NASH EQUILIBRIUM PROBLEMS IN POWER MARKETS AND PRODUCT DESIGN: ANALYSIS AND ALGORITHMS

BY

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THESIS

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Abstract

The focus of this research is on the analysis and computation of equilibria in noncooperative Cournot and Bertrand games. The application of focus for Cournot competition is power markets while that for Bertrand competition is product design. We consider Cournot-based models for strategic behavior in power markets while Bertrand-based models are employed for analyzing the behavior of price-based competition in product design. This thesis is partitioned into three parts. Of these, the first two parts focus on power market applications while the third part focuses on product design.

Motivated by the risk of capacity shortfall faced by market participants with uncertain generation assets, the first part considers a game where agents are assumed to be risk-averse optimizers, using a conditional value-at-risk (CVaR) measure. The resulting game-theoretic problem is a two-period risk-based stochastic Nash game with shared strategy sets. In general, this stochastic game has nonsmooth objectives and standard existence and uniqueness results cannot be leveraged for this class of games, given the lack of compactness of strategy sets and the absence of strong monotonicity in the gradient map of the objectives. However, when the risk-measure is independent of competitive interactions, a subset of equilibria to the risk-averse game are shown to be characterized by a solvable monotone single-valued variational inequality. If the risk-measures are generalized to allow for strategic interactions, then the characterization is through a multi-valued variational inequality. Both this object and its single-valued counterpart, arising from the smoothed game, are shown to admit solutions. The equilibrium conditions of the game grow linearly in size with the the sample-space, network size and the number of participating firms. Consequently, direct schemes are inadvisable for most practical problems and instead, we present a distributed regularized primal-dual and dual projection scheme where both primal and dual iterates are computed separately. Rate of convergence estimates are provided and error bounds are developed for inexact extensions of the dual scheme. Unlike projection schemes for deterministic problems, here the projection step requires the solution of a possibly massive stochastic program. By utilizing cutting plane methods, we ensure that the complexity of the projection scheme scales slowly with the size of the sample-space. Insights regarding market design and operation are obtained after testing the model on a 53-node electricity network.

The second part extends this model by considering the grid operator to be a profit maximizer. However the

effect of risk is neglected in this model. The resulting problem is a quasi variational inequality. An analysis of the equivalent complementarity problem (CP) allows us to claim that the game does admit an equilibrium. By observing that the CP is monotone, we are in a position to employ a class of iterative regularization techniques namely the iterative Tikhonov and the iterative proximal algorithms. The algorithms are seen to scale well with the size of the problem. The model is employed for examining strategic behavior on a twelve node network and several economic insights are drawn.

The third part of this thesis deals with Bertrand competition in a product design regime. With due consideration to the attribute dimension in addition to price competition, more specifically for design and consumer service industries, a game theoretic model is formulated. The logit model, in lieu of some of its tractable properties, is deployed to capture consumer preferences and thereby the demand. Subsequently the variational formulations corresponding to the game are analyzed for existence of solutions. The lack of convexity of objectives, analytical intractability of the variational formulations corresponding to the game state some drawbacks of the logit model. Several projection and interior point schemes are deployed for solving these classes of problems. Numerical results for smaller instances of these games are illustrated by means of a painkiller example. Suggestions on alternate revenue maximization models are presented.

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Chapter 1

Introduction

The realm of optimization addresses design and operational questions in a range of problems lying at the interstices of engineering and economics, such as product design, power markets, logistics, pricing and revenue management [74, 20, 67]. A general optimization problem (OP) in a finite-dimensional space [71, 11] can be stated as requiring the minimization of a function $f : \mathbb{R}^n \to \mathbb{R}$ over a set X:

$$\boxed{\begin{array}{lll} \text{OP} \qquad\qquad & \text{minimize} & f(x) \\ & \text{subject to} & x \in X. \end{array}}$$

In vast generality, optimization of a system necessitates that there is a centralized control of the system. Yet, in many settings, where a number of users are competing for a finite set of resources, a natural extension to this model lies in the employment of a game-theoretic framework. When such players are selfish, a relevant model is that introduced by Nash [69] in which players compete in a noncooperative setting. The resulting Nash equilibrium of the above game may be defined as the set of decisions of all the agents from which no agent can improve his profit by unilaterally changing his decisions. Consider an oligopoly with a set of firms denoted by $\mathcal{J}$ in which agent j's problem (EP) can be mathematically represented as follows:

$$\boxed{\begin{array}{lll} \text{EP}(z^{-j}) \qquad\qquad & \text{minimize} & f_j(z_j; z_{-j}) \\ & \text{subject to} & z_j \in \mathbf{Z}_j, \end{array}}$$

where z_{-j} denotes the decisions of all the other agents excepting j. The importance of such problems may be understood by observing all competitive markets where the intent is on profit maximization. A few applications of such game theoretic regimes may be found in manufacturing segments, congestion modeling, traffic management, flow control problems and power markets.

1.1 Applications

In this thesis, we consider noncooperative game-theoretic problems arising in power markets and product design. Such problems may take on a variety of forms. Two notable instances are Cournot and Bertrand models [4, 3, 35]. A Nash-Cournot game [44, 45, 65] is one where firms make quantity decisions pertaining to a product (homogenous) whose price in the market is given by a function of the aggregate quantity produced. A Bertrand model [26, 46], on the other hand, is one where firms make pricing decisions and the consumer demand is a consequence of the individual prices set by the firms. Chapters 2 and 3 consider the application of the Nash-Cournot models to a setting arising in the design of power markets while chapter 4 applies Bertrand models to deriving insights in product design in competitive regimes.

1.1.1 Power market structure

A typical power market consists of a set of firms with physical generation assets and a grid operator [9, 82, 51, 87]. In many markets, there may also be a collection of arbitrageurs [45]. A grid operator is responsible for maintenance, allocation and dispatch of power while arbitragers buy power at lower prices from the firms and sell it to the customers or other entities at higher prices. In several settings the grid operator (also called the Independent System Operator or the ISO) may also be responsible for electricity pricing. We restrict our attention to models where the price of electricity [80] is defined via an affine function whose structure is known to all the market participants. The function of the ISO varies from one market to another. The ISO may be modeled as being a firm intent on maximizing social welfare [91, 92] or possibly transmission revenue [48, 45]. The strategic behavior of the market participants and the ISO may be articulated by a noncooperative Nash-Cournot game. Our focus is on extensions of such games where costs and prices are uncertain and possibly nonsmooth with a twofold intent. First, we aim at providing rigorous existence and uniqueness statements; Our second goal lies in the development of scalable computational schemes for obtaining equilibria in stochastic settings.

1.1.2 Product positioning

When the sold commodity is not homogenous amongst all agents, then the notion of price differences gains importance along with factors like product quality, reliability, reusability etc. This leads to a Bertrand framework. Almost all non-service oriented industries operate on the lines of price-based or Bertrand competition. This would include a wide spectrum of industries from automation and manufacturing to food processing and clothing. For instance a customer purchasing a gear might be interested in the size and weight of the gear, raw material for the gear and the number of teeth in addition to the gear's final sale price. The dimensions stated above apart from price may be referred to as product attributes. Manufacturers aim to maximize their profits by changing prices and these attributes. In the presence of competition, the customer demand towards a particular product would also depend on the attribute dimensions and prices of the other products. The resulting game where firms compete in price and product attributes is examined in chapter 4.

1.2 Analysis and algorithms

In general, Nash games over general strategy sets may not admit tractable equilibrium conditions. However, when the strategy sets are continuous and the user payoffs are differentiable, then variational formulations prove to be useful. Under suitable convexity assumptions on the objective and strategy sets, the first-order conditions of the optimization problem (OP) may be represented in terms of a variational inequality (VI). Recall that a variational inequality is the problem of finding an $x^* \in X$ such that

$$\nabla f(x^*)^T (x - x^*) \geq 0, \quad \forall x \in X.$$

Extending this to a game-theoretic regime, the first order conditions for the game (EP) (stated previously) may be represented as follows:

$$\mathbf{F}(z)^T (z - z^*) \geq 0, \quad \forall z \in Z,$$

where

$$\mathbf{F} \triangleq \begin{pmatrix} \nabla f_1(z_1, z_{-1}) \\ \vdots \\ \nabla f_J(z_J, z_{-J}) \end{pmatrix} \quad \text{and} \quad Z \triangleq \prod_{j=1}^{J} Z_j.$$

Optimization problems are guaranteed to have optimal solutions under suitable convexity assumptions on the function or compactnenss requirements on the set X. Analogous conditions in a game-theoretic regime require the analsis of the corresponding VI. This is often complicated in regimes where strategy sets are coupled. A common example would be a congestion modeling problem or a traffic problem where firms compete over networked resources. This gives rise to another class of problems called quasi-variational inequalities or QVIs. Such problems are generalizations of variational inequalities where the set Z could be a set-valued map. Less can be said about QVIs, in comparison with VIs, particularly when the games have unbounded strategy sets and nonsmooth objectives.

If an equilibrium exists, the next question arises as to how equilibria may be computed using convergent schemes. Nash games in stochastic regimes are given by a setting where the ith solves a parameterized stochastic optimization problem:

$$\boxed{\begin{array}{lll} \mathrm{SEP}(x_{-i}, y_{-i}) & \text{minimize} & \mathbb{E} f_\omega(x_i, y_i^\omega, x_{-i}, y_{-i}^\omega) \\ & & y_i^\omega \in K_i^\omega, \quad \forall \omega \in \Omega \\ & \text{subject to} & x_i, y_i^\omega \in L_i^\omega, \quad \forall \omega \in \Omega \\ & & x_i \in P_i, \end{array}}$$

where K_i^ω, L_i^ω and P_i represent closed and convex sets and $\omega \in \Omega$ refers to the sample space corresponding to uncertainty. The size of the problem grows massively with the cardinality of the sample space. Therefore as $|\Omega|$ becomes large, it becomes difficult to compute equilibria using conventional algorithms and requires the development of scalable schemes. With reference to all the above stated issues, the following section highlights the major contributions of this research.

1.3 Contributions

This thesis is partitioned into three chapters. Chapters 2 and 3 focus on power markets and chapter 4 focuses on product design. Our goal lies in providing theoretical guarantees for existence and subsequently developing scalable convergent algorithms. The following subsections highlight the key contributions in each of the individual chapters.

1.3.1 Risk-based generalized Nash games in power markets

A two-period setting where the ISO is welfare maximizing and firms bidding successively in the day ahead and real time markets is considered. A stochastic risk averse game theoretic model is formulated (Nash-Cournot). Risk is studied in the context of shortfall and is quantified by means of Cvar, a non-smooth measure. Two different risk averse settings are analyzed. The first setting leads to a variational formulation. Under some weak assumptions convexity of the agent objectives and existence of solutions to the game are guaranteed. An approximation to the original mapping is seen to possess some tractable properties. Under the presence of coupled strategy sets traditional projection schemes do not prove to be effective. In process two convergent schemes namely the primal-dual method and the dual method are deployed to compute equilibria. A hybrid cutting plane scheme is deployed to solve projection sub-problems at every step. An inexact version of the dual scheme is analyzed and error bounds are obtained. The schemes scale very well with the problem size indicating almost linear growth. The second risk averse setting leads to a nonsmooth game with coupled strategy sets. A smoothing technique to eliminate this nonsmoothness is presented. Under weak assumptions existence of equilibria is guaranteed to both the smooth game and its nonsmooth counterpart. Smoothing allows direct use of existing solvers to compute equilibria. Lastly, both the risk averse models are tested on a 53-noded electrical network namely the Belgian grid and economic interpretations are obtained.

1.3.2 Strategic behavior in power markets under uncertainty

This setting stems from the previous model with the major difference of the ISO being a profit maximizer. The ISO is assumed to earn revenue from transmission or wheeling. This setting leads to a game with coupled strategy sets that is generally less tractable. However a complementarity reformulation leads to a more tractable problem that possesses some monotonicity properties. Note that a complementarity problem is of the form:

$$0 \leq z \perp F(z) \geq 0 \qquad \text{or} \quad z \geq 0, \quad F(z) \geq 0, \quad z^T F(z) = 0.$$

The resulting mapping, $F(z)$ is seen to possess some tractable properties. The game and the equivalent complementarity formulations are seen to admit solutions based on some weak assumptions. However, the mapping is not seen to possess sufficient properties for traditional projection schemes to be deployed. In process, two convergent schemes namely the iterative Tikhonov regularization (ITR) and the iterative proximal point (IPP) schemes are deployed to compute equilibria. The schemes turn out to be parallelizable and scale well with the problem size. A simulated twelve node electrical network is taken as a case study for the stochastic equilibrium problem and economic insights are obtained.

1.3.3 A complementarity approach for game theoretic discrete choice models

In contrast to the former application, this portion analyzes a Bertrand framework. Several demand models that capture consumer preferences are studied. A stylized version of the logit model is chosen for modeling and analysis. A Nash-Bertrand model is constructed where agents compete in prices and attributes. The objective functions are observed to be non-convex. Though this leads to no guarantee on the existence of solutions to the original game, the corresponding variational formulations are analyzed and seen to admit solutions. An automotive design game is formulated and its variational and complementarity forms are obtained. Several projection and interior point methods were tested on this problem. A relatively easier setting with regard to pain killers is examined as a park of the case study. The complementarity formulation corresponding to the toy problem is solved. Decomposition schemes on the lines of other previous works are also deployed and seen to admit similar solutions.

Chapter 2

Risk-based Generalized Nash Games in Power Markets: Characterization and Computation of Equilibria

2.1 Introduction

As electricity markets gravitate towards regimes where intermittent renewables such as windpower are an integral part of a firm's generation mix, multiple questions persist regarding how markets should evolve to accommodate such assets. Crucial to answering such questions is the development of a new generation of game-theoretic models that can contend with the uncertainty and risk, in the context of sequential electricity markets. In the past, deterministic variants have proved useful in analyzing a range of questions in the design and operation of markets, both in a single-settlement [44, 45, 65] and a two-settlement framework [13, 47, 49, 91]. Yet, past work provides little from the standpoint of characterizing and computing equilibria, particularly in settings complicated by risk and uncertainty. The current paper is fueled by natural questions arising from the resulting two-period risk-averse stochastic Nash games: (a) Can one *characterize* equilibria in such games; and (b) can such equilibria be computed via *efficient scalable and convergent* schemes?

This range of questions falls at the interstices of stochastic programming and continuous-strategy Nash games. Of these, the former is a subclass of mathematical programming first discussed by Dantzig [25] and Beale [7] and allows for both adaptive [86, 50, 10, 78] (such as models allowing for recourse actions in the second-period, contingent on first-period decisions) and anticipative models [79, 15, 10] (such as chance-constrained models that impose a probabilistic or reliability constraint on the underlying optimization model), amongst others. Game theory [36, 72] has its roots in the work by von Neumann and Morgenstern [89] while the Nash-equilibrium solution concept was forwarded by Nash in 1950 [69].

In this paper, the focus is on N-person risk-averse stochastic Nash games over continuous strategy sets and are inspired by settings where agents make simultaneous bids in the first (such as a forward market) period followed by *recourse* bids in the second (such as a real-time market) period. We use a conditional value-at-risk (CVaR) measure [76] to capture the risk associated with bidding with assets whose availability is uncertain in the real-time market. The class of games under consideration depart from canonical models in at least three ways: first, the strategy sets of the players are coupled, implying that the agents are competing in a generalized Nash game [32, 41];

second, the objectives may possibly be nonsmooth; and finally, each player solves a two-period recourse-based risk-averse optimization problem. The resulting equilibrium problem requires addressing (i) the inherently dynamic competitive framework arising from the two-settlement structure inherent to power markets, (ii) the underlying uncertainty associated with the second-period market and finally (iii) the possible nonsmoothness emerging from risk-averseness of certain participants.

Several challenges are encountered in addressing the characterization and computational questions fueling this paper. In the context of the former (as denoted by (a)), in the realm of continuous strategy games, a common avenue relies on the analysis of the sufficient equilibrium conditions, namely a variational inequality or a complementarity problem, arising from the game. Yet, in this setting, this approach is fraught with several difficulties. First, the strategy sets across agents are coupled when one works within a regime of a networked electricity market, implying that the equilibrium conditions lead to a quasi-variational inequality [73, 14], generally a less tractable object. Second, given that risk-averse agents employ CVaR measures, the resulting objectives are possibly nonsmooth and the resulting variational inequality can be multivalued. Third, even under very strong assumptions, neither are the mappings of the resulting variational inequalities strongly monotone nor are the strategy sets compact. In short, a direct conclusion regarding existence or even uniqueness of equilibria is unavailable.

When considering the computational question (as denoted by (b)), the solution of the resulting complementarity problems in practical settings is constrained by several issues. While a direct application of a solver such as PATH [30] is clearly the best choice for solving such problems, its unlikely that the computational effort will scale well with growth in problem size. Consequently, the solution of truly large-scale instances via direct schemes becomes increasingly difficult and suggests the construction of distributed schemes. Motivated by these challenges, the present work makes the following contributions:

1. Analysis of equilibria arising in risk-based generalized Nash games: In a setting where agents are faced by a forward and an uncertain real-time market, we employ a two-period generalized Nash model and notice that the coupled constraints are shared. In general, the presence of the CVaR measure implies that the agent objectives are nonsmooth. Yet, when the CVaR measure is not parameterized by strategic interactions, by a suitable reformulation, an equilibrium to the game is given by a single-valued variational inequality [31]. Under differing assumptions on forward price specification and risk-aversion, we show that the resulting variational inequalities are monotone and admit compact nonempty solution sets. The monotonicity allows us to claim that the regularized games admit unique solutions. When strategic interactions are allowed in the risk-measures (called the shared risk model), a reformulation is not possible and the game leads to a multivalued stochastic variational inequality. Both this object, and its single-valued counterpart arising from the smoothed game, are shown to be solvable by an analysis of the coercivity properties of the mapping.

2. Convergent scalable schemes with error bounds: We present two distributed projection-based cutting-plane scheme for computing equilibria, the first a single timescale (primal-dual) method while the second is a two timescale (dual) scheme. For the dual scheme, we develop estimates of the convergence rate and extend the analysis to contend with practical implementations. In particular, we analyze the error associated with bounded complexity implementations where the underlying primal scheme is run for a finite number of steps. The ability of the scheme to contend with the size arising from the uncertainty rests on being able to solve the projection problems effectively. By observing that these problems are two-period stochastic convex programs with complete recourse, we employ a cutting-plane method whose effort grows linearly with the cardinality of the sample-space. Numerical results suggest that the overall scheme scales well with problem size.

3. Insights for market design and operation: A numerical implementation on a 53-node model of Belgian network provides numerous insights for market design. For instance, we observe that higher levels of risk-aversion lead to lower participation in the forward markets while higher level of wind penetration leads to greater participation in the forward markets.

The paper is organized into five sections. Section 2 introduces the stochastic two-settlement electricity market model and defines the related shared-constraint games and the resulting variational inequalities. In section 3, we analyze the properties of equilibria arising in such games. A novel hybrid distributed scheme that combines projection methods with cutting-plane algorithms is presented in section 4. In section 5, we obtain insights through a two-settlement networked electricity market model via a risk-based stochastic generalized Nash game. We conclude in section 6.

2.2 A two-settlement electricity market model

Table 2.1: Notation

x_{ij}	Forward decision of generation from firm j at node i
$u_{ij}^{\omega}, v_{ij}^{\omega}$	Positive and negative deviations respectively at scenario ω from firm j at node i
$y_{ij}^{\omega}, cap_{ij}^{\omega}$	Total spot generation decision and total generation capacity at scenario ω for firm j at node i
r_i^{ω}	ISO's spot decision at scenario ω at node i
n,Ω, ρ^{ω}	Number of scenarios, set of all scenarios and probability of scenario ω
p_i^{ω}	Nodal demand function or price at scenario ω at node i
$c_{ij}^{\omega}, d_{ij}^{\omega}$	Coefficient of linear and quadratic terms in the cost function at scenario ω for firm j at node i
f_p, f_n	Penalty functions for positive and negative deviations
N_g,N	Number of generating nodes and total nodes in the network
a_i^0, b_i^0	Intercept and Slope respectively at node i in the forward market
$a_i^{\omega}, b_i^{\omega}$	Intercept and Slope respectively at node i at scenario ω
$g+1$	Number of agents including g firms and the ISO - $(g+1)^{th}$ agent
$Q_{l,i}$	Power flowing across line l due to unit injection/withdrawal of power at node i
$\kappa_j, \forall j \in \mathcal{J}$	Risk factor or risk aversion parameter for firm j
$\mathcal{N}_j, \mathcal{N}_j^c$	Set of all generating nodes and non-generating nodes for firm j respectively
$\mathcal{J}_i$	Set of all generating firms at node i
$\mathcal{L}, \mathcal{N}$	Set of all transmission lines and set of all nodes respectively
G, G^c	Set of all generating nodes and load nodes respectively
$\mathcal{J}, \mathcal{A}$	Set of all generating firms and set of all agents (firms and the ISO) respectively

While extant research has laid the foundation for drawing insights pertaining to agent behavior in power markets [91, 44, 52, 45], these models and the consequent solution concepts are inadequate from at least two standpoints: (1) First, the majority of the past effort has presented a largely deterministic viewpoint, barring [91, 85], ignoring the uncertainty in fuel costs and demand as well as possible risk-averseness. As markets, and the underlying grid infrastructure, evolve rapidly towards the envisaged smart grid, the accommodation of heterogeneous generation resources, such as windpower, becomes paramount. However, the variability inherent in such forms of generation implies that participants, particularly those with wind resources, are faced with significant risk; (2) Much of the past research on bidding in two-period markets assumes a fully-rational model. For instance in [91], firms participating in the forward market compete subject to equilibrium in the spot-market. This in itself is not a shortcoming but the resulting agent problems problems are given by mathematical programs with equilibrium constraints (MPECs) [63], a class of ill-posed nonconvex nonlinear programs. Little existence theory exists for the resulting games, called multi-leader multi-follower games [61], barring results in either conjectured settings [85] or under rather strong assumptions [3, 83]. Further, even when equilibria are known to exist, there are no known convergent algorithms for computing these equilibria. Both shortcomings become even more pronounced when one considers the addition of risk and uncertainty.

The present work is principally motivated by analyzing a class of game-theoretic models that can overcome some of the shortcomings described in (1) and (2). We address (1) through a stochastic game-theoretic framework in which agents have heterogeneous risk preferences and employ a conditional value-at-risk metric to capture the risk of capacity shortfall. This can be viewed as an adapted open-loop game, first studied in an optimization setting by Haurie and his coauthors [42, 43] for modeling multistage decision-making problems. This avenue alleviates some of the challenges articulated in (2), namely from the standpoint of characterizing and computing equilibria. In particular, we consider a simpler question of agents making simultaneous bids in the forward market and the recourse-based bids in the spot-market. Two interpretations of the resulting game can be given: (i) *Economics:* it can be viewed as a bounded-rationality simplification of the fully-rational game in which firms compete in Nash with respect to the ISO, rather than assuming a leadership role, a model studied by Hobbs, amongst others [44]; (ii) *Mathematical programming:* it can also be viewed as a Nash game played at the forward market by agents solving two-period stochastic programs. In particular, agents play a game in the first period and for every scenario in the second period, where recourse decisions may be taken.

Using the model suggested by Yao et al. [91] as a basis, we now describe some features of our framework that are common to the models we introduce in the sections to follow. Suppose the uncertainty in the second-stage is captured by the random vector ξ and $\xi : \Omega \to \mathbb{R}^{\bar{n}}$ is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where Ω is finite in cardinality. Throughout this paper, we refer to components of $\xi(\omega)$ (arising from a sample point $\omega \in \Omega$) by using a subscript ω. Consider a market in which g firms compete in an electricity network where inflow/outflow decisions are managed by the ISO. Let $\mathcal{N}$ and $\mathcal{N}_j$ denote the set of nodes in this network and the set at which firm j owns generation facilities where $j = 1, \ldots, g$.Two-settlement markets are constructed around a sequence of clearings, in which the first settlement specifies the forward price while the second is a consequence of physical transactions and determines the real-time price. We denote by x_{ij} the forward position at node i corresponding to firm j while the corresponding physical generation in scenario ω is denoted by y_{ij}^{ω}. Further, the forward and real-time prices (in scenario ω) at node i are denoted by p_i^0 and p_i^{ω}, respectively. The ISO manages injections and outflows at all nodes, where the inflow at the ith node under scenario ω is denoted by r_i^{ω} where $i \in \mathcal{N}$. Note that a positive (negative) value of r_i^{ω} marks an inflow (outflow).

Our first model (section 2.1) captures a setting where agents compete within a no-arbitrage (nodal forward price is equal to expected spot price) risk-neutral setting and are faced by deviation costs when their real-time generation levels differ from their forward bids. Such models may require modification in several ways. We consider two key changes in section 2.2. The first pertains to the cost of deviations which we replace through a risk-based metric. This risk measure, weighed by the risk-aversenenss levels, provides an ex-ante metric of risk exposure caused by forward decisions of a firm, in contrast with an ex-post deviation cost. A second modification is introduced in the nature of forward price specification for which we prescribe a market-clearing model. This was first suggested by Kamat and Oren [52] and requires that forward prices may be set independently through a clearing. The games in sections 2.1 and 2.2 lead to generalized Nash games, extensions of Nash games in which the strategy sets are coupled across players. In section 2.3, we show that the generalized Nash game is observed to be a Nash game with shared constraints and an equilibrium is given by a specified variational inequality. Note that variable and parameter definitions are summarized in Table 3.1 in the appendix.

2.2.1 No-arbitrage risk neutral model

Our first model assumes that forward prices are specified by expected spot prices and departures from forward positions are discouraged through convex penalization costs. Given positive scalars $(a_i^{\omega}, b_i^{\omega})$, we define the nodal spot prices at scenario ω as an affine function of nodal consumption at that node, given by the total generation by all firms at node i modified by the ISO's injection, denoted by r_i^{ω}.

$$p_i^{\omega}(y^{\omega}, r_i^{\omega}) \triangleq a_i^{\omega} - b_i^{\omega} \left(\sum_{j \in \mathcal{J}} y_{ij}^{\omega} + r_i^{\omega} \right), \quad \forall i \in \mathcal{N}. \tag{2.1}$$

The arbitrage-free model requires that the nodal forward price is equal to the expected nodal spot prices.

$$p_i^0 \triangleq \mathbb{E} p_i^\omega, \quad \forall i \in \mathcal{N}. \tag{2.2}$$

Furthermore, during scenario ω, we denote the the cost of generation of firm j at node i by $\zeta_{ij}^\omega(y_{ij}^\omega)$ and the positive and negative deviation costs by $f_p(u_{ij}^\omega)$ and $f_n(v_{ij}^\omega)$, respectively where u_{ij}^ω and v_{ij}^ω are the positive and negative deviation levels from the forward positions x_{ij}. The generation in the real-time market by firm j at node i is denoted by y_{ij}^ω and is defined by

$$y_{ij}^\omega = x_{ij} + u_{ij}^\omega - v_{ij}^\omega.$$

The profit of firm j, given by the sum of forward and spot market revenues less generation and deviation costs, is defined as

$$\begin{aligned}\pi_j^a(z_j; z_{-j}) &\triangleq \sum_{i \in \mathcal{N}_j} \left(p_i^0 x_{ij} + \mathbb{E}\left(p_i^\omega(y_{ij}^\omega - x_{ij}) - \zeta_{ij}^\omega(y_{ij}^\omega) - (f_p(u_{ij}^\omega) + f_n(v_{ij}^\omega))\right)\right), \\ &= \sum_{i \in \mathcal{N}_j} \underbrace{\mathbb{E}\left(p_i^\omega y_{ij}^\omega - \zeta_{ij}^\omega(y_{ij}^\omega)\right)}_{Mean\,Profit} - \underbrace{\mathbb{E}(f_p(u_{ij}^\omega) + f_n(v_{ij}^\omega))}_{Mean\,deviation\,costs},\end{aligned}$$

where $j = 1, \ldots, g$, z_j is defined as $z_j := \Big(y_j, u_j, v_j, x_j\Big)$ and y_j, u_j, v_j and x_j are given by

$$x_j = (x_{ij})_{i \in \mathcal{N}}, u_j = (u_{ij}^\omega)_{i \in \mathcal{N}, \omega \in \Omega}, v_j = (v_{ij}^\omega)_{i \in \mathcal{N}, \omega \in \Omega}, y_j = (y_{ij}^\omega)_{i \in \mathcal{N}, \omega \in \Omega}, \forall j \in \mathcal{J}.$$

If $\mathcal{J}_i$ denotes the set of firms that have generation at node i and cap_{ij}^ω denotes the capacity of firm i's plant at node j in the second period, then the feasible region of the jth firm's problem is given by $\mathbf{Z}_j \cap \mathcal{D}_j(z_{-j})$ where

$$\mathbf{Z}_j \triangleq \left\{ z_j : \left\{ \begin{array}{l} y_{ij}^\omega = x_{ij} + u_{ij}^\omega - v_{ij}^\omega \\ y_{ij}^\omega \leq cap_{ij}^\omega \\ x_{ij}, u_{ij}^\omega, v_{ij}^\omega, y_{ij}^\omega \geq 0, \end{array} \right\} \quad \forall i \in \mathcal{N}, \forall \omega \in \Omega \right\},$$

$$\text{and} \quad \mathcal{D}_j(z_{-j}) \triangleq \left\{ z_j : \left\{ \sum_{j \in \mathcal{J}_i} y_{ij}^\omega + r_i^\omega \geq 0 \right\}, \forall i \in \mathcal{N}, \forall \omega \in \Omega \right\}, \quad \text{respectively.}$$

In the definition of $\mathbf{Z}_j$, the first set of constraints relate real-time generation to the forward positions through the deviation levels while the second set of constraints impose a bound on real-time generation based on available capacity. The mapping $\mathcal{D}_j(z_{-j})$ specifies that the net outflow at any node is nonnegative.

The ISO maximizes expected social welfare in the spot market subject to network and flow constraints. The network constraints are modeled by means of a DC approximation of Kirchhoff's laws. If Q denotes the power transfer distribution factor matrix and $\bar{\mathcal{N}}$ represents the set of nodes in the network less the slack node, then the feasible set faced by the ISO is given by $\mathbf{Z}_{g+1} \cap \mathcal{D}_{g+1}(z_{-(g+1)})$ where

$$\mathbf{Z}_{g+1} \triangleq \left\{ z_{g+1} : \left\{ \begin{array}{l} \sum_{i\in\mathcal{N}} r_i^\omega = 0 \\ \sum_{i\in\bar{\mathcal{N}}} Q_{l,i} r_i^\omega \leq K_l^\omega \\ \sum_{i\in\bar{\mathcal{N}}} Q_{l,i} r_i^\omega \geq -K_l^\omega, \end{array} \right\} \quad \forall i \in \mathcal{N}, \forall l \in \mathcal{L}, \forall \omega \in \Omega \right\},$$

$$\text{and} \qquad \mathcal{D}_{g+1}(z_{-(g+1)}) \triangleq \left\{ r_i^\omega : \left\{ \sum_{j\in\mathcal{J}_i} y_{ij}^\omega + r_i^\omega \geq 0 \right\}, \forall i \in \mathcal{N}, \forall \omega \in \Omega \right\}, \qquad \text{respectively.}$$

By assumption, injection or withdrawal of power at a slack node does not induce flow on any line in the network. Note that in $\mathbf{Z}_{g+1}$, the first set of constraints are the power balance requirements, while the second and third represent the transmission capacity constraints. The social welfare is given by the expectation of the spot-market revenue less generation cost or

$$\pi_{g+1}^a(z_{g+1}; z_{-(g+1)}) \triangleq \sum_{i\in\mathcal{N}} \mathbb{E}\left(\int_0^{\sum_{j\in\mathcal{J}} y_{ij}^\omega + r_i^\omega} p(\tau)d\tau - \sum_{j\in\mathcal{J}} \zeta_{ij}^\omega(y_{ij}^\omega) \right).$$

The resulting parameterized optimization problem faced by the firms and the ISO is given by

$$\boxed{\begin{array}{lll} \mathrm{Ag}^a(z_{-(j)}) & \text{maximize} & \pi_j^a(z_j; z_{-j}) \\ & \text{subject to} & z_j \in \mathbf{Z}_j \cap \mathcal{D}_j(z_{-j}), \end{array}}$$

where $j = 1, \ldots, g+1$. If $\Pi^a = (\pi_j^a)_{j\in\mathcal{A}}$, $\mathbf{C} = \{\mathbf{Z}_j, D_j\}_{j\in\mathcal{A}}$ and $\mathcal{A}$ is the set of firms and the ISO, the risk-neutral deviation cost generalized Nash game is defined as follows:

Definition 1 (No-arbitrage risk neutral Nash game) *The risk-neutral deviation cost generalized Nash game, denoted by $\mathcal{G}^a$, is given by a triple $(\Pi^a, \mathbf{C}, \mathcal{A})$ and an equilibrium to this game is given by a tuple $\left\{z_j^*\right\}_{j\in\mathcal{A}}$ where z_j^* solves the problem $Ag^a(z_{-j}^*)$ for all $j \in \mathcal{A}$ or*

$$z_j^* \in SOL(Ag^a(z_{-j}^*)), \forall j \in \mathcal{A}.$$

2.2.2 Risk averse market clearing model

Several questions emerge as a consequence of the model suggested in section 2.1. We make two modifications to this model, the first of which pertains to reliability concerns in markets with uncertain generation assets while the second considers the use of an alternate specification of forward prices.

Shortfall in real-time generation capacity is penalized through a deviation cost, implying that the total cost of negative deviation arising for capacity shortfalls provides an estimate of the reliability of the market. For instance, if generators make low forward bids, then the likelihood of real-time shortfall is correspondingly lower. Unfortunately, such a measure of reliability is available upon the settlement of the real-time market, in effect an ex-post measure. Unfortunately, deviation cost models as specified in the earlier section are risk-neutral in that firms minimize the expected cost of deviation. In this subsection, we consider a modified model that replaces deviation costs with a risk measure that incorporates the losses associated with shortfall in real-time generation. Such a modification has several benefits. First, it allows firms to compete with heterogeneous risk preferences where the risk corresponds to the losses associated with capacity shortfall in the real-time market. Second, the risk measure provides an ex-ante measure of reliability of the market.

The second modification pertains to the arbitrage-free model built on the assumption that forward prices are given by expected spot prices. In practice, forward prices are a consequence of a market clearing and need not necessarily match expected spot prices, as discussed by Kamat and Oren [52]. Alternate models of forward pricing [52] point to the non-storability of electricity as being one reason for why the no-arbitrage condition may not hold. In accordance with Kamat and Oren [52], in one of our models, we employ a Cournot-based price function in the forward market. In section 2.2.1, we introduce the risk measure employed within our formulation while in section 2.2.2, we describe the alternate model for specifying forward prices. Section 2.2 concludes with a definition of the risk-based game.

Shortfall risk measures

Current market models discourage deviations from forward positions through the imposition of convex costs on deviations. As a consequence, the firms minimize their expected revenue less their expected cost of generation and deviation. For instance, if $X(\omega; y)$ represents the random loss under realization ω, given forward decision y, then an expected-value approach would address $\min_{y \in Y} \mathbb{E}X(\omega; y)$. However, such a model focuses on the *average* and does not consider the possibility that levels of real-time capacity may result in massive deviation costs. In effect, the expected-value approach does not allow for capturing risk-averseness.

Classical approaches to modeling risk preferences require the use of expected utility theory leading to agents maximizing their expected utility. In particular if $\mathbf{u} : \mathbb{R} \to \mathbb{R}$ is a concave utility function, then a risk-averse firm would maximize $\mathbb{E}(\mathbf{u}(X(\omega); y))$. Unfortunately, eliciting the utility functions of the agents remains rather challenging and often arbitrarily selected utility functions lead to solutions that are difficult to interpret. More recently, an approach for addressing risk aversion is through the use of risk measures. Recently, the value-at-risk

(VaR) measure has gained popularity in the financial industry is defined as

$$\mathrm{VaR}_\alpha(X;y) \triangleq H_X^{-1}(1-\beta),$$

where $H_X(x;y) = \mathbb{P}(X(\omega;y) \le x)$. Unfortunately, the VaR measure does not satisfy the properties of coherence [6]. Additionally, unless the distribution is Gaussian, the VaR measure is nonconvex. Finally, the VaR measure ignores losses beyond the $\mathrm{VaR}_\beta(X)$ level and consequently these can be arbitrarily large. The conditional value-at-risk or CVaR measure is coherent, convex and does consider the expectation of the losses beyond the VaR level and is defined as

$$\mathrm{CVaR}_\tau(X;y) \triangleq \min_{m\in\mathbb{R}} \left\{ m + \frac{1}{1-\tau}\mathbb{E}(X(\omega;y)-m)^+ \right\}, \tag{2.3}$$

where $w^+ = max(w,0)$. In the past, CVaR measures, and more generally coherent risk measures, have been employed in the context of risk management in a power setting [23] as well as an inventory control context [2].

Here, we consider two forms of loss functions $X(\omega;y)$ that are intended to replace the expected deviation costs from forward positions with measures that capture the *risk of shortfall.* These measures have a particular relevance when generation firms have uncertain capacity (as arising from wind-based generation, for instance). In determining the risk of shortfall, we define the non-shared and shared risk measure as

$$\mathrm{CVaR}(cap_i;x_i) \triangleq \begin{cases} m_{ij} + \frac{1}{1-\tau_j}\mathbb{E}(\varrho_{ij}^{NS}(cap_{ij}^\omega;x_{ij}) - m_{ij})^+, & \text{Non-shared measures} \\ m_{ij} + \frac{1}{1-\tau_j}\mathbb{E}(\varrho_{ij}^{S}(cap_{i}^\omega;x_{i}) - m_{ij})^+, & \text{Shared measures} \end{cases}$$

where ϱ_{ij}^{NS} and ϱ_{ij}^{S} denote, non-shared and shared loss functions, $cap_i^\omega = \sum_{j\in\mathcal{J}} cap_{ij}^\omega$, and $x_i = (x_{ij})_{j\in\mathcal{J}}$. Note that the shared-risk measure represent a means for allocating risk to firms when their decisions collectively contribute to the risk at a particular node.

Market clearing model for forward prices

In contrast with more standard arbitrage-free models in which the forward prices are given by expected spot prices, we assume a setting whether forward prices are determined via a market clearing, similar to the way in which spot prices are specified. Specifically, p_i^0 the forward price at node i is given by

$$p_i^0 = a_i^0 - b_i^0(\sum_{j\in\mathcal{J}_i} x_{ij}), \tag{2.4}$$

where a_i^0 and b_i^0 are positive scalars for all $i \in \mathcal{N}$. The resulting profit functions of firm j in the instance of non-shared and shared-risk are given by

$$\pi_j^b(z_j; z_{-j}) \triangleq \sum_{i\in\mathcal{N}} \underbrace{p_i^0 x_{ij} + \mathbb{E}\left(p_i^\omega (y_{ij}^\omega - x_{ij}) - \zeta_{ij}^\omega (y_{ij}^\omega)\right)}_{Mean\,profit} - \kappa_j \underbrace{\mathrm{CVaR}_{\tau_j}(\varrho^{NS}(cap_{ij}^\omega; x_{ij}))}_{Nonshared\,shortfall\,risk}, \tag{2.5}$$

$$\pi_j^c(z_j; z_{-j}) \triangleq \sum_{i\in\mathcal{N}} \underbrace{p_i^0 x_{ij} + \mathbb{E}\left(p_i^\omega (y_{ij}^\omega - x_{ij}) - \zeta_{ij}^\omega (y_{ij}^\omega)\right)}_{Mean\,profit} - \kappa_j \underbrace{\mathrm{CVaR}_{\tau_j}(\varrho^{S}(cap_{i}^\omega; x_{ij}))}_{Shared\,shortfall\,risk}, \tag{2.6}$$

where κ_j represents the risk-aversion parameter of agent j. Note that the profit function of the ISO remains unchanged implying that $\pi_{g+1}^c = \pi_{g+1}^b = \pi_{g+1}^a$. If Π^b and Π^c are defined analogously to Π^a and let Ag^b and Ag^c denote the agent problems for games $\mathcal{G}^b$ and $\mathcal{G}^c$, respectively. Then the risk-based games are defined as follows.

