_1(4) = F_{r+2}.$$

So $b_1(3 \cdot 2^r) = 1$ only when $r = 0$, which means $2 \in \mathcal{P}_{1,0}(1)$.

Case 2. $b_1(n+1) = 0$. Similarly, by Lemma 2.4.4, $n = 3 \cdot 2^r - 2$. By Theorem 2.4.1(3) we know that for $r \geq 1$,

$$b_1(3 \cdot 2^r - 2) = F_r b_1(3) + F_{r-1} b_1(4) + b_1(2) = F_{r+1}.$$

So $b_1(3 \cdot 2^r - 2) = 1$ only when $r = 1$, which means $4 \in \mathcal{P}_{1,0}(1)$. ■

In a similar way we can determine $\mathcal{P}_{\alpha,\beta}(k)$ for many integers $k > 0$. A few are listed below.

Theorem 7.4.2. *For $b_0(n)$ we have the following,*

(1). $\mathcal{P}_{0,1}(0) = \{0\}$,

(2). $\mathcal{P}_{0,1}(1) = \{2, 4, 6, 12\}$,

(3). $\mathcal{P}_{0,1}(2) = \{8, 10, 14, 22, 24, 28\}$,

(4). $\mathcal{P}_{0,1}(3) = \{16, 18, 20, 26, 30, 44, 46, 48, 54, 60\}$,

(5). $\mathcal{P}_{0,1}(4) = \{36, 38, 42, 50, 52, 56, 92, 94, 108\}$,

(6). $\mathcal{P}_{0,1}(5) = \{32, 34, 40, 58, 62, 76, 86, 88, 90, 96, 100, 102, 110, 118, 124, 188\}$,

(7). $\mathcal{P}_{0,1}(6) = \{70, 78, 84, 98, 106, 112, 120, 182, 190, 204, 220\}$,

(8). $\mathcal{P}_{0,1}(7) = \{68, 72, 74, 82, 104, 114, 116, 156, 172, 174, 180, 184, 186, 198, 214, 216, 222, 236, 380\}$,

(9). $\mathcal{P}_{0,1}(8) = \{64, 66, 80, 122, 126, 140, 150, 176, 178, 192, 196, 206, 218, 238, 246, 252, 364, 374, 444\}$,

(10). $\mathcal{P}_{0,1}(9) = \{142, 152, 158, 166, 170, 194, 200, 202, 212, 228, 230, 240, 348, 366, 376, 382, 396, 412, 438\}$,

(11). $\mathcal{P}_{0,1}(10) = \{134, 148, 154, 164, 168, 210, 224, 234, 248, 316, 350, 358, 372, 378, 406, 428, 430, 476, 764\}$.

Theorem 7.4.3. *For $b_1(n)$ we have the following,*

(1). $\mathcal{P}_{1,0}(0) = \{0, 2\}$,

(2). $\mathcal{P}_{1,0}(1) = \{4, 8\}$,

(3). $\mathcal{P}_{1,0}(2) = \{6, 10, 14, 16, 20\}$,

(4). $\mathcal{P}_{1,0}(3) = \{12, 18, 22, 28, 30, 32, 38, 44\}$,

(5). $\mathcal{P}_{1,0}(4) = \{26, 34, 36, 40, 60, 62, 76\}$,

(6). $\mathcal{P}_{1,0}(5) = \{24, 42, 46, 54, 56, 58, 64, 68, 70, 78, 86, 92, 124\}$,

(7). $\mathcal{P}_{1,0}(6) = \{52, 66, 74, 80, 88, 118, 126, 140, 156\}$,

(8). $\mathcal{P}_{1,0}(7) = \{50, 72, 82, 84, 108, 110, 116, 120, 122, 134, 150, 152, 158, 172, 252\}$.

Theorem 7.4.4. *For $b_{1,1}(n)$ we have the following,*

(1). $\mathcal{P}_{1,1}(0) = \{0\}$,

(2). $\mathcal{P}_{1,1}(1) = \{2\}$,

(3). $\mathcal{P}_{1,1}(2) = \{4\}$,

(4). $\mathcal{P}_{1,1}(3) = \{6, 8\}$,

(5). $\mathcal{P}_{1,1}(4) = \{10, 12, 14\}$,

(6). $\mathcal{P}_{1,1}(5) = \{16, 20, 22, 28\}$,

(7). $\mathcal{P}_{1,1}(6) = \{18, 30, 44\}$, *(8).* $\mathcal{P}_{1,1}(7) = \{24, 26, 38, 60\}$.

Question. *For a given pair* (α, β)*, what are the maximum and minimum values in* $\mathcal{P}_{\alpha,\beta}(k)$*? What is* $|\mathcal{P}_{\alpha,\beta}(k)|$*?*

We can determine a bound for the minimum value of $b_{\alpha,\beta}(2n)$, with $n \in I_r$, and $\alpha, \beta \geq 1$. In fact, this value increases as r increases and we get the following result.

Theorem 7.4.5. *For* $2n \in I_r$ *and* $\alpha, \beta \geq 1$, $\min b_{\alpha,\beta}(2n) \to \infty$ *as* $r \to \infty$.

Proof. Consider the minimum even term on the interval I_r,

$$\begin{aligned} \min_{2n \in I_r} b_{\alpha,\beta}(2n) &= \min_{n \in I_{r-1}} (b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1)) \\ &\geq \min_{n \in I_{r-1}} b_{\alpha,\beta}(n) + 1. \end{aligned}$$

Thus as r increases, $\min_{2n \in I_r} b_{\alpha,\beta}(2n)$ increases, and thus as $r \to \infty$,

$$\min_{2n \in I_r} b_{\alpha,\beta}(2n) \to \infty.$$

■

Corollary 7.4.6. *Given* $k \geq 0$ *and* $\alpha, \beta \geq 1$, $\mathcal{P}_{\alpha,\beta}(k)$ *is finite.*

Theorem 7.4.7. *For* $k \geq 0$, $\mathcal{P}_{0,1}(k)$ *and* $\mathcal{P}_{1,0}(k)$ *are finite.*

Proof. Consider the minimum even term on the interval I_r,

$$\min_{2n \in I_r} b_{\alpha,\beta}(2n) = \min_{n \in I_{r-1}} (b_{\alpha,\beta}(n) + b_{\alpha,\beta}(n+1)).$$

But by Lemmas 2.4.3 and 2.4.4 we know exactly which terms are 0, and with the exception of $b_0(0) = b_0(1) = 0$, there are never two adjacent zeroes. Thus at least one of $b_{\alpha,\beta}(n)$ and $b_{\alpha,\beta}(n+1)$ must be positive. Therefore

$$\min_{2n \in I_r} b_{\alpha,\beta}(2n) \geq \min_{n \in I_{r-1}} \{b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1)\} + 1.$$

So as r increases, $\min_{2n \in I_r} b_{\alpha,\beta}(2n)$ increases, and thus as $r \to \infty$,

$$\min_{2n \in I_r} b_{\alpha,\beta}(2n) \to \infty.$$

Thus for $k \geq 0$, $\mathcal{P}_{0,1}(k)$ and $\mathcal{P}_{1,0}(k)$ are finite. ■

Remark. If $\alpha, \beta \geq 0$, with at least one of $\{\alpha, \beta\}$ positive, then $\mathcal{P}_{\alpha,\beta}(k)$ is finite. However, this is not true when we allow α or β to be negative. In fact, in Corollary 2.4.2 we showed that $b_0(9 \cdot 2^r) = b_1(9 \cdot 2^r)$. So $\mathcal{P}_{1,-1}(0)$ contains all integers of the form $9 \cdot 2^r$, and is infinite.

Surprisingly, when searching for primitive values, we find that some kinds of pairs always occur, while other pairs never occur at all. Here is a partial list of pairs which do not occur for $b_0(n)$.

Theorem 7.4.8. *For $b_0(n)$, the following pairs $(b_0(n), b_0(n+1))$ never occur:*

$$(1,9),\ (10,1),\ (8,4),\ (2,10),\ (1,12),\ (6,8),\ (12,2),$$
$$(13,1),\ (1,14),\ (1,15),\ (3,13),\ (15,1),\ \text{and } (16,1).$$

We have checked that these pairs never occur by looking at the pairs $(b_0(n), b_0(n+1))$ for $1 \leq n \leq 10^6$ with *Mathematica*. If $n > 10^6$, then the even-indexed term of $(b_0(n), b_0(n+1))$ is greater than 16.

We have proved that the following pairs never occur.