Definition 2 (Risk-based market clearing Nash game) *The generalized Nash games with nonshared and shared risk are denoted by $\mathcal{G}^b$ and $\mathcal{G}^c$, respectively and are given by the triples $(\Pi^b, \mathbf{C}, \mathcal{A})$ and $(\Pi^c, \mathbf{C}, \mathcal{A})$, respectively. Furthermore, an equilibrium to $\mathcal{G}^b$ is given by a tuple $\left\{z_j^*\right\}_{j\in\mathcal{A}}$ where z_j^* solves the problem $Ag^b(z_{-j}^*)$ for all $j \in \mathcal{A}$ and an equilibrium to $\mathcal{G}^c$ is given by a tuple $\left\{z_j^*\right\}_{j\in\mathcal{A}}$ where z_j^* solves the problem $Ag^c(z_{-j}^*)$ for all $j \in \mathcal{A}$.*

2.2.3 Shared-constraint generalized Nash game

The classical Nash solution concept does not allow for an interaction in the strategy sets. Yet in our setting, we observe that the strategy sets are indeed coupled, leading to a generalized Nash game. In general, under suitable convexity and differentiability assumptions, the resulting equilibrium conditions of the shared-constraint Nash game are given by a quasi-variational inequality, an extension of the variational inequality [77, 40]. Recent work by Facchinei et al. [31] has shown that if the strategy sets are coupled through a shared constraint, an equilibrium of the game is given by the solution of an appropriately defined scalar variational inequality. This holds in our setting where the firms and the ISO are coupled through

$$\sum_{j\in\mathcal{J}_i} y_{ij}^\omega + r_i^\omega \geq 0, \forall i \in \mathcal{N}, \forall \omega \in \Omega.$$

The analysis of generalized Nash equilibrium problems with a set of convex shared constraints has been studied recently in [31, 33, 32]. Consider a mapping $\mathbf{F}$ and a set $\mathbf{Z}$ given by[1]

$$\mathbf{F}(z) \triangleq \left(\nabla_{z_j}\pi_j(z_j; z_{-j})\right)_{j=1}^{g+1}, \mathbf{Z} = \left(\prod_{j=1}^{g+1}\mathbf{Z}_j\right) \cap \mathcal{D}, \tag{2.7}$$

$$\mathbf{Z}_j = \left\{ z_j : \left\{ \begin{array}{l} y_{ij}^{\omega} = x_{ij} + u_{ij}^{\omega} - v_{ij}^{\omega} \\ y_{ij}^{\omega} \leq cap_{ij}^{\omega} \\ x_{ij}, u_{ij}^{\omega}, v_{ij}^{\omega}, y_{ij}^{\omega} \geq 0 \end{array} \right\} \quad \forall i \in \mathcal{N}, \forall \omega \in \Omega \right\},$$

$$\mathbf{Z}_{g+1} = \left\{ z_{g+1} : \left\{ \begin{array}{l} \sum_{i \in \mathcal{N}} r_i^{\omega} = 0 \\ \sum_{i \in \mathcal{N}} Q_{l,i} r_i^{\omega} \leq K_l^{\omega} \\ \sum_{i \in \mathcal{N}} Q_{l,i} r_i^{\omega} \geq -K_l^{\omega} \end{array} \right\} \quad \forall i \in \mathcal{N}, \forall l \in \mathcal{L}, \forall \omega \in \Omega \right\},$$

$$\text{and} \quad \mathcal{D} = \left\{ z : \sum_{j \in \mathcal{J}_i} y_{ij}^{\omega} + r_i^{\omega} \geq 0, \forall i \in \mathcal{N}, \forall \omega \in \Omega \right\}.$$

Then the key result in [31] proves that the solvability of VI($\mathbf{Z}, \mathbf{F}$) suffices for ensuring that the original shared-constraint game admits an equilibrium. Recall that VI($\mathbf{Z}, \mathbf{F}$) is defined as the problem of finding a vector $z^* \in \mathbf{Z}$ such that,

$$\mathbf{F}(z^*)^T(z - z^*) \geq 0, \qquad \forall z \in \mathbf{Z}.$$

The equilibrium corresponding to a solution of this variational problem is referred to as the normalized equilibrium [77] or the variational equilibrium [31] (VE) and its relationship to the shared-constraint game is given by the following.

Theorem 3 *Suppose the objective function $\pi_j(z_j; z_{-j})$ is concave and differentiable in z_j for all z_{-j} for all $j \in \mathcal{A}$ and $\mathcal{D}$, $\mathbf{Z}_1, \ldots, \mathbf{Z}_{g+1}$ are closed and convex sets. Then every solution to VI($\mathbf{Z}, \mathbf{F}$) is a solution to the shared-constraint game.*

A similar result is available when π_j can only shown to be continuous for all $j \in \mathcal{A}$; specifically every solution to an appropriately defined multi-valued variational inequality is a solution to the game [33].

2.3 Characterization of equilibria

The analysis of both the single-valued and the multi-valued variational inequalities is our next goal, an obvious motivation being the need to provide existence and uniqueness statements for the variational equilibria. Additionally, such an analysis is relevant from the standpoint of developing convergent schemes. In particular, the convergence of

[1]Note: $y_{ij}^{\omega}, u_{ij}^{\omega}, v_{ij}^{\omega}, s_{ij}^{\omega}, m_{ij}, x_{ij} \equiv 0, \forall i \in G^c, \forall \omega \in \Omega, \forall j \in \mathcal{J}$. Also, the above holds $\forall i \in \mathcal{J}_i^c$.

projection-based methods is intimately tied to the monotonicity of the mapping.

The analysis of the VE rests on the properties of the variational object, denoted in the single-valued settings by the variational inequality denoted by VI($\mathbf{F}, \mathbf{Z}$). When $\mathbf{Z}$ is closed and convex and $\mathbf{F}$ is continuous, compactness of $\mathbf{Z}$ suffices for existence [34]. Similarly, uniqueness follows if $\mathbf{F}$ is strongly monotone over $\mathbf{Z}$, which requires that there exists a $\nu > 0$ such that

$$(\mathbf{F}(x) - \mathbf{F}(y))^T (x - y) \geq \nu \|x - y\|^2, \quad \forall x, y \in \mathbf{Z}.$$

Unfortunately, in the current setting, neither compactness of $\mathbf{Z}$ nor continuity of $\mathbf{F}$ holds. Furthermore, in risk-averse settings, $\mathbf{F}$ fails to even be single-valued. These complications motivate a deeper analysis of VI($\mathbf{Z}, \mathbf{F}$) and represent the core of this section. Note that the analysis of variational inequalities enjoys a long history and an expansive discussion of these topics may be found in [34, Ch.2,3].

In section 3.1, we focus on the no-arbitrage risk-neutral deviation cost game and show that a unique variational equilibrium exists for such a game. While a similar existence result is shown for the risk-averse market-clearing model in section 3.2, a corresponding uniqueness result is shown for an appropriately defined ϵ-Nash equilibrium. Finally, the shared-risk extension leads to a nonsmooth Nash game whose equilibrium conditions are captured by a multivalued variational inequality. In section 3.3, we show that a solution to this variational inequality and its single-valued counterpart exist. Note that the strategy sets $\mathbf{Z}_j \subseteq \mathbb{R}^M$ are closed and convex for all $j = 1, \ldots, g+1$. We state the following assumptions on costs and prices and invoke them when necessary.

Assumption 4

(A1) The cost of generation ζ_{ij}^{ω} is a convex twice-continuously differentiable function of y_{ij}^{ω} for all $i \in \mathcal{N}$, $j \in \mathcal{J}$ and for all $\omega \in \Omega$.

(A2) The nodal spot-market price is defined by the affine price function (2.1) *for all $i \in \mathcal{N}$ and for all $\omega \in \Omega$.*

2.3.1 No-arbitrage risk-neutral game

As mentioned earlier, the existence of a solution to a variational inequality is immediate when either the mapping is strongly monotone or the set is compact. However, existence of a solution may also be deduced by ensuring that a suitable coercivity requirement can be shown to hold. In particular, we have the following from [34]:

Theorem 5 *Let $\mathbf{Z}$ be closed and convex and $\mathbf{F} : \mathbf{Z} \to \mathbb{R}^M$ be a continuous mapping. If there exists a vector $z^{ref} \in \mathbf{Z}$ such that*

$$\liminf_{z \in \mathbf{Z}, \|z\| \to \infty} \mathbf{F}(z)^T (z - z^{ref}) > 0,$$

then the VI($\mathbf{Z}, \mathbf{F}$) has a nonempty compact solution set.

We begin our discussion by showing that for a feasible tuple of decisions, the ISO's decisions lie in a compact set. Here, we denote the set of nodes housing generation facilities by G and its complement by G^c.

Lemma 6 *Consider a tuple $z_1, \ldots, z_{g+1}$ such that $z_j \in \mathbf{Z}_j \cap \mathcal{D}$ for all $j \in \mathcal{J}$. Then the import/export decisions r_i^ω are bounded for all $i \in \mathcal{N}$.*

Proof : Recall that from the feasibility of the tuple, we have $\sum_{i \in \mathcal{J}_i} y_{ij}^\omega + r_i^\omega \geq 0$ for all $i \in \mathcal{N}$. By the feasibility of the real-time generation, we have $y_{ij}^\omega \leq cap_{ij}^\omega$ for all $j \in \mathcal{J}_i, i \in \mathcal{N}$. This implies that, $r_i^\omega \geq -\sum_{j \in \mathcal{J}_i} cap_{ij}^\omega$ at all generation nodes, namely for all $i \in G$. But

$$\sum_{i \in \mathcal{N}} r_i^\omega = 0 \text{ and } r_i^\omega \geq 0, \forall i \in G^c \implies \sum_{i \in G} r_i^\omega + \sum_{i \in G^c} r_i^\omega = 0.$$

This implies that, $r_i^\omega + \sum_{k \in G, k \neq i} r_k^\omega \leq 0, \forall i \in G$. But, $r_k^\omega \geq -\sum_{j \in \mathcal{J}_k} cap_{kj}^\omega, \forall k \in G$. It follows that

$$-\sum_{j \in \mathcal{J}_i} cap_{ij}^\omega \leq r_i^\omega \leq \sum_{k \in G, k \neq i} \sum_{j \in \mathcal{F}} cap_{kj}^\omega, \qquad \forall i \in G.$$

Since the total import at the load nodes cannot be greater than the total capacity and since no export is also possible at these nodes, it follows that $r_i^\omega \geq 0, \forall i \in G^c$. It follows that

$$0 \leq r_i^\omega \leq \sum_{i \in G^c} \sum_{j \in \mathcal{J}_i} cap_{ij}^\omega, \qquad \forall i \in G^c.$$

The boundedness of r_i^ω for all $i \in \mathcal{N}$ can then be concluded. ∎

Using the boundedness of the r_i^ω and y_i^ω, we proceed to show that $\mathcal{G}^a$ admits an equilibrium by proving that VI$(\mathbf{Z}, \mathbf{F})$ satisfies a coercivity property under the additional assumption on the deviation cost functions.

Assumption 7 *(A3) The deviation cost functions $f_p^\omega(u_{ij}^\omega)$ and $f_n^\omega(v_{ij}^\omega)$ are strictly convex, twice continuously differentiable and increasing in u_{ij}^ω and v_{ij}^ω for all $\omega \in \Omega$.*

Proposition 8 (Existence of Nash equilibrium to $\mathcal{G}^a$) *Consider the game $\mathcal{G}^a$ and let assumptions (A1)–(A3) hold. Then $\mathcal{G}^a$ has a nonempty compact set of equilibria.*

Proof : Based on Theorem 3, it suffices to prove the existence of a solution to VI$(\mathbf{Z}, \mathbf{F})$. By Lemma 41, this variational inequality is solvable if there exists a $z^{\text{ref}} \in \mathbf{Z}$ such that the expression in Theorem 41 is satisfied.

First, it is observed that $\mathbf{0} \in \mathbf{Z}$ and z^{ref} is chosen to be $z^{\text{ref}} \triangleq \mathbf{0}$. It suffices to show that Theorem 41 holds. By our choice of z^{ref}, the term $\mathbf{F}(z)^T(z)$ can be written as

$$\begin{aligned}
\mathbf{F}(z)^T(z) \quad = \quad & \underbrace{\sum_{\omega\in\Omega}\sum_{i\in N}\rho^\omega(-a_i^\omega + b_i^\omega(\sum_{j\in e\mathcal{J}_i} y_{ij}^\omega + r_i^\omega))r_i^\omega}_{term1\,(or)\,(\mathbf{F}_r(z))^T(r)} + \underbrace{\sum_{j\in\mathcal{J}}\sum_{\omega\in\Omega}\sum_{i\in\mathcal{N}_j}\rho^\omega(f_p'(u_{ij}^\omega)u_{ij}^\omega + f_n'(v_{ij}^\omega)v_{ij}^\omega)}_{term\,2\,(or)\,\sum_{j\in\mathcal{J}}\mathbf{F}_{u_j}(z)^T(u_j)+\mathbf{F}_{v_j}(z)^T(v_j)} \\
+ \quad & \underbrace{\sum_{j\in\mathcal{J}}\sum_{\omega\in\Omega}\sum_{i\in\mathcal{N}_j}\rho^\omega(-a_i^\omega + \zeta_{ij}^{'\,\omega}(y_{ij}^\omega) + b_i^\omega y_{ij}^\omega + b_i^\omega(\sum_{j\in\mathcal{J}} y_{ij}^\omega + r_i^\omega))y_{ij}^\omega}_{term\,3\,(or)\,\sum_{j\in\mathcal{J}}\mathbf{F}_{y_j}(z)^T(y_j)}.
\end{aligned}$$

Consider any sequence $\{z^k\} \in \mathbf{Z}$ such that $\lim_{k\to\infty}\|z^k\| = \infty$. Along any such sequence, the feasibility of y^k with respect to the capacity constraint implies that y^k stays bounded. By Lemma 6, r^k stays bounded as well implying that along any sequence term 3 stays bounded. It that one of x^k, u^k or v^k tend to infinity and suffices to consider terms 1 and 2 through the following two cases.

Case 1: Suppose $x^k \to \infty$. Since y^k is bounded, either v_k or both v_k and u_k are growing to infinity. This ensures that term 2 tends to $+\infty$.

Case 2: Suppose the positive deviation u^k or the negative deviation v^k or both tend to infinity. Consequently, term 2 tends to $+\infty$. This completes the proof. ∎

It remains to show that the VE corresponding to the solution of VI($\mathbf{Z}, \mathbf{F}$) is unique. Note that this does not extend to claiming that the original generalized Nash game has a unique solution but merely allows us to claim that the variational equilibrium is unique.

Proposition 9 (Uniqueness of variational Nash equilibrium to $\mathcal{G}^a$) *Consider the game $\mathcal{G}^a$ and let assumptions (A1)–(A3) hold. Then the VI($\mathbf{Z}, \mathbf{F}$) corresponding to $\mathcal{G}^a$ has a unique solution.*

Proof : We have proved that VI($\mathbf{Z}, \mathbf{F}$) is solvable. It suffices to show that the gradient mapping $\nabla\mathbf{F}$ is strictly monotone definite implying that the variational inequality has at most one solution. Since the player objectives are nodally decomposable, $\nabla\mathbf{F}$ and $\nabla_i\mathbf{F}_i$ are given by

$$\nabla\mathbf{F} = \begin{pmatrix} \nabla_1\mathbf{F}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \nabla_2\mathbf{F}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \nabla_N\mathbf{F}_N \end{pmatrix} \text{ and } \nabla\mathbf{F}_i = \begin{pmatrix} A_i^1 & \dots & 0 & C_i^1 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & A_i^n & C_i^n & \vdots \\ B_i^1 & \dots & B_i^n & D_i & 0 \\ 0 & \dots & \dots & 0 & E_i \end{pmatrix},$$

where $\nabla_i \mathbf{F}_i$ is the gradient of the $\mathbf{F}_i$ with respect to the variables corresponding to node i and the submatrices of $\nabla_i \mathbf{F}_i$ are specifed as follows:

$$A_i^\omega = \begin{pmatrix} \rho^\omega(2b_i^\omega + d_{i1}^\omega) & \dots & \rho^\omega b_i^\omega \\ \vdots & \ddots & \vdots \\ \rho^\omega b_i^\omega & \dots & \rho^\omega(2b_i^\omega + d_{ig}^\omega) \end{pmatrix}, \forall \omega \in \Omega, C_i^1 = \begin{pmatrix} \rho^1 b_i^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \rho^1 b_i^1 & \dots & 0 \end{pmatrix}, C_i^n = \begin{pmatrix} 0 & \dots & \rho^n b_i^n \\ \vdots & \ddots & \vdots \\ 0 & \dots & \rho^n b_i^n \end{pmatrix},$$

$$E_i = \begin{pmatrix} E_i^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & E_i^J \end{pmatrix}, E_i^j = \begin{pmatrix} f_p''(u_{ij}^1) & 0 & \dots & \dots & 0 \\ 0 & f_n''(v_{ij}^1) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & f_p''(u_{ij}^n) & 0 \\ 0 & \dots & \dots & 0 & f_n''(v_{ij}^n) \end{pmatrix}, \forall j \in \mathcal{J}$$

$$B_i^1 = \begin{pmatrix} \rho^1 b_i^1 & \dots & \rho^1 b_i^1 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}, B_i^n = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \rho^n b_i^n & \dots & \rho^n b_i^n \end{pmatrix}, \text{and } D_i = \begin{pmatrix} \rho^1 b_i^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \rho^n b_i^n \end{pmatrix}.$$

Note that the gradient map $\nabla_f \mathbf{F}_i$ is defined in the order y, r, u and u respectively. From the strict convexity of the deviation penalties, the matrix H is positive definite. It suffices to show that the submatrix

$$\nabla \bar{F}_i = \begin{pmatrix} A_i^1 & \dots & 0 & C_i^1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & A_i^n & C_i^n \\ B_i^1 & \dots & B_i^n & D_i \end{pmatrix}$$

is positive definite which follows if $s^T \nabla \bar{F}_i s > 0$ for all $s \neq 0$. Specifically, for all $i \in G$, we have

$$\begin{aligned} s^T \nabla \bar{F}_i s &= \sum_{\omega=1}^n \rho^\omega b_i^\omega \left(\sum_{k=1}^g s_{(\omega-1)g+k} \right)^2 + \sum_{\omega=1}^n \rho^\omega \sum_{k=1}^g (b_i^\omega + d_{ik}^\omega) s_{(\omega-1)g+k}^2 \\ &+ \sum_{\omega=1}^n \rho^\omega b_i^\omega s_{ng+\omega} \sum_{k=1}^g s_{(\omega-1)g+k} + \sum_{\omega=1}^n \rho^\omega b_i^\omega s_{ng+\omega} \sum_{k=1}^J s_{(\omega-1)g+k} + \sum_{\omega=1}^n \rho^\omega b_i^\omega s_{ng+\omega}^2 \\ &= \sum_{\omega=1}^n \rho^\omega b_i^\omega \left(\left(\sum_{k=1}^g s_{(\omega-1)g+k} \right) + s_{ng+\omega} \right)^2 + \sum_{\omega=1}^n \rho^\omega \sum_{k=1}^g (b_i^\omega + d_{ik}^\omega) s_{(\omega-1)g+k}^2 > 0. \end{aligned}$$

When i is a load node, namely $i \in G^c$, then ∇F_i consists of just D_i, which is positive definite. It follows that $\nabla \mathbf{F}$ is positive definite completing the proof. But this implies that $\mathbf{F}$ is a strictly monotone mapping and at most one VE exists. The required uniqueness result can be concluded by the earlier existence result. ∎

2.3.2 Risk-based market-clearing Nash game

Next, we consider the game denoted by $\mathcal{G}^b$. Invoking the definition of the conditional value at risk, we can reformulate the nonsmooth firm problem as a smooth convex program by the addition of a set of convex constraints, each corresponding to one realization of uncertainty. Effectively the problem for agent $j \in \mathcal{J}$, we have

$$\begin{array}{lll} \mathrm{Ag}^b(z_{-j}) & \text{maximize} & \displaystyle\sum_{i \in G} \left(\pi_{ij}(x_{ij}) + \mathbb{E}(\pi_{ij}^{\omega}(y_{ij}^{\omega}; r_i^{\omega})) - \kappa_j(m_{ij} + \sum_{\omega \in \Omega} \rho^{\omega} \frac{s_{ij}^{\omega}}{1-\tau_j}) \right) \\ & \text{subject to} & \left\{ \begin{array}{l} y_{ij}^{\omega} = x_{ij} + u_{ij}^{\omega} - v_{ij}^{\omega} \\ y_{ij}^{\omega} \leq cap_{ij}^{\omega} \\ s_{ij}^{\omega} \geq \varrho_{ij}(x_{ij}, cap_{ij}^{\omega}) - m_{ij} \\ \sum_{j \in \mathcal{J}_i} y_{ij}^{\omega} + r_i^{\omega} \geq 0 \\ x_{ij}, u_{ij}^{\omega}, v_{ij}^{\omega}, y_{ij}^{\omega}, s_{ij}^{\omega} \geq 0 \end{array} \right\}, \forall i \in \mathcal{N} \quad \forall \omega \in \Omega. \end{array}$$

Based on the redefinition of the agent problems, in this subsection, the mapping $\mathbf{F}$ is appropriately redefined to include the gradients of s_{ij}^{ω} and m_{ij}. Similarly, $\mathbf{Z}_j$ is extended to account for s_{ij}^{ω} and m_{ij}. The characterization of equilibria to $\mathcal{G}^b$ requires the following assumption on the loss function as well as a relationship between the slopes of the real-time and forward-market price functions.

Assumption 10 *(A4) The loss function $\varrho_{ij}(x_{ij}, cap_{ij}^{\omega})$ is convex and increasing in x_{ij} and $\mathbb{E}b_i^{\omega} \leq 4b_i^0$ for all $i \in \mathcal{N}$.*

The above assumption ensures the convexity of the problem and allows for showing that $\mathcal{G}^b$ admits an equilibrium. We begin by proving an intermediate result that shows that the objective function is convex under a mild assumption on the slopes of the price functions.

Lemma 11 *Suppose assumptions (A1)–(A2), (A4) hold. Then the objective functions of the firms and the ISO are concave.*

Proof : It suffices to prove the convexity of the expectation term of every agent's objective, given by $\eta_{ij}(x_{ij}, y_{ij}; y_{i,-j})$, defined as

$$\eta_{ij}(x_{ij}, y_{ij}; x_{i,-j}, y_{i,-j}) = -(a_i^0 - b_i^0 \sum_{j \in \mathcal{J}} x_{ij})x_{ij} - \sum_{\omega \in \Omega} \rho^{\omega}(a_i^{\omega} - b_i^{\omega}(\sum_{j \in \mathcal{J}} y_{ij}^{\omega} + r_i^{\omega}))(y_{ij}^{\omega} - x_{ij}).$$

The gradient and Hessian of this function are given by

$$\nabla \eta_{ij} = \begin{pmatrix} b_i^0 x_{ij} + b_i^0 \sum_{j\in\mathcal{J}} x_{ij} - a_i^0 + \sum_{\omega\in\Omega} \rho^\omega a_i^\omega - \sum_{\omega\in\Omega} \rho^\omega b_i^\omega (\sum_{j\in\mathcal{J}} y_{ij}^\omega + r_i^\omega) \\ \rho^\omega(-a_i^1 + b_i^1(y_{ij}^1 + \sum_{j\in\mathcal{J}} y_{ij}^1) + b_i^1 r_i^1 - b_i^1 x_{ij}) \\ \vdots \\ \rho^n(-a_i^n + b_i^n(y_{ij}^n + \sum_{j\in\mathcal{J}} y_{ij}^n) + b_i^n r_i^n - b_i^n x_{ij}) \end{pmatrix},$$

$$\text{and } \nabla^2 \eta_{ij} = \begin{pmatrix} 2b_i^0 & -\rho^1 b_i^1 & \dots & -\rho^n b_i^n \\ -\rho^1 b_i^1 & 2\rho^1 b_i^1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\rho^n b_i^n & 0 & \dots & 2\rho^n b_i^n \end{pmatrix}, \text{respectively.}$$

Let s be an arbitrary nonzero vector. Then by adding and subtracting terms, we have

$$\begin{aligned} s^T \nabla^2 \eta_{ij} s &= 2b_i^0 s_1^2 - 2s_1 \sum_{\omega=1}^n \rho^\omega b_i^\omega s_{\omega+1} + 2\sum_{\omega=1}^n \rho^\omega b_i^\omega s_{\omega+1}^2 \\ &= (2b_i^0 - \sum_{\omega=1}^n \rho^\omega \frac{b_i^\omega}{2}) s_1^2 + \sum_{\omega=1}^n \rho^\omega \frac{b_i^\omega}{2} s_1^2 - 2s_1 \sum_{\omega=1}^n \rho^\omega b_i^\omega s_{\omega+1} + 2\sum_{\omega=1}^n \rho^\omega b_i^\omega s_{\omega+1}^2 \\ &= (2b_i^0 - \sum_{\omega=1}^n \rho^\omega \frac{b_i^\omega}{2}) s_1^2 + \sum_{\omega=1}^n \rho^\omega b_i^\omega \left(\frac{s_1}{\sqrt{2}} - \sqrt{2} s_{\omega+1} \right)^2. \end{aligned}$$

By assumption $\mathbb{E}(b_i^\omega) \le 4b_i^0$ implying that $s^T \nabla^2 \eta_{ij} s > 0$ for all nonzero s and $\eta_{ij}(x_{ij}, y_{ij}; y_{i,-j})$ is a strictly convex function in x_{ij} and y_{ij} for all $x_{i,-j}$ and $y_{i,-j}$. The convexity of π_j in z_j follows from recalling that the generation costs and the conditional value at risk (CVaR) measure are known to be convex. ∎

Proposition 12 (Existence of a Nash equilibrium to $\mathcal{G}^b$) *Consider the nonshared risk-based game $\mathcal{G}^b$ and let assumptions (A1)–(A2),(A4) hold. Then $\mathcal{G}^b$ admits a nonempty compact set of equilibria.*

Proof : Based on Theorem 3, it suffices to prove the existence of a solution to VI($\mathbf{Z}, \mathbf{F}$). By Theorem 41, this variational inequality is solvable if there exists a $z^{\text{ref}} \in \mathbf{Z}$ such that the expression in Theorem 41 holds. If we set $(s_{ij}^\omega)^{\text{ref}} \triangleq \varrho_{ij}(0, cap_{ij}^\omega)$, and $x^{\text{ref}}, y^{\text{ref}}, r^{\text{ref}}, m^{\text{ref}} \triangleq 0$, then $z^{\text{ref}} \in \mathbf{Z}$. By our choice of z^{ref}, the term $\mathbf{F}(z)^T(z)$ can be written as:

$$
\begin{aligned}
\mathbf{F}(z)^T(z) \quad &= \quad \underbrace{\sum_{\omega\in\Omega}\sum_{i\in\mathcal{N}}\rho^\omega(-a_i^\omega + b_i^\omega(\sum_{j\in\mathcal{J}} y_{ij}^\omega + r_i^\omega))r_i^\omega}_{(\mathbf{F}_r(z))^T(r)} + \sum_{j\in\mathcal{J}}\kappa_j \underbrace{\sum_{i\in\mathcal{N}_j}\sum_{\omega\in\Omega}\rho^\omega(\frac{s_{ij}^\omega - (s_{ij}^\omega)^{ref}}{1-\tau} + m_{ij})}_{(\mathbf{F}_{s_j}(z))^T(s_j) + (\mathbf{F}_{m_j}(z))^T(m_j)}\,. \\
&+ \quad \sum_{j\in\mathcal{J}} \underbrace{\sum_{\omega\in\Omega}\sum_{i\in\mathcal{N}_j}\rho^\omega(-a_i^\omega + \frac{\partial\zeta_{ij}{}^\omega}{\partial y_{ij}^\omega} + b_i^\omega y_{ij}^\omega + b_i^\omega(\sum_{j\in\mathcal{J}} y_{ij}^\omega + r_i^\omega) - b_i^\omega x_{ij})y_{ij}^\omega}_{\mathbf{F}_{y_j}(z)^T(y_j)} \\
&+ \quad \sum_{j\in\mathcal{J}} \underbrace{\sum_{i\in\mathcal{N}_j}(b_i^0 x_{ij} + b_i^0\sum_{j\in\mathcal{J}} x_{ij} - a_i^0 + \sum_{\omega\in\Omega}\rho^\omega a_i^\omega - \sum_{\omega\in\Omega}\rho^\omega b_i^\omega(\sum_{j\in\mathcal{J}} y_{ij}^\omega + r_i^\omega))x_{ij}}_{\mathbf{F}_{x_j}(z)^T(x_j)}\,.
\end{aligned}
$$

The term $\mathbf{F}(z)^T z$ may be rewritten as

$$
\begin{aligned}
\mathbf{F}(z)^T(z) \quad &= \quad \underbrace{\sum_{j\in\mathcal{J}}\sum_{\omega\in\Omega}\sum_{i\in\mathcal{N}_j}\rho^\omega(-a_i^\omega + \frac{\partial\zeta_{ij}{}^\omega}{\partial y_{ij}^\omega} + b_i^\omega y_{ij}^\omega + b_i^\omega(\sum_{j\in\mathcal{J}} y_{ij}^\omega + r_i^\omega))y_{ij}^\omega}_{term\,1} \\
&+ \quad \underbrace{\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{N}_j}(b_i^0 x_{ij} + b_i^0\sum_{j\in\mathcal{J}} x_{ij} - a_i^0 + \sum_{\omega\in\Omega}\rho^\omega a_i^\omega - \sum_{\omega\in\Omega}\rho^\omega b_i^\omega(\sum_{j\in\mathcal{J}} 2y_{ij}^\omega + r_i^\omega))x_{ij}}_{term\,2} \\
&+ \quad \underbrace{\sum_{\omega\in\Omega}\sum_{i\in\mathcal{N}}\rho^\omega(-a_i^\omega + b_i^\omega(\sum_{j\in\mathcal{J}} y_{ij}^\omega + r_i^\omega))r_i^\omega}_{term\,3} + \underbrace{\sum_{j\in\mathcal{J}}\kappa_j\sum_{i\in\mathcal{N}_j}\sum_{\omega\in\Omega}\rho^\omega(\frac{s_{ij}^\omega - (s_{ij}^\omega)^{\mathrm{ref}}}{1-\tau} + m_{ij})}_{term\,4}\,.
\end{aligned}
\tag{2.8}
$$

From Lemma 6, we may conclude that terms 1 and 2 are bounded for any sequence, $\{z_k\}$, such that $\|z_k\| \to \infty$. It follows that one of the sequences $\{\|x_k\|\}, \{\|s_k\|\}$ and $\{|m_k|\}$ are tending to $+\infty$.[2]

Case 1: Suppose the forward generation bid x^k tends to infinity implying that term 2 tends to $+\infty$ at a quadratic rate.

Case 2: Suppose either (or both) s^k or $|m^k|$ tend to $+\infty$. $s^k \in \mathbf{Z}$, $m^k \in \mathbf{Z}$, $s^k \geq 0$ and $s^k + m^k$ is bounded from below. If, m^k tends to $-\infty$, then s^k tends to $+\infty$. Hence, term 4 grows to $+\infty$. If m^k or s^k tend to $+\infty$, then term 4 tends to $+\infty$. [3]

Case 3: Suppose x^k tends to $+\infty$ and any combination of s^k, and $|m^k|$ tends to $+\infty$. If m^k alone tends to $-\infty$, then term 4 tends to $-\infty$ and term 2 tends to $+\infty$ at a quadratic rate. Consequently, the entire sum tends to $+\infty$.

[2]In this model we do not consider deviation penalties and terms u and v are automatically dropped.
[3]Here s^{ref} is finite because *cap* is finite and $\varrho_{ij}(0, cap_{ij}^\omega)$ is also finite.

If s^k tends to $+\infty$, m^k reduces to $-\infty$ and x^k tends to $+\infty$ then Cases 1 and 3 can be used in conjunction. The other possibilities lead to immediate results of the sequence tending to $+\infty$.

Consider any sequence $\{z^k\} \in \mathbf{Z}$ such that $\lim_{k\to\infty} \|z^k\| = \infty$. Since none of the terms tend to $-\infty$ and at least one of the terms tend to ∞, it follows that

$$\liminf_{z\in\mathbf{Z},\|z\|\longrightarrow\infty} \mathbf{F}(z)^T(z) = \infty.$$

This completes the proof. ∎

A uniqueness result rests on being able to show that the mapping is strictly monotone. However, in the current setting, the mapping arising from the nonshared risk-based game can only be shown to be monotone, as the next result shows. As a consequence, the regularized game, denoted by $\mathcal{G}^b_\epsilon$ is solvable. This requires showing $\nabla\mathbf{F}$, given by

$$\nabla\mathbf{F} = \begin{pmatrix} \nabla_1\mathbf{F}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \nabla_2\mathbf{F}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \nabla_N\mathbf{F}_N \end{pmatrix},$$

is positive semidefinite, where $\nabla_i F_i$ represents the gradient mapping (in the order $x_i, y_i, r_i, s_i, m_i, u_i$ and v_i) with respect to the nodal variables corresponding to node i. The matrix $\nabla F_i, \forall i \in G$ is given by

$$\nabla\mathbf{F}_i = \begin{pmatrix} P_i^0 & P_i^1 & \dots & P_i^n & H_i & 0 \\ R_i^1 & S_i^1 & \dots & 0 & F_i^1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ R_i^n & 0 & \dots & S_i^n & F_i^n & 0 \\ 0 & T_i^1 & \dots & T_i^n & K_i & 0 \\ 0 & 0 & \dots & \dots & 0 & V_i \end{pmatrix}, \text{ where } P_i^\omega = \begin{pmatrix} -\rho^\omega b_i^\omega & \dots & -\rho^\omega b_i^\omega \\ \vdots & \ddots & \vdots \\ -\rho^\omega b_i^\omega & \dots & -\rho^\omega b_i^\omega \end{pmatrix},$$

$$R_i^\omega = \begin{pmatrix} -\rho^\omega b_i^\omega & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & -\rho^\omega b_i^\omega \end{pmatrix}, S_i^\omega = \begin{pmatrix} \rho^\omega(2b_i^\omega + d_{i1}^\omega) & \dots & \rho^\omega b_i^\omega \\ \vdots & \ddots & \vdots \\ \rho^\omega b_i^\omega & \dots & \rho^\omega(2b_i^\omega + d_{ig}^\omega) \end{pmatrix}, \forall\omega \in \Omega$$

$$P_i^0 = \begin{pmatrix} 2b_i^0 & \dots & b_i^0 \\ \vdots & \ddots & \vdots \\ b_i^0 & \dots & 2b_i^0 \end{pmatrix}, H_i = \begin{pmatrix} -\rho^1 b_i^1 & \dots & -\rho^n b_i^n \\ \vdots & \ddots & \vdots \\ -\rho^1 b_i^1 & \dots & -\rho^n b_i^n \end{pmatrix}, K_i = \begin{pmatrix} \rho^1 b_i^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \rho^n b_i^n \end{pmatrix}, V_i = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

$$F_i^1 = \begin{pmatrix} \rho^1 b_i^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \rho^1 b_i^1 & \dots & 0 \end{pmatrix}, F_i^n = \begin{pmatrix} 0 & \dots & \rho^n b_i^n \\ \vdots & \ddots & \vdots \\ 0 & \dots & \rho^n b_i^n \end{pmatrix}, T_i^1 = \begin{pmatrix} \rho^1 b_i^1 & \dots & \rho^1 b_i^1 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{pmatrix}, T_i^n = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ \rho^n b_i^n & \dots & \rho^n b_i^n \end{pmatrix}.$$

Note that V_i refers to the zero matrix representing the second order derivatives with respect to s, m, u and v. The uniqueness of the VE corresponding to the regularized game can then be shown.

Proposition 13 (Uniqueness of variational Nash equilibrium to $\mathcal{G}_\epsilon^b$) *Consider the nonshared risk-based game $\mathcal{G}^b$ and let (A1)–(A2), (A4) hold. Then the resulting mapping $\mathbf{F}(z)$ is monotone over $\mathbf{Z}$. Furthermore $\mathcal{G}_\epsilon^b$ has a unique solution.*

Proof : Since, the matrix V_i is a zero matrix, it suffices to show that principal submatrix of ∇F_i, without the last row and column corresponding to V_i, is positive semidefinite. If $\hat{\mathbf{F}}$ represents this mapping in the reduced space, it suffices to show that for all $s \neq 0$ we have $s^T \nabla \hat{\mathbf{F}}_i s > 0$ where $s^T \nabla \hat{\mathbf{F}}_i s$ is given by

$$\begin{aligned} s^T \nabla \hat{\mathbf{F}}_i s &= b_i^0 \sum_{k=1}^g s_k^2 + b_i^0 (\sum_{k=1}^g s_k)^2 - \sum_{\omega=1}^n \rho^\omega b_i^\omega \sum_{k=1}^g s_k s_{\omega g+k} \\ &- \sum_{\omega=1}^n \rho^\omega b_i^\omega \sum_{k=1}^g s_k \sum_{k=1}^g s_{\omega g+k} + \sum_{\omega=1}^n \rho^\omega b_i^\omega (\sum_{k=1}^g s_{\omega g+k}^2 + (\sum_{k=1}^g s_{\omega g+k})^2) + \sum_{\omega=1}^n \rho^\omega (\sum_{k=1}^g d_{ik}^\omega s_{\omega g+k}^2) \end{aligned}$$

$$\begin{aligned} &+ \sum_{\omega=1}^n \rho^\omega b_i^\omega \left(s_{(n+1)g+\omega} \sum_{k=1}^g s_{\omega g+k} \right) - \sum_{\omega=1}^n \rho^\omega b_i^\omega \left(s_{(n+1)g+\omega} \sum_{k=1}^g s_k \right) \\ &+ \sum_{\omega=1}^n \rho^\omega b_i^\omega \left(s_{(n+1)g+\omega} \sum_{k=1}^g s_{\omega g+k} \right) + \sum_{\omega=1}^n \rho^\omega b_i^\omega \left(s_{(n+1)g+\omega} \right)^2. \end{aligned}$$

Adding and subtracting terms, the right-hand side is given by

$$\begin{aligned} s^T\nabla\hat{\mathbf{F}}_i s &= (b_i^0 - \sum_{\omega=1}^n \frac{\rho^\omega b_i^\omega}{4})\sum_{k=1}^g s_k^2 + (b_i^0 - \sum_{\omega=1}^n \frac{\rho^\omega b_i^\omega}{4})(\sum_{k=1}^g s_k)^2 - \sum_{\omega=1}^n \rho^\omega b_i^\omega \sum_{k=1}^g s_k s_{\omega g+k} \\ &+ \sum_{\omega=1}^n \frac{\rho^\omega b_i^\omega}{4}\sum_{k=1}^g s_k^2 + \sum_{\omega=1}^n \frac{\rho^\omega b_i^\omega}{4}(\sum_{k=1}^g s_k)^2 - \sum_{\omega=1}^n \rho^\omega b_i^\omega \sum_{k=1}^g s_k \sum_{k=1}^g s_{\omega g+k} \\ &+ \sum_{\omega=1}^n \rho^\omega b_i^\omega (\sum_{k=1}^g s_{\omega g+k}^2 + (\sum_{k=1}^g s_{\omega g+k})^2) + \sum_{\omega=1}^n \rho^\omega (\sum_{k=1}^g d_{ik}^\omega s_{\omega g+k}^2) \\ &+ 2\sum_{\omega=1}^n \rho^\omega b_i^\omega \left(s_{(n+1)g+\omega} \sum_{k=1}^g s_{\omega g+k} \right) - \sum_{\omega=1}^n \rho^\omega b_i^\omega \left(s_{(n+1)g+\omega} \sum_{k=1}^g s_k \right) + \sum_{\omega=1}^n \rho^\omega b_i^\omega \left(s_{(n+1)g+\omega}\right)^2 . \end{aligned}$$

On rearranging, $s^T\nabla\hat{\mathbf{F}}_i s$ is given by

$$\begin{aligned} &(b_i^0 - \sum_{\omega=1}^n \frac{\rho^\omega b_i^\omega}{4})\sum_{k=1}^g s_k^2 + (b_i^0 - \sum_{\omega=1}^n \frac{\rho^\omega b_i^\omega}{4})(\sum_{k=1}^g s_k)^2 + \sum_{\omega=1}^n \left(\rho^\omega b_i^\omega \sum_{k=1}^g \left(\frac{s_k}{2} - s_{\omega g+k} \right)^2 \right) \\ &+ \sum_{\omega=1}^n \rho^\omega (\sum_{k=1}^g d_{ik}^\omega s_{\omega g+k}^2) + \sum_{\omega=1}^n \rho^\omega b_i^\omega \left(s_{(n+1)g+\omega} + \sum_{k=1}^g s_{\omega g+k} - \sum_{k=1}^g \frac{s_k}{2} \right)^2 . \end{aligned}$$

Since $4b_i^0 \geq \mathbb{E} b_i^\omega$ holds by assumption, it follows that $s^T\nabla\hat{\mathbf{F}}_i s \geq 0$ for all $i \in G$ implying that ∇F_i is also positive semidefinite for all $i \in G$. The gradient mapping for all $i \in G^c$ is given by a mapping with all zeros except for the block K_i that is positive semi-definite. Since the gradient mappings corresponding to the load nodes are positive semidefinite, the positive semidefiniteness of the entire gradient mapping $\nabla\mathbf{F}$ follows. Consequently, $\mathbf{F}$ is a monotone mapping and its regularization, namely $\mathbf{F}_\epsilon = \mathbf{F} + \epsilon\mathbf{I}$, is a strongly monotone mapping. It follows that a unique solution to VI$(\mathbf{Z}, \mathbf{F}_\epsilon)$ exists, allowing us to conclude a unique ϵ-Nash equilibrium exists. ∎

Before proceeding to discuss a more general class of risk-based games, it is worth commenting on whether the uniqueness result may be strengthened. In the above result, $\mathbf{F}$ is a continuous monotone map, implying that it is a continuous $\mathbf{P}_0$ map. If a solution to VI$(\mathbf{Z}, \mathbf{F})$ is shown to be isolated, then uniqueness follows [34, Th. 3.6.6]. This requires utilizing the structure of $\mathbf{F}$ more carefully and is considered in future work.

2.3.3 Risk-based market clearing game with shared risk

When the risk measure is parameterized by competitive decisions, one cannot replace the measure by a set of linear constraints. A consequence of adopting such a direction is that it complicates the coupling between the strategy sets; specifically, it ceases to have the attractive shared property preventing us from using a variational inequality for purposes of analysis and have to resort to the less tractable quasi-variational inequality. Instead, we retain the nonsmooth risk measure in the objective and attempt to show that the resulting multivalued variational inequality is solvable. Furthermore, we prove that a smoothed counterpart is solvable, the latter being used for purposes of computation.