Theorem 7.4.9. *If a is not a Fibonacci number, then $(b_0(n), b_0(n+1))$ is never equal to $(0, a)$ or $(a, 0)$.*

Proof. We know by Lemma 2.4.3 that $b_0(2^r - 1) = 0$, and by Theorem 2.3.4, $b_0(2^r) = F_r$. By the recursion $b_0(2^r - 2) = b_0(2^{r-1} - 1) + b_0(2^{r-1})$. So by Lemma 2.4.3 and Theorem 2.3.4, $b_0(2^r - 2) = F_{r-1}$ for $r \geq 1$. Moreover, by Lemma 2.4.3, $b_0(n) = 0 \iff n = 2^r - 1$. Thus we can see that the pairs $(0, a)$ and $(a, 0)$ always occur for $a = F_j$, $j \geq 0$, but never occur for $a \neq F_j$. ■

We know that some kinds of pairs always occur. Here is a partial list of pairs that always occur for $b_0(n)$.

Theorem 7.4.10. *For $b_0(n)$, the pairs $\{b_0(n), b_0(n+1)\} = \{a, b\}$ always occur for the following values of a and b:*

(1). $a = F_j$, $b = F_j$, $j \geq 1$*;*

(2). $a = 0$, $b = F_j$, $j \geq 1$;

(3). $a = F_j$, $b = 0$, $j \geq 1$.

Proof. We know by Lemma 2.4.3 that $b_0(2^r - 1) = 0$, and by Theorem 2.3.4 $b_0(2^r) = F_r$. By the recursion $b_0(2^r - 2) = b_0(2^{r-1} - 1) + b_0(2^{r-1})$. So by Lemma 2.4.3 and Theorem 2.3.4, $b_0(2^r - 2) = F_{r-1}$ for $r \geq 1$. Thus we can see that the pairs $(0, a)$ and $(a, 0)$ always occur for $a = F_j$, $j \geq 0$.

Given that the pair $(F_j, 0)$ occurs, we have some n for which $b_0(n) = F_j$ and $b_0(n+1) = 0$. Thus $b_0(2n) = F_j$ and $b_0(2n+1) = F_j$ by the recurrence. Thus all pairs (F_j, F_j) also occur, and we have

$$(b_0(2^r - 4), b_0(2^r - 3)) = (F_{r-2}, F_{r-2}).$$

■

Remark. Not only do these pairs occur, but they each always occur exactly once, except for the first case, which occurs exactly twice.

We would like to prove the following conjecture:

Conjecture 4. For $n \geq 0$, if $b_0(n) = b_0(n+1)$ then $b_0(n) = F_r$.

The conjecture holds for n even. Let $n = 2k$, then $b_0(2k) = b_0(2k+1)$. By the recursion,

$$b_0(k) + b_0(k+1) = b_0(k).$$

This is only true when $b_0(k+1) = 0$. Thus $k = 2^r$ by Lemma 2.4.3 and $b_0(k) = F_r$ by Theorem 2.3.4.

In fact, up to 10^6, the only values of n for which $b_0(n) = b_0(n+1)$ have the form $n = 3 \cdot 2^4 - 11$ and $3 \cdot 2^r - 4$, or $n = 7$. The conjecture also appears to be valid for $b_1(n)$ up to 10^6, where $n = 2^r - 11$ and $n = 2^r - 4$, with the exception of $n = 11$. On the other hand, for $b_{1,1}(n)$, the only equal consecutive terms up to $2 \cdot 10^6$ are $n = 1, 2$.