We begin by noting that the risk-measure represents the minimal value of the optimization problem

$$\mathrm{CVaR}_{\tau_j}(\varrho_{ij}^S) \triangleq \min_{m_{ij}\in\mathbb{R}} \left\{ m_{ij} + \frac{1}{1-\tau_i}\max(\varrho_{ij}^S - m_{ij}, 0)\right\},$$

where the dependence of ϱ_{ij} on x_{ij} is suppressed. We may then consider the game as being one over the larger space given by $\mathbf{Z}\times\mathbb{R}^{\bar{g}}$, where $\bar{g} = \sum_{j\in\mathcal{J}} |\mathcal{N}_j|$, where $\mathbf{Z}$ is as defined in (2.7). The corresponding multivalued variational inequality arising from the equilibrium conditions of the nonsmooth game is given by

$$\partial_z \Pi^c(z^*, m^*)^T(z - z^*) + \partial_m \Pi^c(z^*, m^*)^T(m - m^*) \geq 0 \quad \forall (z\times m) \in \mathbf{Z}\times\mathbb{R}^{\bar{g}}, \tag{2.9}$$

where $\partial_z \Pi^c(z,m)$ and $\partial_m \Pi^c(z,m)$ are given by

$$\partial_z \Pi^c(z,m) \triangleq \prod_{j=1}^{g+1} \partial_{z_j}\pi_j^c(z_j, m_j; z_{-i}) \text{ and } \partial_m \Pi^c(z,m) \triangleq \prod_{j=1}^{g} \partial_{m_j}\pi_j^c(z_j, m_j; z_{-j}),$$

respectively. Furthermore, we have that $\partial_{ij}\pi_j = \partial_{z_{ij}}\pi_j \times \partial_{m_{ij}}\pi_j$. The generalized Clarke gradient $\partial_{ij}\pi_i$ is defined as

$$\begin{aligned}\partial_{ij}\pi_j^c(z_j, m_j; z_{-j}) &= \partial_{ij}\left(p_i^0 x_{ij} + \mathbb{E}\left(p_i^\omega(y_{ij}^\omega - x_{ij}) - \zeta_{ij}^\omega(y_{ij}^\omega)\right) - \kappa_j \mathbb{E}\left(\underbrace{m_{ij} + \frac{1}{1-\tau}\max(\varrho_{ij}^S - m_{ij}, 0)}_{\upsilon_{ij}^\omega} \right)\right) \\ &= \nabla_{ij}\left(p_i^0 x_{ij} + \mathbb{E}\left(p_i^\omega(y_{ij}^\omega - x_{ij}) - \zeta_{ij}^\omega(y_{ij}^\omega)\right)\right) - \kappa_i \mathbb{E}\partial_{ij}\upsilon_{ij}^\omega\end{aligned}$$

where the interchange of the expectation and derivative follows immediately since the sample-space is finite and the summation is finite. Note that the equality holds in the second expression since all the terms except one are continuously differentiable. Finally, it is recalled from nonsmooth Clarke calculus [21] that

$$\partial_{x_{ij}}\upsilon_{ij}^\omega = \partial_{x_{ij}}\left(\max(\varrho_{ij}^S - m_{ij}, 0)\right) = \begin{cases} (\varrho_{ij}^S)', & \text{if } \varrho_{ij}^\omega - m_{ij} > 0, \\ \mathrm{conv}\{(\varrho_{ij}^S)', 0\} & \text{if } \varrho_{ij}^\omega - m_{ij} = 0, \\ 0, & \text{if } \varrho_{ij}^\omega - m_{ij} < 0, \end{cases}$$

In our model, we employ a shared risk measure in which ϱ^S is defined as

$$\varrho_{ij}^S(cap_i^\omega; x_{ij}) \triangleq \varsigma_{ij} x_{ij} \left(\frac{\sum_{j \in \mathcal{J}} x_{ij}}{cap_i^\omega} + \psi \right)^t, \tag{2.10}$$

where ς_{ij} is a strictly positive constant for all $i \in \mathcal{N}$ and $j \in \mathcal{J}$ and t is a scalar satisfying $0 < t < 1$. In the forthcoming sections we denote non-shared risk by ϱ_{ij} and shared risk by ϱ_{ij}^S. Based on (2.10), we have

$$\frac{\partial \varrho_{ij}^S}{\partial x_{ij}} = \varsigma_{ij} \left(\left(\frac{\sum_{j \in \mathcal{J}} x_{ij}}{cap_i^\omega} + e \right)^t + \frac{t x_{ij}}{cap_i^\omega} \left(\frac{\sum_{j \in \mathcal{J}} x_{ij}}{cap_i^\omega} + e \right)^{t-1} \right).$$

Similarly, the generalized Clarke gradient of the risk measure with respect to m_{ij} is given by

$$\begin{aligned}
\partial_{m_{ij}} \upsilon_{ij}^\omega = \partial_{m_{ij}} \left(m_{ij} + \frac{1}{1-\tau_j} \max(\varrho_{ij}^\omega - m_{ij}, 0) \right) &= 1 + \frac{1}{1-\tau_j} \partial_{m_{ij}} \max\{(\varrho_{ij}^\omega - m_{ij}), 0\} \\
&= \begin{cases} 1 - \frac{1}{1-\tau_j}, & \text{if } \varrho_{ij}^\omega - m_{ij} > 0, \\ 1 + \frac{1}{1-\tau_j} \operatorname{conv}\{-1, 0\}, & \text{if } \varrho_{ij}^\omega - m_{ij} = 0, \\ 1, & \text{if } \varrho_{ij}^\omega - m_{ij} < 0. \end{cases}
\end{aligned}$$

It follows that if $\vartheta_{ij}^\omega \in \partial_{ij} \upsilon_{ij}^\omega$, then its component ϑ_{ij}^ω is defined as

$$\vartheta_{ij}^\omega \triangleq \begin{pmatrix} \vartheta_{ij}^{z,\omega} \\ \vartheta_{ij}^{m,\omega} \end{pmatrix} = \begin{pmatrix} \alpha_{ij}^{z,\omega} \frac{\partial \varrho_{ij}^S}{\partial x_{ij}} \\ 1 - \frac{\alpha_{ij}^{m,\omega}}{1-\tau_j} \end{pmatrix} \text{ where } (\alpha_{ij}^{z,\omega}, \alpha_{ij}^{m,\omega}) \text{ is given by}$$

$$\begin{cases} (\alpha_{ij}^{z,\omega}, \alpha_{ij}^{m,\omega}) = (1,1) & \varrho_{ij}^\omega - m_{ij} > 0, \\ (\alpha_{ij}^{z,\omega}, \alpha_{ij}^{m,\omega}) \in [0,1] \times [0,1] & \varrho_{ij}^\omega - m_{ij} = 0, \\ (\alpha_{ij}^{z,\omega}, \alpha_{ij}^{m,\omega}) = (0,0) & \varrho_{ij}^\omega - m_{ij} < 0. \end{cases} \tag{2.11}$$

We prove the existence of a shared-constraint Nash equilibrium in the nonsmooth settings by showing that the following sufficiency condition from [33] is satisfied. In proving this result, through a slight abuse of notation, we extend $\mathbf{Z}$ to include m and correspondingly expand $\mathbf{F}$ to include the gradient information of m.

Theorem 14 *Consider the generalized Nash game denoted by $\mathcal{G}^c$ and given by $(\Pi^c, \mathbf{C}, \mathcal{A})$. If there exists a bounded open set D and a vector $z^{ref} \in \mathbf{Z}$ such that $L_< \cap bd(D) = \emptyset$, where*

$$L_< \triangleq \left\{ z \in \mathbf{Z} : \exists \vartheta \in \partial\pi(z), (z - z^{ref})^T \vartheta < 0 \right\},$$

then $\mathcal{G}^c$ admits a Nash equilibrium in $\mathbf{Z}$.

It can be seen that $L_<$, defined above, is nonempty, if the following holds:

$$\liminf_{\substack{\|z\| \to \infty,\, z \in \mathbf{Z} \\ \vartheta \in \partial\pi(z)}} \vartheta^T(z - z^{\rm ref}) > 0.$$

We prove that precisely such a condition holds in showing that the nonsmooth shared-risk generalized Nash game admits an equilibrium under assumptions (A1), (A2) and the following assumption.

Assumption 15 (A5) *The loss function ϱ^S used in the shared risk measure is given by* (2.10) *where* $0 < t < 1$ *and* $\psi, \varsigma_{ij} > 0$.

We begin by proving the convexity of the loss function stated in equation 2.10 which is an implicit assumption in showing that the CVaR measure is convex and that the equilibrium conditions are sufficient.

Lemma 16 *Consider the loss function stated in* (2.10). *Suppose (A5) holds. Then ϱ^S is convex in x_{ij}.*

Proof : We begin by noting that $\varrho^S_{ij}(x_{ij}, cap_i^\omega)$ can be rewritten as

$$\varrho^S_{ij}(x_{ij}, cap_i^\omega) = \frac{\varsigma_{ij}}{(cap_i^\omega)^t} h(x_{ij}), \quad \text{where} \quad h(x_{ij}) = x_{ij}\left(x_{ij} + \psi^0\right)^t, \quad \psi^0 = \sum_{k \neq j, k \in \mathcal{J}} x_{ik} + \psi cap_i^\omega.$$

It suffices to prove that $h(x_{ij})$ is convex. The first and second order derivatives are given by,

$$\begin{aligned} h'(x_{ij}) &= \left(x_{ij} + \psi^0\right)^t + t x_{ij}\left(x_{ij} + \psi^0\right)^{t-1} \\ h''(x_{ij}) &= t\left(x_{ij} + \psi^0\right)^{t-2}\left(x_{ij}(t+1) + 2\psi^0\right). \end{aligned}$$

Under (A5), we have that ψ, t and ς_{ij} are positive implying that $h''(x_{ij}) > 0$ for nonnegative x_{ij}, giving us the result. ∎

Theorem 17 (Existence of Nash equilibrium to $\mathcal{G}^c$) *Consider the game $\mathcal{G}^c$ denoted by $(\Pi^c, \mathbf{C}, \mathcal{A})$ and suppose (A1)–(A2), (A5) hold. Then $\mathcal{G}^c$ admits an equilibrium.*

Proof : We define a vector $w^{\rm ref} \triangleq \mathbf{0} \in \mathbf{Z}$ and proceed to show

$$\lim_{\substack{\|w\| \to \infty,\, w \in \mathbf{Z} \\ \vartheta \in \partial\pi(z)}} w^T\vartheta = \infty$$

where the components of ϑ, namely ϑ_{ij} are defined in (2.11). The expression $w^T\vartheta$ can be written as the sum of terms 1, 2 and 3 defined in (2.8) and the following two terms:

$$\underbrace{\sum_{j\in\mathcal{J}}\left(\sum_{i\in\mathcal{N}_j}\sum_{\omega\in\Omega}\rho^\omega\vartheta_{ij}^{m,\omega}m_{ij}\right)}_{term\,4}+\underbrace{\sum_{j\in\mathcal{J}}\left(\sum_{i\in\mathcal{N}_j}\sum_{\omega\in\Omega}\rho^\omega\vartheta_{ij}^{z,\omega}x_{ij}\right)}_{term\,5}.$$

By noting that term 5 is nonnegative for all z, we consider the following cases:

Case 1: Suppose $x_{ij}\to+\infty$ implying that $\vartheta_{ij}^{z,\omega}\to+\infty$. Consequently, terms 2 and 5 tend to $+\infty$.

Case 2: Suppose $m_{ij}\to+\infty$. Then, $\vartheta_{ij}^{m,\omega}\to 1$. Consequently, term 4 tends to $+\infty$. Consider instead a sequence along which $m_{ij}\to-\infty$. Then, $\vartheta_{ij}^{m,\omega}\to\frac{-\tau_j}{1-\tau_j}$. Consequently term 4 tends to $+\infty$.

Case 3: Suppose $x_{ij}\to+\infty$ and $m_{ij}\to+\infty$. In this case $\varrho_{ij}^\omega>0$ and tends to $+\infty$ at a superlinear rate. This implies that $\vartheta_{ij}^{z,\omega}\to+\infty$ and $\vartheta_{ij}^{m,\omega}\to\frac{-\tau_j}{1-\tau_j}$. Consequently term 2 tends to $+\infty$ quadratically while term 4 tends to $-\infty$ linearly. Finally, term 5 tends to $+\infty$ at a superlinear rate implying that sum tends to $+\infty$.

Case 4: In this case, we consider possible sequences tending to infinity while $\varrho_{ij}^\omega=m_{ij}$. Suppose m_{ij} tends to $+\infty$ linearly, at a rate given by $O(m)$. To ensure that $\varrho_{ij}=m_{ij}$, based on the definition of ϱ_{ij}, there are two possibilities

1. If $m_{ij}\to\infty$ at a linear rate, one possibility is that x_{ij} tends to $+\infty$ at a superlinear rate given by $O(m^{\frac{1}{1+t}})$. As stated previously term 5 is nonnegative for all z. It can be seen that term 2 tends to $+\infty$ at a superlinear rate (since, $t<1$) given by $O(m^{\frac{2}{1+t}})$ while term 4 tends to $\pm\infty$ at a linear rate ($O(m)$). Hence the sum tends to $+\infty$.

2. If $m_{ij}\to\infty$, then the loss function grows linearly if $x_{ik}\to+\infty$ at $O(m^{\frac{1}{t}})$ for $k\in\mathcal{J},k\neq j$. Similarly term 5 is nonnegative and term 2 tends to $+\infty$ at a superlinear rate (since, $t<1$) (i.e) at $O(m^{\frac{2}{t}})$ and term 4 tends to $\pm\infty$ at a linear rate ($O(m)$). Hence the sum tends to $+\infty$.

Other cases of more than one sequence tending to infinity may be considered to be a combination of the other cases mentioned above. Consider any sequence $\{z^k\}$ such that $\lim_{k\to\infty}\|z^k\|=\infty$ and $z_k\in\mathbf{Z}$. From the above cases, either (i) at least one of the terms converges to ∞ or (ii) one term converges to $-\infty$ linearly while at least one converges to $+\infty$ at a quadratic rate. The required result follows. ∎

As we emphasized earlier in this section, the nonsmooth game leads to a multivalued variational inequality that is less easy to solve in practice. However, the smoothed problem leads to a variational inequality for which a solution can be shown to exist. Often, the solvability of the smoothed problem can be directly concluded through a degree-theoretic approach (see [34]). Instead, we use the avenue that has been followed through this paper in claiming the existence of a solution to the smoothed problem, namely through an analysis of the coercivity properties of the variational inequality. Consider the following approximation of the max-function:

$$q_{ij}^{\omega} = \varrho_{ij}^{S}(x_{ij}, cap_i^{\omega}) - m_{ij}.$$

Consider a function $f(q_{ij}^{\omega}) = \max(q_{ij}^{\omega}, 0)$ which can be can be written as,

$$f(q_{ij}^{\omega}) = \frac{q_{ij}^{\omega} + |q_{ij}^{\omega}|}{2},$$

where $|q_{ij}^{\omega}|$ represents the absolute value of q_{ij}^{ω}. The absolute value function is not differentiable at zero and can be approximated by a globally smooth function given by

$$|q_{ij}^{\omega}| \approx \sqrt{(q_{ij}^{\omega})^2 + \epsilon},$$

implying that the smooth approximation $f_\epsilon(q_{ij}^{\omega})$ and its first and second derivatives are given by,

$$f_\epsilon(q_{ij}^{\omega}) = \frac{q_{ij}^{\omega} + \sqrt{(q_{ij}^{\omega})^2 + \epsilon}}{2}, f'_\epsilon(q_{ij}^{\omega}) = \frac{1}{2}\left(1 + \frac{q_{ij}^{\omega}}{\sqrt{(q_{ij}^{\omega})^2 + \epsilon}}\right), f''_\epsilon(q_{ij}^{\omega}) = \frac{\epsilon}{2((q_{ij}^{\omega})^2 + \epsilon)^{\frac{3}{2}}}.$$

As seen from the above expressions, the function lies in C^∞ and is clearly convex. Using this function, we approximate υ_{ij}^{ω} by

$$\upsilon_{ij,\epsilon}^{\omega} = \left(m_{ij} + \frac{f_\epsilon(q_{ij}^{\omega})}{1 - \tau_j}\right).$$

It follows that the

$$\nabla_{x_{ij}} \upsilon_{ij,\epsilon}^{\omega} = \frac{f'(q_{ij}^{\omega})}{1 - \tau_j}.\frac{\partial \varrho_{ij}^{S}(x_{ij}, cap_i^{\omega})}{\partial x_{ij}}, \quad \nabla_{m_{ij}} \upsilon_{ij,\epsilon}^{\omega} = \left(1 - \frac{f'_\epsilon(q_{ij}^{\omega})}{1 - \tau_j}\right).$$

This allows us to define a smoothed mapping, denoted by $\mathbf{F}_\epsilon$, which is then employed in developing an existence analysis for the smoothed game $\mathcal{G}_\epsilon^c$.

Proposition 18 (Existence of equilibrium to $\mathcal{G}_\epsilon^c$) *Consider the game $\mathcal{G}_\epsilon^c$ denoted by $(\Pi_\epsilon^c, \mathbf{C}, \mathcal{A})$ and suppose assumptions (A1)–(A2), (A5) hold. Then $\mathcal{G}_\epsilon^c$ admits an equilibrium.*

Proof : Consider an $z^{\text{ref}} \equiv 0 \in \mathbf{Z}$. As shown earlier, it suffices to show that,

$$\liminf_{z \in \mathbf{Z}, \|z\| \longrightarrow \infty} \mathbf{F}^\epsilon(z)^T(z) > 0,$$

where $\mathbf{F}_\epsilon(z)^T(z)$ is given by the sum of terms 1, 2 and 3 from (2.8) and the following two terms:

$$\underbrace{\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{N}_j}\sum_{\omega\in\Omega}\rho^\omega\nabla_{m_{ij}}\upsilon^\omega_{ij,\epsilon}(m_{ij})}_{term\,4}+\underbrace{\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{N}_j}\sum_{\omega\in\Omega}\rho^\omega\nabla_{x_{ij}}\upsilon^\omega_{ij,\epsilon}(x_{ij})}_{term\,5}.$$

Consider the following cases and recall that φ is positive and an increasing function:

Case 1: Suppose $x_{ij}\to+\infty$. This implies that $q^\omega_{ij}\to+\infty$. Consequently, terms 2 and 5 tend to $+\infty$.

Case 2: Suppose $m_{ij}\to+\infty$. Then, $q^\omega_{ij}\to-\infty$ implying that term 4 tends to $+\infty$. Consider $m_{ij}\to-\infty$. Then, $q^\omega_{ij}\to+\infty$ and term 4 tends to $+\infty$.

Case 3: Suppose $x_{ij}\to+\infty$ and $m_{ij}\to+\infty$. In this case $q^\omega_{ij}>0$ and tends to $+\infty$ at a superlinear rate. This implies that term 2 tends to $+\infty$ quadratically and term 4 tends to $-\infty$ linearly. term 5 tends to $+\infty$ at a superlinear rate. Effectively the sum tends to $+\infty$.

Case 4: In this case, we consider possible sequences tending to infinity along which $q^{k,\omega}_{ij}=0$ for $k\geq\mathcal{K}$. Suppose m_{ij} tends to $+\infty$ linearly, denoted by $O(m)$. Then there are two possibilities:

1. Let x_{ij} tend to $+\infty$ at $O(m^{\frac{1}{1+t}})$. As stated previously term 5 is nonnegative for all z. It can be seen that term 2 tends to $+\infty$ at a superlinear rate (since, $t<1$) (i.e) at $O(m^{\frac{2}{1+t}})$ and term 4 tends to $\pm\infty$ ($\tau_j<0.5$ and $\tau_j>0.5$ respectively) at a linear rate ($O(m)$). Hence the sum tends to $+\infty$.

2. Alternatively, let $x_{ik}\to+\infty$ at $O(m^{\frac{1}{t}})$ for $k\in\mathcal{J},k\neq j$ and $m_{ij}\to+\infty$ linearly (i.e) $O(m)$. Similarly term 5 is nonnegative and term 2 tends to $+\infty$ at a superlinear rate (since, $t<1$) (i.e) at $O(m^{\frac{2}{t}})$ and term 4 tends to $\pm\infty$ ($\tau_j<0.5$ and $\tau_j>0.5$ respectively) at a linear rate ($O(m)$). Hence the sum tends to $+\infty$.

Other cases of more than one sequence tending to infinity may be considered to be a combination of the other cases mentioned above. Consider any sequence $\{z^k\}$ such that $\lim_{k\to\infty}\|z^k\|=\infty$ and $z_k\in\mathbf{Z}$. In this case, either (i) one of the terms converges to ∞ or (ii) one term converges to $-\infty$ linearly and the other converges to ∞ at a quadratic rate. This concludes the proof. ∎

2.4 Cutting-plane projection methods

The game-theoretic problems, denoted by $\mathcal{G}^a$ and $\mathcal{G}^b$, introduced in the section 2 lead to stochastic variational problems that are shown to be monotone. While significant theory exists for solving monotone variational inequalities [34, 55], unfortunately most schemes can neither be implemented in a distributed setting (since the constraint sets are coupled) nor do they possess the scalability required to address this class of problems since our class of problems can be arbitrarily large in terms of the number of agents, the size of the network and the cardinality of the sample-space.

It should be remarked that there have been relatively few attempts to examine the class of generalized Nash games while even fewer have considered their stochastic counterparts. Fukushima and Pang [73] suggested a sequential penalization approach for solving such problems while a review of approaches can be found in [32]. Of note is a recent approach that uses a Nikaido-Isoda (NI) function by Von Heusinger and Kanzow [88] and a relaxation algorithm using the NI function by Krawczyk and Uryasev [58]. In [27, 90], sample-average approximation schemes are suggested and accommodate neither the coupled nature of their strategy sets nor their semi-infinite nature. Note that it may be possible to employ their approach on the complementarity problem that emerges from this setting. An alternate scheme that relies on matrix splitting techniques is suggested in [85].

Accordingly, our focus is on developing convergent algorithms with suitable error bounds, for addressing this class of problems. We place an emphasis on the construction of distributed schemes that scale with the cardinality of the sample-space, namely $|\Omega|$, the number of agents $|\mathcal{J}|$ and the size of the network $|\mathcal{N}|$. To address these needs, we develop a *distributed projection-based method* that employs a *cutting-plane method* for solving the agent-specific projection problems.

In section 4.1, we describe a dual and a primal-dual projection method for the solution of shared-constraint stochastic Nash games. At the heart of these schemes is a projection step which in general leads to a massive stochastic convex program. In section 4.2, we employ a cutting-plane method for the solution of such problems that scales with $|\Omega|$. Rate estimates and error bounds, particularly for inexact generalizations, are presented for the projection schemes in section 4.3. Finally, in section 4.4, in examining the numerical behavior of the schemes, we observe that the schemes display the desired scalability properties and the inexact generalizations prove to have significant benefits.

2.4.1 Distributed primal-dual and dual projection methods

Consider an $N-$player deterministic Nash game in which the jth agent solves the parameterized convex optimization problem given by

$$\boxed{\begin{array}{lll} \mathrm{A}(z_{-j}) \qquad\qquad & \text{maximize} & \pi_j(z_j; z_{-j}) \\ & \text{subject to} & z_j \in \mathbf{Z}_j, \end{array}}$$

where $\pi_j(z_j; z_{-j})$ is a convex differentiable function of z_j for all z_{-j} and $\mathbf{Z}_j$ is a closed and convex set. Then a standard distributed projection scheme is given by

$$z_j^{k+1} := \Pi_{\mathbf{Z}_j}(z_j^k + \gamma \nabla \pi_j(z_j^k; z_{-j}^k)), \text{ for all } j = 1, \ldots, N,$$

where γ is a fixed steplength. Yet, the convergence of such a scheme relies on two properties: First, the gradient mapping given by $F(z)$ needs to satisfy strict monotonicity, strong monotonicity or co-coercivity property [34] over a set $\mathbf{Z}$ where $F(z)$ and $\mathbf{Z}$ are defined as

$$F(z) := -\begin{pmatrix} \nabla_{z_1} \pi_1^T, & \ldots, & \nabla_{z_{g+1}} \pi_{g+1}^T \end{pmatrix}^T \text{ and } \mathbf{Z} \triangleq \prod_{j=1}^{g+1} \mathbf{Z}_j.$$

Second, the strategy sets across agents cannot be coupled. In our setting, neither assumption holds and therefore a direct application of the primal approach cannot be employed.

Instead, we observe that the shared-constraint game can be cast as a monotone complementarity problem in the primal-dual space. By solving a sequence of regularized (and therefore strongly monotone) complementarity problems through a Tikhonov regularization scheme [34], we obtain a solution to the original problem. Note that the monotonicity of the mapping in the primal-dual space suffices for the Tikhonov trajectory to converge to the solution of the original problem [34, Ch. 12]. This avenue allows us to leverage fixed steplength projection schemes for the solution of each regularized complementarity problem. Importantly, each subproblem can be massive, with a size proportional to $|\Omega| \times |\mathcal{J}| \times |\mathcal{N}|$, and a direct solution of such problems is only possible in modest settings. To cope with such a challenge, we develop a distributed framework that relies on decomposition methods that scale well with all three sources of complexity.

We now proceed to describe the distributed projection framework. If the Lagrange multipliers corresponding to the shared constraint, denoted by $d(z) \geq 0$, are denoted by λ, then it follows that (z^*, λ^*) is an equilibrium of shared-constraint Nash game if and only if (z^*, λ^*) is a solution of set of coupled fixed-point problems:

$$z = \Pi_{\mathbf{Z}}(z - \gamma \mathbf{F}_z(z, \lambda)) \tag{2.12}$$

$$\lambda = \Pi_{\mathbb{R}_m^+}(\lambda - \gamma \mathbf{F}_\lambda(z, \lambda)), \tag{2.13}$$

where

$$\mathbf{F}_z(z,\lambda) = \begin{pmatrix} -\nabla_{z_1}\pi_1 - \nabla_{z_1} d(z)^T\lambda \\ \vdots \\ -\nabla_{z_N}\pi_N - \nabla_{z_N} d(z)^T\lambda \end{pmatrix} \text{ and } \mathbf{F}_\lambda(z,\lambda) = d(z). \tag{2.14}$$

The fixed-point representations motivate a *primal-dual* method that requires constructing a primal and dual method on the same timescale with a fixed steplength γ_{pd} in a regularized setting. Specifically, this entails the following set of regularized primal and dual steps for $k \geq 0$:

$$z_j^{k+1} = \Pi_{\mathbf{Z}_j}(z_j^k - \gamma_{pd}(\mathbf{F}_z(z_j^k; z_{-j}^k, \lambda^k) + \epsilon^\ell z_j^k)), \text{ for all } j \tag{2.15}$$

$$\lambda^{k+1} = \Pi_{\mathbb{R}_m^+}(\lambda^k - \gamma_{pd}(\mathbf{F}_\lambda(z^k, \lambda^k) + \epsilon^\ell \lambda^k)), \tag{2.16}$$

where ϵ^ℓ is the regularization parameter at the ℓth iteration of the outer Tikhonov scheme. In the regularized primal-dual approach, the steplength γ_{pd} has to be chosen in accordance with the monotonicity and Lipschitz constant of the appropriate mappings in *both* the primal and dual spaces (see section 4.3 for more details). In effect, if the mappings in one of the spaces has a large Lipschitz constant (or alternately a low monotonicity constant), the progress of the entire algorithm may be hampered.

A dual method for solving the monotone complementarity problem does not tie these two steplengths together and can be employed instead. This requires that for every update in the dual space, an *exact* primal solution is required. In particular, for $k \geq 0$, this leads to a set of iterations given by

$$z_j^k = \Pi_{\mathbf{Z}_j}(z_j^k - \gamma_d(\mathbf{F}_z(z_j^k; z_{-j}^k, \lambda^k) + \epsilon^\ell z_j^k)), \text{ for all } j \tag{2.17}$$

$$\lambda^{k+1} = \Pi_{\mathbb{R}_m^+}(\lambda^k - \gamma_p(\mathbf{F}_\lambda(z^k, \lambda^k) + \epsilon^\ell \lambda^k)), \tag{2.18}$$

where γ_p and γ_d are the primal and dual steplengths, respectively. The termination of the inner scheme occurs when the error in the fixed-point problem falls within a threshold and is ensured by the following for the primal-dual scheme

$$\left\| \left(\frac{\|z^{k+1} - z^k\|}{1 + \|z^k\|}, \frac{\|\lambda^{k+1} - \lambda^k\|}{1 + \|\lambda^k\|} \right) \right\| \leq \epsilon^{inner}, \tag{2.19}$$

and the dual scheme

$$\frac{\|\lambda^{k+1} - \lambda^k\|}{1 + \|\lambda^k\|} \leq \epsilon^{inner}. \tag{2.20}$$

Note that within the dual scheme, every dual step requires an *exact* solution of the primal fixed-point problem. Naturally, the exact solution of such a problem may prove difficult, suggesting instead that we may need to employ inexact or approximate solutions. Expectedly, this would lead to errors that require quantification. This analysis is provided, along with suitable convergence results, in section 4.3. We conclude this subsection with an algorithm statement for the projection-based schemes.

Algorithm 1: Distributed Primal-dual and Dual Projection Methods

initialization $k = 0, \ell = 0$;
choose constants $\epsilon^0, \epsilon^{inner}, \epsilon^{outer} > 0$ and γ_{pd}, γ_p and γ_d, initial solution (z^0, λ^0), $\bar{\gamma} < 1$;
while $\epsilon^\ell > \epsilon^{outer}$ **do**
    while conditions (2.19) or (2.20) are not satisfied do
        Let λ^{k+1} be given by (2.16) (Primal-dual) or (2.18) (Dual) ;
        Let z^{k+1} be given by (2.15) (Primal-dual) or the solution of (2.17) (Dual) ;
        $k := k + 1$;
    end
    Update regularization $\epsilon^{\ell+1} := \bar{\gamma}\epsilon^\ell$;
    $\ell := \ell + 1$;
end

2.4.2 A scalable cutting-plane method for the projection problem

In the projection schemes presented in the earlier section, the solution of the primal projection step, as denoted by (2.15) and (2.17), requires the solution of a large convex program of size $O(|\Omega|)$. This is generally only possible via direct solvers for modest sample-spaces and in this subsection, we discuss how one may solve such problems in a scalable fashion for arbitrarily large sample-spaces.

In the current setting, $\mathbf{Z}_j$ is a polyhedral set implying that the projection problem is a quadratic program (QP) and, given that the problem originates from a projection problem, this QP is, in fact, strongly convex. In the past, QPs has been solved by a variety of schemes, such as interior-point methods, active-set methods and others [71]. All of these schemes are necessarily direct approaches in that they make no obvious effort to utilize the structure of the problem. However, in this instance, the problems belong to a class of *recourse-based stochastic quadratic programs* [10]. The key computational challenge in solving recourse-based stochastic optimization problems lies in ensuring that scenario-specific second-stage problems are solved in parallel, effectively allowing for a scalable method. In 1969, based on a decomposition scheme suggested by Benders [8], Van-Slyke and Wets [86] suggested a cutting-plane method for the solution of recourse-based stochastic linear programs (LPs) that allows for precisely such a parallelization. While much has been done on the solution of stochastic LPs (cf. [50, 10]), stochastic convex programming has been less studied in general [75]. Parallel schemes for the solution of stochastic QPs via splitting and projection methods were discussed by Womersley and Chen [16] while extensions to the L-shaped cutting-plane method have been suggested by Zakeri et al. [94]. More recently, Kulkarni and Shanbhag developed an inexact-cut and a trust-region L-shaped method for solving stochastic QPs that was subsequently employed as a QP solver within a more general sequential quadratic programming method for solving nonconvex stochastic NLPs [59, 60]. We employ a similar L-shaped scheme for solving the stochastic quadratic program arising from the projection problem.

Computing the projection in the primal space (2.17) and (2.15), requires solving a stochastic program given by

$$\boxed{\begin{array}{ll} \text{minimize} & \frac{1}{2}(\widehat{z}_j - \bar{z}_j^k)^T(\widehat{z}_j - \bar{z}_j^k) \\ \text{subject to} & \widehat{z}_j \in \mathbf{Z}_j, \end{array}}$$

where

$$\bar{z}_j^k = (z_j^k - \gamma F_{z_j}(z_j; z_{-j}^k, \lambda^k)), \widehat{z}_j = \begin{pmatrix} \widehat{x}_j \\ (\widehat{y}_j^\omega)_{\omega\in\Omega} \end{pmatrix}, \widehat{x}_j = \begin{pmatrix} x_j \\ m_j \end{pmatrix}, \widehat{y}_j^\omega = \begin{pmatrix} u_j^\omega \\ v_j^\omega \\ y_j^\omega \\ s_j^\omega \end{pmatrix}, \forall j \in \mathcal{J}, \widehat{x}_{g+1} = \begin{pmatrix} 0 \end{pmatrix}, \widehat{y}_{g+1}^\omega = \begin{pmatrix} r_1^\omega \\ \vdots \\ r_N^\omega \end{pmatrix}.$$

In settings where the loss function in the risk measure is affine (or in the risk-neutral deviation cost setting), the projection problem reduces to a stochastic quadratic program given by:

$$\boxed{\begin{array}{ll} \text{minimize} & \frac{1}{2}\widehat{x}_j^T\widehat{x}_j + \widehat{x}_j^T\bar{x}_j + \sum_{\omega\in\Omega}\left(\frac{1}{2}(\widehat{y}_j^\omega)^T(\widehat{y}_j^\omega) - (\widehat{y}_j^\omega)^T\bar{y}_j^\omega\right) \\ \text{subject to} & (\widehat{x}_j, \widehat{y}_j^\omega) \in \mathbf{Z}_j = \left\{ (\widehat{x}_j, \widehat{y}_j^\omega) : \left\{ \begin{array}{c} A_1\widehat{x}_j + A_2\widehat{y}_j^\omega = \widehat{b}_j^\omega \\ A_3\widehat{y}_j^\omega \le cap_{ij}^\omega, \\ A_4\widehat{x}_j \ge 0, \\ \widehat{y}^\omega \ge 0, \end{array} \right\} \forall\omega\in\Omega \right\}, \end{array}}$$

where A_1, A_2, A_3, and A_4 are defined appropriately. As Ω grows in cardinality, a direct solution of the quadratic program becomes challenging. Instead, we pursue a stochastic programming avenue by noting that the constraint structure allows one to cast the problem as a recourse-based stochastic program. Specifically, we have

$$\boxed{\begin{array}{ll} \text{minimize} & \frac{1}{2}\widehat{x}_j^T\widehat{x}_j + \widehat{x}_j^T\bar{x}_j + \mathcal{Q}(\widehat{x}_j) \\ \text{subject to} & A_4\widehat{x}_j \ge 0 \end{array}}$$

where $\mathcal{Q}(\widehat{x}_j)$, the cost of recourse is given by $\mathcal{Q}(\widehat{x}_j) = \mathbb{E}\mathcal{Q}(\widehat{x}_j;\omega)$ and $\mathcal{Q}(\widehat{x}_j;\omega)$ is the optimal value of the scenario-specific quadratic program:

$$\boxed{\begin{array}{lll} \mathrm{Sub}(\widehat{x}_j;\omega) & \text{minimize} & \left(\frac{1}{2}(\widehat{y}_j^\omega)^T(\widehat{y}_j^\omega) - (\widehat{y}_j^\omega)^T\bar{y}_j^\omega\right) \\ & \text{subject to} & y_j^\omega \in \mathbf{Y}_j^\omega(\widehat{x}_j) \end{array}}$$

and

$$\mathbf{Y}^\omega(\widehat{x}_j) = \left\{ \widehat{y}_j^\omega : \left\{ \begin{array}{c} A_2\widehat{y}_j^\omega = \widehat{b}_j^\omega - A_1\widehat{x}_j \\ A_3\widehat{y}_j^\omega \le cap_{ij}^\omega, \\ \widehat{y}^\omega \ge 0 \end{array} \right\} \right\}.$$

It should be emphasized that in general, a first-stage decision $\widehat{x}$ might render the $\mathbf{Y}^\omega(\widehat{x})$ empty. However, in this particular case, the nonnegative deviation levels u^ω and v^ω can be made arbitrarily large to ensure that the second-stage problem is *always feasible* and the resulting problem is said to possess complete recourse.

The L-shaped method for the solution of stochastic QPs requires solving a sequence of increasingly constrained (QPs) (called the master problem) where the additional constraints, termed as cuts, arise from the solution of the set of scenario-specific second-stage problems. The master problem is given by

$$\boxed{\begin{array}{lll} \mathrm{Master}_k & \underset{\widehat{x}_j,\theta_j}{\text{minimize}} & \frac{1}{2}\widehat{x}_j^T\widehat{x}_j + \widehat{x}_j^T\bar{x}_j + \theta \\ & \text{subject to} & \begin{array}{c} A_4\widehat{x}_j \ge 0 \\ \theta - G_{j,i}^T\widehat{x}_j \ge g_{j,i}, \quad i = 1,\ldots,k. \end{array} \end{array}}$$

where $(G_{j,i}, g_{j,i})$ are the coefficients of the ith (see [81] for more details) defined as:

$$G_{j,i} \triangleq -\sum_{\omega\in\Omega} A_1^T\pi^\omega \quad \text{and} \quad g_{j,i} \triangleq \sum_{\omega\in\Omega}(\pi^\omega)^T\hat{b}_j^\omega - \frac{1}{2}\sum_{\omega\in\Omega}(\hat{y}^\omega)^T\hat{y}^\omega,$$

where π^ω represents the vector of dual variables corresponding to the sub problem (scenario ω) and I represents the identity matrix. Note that the ith cut associated with the jth agent requires the solution of $\mathrm{Sub}(\widehat{x}_j^i)$. It is worth reiterating that the complexity arising from a massive sample-space is addressed by decomposing what is a potentially massive QP into a set of $|\Omega|$ smaller QPs. In the L-shaped method, the termination is contingent on the lower bound L_j^k and upper bound U_j^k are sufficiently close where L_j^k and U_j^k are defined as

$$L_j^k \equiv \tfrac{1}{2}(\widehat{x}_j^k)^T(\widehat{x}_j^k) + (\widehat{x}_j^k)^T\bar{x}_j^k + \theta_j^k \text{ and } U_j^k \equiv \min\{U_j^{k-1}, \tfrac{1}{2}(\widehat{x}_j^k)^T(\widehat{x}_j^k) + (\widehat{x}_j^k)^T\bar{x}_j^k + \mathcal{Q}(\widehat{x}_j^k)\}, \text{ respectively.}$$

Notice that $\{L_j^k\}$ is a monotonically increasing sequence while $\{U_j^k\}$ is a monotonically decreasing sequence. Algorithm 2) provides a formal statement of the L-shaped method [81] and its convergence is easily proved and can be found in [10, 78].

Algorithm 2: L–shaped method

initialization $k = 1, j \in \mathcal{J}, U_j^k = \infty, L_j^k = -\infty$;
choose $\epsilon_1, \tau, u > 1$;
while $|U_j^k - L_j^k| > \tau$ **do**
    Solve (Master$_k$) to get $\left(\widehat{x}_j^k, \theta_j^k\right)$;
    Update lower bound L_j^k;
    Solve Sub$(\widehat{x}_j^k; \omega)$ for all $\omega \in \Omega$;
    Construct $\left(G_I^k, g_I^k\right)$;
    Update upper bound U_j^k and add optimality cut $\left(G_I^k, g_I^k\right)$ to (Master$_k$);
    $k = k + 1$
end

2.4.3 Convergence and error analysis of projection methods

Convergence of projection schemes is reliant on the underlying mappings satisfying a strict or strong monotonicity property. The absence of such a property may be addressed through a Tikhonov-based regularization scheme [34]. Each iterate of the Tikhonov scheme may be solved efficiently and in this subsection, we provide the convergence theory for the suggested dual and primal-dual schemes for solving precisely such problems. In this section, we present three sets of results. First, our convergence statements require a precise specification of the Lipschitz and monotonicity constants of the relevant mapping and represents our first result. Second, we present a convergence result for the dual scheme in a regularized setting and further equip this result with rate estimates. The *exact* form of the dual scheme requires exact primal iterates for a given dual solution. In a regime where a bound on the primal strategy sets is assumed to be available, we relax this requirement in constructing an *inexact dual method* and allow for *inexact* primal solutions. The third set of results focus on developing error bounds for the inexact dual scheme in this setting along with suitable bounds on the primal suboptimality and primal infeasibility.

Before proving the Lipschitzian and monotonicity properties of $\mathbf{F}^\epsilon$, we consider the polyhedral shared constraint denoted by $Bz \geq 0$ and provide a precise relationship between $\|B\|$ and the problem size, where z is specified as follows:

$$z = \begin{pmatrix} \bar{p}_1 \\ \vdots \\ p_{\bar{N}_g} \\ \bar{p}_0 \end{pmatrix}, \bar{p}_i = \begin{pmatrix} \bar{p}_i^1 \\ \vdots \\ \bar{p}_i^n \end{pmatrix}, \bar{p}_i^\omega = \begin{pmatrix} y_{i1}^\omega \\ \vdots \\ y_{iJ}^\omega \\ r_i^\omega \end{pmatrix}, \qquad \forall i \in G, \forall \omega \in \Omega,$$

and $\bar{p}_0$ represents the other components of the vector z, not indicated above. Consequently, the matrix B is defined as

$$B = \begin{pmatrix} B_1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & B_{N_g} & 0 \end{pmatrix}, \text{ where } B_i = \begin{pmatrix} B_i^1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_i^n \end{pmatrix}, B_i^\omega = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}, \ \forall i \in G, \forall \omega \in \Omega. \tag{2.21}$$

The following result gives a bound on $\|B\|$, that is subsequently employed in our rate analysis.

Lemma 19 *Consider the matrix B defined in (2.21). If N_f and N_g are the total number of players in the game and the number of generating nodes, respectively, then $\|B\|_2 \leq \sqrt{N_f N_g n}$.*

Proof : Recall that $\|B\|_2 \leq \|B\|_F$, where $\|A\|_F$ represents the Froebenius norm of the matrix (see [39]). When B is given by (2.21), then

$$\|B\|_F = \sqrt{\sum_{\omega \in \Omega} \sum_{i \in \mathcal{N}_g} (\|\mathcal{J}\| + 1)} = \sqrt{N_f N_g n}. \quad \blacksquare$$

By recalling the definitions of $\mathbf{F}_z$ and $\mathbf{F}_\lambda$ in (2.14), we further define $\mathbf{F}_z^\epsilon, \mathbf{F}_\lambda^\epsilon, \mathbf{F}_f^\epsilon$ and $\mathbf{F}_d$ as $\mathbf{F}_z^\epsilon := \mathbf{F}_z + \epsilon z, \mathbf{F}_\lambda^\epsilon := \mathbf{F}_\lambda + \epsilon\lambda$ and

$$\mathbf{F}_f^\epsilon := \begin{pmatrix} \nabla_{z_1}\pi_1 + \epsilon z_1 \\ \vdots \\ \nabla_{z_{g+1}}\pi_{g+1} + \epsilon z_{g+1} \end{pmatrix}, \mathbf{F}_d := \begin{pmatrix} \nabla_{z_1} d^T\lambda \\ \vdots \\ \nabla_{z_{g+1}} d^T\lambda \end{pmatrix}, \mathbf{F}_z^\epsilon := \mathbf{F}_f^\epsilon - \mathbf{F}_d. \tag{2.22}$$

Furthermore, we define $z, z_i, l_i^\omega, l_i\ u_i, v_i, s_i, m_i$ and x_i as follows:

$$z = \begin{pmatrix} z_1 \\ \vdots \\ z_{g+1} \end{pmatrix}, z_i = \begin{pmatrix} l_i \\ u_i \\ v_i \\ s_i \\ m_i \\ x_i \end{pmatrix}, l_i^\omega = \begin{pmatrix} y_{i1}^\omega \\ \vdots \\ y_{ig}^\omega \\ r_i^\omega \end{pmatrix}, l_i = \begin{pmatrix} l_i^1 \\ \vdots \\ l_i^n \end{pmatrix},$$

$$u_i = \begin{pmatrix} u_{i1}^1 \\ \vdots \\ u_{ig}^n \end{pmatrix}, v_i = \begin{pmatrix} v_{i1}^1 \\ \vdots \\ v_{ig}^n \end{pmatrix}, s_i = \begin{pmatrix} s_{i1}^1 \\ \vdots \\ s_{ig}^n \end{pmatrix}, m_i = \begin{pmatrix} m_{i1} \\ \vdots \\ m_{ig} \end{pmatrix}, x_i = \begin{pmatrix} x_{i1} \\ \vdots \\ x_{ig} \end{pmatrix}.$$

Using these definitions, the Lipschitz continuity and strong monotonicity constants of $\mathbf{F}_\epsilon$ can be derived. Note that we provide a result for the game $\mathcal{G}^b$. The mapping for $\mathcal{G}^a$ is over a smaller space and requires an appropriate Lipshitz assumption on the deviation costs.

Lemma 20 *Consider the mapping $\mathbf{F}_\epsilon(z,\lambda)$, defined in (2.22), arising from the Nash game $\mathcal{G}^b$. Suppose assumptions (A1)–(A2), (A4) hold and suppose the cost functions ζ_{ij}^ω are Lipschitz continuous with constants $L_\zeta^{ij,\omega}$, for all $i \in \mathcal{N}$, $j \in \mathcal{J}$ and for all $\omega \in \Omega$. Then this mapping is Lipschitz continuous and strongly monotone with constants L and ϵ, respectively where*

$$L \triangleq (M + \|B\| + \epsilon), M \triangleq \max_{i \in G} \left(2N_f^2(b_i^0 + \mathbb{E}(b_i^\omega + \bar{L}_\zeta^{i,\omega})) \right),$$

and $\|B\| \leq \sqrt{N_f N_g n}$.

Proof : We first derive the Lipschitz constant for $\mathbf{F}_\epsilon$. This requires analyzing each of the three terms.

$$\|\mathbf{F}^\epsilon(z^1,\lambda^1) - \mathbf{F}^\epsilon(z^2,\lambda^2)\| = \left\| \begin{pmatrix} \mathbf{F}_f^\epsilon(z^1,\lambda^1) - \mathbf{F}_f^\epsilon(z^2,\lambda^2) + \mathbf{F}_d(z^1,\lambda^1) - \mathbf{F}_d(z^2,\lambda^2) \\ \mathbf{F}_\lambda(z^1,\lambda^1) - \mathbf{F}_\lambda(z^2,\lambda^2) \end{pmatrix} \right\| \tag{2.23}$$

$$\leq \underbrace{\|\mathbf{F}_f^\epsilon(z^1,\lambda^1) - \mathbf{F}_f^\epsilon(z^2,\lambda^2)\|}_{\textbf{Term 1}} + \underbrace{\|\mathbf{F}_d(z^1,\lambda^1) - \mathbf{F}_d(z^2,\lambda^2)\|}_{\textbf{Term 2}} + \underbrace{\|\mathbf{F}_\lambda^\epsilon(z^1,\lambda^1) - \mathbf{F}_\lambda^\epsilon(z^2,\lambda^2)\|}_{\textbf{Term 3}}. \tag{2.24}$$

We bound each of the three terms as follows:

Term 1: Given two vectors z^1 and z^2, we may decompose $\mathbf{F}$ into $H + B$ allowing term 1 to be expressed as

$$\mathbf{F}_i(z^1) - \mathbf{F}_i(z^2) = \begin{pmatrix} F_i^l(z^1) - F_i^l(z^2) \\ F_i^s(z^1) - F_i^s(z^2) \\ F_i^m(z^1) - F_i^m(z^2) \\ F_i^x(z^1) - F_i^x(z^2) \end{pmatrix} = \underbrace{\begin{pmatrix} H_i^l(z^1) - H_i^l(z^2) \\ H_i^s(z^1) - H_i^s(z^2) \\ H_i^m(z^1) - H_i^m(z^2) \\ H_i^x(z^1) - H_i^x(z^2) \end{pmatrix}}_{\textbf{term 4}} + \underbrace{\begin{pmatrix} B_i^l(z^1) - B_i^l(z^2) \\ B_i^s(z^1) - B_i^s(z^2) \\ B_i^m(z^1) - B_i^m(z^2) \\ B_i^x(z^1) - B_i^x(z^2) \end{pmatrix}}_{\textbf{term 5}}.$$

Terms in l and x are nonzero in the specification of term 4 and the first of these for $i \in G$ is bounded as shown next.

$$\left(H_i^l(z^1) - H_i^l(z^2)\right)_\omega = \begin{pmatrix} 2\rho^\omega b_i^\omega(y^1_{i1,\omega} - y^2_{i1,\omega}) + (\zeta^\omega_{i1}(y^1_{i1,\omega}) - \zeta^\omega_{i1}(y^2_{i1,\omega})) \\ \vdots \\ 2\rho^\omega b_i^\omega(y^1_{ig,\omega} - y^2_{ig,\omega}) + (\zeta^\omega_{ig}(y^1_{ig,\omega}) - \zeta^\omega_{ig}(y^2_{ig,\omega})) \\ \rho^\omega b_i^\omega(r^1_{i,\omega} - r^2_{i,\omega}) \end{pmatrix} \leq \begin{pmatrix} 2\rho^\omega b_i^\omega(y^1_{i1,\omega} - y^2_{i1,\omega}) \\ \vdots \\ 2\rho^\omega b_i^\omega(y^1_{ig,\omega} - y^2_{ig,\omega}) \\ \rho^\omega b_i^\omega(r^1_{i,\omega} - r^2_{i,\omega}) \end{pmatrix}$$

$$+ \begin{pmatrix} \rho^\omega(\zeta^\omega_{i1}(y^1_{i1,\omega}) - \zeta^\omega_{i1}(y^2_{i1,\omega})) \\ \vdots \\ \rho^\omega(\zeta^\omega_{ig}(y^1_{ig,\omega}) - \zeta^\omega_{ig}(y^2_{ig,\omega})) \end{pmatrix} \leq \begin{pmatrix} \rho^\omega(2b_i^\omega + L_\zeta^{i1,\omega})(y^1_{i1,\omega} - y^2_{i1,\omega}) \\ \vdots \\ \rho^\omega(2b_i^\omega + L_\zeta^{ig,\omega})(y^1_{ig,\omega} - y^2_{ig,\omega}) \\ \rho^\omega b_i^\omega(r^1_{i,\omega} - r^2_{i,\omega}) \end{pmatrix} \leq M_i^{l,\omega} \begin{pmatrix} (y^1_{i1,\omega} - y^2_{i1,\omega}) \\ \vdots \\ (y^1_{ig,\omega} - y^2_{ig,\omega}) \\ (r^1_{i,\omega} - r^2_{i,\omega}) \end{pmatrix},$$

where $M_i^{l,\omega} = \rho^\omega(2b_i^\omega + \max_{j\in\mathcal{J}} L_\zeta^{ij,\omega})$. Similarly, for $i \in G$, the other nonzero term in term 4 is bounded as follows:

$$H_i^x(z^1) - H_i^x(z^2) = M_i^x \begin{pmatrix} (x^1_{i1} - x^2_{i1}) \\ \vdots \\ (x^1_{ig} - x^2_{ig}) \end{pmatrix},$$

where $M_i^x = 2b_i^0$. By noting that when $i \in G^c$, $M_i^{l,\omega} = \rho^\omega b_i^\omega$, the Lipschitz constant for term 4, denoted by M_4, is given by

$$M_4 = \max_{i\in G\cup G^c} \left(\sum_{\omega\in\Omega} M_i^{l,\omega} + M_i^x \right) \leq \max_{i\in G}(2(\mathbb{E}b_i^\omega + b_i^0) + \mathbb{E}\bar{L}_\zeta^{i,\omega}), \bar{L}_\zeta^{i,\omega} = \max_{j\in\mathcal{J}} L_\zeta^{ij,\omega}.$$

Similarly for $i \in G$, the norms of the two nonzero terms in term 5, may be bounded through the use of the triangle inequality in the following fashion:

$$\begin{aligned} \left\| \left(B_i^l(z^1) - B_i^l(z^2)\right)_\omega \right\| &= \left\| \begin{pmatrix} \rho^\omega \left(b_i^\omega(\sum_{j\in\mathcal{J}, j\neq 1}(y^1_{ij,\omega} - y^2_{ij,\omega})) + b_i^\omega(r^1_{i,\omega} - r^2_{i,\omega}) - b_i^\omega(x^1_{i1} - x^2_{i1}) \right) \\ \vdots \\ \rho^\omega \left(b_i^\omega(\sum_{j\in\mathcal{J}, j\neq g}(y^1_{ij,\omega} - y^2_{ij,\omega})) + b_i^\omega(r^1_{i,\omega} - r^2_{i,\omega}) - b_i^\omega(x^1_{iJ} - x^2_{iJ}) \right) \\ \rho^\omega \left(b_i^\omega(\sum_{j\in\mathcal{J}}(y^1_{ij,\omega} - y^2_{ij,\omega})) \right) \end{pmatrix} \right\| \\ &\leq \bar{M}_i^{l,\omega} \left\| z^1 - z^2 \right\|, \end{aligned}$$

$\bar{M}_i^{l,\omega} = \rho^\omega b_i^\omega (1 + (g+1)^2)$ and

$$\begin{aligned}\left\|B_i^x(z^1) - B_i^x(z^2)\right\| &= \left\| \begin{pmatrix} b_i^0 \sum_{j\in\mathcal{J}, j\neq 1}(x_{ij}^1 - x_{ij}^2) - \sum_{\omega\in\Omega}\rho^\omega b_i^\omega (\sum_{j\in\mathcal{J}}(y_{ij,\omega}^1 - y_{ij,\omega}^2) + r_{i,\omega}^1 - r_{i,\omega}^2) \\ \vdots \\ b_i^0 \sum_{j\in\mathcal{J}, j\neq g}(x_{ij}^1 - x_{ij}^2) - \sum_{\omega\in\Omega}\rho^\omega b_i^\omega (\sum_{j\in\mathcal{J}}(y_{ij,\omega}^1 - y_{ij,\omega}^2) + r_{i,\omega}^1 - r_{i,\omega}^2) \end{pmatrix} \right\| \\ &\leq \bar{M}_i^x \left\|z^1 - z^2\right\|.\end{aligned}$$

where $\bar{M}_i^x = g^2(b_i^0 + \mathbb{E}b_i^\omega)$. The corresponding constant for $i \in G^c$ is seen to be zero allowing us to define M_5, the Lipschitz constant for term 5, by

$$M_5 = \max_{i\in G\cup G^c}\left(\sum_{\omega\in\Omega}\bar{M}_i^{l,\omega} + \bar{M}_i^x\right) = \max_{i\in G}\left(2(g+1)^2(b_i^0 + \mathbb{E}b_i^\omega)\right).$$

If $N_f = (g+1)$, then the overall Lipschitz constant for term 1 is given by

$$M \triangleq \max_{i\in G}\left(2N_f^2(b_i^0 + \mathbb{E}(b_i^\omega + \bar{L}_\zeta^{i,\omega}))\right).$$

Term 2: Term 2 may be bounded as

$$\begin{aligned}\|\mathbf{F}_d(z^1,\lambda^1) - \mathbf{F}_d(z^2,\lambda^2)\| &= \|\nabla d(z^1)^T\lambda_1 - \nabla d(z^2)^T\lambda^2\| \\ &\leq \|\nabla d(z^1)^T\lambda_1 - \nabla d(z^2)^T\lambda^1\| + \|\nabla d(z_2)^T(\lambda^1 - \lambda^2)\| \\ &\leq \|\nabla d(z^1) - \nabla d(z^2)\|\|\lambda^1\| + \|\nabla d(z^2)\|\|\lambda^1 - \lambda^2\|,\end{aligned}$$

where the inequalities follows from the application of the triangle inequality and the Cauchy-Schwartz inequality. Furthermore, $\nabla d(z)$ is a constant since $d(z)$ is a polyhedral constraint given by $d(z) = Bz$ implying that $\|\nabla d(z^1) - \nabla d(z^2)\| = 0$, allowing us to conclude that

$$\|\mathbf{F}_d(z^1,\lambda^1) - \mathbf{F}_d(z^2,\lambda^2)\| \leq \|B\|\|\lambda^1 - \lambda^2\|.$$

Term 3: Term 3 may be bounded by recalling that $d(z)$ is polyhedral, allowing us to proceed as follows

$$\begin{aligned}\|\mathbf{F}_\lambda(z_1,\lambda_1) - \mathbf{F}_\lambda(z_2,\lambda_2)\| &\leq \|d(z^1) - d(z^2)\| + \epsilon\|\lambda^1 - \lambda^2\| \\ &\leq \|B\|\|z^1 - z^2\| + \epsilon\|\lambda^1 - \lambda^2\|,\end{aligned}$$

where the inequalities follow again from the triangle inequality, the Cauchy-Schwartz inequality and the functional form of $d(z)$. It follows that the Lipschitz constant for the overall mapping is given by $(M + \|B\| + \epsilon)$.

The strong monotonicity of the mapping $\mathbf{F}^\epsilon$ with monotonicity constant ϵ can be deduced by noting that $\nabla \mathbf{F}^\epsilon$, given by

$$\nabla \mathbf{F}^\epsilon = \begin{pmatrix} \nabla_z \mathbf{F}_z + \epsilon \mathbf{I} & -\nabla d^T \\ \nabla d & \epsilon \mathbf{I} \end{pmatrix},$$

is positive definite since $\nabla_z \mathbf{F}_z$ is positive semidefinite for all z. ∎

Primal-dual scheme

When the mapping $\mathbf{F}_\epsilon(z,\lambda)$ is Lipschitz continuous and strongly monotone, the convergence of the primal-dual scheme can be claimed. Note that weaker conditions such as strict monotonicity can also be used to be guarantee convergence while mere monotonicity requires alternate schemes (such as two-step methods) (See [34, Ch. 12]).

Proposition 21 (Convergence of primal-dual scheme [34]) *Consider the primal-dual scheme given by* (2.15) *and* (2.16). *Suppose assumptions (A1)–(A2), (A4) hold. If the steplength* $\gamma^{pd} \leq 2\epsilon/L^2$, *then the sequence* $\{(z^k,\lambda^k)\}$ *converges to* $(z^*_\epsilon,\lambda^*_\epsilon)$, *an* $\epsilon-$*Nash equilibrium of* $\mathcal{G}^b$.

Note that an analogous result is available for the risk-neutral no-arbitrage game $\mathcal{G}^a$.

Exact and inexact dual schemes

In this subsection, we consider the dual scheme in both its exact and inexact forms. While a proof for the convergence of the original dual scheme is provided in [55], we present a different argument in a regularized setting. Crucial to this result is the supporting requirement on co-coercivity of $d(z(\lambda))$. We provide a proof that uses the mapping $\mathbf{F}^\epsilon_z, \mathbf{F}^\epsilon_f$ and $\mathbf{F}_d$ as defined in (2.24), adapted from a result in [55]. It must be emphasized that the inexact dual has been studied recently by the second author in a multiuser optimization setting [56, 57] and our results, while couched in a stochastic game-theoretic setting, are closely related. Yet, given that they have never been proved for equilibrium problems, we see the results here being of relevance. Furthermore, the polyhedral nature of $d(z)$ simplifies some of the proofs are often simpler and allows for somewhat different yet more refined bounds that relate the error directly to *problem size.*

Lemma 22 *Consider the function $d(z(\lambda))$ where $z(\lambda)$ is a solution to the primal problem* (2.12)*. Then $d(z(\lambda)) \equiv Bz$ is co-coercive with constant η_{cc} or*

$$(\lambda_2 - \lambda_1)^T (d(z(\lambda_1)) - d(z(\lambda_2))) \geq \eta_{cc} \| d(z(\lambda_2)) - d(z(\lambda_1)) \|^2 \qquad \textit{for all} \quad \lambda_1, \lambda_2 \in \mathbb{R}^m_+,$$

where $\eta_{cc} = \epsilon/(N_f N_g n)$. Furthermore, we have

$$\| z(\lambda_1) - z(\lambda_2) \| \leq \frac{\sqrt{N_f N_g n}}{\epsilon} \| \lambda_1 - \lambda_2 \| \qquad \textit{for all} \quad \lambda_1, \lambda_2 \in \mathbb{R}^m_+. \tag{2.25}$$

Proof : Let $z_1 \equiv z(\lambda_1)$ and $z_2 \equiv z(\lambda_2)$ represent solutions to $\mathrm{VI}(\mathbf{Z}, \mathbf{F}^\epsilon_z(z; \lambda_1))$ and $\mathrm{VI}(\mathbf{Z}, \mathbf{F}^\epsilon_z(z; \lambda_2))$, respectively. Then, we have

$$(z_2 - z_1)^T \mathbf{F}^\epsilon_z(z_1, \lambda_1) \geq 0 \quad \text{and} \quad (z_1 - z_2)^T \mathbf{F}^\epsilon_z(z_2, \lambda_2) \geq 0.$$

By recalling from (2.14), we have that

$$(z_2 - z_1)^T (\mathbf{F}_d(z_1, \lambda_1) - \mathbf{F}_d(z_2, \lambda_2)) \geq (z_2 - z_1)^T (\mathbf{F}^\epsilon_f(z_2, \lambda_2) - \mathbf{F}^\epsilon_d(z_1, \lambda_1)) \geq \epsilon \| z_2 - z_1 \|^2, \tag{2.26}$$

where the second inequality follows from the strong monotonicity of $\mathbf{F}^\epsilon_f$ with constant ϵ. It follows from the definition of $d(z)$ that

$$\begin{aligned} &(z_2 - z_1)^T (\mathbf{F}_d(z_1, \lambda_1) - \mathbf{F}_d(z_2, \lambda_2)) = (z_2 - z_1)^T (-B^T \lambda_1 + B^T \lambda_2) \\ &= (Bz_2 - Bz_1)^T (-\lambda_1 + \lambda_2) \geq \epsilon \| z_2 - z_1 \|^2 \geq \frac{\epsilon}{\|B\|^2} \| d(z_1) - d(z_2) \|^2, \end{aligned}$$

where the last two inequalities follow from (2.26) and the Lipschitz continuity of $d(z)$ with constant $\|B\|$. Finally by applying the Cauchy-Schwartz inequality to the first inequality above, the second result (2.25) may be obtained as follows:

$$\| z_2 - z_1 \|^2 \leq \frac{1}{\epsilon} (d(z_2) - d(z_1))^T (\lambda_2 - \lambda_1) \leq \frac{1}{\epsilon} \|B\| \| z_2 - z_1 \| \lambda_2 - \lambda_1 \|$$

$$\text{giving us} \quad \| z_2 - z_1 \| \leq \frac{\|B\|}{\epsilon} \| \lambda_2 - \lambda_1 \| \leq \frac{\sqrt{N_f N_g n}}{\epsilon} \| \lambda_2 - \lambda_1 \|. \quad \blacksquare$$

Using the co-coercivity of $d(z(\lambda))$, the convergence of the iterates constructed from regularized dual scheme can be shown to converge to λ^*_ϵ, a dual solution to the regularized problem.

Proposition 23 (Convergence of exact dual scheme) *Consider the dual scheme given by* (2.17) *and* (2.18)*. If* $d(z(\lambda))$ *is co-coercive with constant* $\eta_{cc} = \frac{\epsilon}{N_f N_g n}$ *and* γ_d *satisfies*

$$\gamma_d < \frac{2\epsilon}{2\epsilon^2 + N_f N_g n}, \tag{2.27}$$

then the sequence

$$\|\lambda^{k+1} - \lambda_\epsilon^*\| \leq q_d^k \|\lambda^0 - \lambda_\epsilon^*\| \quad \textit{where} \quad q_d = (1 - \gamma_d \epsilon).$$

Proof : By invoking the definition of λ^{k+1}, noting that λ^* is a fixed-point of (2.13) and the non-expansivity of the Euclidean projector, we have

$$\begin{aligned}
\|\lambda^{k+1} - \lambda_\epsilon^*\| &= \left\| \Pi_{\mathbb{R}_m^+}(\lambda^k - \gamma_d d(z^k) - \gamma_d \epsilon \lambda^k) - \lambda_\epsilon^* \right\| \\
&= \left\| \Pi_{\mathbb{R}_m^+}(\lambda^k - \gamma_d d(z^k) - \gamma_d \epsilon \lambda^k) - \Pi_{\mathbb{R}_m^+}(\lambda_\epsilon^* - \gamma_d d(z_\epsilon^*) - \gamma_d \epsilon \lambda_\epsilon^*) \right\| \\
&\leq \left\| (\lambda^k - \gamma_d d(z^k) - \gamma_d \epsilon \lambda^k) - (\lambda_\epsilon^* - \gamma_d d(z_\epsilon^*) - \gamma_d \epsilon \lambda^*) \right\| \\
&= \left\| (1 - \gamma_d \epsilon)(\lambda^k - \lambda_\epsilon^*) - \gamma_d (d(z^k) - d(z_\epsilon^*)) \right\|.
\end{aligned}$$

Then, by expanding the square of the expression on the right hand side and by leveraging the co-coercivity of $d(\lambda(z))$ with respect to z, we have the following inequality:

$$\begin{aligned}
\|\lambda^{k+1} - \lambda_\epsilon^*\|^2 &\leq (1 - \gamma_d \epsilon)^2 \|\lambda^k - \lambda_\epsilon^*\|^2 + (\gamma_d)^2 \|d(z^k) - d(z_\epsilon^*)\|^2 - 2\gamma_d (1 - \gamma_d \epsilon)(\lambda^k - \lambda_\epsilon^*)^T (d(z^k) - d(z_\epsilon^*)) \\
&\leq (1 - \gamma_d \epsilon)^2 \|\lambda^k - \lambda_\epsilon^*\|^2 + (\gamma_d^2 - 2\gamma_d \eta_{cc}(1 - \gamma_d \epsilon)) \|d(z^k) - d(z_\epsilon^*)\|^2,
\end{aligned}$$

where the second inequality follows from the co-coercivity of $d(z(\lambda))$ with a constant η_{cc}. Convergence of the scheme follows if γ_d is chosen in accordance with

$$\gamma_d < \min\left\{ \frac{1}{\epsilon}, \frac{2\eta_{cc}}{1 + 2\eta_{cc}\epsilon} \right\}, \quad \text{where} \quad \eta_{cc} = \frac{\epsilon}{N_f N_g n}.$$

But we have

$$\frac{2\eta_{cc}}{1 + 2\eta_{cc}\epsilon} = \frac{1}{\frac{N_f N_g n}{2\epsilon} + \epsilon} < \frac{1}{\epsilon} \quad \text{implying that} \quad \gamma_d < \frac{2\epsilon}{2\epsilon^2 + N_f N_g n}. \quad \blacksquare$$

The convergence of λ^k to λ_ϵ^* allows for deriving similar statements for z^k and the infeasibility, namely $\max(0, d(z^k))$.

Lemma 24 *Consider the dual scheme given by* (2.17) *and* (2.18) *and suppose* $d(z(\lambda))$ *is co-coercive with constant*

$\epsilon/\|B\|^2$. *Then for any* $k \geq 0$ *we have*

$$\|z^k - z^*\| \leq \frac{\sqrt{N_f N_g n}}{\epsilon}\|\lambda^k - \lambda_\epsilon^*\| \quad \textit{and} \quad \max(0, -d(z^k)) \leq \frac{N_f N_g n}{\epsilon}\|\lambda^k - \lambda_\epsilon^*\|.$$

Proof : A bound on the suboptimality may be directly obtained from Lemma 22. The infeasibility in the constraint $d(z) \geq 0$, namely $\max(0, -d(z))$, is bounded as shown through the following sequence of relationships, that use the Cauchy-Schwartz inequality and the bound on the suboptimality of z^k:

$$\max(0, -d(z^k)) \leq -Bz^k = -B(z^k + z_\epsilon^* - z_\epsilon^*) \leq B(z_\epsilon^* - z^k) \leq \|B\|\|z_\epsilon^* - z^k\| \leq \frac{N_f N_g n}{\epsilon}\|\lambda_\epsilon^* - \lambda^k\|. \quad \blacksquare$$

A shortcoming of the dual scheme is the need for exact primal solutions for every dual solution. Since this requires iteratively solving a fixed-point problem, it can prove to be an inordinately expensive component of the algorithm. Our intent is in constructing a *bounded complexity* variant that requires that only K iterations of the primal scheme be made for a given value of the dual iterates. This is given by

$$z_j^{t+1} = \Pi_{\mathbf{Z}_j}(z_j^t - \gamma_d(\mathbf{F}_z(z_j^t; z_{-j}^t, \lambda^k) + \epsilon^\ell z_j^t)), \text{ for all } j, t = 0, \ldots, K-1. \tag{2.28}$$

However, in obtaining error bounds, we require that the primal strategy sets be bounded. It is worth remarking that in general this bound may be difficult to obtain in closed-form but we assume that such a bound is available for purposes of this analysis. In the current setting, one avenue for deriving such a bound would be through imposing a bound on forward positions. In the remainder of this section, we assume that $\|z\| \leq M_z$ throughout the remainder of this section. Finally, the strong monotonicity of the primal problem implies that $\|z^t - z^*\| \leq q_p^{t/2}\|z^0 - z^*\|$, where $q_p = 2\epsilon/M^2 < 1$ where M is the Lipschitz constant of the primal mapping $\mathbf{F}_f(z)$, as specified in Lemma 20.

Proposition 25 (Error bounds for inexact-dual scheme) *Consider the inexact dual scheme given by* (2.28) *and* (2.18). *If* $d(z(\lambda))$ *is co-coercive with constant* $\epsilon/\|B\|^2$, $\|z\| \leq M_z$ *and* γ_d *satisfies*

$$\gamma_d < \frac{2\epsilon}{2\epsilon^2 + N_f N_g n},$$

then we have

$$\|\lambda^k - \lambda_\epsilon^*\| \leq q_d^k\|\lambda^0 - \lambda_\epsilon^*\|^k + \left(\frac{1 - q_d^k}{1 - q_d}\right)\left(\left(\frac{2}{\epsilon^2} + 4\right)(N_f N_g n)^{1/2} q_p^{K/2} M_z^2 (1 + (N_f N_g n)^{1/2} q_p^{K/2})\right).$$

Proof : As earlier, the definition of λ^{k+1} and the fixed-point property of λ^*_ϵ, we have the following inequality:

$$\begin{aligned}\|\lambda^{k+1}-\lambda^*_\epsilon\| &= \|\Pi_{\mathbb{R}^+_m}(\lambda^k-\gamma_d(d(z^k_K)+\epsilon_k\lambda^k))-\Pi_{\mathbb{R}^+_m}(\lambda^*_\epsilon-\gamma_d(d(z^*)+\epsilon_k\lambda^*_\epsilon))\| \\ &\le \|(\lambda^k-\gamma_d(d(z^k_K)+\epsilon_k\lambda^k))-(\lambda^*_\epsilon-\gamma_d(d(z^*)+\epsilon_k\lambda^*_\epsilon))\|\end{aligned}$$

By adding and subtracting terms and by using the triangle inequality, the right-hand side can be shown to be

$$\begin{aligned}&\|(\lambda^k-\gamma_d(d(z^k_K)+\epsilon\lambda^k))-(\lambda^*-\gamma_d(d(z^*)+\epsilon_k\lambda^*_\epsilon))\|^2 \\ &= \|(1-\gamma_d\epsilon)(\lambda^k-\lambda^*_\epsilon)-\gamma_d(d(z^k_K)-d(z^*_\epsilon))\|^2 \\ &= (1-\gamma_d\epsilon)^2\|\lambda^k-\lambda^*_\epsilon\|^2+\gamma_d^2\underbrace{\|d(z^k_K)-d(z^*_\epsilon)\|^2}_{\textbf{term 1}}\underbrace{-2\gamma_d(1-\gamma_d\epsilon)(\lambda^k-\lambda^*_\epsilon)^T(d(z^k_K)-d(z^*_\epsilon))}_{\textbf{term 2}}.\end{aligned}$$

By noting that $d(z_k)$ is given by $Bz_k \ge 0$ for some matrix B, it follows that term 1 can be bounded by

$$\|d(z^k_K)-d(z^*_\epsilon)\|^2 \le \|d(z^k_K)-d(z^k)\|^2+\|d(z^k)-d(z^*_\epsilon)\|^2+2\|d(z^k)-d(z^k_K)\|\|d(z^k)-d(z^*_\epsilon)\|.$$

Furthermore, by using the co-coercivity of $d(x(\lambda))$, term 2 may be bounded in the following fashion:

$$\begin{aligned}&-2\gamma_d(1-\gamma_d\epsilon)(\lambda^k-\lambda^*_\epsilon)^T(d(z^k_K)-d(z^*_\epsilon)) \\ &= -2\gamma_d(1-\gamma_d\epsilon)(\lambda^k-\lambda^*_\epsilon)^T(d(z^k)-d(z^*_\epsilon))-2\gamma_d(1-\gamma_d\epsilon)(\lambda^k-\lambda^*_\epsilon)^T(d(z^k_K)-d(z^k)) \\ &\le -2\gamma_d(1-\gamma_d\epsilon)\frac{\epsilon}{\|B\|^2}\|d(z^k)-d(z^*_\epsilon)\|^2+\gamma_d^2\|\lambda^k-\lambda^*_\epsilon\|^2+(1-\gamma_d\epsilon)^2\|d(z^k_K)-d(z^k)\|^2.\end{aligned}$$

Using the bounds on terms 1 and 2, we have the following:

$$\begin{aligned}&(1-\gamma_d\epsilon)^2\|\lambda^k-\lambda^*_\epsilon\|^2+\gamma_d^2\|d(z^k_K)-d(z^*_\epsilon)\|^2-2\gamma_d(1-\gamma_d\epsilon)(\lambda^k-\lambda^*_\epsilon)^T(d(z^k_K)-d(z^*_\epsilon)) \\ &\le (1-\gamma_d\epsilon)^2\|\lambda^k-\lambda^*_\epsilon\|^2+\gamma_d^2\left(\|d(z^k_K)-d(z^k)\|^2+\|d(z^k)-d(z^*_\epsilon)\|^2+2\|d(z^k)-d(z^k_K)\|\|d(z^k)-d(z^*_\epsilon)\|\right) \\ &-2\gamma_d(1-\gamma_d\epsilon)\frac{\epsilon}{\|B\|^2}\|d(z^k)-d(z^*_\epsilon)\|^2+\gamma_d^2\|\lambda^k-\lambda^*_\epsilon\|^2+(1-\gamma_d\epsilon)^2\|d(z^k_K)-d(z^k)\|^2 \\ &= \underbrace{\left((1-\gamma_d\epsilon)^2+\gamma_d^2\right)\|\lambda^k-\lambda^*_\epsilon\|^2+\left(\gamma_d^2-2\gamma_d\frac{\epsilon}{\|B\|^2}(1-\gamma_d\epsilon)\right)\|d(z_k)-d(z^*_\epsilon)\|^2}_{\textbf{term 3}} \\ &+\underbrace{(\gamma_d^2+(1-\gamma_d\epsilon)^2)\|d(z^k_K)-d(z^k)\|^2+2\gamma_d^2\|d(z^k)-d(z^k_K)\|\|d(z^k)-d(z^*_\epsilon)\|}_{\textbf{term 4}}.\end{aligned}$$

If γ_d is chosen in accordance with

$$\begin{array}{lll} \left((1-\gamma_d\epsilon)^2+\gamma_d^2\right) & <1, & \gamma_d < \frac{1+\epsilon^2}{2\epsilon} \\ \left(\gamma_d^2 - 2\gamma_d\eta_{cc}(1-\gamma_d\epsilon)\right) & <0, & \gamma_d < \frac{2\eta_{cc}}{1+2\eta_{cc}\epsilon}, \end{array} \implies \gamma_d < \min\left(\frac{1+\epsilon^2}{2\epsilon}, \frac{2\eta_{cc}}{1+2\eta_{cc}\epsilon}\right).$$

then term 3 would lead to a contraction. However, it can be seen that

$$\frac{2\eta_{cc}}{1+2\eta_{cc}\epsilon} = \frac{1}{\frac{N_f N_g n}{2\epsilon}+\epsilon} < \frac{1}{2\epsilon} < \frac{1+\epsilon^2}{2\epsilon},$$

if $\frac{N_f N_g n}{2\epsilon} > \epsilon$ or $N_f N_g n > 2\epsilon^2$. It suffices that

$$\gamma_d < \frac{2\epsilon}{2\epsilon^2 + N_f N_g n}.$$

Note that the error arising from term 4 may be bounded by recalling that $d(z) = Bz$ is a Lipschitz continuous mapping implying that

$$\begin{aligned} &(\gamma_d^2 + (1-\gamma_d\epsilon)^2)\|d(z_K^k) - d(z^k)\|^2 + 2\gamma_d^2\|d(z^k) - d(z_K^k)\|\|d(z^k) - d(z_\epsilon^*)\| \\ &\le (\gamma_d^2 + (1-\gamma_d\epsilon)^2)\|B\|^2\|z_K^k - z^k\|^2 + 2\gamma_d^2\|B\|\|z^k - z_K^k\|M_z. \end{aligned}$$

Then by observing that $\|z^k - z_K^k \le \|z^k - z_0^k\|q_p^{K/2} \le M_z q_p^{K/2}$, where the first inequality follows from geometric convergence of the sequence $\{z_K^k\}$ to z^k as $K \to \infty$ and the second follows from the boundedness of the primal space with bound M_z. It follows that

$$\begin{aligned} &(\gamma_d^2 + (1-\gamma_d\epsilon)^2)\|B\|^2\|z_K^k - z^k\|^2 + 2\gamma_d^2\|B\|\|z^k - z_K^k\|M_z \\ &\le (\gamma_d^2 + (1-\gamma_d\epsilon)^2)\|B\|q_p^K M_z^2 + 2\gamma_d^2\|B\|^2 q_p^{K/2} M_z^2. \end{aligned}$$

Finally, by observing that $(\gamma_d^2 + (1-\gamma_d\epsilon)^2) \le (\gamma_d^2 + (1+\gamma_d\epsilon)^2)$ which is further bounded by $(\frac{1}{\epsilon^2}+4)$ and $\gamma_d^2 \le \frac{1}{\epsilon^2}$, we have

$$\begin{aligned} &(\gamma_d^2 + (1-\gamma_d\epsilon)^2)\|B\|q_p^K M_z^2 + 2\gamma_d^2\|B\|^2 q_p^{K/2} M_z^2 \le (\frac{1}{\epsilon^2}+4)\|B\|q_p^K M_z^2 + \frac{2}{\epsilon^2}\|B\|q_p^{K/2}M_z^2 \\ &\le (\frac{2}{\epsilon^2}+4)\|B\|q_p^{K/2}M_z^2(1+\|B\|q_p^{K/2}) \\ &\le (\frac{2}{\epsilon^2}+4)(N_f N_g n)^{1/2} q_p^{K/2} M_z^2 (1 + (N_f N_g n)^{1/2} q_p^{K/2}) \end{aligned}$$

Then given a starting point λ^0, we have

$$\|\lambda^k - \lambda_\epsilon^*\| \leq q_d^k \|\lambda^0 - \lambda_\epsilon^*\|^k + \underbrace{\left(\frac{1-q_d^k}{1-q_d}\right)\left(\left(\frac{2}{\epsilon^2}+4\right)(N_f N_g n)^{1/2} q_p^{K/2} M_z^2 (1+(N_f N_g n)^{1/2} q_p^{K/2})\right)}_{\textbf{Error from inexact solution of primal}}. \quad \blacksquare$$

It can be seen that the error term arising from inexact primal solutions converges to zero as $K \to \infty$. We conclude this section with a bound on the suboptimality of z^k and infeasibility associated with $d(z^k)$ if the dual scheme terminates prematurely.

Lemma 26 *Consider the inexact dual scheme given by* (2.28) *and* (2.18). *If $d(z(\lambda))$ is co-coercive with constant $\epsilon/\|B\|^2$, $\|z\| \leq M_z$ and γ_d satisfies*

$$\gamma_d < \frac{2\epsilon}{2\epsilon^2 + N_f N_g n}.$$

Then for any nonnegative integers $k, K \geq 0$, we have

$$\|z_K^k - z_\epsilon^*\| \leq q_p^{K/2} M_z + \frac{\sqrt{N_f N_g n}}{\epsilon}\|\lambda^k - \lambda_\epsilon^*\|,$$
$$\max(0, -d(z_K^k)) \leq \sqrt{N_f N_g n}\left(q_p^{K/2} M_z + \frac{\sqrt{N_f N_g n}}{\epsilon}\|\lambda^k - \lambda_\epsilon^*\|\right).$$

Proof : The first result follows easily by using the triangle inequality and employing the earlier result.

$$\begin{aligned}\|z_K^k - z_\epsilon^*\| &\leq \|z_K^k - z^k\| + \|z^k - z_\epsilon^*\| \\ &\leq q_p^{K/2} M_z + \frac{\sqrt{N_f N_g n}}{\epsilon}\|\lambda^k - \lambda_\epsilon^*\|.\end{aligned}$$

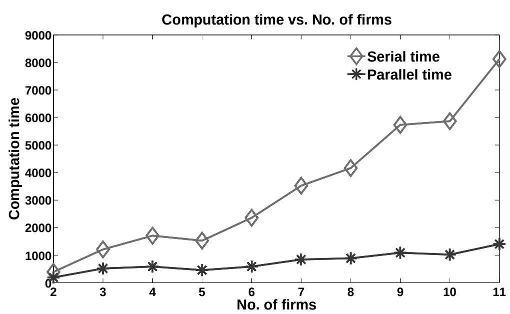
Similarly, the bound on the infeasibility at a point z_K^k is provided by adding and subtracting $d(z_\epsilon^*)$, applying the triangle and Cauchy-Schwartz inequality:

$$\begin{aligned}\max(0, -d(z_K^k)) &\leq -d(z_K^k) = -d(z_K^k) + d(z_\epsilon^*) - d(z_\epsilon^*) \\ &\leq -B(z_K^k - z_\epsilon^*) \leq \|B\|\|z_K^k - z^k\| + \|B\|\|z^k - z_\epsilon^*\| \\ &\leq (N_f N_g n)^{1/2} q_p^{K/2} M_z + \frac{N_f N_g n}{\epsilon}\|\lambda^k - \lambda_\epsilon^*\|. \quad \blacksquare\end{aligned}$$

2.4.4 Numerical performance

In this section, we examine the performance of our hybrid projection-based cutting-plane scheme with a focus on several questions. First, we consider whether the scheme scales with $|\Omega|$, $|\mathcal{J}|$ and $|\mathcal{G}|$. Second, we examine the relative performance of the primal-dual versus the dual scheme. Finally, we examine the benefits arising from inexact solutions of the primal problem.

We confine our discussion to the game $\mathcal{G}^b$ and examine the behavior of the scheme on a regularized game with $\epsilon = 1\mathtt{e}^{-3}$. In our computational results we define the loss function to be of the form: $\rho_{ij}^{\omega} = \chi(x_{ij} - cap_{ij}^{\omega})^{+}$, where $\chi = 0.5$. Therefore, in addition to the earlier set of constraints we have another constraint $s_{ij}^{\omega} \geq -m_{ij}$. Furthermore, we maintain χ to be the same across all agents. The risk aversion parameters are assumed to be 0.5 for all the agents unless specified otherwise. The nodal demand function intercepts were taken to be 150 and 200 for the spot and forward markets respectively across all nodes while the slopes of the spot-market price functions are specified to be normally distributed as per $N(1, 0.02)$ in the forward and spot markets.The algorithm was implemented on a Matlab 7.0 on a Linux OS with a processor with a clockspeed of 2.39 GHZ and a memory of 16 GB.