Chapter 8

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Appendix A

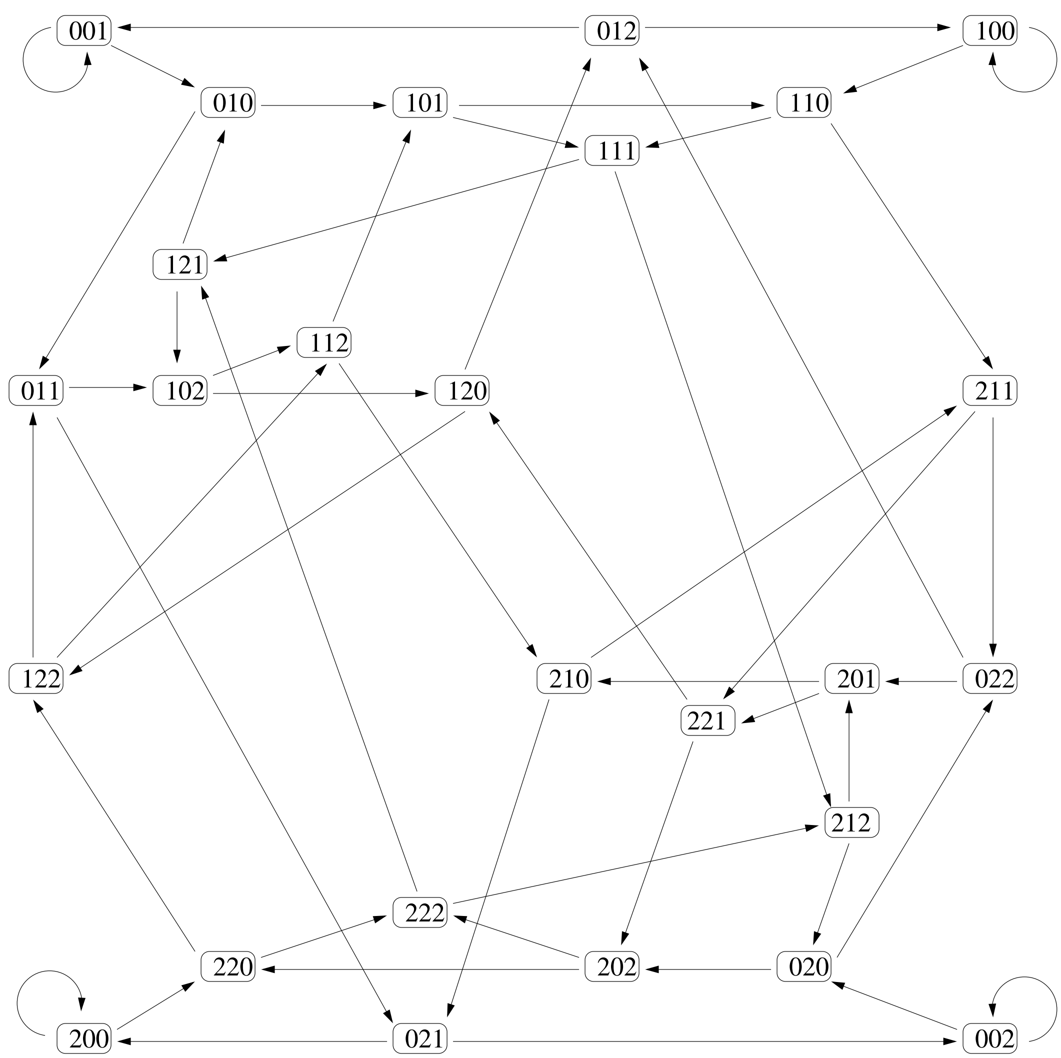

Figure A.1: Directed graph G^3

$$
\begin{pmatrix}
12 & 11 & 8 & 9 & 10 & 8 & 11 & 9 & 11 & 9 & 13 & 10 & 6 & 12 & 10 & 9 & 11 & 11 & 8 & 8 & 10 & 11 & 9 & 10 & 9 & 11 \\
10 & 10 & 8 & 9 & 12 & 7 & 11 & 9 & 10 & 11 & 15 & 11 & 6 & 11 & 12 & 8 & 11 & 9 & 8 & 9 & 11 & 11 & 8 & 9 & 10 & 10 \\
10 & 11 & 13 & 10 & 9 & 12 & 8 & 10 & 10 & 11 & 15 & 10 & 3 & 12 & 12 & 8 & 9 & 11 & 10 & 9 & 10 & 9 & 8 & 9 & 10 & 7 \\
9 & 9 & 8 & 10 & 12 & 9 & 12 & 9 & 9 & 10 & 13 & 10 & 6 & 12 & 12 & 7 & 11 & 9 & 10 & 11 & 10 & 11 & 7 & 11 & 8 & 11 \\
13 & 9 & 10 & 10 & 12 & 7 & 11 & 9 & 13 & 11 & 14 & 13 & 6 & 12 & 11 & 6 & 10 & 9 & 8 & 8 & 10 & 12 & 6 & 9 & 9 & 8 \\
12 & 11 & 10 & 10 & 9 & 11 & 9 & 9 & 12 & 8 & 13 & 10 & 5 & 15 & 13 & 8 & 10 & 11 & 9 & 8 & 8 & 11 & 8 & 9 & 9 & 8 \\
8 & 11 & 7 & 10 & 10 & 9 & 12 & 10 & 8 & 10 & 13 & 10 & 7 & 13 & 13 & 7 & 10 & 11 & 9 & 10 & 10 & 11 & 7 & 10 & 10 & 10 \\
11 & 9 & 9 & 10 & 11 & 7 & 11 & 10 & 11 & 13 & 15 & 12 & 6 & 12 & 11 & 7 & 11 & 9 & 7 & 8 & 12 & 11 & 7 & 8 & 10 & 8 \\
13 & 9 & 11 & 11 & 10 & 7 & 9 & 10 & 14 & 13 & 16 & 14 & 5 & 12 & 13 & 5 & 9 & 9 & 7 & 8 & 10 & 12 & 5 & 7 & 11 & 6 \\
10 & 10 & 10 & 12 & 10 & 10 & 8 & 12 & 10 & 15 & 15 & 11 & 5 & 12 & 13 & 3 & 10 & 10 & 8 & 11 & 12 & 9 & 3 & 9 & 11 & 7 \\
13 & 10 & 9 & 9 & 11 & 7 & 8 & 10 & 13 & 10 & 14 & 9 & 3 & 11 & 8 & 10 & 13 & 10 & 7 & 7 & 12 & 9 & 10 & 10 & 11 & 12 \\
10 & 9 & 7 & 9 & 13 & 7 & 12 & 9 & 10 & 10 & 14 & 13 & 8 & 13 & 14 & 6 & 11 & 9 & 8 & 10 & 9 & 12 & 6 & 10 & 8 & 9 \\