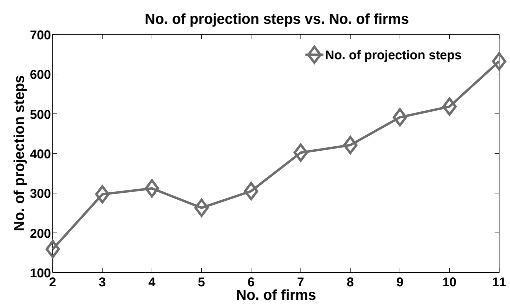


Figure 2.1: Scalability of effort with number of firms

Scalability: The algorithm is implemented in a distributed fashion with each agent solving his projection problem independently. As a consequence, we expect that the effort should scale with the number of agents. In fact, when the number of firms is raised from 2 to 11, the variation of serial and parallel times are shown in Figure 2.1. Note that the parallel time is computed assuming that there are as many processors as there are agents. The variation in the number of overall projection steps with increase in the number of firms is also shown in Figure 2.1. The projection scheme is terminated when $\epsilon^{inner} = 5e^{-3}$. Both graphs show that the effort, both in terms of CPU time and projection steps, grows slowly with the number of firms.

If an analogous question is studied when the number of generating nodes is varied, we observe similar results, as shown in Figure 2.2. Note that the the nodal problems decompose at the firm level implying that large networks, while computationally expensive, will not lead to rapid growth in effort. Instead, such settings will necessitate the solution of a larger number of separable nodal problems.

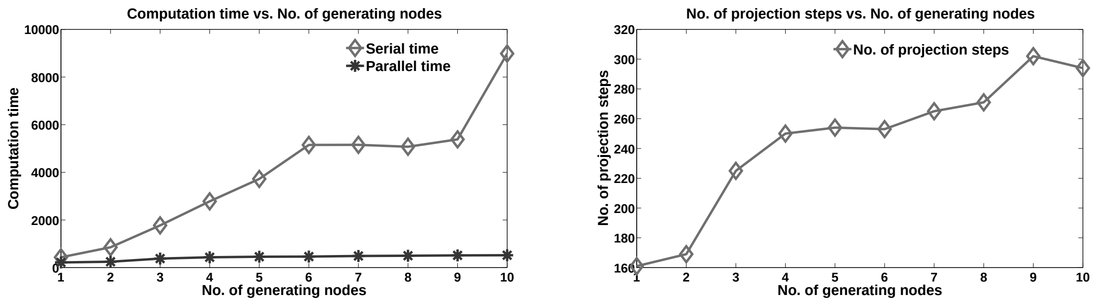


Figure 2.2: Scalability of effort with number of generating nodes

Perhaps the most challenging source of complexity is the uncertainty. This leads to arbitrarily large projection problems which are addressed through a cutting-plane method. If the number of scenarios from 30 to 240, then the variation of serial times are as seen in Figure 2.3. Additionally, the variation in the number of overall projection steps is also shown in Figure 2.3. It is seen that the effort grows slowly with an increase in the size of the sample-space, suggesting that the decomposition scheme for solving the projection problem proves efficient.

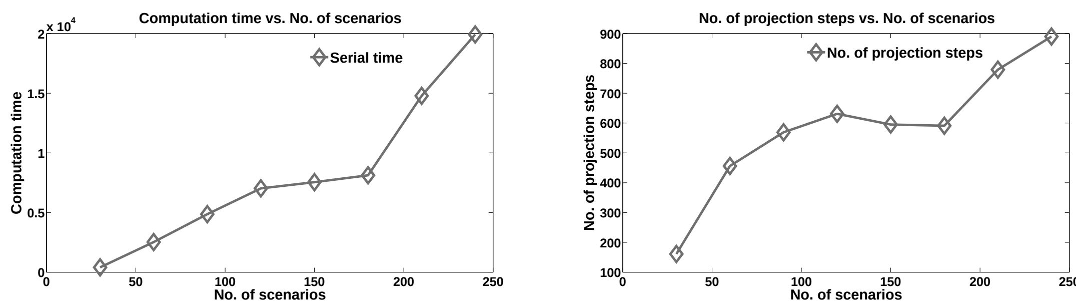


Figure 2.3: Scalability of effort with sample-size

Comparison between primal-dual and dual schemes: A two firm problem, under the setting of one generating node was taken as a case study to compare the primal-dual and inexact dual schemes. The primal and dual step lengths were taken to be 2 for all the cases. Different instances of the above problem were solved by varying the demand and generation capacities. Instances I to VI represent increasing values of (a, a^0) from (150, 200) to (400,450) respectively in steps of 50. Generation capacities were correspondingly increased from $(N(100, 0.5))$ to $(N(162.5, 0.5))$ in steps of 12.5. The above set of problems were solved for ten, fifteen, twenty and twenty-five scenarios. The table below shows the number of iterations and the time taken to solve each problem by means of the primal dual and inexact dual methods. In the case of inexact dual methods, we show results for one, five and nine inexact primal steps. It can be seen that the primal-dual schemes tend to be more efficient than dual scheme while fewer inner primal steps are generally advisable in the context of dual schemes.

Table 2.2: Comparison: primal dual and inexact dual algorithms

n	Inst.	Primal-dual		Dual-1			Dual-5			Dual-9		
		Steps	Time (s)	Dual steps	Total	Time (s)	Dual steps	Total	Time (s)	Dual steps	Total	Time (s)
10	1	53	49.34	53	53	50.50	14	70	79.80	9	81	98.18
	2	50	45.12	50	50	46.18	12	60	53.07	9	81	81.54
	3	50	47.72	50	50	48.64	12	60	46.94	8	72	57.00
	4	49	55.92	49	49	57.06	12	60	58.00	9	81	89.13
	5	50	54.36	50	50	55.56	14	70	81.56	10	90	110.53
	6	50	48.76	50	50	49.71	12	60	53.60	8	72	63.84
15	1	81	88.24	81	81	89.95	19	95	105.49	12	108	129.69
	2	76	101.36	76	76	103.00	18	90	121.50	11	99	127.95
	3	74	95.94	74	74	97.91	18	90	110.55	13	117	146.83
	4	74	103.84	74	74	105.47	18	90	122.76	11	99	127.25
	5	75	120.41	75	75	122.85	18	90	133.81	12	108	171.43
	6	75	122.29	75	75	124.70	18	90	149.65	11	99	149.65
20	1	108	180.32	108	108	183.22	25	125	207.85	16	144	249.98
	2	99	202.51	99	99	206.86	24	120	252.98	15	135	282.00
	3	97	173.93	97	97	176.48	24	120	220.43	14	126	209.15
	4	99	191.78	99	99	194.93	24	120	208.75	14	126	208.99
	5	98	183.08	98	98	186.31	24	120	212.35	14	126	202.91
	6	100	191.86	100	100	195.46	25	125	227.13	17	153	244.62
25	1	134	328.33	134	134	337.08	32	160	391.47	19	171	420.41
	2	122	289.25	122	122	294.83	29	145	337.35	18	162	388.56
	3	122	302.28	122	122	310.46	29	145	331.56	22	198	488.53
	4	120	289.69	120	120	294.15	29	145	343.48	21	189	381.28
	5	121	275.68	121	121	280.89	29	145	307.61	19	171	364.32
	6	121	272.19	121	121	274.31	29	145	309.52	19	171	366.11

2.5 Insights for market design and operations

In this section, we provide some insights for market design and operations by examining the strategic behavior of agents in the setting of a 53-node network, referred to as the Belgian grid and shown in Figure A.1 in the appendix. This network has provided the basis for prior studies [92, 91] and the line impedances and capacities are listed in Table A.1.3. We assume that nodes 7, 9, 10, 11, 14, 22, and 24 house generation facilities. We assume that the generation mix at each of these nodes is identical and is specified by Table 4.2. Here, the generation capacities and costs are assumed to be normally distributed across thirty scenarios ($n = 30$). Demand at all the nodes is articulated through affine functions. In the forward-clearing model, the intercepts in the forward and spot markets are taken to be fixed at 1500 at all nodes while the slopes in the spot market are assumed to vary normally with a mean of 1 and a standard deviation of 0.02. The parameter τ_j is taken to be 0.9 for all the firms and $\chi = 40$.

Table 2.3: Generator details

Generator type	Capacity	Linear cost	Quadratic cost
Oil 1	$N(2000, 10)$	$N(10, 1)$	$N(0.3, 0.01)$
Oil 2	$N(2000, 10)$	$N(10, 1)$	$N(0.3, 0.01)$
Wind 3	$N(650, 270)$	$N(0, 0)$	$N(0, 0)$
Wind 4	$N(730, 320)$	$N(0, 0)$	$N(0, 0)$
Coal 5	$N(1400, 10)$	$N(12, 1)$	$N(0.25, 0.01)$
Coal 6	$N(1400, 10)$	$N(12, 1)$	$N(0.25, 0.01)$

Our intent lies in ascertaining the relationship of a variety of parameters, such as risk-aversion, uncertainty and demand levels, on market outcomes such as forward market participation and penetration levels of wind resources.

The complementarity problems are solved via KNITRO [12].[4] The detailed formulation of the complementarity problems can be found in the appendix.

Risk aversion: In this setting, we vary risk aversion parameter κ_i for all the firms from 0 to 3 in steps of 0.5. As shown in Figure 2.4, we find that the forward bids drop for the wind generators and increase for the coal and oil generators. This behavior suggests that as generators become risk-averse, firms with a larger number of wind-based assets tend to be conservative in forward market bidding. This is primarily because firms with uncertain generation face much higher risk of shortfall. As they are penalized higher amounts for exposing the market to such risk, firms tend to bid lower, reducing their risk exposure. This is manifested through lower participation in the forward market by wind-based generators. In a prisoner's dilemma-type effect, generators exposed to less risk tend to increase their positions in the forward market. Figure 2.4 also shows that the excess of forward price over expected spot price (risk premium) increases with risk aversion. This is expected as total forward participation reduces, thereby raising forward prices, and leading to higher premiums.

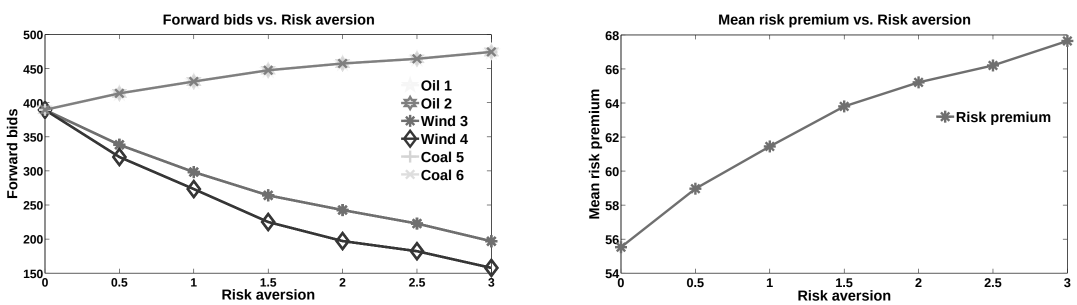


Figure 2.4: Impacts of increasing risk aversion

Uncertainty in generation capacity: Under the assumption that firms are assumed to have a constant risk aversion (fixed at 1 for all firms), we examine the relationship between uncertainty in capacity and risk exposure and level of forward participation. While coal and oil generators are expected to be close to deterministic in their availability, we assume that wind generators are faced with far greater uncertainty. In our numerical experiments, we vary the standard deviation of the wind generators (Wind 3 and Wind 4) from 10 to 885 in steps of 175. Expectedly, the risk exposure increases as the variability in wind assets grows (Figure 2.5). Moreover, while the general belief would be that participation in the forward markets would aid in hedging spot-market uncertainty, when risk-based penalties are introduced, we observe that wind-based generators are less inclined to participate. It should be emphasized that the deviation costs, arising from $\mathcal{G}^a$, tend to have a similar impact on behavior. Note that drops in forward market participation lead to higher prices in the forward market with respect to the spot and are are captured by an increase in risk-premium with higher uncertainty in wind assets.

[4] Note that the PATH solver would have proved to be a better choice but was not available to us on the Tomlab environment on Linux.

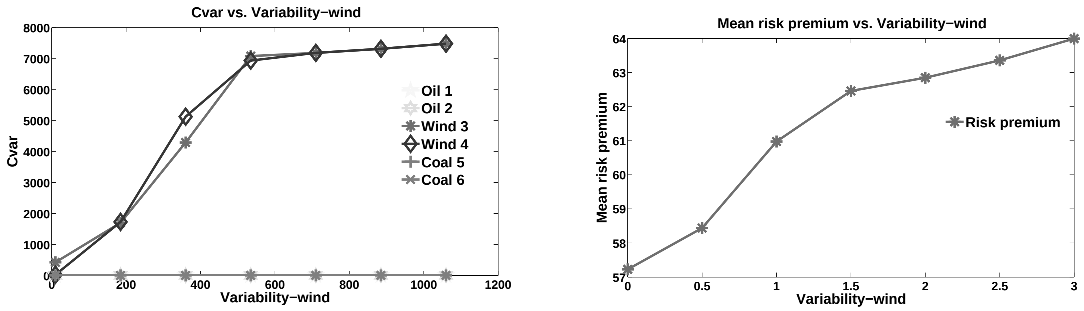


Figure 2.5: Impact of increasing uncertainty in wind-based capacity

Specification of forward price functions: A crucial question is how the choice of forward price function influences the results. In no-arbitrage models, this problem does not appear since the forward price function is not explicitly defined. In our market clearing models, we expect that our assumption on forward price function have significant impact on the results that emerge. Yet, it appears that for sufficiently low forward price intercepts, there is no forward market participation since the revenues garnered through participation are not sufficient. However, beyond a certain level, forward market participation becomes positive. Therefore, while the precise level of the forward market intercept is not as relevant, if the prices are set too low (a consequence of low intercepts), then this adversely affects bidding in this market. In our experiments, we fix the spot intercepts, slopes and forward slopes and vary the forward intercepts from 150 to 1800 in steps of 150. We find that there are no forward bids till a particular threshold of the forward intercept. Beyond this level, the forward bids and the premium increases as the forward intercept increases. Table 2.4 shows the variation of the forward bids and premium across nodes 7,10 and 11. We find that when there is no risk premium, there is no forward participation (when the expected spot prices are greater than the forward prices). When the risk premium is positive, there is an incentive for bidding in the forward market.

Table 2.4: Relationship of forward participation and risk premiums to forward price functions

Intercepts	Node 7		Node 10		Node 11	
	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$
450	0	-621.64	0	-689.73	0	-424.67
600	0	-471.64	0	-539.73	0	-274.67
750	0	-321.64	0	-389.73	0	-124.67
900	0	-171.64	0	-239.74	22.20	3.53
1050	0	-21.64	0	-89.73	149.35	28.71
1200	107.48	19.62	52.73	8.52	275.17	55.20
1350	229.01	46.68	178.06	121.77	403.47	35.55
1500	351.73	72.54	303.77	62.19	534.44	100.68
1650	478.82	93.97	434.17	84.24	665.41	122.11

Increasing penetration of wind: As the role of renewables in the nation's fuel mix grows, a question that remains is whether forward markets will continue attract participation. We investigate this question by increasing the mean of the capacity of the wind generators from 300 to 2050 in steps of 350 and also raise the standard deviations in

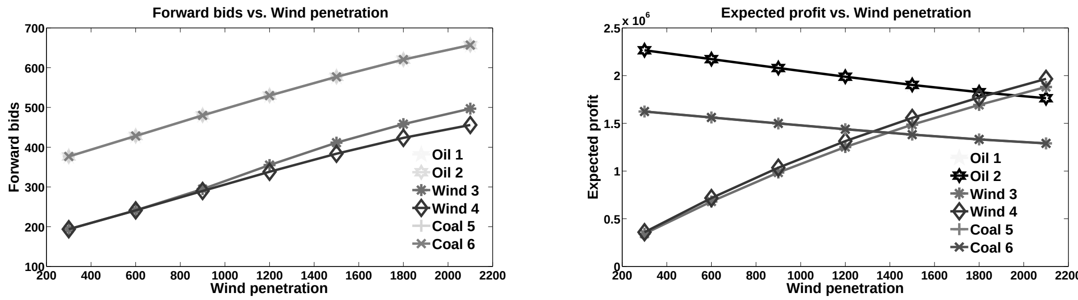


Figure 2.6: Increasing Penetration-Wind

availability from 150 to 1025 in steps of 175. We observe that for a fixed level of risk aversion, the forward bids of all the firms increase with increasing wind power penetration. This is in response to the volatility in the spot market with wind power penetration (Figure 3.1). It is also observed that with increasing wind power penetration, there is a significant increase in profits of wind generators at the expense of the profits of firms with no wind assets as shown.

Shared risk measures: We use the Belgian grid as our network and solve the smoothed game denoted by $\mathcal{G}^c_\epsilon$. We assume generation takes place at three nodes 7,9 and 10. The values of the forward and spot intercepts are maintained at 300 and 360, respectively for all scenarios, across all nodes. We solve a 12 scenario problem, with the smoothing parameter $\epsilon = 1, \tau = 0.9$ for all the firms and ψ is 1e-3. The slopes are normally distributed with $N(1, 0.02)$ and in the specification of the shared loss function, we assume that $t = 0.5$ and $\psi = 1\mathrm{e}{-3}$. Since, wind generators at a node will have the same windflow pattern, we assume that wind generators will have identical capacity distributions at a node. The generator details are given in Table 4.4. We fix the risk aversion κ for Wind-2 to be 0.5 and vary the risk aversion parameter for Wind-1 from 0.25 to 1.5 in steps of 0.25. implying that the ratio $\frac{\kappa_1}{\kappa_2}$ varies from 0.5 to 3 in steps of 0.5. The constant ς_{ij} is taken to be 20 across all nodes for all firms. In effect, we examine whether at equilibrium this risk measure penalizes risk-averse generators less for exposure in comparison with higher levels of exposure.

Table 2.5: Generator details: shared risk measures

Generator type	Capacity	Linear cost	Quadratic cost
Wind 1	$N(300, 180)$	$N(0, 0)$	$N(0, 0)$
Wind 2	$N(315, 189)$	$N(0, 0)$	$N(0, 0)$

With increasing values of the risk aversion ratio, we find that the forward participation of Wind-2 increases while that of Wind-1 drops (Figure 2.7). Interestingly, the total forward participation tends to drop as shown by Figure 2.7. Equilibrium values of total risk exposure follow a similar pattern as that of the forward bids.

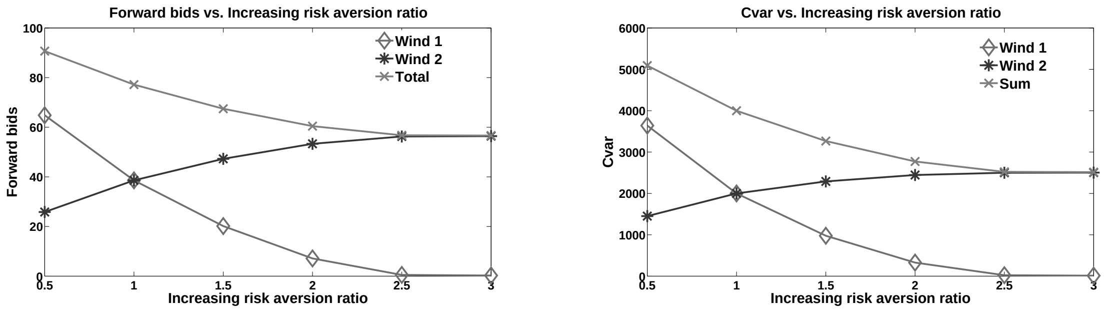


Figure 2.7: Shared risk measures

2.6 Summary

In this paper, we consider an uncertain two-period stochastic game with risk-averse agents arising in power markets where firms make first and second-period recourse decisions. The resulting game-theoretic problem can be viewed as a generalized Nash game under uncertainty. By observing that the coupling between the strategy sets is through a set of shared constraints, a subset of equilibria to the original game are given by the solution to an appropriately defined variational inequality.

Risk-averseness in the agent problems is captured through a conditional value-at-risk (CVaR) measure that leads to nonsmoothness. In fact, when these agent-specific measures are independent of competitive interactions, the related smooth games are shown to lead to solvable monotone variational inequalities. However when these risk measures are parameterized by the decisions of one's competitors, then such a reformulation does not lead to a tractable game. Instead, the multivalued variational inequality, as well its single-valued counterpart arising from the smoothed game, are shown to be solvable.

The monotonicity of the mapping in the variational problem allows for the use of regularized distributed projection schemes, both in a single time-scale (primal-dual) setting and a two time-scale (dual) setting. Rate of convergence estimates are provided for the dual scheme when the primal solution is computed exactly. A bounded complexity extension that allows for inexact computations of primal solution is also studied and leads to the provision of error bounds for the primal solution, dual solution and the infeasibility. The scalability of the projection scheme with $|\Omega|$, the cardinality of the sample space is contingent on effective solution of the projection step. In fact, we observe that this step essentially requires the solution of a strongly convex stochastic program and can be solved through a cutting-plane method that scales well with the cardinality of the sample-space. Numerical results support that the scheme scales well with the size of the network, the number of firms and the size of the sample-space.

The paper concludes with a discussion of insights for market design and operation by applying the model to a 53-node network drawn from the Belgian grid. Through this model, we observe that higher levels of risk-aversion lead to lower participation in the forward markets by agents with uncertain assets. Furthermore, higher levels of uncertainty in generation capacity leads to lower levels of forward participation. When forward price intercepts are sufficiently high, firms have incentives to participate in the forward market leading to a positive premium. Finally, the utility of the shared risk measure is found to aid in risk allocation, particularly when firms have diverse risk preferences.

Chapter 3

Strategic Behavior in Power Markets Under Uncertainty

3.1 Introduction

With increasing concerns of pollution and environmental impacts from fossil fuels, attention has shifted towards renewable sources of energy. Particularly, wind power is gaining prominence amidst several system operators throughout the United States. This immediately raises questions on the effect of wind power penetration on reliability, shortfall and consumer welfare in power markets. With growing uncertainty in the generation mix, power market games can no longer be addressed from a deterministic standpoint and stochastic counterparts need to be analyzed. However such models lead to necessarily large-scale problems that are less easy to solve directly. Therefore, in addition to analytical tractability development of efficient schemes and scalable algorithms gains relevance.

Game theoretic formulations have so far proved to be useful tools for analyzing power market designs. Game theory [36, 72] has its roots in the work by von Neumann and Morgenstern [89] while the Nash-equilibrium solution concept was forwarded by Nash in 1950 [69].

Market designs are generally characterized by settlements. A single settlement structure refers to one where firms bid in the real time market and are paid at current prices for their bids/ delivery [44, 45, 65]. A two settlement structure is characterized by firms bidding successively in the forward or day ahead market and the real time or spot market [13, 47, 49, 91]. Firms are paid at the forward price for their forward bids (promised generation levels). The deviation in the real time market is compensated at the real time price. These games may be analyzed under different levels of rationality. A completely rational model would yield in a game where agents compete in the first period market subject to spot market equilibrium. This leads to the most challenging class of problems namely the EPEC (Equilibrium problem with Equilibrium Constraints), where in each agent solves an MPEC (Mathematical Problem with Equilibrium Constraints) [92, 91, 85]. A bounded rationality framework leads to a game where forward and spot decisions are made simultaneously, leading to variational or complementarity formulations [66, 65, 45, 54], thereby leading to more tractable problems. Lastly models may also be specified by the ISO's objective which may be maximization of social welfare [54, 91, 92] or maximization of wheeling or transmission revenue [44, 65]. It may also be characterized by settings where ISO is not an active player [85].

This work employs the framework proposed in [45] and extends the methodology on the lines of [54]. As opposed to the ISO maximizing social welfare in [54] this work focuses on a setting where the ISO maximizes wheeling revenue. The model also gives an additional provision of selling power at multiple nodes that is independent of individual generation levels at a particular generation facility. We observe that the resulting game is a Nash game with coupled strategy sets. With regard to these important features of the model, the equivalent mathematical formulation yields a QVI (quasi variational inequality). With inherent difficulties in both computation and theoretical analysis, we move to the primal-dual space where the equilibrium conditions are given by a complementarity framework that is more tractable. This work also aims at solving the actual complementarity formulation by iterative regularization as opposed to solving a penalized/regularized problem in [54]. Lastly and most importantly, the model is a two settlement model as opposed to the single settlement model in [45].

Motivated by questions pertaining to penetration of renewable energy sources and uncertainty, a scheme is developed that enables solving truly large instances of this class of problems in a distributed fashion. Lastly by testing the model and scheme on a power grid, insights on the next generation market design are obtained.

The remaining of the paper is organized into five sections. Section 2 introduces the stochastic two-settlement electricity market model with market clearing conditions. In section 3, we analyze the properties of equilibria arising in such games by examining the properties of the complementarity formulation. A distributed scheme for computing equilibria for this class of problems is derived in Section 4. In section 5, we obtain insights through a two-settlement networked electricity market model. We conclude in section 6.

3.2 Model

Table 3.1: Notation

x_{ij}	Forward decision of sales from firm j at node i
s_{ij}^{ω}	Spot decision of sales from firm j at node i during scenario ω
$u_{ij}^{\omega}, v_{ij}^{\omega}$	Positive and negative deviations respectively at scenario ω from firm j at node i
$y_{ij}^{\omega}, cap_{ij}^{\omega}$	Total spot generation decision and total generation capacity at scenario ω for firm j at node i
r_i^{ω}	Import/export at scenario ω at node i
n, Ω, ρ^{ω}	Number of scenarios, set of all scenarios and probability of scenario ω
p_i^{ω}	Nodal demand function or price at scenario ω at node i
$c_{ij}^{\omega}, d_{ij}^{\omega}$	Coefficient of linear and quadratic terms in the cost function at scenario ω for firm j at node i
f_p, f_n	Penalty functions for positive and negative deviations
N	Total number of nodes in the network
a_i^0, b_i^0	Intercept and Slope respectively at node i in the forward market
$a_i^{\omega}, b_i^{\omega}$	Intercept and Slope respectively at node i at scenario ω
J	Total number of of firms
$Q_{l,i}$	Power flowing across line l due to unit injection/withdrawal of power at node i
K_l^{ω}	Transmission capacity of line l at scenario ω
$\mathcal{N}_j, \mathcal{N}_j^c$	Set of all generating nodes and non-generating nodes for firm j respectively
$\mathcal{J}_i$	Set of all generating firms at node i
$\mathcal{L}, \mathcal{N}$	Set of all transmission lines and set of all nodes respectively
G, G^c	Set of all generating nodes and load nodes respectively
$\mathcal{J}$	Set of all generating firms

A variety of settings have dealt with game theoretic power market problems. A prominent deterministic setting is one where firms compete in just a spot market or a single settlement market. However in power market regimes, the market is characterized by a two settlement framework, where firms bid quantities in the forward market and deviate in the real time market. Most notable two settlement classifications may be specified with regard to assumptions on rationality. Under complete rationality, firms compete in the forward market, subject to spot market equilibrium. Effectively the problem solved is an Equilibrium Problem with Equilibrium Constraints (EPEC), where each firm solves a Mathematical Program with Equilibrium Constraints (MPEC) [63]. Under the setting of bounded rationality firms are assumed to simultaneously take decisions in the spot and forward markets. This leads to variational and complementarity formulations.

We consider a bounded rationality framework under a Nash-cournot setting and extend the realm of the single-settlement model in [45] to a two settlement setting. We consider a network where nodes and transmission lines are denoted by $i \in \mathcal{N}$ and $l \in \mathcal{L}$. Firm $j \in \mathcal{J}$ bids x_{ij} at node i in the forward market and is paid at a common forward price p_i^0. The spot sales and generation are denoted by s_{ij}^ω and y_{ij}^ω respectively. The spot price may be denoted by p_i^ω. Positive and negative deviations are settled at the respective spot prices p_i^ω. This immediately raises a question of arbitrage. A no-arbitrage assumption specifies that the forward price equals expected spot price.

$$p_i^0 = \mathbb{E}(p_i^\omega).$$

In practice, arbitrage opportunities exist in power markets and the forward market is cleared independently. We assume that the forward and spot prices are defined by

$$p_i^0 \triangleq a_i^0 - b_i^0 \sum_{j \in \mathcal{J}} x_{ij}, \qquad p_i^\omega \triangleq a_i^\omega - b_i^\omega \sum_{j \in \mathcal{J}} s_{ij}^\omega,$$

where a_i^ω, a_i^0 and b_i^ω, b_i^0 denote the respective intercepts and slopes. In order to avoid settings where firms manipulate the market to earn unreasonable incomes, an additional layer of deviation penalties is added. This model allows for smooth penalties for positive and negative deviations. In addition, each firm incurs a generation cost at a generation facility. Firms are permitted to transfer power from the actual generation facilities to other nodes. The ISO is responsible for setting the transmission or wheeling at a price w_i^ω. If a firm generates more power than it sells at a particular node it exports power at a particular price and earns a revenue. In the other case, it pays the system operator for importing power. Giving allowance to the fact that multiple firms operate at several nodes in the network, the revenue of agent j may be written as follows:

$$\pi_j = \sum_{i \in \mathcal{N}} \left(\mathbb{E}(\pi_{ij}^{\omega}) + \pi_{ij}^{0} \right),$$

where

$$\pi_{ij}^{\omega} = \underbrace{(a_i^{\omega} - b_i^{\omega} \sum_{j \in \mathcal{J}} s_{ij}^{\omega}) s_{ij}^{\omega}}_{\text{Spot revenue}} - \underbrace{C_j^{\omega}(s_{ij}^{\omega})}_{\text{Costs}} - \underbrace{(f_{ij,p}^{\omega}(u_{ij}^{\omega}) + f_{ij,n}^{\omega}(v_{ij}^{\omega}))}_{\text{Deviation penalties}} - \underbrace{w_i^{\omega}(s_{ij}^{\omega} - y_{ij}^{\omega})}_{\text{Wheeling costs}},$$

$$\pi_{ij}^{0} = \left((a_i^0 - b_i^0 \sum_{j \in \mathcal{J}} x_{ij}) - \mathbb{E}((a_i^{\omega} - b_i^{\omega} \sum_{j \in \mathcal{F}} s_{ij}^{\omega})) \right) x_{ij}.$$

The generation levels of every firm are bounded by their capacities. The total sales across different nodes at the same time has to equal to the total quantity of generation for every firm. The forward and spot sales are related through

$$s_{ij}^{\omega} = x_{ij} + u_{ij}^{\omega} - v_{ij}^{\omega}, \quad \forall i \in \mathcal{N}, \forall j \in \mathcal{J}, \forall \omega \in \Omega.$$

Then, agent j's problem may be compactly represented as follows:

$$\boxed{\begin{array}{lll} \mathrm{Ag}^b(z_{-j}) & \text{maximize} & \pi_j^b(z_j; z_{-j}) = \sum_{i \in \mathcal{N}} \left\{ \mathbb{E}(\pi_{ij}^{\omega}) + \pi_{ij}^{0} \right\} \\ & \text{subject to} & \left\{ \begin{array}{cc} y_{ij}^{\omega} \leq cap_{ij}^{\omega} & (\alpha_{ij}^{\omega}) \\ s_{ij}^{\omega} - x_{ij} - u_{ij}^{\omega} + v_{ij}^{\omega} \leq 0 & (\beta_{ij}^{\omega}) \\ -s_{ij}^{\omega} + x_{ij} + u_{ij}^{\omega} - v_{ij}^{\omega} \leq 0 & (\gamma_{ij}^{\omega}) \\ \sum_{i \in \mathcal{N}} y_{ij}^{\omega} - \sum_{i \in \mathcal{N}} s_{ij}^{\omega} \leq 0 & (\delta_j^{\omega}) \\ \sum_{i \in \mathcal{N}} s_{ij}^{\omega} - \sum_{i \in \mathcal{N}} y_{ij}^{\omega} \leq 0 & (\phi_j^{\omega}) \\ x_{ij}, s_{ij}^{\omega}, y_{ij}^{\omega}, u_{ij}^{\omega}, v_{ij}^{\omega} \geq 0 & \end{array} \right\}, \forall i \in \mathcal{N}, \quad \forall l \in \mathcal{L}, \quad \forall \omega \in \Omega. \end{array}}$$

Note that the set of equality constraints with respect to s, u, v and x are written as two sets of inequality constraints. Similarly the equality constraints with regard to s and y are written as two sets of inequality constraints. This allows for the formulation of a pure complementarity problem. The symbols $\alpha, \beta, \gamma, \delta$ and ϕ represent Lagrange multipliers corresponding to the constraints.

3.2.1 ISO's problem

The ISO is responsible for allocation, dispatch and grid maintenance. The role of the ISO differs from one market to another. The ISO may refer to a non-profit organization that maximizes social welfare [54, 91, 92]. In certain settings, the ISO levies transmission or wheeling charges on the firms and thereby gains a revenue. We focus on the latter case where the ISO sets the wheeling charges [65, 45]. Subsequently the ISO needs to make sure that transmission constraints are not violated. Power distribution factors, or more specifically the Injection Shift Factor(ISF) may be used to quantify the power flowing across lines in a network. Let Q represent the power distribution factor matrix. Then the power flowing in transmission line l, due to unit injection or withdrawal of power at node i may be denoted by $Q_{l,i}$. The power distribution factor (ISF) is independent of uncertainty and is purely dependent on the network and the choice of the slack node. The details regarding the computation is presented in [62][1]. The ISO's problem may hence be defined as follows:

$$\boxed{\begin{array}{lll} \text{ISO} & \text{maximize} & \displaystyle\sum_{\omega\in\Omega}\sum_{i\in\mathcal{N}} \rho^\omega w_i^\omega r_i^\omega \\ & & \\ & \text{subject to} & r_i^\omega = \sum_{j\in\mathcal{J}}(s_{ij}^\omega - y_{ij}^\omega) \\ & & \sum_{i\in\mathcal{N}} Q_{l,i} r_i^\omega \leq K_l^\omega \quad (\mu_l^\omega) \\ & & -\sum_{i\in\mathcal{N}} Q_{l,i} r_i^\omega \leq K_l^\omega \quad (\eta_l^\omega) \end{array}}$$

Note that r_i refers to the inflow or outflow corresponding to node i. The strategy set of the ISO consists o the decision variables of the generating firms. However the firms' constraints are independent of the ISO's decisions. This leads to a generalized Nash game with coupled constraints. Due to the presence of coupled strategy sets we consider a primal-dual setting, where the KKT conditions of the ISO's problem may be compactly represented as follows[2]:

$$\begin{aligned} \text{free} \perp -\rho^\omega w_i^\omega + \sum_{l\in\mathcal{L}} Q_{l,i}\,(\mu_l^\omega - \eta_l^\omega) &= 0 \\ 0 \leq \mu_l^\omega \perp K_l^\omega - \sum_{i\in\mathcal{N}} Q_{l,i} \sum_{j\in\mathcal{J}} (s_{ij}^\omega - y_{ij}^\omega) &\geq 0 \\ 0 \leq \eta_l^\omega \perp K_l^\omega + \sum_{i\in\mathcal{N}} Q_{l,i} \sum_{j\in\mathcal{J}} (s_{ij}^\omega - y_{ij}^\omega) &\geq 0 \end{aligned}$$

[1] A slack node may be one, where injection or withdrawal of power is assumed to have no impact on any line in the network.

[2] $0 \leq x \perp y \geq 0$ implies that, $x, y \geq 0$ and $x^T y = 0$

3.3 Analysis

To move from the setting of the game to its variational/ complementarity formulation, the agent objectives have to be convex. Note that the ISO's objective is linear. We therefore go ahead to prove the convexity of the firms' objectives under the following assumptions.

Assumption 27

(A1) The cost of generation C_{ij}^{ω} is an increasing convex function of y_{ij}^{ω} for all $i \in \mathcal{N}$, $j \in \mathcal{J}$ and for all $\omega \in \Omega$.

(A2) The nodal forward and spot-market prices are defined by affine price functions (3.2) *for all $i \in \mathcal{N}$ and for all $\omega \in \Omega$.*

(A3) The forward slopes for all $i \in \mathcal{N}$ are defined such that, $b_i^0 \geq \frac{1}{4}\mathbb{E}b_i^{\omega}$.

(A4) The deviation penalty functions $f_{ij,p}$ and $f_{ij,n}$ are convex increasing functions of u_{ij}^{ω} and v_{ij}^{ω} for all $i \in \mathcal{N}, j \in \mathcal{J}$ and $\omega \in \Omega$.

Lemma 28 *Suppose (A1)–(A4) hold. Then the objective functions of the firms are convex.*

Proof : With convex generation costs and linear wheeling charges it suffices to prove the convexity of the expectation term of every agent's objective, given by $\eta_{ij}(x_{ij}, y_{ij}; y_{i,-j})$, defined as

$$\eta_{ij}(x_{ij}, s_{ij}; x_{i,-j}, s_{i,-j}) = -(a_i^0 - b_i^0 \sum_{j \in \mathcal{J}} x_{ij})x_{ij} - \sum_{\omega \in \Omega} \rho^{\omega}(a_i^{\omega} - b_i^{\omega}(\sum_{j \in \mathcal{J}} s_{ij}^{\omega}))(s_{ij}^{\omega} - x_{ij}).$$

The gradient and Hessian of this function are given by

$$\nabla \eta_{ij} = \begin{pmatrix} b_i^0 x_{ij} + b_i^0 \sum_{j \in \mathcal{J}} x_{ij} - a_i^0 + \sum_{\omega \in \Omega} \rho^{\omega} a_i^{\omega} - \sum_{\omega \in \Omega} \rho^{\omega} b_i^{\omega}(\sum_{j \in \mathcal{J}} s_{ij}^{\omega}) \\ \rho^{\omega}(-a_i^1 + b_i^1(s_{ij}^1 + \sum_{j \in \mathcal{J}} s_{ij}^1) - b_i^1 x_{ij}) \\ \vdots \\ \rho^n(-a_i^n + b_i^n(s_{ij}^n + \sum_{j \in \mathcal{J}} s_{ij}^n) - b_i^n x_{ij}) \end{pmatrix},$$

$$\text{and } \nabla^2 \eta_{ij} = \begin{pmatrix} 2b_i^0 & -\rho^1 b_i^1 & \dots & -\rho^n b_i^n \\ -\rho^1 b_i^1 & 2\rho^1 b_i^1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\rho^n b_i^n & 0 & \dots & 2\rho^n b_i^n \end{pmatrix}, \text{respectively.}$$

Let m be an arbitrary nonzero vector. Then by adding and subtracting terms, we have

$$\begin{aligned} m^T \nabla^2 \eta_{ij} m &= 2b_i^0 m_1^2 - 2m_1 \sum_{\omega=1}^{n} \rho^\omega b_i^\omega m_{\omega+1} + 2\sum_{\omega=1}^{n} \rho^\omega b_i^\omega m_{\omega+1}^2 \\ &= (2b_i^0 - \sum_{\omega=1}^{n} \rho^\omega \frac{b_i^\omega}{2}) m_1^2 + \sum_{\omega=1}^{n} \rho^\omega \frac{b_i^\omega}{2} m_1^2 - 2m_1 \sum_{\omega=1}^{n} \rho^\omega b_i^\omega m_{\omega+1} + 2\sum_{\omega=1}^{n} \rho^\omega b_i^\omega m_{\omega+1}^2 \\ &= (2b_i^0 - \sum_{\omega=1}^{n} \rho^\omega \frac{b_i^\omega}{2}) m_1^2 + \sum_{\omega=1}^{n} \rho^\omega b_i^\omega \left(\frac{m_1}{\sqrt{2}} - \sqrt{2} m_{\omega+1} \right)^2 . \end{aligned}$$

By assumption $\mathbb{E}\left(b_i^\omega\right) \le 4b_i^0$. This implies that $m^T \nabla^2 \eta_{ij} m \ge 0$ for all nonzero m and $\eta_{ij}(x_{ij}, s_{ij}; s_{i,-j})$ is a convex function in x_{ij} and s_{ij} for all fixed $x_{i,-j}$ and $s_{i,-j}$. The convexity of the agent objectives follow. ∎

For the sake of simplicity quadratic deviation penalties are assumed. Eliminating w_i^ω and adding the ISO's market clearing conditions to the equilibrium conditions of the agents, the entire game may be represented as a linear complementarity problem as follows:

$$0 \le x_{ij} \perp \sum_{j \in \mathcal{J}} b_i^0 x_{ij} + b_i^0 x_{ij} - a_i^0 + \sum_{\omega \in \Omega} \rho^\omega (a_i^\omega - b_i^\omega \sum_{j \in \mathcal{J}} s_{ij}^\omega) - \sum_{\omega \in \Omega} \beta_{ij}^\omega + \sum_{\omega \in \Omega} \gamma_{ij}^\omega \ge 0$$

$$0 \le s_{ij}^\omega \perp \rho^\omega \left(b_i^\omega \sum_{j \in \mathcal{J}} s_{ij}^\omega + b_i^\omega s_{ij}^\omega - b_i^\omega x_{ij} - a_i^\omega \right) + \sum_{l \in \mathcal{L}} Q_{l,i} (\mu_l^\omega - \eta_l^\omega) - \delta_j^\omega + \phi_j^\omega + \beta_{ij}^\omega - \gamma_{ij}^\omega \ge 0$$

$$0 \le y_{ij}^\omega \perp \rho^\omega (d_{ij}^\omega y_{ij}^\omega + e_{ij}^\omega) - \sum_{l \in \mathcal{L}} Q_{l,i} (\mu_l^\omega - \eta_l^\omega) + \alpha_{ij}^\omega + \delta_j^\omega - \phi_j^\omega \ge 0$$

$$0 \le u_{ij}^\omega \perp \rho^\omega \left(e_{ij}^\omega u_{ij}^\omega + h_{ij}^\omega \right) - \beta_{ij}^\omega + \gamma_{ij}^\omega \ge 0$$

$$0 \le v_{ij}^\omega \perp \rho^\omega \left(o_{ij}^\omega v_{ij}^\omega + t_{ij}^\omega \right) + \beta_{ij}^\omega - \gamma_{ij}^\omega \ge 0$$

$$0 \le \alpha_{ij}^\omega \perp cap_{ij}^\omega - y_{ij}^\omega \ge 0$$

$$0 \le \beta_{ij}^\omega \perp x_{ij} + u_{ij}^\omega - v_{ij}^\omega - s_{ij}^\omega \ge 0$$

$$0 \le \gamma_{ij}^\omega \perp -x_{ij} - u_{ij}^\omega + v_{ij}^\omega + s_{ij}^\omega \ge 0$$

$$0 \le \delta_j^\omega \perp \sum_{i \in \mathcal{N}} s_{ij}^\omega - \sum_{i \in \mathcal{N}} y_{ij}^\omega \ge 0$$

$$0 \le \phi_j^\omega \perp \sum_{i \in \mathcal{N}} y_{ij}^\omega - \sum_{i \in \mathcal{N}} s_{ij}^\omega \ge 0$$

$$0 \le \mu_l^\omega \perp K_l^\omega - \sum_{i \in \mathcal{N}} Q_{l,i} \sum_{j \in \mathcal{J}} (s_{ij}^\omega - y_{ij}^\omega) \ge 0$$

$$0 \le \eta_l^\omega \perp K_l^\omega + \sum_{i \in \mathcal{N}} Q_{l,i} \sum_{j \in \mathcal{J}} (s_{ij}^\omega - y_{ij}^\omega) \ge 0$$

Note that the above can be represented by the pair LCP (Z, M), where Z denotes $\mathbb{R}^{N+}$ (set of positive reals) and M represents the LCP matrix. Let

$$z = \begin{pmatrix} p \\ d \end{pmatrix}, p = \begin{pmatrix} p_1 \\ \vdots \\ p_N \end{pmatrix}, p_i = \begin{pmatrix} x_i \\ s_i \\ y_i \\ u_i \\ v_i \end{pmatrix}, s_i = \begin{pmatrix} s_i^1 \\ \vdots \\ s_i^n \end{pmatrix}, s_i^\omega = \begin{pmatrix} s_{i1}^\omega \\ \vdots \\ s_{iJ}^\omega \end{pmatrix}, d = \begin{pmatrix} d_1 \\ \vdots \\ d_N \\ \delta \\ \phi \\ \mu \\ \eta \end{pmatrix}, d_i = \begin{pmatrix} \alpha_i \\ \beta_i \\ \gamma_i \end{pmatrix}, \alpha_i = \begin{pmatrix} \alpha_i^1 \\ \vdots \\ \alpha_i^n \end{pmatrix},$$

$$\alpha_i^\omega = \begin{pmatrix} \alpha_{i1}^\omega \\ \vdots \\ \alpha_{iJ}^\omega \end{pmatrix}, \delta = \begin{pmatrix} \delta^1 \\ \vdots \\ \delta^n \end{pmatrix}, \delta^\omega = \begin{pmatrix} \delta_1^\omega \\ \vdots \\ \delta_J^\omega \end{pmatrix}, \mu = \begin{pmatrix} \mu^1 \\ \vdots \\ \mu^n \end{pmatrix}, \mu^\omega = \begin{pmatrix} \mu_1^\omega \\ \vdots \\ \mu_L^\omega \end{pmatrix}.$$

The notation for x, y, u and v follows that of s. Note that x is defined only for the forward market. The problem may be written as

$$0 \leq z \perp Mz + q \geq 0, \text{where}$$

$$M = \begin{pmatrix} M_p & -M_d^T \\ M_d & 0 \end{pmatrix}.$$

The matrices M_p and M_d may be written as follows:

$$M_p = \begin{pmatrix} M_{p,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & M_{p,N} \end{pmatrix}, \quad M_{p,i} = \begin{pmatrix} \bar{M}_{p,i} & 0 \\ 0 & T_i \end{pmatrix}, \quad \bar{M}_{p,i} = \begin{pmatrix} N_i^0 & P_i^1 & \dots & P_i^n \\ R_i^1 & N_i^1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ R_i^n & 0 & \dots & N_i^n \end{pmatrix}, \quad T_i = \begin{pmatrix} T_{y,i} & 0 & 0 \\ 0 & T_{u,i} & 0 \\ 0 & 0 & T_{v,i} \end{pmatrix}.$$

It can be seen that the matrix M_p is the coefficient matrix corresponding to the primal constraints and primal variables. It is to be noted that M_p has been broken up with regard to nodes and written accordingly. Matrices $T_{u,i}$,

$T_{v,i}$ and $T_{y,i}$ represent diagonal matrices. These matrices may be further broken down with regard to scenarios (ω).

$$T_{u,i} = \text{diag}\begin{pmatrix} T^1_{u,i} & \dots & T^n_{u,i} \end{pmatrix}, \quad T_{v,i} = \text{diag}\begin{pmatrix} T^1_{v,i} & \dots & T^n_{v,i} \end{pmatrix}, \quad T_{y,i} = \text{diag}\begin{pmatrix} T^1_{y,i} & \dots & T^n_{y,i} \end{pmatrix},$$
$$T^\omega_{u,i} = \text{diag}\begin{pmatrix} \rho^\omega e^\omega_{i1} & \dots & \rho^\omega e^\omega_{iJ} \end{pmatrix}, \quad T^\omega_{v,i} = \text{diag}\begin{pmatrix} \rho^\omega o^\omega_{i1} & \dots & \rho^\omega o^\omega_{iJ} \end{pmatrix}, \quad T_{y,i} = \text{diag}\begin{pmatrix} \rho^\omega d^\omega_{i1} & \dots & \rho^\omega d^\omega_{iJ} \end{pmatrix},$$
$$P^\omega_i = -\rho^\omega b^\omega_i ee^T, \quad N^\omega_i = \rho^\omega b^\omega_i (I + ee^T), \quad N^0_i = b^0_i (I + ee^T), \quad R^\omega_i = -\rho^\omega b^\omega_i I.$$

Note that I refers to an identity matrix and e represents a column vector of ones. The matrix M_d refers to the coefficient matrix corresponding to the dual constraints and primal variables. Note that $-M_d^T$ refers to the coefficient matrix corresponding to the primal constraints and dual variables.

$$M_d = \begin{pmatrix} D_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & D_N \\ E_1 & \dots & E_N \\ -E_1 & \dots & -E_N \\ F_1 & \dots & F_N \\ -F_1 & \dots & -F_N \end{pmatrix}, \quad D_i = \begin{pmatrix} 0 & 0 & -I & 0 & 0 \\ I & -I & 0 & I & -I \\ -I & I & 0 & -I & I \end{pmatrix}, \quad E_i = \begin{pmatrix} 0 & B_i & -B_i & 0 & 0 \end{pmatrix}, \quad B_i = I,$$

$$F_i = \begin{pmatrix} 0 & -K_i & K_i & 0 & 0 \end{pmatrix}, \quad K_i = \begin{pmatrix} Q_i & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & Q_i \end{pmatrix}, \quad Q_i = \begin{pmatrix} Q_{1,i} & \dots & Q_{1,i} \\ \vdots & \ddots & \vdots \\ Q_{L,i} & \dots & Q_{L,i} \end{pmatrix}.$$

Lemma 29 *Suppose (A1)–(A4) hold. Then M is positive semidefinite.*

Proof : With assumptions on convexity of costs and deviation it clearly suffices to show that $\bar{M}_{p,i}$ is positive semi-definite. Let m be an arbitrary column vector. Then,

$$\begin{aligned} m^T \bar{M}_{p,i} m &= (b^0_i - \sum_{\omega=1}^n \frac{\rho^\omega b^\omega_i}{4}) \sum_{k=1}^g m_k^2 + (b^0_i - \sum_{\omega=1}^n \frac{\rho^\omega b^\omega_i}{4})(\sum_{k=1}^g m_k)^2 - \sum_{\omega=1}^n \rho^\omega b^\omega_i \sum_{k=1}^g m_k m_{\omega g+k} \\ &+ \sum_{\omega=1}^n \frac{\rho^\omega b^\omega_i}{4} \sum_{k=1}^g m_k^2 + \sum_{\omega=1}^n \frac{\rho^\omega b^\omega_i}{4} (\sum_{k=1}^g m_k)^2 - \sum_{\omega=1}^n \rho^\omega b^\omega_i \sum_{k=1}^g m_k \sum_{k=1}^g m_{\omega g+k} \\ &+ \sum_{\omega=1}^n \rho^\omega b^\omega_i (\sum_{k=1}^g m^2_{\omega g+k} + (\sum_{k=1}^g m_{\omega g+k})^2) \end{aligned}$$

Combining terms and completing the squares we get the following expression:

$$\begin{aligned} m^T \bar{M}_{p,i} m = \quad & (b^0_i - \sum_{\omega=1}^n \frac{\rho^\omega b^\omega_i}{4}) \sum_{k=1}^g m_k^2 + (b^0_i - \sum_{\omega=1}^n \frac{\rho^\omega b^\omega_i}{4})(\sum_{k=1}^g m_k)^2 + \sum_{\omega=1}^n \left(\rho^\omega b^\omega_i \sum_{k=1}^g \left(\frac{m_k}{2} - m_{\omega g+k} \right)^2 \right) \\ & + \sum_{\omega=1}^n \rho^\omega b^\omega_i \left(\sum_{k=1}^g m_{\omega g+k} - \sum_{k=1}^g \frac{m_k}{2} \right)^2 \geq 0. \end{aligned}$$

This proves the fact that $\bar{M}_{p,i}$ is positive semidefinite. This completes the proof. ∎

This property of monotonicity in conjunction with Theorem 2.4.7 stated in [34] enables us to claim that solutions exist to the above CP and therefore the game.

Theorem 30 *Let Z be a polyhedral cone in $\mathbb{R}^N$ and F be a monotone affine map from $\mathbb{R}^N$ into itself. The CP(Z,F) is solvable if and only if it is feasible.*

Theorem 31 *Let assumptions (A1)–(A4) hold. Then $\mathcal{G}$ admits an equilibrium.*

Proof : It has already been proved that the matrix M is positive semi-definite. It is also clear that Z is a polyhedral cone in $\mathbb{R}^{N+}$.Therefore it suffices to show that, there exists a vector $z^{\text{ref}} \in Z$ such that $(Mz^{\text{ref}} + q) \geq 0$. Let

$$s^{\text{ref}} \triangleq 0, \quad y^{\text{ref}}, u^{\text{ref}}, \beta^{\text{ref}}, \gamma^{\text{ref}}, \delta^{\text{ref}}, \mu^{\text{ref}}, \eta^{\text{ref}} \triangleq 0. \quad \text{Furthermore let,} \quad \alpha^{\text{ref}}_{ij,\omega} \triangleq \max_{i\in\mathcal{N}} \rho^\omega a_i^\omega + \max_{i\in\mathcal{N}} \rho^\omega b_i^\omega \frac{a_i^0}{b_i^0} + 2\Delta,$$
$$x^{\text{ref}}_{ij} \triangleq \frac{a_i^0}{b_i^0}, \quad v^{\text{ref}}_{ij,\omega} = x^{\text{ref}}_{ij}, \quad \phi^{\text{ref}}_{j,\omega} \triangleq \max_{i\in\mathcal{N}} \rho^\omega a_i^\omega + \max_{i\in\mathcal{N}} \rho^\omega b_i^\omega \frac{a_i^0}{b_i^0} + \Delta, \text{where} \Delta > 0.$$

Let $\bar{N}$ refer to the size of the vector z. It is seen that z^{ref} satisfies both $(Mz^{\text{ref}} + q)_i \geq 0, \quad z_i^{\text{ref}} \geq 0, \quad$ forall $\quad i = 1, 2, ..., \bar{N}$ and is a feasible point to the CP(Z,F). This completes the proof. ∎

3.4 Distributed schemes

As stated previously, the variational formulation of the game leads to moving strategy sets that prevents any suitable schemes to be deployed. Whereas, the deeper analysis of its complementarity counterpart has lead to a monotone problem that is more tractable. However to build convergent schemes, the resulting mapping needs to be strongly monotone. From the algorithmic standpoint, regularization of the original problem may be an immediate fix. But, operating with higher values of regularization, in spite of the advantage of faster convergence, may lead to highly inexact solutions. Lesser the penalty due to regularization, slower is the convergence. Motivated by the above issues, we come up with two different schemes namely the iterative-Tikhonov regularization scheme (ITR) and the iterative proximal point (IPP) scheme. Before proceeding to the algorithmic description and convergence theory, we review a few basic concepts.

Theorem 32 *Consider the CP(Z,F), where $F = Mz + q$ and $Z \triangleq R^{N+}$. Then, z^* is a solution to the CP(Z,F) if and only if z^* solves the VI(Z,F) (Karmarkar's result).*

Note that z^* is a solution to the VI(Z,F) if and only if,

$$z^* = P_Z(z^* - \gamma F(z^*)), \tag{3.1}$$

a solution to the fixed point problem. A regularization to a monotone mapping would yield a strongly monotone mapping. The standard Tikhonov scheme rests on solving a sequence of such well posed regularized problems. The sequence of such iterates tends to the solution of the original problem. Mathematically the iterates are defined by

$$z^k = \Pi_{\mathbf{Z}}\left(z^k - \gamma(F(z^k) + \epsilon^k z^k)\right), \quad \lim_{k\to\infty} \epsilon_k = 0, \quad \lim_{k\to\infty} z^k = z^*.$$

There are several exact and inexact variants of the Tikhonov scheme. However the Tikhonov scheme faces computational difficulties as the regularization$\left\{\epsilon^k\right\}$ drops to zero.

An alternative to this scheme is the proximal scheme that stems from a similar concept of solving regularized/penalized problems. However, in this scheme strong monotonicity is maintained (usually fixed) at every step. Each step solves a subproblem which may be defined as follows:

$$F^k = F(z) + \theta(z - z^{k-1}), \quad z^k = \Pi_Z\left(z^k - \gamma F(z^k) + \theta(z^k - z^{k-1})\right) \quad \text{or} \quad z^k = \mathrm{SOL}(\mathrm{VI}(K, F^k)).$$

The regularization $\theta > 0$, may be a fixed or variable parameter. The fundamental idea is that as the iterates converge($\|z^k - z^{k-1}\| \to 0$), F^k converges to $F(z^k)$, thereby solving the original fixed point problem. The convergence of this scheme is well established for monotone variational inequalities and may be found in [34, 5].

3.4.1 Iterative Tikhonov scheme

The original Tikhonov scheme is a two timescale scheme that rests on solving a sub problem given by solving a strongly monotone variational inequality, at every step. In general this may be a computationally challenging requirement. We alleviate this problem by introducing an iterative projection scheme that is essentially of single time order. The scheme may be stated as follows: Note that the value of ψ is taken to be 1 in general. Δ refers

Algorithm 3: Iterative Tikhonov Algorithm

```
0  initialization k = 0;
   choose constants ψ, Δ > 0  and  ε_0, γ_0 > 0 and α ∈ (0.5, 1), β ∈ (0, 0.5), initial point (z^0) ;
   while ||F̄^nat(z^k, F^k)|| > Δ do
1      z^{k+1} = Π_Z ( z^k − γ_k(F(z^k) + ε_k z^k) ) ;
2      Update regularization ε_{k+1} := ε_0 / (k+1)^β ;
3      Update step size γ_{k+1} := γ_0 / (k+1)^α ;
4      Compute ||F̄^nat(z^k, F^k)|| = ||z^k − Π_Z ( z^k − ψ(F(z^k)) )||;
5      k := k + 1;
   end
```

to the stopping criterion. Ideally $\Delta = 0$ implies that the solution is a fixed point of the problem. The basic idea behind this scheme is that as the regularization drops to zero and if the iterates converge, the limit point would solve the fixed point problem. The following theorem from [93] specifies the requirements on the step size and the regularization parameter to obtain a convergent solution.

Theorem 33 *Consider the VI($\mathbf{Z}, \mathbf{F}$). Let the step sizes be defined such that $\sum_{k=1}^{\infty} \gamma_k = \infty$ and $\sum_{k=1}^{\infty} \gamma_k^2 < \infty$. In addition, let $\sum_{k=1}^{\infty} \epsilon^k \gamma_k = \infty$. Further let $\epsilon^k \to 0$ and $\lim_{k \to \infty} \frac{\gamma_k}{\epsilon_k} = 0$. Let F be monotone on Z. Then, the sequence z^k converges to z^*, where z^* is given by the fixed point relation in* (3.1).

Lemma 34 *Consider the choice of step sizes and regularizations of the form $\gamma_k = \frac{\gamma_0}{k^\alpha}$ and $\epsilon^k = \frac{\epsilon_0}{k^\beta}$. Let $0.5 < \alpha < 1$ and $0 < \beta < 0.5$. Then the parameters satisfy the requirements stated in Theorem 33*

Proof : The proof has been discussed in [93].

3.4.2 Iterative proximal scheme

We also present a modified single timescale version of the proximal scheme. Just as the former ITR scheme, each iterate is characterized by a projection step. A proximal term marked by a scaled difference between the two previous terms is added to the existing term. Mathematically,

$$z^{k+1} = \Pi_Z \left(z^k - \gamma_k (F(z^k) + \theta(z^k - z^{k-1})) \right).$$

The fundamental idea is that as the iterates converge, for a standard fixed step length, the limit point solves the

Algorithm 4: Iterative Proximal Point Algorithm

```
initialization k = 0;
choose constants ψ, Δ > 0  and  θ, γ_0 > 0 and α ∈ (0.5, 1), initial point (z^0) ;
while ||F̄^nat(z^k, F^k)|| > Δ do
    z^{k+1} = Π_Z ( z^k − γ_k (F(z^k) + θ(z^k − z^{k−1})) ) ;
    Update step size γ_{k+1} := γ_0 / (k+1)^α ;
    Compute ||F̄^nat(z^k, F^k)|| = ||z^k − Π_Z ( z^k − ψ(F(z^k) + θ(z^k − z^{k−1})) )||;
    k := k + 1;
end
```

fixed point problem. However, the theoretical convergence for the fixed step length case is yet to be analyzed. The convergence of the scheme for diminishing step-lengths has been proved in [53] and the formal result is stated as follows:

Theorem 35 *Let the mapping F be strictly monotone on Z that is convex. In addition, let Z be closed and compact. Let $\sum_{k=1}^{\infty} \gamma_k = \infty$ and let $\sum_{k=1}^{\infty} \gamma_k^2 < \infty$. Let $\theta > 0$ be a fixed parameter. Then the sequence $\left\{ z^k \right\} \to z^*$, where z^* refers to a solution to the VI(Z, F).*

3.4.3 Numerical experiments

This section analyzes the performance of the proposed algorithms. The scalability of the iterative Tikhonov regularization scheme was tested by applying it to large scale stochastic problems (large ω). The iterative Tikhonov and the iterative proximal schemes were compared for different test cases. The discussion was confind to a simulated grid that is shown below. The number of nodes and transmission lines were twelve and thirteen respectively. The grid details are shown in Table 3.2. Node 12 was chosen to be the slack node. Four generators were assumed to compete in the market, the details of which are mentioned in Table 3.3. The spot intercepts were taken to be 700 across all nodes and all scenarios. The forward and spot slopes were taken to be $N\ (1, 0.02)$ across all nodes and all scenarios. Linear and quadratic deviation penalties were taken to be $N\ (8, 0)$ and $N\ (8, 0)$ respectively for all generators at all nodes and scenarios. The schemes were implemented on Matlab 7.0 on a Linux OS machine with a clockspeed of 2.39 GHZ and a memory of 16GB.

Table 3.2: Network details

Line	Imp. (Ohm)	Cap.(MW)
1-2	11000	400
2-3	8500	480
3-4	8000	440
4-5	7000	440
1-3	9000	480
1-6	10000	520
6-7	6000	360
7-8	8000	400
8-9	6500	340
9-10	9500	380
4-10	8500	420
9-11	8000	460
10-12	7000	500

Table 3.3: Generator details

Generator type	Capacity	Linear costs	Quadratic costs
1	$N(2000, 10)$	$N(2, 0)$	$N(8, 0)$
2	$N(2000, 10)$	$N(2, 0)$	$N(8, 0)$
3	$N(650, 270)$	$N(2, 0)$	$N(8, 0)$
4	$N(730, 320)$	$N(2, 0)$	$N(8, 0)$

Scalability: The initial step lengths and regularizations were taken to be, $\gamma^0 = 0.15$ and $\epsilon^0 = 0.25$ respectively for all the runs. The order of decrease of step lengths was taken to be $\beta = 0.5001$. The order of decrease of the regularization parameter was taken to be $\alpha = 0.498$. For the first set, we fixed the number of firms to be three (Firms 1,2 and 3). For two different values of the forward intercepts (a^0), we varied the number of scenarios from 5 to 60 in steps of 5. It is to be noted that the scheme is distributed and computation can be done in parallel. However, we proceed to show that even serial times scale well with the size of the problem. The stopping tolerance Δ was taken to be proportional to the problem size.

$$\Delta = \|F^{nat}(z)\| = \|z - \max(z - F(z), 0)\| \leq \frac{|\Omega|}{10}.$$

Table 3.4 reports the corresponding serial computation time and final regularization values for all instances and scenarios.

Table 3.4: Scalability:scenarios

	No. of scenarios	Variables	Serial time (s)	Iterations
$a^0 = 900$	5	1456	123.48	596251
	10	2876	139.24	272007
	15	4296	181.96	178963
	20	5716	195.36	135193
	25	7136	248.77	124511
	30	8556	379.70	162553
	35	9976	576.26	210889
	40	11396	875.62	278733
	45	12816	1296.48	366574
	50	14236	1814.03	443753
	55	15656	2331.13	528315
	60	17076	3069.52	607701
$a^0 = 950$	5	1456	139.10	665701
	10	2876	138.98	300640
	15	4296	200.14	197092
	20	5716	214.89	148469
	25	7136	259.14	130812
	30	8556	383.25	164232
	35	9976	584.05	212735
	40	11396	873.52	279423
	45	12816	1307.09	369709
	50	14236	1833.24	449557
	55	15656	2361.93	538197
	60	17076	3122.51	617349

Comparison between the iterative Tikhonov and iterative proximal schemes: A four firm problem, under the same setting was taken as a case study to compare the two schemes. The schemes were tested by varying the number of scenarios from 10 to 20 in steps of 5 for three different instances. The initial step size was the same ($\gamma_0 = 0.15$) for both the schemes. θ was taken to be 10 for the IPP scheme. The stopping criterion was taken to be the same in both the cases. The results are reported in Table 3.5. The IPP scheme shows a better performance terms of the number of iterations. A better version with different parametrization of the step lengths and the regularization parameters may significantly improve the scheme.

3.5 Insights

Though the above stated schemes perform well with regard to large problems, second order solvers prove to be more efficient for smaller scale problems. In this section, we use KNITRO (v 5.0) as our solver to solve the exact LCP(q, M). This case study uses the same generator details mentioned in Table 3.3. The number of scenarios was

Table 3.5: Comparison: iterative Tikhonov and proximal schemes

	No. of scenarios	Iterations	
		Iterative Tikhonov	Iterative Proximal
$a^0 = 900$	10	511402	46273
	15	472182	59406
	20	1029679	66484
$a^0 = 950$	10	532134	49293
	15	481456	61836
	20	1032890	71708

taken to be twenty ($n = 20$). However, in this case, the linear and quadratic costs were taken to be $N(2, 0)$ and $N(0.2, 0)$ respectively. Unless stated, the linear and quadratic deviation penalties (positive and negative) were taken to be $N(2, 0)$ and $N(0.2, 0)$ respectively. We focus on two major questions pertaining to two settlement markets, namely, impact of prices and premia on forward commitments and impact of wind power penetration.

3.5.1 Forward commitments

We define a term called nodal premium, that may be quantified by the difference between the forward price and the expected spot price at that node. The deviation penalties were set to be zero and the spot intercepts were fixed to be 700. The forward intercepts across all nodes were varied from 50 to 500 in steps of 50. Table 3.6 shows the variation of forward bids with increasing forward intercepts across nodes 1,2 and 3. Firms do not bid in the forward market, till a particular level is reached where they find a positive premium. The same behavior is seen across the other nodes. However with sufficiently high deviation penalties, the behavior is not the same. Firms bid in the forward market even when they do not find a premium, in order to decrease losses due to positive deviation. Results with the above assumed deviation penalties are reported in Table 3.7.

Table 3.6: Forward participation and premium- no deviation penalties

Intercepts	Node 1		Node 2		Node 3	
	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$
50	0.00	-183.99	0.00	-183.99	0.00	-183.99
100	0.00	-133.99	0.00	-133.99	0.00	-133.99
150	0.00	-83.99	0.00	-83.99	0.00	-83.99
200	0.00	-33.99	0.00	-33.99	0.00	-33.99
250	12.95	3.21	13.35	3.23	12.77	3.20
300	53.37	13.23	55.03	13.30	52.65	13.19
350	93.80	23.25	96.72	23.37	92.52	23.19
400	134.22	33.27	138.40	33.45	132.40	33.18
450	174.65	43.28	180.09	43.52	172.27	43.18
500	215.08	53.30	221.77	53.60	212.15	53.17

Table 3.7: Forward participation and premium- quadratic deviation penalties

Intercepts	Node 1		Node 2		Node 3	
	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$	Total Bids	$p_i^0 - \mathbb{E}p_i^\omega$
50	0.00	-183.97	0.00	-183.93	0.00	-184.01
100	0.00	-133.97	0.00	-133.93	0.00	-134.01
150	0.00	-83.97	0.00	-83.93	0.00	-84.01
200	0.00	-33.97	0.00	-33.93	0.00	-34.01
250	31.36	-14.98	32.32	-14.85	30.95	-15.06
300	70.23	-3.41	72.39	-3.18	69.29	-3.53
350	109.10	8.16	112.46	8.49	107.63	8.00
400	147.97	19.73	152.53	20.15	145.98	19.53
450	186.84	31.30	192.60	31.82	184.32	31.06
500	225.70	42.88	232.67	43.49	222.66	42.59

3.5.2 Wind power penetration

For this study, the generation levels of generators 1,2 and 3 were fixed. and the generation levels of the wind (fourth) generator were varied from $\mathcal{N}$ $(30, 10)$ to $\mathcal{N}$ $(300, 90)$. It is seen that the forward commitments of the firms tend to increase (Figure 3.1). Increased volatility and reduced spot prices may be attributed to be reasons for this behavior. In addition, it is seen that the mean premium tends to increase.

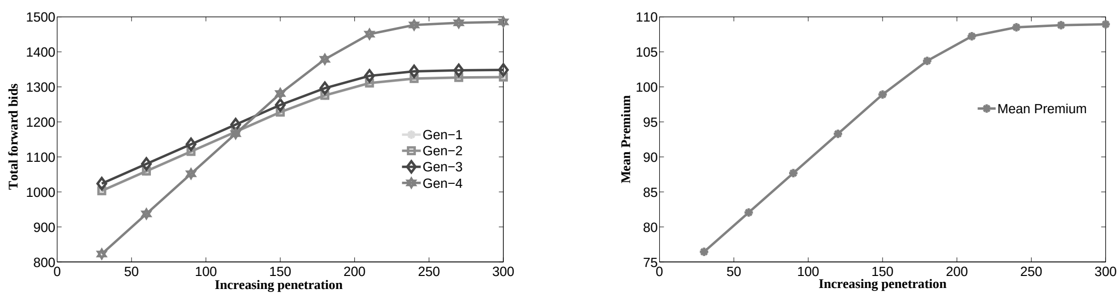


Figure 3.1: Penetration of wind power

3.6 Summary

A two settlement structure with uncertainty is considered where agents compete in the forward and spot markets. With an assumption of bounded rationality it is assumed that agents take simultaneous decisions in the forward and spot markets. The model as stated previously gives a great flexibility with regard to generation, transmission and sales. As opposed to models where social welfare is maximized, the ISO maximizes wheeling revenue. This setting leads to a generalized Nash game with non-shared constraints.

The agent objective functions are shown to be convex and the resulting complementarity formulation proves to be more tractable from theoretical and analytical standpoints. In fact, the mapping of the LCP proves to be monotone. This property aims in claiming an existence statement.

However the mapping proves to be just monotone and not strongly monotone, thereby ruling out the avenue of traditional projection algorithms. Motivated by this question this chapter discusses two different convergent schemes namely the iterative Tikhonov regularization and the iterative proximal point algorithms. It is seen from the numerical results that the algorithms scale very well with the problem size and comparison tests show that the IPP algorithm is more effective. However it is to be noted that the convergence result for the proximal point algorithm would hold only for strictly monotone mappings, with the strategy sets being compact.

Lastly some economic interpretations are obtained from the above model by applying it to a simulated 12 noded network. It is also observed that in the absence of exogenous deviation penalties, firms do not bid in the forward market unless they see an incentive. The same is not seen to be the case in a setting with deviation penalties. Moreover with increasing wind penetration it is seen that the market becomes more volatile and the firms bid more in the forward market. It is also seen that with increasing volatility due to wind assets, the risk premium tends to increase.

Chapter 4

A Complementarity Approach for Game Theoretic Discrete Choice Models

4.1 Introduction

Regardless of the industry or the type of sector, the present day market is marked by several firms competing amongst themselves. This can be termed as an oligopoly. Each firm defines its own price and product attribute dimensions that depend on manufacturing cost and customer demand. Extant of previous research has focussed on finding optimal configurations of products that maximize an individual firm's revenue. A key note that is worth mentioning with regard to the aforementioned models is the exclusion of competitors' reactions while maximizing an individual firm's revenue. One of the first models to include such competitor reactions was [18], the prime focus of which was bertrand or price competition. This may also be referred to as short-run competition. In contrary, long-run competition may refer to one where firms simultaneously change product attributes and prices. Subsequent research has also been done on new product positioning, with only the new entrant focussing on product attribute dimensions and price [17]. Some research has also dealt with special instances of long run competition with simple practical examples [19]. It can be clearly stated that a firm's profit depends on the consumer demand, the prices set by the firm and the manufacturing costs. Manufacturing costs and price can be quantified by closed form expressions. On the other hand, consumer demand remains to be the most difficult to predict and model. A lot of previous work has focused on obtaining some closed form approximations of this demand. One easy approximation is the linear demand function, where the consumer utility / demand is a linear function with respect to prices and attributes. In competitive settings the consumer demand may be taken to be a function of several firms' prices and attributes. One such model is the Multiplicative Competitive Interaction(MCI) model that accounts for competition in a multi dimensional space and quantifies the probability for a particular product to be purchased [22, 68]. This model, though highly effective, lacks several mathematical properties like convexity. The new generation models, namely the logit and probit models seem to be more tractable from a mathematical standpoint. With the logit model claiming acclaim in several facets, it continues to be employed as a widely accepted universal model from a competitive standpoint.

In practical settings this competition may be marked by a framework where all the firms move simultaneously. In other words, this refers to firms taking decisions simultaneously with no prior knowledge of the final decisions of the competitive firms. In this market scenario, the agents are said to be competing in a "Nash game". In some instances, markets are also characterized by firms taking decisions that depend on final decisions taken apriori by one or more firms. For instance, if a new market player enters the market, the veteran firms modify their decisions based on the decisions of the entering firm. In this context, the new firm may be referred to as a follower and the veteran firms as leaders. There exist markets where the framework may be characterized by several such leaders and followers. In these settings the agents are said to be competing in the "Nash-Stackelberg" game. An equilibrium to any one of the above mentioned games involving two or more players, represents the set of final decisions of all the players where no player can maximize his profit or do better by changing only his or her own strategy/decisions unilaterally. Analysis and computation of equilibria refers to solving of multiple agent problems simultaneously. This multi agent problem may be compactly represented as a Mathematical program with complementarity constraints (MPCC) [44, 52, 45]. In the case of Nash-stackelberg games, every leader knows the followers' equilibrium decisions. Hence, for a single firm this may refer to solving of a profit maximization problem that is subject to followers' equilibrium decisions. The entire game therefore can be referred to as an EPEC(Equilibrium Problem with Equilibrium Constraints) [85, 91, 92] where each agent solves an MPEC (Mathematical Program with Equilibrium Constraints). In general the analysis of the latter framework (EPEC) is highly complicated. Little theory exists on the existence of solutions to such games. Computation of equilibria become highly impractical in several such settings.

In this paper, we focus on entirely a Nash framework, where in several firms compete with respect to price and attributes. Variational formulations have been useful tools in analyzing such Nash-equilibria. With the model accounting for unbounded strategy sets, we theoretically prove the existence of solutions to the variational formulations in both cases of the short and long run competition under some weak assumptions. For ease with regard to computation we transform these variational inequalities to equivalent complementarity problems. We formulate a typical Nash-Bertrand automobile game involving several agents and propose schemes to solve this game. We also present some numerical results for an example with painkillers which has been analyzed previously.

The reminder of the paper is organized in six sections. Section 4.2 elaborates on various choice models. Section 4.3 focuses on modeling a general long-run competitive game, section 4.4 throws light on existence of solutions to the corresponding VIs. An automobile design game is formulated in section 4.5 . Section 4.6 focuses on formulating and solving a painkiller market example. We conclude in section 4.7.

Notation

Π_j	-	Profit of product j
θ_j	-	Minimization function for firm j (negative of profit)
K_j	-	Set of all constraints for firm j (Bertrand competition)
X_j	-	Set of all constraints for firm j (Price and attribute competition)
C_j	-	Cost of product j
$\mathcal{J}$	-	Set of all firms
a_{ij}	-	$i^t h$ attribute's value for firm j
α_i	-	Coefficient of attribute i in the conditional logit model
p_j	-	Price of product j
α_p	-	Importance weight for price in the conditional logit model
U_j	-	Utility value of the product j
ρ_j	-	Probability of product j to be chosen
n	-	Number of attributes
J	-	Number of firms

Assumption 36

(A1) The price set by firm j is greater than the production cost (i.e) $p_j > C_j$.

(A2) The cost function with regard to attributes is a strictly convex function with regard to the attributes. That is, $\frac{\partial C_j}{\partial a_{ij}} > 0$.

(A3) The order of decrease of customer utility with regard to price is not less than the order of increase of customer utility with regard to attributes. That is, $O(f_p(p_j)) \geq O(f_i(a_{ij}))$. *Moreover the functions* $f_p(p_j)$ *and* $f_i(a_{ij})$ *are smooth, non-negative and strictly increasing in* R^{N^+}.

(A4) The agent constraint sets X_j *and* K_j *are nonempty, closed and convex for all* $j \in \mathcal{J}$.

4.2 Choice models

Obtaining closed form expressions for consumer demand has proved to be great challenge in the community of discrete choice analysis. The simplest function that can approximate quantity may be given by a linear function.

$$q_j = \frac{z_j^T w}{\sum_{j \in \mathcal{J}} z_j^T w}.$$

A linear demand (price competition) may also take the form [35],

$$\mathbf{q} = \mathbf{q^0} - \mathbf{Bp},$$

where $\mathbf{q}$ represents the quantity vector and $\mathbf{B}, \mathbf{p}$ represent the sensitivity matrix and the price vector respectively. Newer models quantify consumer demand in a better fashion and are briefly explained in the next few subsections.

4.2.1 Generalized extreme value model

The generalized extreme value model [24, 70] proposed by Mcfadden in 1978 has been the basis for most of the discrete choice models and random utility theories. With n alternatives, each marked by utility V_i, the probability that the alternative i is chosen is given by,

$$\rho_i = \rho(i|C) = \frac{y_i \frac{\partial G}{\partial y_i}(y_1, .., y_n)}{\mu G(y_1, .., y_n)},$$

where, $y_i = e^{V_i}$. In addition, the following conditions have to be met for a model to be classified as a GEV model:

1. $G(y) \geq 0$

2. G is homogenous of degree $\mu > 0$. That is, $G(\alpha y) = \alpha^{\mu} G(y)$.

3. G is coercive. That is, $\lim_{y_i \to \infty} G(y_1, .., y_n) = \infty, \quad \forall i = 1, .., n$.

4. The partial derivatives of G with respect to k distinct y_i 's is non-negative for odd k and non-positive for even k.

$$(-1)^k \frac{\partial^k G}{\partial y_m ... \partial y_n} \leq 0.$$

A special case of the GEV model is the logit model which is obtained when

$$G(y) = \sum_{i=1}^{n} y_i.$$

4.2.2 Multinomial logit model

The logit model [64] as stated previously is derived from the NETGEV model. However, there are several variants of the logit model. The most common is the multinomial logit model (MNL). This is a general model that gives way to differences across customers in terms of their preferences. Mathematically,

$$\rho_{ji} = \frac{e^{V_{ji}}}{\sum_{k=1}^{n} e^{V_{jk}}}.$$

Note that ρ_{ji} represents the probability of customer i choosing alternative j and V_{ji} represents the utility of product i with respect to customer j. The utility may be chosen to be any function as long as it follows the norms of the GEV model. In practice, utility functions are chosen to be linear and may be given by,

$$V_{ji} = \alpha_i^T w_j,$$

where w_j represents the vector of customer attributes and α represents weighing coefficients.

4.2.3 Conditional logit

The conditional logit model [84] is similar to the MNL except for the fact that it does not account for differences between customers. In turn, the alternatives are weighed in accordance to the alternative's attributes or dimensions. More specifically, if alternative i has d dimensions, then

$$\rho_i = \frac{e^{V_i}}{\sum_{k=1}^{n} e^{V_k}}, \quad V_i = \sum_{l=1}^{d} \alpha_l a_i^l.$$

4.2.4 Mixed logit model

The mixed logit model is the most general form of the logit model and encompasses the features of both the multinomial logit and the conditional logit models. In other words, ρ_{ji} will be a combination of both the terms. Mathematically,

$$\rho_{ji} = \frac{e^{V_{ji}}}{\sum_{k=1}^{n} e^{V_{jk}}}, \quad V_{ji} = \sum_{k=1}^{f} \alpha^k w_j^k + \sum_{l=1}^{d} \beta_l a_i^l.$$

4.2.5 Multiplicative competitive interaction

The multiplicative competition interaction model [22, 68] does not come under the set of GEV models. The model is similar in terms of construction. However, the concept of exponential utility does not gain prominence in this case. This was one of the old models used in discrete choice analysis. Mathematically it can be expressed as follows:

$$\rho_i = \frac{V_i}{\sum_{k=1}^{n} V_k}.$$

This model is however restricted to competitive analysis and does not gain great prominence in other discrete choice analysis. As the name goes, V is not a linear combination of the attributes or the dimensions of product i. That is,

$$V_i = \Pi_{r=1}^{d} a_{ri}^{\alpha^r},$$

where a_{ri} represents the $r^t h$ dimension of product i. Notably, unlike the logit model, this is not convex due to the presence of bilinear terms.

4.3 Competition modeling

We consider the case of a market with homogenous firms, (i.e) firms producing the same product. From an economic standpoint, competition can be either of the Cournot type or the Bertrand type. A Cournot model is one where firms compete in terms of quantity and are paid at a common price that depends on the total quantity. The latter refers to one where individual firms change prices in the course of competition. Simultaneous changes in quantity also fall under this regime. Each firm's revenue may be defined to be a product of price and demand. In simple terms

$$\pi_j \triangleq p_j q_j.$$

The above is valid in a static setting where other firms are assumed to be fixed in terms of their decisions and strategies. But in practice when other competing firms change their decisions, the variable q becomes a moving quantity that depends on other agents' decisions. That is,

$$\pi_j = p_j q_j(p_j, p^{-j}).$$

Note thatp^{-j} refers to the price decisions of the competing agents. This setting is profound in several airline industries and other transportation sectors [38, 37]. This competition gains more prominence in other engineering sectors where the focus is not restricted to price alone. Several automobile and manufacturing industries add another dimension to the above spectrum, by bringing in design attributes. In the latter case, the quantity or the demand is a function of the price and attributes and may be given by,

$$\pi_j = p_j q_j(z_j, z^{-j}),$$

where z_j is the vector of price and attributes.

Our mathematical formulation deploys the conditional logit model that operates on the assumption that the weighting coefficients of a population of customers can be replaced by a standard coefficient that is common to that market segment. We also operate on the assumption that firms produce exactly the same quantity as demanded. In mathematical terms, the utility of product j may be defined as,

$$U_j = -f_p(p_j) + \sum_{i \in \mathcal{N}} f_a^i(a_{ij}), \qquad \forall j \in \mathcal{J}$$

The probability of a customer choosing a product j, is given by the conditional logit model as follows:

$$\rho_j = \frac{exp(U_j)}{\sum_{j \in \mathcal{J}} exp(U_j) + 1} \tag{4.1}$$

Here 1 is added to the denominator to indicate the no-purchase option. Let z_j refer to the vector of price and attributes for firm j.

$$z_j = \begin{pmatrix} p_j \\ a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$$

With the above definition, the problem for firm j may be defined as:

$$\boxed{\text{Ag}(z^{-j}) \qquad \begin{array}{ll} \text{maximize} & (p_j - C_j)\,\rho_j \\ \text{subject to} & \left\{ \begin{array}{l} p_j - C_j \geq 0 \\ a_{ij} \geq 0, \quad \forall i \in \mathcal{N} \\ z_j \in K_j \end{array} \right\} \end{array}}$$

K_j refers to a convex set of design constraints. Some markets may impose an upper bound on prices. This can also be assumed to fall under the set K_j. However in our work, we do not assume the boundedness of price or attributes.