10 & 9 & 6 & 11 & 12 & 6 & 12 & 10 & 10 & 12 & 12 & 12 & 7 & 12 & 12 & 6 & 11 & 9 & 7 & 11 & 11 & 11 & 6 & 11 & 9 & 11 \\
11 & 10 & 15 & 12 & 6 & 12 & 10 & 8 & 11 & 11 & 17 & 12 & 4 & 14 & 13 & 8 & 6 & 10 & 11 & 8 & 8 & 10 & 8 & 7 & 8 & 6 \\
13 & 8 & 8 & 10 & 12 & 8 & 11 & 8 & 13 & 9 & 11 & 11 & 6 & 13 & 12 & 7 & 12 & 8 & 9 & 10 & 10 & 11 & 7 & 10 & 8 & 11 \\
13 & 10 & 8 & 9 & 10 & 8 & 11 & 8 & 13 & 6 & 12 & 11 & 6 & 13 & 12 & 9 & 11 & 10 & 9 & 9 & 7 & 12 & 9 & 10 & 8 & 12 \\
12 & 10 & 10 & 10 & 8 & 8 & 9 & 10 & 12 & 10 & 14 & 9 & 3 & 11 & 8 & 11 & 12 & 10 & 7 & 8 & 12 & 9 & 11 & 9 & 11 & 12 \\
8 & 13 & 7 & 8 & 11 & 11 & 9 & 11 & 8 & 9 & 12 & 7 & 4 & 12 & 11 & 9 & 12 & 14 & 9 & 10 & 11 & 9 & 9 & 12 & 10 & 10 \\
9 & 11 & 8 & 10 & 9 & 9 & 11 & 9 & 9 & 10 & 15 & 11 & 7 & 14 & 14 & 8 & 9 & 11 & 9 & 9 & 9 & 12 & 8 & 8 & 9 & 8 \\
10 & 11 & 11 & 10 & 11 & 12 & 7 & 12 & 10 & 12 & 13 & 8 & 2 & 11 & 11 & 6 & 11 & 11 & 10 & 12 & 12 & 7 & 6 & 11 & 10 & 9 \\
7 & 10 & 9 & 9 & 11 & 8 & 12 & 10 & 7 & 14 & 17 & 11 & 6 & 11 & 12 & 7 & 10 & 10 & 8 & 9 & 13 & 11 & 7 & 9 & 10 & 8 \\
10 & 11 & 7 & 9 & 12 & 8 & 10 & 9 & 10 & 10 & 14 & 11 & 7 & 14 & 14 & 7 & 11 & 11 & 8 & 8 & 9 & 13 & 7 & 9 & 9 & 8 \\
9 & 11 & 8 & 8 & 12 & 7 & 11 & 9 & 9 & 12 & 17 & 11 & 6 & 10 & 9 & 9 & 12 & 11 & 7 & 6 & 12 & 11 & 9 & 10 & 10 & 10 \\
11 & 9 & 10 & 11 & 11 & 11 & 10 & 9 & 11 & 10 & 12 & 10 & 5 & 12 & 11 & 5 & 11 & 9 & 11 & 11 & 10 & 9 & 5 & 12 & 8 & 12 \\
11 & 12 & 8 & 8 & 12 & 11 & 7 & 11 & 11 & 8 & 12 & 8 & 4 & 13 & 12 & 8 & 12 & 12 & 9 & 9 & 10 & 9 & 8 & 11 & 11 & 9 \\
10 & 9 & 6 & 11 & 12 & 6 & 12 & 10 & 10 & 12 & 12 & 12 & 7 & 12 & 12 & 6 & 11 & 9 & 7 & 11 & 11 & 11 & 6 & 11 & 9 & 11
\end{pmatrix}
$$

Figure A.2: $(A_{G^3})^8$

Appendix B

We have calculated the number of triples

$$(b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) \equiv (i,j,k) \pmod{d}$$

for $4 \leq d \leq 10$, and $(\alpha, \beta) = (1,0)$, and $(0,1)$. Column two corresponds to the size of the set

$$T^d_{\alpha,\beta} = \{n : (b_{\alpha,\beta}(n), b_{\alpha,\beta}(n+1), b_{\alpha,\beta}(n+2)) \equiv (i,j,k) \pmod{d}, 1 \leq n \leq 10^6\},$$

which has the same size for $b_0(n)$ and $b_1(n)$. Column two gives the size $|T^d_{0,1}| = |T^d_{1,0}|$. Columns three and four give the minimum and maximum frequency for the congruence classes. Column five gives the expected density of the congruence classes in decimal form. As we can see from the Table below, all of the expected congruence classes occur.

d	$\|T^d_{0,1}\|$	$b_0(n)$	$b_1(n)$	Expected Density
2	7	$142857 \to 142858$	$142850 \to 142863$	0.142857
3	26	$38408 \to 38504$	$38397 \to 38504$	0.038461
4	56	$17716 \to 17999$	$17739 \to 18014$	0.017857
5	124	$7931 \to 8212$	$7895 \to 8214$	0.008064
6	182	$5295 \to 5677$	$5329 \to 5675$	0.005494
7	342	$2735 \to 3058$	$2815 \to 3035$	0.002923
8	448	$2007 \to 2497$	$2037 \to 2459$	0.002232
9	702	$1311 \to 1524$	$1343 \to 1521$	0.001424
10	868	$1040 \to 1312$	$1058 \to 1275$	0.001152

Table B.1: Frequencies of congruence classes of triples modulo d

Appendix C

We have calculated the number of terms of $b_{\alpha,\beta}(n)$ congruent to a modulo d for $d \leq 10$ and $(\alpha, \beta) = (0, 1),\ (1, 0)$, and $(1, 1)$. Columns two through four correspond to the size of the sets

$$\{n : b_{\alpha,\beta}(n) \equiv a \pmod{d}, 1 \leq n \leq 10^6\},$$

while the last column gives the conjectured density in decimal form.