Definition 37 *A Nash equilibrium for the above game can be defined as a setting, where no firm can can benefit by changing its strategy regarding the others "as committed to their choices"*

Mathematically, the Nash equilibrium can be stated as

$$\Pi_j(p_j^*, a_j^*) \geq \Pi_j(p_j, a_j, (p^{-j})^*, (a^{-j})^*), \qquad \forall j \in \mathcal{J}$$

4.3.1 Game theoretic formulation

Definition 38 (Nash price equilibrium) *Consider a set of $\mathcal{J}$ firms . Suppose all the firms compete with regard to price. Let the jth agent solve the problem*

$$\begin{array}{lll} Ag^p(p^{-j}) & \min & \theta_j(p_j; p^{-j}) \\ & \textit{subject to} & p_j \in K_j \end{array}$$

,

where K_j refers to the set of all constraints on p_j (i.e) $p_j > C_j$. θ_j refers to the minimization function of firm j (negative of the profit function).Then the Nash price equilibrium is given by $\{p_j^*\}_{j\in\hat{\mathcal{J}}}$ where

$$p_j^* \in SOL(Ag^p(p^{-j,*})), \qquad \forall j \in \mathcal{J}.$$

Definition 39 (Nash competitive equilibrium) *Consider a set of $\mathcal{J}$ firms . Suppose the firms compete with regard to price and attributes. Let the jth agent solve the problem*

$$\begin{array}{lll} Ag^{ap}(z^{-j}) & \min & \theta_j(z_j; z^{-j}) \\ & \textit{subject to} & z_j \in X_j, \end{array}$$

where X_j refers to the set of all constraints on z_j, both price and attributes (indicated in the previous section). θ_j refers to the minimization function of firm j (negative of the profit function). The Nash competitive equilibrium may be given by $\{z_j^*\}_{j\in\mathcal{J}}$ where

$$z_j^* \in SOL(Ag^{ap}(z^{-j,*})), \qquad \forall j \in \mathcal{J}.$$

4.4 Theoretical results

If the objective functions of all agents are convex, then the necessary conditions for the optimality are given by the set of variational inequalities as follows:

$$\nabla_{z_j}^T \theta_j(z_j^*; z^{-j,*}) \left(z_j - z_j^*\right), \forall z_j \in X_j, \forall j \in \mathcal{J},$$

where $z_j^* \in X_j$ marks the optimal solution for the Variational Inequality. In short, the optimality conditions of the game can be given by

$$F^T(z^*)(z - z^*) \geq 0, \forall z \in X,$$

where

$$F = \begin{pmatrix} \nabla^T_{z_1} \theta_1(z_1, z^{-1}) \\ \vdots \\ \nabla^T_{z_J} \theta_1(z_J, z^{-J}) \end{pmatrix}, U = \Pi_{j=1}^{J} U_j$$

But, the objective functions shown in our case need not be necessarily convex. The above conditions do not guarantee that z^* is indeed the solution of the game. If the sufficiency conditions hold at the solution z^*, then we can ascertain that z^* is indeed a local equilibrium for the game. The rest of the section analyzes the VIs' corresponding to games Ag^p and Ag^{ap}.

Definition 40 *For a sequence* (s_k)*, we write* $\lim s_k = \infty$ *iff for each* $\delta > 0$*, there is a number* M *such that* $k > M$ *implies* $s_k > \delta$

Theorem 41 *Let* $X \subset R^N$ *be closed convex and* $F : X \longrightarrow \mathbb{R}^N$ *be continuous. If there exists a vector* $z^{ref} \in X$ *such that the set,*

$$L_< = \left\{z \in X, F(z)^T(z - z^{ref}) < 0\right\}$$

is bounded, then the $VI(X, F)$ *has a solution.*

Proof : The Proof has been discussed in [34]. We prove in the following theorems that the variational formulations corresponding to the games stated above admit solutions.

Theorem 42 *Consider the Nash game given by* Ag^p *and let assumptions (A1-A4) hold. Then the* $VI(X, F)$ *corresponding to the game,* Ag^p *(i.e) has a solution.*

Proof : We know that the gradient mapping $F_j, \forall j \in \mathcal{J}$ is given by,

$$F_j = \left(-\rho_j - f_p'(p_j)(p_j - C_j)\rho_j(1 - \rho_j)\right)$$

Let us choose some finite $z^{ref} \in \mathbf{X}$. Then,

$$F(z)^T(z - z^{ref}) = \sum_{j=1}^{J} \rho^j g_j,$$

where

$$g_j = \underbrace{\left(-1 - f_p'(p_j)(p_j - C_j)(1-\rho_j)\right)(p_j - p_j^{ref})}_{\text{Term A}}.$$

Consider the case of the following cases of terms approaching infinity:

Case 1: Let $p_j \longrightarrow \infty$.By assumptions $f_p'(p_j)$ is a strictly increasing function. Then Term A tends to infinity. So, g_j grows to infinity.

So, $\lim_{z_j \longrightarrow \infty} g_j = \infty$. From definition 40, we can say that for some value $Q_j > 0$, such that $\| z_j \| \geq Q_j, g_j > Q > 0$. But, we also know that, $\rho_j \geq 0, \forall j \in \mathcal{J}$. Let, $Q = \max(Q_1, Q_2, ..., Q_J)$. Then,

$$F(z)^T(z - z^{ref}) \geq 0, \qquad \forall z \in K, \| z \| \geq Q.$$

Thus we can see that the set $L_<$ is bounded from above by R. Thus we can conclude that the VI(K,F) has a solution. This completes the proof. ∎

Theorem 43 *Consider the Nash game given by Ag^{ap} and let assumptions (A1-A4) hold. Then the VI(X,F) corresponding to the game, Ag^{ap} has a solution.*

Proof : We know that the gradient mapping $F_j, \forall j \in \mathcal{J}$ is given by,

$$F_j = \begin{pmatrix} -\rho_j - f_p'(p_j)(p_j - C_j)\rho_j(1-\rho_j) \\ -(p_j - C_j)f_1'(a_{1j})\rho_j(1-\rho_j) + \frac{\partial C_j}{\partial a_{1j}}\rho_j \\ \vdots \\ -(p_j - C_j)f_n'(a_{nj})\rho_j(1-\rho_j) + \frac{\partial C_j}{\partial a_{nj}}\rho_j \end{pmatrix}$$

Let us choose some finite $z^{ref} \in X$. Let, $F(z)^T(z - z^{ref}) = \sum_{j=1}^{J} \rho^j g_j$, where

$$g_j = \underbrace{\left(-1 - f_p'(p_j)(p_j - C_j)(1-\rho_j)\right)(p_j - p_j^{ref})}_{\text{Term A}} + \underbrace{\sum_{i=1}^{n}(-(p_j - C_j)f_i'(a_{ij})(1-\rho_j))(a_{ij} - a_{ij}^{ref})}_{\text{Term B}}$$
$$+ \underbrace{\sum_{i=1}^{n}(\frac{\partial C_j}{\partial a_{ij}})(a_{ij} - a_{ij}^{ref})}_{\text{Term C}}.$$

Consider the case of the following cases of terms approaching infinity:

Case 1: Let $p_j \longrightarrow \infty$. Then Term A tends to infinity and Term B tends to negative infinity. But, Term A tends to infinity at a higher order. So, effectively g_j tends to infinity.

Case 2: Let $a_{ij} \longrightarrow \infty$. From assumptions (A1-A6), we know that C_j grows super-linearly to infinity (strict convexity). But, we also know that $p_j \geq C_j, \forall p_j \in K_j$ [1]. So, p_j also grows to infinity at least at a linear rate. So, Term A grows to infinity and Term B tends to negative infinity slower than Term A. Term C tends to infinity at least at a linear rate. Effectively, g_j tends to infinity.

Therefore $\lim_{z_j \longrightarrow \infty} g_j = \infty$. From definition 40, we can say that for some value R_j, such that $\| z_j \| \geq R_j, g_j > 0$. But, we also know that, $\rho_j \geq 0, \forall j \in \mathcal{J}$. Let $R = \max(R_1, R_2, ..., R_J)$. Then,

$$F(z)^T(z - z^{ref}) \geq 0, \qquad \forall z \in X, \| z \| \geq R.$$

Thus we can see that the set $L_<$ is bounded from above by R. Thus we can conclude that the VI(X, F) has a solution. This completes the proof. ∎

4.5 Automotive design

Automotive design involves the careful consideration of several design factors and parameters. Of several parameters and attributes, the customer knows about a few attributes and is inclined to purchase a particular product only based on these attributes. Color, transmission type, number of doors etc. are some of these relevant discrete attributes. In general this would leave us with little existence theory in addition to lack of convergent schemes. Therefore this model sticks to a subset of relevant continuous attributes that are defined as follows:

1. Miles Per Gallon (η_j)
2. Acceleration (A_j)
3. Horsepower(H_j)
4. Length(L_j)
5. Breadth(B_j)
6. Height(h_j)

The mileage of an automobile drops with increasing horsepower and weight. We come up with a linear relationship between the mileage and other attributes for the sake of simplicity and convexity. In addition to these attributes, the mileage is affected by several design parameters and attributes that are absent in the model. This error or constant

[1] C_j is a convex function in terms of attributes, a_{ij}. So, the constraint $p_j \geq C_j$ marking the epigraph of this convex function is also convex.

is accounted by a_j^1.

$$\eta_j = a_j^1 - b_j^1 H_j - c_j^1 L_j - d_j^1 B_j - e_j^1 h_j$$

The acceleration increases with horsepower and decreases with curb weight. Similarly a_j^2 accounts for a different error term.

$$A_j = a_j^2 + b_j^2 H_j - c_j^2 L_j - d_j^2 B_j - e_j^2 h_j$$

We employ a logit model with linear utility functions where the utility is given by,

$$U_j = \alpha_H H_j + \alpha_A A_j + \alpha_\eta \eta_j + \alpha_L L_j + \alpha_B B_j + \alpha_h h_j - \alpha_p p_j$$

The maximization problem for agent j may be defined as follows:

$$\boxed{\begin{array}{lll} Ag^{ag}(z^{-j}) & \text{minimize} & \theta_j^{ap} = -(p_j - C_j)\rho_j \\ & \text{subject to} & \left\{ \begin{array}{ll} C_j - p_j \leq 0 & (\gamma_j) \\ \eta_j = a_j^1 - b_j^1 H_j - c_j^1 L_j - d_j^1 B_j - e_j^1 h_j & (\delta_j) \\ A_j = a_j^2 + b_j^2 H_j - c_j^2 L_j - d_j^2 B_j - e_j^2 h_j & (\phi_j) \\ p_j, A_j, H_j, \eta_j, L_j, B_j, h_j \geq 0 & \end{array} \right\}. \end{array}}$$

4.5.1 The complementarity problem

The KKT conditions for the individual firms may be unified to form a complementarity problem stated as follows:

$$
\begin{aligned}
&0 \le p_j \perp -\rho_j + \alpha_p(p_j - C_j)\rho_j(1-\rho_j)\beta_j - \gamma_j \ge 0\\
&0 \le H_j \perp (\rho_j + \gamma_j)\frac{\partial C_j}{\partial H_j} - \alpha_H(p_j - C_j)\rho_j(1-\rho_j) + \delta_j b_j^1 - \phi_j b_j^2 \ge 0\\
&0 \le A_j \perp (\rho_j + \gamma_j)\frac{\partial C_j}{\partial A_j} - \alpha_A(p_j - C_j)\rho_j(1-\rho_j) + \phi_j \ge 0\\
&0 \le \eta_j \perp (\rho_j + \gamma_j)\frac{\partial C_j}{\partial \eta_j} - \alpha_\eta(p_j - C_j)\rho_j(1-\rho_j) + \delta_j \ge 0\\
&0 \le L_j \perp (\rho_j + \gamma_j)\frac{\partial C_j}{\partial L_j} - \alpha_L(p_j - C_j)\rho_j(1-\rho_j) + \delta_j c_j^1 + \phi_j c_j^2 \ge 0\\
&0 \le B_j \perp (\rho_j + \gamma_j)\frac{\partial C_j}{\partial B_j} - \alpha_B(p_j - C_j)\rho_j(1-\rho_j) + \delta_j d_j^1 + \phi_j d_j^2 \ge 0\\
&0 \le h_j \perp (\rho_j + \gamma_j)\frac{\partial C_j}{\partial h_j} - \alpha_h(p_j - C_j)\rho_j(1-\rho_j) + \delta_j e_j^1 + \phi_j e_j^2 \ge 0\\
&0 \le \gamma_j \perp p_j - C_j \ge 0\\
&\text{free}(\delta_j) \perp -\left(\eta_j - a_j^1 + b_j^1 H_j + c_j^1 L_j + d_j^1 B_j + e_j^1 h_j\right) = 0\\
&\text{free}(\phi_j) \perp -\left(A_j - a_j^2 - b_j^2 H_j + c_j^2 L_j + d_j^2 B_j + e_j^2 h_j\right) = 0
\end{aligned}
$$

Let this problem in the vector form be denoted as,

$$
\begin{aligned}
&0 \le z_a \perp F_a(z) \ge 0\\
&\text{free}(z_b) \perp F_b(z) = 0,
\end{aligned}
$$

$$
F(z) = \begin{pmatrix} F_a(z) \\ F_b(z) \end{pmatrix}.
$$

Lemma 44 *Consider the mapping $F(z)$. ∇F is not positive semi-definite and hence F is not guaranteed to be monotone.*

Proof : Let the gradient elements of F be denoted by G. Let the diagonal elements of G_{ii} be denoted by a column vector DG_{ii}. That is,

$$\nabla F = \begin{pmatrix} G_{11} & \dots & G_{1J} \\ \vdots & \ddots & \vdots \\ G_{J1} & \dots & G_{JJ} \end{pmatrix}, \quad DG_{ii} = \begin{pmatrix} \alpha_p \rho_j (1-\rho_j)\left(2 - \alpha_j (p_j - C_j)(1-2\rho_j)\right) \\ \alpha_H \rho_j (1-\rho_j)\left(2\frac{\partial C_j}{\partial H_j} - \alpha_H (p_j - C_j)(1-2\rho_j)\right) + (\rho_j + \gamma_j)\frac{\partial^2 C_j}{\partial H_j^2} \\ \vdots \\ \alpha_h \rho_j (1-\rho_j)\left(2\frac{\partial C_j}{\partial h_j} - \alpha_h (p_j - C_j)(1-2\rho_j)\right) + (\rho_j + \gamma_j)\frac{\partial^2 C_j}{\partial h_j^2} \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It can be seen that the sign of the elements of G_{ii} depend on the sign of $1-2\rho_j$. It is to be noted that $1-2\rho_j$ can be positive and correspondingly the diagonal elements can be negative for some values of the variables. Therefore ∇F is indeterminate and hence F is not guaranteed to be monotone.

4.5.2 Algorithmic trials

Projection schemes-VI

The game's representation as a variational inequality is shown in Section 2. But, under the absence of convexity, the solution to the VI is not necessarily guaranteed to be the solution to the original game. In addition to necessary conditions solved by the VI, second order sufficiency conditions have to be satisfied. However, with the inherent assumption that second order conditions are satisfied, we try to compute a local equilibrium to the game. The solution to a VI(Z, F) may be defined as the problem of finding a $z^* \in Z$ such that,

$$F(z^*)^T(z - z^*) \geq 0, \quad \forall z \in Z.$$

This can also be represented as a fixed point relation. That is,

$$z^* = \Pi_Z(z^* - \gamma F(z^*)).$$

Π represents projection. The projection of a point A on a set is the point B in the set that is of least distance from the point A. If $\bar{z} = \Pi_Z(y)$, then $\bar{z}$ is the solution to the problem defined below:

$$\min_z \quad (z-y)^T(z-y)$$
$$\text{subject to} \quad z \in Z$$

Then, A standard projection algorithm to compute the solution of the VI would be,

$$z^{k+1} = \Pi_Z(z^k - \gamma F(z^k)).$$

As z^{k+1} converges to z^k for fixed γ, it is easy to find that, $z^k \to z^*$, where z^* is the solution to the VI. However, the scheme is not guaranteed to converge unless F is Lipschitz continuous (constant L) and strongly monotone (constant η).

$$\|\mathbf{F}(x) - \mathbf{F}(y)\| \leq L\|x - y\|, \quad (\mathbf{F}(x) - \mathbf{F}(y))^T (x - y) \geq \eta\|x - y\|^2.$$

In addition the step size is governed by, $\gamma < \frac{2\eta}{L^2}$. With the mapping $\mathbf{F}$ lacking monotonicity properties, we do not have any guarantee that the sequence converges to the solution. The above scheme was employed in conjunction with SQOPT (quadratic problem solver) to solve the current automotive example.

Projection schemes-CP

With the inherent difficulty to solve a quadratic problem at every projection in the variational inequality, we transform the variational inequality into a pure nonlinear complementarity problem, formulated in the previous subsection. A stylized version of the projection algorithm is employed to solve this complementarity problem. It may be stated as follows:

$$z_{k+1} = \max(z_k - \gamma_k F(z_k), 0).$$

The advantage of this scheme is that the exact values of the iterates are obtained at every step. However like the previous example this requires strong monotonicity of the mapping to ensure convergence of the iterates.

Diminishing projection scheme

A variant of the projection scheme uses diminishing step lengths rather than a fixed step length. The scheme may be defined to be as follows:

$$z_{k+1} = \max(z_k - \gamma_k F(z_k), 0), \qquad \text{where,}$$

$$\sum_{k=1}^{\infty} \gamma_k = \infty, \quad \sum_{k=1}^{\infty} \gamma_k^2 < \infty.$$

The convergence for strictly monotone VIs is discussed in [55].

Proximal schemes

An alternative iterative regularization technique to solve variational inequalities is obtained by employing proximal sequences. This requires solving a strongly monotone variational inequality at every step. The scheme may be mathematically expressed as follows.

$$z^{k+1} = \mathrm{SOL}(Z, F^k), \quad F^k = F(z) + \theta(z - z^k)$$

If the sequence z^k converges, then it is easy to note that it converges to the fixed point of the original problem. A larger θ makes it easy to solve the subproblems. However this scheme is guaranteed to converge only if the original mapping F is monotone [34, 5].

Iterative proximal schemes

A new extension of the proximal scheme is the iterative proximal scheme [53]. The scheme lies on solving the sub-problem inexactly at every step. Instead of taking several projection steps to solve the sub-problem, a single step is taken that improves the iterate. A diminishing step size employed with increasing regularization would prove to be an effective technique for strictly monotone variational inequalities. In the case of general VIs, the iterates can be proved to converge to a particular point. However, there is no guarantee that this point is indeed the optimal solution.

Algorithm 5: Iterative Proximal Scheme

initialization $k = 1$;
choose constants $\theta, \bar{\psi} > 0$ and $0 < \gamma \leq \bar{\gamma}$ and initial points (z^0), $z^1 = z^0$. ;
while $\psi^k > \bar{\psi}$ **do**
  $z^{k+1} = \max(z^k - \gamma_k(F(z^k) + \theta_k(z^k - z^{k-1})), 0)$;
  $\psi^k = \|z^{k+1} - z^k\|$;
  $k := k + 1$;
end

A standard way to define γ_k and θ_k is

$$\gamma_k = \frac{1}{k}, \quad \theta_k = \frac{0.5}{\gamma_k}.$$

When tested on the automotive example the above stated projection schemes did not prove to be convergent.

Interior point methods

Under the absence of monotonicity Newton schemes prove to be better when compared to projection type schemes. Under the assumptions of regularity, we analyze the equivalent complementarity problem. The CP(Z, F) may be written as follows:

$$\begin{aligned} z^T F(z) &= 0 \\ z &\geq 0 \\ F(z) &\geq 0 \end{aligned}$$

If $z^T F$ is convex and F is linear, the problem can be reformulated as,

$$\begin{aligned} \min_{z} \quad & z^T F(z) \\ \text{subject to} \quad & z \geq 0 \\ & F(z) \geq 0 \end{aligned}$$

However it cannot be ascertained that $z^T F$ is convex. F is also non- linear. Thus the problem may be written as an equality constrained problem with a log barrier objective. That is,

$$\begin{aligned} \min_{z} \quad & G(z) = -\mu \sum_{i=1}^{n} (\log z_i + \log s_i) \\ \text{subject to} \quad & z^T F(z) = 0 \\ & F(z) - s = 0 \end{aligned}$$

An exact version of this scheme would mean solving the above problem repeatedly for decreasing μ's. An inexact variant of this scheme refers to taking a Newton's direction of the above problem (KKT conditions) at every step. Simultaneously μ is taken to zero. The detailed convergence theory has been discussed in [71].

Smoothing techniques

An advantage of moving from the VI's setting to the CP's setting is that the fixed point problem corresponding to the CP represents a non-smooth equation. We consider the CP in its standard form:

$$0 \leq z \perp F(z) \geq 0, \quad z_{k+1} = \max(z_k - \gamma_k F(z_k), 0).$$

The entity causing the non-smoothness is the max function.The max function may be written as,

$$\begin{aligned}\max(y,0) &= \frac{1}{2}(y+|y|)\\ &\approx \frac{1}{2}\left(y+\sqrt{y^2+\epsilon}\right)\end{aligned}$$

Thus the solution for the CP may be approximated as follows:

$$z_i^* \approx \frac{1}{2}\left(z_i^* - F_i(z^*) + \sqrt{(z_i^* - F_i(z^*))^2+\epsilon}\right).$$

$$z_i^* + F_i(z^*) \approx \sqrt{(z_i^* - F_i(z^*))^2+\epsilon}.$$

To find an approximate solution, it suffices to find the zeros of the vector function $H(z)$ as ϵ is sufficiently close to 0.

$$H(z) = \begin{pmatrix} H_1(z) \\ \vdots \\ H_n(z)\end{pmatrix} \quad H_i(z) = z_i + F_i(z) - \sqrt{(z_i - F_i(z))^2+\epsilon}, \quad H(z^{*,\epsilon}) = \bar{0}.$$

Both the Newton type schemes discussed did not prove to be effective to solve the automobile design problem.

Iterative optimization scheme

In lieu of several algorithmic challenges, we move over to an iterative gaussian setting where a set of optimization problems are solved sequentially. Consider a game with N agents. If all but agent i are assumed to be stationary, then agent i would solve an optimization problem to maximize his profits. In other words, this can be termed as "the best response" from agent i's standpoint. Consider an instance where all agents follow this procedure given the decisions of the other agents at the previous instance. In this setting all agents are said to be going by their best responses. If the set of decisions of all agents converge, then no agent is said to perform better given other agents' decisions. This would immediately imply a Nash equilibrium.

We present this from a mathematical standpoint. Let z_i^0 refer to the initial iterate for agent i. At every iteration agent i solves an optimization problem given z_{-i}^{k-1} (other agent decisions fixed) to compute optimal $z_i^k \in Z_i$.

$$\begin{aligned} z_i^k &= SOL(Ag(z_{-i}^{k-1})),\\ z_i^k &= \Pi_{Z_i}\left(z_i^k - \gamma_k \nabla f_i(z_i^k, z_{-i}^{k-1})\right).\end{aligned}$$

If f_i is convex for all $i \in \mathcal{N}$ and K_i is convex and compact, then it is easy to claim that there exists a solution to f_i

being minimized for fixed z_{-i}. Therefore it is easy to claim that all iterates are defined. Moreover, if z^k converges to some z^*, then

$$\lim_{k\to\infty} z_i^k = \Pi_{Z_i}\left(\lim_{k\to\infty} z_i^k - \lim_{k\to\infty} \gamma_k \nabla f_i(z_i^k, z_{-i}^{k-1})\right), z_i^* = \Pi_{Z_i}\left(z_i^* - \gamma^* \nabla f_i(z_i^*, z_{-i}^*)\right).$$

If the sequence converges, it would converge to z^*, the solution to the game $\mathcal{G}$. However there do not exist concrete theoretical results to state that the iterates converge for all the problems.

4.5.3 Challenges

One major problem faced in the above problem, is the presence of highly nonlinear and bilinear terms. The game characterized by price and attributes is seen to be non-convex. Though we prove that the equivalent VI is solvable, this leads to no guarantee that the game has a solution. In addition, the mapping obtained from the VI happens to be non-monotone. Projection schemes and interior point methods therefore do not seem to be effective. An alternative way to deal with the nonlinear terms is by moving them to the constraint set, maintaining convexity of the constraint set. Since this automotive design is analyzed from a game theoretic standpoint, this gives rise to a game with coupled strategy sets or more specifically a quasi-variational-inequality that is harder from a computational standpoint.

4.6 Pain killer-case study

In this section, we solve a Bertrand game with reference to pain killers. The first part of the analysis deals with purely the pricing game or short run competition. The second part focuses on the positioning of a new product with regard to its attribute dimensions and price. This analysis is based on the data set collected by a group of thirty undergraduate business students from the University of Pennsylvania:[29]. Note that this problem has already been analyzed by [18, 17]. However, it is to be noted that earlier works have solved this problem by means of iterative optimization and diagonalization approaches. Our solution procedure is different in that we aim to solve the actual variational/ complementarity problem arising from the game.

The market is assumed to have fourteen manufacturers of which brand eight is generic and is not assumed to take part in the competition. Table A.2(Appendix) shows the different attribute values of the manufacturers. Table A.3(Appendix) shows the different customer preferences for those attributes. We employ the multinomial logit model as opposed to the conditional logit in the previous section. We also employ a disutility function that employs a quadratic function with respect to attributes and a linear function with respect to price. Mathematically,

$$DU_{ij} = \sum_{i=1}^{N} \alpha_{in}(a_{jn} - b_{in})^2 + \alpha_{ip} p_j + e_i,$$

where i refers to the customer and j refers to the product. The product is marked by n attribute dimensions and e_i refers to an error term that is independent of the price, attributes and the product. It is to be noted that b_{in} refers to customer i's preference for the nth attribute. Any deviation from this point is penalized in terms of disutility. The coefficients α may be found by means of an MDS procedure [28], [29] or maximum likelihood estimation. The multinomial logit model is given by,

$$\rho_{ij} = \frac{exp(-\chi DU_{ij})}{\sum_{j=1}^{J} exp(-\chi DU_{ij}) + 1},$$

where χ is some characteristic constant and the constant 1 is introduced in the denominator to indicate the no purchase option. Let I refer to the total number of customers(30 in this case). Then, the optimization problem for agent i may be stated as follows:

$$\boxed{\begin{array}{lll} \text{Ag}(z^{-j}) & \text{maximize} & (p_j - C_j)\displaystyle\sum_{i=1}^{I} \rho_{ij} \\ & \text{subject to} & \left\{ \; p_j, a_{nj} \geq 0, \quad \forall n \in \mathcal{N} \; \right\} \end{array}}$$

With the presence of non-negativity constraints on the variables, the KKT conditions of the above problem represent complementarity constraints. The complementarity problem may be stated as follows:

$$\begin{aligned} &0 \leq p_j \perp -\left(\sum_{i=1}^{I} \rho_{ij} - (p_j - C_j)\sum_{i=1}^{I} \chi\alpha_{ip}\rho_{ij}(1-\rho_{ij})\right) \geq 0, \quad \forall j \in \mathcal{J} \\ &0 \leq a_{nj} \perp (p_j - C_j)\left(\sum_{i=1}^{I} 2x\alpha_{in}(a_{nj} - b_{in})\rho_{ij}(1-\rho_{ij})\right) + \frac{\partial C_j}{\partial a_{nj}}\sum_{i=1}^{I}\rho_{ij} \geq 0, \quad \forall j \in \mathcal{J}, \forall n \in \mathcal{N} \end{aligned} \tag{4.2}$$

4.6.1 Numerical results

4.6.2 Nash price equilibrium

The coefficients of the disutility function have already been determined by previous works and are also reported in Table A.4. The online version may be found at [1]. Table 10 shows the values of the coefficients of attributes and price in the disutility function. The value of the parameter χ has been reported to be equal to three. This part analyzes the price competition arising in this setting. As stated earlier brand 8 is assumed to be generic. Since, attributes do not change in this short run competition, we solve for the equilibrium corresponding to the first set of constraints (with respect to p) stated in (4.2). The nonlinear part of the disutility function is over ruled in this analysis. However the complementarity problem is highly nonlinear due the presence of logit terms.

All in one approach

We form fourteen complementarity constraints. The complementarity problem is invoked in KNITRO (version 5). KNITRO uses a modified version of the interior point methods stated previously. All the complementarity conditions are solved simultaneously. Table A.1.3 reports the results for this all in one approach.

Table 4.1: Nash-price competition-all in one approach

Market Player	Initial Price	Nash price equilibrium
1	.6990	0.6113
2	.3970	0.2269
3	.5290	0.6113
4	.3290	0.2298
5	.2690	0.2004
6	.3890	0.2210
7	.5310	0.2484
8	.1990	0.1990
9	.5750	0.6113
10	.4990	0.5151
11	.7590	0.6113
12	.4990	0.6113
13	.3690	0.4425
14	.4990	0.4089

Decomposition method

We also use the decomposition technique to split the original poblem into fourteen different subproblems. At each iteration the solver solves for every agent's KKT conditions (complementarity problem) by considering the parameters with respect to the competitors fixed at the previous iteration's value. As discussed previously this gaussian framework refers to the best response scheme. This process was continued till convergence was obtained. Table 4.2 shows the values of the price vector at every iteration. The last iteration marks the point of convergence or in other words, the Nash price equilibrium.

Table 4.2: Nash-price competition-decomposition method

		Iteration								
Agent	Initial Price	1	2	3	4	5	6	7	8	$Nash$
1	0.699	0.64	0.6057	0.6113	0.612	0.6112	0.6113	0.6114	0.6113	0.6113
2	0.397	0.235	0.227	0.2268	0.2269	0.2269	0.2269	0.2269	0.2269	0.2269
3	0.529	0.6415	0.6057	0.6113	0.612	0.6112	0.6113	0.6114	0.6113	0.6113
4	0.329	0.2423	0.2298	0.2296	0.2298	0.2298	0.2298	0.2298	0.2298	0.2298
5	0.269	0.2193	0.2016	0.2003	0.2004	0.2004	0.2004	0.2004	0.2004	0.2004
6	0.389	0.233	0.2214	0.2208	0.221	0.221	0.221	0.221	0.221	0.221
7	0.531	0.2521	0.248	0.2483	0.2484	0.2484	0.2484	0.2484	0.2484	0.2484
8	0.199	0.199	0.199	0.199	0.199	0.199	0.199	0.199	0.199	0.199
9	0.575	0.6425	0.6057	0.6113	0.612	0.6112	0.6113	0.6114	0.6113	0.6113
10	0.499	0.5048	0.5054	0.5141	0.5151	0.5151	0.5151	0.5151	0.5151	0.5151
11	0.759	0.6382	0.6056	0.6113	0.612	0.6112	0.6113	0.6114	0.6113	0.6113
12	0.499	0.6388	0.6056	0.6113	0.612	0.6112	0.6113	0.6114	0.6113	0.6113
13	0.369	0.4476	0.4467	0.4409	0.4424	0.4426	0.4424	0.4425	0.4425	0.4425
14	0.499	0.4054	0.4009	0.4075	0.4092	0.4089	0.4089	0.4089	0.4089	0.4089

4.6.3 Nash equilibrium with a new product

This part focuses on positioning a new product in terms of its price and attribute dimensions. The other agents are allowed to change their prices. In this example we have four different attributes for the new product. Let the new product be indexed by k. Moreover, two additional constraints are imposed on the new product.

$$
\begin{aligned}
-a_{1k} - a_{2k} &\leq -0.325 \\
a_{1k} + a_{2k} &\leq 0.5
\end{aligned}
$$

Let λ_1 and λ_2 be the respective multipliers to the above constraints. The KKT conditions for this game may be stated as follows:

$$
\begin{aligned}
& 0 \leq p_j \perp -\left(\sum_{i=1}^{I} \rho_{ij} - (p_j - C_j)\sum_{i=1}^{I} \chi\alpha_{ip}\rho_{ij}(1-\rho_{ij})\right) \geq 0, \quad \forall j \in \mathcal{J} \\
& 0 \leq a_{nk} \perp (p_k - C_k)\left(\sum_{i=1}^{I} 2x\alpha_{in}(a_{nk} - b_{in})\rho_{ik}(1-\rho_{ik})\right) + \frac{\partial C_k}{\partial a_{nk}}\sum_{i=1}^{I}\rho_{ik} - \lambda_1 + \lambda_2 \geq 0, \quad n = 1,2 \\
& 0 \leq a_{nk} \perp (p_k - C_k)\left(\sum_{i=1}^{I} 2x\alpha_{in}(a_{nk} - b_{in})\rho_{ik}(1-\rho_{ik})\right) + \frac{\partial C_k}{\partial a_{nk}}\sum_{i=1}^{I}\rho_{ik} \geq 0, \quad n = 3,4 \\
& 0 \leq \lambda_1 \perp a_{1k} + a_{2k} - 0.325 \geq 0 \\
& 0 \leq \lambda_2 \perp 0.5 - a_{1k} + a_{2k} \geq 0
\end{aligned}
$$

Remark: It is easy to note that the profit function increases monotonically with respect to the attributes (1 and 2) if the value of the attribute is greater than all the customers' perceptions of the same. The constraint marking the lower bound decreases monotonically with respect to the attributes and the constraint marking the upper bound increases monotonically with respect to the attributes. Therefore the lower bound would become active if the former happens and the upper bound constraint would become active in the latter case. However one cannot theoretically ascertain whether the attribute value is greater or lesser than the customers' perceptions at optimality.

All in one approach

The above complementarity problem with twenty one complementarity constraints was solved by using KNITRO (version 5). The results for the all in one approach are reported in tables 4.3 and 4.4. It is to be mentioned that different starting points, different solvers and different algorithms yield different solutions for this complementarity problem. It should also be noted that the solution reported above was obtained with the initial starting points being set to the "traditional or initial" values of the prices and attributes.

Table 4.3: Nash equilibrium-attribute values of the new product

Attribute	Traditional	NASH Equilibrium
Asprin	0.1239	0.1026
Asprin substitute	0.2011	0.2224
Caffeine	0	0
Other ing.	0	0

Table 4.4: Nash equilibrium-price values

Brand	Traditional	Nash equilibrium
A	0.699	0.6268
B	0.397	0.2259
C	0.529	0.6268
D	0.329	0.2276
E	0.269	0.197
F	0.389	0.2184
G	0.531	0.2472
H	0.199	0.199
I	0.575	0.6268
J	0.499	0.4762
K	0.759	0.6268
L	0.499	0.6268
M	0.369	0.426
N	0.499	0.3926
New Product	0.374	0.3853

Decomposition method

The above problem was also solved by means of the iterative decomposition method (best response). Tables 4.5 and 4.6 show the convergence of the iterates to the same solution.

Table 4.5: Convergence to Nash equilibrium-attribute values of the new product

		Iteration						
Attribute	Traditional	1	2	3	4	5	6	7-NASH
asprin	0.1239	0.1239	0.1028	0.1021	0.1027	0.1027	0.1026	0.1026
asprin substitute	0.2011	0.2011	0.2222	0.2229	0.2223	0.2223	0.2224	0.2224
caffeine	0	0	0	0	0	0	0	0
other ing	0	0	0	0	0	0	0	0

4.7 Summary

A long run Bertrand competition with regard to price and attributes has been formulated. The logit model has been deployed to capture consumer preferences. However the non-convexity arising from the interaction terms (logit and price) proves to be a deterrent in establishing any tractable properties for the game. Under some assumptions it is claimed that the equivalent variational forms admit solutions. It is also seen that the variational and the

Table 4.6: Convergence to Nash equilibrium-price values

		Iteration						
Brand	Traditional	1	2	3	4	5	6	7-NASH
A	0.699	0.6454	0.6293	0.6253	0.6268	0.6269	0.6268	0.6268
B	0.397	0.2332	0.2259	0.2259	0.2259	0.2259	0.2259	0.2259
C	0.529	0.6471	0.6293	0.6253	0.6268	0.6269	0.6268	0.6268
D	0.329	0.2381	0.2272	0.2275	0.2276	0.2276	0.2276	0.2276
E	0.269	0.2151	0.1976	0.197	0.197	0.197	0.197	0.197
F	0.389	0.2289	0.2184	0.2183	0.2184	0.2184	0.2184	0.2184
G	0.531	0.2504	0.2465	0.2472	0.2472	0.2472	0.2472	0.2472
H	0.199	0.199	0.199	0.199	0.199	0.199	0.199	0.199
I	0.575	0.648	0.6293	0.6253	0.6268	0.6269	0.6268	0.6268
J	0.499	0.4654	0.4742	0.4756	0.4762	0.4762	0.4762	0.4762
K	0.759	0.6437	0.6293	0.6253	0.6268	0.6269	0.6268	0.6268
L	0.499	0.6447	0.6293	0.6253	0.6268	0.6269	0.6268	0.6268
M	0.369	0.4166	0.4311	0.4264	0.4258	0.426	0.4261	0.426
N	0.499	0.3797	0.3878	0.392	0.3926	0.3926	0.3926	0.3926
New Product	0.374	0.3744	0.3838	0.3854	0.3852	0.3853	0.3853	0.3853

complementarity forms lack monotonicity. A practical automotive game is formulated and analyzed. The use of projection and iterative regularization schemes do not seem to be effective. It is also seen that interior point methods and smoothing techniques also do not perform greatly with these problems. Interior point methods (KNITRO-solver) and iterative gaussian methods were tested on a smaller pain killer toy problem. Results proved to be convergent and the final solutions were seen to be the same with regard to both the schemes. Though the logit seems to be a great model to capture consumer preferences, its analytical properties are not seen to be the best from the standpoint of revenue maximization. Future work may therefore focus on a betterment of this logit model from the perspective of revenue management in a game theoretic setting in addition to development of suitable convergent schemes for these classes of problems.

Appendix A

Appendix

A.1 Risk-based generalized Nash games in power markets: characterization and computation of equilibria

With the respective dual variables, the variational inequality can be written in the form of a complementarity problem. The solution to the complementarity problem is the same as that of the VI and in turn a solution for the original GNP. For computation, we solve the complementarity problem to get the solution of the VI and in turn a solution to the original GNP.