$a \pmod{d}$	$b_0(n)$	$b_1(n)$	$b_{1,1}(n)$	conjectured density
$0 \pmod{2}$	$428,572$	$428,578$	$428,564$	0.428571
$1 \pmod{2}$	$571,428$	$571,422$	$571,436$	0.571429
$0 \pmod{3}$	$307,694$	$307,616$	$307,779$	0.307692
$1 \pmod{3}$	$346,135$	$346,152$	$346,192$	0.346154
$2 \pmod{3}$	$346,171$	$346,232$	$346,029$	0.346154
$0 \pmod{4}$	$214,589$	$214,564$	$213,873$	0.214286
$1 \pmod{4}$	$285,856$	$285,862$	$285,733$	0.285714
$2 \pmod{4}$	$213,983$	$214,014$	$214,691$	0.214286
$3 \pmod{4}$	$285,572$	$285,560$	$285,703$	0.285714
$0 \pmod{5}$	$193,804$	$193,586$	$193,358$	0.193548
$1 \pmod{5}$	$201,985$	$202,039$	$201,335$	0.201613
$2 \pmod{5}$	$201,043$	$201,149$	$201,632$	0.201613
$3 \pmod{5}$	$201,025$	$200,852$	$201,987$	0.201613
$4 \pmod{5}$	$202,143$	$202,374$	$201,688$	0.201613
$0 \pmod{6}$	$131,111$	$131,214$	$131,579$	0.131868
$1 \pmod{6}$	$197,165$	$197,302$	$198,042$	0.197802
$2 \pmod{6}$	$148,491$	$148,514$	$148,835$	0.148352
$3 \pmod{6}$	$176,583$	$176,402$	$176,200$	0.175824
$4 \pmod{6}$	$148,970$	$148,850$	$148,150$	0.148352
$5 \pmod{6}$	$197,680$	$197,718$	$197,194$	0.197802

Table C.1: Congruence classes of $b_{\alpha,\beta}(n)$ modulo d with $d \leq 6$

$a \pmod{d}$	$b_0(n)$	$b_1(n)$	$b_{1,1}(n)$	conjectured density
$0 \pmod{7}$	$140,424$	$140,326$	$140,504$	0.140351
$1 \pmod{7}$	$142,844$	$142,903$	$143,199$	0.143275
$2 \pmod{7}$	$143,721$	$143,697$	$143,373$	0.143275
$3 \pmod{7}$	$143,322$	$142,993$	$143,407$	0.143275
$4 \pmod{7}$	$142,940$	$143,092$	$143,030$	0.143275
$5 \pmod{7}$	$143,372$	$143,287$	$143,071$	0.143275
$6 \pmod{7}$	$143,377$	$143,702$	$143,416$	0.143275
$0 \pmod{8}$	$106,866$	$106,833$	$107,143$	0.107143
$1 \pmod{8}$	$142,937$	$143,421$	$142,909$	0.142857
$2 \pmod{8}$	$107,784$	$107,709$	$107,207$	0.107143
$3 \pmod{8}$	$142,648$	$142,453$	$143,711$	0.142857
$4 \pmod{8}$	$107,723$	$107,681$	$106,730$	0.107143
$5 \pmod{8}$	$142,919$	$142,441$	$142,824$	0.142857
$6 \pmod{8}$	$106,199$	$106,305$	$107,484$	0.107143
$7 \pmod{8}$	$143,107$	$142,924$	$141,992$	0.142857
$0 \pmod{9}$	$102,962$	$102,865$	$102,623$	0.102564
$1 \pmod{9}$	$115,681$	$115,843$	$115,903$	0.115385
$2 \pmod{9}$	$115,897$	$115,821$	$115,718$	0.115385
$3 \pmod{9}$	$102,366$	$102,614$	$102,946$	0.102564
$4 \pmod{9}$	$115,380$	$114,832$	$115,053$	0.115385
$5 \pmod{9}$	$115,114$	$115,148$	$114,923$	0.115385
$6 \pmod{9}$	$102,366$	$102,137$	$102,210$	0.102564
$7 \pmod{9}$	$115,074$	$115,477$	$115,236$	0.115385
$8 \pmod{9}$	$115,160$	$115,263$	$115,388$	0.115385
$0 \pmod{10}$	$83,761$	$83,690$	$83,166$	0.082949
$1 \pmod{10}$	$115,577$	$115,559$	$115,074$	0.115207
$2 \pmod{10}$	$86,186$	$86,077$	$86,378$	0.086405
$3 \pmod{10}$	$115,524$	$115,498$	$115,611$	0.115207
$4 \pmod{10}$	$86,716$	$86,977$	$86,383$	0.086405
$5 \pmod{10}$	$110,043$	$109,896$	$110,192$	0.110599
$6 \pmod{10}$	$86,408$	$86,480$	$86,261$	0.086405
$7 \pmod{10}$	$114,857$	$115,072$	$115,254$	0.115207
$8 \pmod{10}$	$85,501$	$85,354$	$86,376$	0.086405
$9 \pmod{10}$	$115,427$	$115,397$	$115,305$	0.115207

Table C.2: Congruence classes of $b_{\alpha,\beta}(n)$ modulo d with $7 \leq d \leq 10$