A.1.1 Nonshared risk

For our computation, we assume the loss function to be linear. Let us assign the multipliers $\alpha_{ij}^{\omega}, \beta_{ij}^{\omega}$ to equality and capacity constraints respectively for the firms' problems.Let $\gamma_{ij}^{\omega}, \delta_{ij}^{\omega}$ refer to the constraints with respect to s_{ij}^{ω}(firms' problems). Let, $\mu^{\omega}, \sigma_l^{\omega}, \eta_l^{\omega}$ be the multipliers assigned to the power balance/ equality and transmission constraints of the Independent System Operator. Let ϕ_i^{ω} represent the multiplier for the shared constraint. Since, there are no deviation penalties, $f'_p(u^\omega_{ij}) = f'_n(v^\omega_{ij}) = 0, \forall i \in G, \forall j \in \mathcal{J}, \forall \omega \in \Omega$. Then, the complementarity problem is given by:

$$
\begin{aligned}
&0 \leq x_{ij} \perp b^0_i x_{ij} + b^0_i \sum_{j\in\mathcal{J}} x_{ij} - a^0_i + \sum_{\omega\in\Omega} \rho^\omega a^\omega_i - \sum_{\omega\in\Omega} \rho^\omega b^\omega_i (\sum_{j\in\mathcal{J}} y^\omega_{ij} + r^\omega_i) - \sum_{\omega\in\Omega} \alpha^\omega_{ij} + \chi \sum_{\omega\in\Omega} \delta^\omega_{ij} \geq 0 \\
&0 \leq y^\omega_{ij} \perp \rho^\omega(-a^\omega_i + c^\omega_i + (b^\omega_i + d^\omega_{ij}) y^\omega_{ij} + b^\omega_i (\sum_{j\in\mathcal{J}} y^\omega_{ij}) + b^\omega_i r^\omega_i - b^\omega_i x_{ij}) + \alpha^\omega_{ij} + \beta^\omega_{ij} - \phi^\omega_i \geq 0 \\
&0 \leq u^\omega_{ij} \perp f'_p(u^\omega_{ij}) - \alpha^\omega_{ij} \geq 0 \\
&0 \leq v^\omega_{ij} \perp f'_n(v^\omega_{ij}) + \alpha^\omega_{ij} \geq 0 \\
&0 \leq s^\omega_{ij} \perp \frac{\kappa_j \rho^\omega}{1-\tau} - \gamma^\omega_{ij} - \delta^\omega_{ij} \geq 0 \\
&\quad free \perp \kappa_j - \sum_{j\in\mathcal{J}} \gamma^\omega_{ij} - \sum_{j\in\mathcal{J}} \delta^\omega_{ij} = 0 \\
&0 \leq \beta^\omega_{ij} \perp cap^\omega_{ij} - y^\omega_{ij} \geq 0 \\
&0 \leq \gamma^\omega_{ij} \perp s^\omega_{ij} + m_{ij} \geq 0 \\
&0 \leq \delta^\omega_{ij} \perp s^\omega_{ij} + m_{ij} - \chi(x_{ij} - cap^\omega_{ij}) \geq 0 \\
&\quad free \perp y^\omega_{ij} - x_{ij} + u^\omega_{ij} - v^\omega_{ij} = 0 \\
&0 \leq \phi^\omega_i \perp \sum_{j\in\mathcal{J}} y^\omega_{ij} + r^\omega_i \geq 0 \\
&\quad free \perp -\rho^\omega a^\omega_i + \rho^\omega b^\omega_i (\sum_{j\in\mathcal{J}} y^\omega_{ij} + r^\omega_i) + \mu^\omega + \sum_{l\in\mathcal{L}} Q_{l,i}(\sigma^\omega_l - \eta^\omega_l) - \phi^\omega_i = 0, \qquad i \in G \\
&\quad\quad r^\omega_i \perp -\rho^\omega a^\omega_i + \rho^\omega b^\omega_i r^\omega_i + \mu^\omega + \sum_{l\in\mathcal{L}} Q_{l,i}(\sigma^\omega_l - \eta^\omega_l) \geq 0, \qquad i \in (G^c - \{51\}) \\
&\quad\quad r^\omega_i \perp -\rho^\omega a^\omega_i + \rho^\omega b^\omega_i r^\omega_i + \mu^\omega \geq 0, \qquad \text{slack node-51} \\
&\quad free \perp \sum_{i\in\mathcal{N}} r^\omega_i = 0 \\
&\quad\quad \sigma^\omega_l \perp K^\omega_l - \sum_{i\in\mathcal{N}} Q_{l,i} r^\omega_i \geq 0 \\
&\quad\quad \eta^\omega_l \perp K^\omega_l + \sum_{i\in\mathcal{N}} Q_{l,i} r^\omega_i \geq 0.
\end{aligned}
$$

A.1.2 Shared risk

The multipliers are the same as before(except that γ and δ are absent). Similarly, $f'_p(u^{\omega}_{ij}) = f'_n(v^{\omega}_{ij}) = 0, \forall i \in G, \forall j \in \mathcal{J}, \forall \omega \in \Omega$. Here, we write the complementarity problem for the smooth problem as indicated in section 3.

$$0 \le x_{ij} \perp b^0_i(x_{ij} + \sum_{j\in\mathcal{J}} x_{ij}) - a^0_i + \sum_{\omega\in\Omega} \rho^{\omega}(a^{\omega}_i - b^{\omega}_i(\sum_{j\in\mathcal{J}} y^{\omega}_{ij} + r^{\omega}_i)) - \sum_{\omega\in\Omega} \alpha^{\omega}_{ij} + \kappa_j \sum_{\omega\in\Omega} \rho^{\omega} \frac{f'(q^{\omega}_{ij})}{1-\tau} . \frac{\partial \varrho^S_{ij}(x_{ij}, cap^{\omega}_i)}{\partial x_{ij}} \ge 0$$

$$0 \le y^{\omega}_{ij} \perp \rho^{\omega}(-a^{\omega}_i + c^{\omega}_i + (b^{\omega}_i + d^{\omega}_{ij})y^{\omega}_{ij} + b^{\omega}_i(\sum_{j\in\mathcal{J}} y^{\omega}_{ij}) + b^{\omega}_i r^{\omega}_i - b^{\omega}_i x_{ij}) + \alpha^{\omega}_{ij} + \beta^{\omega}_{ij} - \phi^{\omega}_i \ge 0$$

$$0 \le u^{\omega}_{ij} \perp f'_p(u^{\omega}_{ij}) - \alpha^{\omega}_{ij} \ge 0$$

$$0 \le v^{\omega}_{ij} \perp f'_n(v^{\omega}_{ij}) + \alpha^{\omega}_{ij} \ge 0$$

$$free \perp \kappa_j(1 - \sum_{\omega\in\Omega} \rho^{\omega} \frac{f'(q^{\omega}_{ij})}{1-\tau}) = 0$$

$$0 \le \beta^{\omega}_{ij} \perp cap^{\omega}_{ij} - y^{\omega}_{ij} \ge 0$$

$$free \perp y^{\omega}_{ij} - x_{ij} + u^{\omega}_{ij} - v^{\omega}_{ij} = 0$$

$$0 \le \phi^{\omega}_i \perp \sum_{j\in\mathcal{J}} y^{\omega}_{ij} + r^{\omega}_i \ge 0$$

$$free \perp -\rho^{\omega} a^{\omega}_i + \rho^{\omega} b^{\omega}_i(\sum_{j\in\mathcal{J}} y^{\omega}_{ij} + r^{\omega}_i) + \mu^{\omega} + \sum_{l\in\mathcal{L}} Q_{l,i}(\sigma^{\omega}_l - \eta^{\omega}_l) - \phi^{\omega}_i = 0, \qquad i \in G$$

$$r^{\omega}_i \perp -\rho^{\omega} a^{\omega}_i + \rho^{\omega} b^{\omega}_i r^{\omega}_i + \mu^{\omega} + \sum_{l\in\mathcal{L}} Q_{l,i}(\sigma^{\omega}_l - \eta^{\omega}_l) = 0, \qquad i \in (G^c - \{51\})$$

$$r^{\omega}_i \perp -\rho^{\omega} a^{\omega}_i + \rho^{\omega} b^{\omega}_i r^{\omega}_i + \mu^{\omega} \ge 0, \qquad \text{slack node-51}$$

$$free \perp \sum_{i\in\mathcal{N}} r^{\omega}_i = 0$$

$$\sigma^{\omega}_l \perp K^{\omega}_l - \sum_{i\in\mathcal{N}} Q_{l,i} r^{\omega}_i \ge 0$$

$$\eta^{\omega}_l \perp K^{\omega}_l + \sum_{i\in\mathcal{N}} Q_{l,i} r^{\omega}_i \ge 0.$$

A.1.3 Network details

The Belgian grid is shown in Figure A.1. The network details (Belgian grid) are reported in Table A.1.3.

A.2 A complementarity approach for game theoretic discrete choice models

Tables A.2, A.3 and A.4 report the requisite data for Chapter 4. A set of codes have been included for Chapter 4 in the following subsections.

Table A.1: Network details

Line	Imp. (Ohm)	Cap.(MW)	Line	Imp. (Ohm)	Cap.(MW)	Line	Imp. (Ohm)	Cap.(MW)
1-2	23716	345	16-17	2633	5154	34-37	7048	1350
1-15	6269	345	17-18	4236	1715	34-52	12234	1350
2-15	8534	345	17-19	1939	5140	35-41	14204	1350
3-4	5339	240	17-20	8071	1179	35-52	9026	1420
3-15	11686	240	18-19	1465	13170	36-41	15777	2770
4-5	6994	510	19-52	11321	1179	36-42	11186	2840
4-12	5887	405	20-23	13165	1316	36-43	15408	2770
4-15	3644	240	21-22	47621	1420	37-39	66471	1420
5-13	6462	510	22-23	11391	1350	37-41	21295	1350
6-7	23987	300	22-49	9138	1350	38-39	10931	1650
6-8	9138	400	23-24	41559	5540	38-51	17168	946
7-21	14885	541	23-25	16982	1420	39-51	8596	1650
7-32	5963	410	23-28	8610	1350	40-41	11113	2770
8-9	45360	400	23-32	33255	1350	41-46	11509	2840
8-10	26541	800	25-26	134987	1420	41-47	13797	1420
8-32	11467	400	25-30	11991	1420	43-45	34468	1350
9-11	20157	410	27-28	64753	1420	44-45	47128	1420
9-32	10012	375	28-29	38569	1350	46-47	34441	1420
11-32	18398	375	29-31	284443	1350	47-48	14942	1420
12-32	4567	405	29-45	14534	1350	48-49	6998	1420
13-14	121410	2700	30-31	269973	1420	49-50	5943	3784
13-15	5094	790	30-43	10268	1420	50-51	2746	5676
13-23	5481	2770	31-52	1453	400	52-53	1279	2840
15-16	8839	400	33-34	40429	1420			

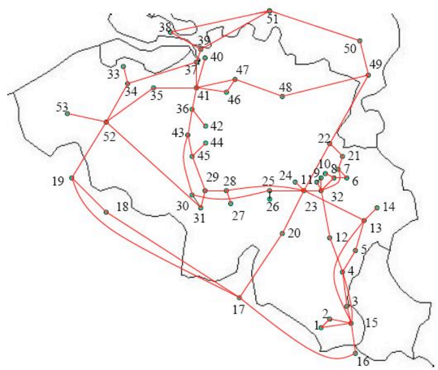


Figure A.1: The Belgian grid

A.2.1 Code-automobile problem-projection–VI

The code that was used to solve the variational formulation corresponding to the automobile game is shown below. Quadratic subproblems at every step are solved by using SQOPT.

```
tic;
% Automobile competition- equivalent VI
% z - vector of decision variables
% z=[z1 ...zJ]';
% zi - order- p,H,A,eta,L,B,He
% Equivalent VI is written
% Attempt to solve the epsilon penalized problem
% Primal form
% Use SQOPT as the solver to solve the quadratic projection problems at every step
% Definition of parameters

% No. of firms
J=3;
% Initial vector - Starting point
z0=10000*ones(10*J,1);
for(j=1:J)
z0(10*(j-1)+1,1)=10000;
z0(10*(j-1)+2,1)=100;
```

Table A.2: Attribute values of the manufacturers

Market Player	Asprin	Asprin sub.	Caffeine	Other ing.	Cost	Market price
1	0	0.5	0	0	0.4	0.699
2	0.4	0	0.032	0	0.1328	0.397
3	0	0.5	0	0	0.4	0.529
4	0.325	0	0	0.15	0.1275	0.329
5	0.325	0	0	0	0.0975	0.269
6	0.324	0	0	0.1	0.1172	0.389
7	0.421	0	0.032	0.075	0.1541	0.531
8	0.5	0	0	0.1	0.17	0.199
9	0	0.5	0	0	0.4	0.575
10	0.25	0.25	0.065	0	0.301	0.499
11	0	0.5	0	0	0.4	0.759
12	0	0.5	0	0	0.4	0.499
13	0	0.325	0	0	0.26	0.369
14	0.227	0.194	0	0.075	0.2383	0.499

```
z0(10*(j-1)+3,1)=20;
z0(10*(j-1)+4,1)=20;
z0(10*(j-1)+5,1)=20;
z0(10*(j-1)+6,1)=40;
z0(10*(j-1)+7,1)=20;
end
%z0=[];
z0=10000*ones(10*J,1);

% Regularization

epsilon=0.001;
epsilon1=0.001;
eps=epsilon;
% step size
step1=0.00025;
gamma=step1;
step2=0.0005;

% Costs

UH=10*0.15*ones(J,1);
UA=0*ones(J,1);
Ueta=0*ones(J,1);
UL=10*0.15*ones(J,1);
UB=10*0.15*ones(J,1);
UHe=10*0.15*ones(J,1);

% Logit coefficients

alphaH=0.5;
alphaA=0.5;
alphaeta=0.5;
alphaL=2;
alphaB=0.2;
alphaHe=0.2;
alphap=0.000005;
wH=alphaH;
wA=alphaA;
weta=alphaeta;
wL=alphaL;
wB=alphaB;
wHe=alphaHe;
wp=alphap;
```

Table A.3: Customer preference data

Customer	Asprin	Asprin sub.	Caffeine	Other ing.
1	0	0.0835	0	0.0331
2	0	0.543	0.0075	0.0204
3	0	0.4889	0.0055	0
4	0.479	0.0568	0	0.0725
5	0.3202	0	0.0013	0
6	0	0.1395	0	0
7	0	0.4805	0	0
8	0.0649	0.3759	0.0022	0
9	0	0.3834	0	0
10	0.3431	0.0908	0	0.0695
11	0.0484	0.3229	0.0351	0
12	0.2696	0.0741	0.0005	0.111
13	0.4348	0.0276	0.0013	0.0605
14	0.2634	0	0.0022	0
15	0.3163	0.0581	0	0
16	0.0859	0.0488	0	0.1355
17	0.3197	0.032	0.0424	0.063
18	0.1872	0.7724	0	0.0186
19	0.4398	0.0235	0.023	0.0765
20	0	0.196	0	0.0604
21	0.0242	0.5938	0.0016	0.0002
22	0.0016	0.5157	0.0399	0.0079
23	0.2584	0.0761	0.0024	0.0065
24	0	0.5171	0	0
25	0.1094	0.1291	0	0.0934
26	0.0153	0.2855	0	0
27	0.1851	0.0874	0.0322	0.0903
28	0.1289	0.262	0.1226	0
29	0.0472	0.2513	0.0059	0
30	0.2752	0.0199	0.0003	0.0224

```
% Design parameters

a1=200*ones(J,1);
a2=140*ones(J,1);
b1=0.4*ones(J,1);
b2=0.4*ones(J,1);
c1=0.4*ones(J,1);
c2=0.4*ones(J,1);
d1=0.4*ones(J,1);
d2=0.4*ones(J,1);
e1=0.4*ones(J,1);
e2=0.4*ones(J,1);



% WHILE LOOP- PROJECTION
tol=100;
i=1;
z=z0;

% Definition of the constraint matrix (since constraints are linear)

for(j=1:J)
    Co(:,:,j)=[0 -b1(j,1) 1 0 c1(j,1) d1(j,1) e1(j,1);0 b2(j,1) 0 1 c2(j,1) d2(j,1) e2(j,1)];
Co1(:,:,j)=[Co(:,:,j);eye(7,7)];
    R1(:,j)=[a1(j,1);a2(j,1);0;0;0;0;0;0;0];
    R2(:,j)=[a1(j,1);a2(j,1);inf;inf;inf;inf;inf;inf;inf];
end

% Actual while loop


while(tol>0.00000001)
```

Table A.4: Coefficients of the disutility function

Customer	Attribute coefficient	Price coefficient	Error term
1	15.1354	3.8655	-4.4286
2	4.6278	1	-2.0476
3	2.2123	1	-1.8206
4	0	4.0706	-3.2257
5	0	2.9537	-2.1314
6	10.5894	1.5244	-2.758
7	5.0178	1	-1.9722
8	3.5191	3.0352	-2.7977
9	9.101	3.0648	-3.1728
10	0	2.6051	-2.228
11	10.5342	7.6762	-5.1675
12	0	7.5246	-4.4067
13	0	5.3952	-3.0808
14	0	5.7735	-3.4689
15	0	3.2881	-2.6675
16	7.4649	4.944	-4.1138
17	0.6457	2.0779	-1.8347
18	4.8654	1	-3.5624
19	0.5351	3.9169	-2.3135
20	5.3182	1.9882	-2.2817
21	6.8606	5.2027	-4.387
22	5.6944	1	-1.8547
23	0	4.7539	-2.755
24	5.986	2.3496	-2.6193
25	14.4747	1	-2.6596
26	13.5548	1	-2.9508
27	13.0129	1	-2.5012
28	22.7317	1.9678	-3.6522
29	5.1373	3.4133	-2.8745
30	0.0755	5.1061	-2.7871

```
for(j=1:J)
    p(j,1)=z(10*(j-1)+1,1);
    H(j,1)=z(10*(j-1)+2,1);
    A(j,1)=z(10*(j-1)+3,1);
    eta(j,1)=z(10*(j-1)+4,1);
    L(j,1)=z(10*(j-1)+5,1);
    B(j,1)=z(10*(j-1)+6,1);
    He(j,1)=z(10*(j-1)+7,1);
    D(j,1)=exp(alphaH*H(j,1)+alphaA*A(j,1)+alphaeta*eta(j,1)+alphaL*L(j,1)+alphaB*B(j,1)+alphaHe*He(j,1)-alphap*p(j,1));
    C(j,1)=0.5*(UH(j,1)*H(j,1)^2+UA(j,1)*A(j,1)^2+Ueta(j,1)*eta(j,1)^2+UL(j,1)*L(j,1)^2+UB(j,1)*B(j,1)^2+UHe(j,1)*He(j,1)^2);
    DHOCH(j,1)=UH(j,1)*H(j,1);
    DHOCA(j,1)=UA(j,1)*A(j,1);
    DHOCeta(j,1)=Ueta(j,1)*eta(j,1);
    DHOCL(j,1)=UL(j,1)*L(j,1);
    DHOCB(j,1)=UB(j,1)*B(j,1);
    DHOCHe(j,1)=UHe(j,1)*He(j,1);
end
Pr(:,1)=D/sum(D);

% Definition of the CP

for(j=1:J)
    F(10*(j-1)+1,1)=-Pr(j,1)+wp*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1));
    F(10*(j-1)+2,1)=(Pr(j,1)+beta(j,1))*DHOCH(j,1)-wH*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1));
    F(10*(j-1)+3,1)=(Pr(j,1)+beta(j,1))*DHOCA(j,1)-wA*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1));
    F(10*(j-1)+4,1)=(Pr(j,1)+beta(j,1))*DHOCeta(j,1)-weta*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1));
    F(10*(j-1)+5,1)=(Pr(j,1)+beta(j,1))*DHOCL(j,1)-wL*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1));
    F(10*(j-1)+6,1)=(Pr(j,1)+beta(j,1))*DHOCB(j,1)-wB*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1));
    F(10*(j-1)+7,1)=(Pr(j,1)+beta(j,1))*DHOCHe(j,1)-wHe*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1));


    z1=z;
    % Projection
```

```
    x0(:,j)=(1-gamma*eps)*z(10*(j-1)+1:10*(j-1)+7,1)-gamma*F(10*(j-1)+1:10*(j-1)+7);
I=2*eye(7,7);
    %zit(:,j)=sqopt(I,-2*x0(:,j),Co(:,:,j),R(:,j));
    %z(10*(j-1)+1:10*(j-1)+7,1)=zit(:,j);
    ss_0=[];
    Prob1=qpAssign(I,-2*x0(:,j),Co1(:,:,j),R1(:,j),R2(:,j),ss_0);

    % same time comp for the iso's problem

    ct=cputime;
    Sol=tomRun('sqopt',Prob1);
     zit(:,j)=Sol.x_k(1:7,1);
     et=cputime;
    z(10*(j-1)+1:10*(j-1)+7,1)=zit(:,j);

end

tol=norm(z-z1);
i=i+1;
TOL(i)=tol;
F2(:,i)=F;
end
toc;


%  Obtain the final solution z if convergence is obtained.
```

A.2.2 Code-automobile problem-projection–CP

This reduces the complexity of solving quadratic subproblems. This focuses on the primal dual form stated previously (complementarity problem). The projection operator is therefore replaced by a max operator at the complementarity constraints. For equality equilibrium conditions, the max term is excluded.

```
tic;

% Automobile competition- equivalent CP
% z - vector of decision variables
% z=[z1 ...zJ]';
% zi - Primal and dual variables for agent i
% zi - order- p,H,A,eta,L,B,He
% Equivalent CP is written with primal and dual variables
% Attempt to solve the epsilon penalized problem
% Primal-dual form
% Use max when the projection is on R^{N+}
% Use pure gradient descent when the projection is over R^{N}

AA=sparse(100000,100000);

% Definition of parameters

% No. of firms

J=4;
epsilon=5e-1;
epsilon1=5e-1;
% step size

step1=.001;
```

```
step2=.001;
eps=epsilon;

% Starting point

z0=0*ones(10*J,1);
z=z0;
%step1=step11;
%step2=step22;
jj=1;
clear TOL;
eps=epsilon;

% Costs

UH=10*0.15*ones(J,1);
UA=0*ones(J,1);
Ueta=0*ones(J,1);
UL=10*0.15*ones(J,1);
UB=10*0.15*ones(J,1);
UHe=10*0.15*ones(J,1);

% Logit coefficients

alphaH=10*0.5;
alphaA=10*0.5;
alphaeta=10*0.5;
alphaL=10*2;
alphaB=10*0.2;
alphaHe=10*0.2;
alphap=100*0.000005;
wH=alphaH;
wA=alphaA;
weta=alphaeta;
wL=alphaL;
wB=alphaB;
wHe=alphaHe;
wp=alphap;

% Design parameters

a1=10*ones(J,1);
a2=7*ones(J,1);
b1=1.1*ones(J,1);
b2=1*ones(J,1);
c1=0.4*ones(J,1);
c2=0.4*ones(J,1);
d1=1*ones(J,1);
d2=0.4*ones(J,1);
e1=1*ones(J,1);
e2=0.4*ones(J,1);

% WHILE LOOP- PROJECTION
tol=100;
i=1;
while(tol>eps*1e-6)

for(j=1:J)
    p(j,1)=z(10*(j-1)+1,1);
    H(j,1)=z(10*(j-1)+2,1);
    A(j,1)=z(10*(j-1)+3,1);
    eta(j,1)=z(10*(j-1)+4,1);
    L(j,1)=z(10*(j-1)+5,1);
    B(j,1)=z(10*(j-1)+6,1);
```

```
    He(j,1)=z(10*(j-1)+7,1);
    beta(j,1)=z(10*(j-1)+8,1);
    gamma(j,1)=z(10*(j-1)+9,1);
    delta(j,1)=z(10*(j-1)+10,1);
    D(j,1)=exp(alphaH*H(j,1)+alphaA*A(j,1)+alphaeta*eta(j,1)+alphaL*L(j,1)+alphaB*B(j,1)+alphaHe*He(j,1)-alphap*p(j,1));
    C(j,1)=0.5*(UH(j,1)*H(j,1)^2+UA(j,1)*A(j,1)^2+Ueta(j,1)*eta(j,1)^2+UL(j,1)*L(j,1)^2+UB(j,1)*B(j,1)^2+UHe(j,1)*He(j,1)^2);
    DHOCH(j,1)=UH(j,1)*H(j,1);
    DHOCA(j,1)=UA(j,1)*A(j,1);
    DHOCeta(j,1)=Ueta(j,1)*eta(j,1);
    DHOCL(j,1)=UL(j,1)*L(j,1);
    DHOCB(j,1)=UB(j,1)*B(j,1);
    DHOCHe(j,1)=UHe(j,1)*He(j,1);
end
Pr(:,1)=D/sum(D);

% Definition of the CP

for(j=1:J)
    F(10*(j-1)+1,1)=-Pr(j,1)+wp*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1))-beta(j,1);
    F(10*(j-1)+2,1)=(Pr(j,1)+beta(j,1))*DHOCH(j,1)-wH*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1))-gamma(j,1)*b1(j,1)+gamma(j,1)*b2(j,1);
    F(10*(j-1)+3,1)=(Pr(j,1)+beta(j,1))*DHOCA(j,1)-wA*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1))+gamma(j,1);
    F(10*(j-1)+4,1)=(Pr(j,1)+beta(j,1))*DHOCeta(j,1)-weta*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1))+delta(j,1);
    F(10*(j-1)+5,1)=(Pr(j,1)+beta(j,1))*DHOCL(j,1)-wL*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1))+gamma(j,1)*c1(j,1)+delta(j,1)*c2(j,1);
    F(10*(j-1)+6,1)=(Pr(j,1)+beta(j,1))*DHOCB(j,1)-wB*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1))+gamma(j,1)*d1(j,1)+delta(j,1)*d2(j,1);
    F(10*(j-1)+7,1)=(Pr(j,1)+beta(j,1))*DHOCHe(j,1)-wHe*(p(j,1)-C(j,1))*Pr(j,1)*(1-Pr(j,1))+gamma(j,1)*e1(j,1)+delta(j,1)*e2(j,1);
    F(10*(j-1)+8,1)=p(j,1)-C(j,1);
    F(10*(j-1)+9,1)= A(j,1)-a1(j,1)-b1(j,1)*H(j,1)+c1(j,1)*L(j,1)+d1(j,1)*B(j,1)+e1(j,1)*He(j,1);
    F(10*(j-1)+10,1)=eta(j,1)-a2(j,1)+b2(j,1)*H(j,1)+c2(j,1)*L(j,1)+d2(j,1)*B(j,1)+e2(j,1)*He(j,1);

    z1=z;
    % Projection

    z(10*(j-1)+1:10*(j-1)+8,1)=max(((1-step1*epsilon)*z(10*(j-1)+1:10*(j-1)+8,1)-step1*F(10*(j-1)+1:10*(j-1)+8,1)),0);
    z(10*(j-1)+9:10*(j-1)+10,1)=((1-step2*epsilon1)*z(10*(j-1)+9:10*(j-1)+10,1)-step2*F(10*(j-1)+9:10*(j-1)+10,1));
end
ZZ(:,i)=z;
tol=norm(z-z1);
i=i+1;
TOL(i,1)=tol;
rr(i)=abs((TOL(i)-TOL(i-1)));
F2(:,i)=F;
end
AA(1:i,jj)=TOL;
jj=jj+1;
toc;
save('codepricetry');
```

A.2.3 Code-automobile problem-solver-KNITRO

The problem was solved using the solver KNITRO (interior point methods). The file call.m calls the solver KNITRO. The complementarity problem is defined in the file constraint.m. The file mpdf.m marks the function to be minimized (0 in this case).

call.m

```
% J No. of firms
% N - No. of attributes
% Solve a CP using KNITRO -auto example- For details see constraint.m
```

```
global J;
global N;

J=2;
N=6;
N2=N+4;
A=[];
b_L=[];
b_U=[];
H_L=100;

% Using KNITRO

ConsPattern = [];

HessPattern = [];

% constraints w.r.t x and m. x-comp cons  m-eq cons.


c_L = zeros((N+4)*J,1);
c_U = inf*ones((N+4)*J,1);

for(j=1:J)
c_U((j-1)*N2+9)=0;
c_U((j-1)*N2+10)=0;
end

% Lower bound on m

x_L=zeros(J*(N+4),1);
x_U=inf*ones(J*(N+4),1);



% Starting point

%x_01=[110; 60; 60; 150; 45; 16; 19800; 0; -5; 5];
%x_0=[x_01;x_01;x_01];
x_0=[];

Name = 'raj';

% Functions for calculating the nonlinear function and derivative values.

f1   = 'mpd_f';
g    = [];
H    = [];
c1   = 'constraint';
dc   = [];
d2c = [];

Prob = conAssign(f1, g, H, HessPattern, x_L, x_U, Name, x_0, ...
   [],[], ...
   A, b_L, b_U, c1, dc, d2c, ConsPattern, c_L, c_U);

Prob.PriLevOpt = 3;

% KNITRO options. Algorithm 1 works good on this problem.

opts     = [];

%opts.FEASTOL=0.0001;
%opts.OPTTOL=0.0001;
opts.ALG = 3;
```

```
Prob.KNITRO.options = opts;

% Definition of MPEC

ff=(N+2)*J;
mpec=zeros(ff,6);
t=1;
for(j=1:J)
for(i=1:N+2)
    mpec(t,1)=(j-1)*N2+i;
    mpec(t,5)=(j-1)*N2+i;
t=t+1;
end
end

% Include MPEC in the Prob structure

Prob = BuildMPEC(Prob,mpec);

% Now solve this problem with KNITRO:

Sol = tomRun('knitro',Prob,2);
x1=Sol.x_k;
```

constraint.m

```
function c1=constraint(x,Prob)

% Primal variables in order l,b,h,H,eta,A
% dual variables in order beta, gamma, delta

% alpha marks the coefficient vector...coefficients of attributes and price in the order of variables

% Length, breadth, height, horsepower, mpg, acceleration and price
% respectively
% H1 represents the coeff of horsepower in the linear constraint with MPG
% H2 represnts - the coeff of horsepower in the linear constraint with acceleration
% l1 represents the coeff of length in the linear constraint with MPG
% l2 represents the coeff of length in the linear constraint with acceleration
% w1,w2 defined accordingly
% h1,h2 defined accordingly
% con1 con2 represent the constants.

global J;
H1=0.5*ones(J,1);
H2=0.05*ones(J,1);
l1=0.1*ones(J,1);
l2=0.01*ones(J,1);
w1=0.1*ones(J,1);
w2=0.01*ones(J,1);
h1=0.1*ones(J,1);
h2=0.01*ones(J,1);
con1=150;
con2=15;

% cost represents cost coefficients

global N;
th=0.5;
N=6;


alpha=[0.5;0.6;0.63;0.5;0.401;0.5;-.1];
```

```
co=0.001*[10 10 10 0.1];
for(j=1:J)
  cost(j,:)=co;
end

% J represents the number of firms
% N represents the number of attributes

N1=N+1;
N2=N+4;

% Cj represents the cost
% dC represents partial derivative
% Wj represents weight
% dW represents partial derivative

W=zeros(J,1);
dWl=zeros(J,1);
dWb=zeros(J,1);
dWh=zeros(J,1);
for(j=1:J)
    C(j)=8+cost(j,1)*x((j-1)*N2+1)+cost(j,2)*x((j-1)*N2+2)+cost(j,3)*x((j-1)*N2+3)+cost(j,4)*(x((j-1)*N2+4));
    dCl(j)=cost(j,1);
    dCb(j)=cost(j,2);
    dCh(j)=cost(j,3);
    dCH(j)=cost(j,4);
    %dCH(j)=2*cost(j,4)*(x((j-1)*N2+4));

end

% Calculation of probability

% exponential Utility

for(j=1:J)
    U(j)= exp(alpha(1,1)*x((j-1)*N2+1)+alpha(2,1)*x((j-1)*N2+2)+alpha(3,1)*x((j-1)*N2+3)+alpha(4,1)*x((j-1)*N2+4)+alpha(5,1)*x((j-1)*N2+5)+alpha(6,1)*x((j-1)*N2+6))-alpha(7,1)*x((j-1)*N2+7);
end



% Logit

Pr=U/sum(U);

for(j=1:J)
    c((j-1)*N2+1)=dCl(j)*(Pr(j)+x((j-1)*N2+8))-(x((j-1)*N2+7)-C(j))*Pr(j)*(1-Pr(j))*(alpha(1,1))+x((j-1)*N2+9)*l1(j)+x((j-1)*N2+10)*l2(j);
    c((j-1)*N2+2)=dCb(j)*(Pr(j)+x((j-1)*N2+8))-(x((j-1)*N2+7)-C(j))*Pr(j)*(1-Pr(j))*(alpha(2,1))+x((j-1)*N2+9)*w1(j)+x((j-1)*N2+10)*w2(j);
    c((j-1)*N2+3)=dCh(j)*(Pr(j)+x((j-1)*N2+8))-(x((j-1)*N2+7)-C(j))*Pr(j)*(1-Pr(j))*(alpha(3,1))+x((j-1)*N2+9)*h1(j)+x((j-1)*N2+10)*h2(j);
    c((j-1)*N2+4)=dCH(j)*(Pr(j)+x((j-1)*N2+8))-(x((j-1)*N2+7)-C(j))*Pr(j)*(1-Pr(j))*(alpha(4,1))+x((j-1)*N2+9)*H1(j)-x((j-1)*N2+10)*H2(j);
    c((j-1)*N2+5)=-(x((j-1)*N2+7)-C(j))*Pr(j)*(1-Pr(j))*(alpha(5,1))+x((j-1)*N2+9);
    c((j-1)*N2+6)=-(x((j-1)*N2+7)-C(j))*Pr(j)*(1-Pr(j))*(alpha(6,1))+x((j-1)*N2+10);
    c((j-1)*N2+7)=-Pr(j)-x((j-1)*N2+8)+alpha(7,1)*(x((j-1)*N2+7)-C(j))*Pr(j)*(1-Pr(j));
    c((j-1)*N2+8)=x((j-1)*N2+7)-C(j);
    c((j-1)*N2+9)=x((j-1)*N2+5)+H1(j)*x((j-1)*N2+4)+l1(j,1)*x((j-1)*N2+1)+w1(j,1)*x((j-1)*N2+2)+h1(j,1)*x((j-1)*N2+3)-con1;
    c((j-1)*N2+10)=x((j-1)*N2+6)-H2(j)*x((j-1)*N2+4)+l2(j,1)*x((j-1)*N2+1)+w2(j,1)*x((j-1)*N2+2)+h2(j,1)*x((j-1)*N2+3)-con2;
end
c=c';
c1=c;
```

mpdf.m

```
function f1 = mpd_f(x,Prob)
f1 = 0;
```

A.2.4 Code-automobile problem -smoothing

This focuses on purely the price competition problem excluding the attributes. The complementarity problem is smoothed using the techniques stated previously. The code is shown as follows:

```
%%% Price competition -- Smooth-- Newton code

% Coefficients and parameters
% No. of firms J
% weighing coefficient w
% C costs
% epsilon smoothing parameter
% F original mapping - CP
% Smooth the max fixed point relation of the CP
% Solve a set of equations.. g=0
% Compute gradg
% Apply Newton's method

J=5;
w=1e-3;
eps=normrnd(0.1,0.02,J,1);
C0=20000;
C=C0*eps;
p=3000*ones(J,1);
tol=100;
beta=1;
kk=1;
TOL=sparse(100000,1);

while(tol>1e-20)
z=p;
barp=p-C;
epsilon=0.01;
U=exp(-w*p+eps);
Pr=U/sum(U);
F=(w*Pr.*(1-Pr)).*((2-w*(barp).*(1-2*Pr)));

barz=z-F;
g=z+F-((barz.*barz)+epsilon).^0.5;

for(i=1:J)
    for(j=1:J)
        nablaF(i,j)=w*Pr(i,1)*Pr(j,1)*(1-w*(p(i,1)-C(i,1))*(1-2*Pr(i,1)));
        if(i==j)
            nablaF(i,j)=w*Pr(i,1)*(1-Pr(j,1))*(2-w*(p(i,1)-C(i,1))*(1-2*Pr(i,1)));
        end
    end
end

gradg=(eye(J,J)+nablaF)-(diag((barz./(barz.*barz+epsilon)))*(eye(J,J)-nablaF));

p1=p;
p=p+beta*inv(gradg)*g;
tol=norm(p-p1);
kk=kk+1;
TOL(kk,1)=tol;

end
```

A.2.5 Code-pain killer problem-price competition-all in one

The code consists of three files price1.m, constraint.m and min.m. price1.m calls the solver KNITRO. constraint.m defines the complementarity constraints. min.m represents the objective to be minimized which is zero in this case.

price1.m

```
% Pain killer problem - price competition 14 firms
% All in One
% 14 complementarity constraints
% 1-14 - Price FOC'S(KKTs)


% No (explicitly) linear constraints.

A   = [];
b_L = [];
b_U = [];

% Provide a pattern for the nonlinear constraints
ConsPattern = [];
HessPattern = [];

% First constraint is equality == 0, the remaining are >= 0
c_L = zeros(14,1);
c_U = inf*ones(14,1);
x_L = zeros(14,1);
x_L(8,1)=0.199;
x_U = inf*ones(14,1);
x_U(8,1)=0.199;

% DO not specify the initial vector

Name = 'pr';
x_0=[];

% Functions for calculating the nonlinear function and derivative values.
f   = 'min';
c   = 'constraint';
g=[];
H=[];
dc=[];
d2c=[];
Prob = conAssign(f, g, H, HessPattern, x_L, x_U, Name, x_0, ...
   [],[], ...
   A, b_L, b_U, c, dc, d2c, ConsPattern, c_L, c_U);

% Brand 8 does not take part in competition

mpec=zeros(13,6);
for(i=1:7)
    mpec(i,1)=i;
    mpec(i,6)=i;
end
for(i=8:13)
    mpec(i,1)=i+1;
    mpec(i,6)=i+1;
end
Prob = BuildMPEC(Prob,mpec);
Result = tomRun('knitro',Prob);
x = Result.x_k
```

constraint.m

```
function c = constraint(x,Prob)
% No of manufacturers f1
f1=14;
c=zeros(14,1);
% m refers to the total number of customers
% Cost refers to the cost vector
% Ca refers to the weighing coefficient of each consumer towards the attributes
% Cp refers to the weighing coefficient with respect to price
% A1,A2,A3,A4 refer to the set of atribute values for m customers
% Af1,Af2,Af3,Af4 refer to the set of attribute values for 14 manufacturers
% Let us take an example of five consumers
m=30;

AA=[0 0.0835 0 0.0331
0 0.543 0.0075 0.0204
0 0.4889 0.0055 0
0.479 0.0568 0 0.0725
0.3202 0 0.0013 0
0 0.1395 0 0
0 0.4805 0 0
0.0649 0.3759 0.0022 0
0 0.3834 0 0
0.3431 0.0908 0 0.0695
0.0484 0.3229 0.0351 0
0.2696 0.0741 0.0005 0.111
0.4348 0.0276 0.0013 0.0605
0.2634 0 0.0022 0
0.3163 0.0581 0 0
0.0859 0.0488 0 0.1355
0.3197 0.032 0.0424 0.063
0.1872 0.7724 0 0.0186
0.4398 0.0235 0.023 0.0765
0 0.196 0 0.0604
0.0242 0.5938 0.0016 0.0002
0.0016 0.5157 0.0399 0.0079
0.2584 0.0761 0.0024 0.0065
0 0.5171 0 0
0.1094 0.1291 0 0.0934
0.0153 0.2855 0 0
0.1851 0.0874 0.0322 0.0903
0.1289 0.262 0.1226 0
0.0472 0.2513 0.0059 0
0.2752 0.0199 0.0003 0.0224
];
A1=AA(:,1);
A2=AA(:,2);
A3=AA(:,3);
A4=AA(:,4);

% Subject's importance

CC=[15.1354 3.8655 -4.4286
4.6278 1 -2.0476
2.2123 1 -1.8206
0 4.0706 -3.2257
0 2.9537 -2.1314
10.5894 1.5244 -2.758
5.0178 1 -1.9722
3.5191 3.0352 -2.7977
9.101 3.0648 -3.1728
0 2.6051 -2.228
10.5342 7.6762 -5.1675
```

```
0 7.5246 -4.4067
0 5.3952 -3.0808
0 5.7735 -3.4689
0 3.2881 -2.6675
7.4649 4.944 -4.1138
0.6457 2.0779 -1.8347
4.8654 1 -3.5624
0.5351 3.9169 -2.3135
5.3182 1.9882 -2.2817
6.8606 5.2027 -4.387
5.6944 1 -1.8547
0 4.7539 -2.755
5.986 2.3496 -2.6193
14.4747 1 -2.6596
13.5548 1 -2.9508
13.0129 1 -2.5012
22.7317 1.9678 -3.6522
5.1373 3.4133 -2.8745
0.0755 5.1061 -2.7871];

Ca=CC(:,1);
Cp=CC(:,2);

%Error term

e=CC(:,3);

% Af defines the attribute values of the manufacturer

Af=[0 0.5 0 0
0.4 0 0.032 0
0 0.5 0 0
0.325 0 0 0.15
0.325 0 0 0
0.324 0 0 0.1
0.421 0 0.032 0.075
0.5 0 0 0.1
0 0.5 0 0
0.25 0.25 0.065 0
0 0.5 0 0
0 0.5 0 0
0 0.325 0 0
0.227 0.194 0 0.075
0.1239 0.2011 0 0
];
Af1=Af(:,1);
Af2=Af(:,2);
Af3=Af(:,3);
Af4=Af(:,4);


% Initial Price values

p=[
   0.6113
    0.2269
    0.6113
    0.2298
    0.2004
    0.2210
    0.2484
    0.1990
    0.6113
    0.5151
```

```
    0.6113
    0.6113
    0.4425
    0.4089
];


% Costs

Cost=[ 0.4
0.1328
0.4
0.1275
0.0975
0.1172
0.1541
0.17
0.4
0.301
0.4
0.4
0.26
0.2383
];

% L-function of error
L=3;
Du=ones(m,f1);
% i refers to the customer
% j refers to the firm
% Con(i,j) refers to the sum of the attribute terms in the dis utility
% function
for(i=1:m)
    for(j=1:f1)
        Con(i,j)=Ca(i)*(Af1(j)-A1(i))^2+Ca(i)*(Af2(j)-A2(i))^2+Ca(i)*(Af3(j)-A3(i))^2+Ca(i)*(Af4(j)-A4(i))^2;
        Du(i,j)=Ca(i)*(Af1(j)-A1(i))^2+Ca(i)*(Af2(j)-A2(i))^2+Ca(i)*(Af3(j)-A3(i))^2+Ca(i)*(Af4(j)-A4(i))^2+Cp(i)*p(j)+e(i);
    end
end
% Summing up the values of the disutility function for the denominator of
% firm j's optimization problem for consumer i ( The process excludes
% disutility of i from the picture). F refers to the summation
F=zeros(m,f1);
for(i=1:m)
    for(j=1:f1)
        for(k=1:f1)
            if(k>j|k<j)
            F(i,j)=F(i,j)+exp(-L*Du(i,k));
            else
                continue
            end
        end
    end
end
for(i=1:m)
    for(j=1:f1)
        Pr(i,j)=exp(-L*(Con(i,j)+Cp(i)*x(j)+e(i)))/(exp(-L*(Con(i,j)+Cp(i)*x(j)+e(i)))+F(i,j));
    end
end
% Sum 1 represents the sum of Prij values for all i.
% Sum 2 represents the sum of all wi*(1+Prij) values for all i
Sum1=zeros(f1,1);
Sum2=zeros(f1,1);
for(j=1:f1)
    for(i=1:m)
```

```
        Sum1(j)=Sum1(j)+Pr(i,j);
        Sum2(j)=Sum2(j)+Cp(i)*Pr(i,j)*(1-Pr(i,j));
    end
end
% Declaration of the constraint vector
for(j=1:f1)
    if(j==8)
        continue
    else
    c(j)=-f1/m*(Sum1(j)-((x(j)-Cost(j))*L*Sum2(j)));
    end
end
```

min.m

```
function f = min(x,Prob)
f=0;
```

A.2.6 Code-pain killer-new product equilibrium-all in one

The code consists of three files price1.m, constraint.m and min.m. price1.m calls the solver KNITRO. constraint.m defines the complementarity constraints. min.m represents the objective to be minimized which is zero in this case.

price1.m

```
% Pain killer problem with the new product

% 21 complementarity constraints
% 1-15 - Price FOC'S(KKTs)
%16-19 attributes for the new product
% Multipliers for 20,21 constraints for the new product


% No (explicitly) linear constraints.

A   = [];
b_L = [];
b_U = [];

% Provide a pattern for the nonlinear constraints

ConsPattern = [];
HessPattern = [];

% Setting lower and upper bounds for the complementarity constraints

c_L = zeros(21,1);
c_U = inf*ones(21,1);
x_L = zeros(21,1);
x_L(8,1)=0.199;
x_U = inf*ones(21,1);
x_U(8,1)=0.199;

% Specify initial vector

x_0=[0.6990 0.3970 0.5290 0.3290 0.2690 0.3890 0.5310 0.1990 0.5750 0.4990 0.7590 0.4990 0.3690 0.4990 0.385 0.102 0.223 0 0 1.6 0]';
Name = 'pr';

% Functions for calculating the nonlinear function and derivative values.
```

```
f   = 'min';
c   = 'constraint';
g=[];
H=[];
dc=[];
d2c=[];
Prob = conAssign(f, g, H, HessPattern, x_L, x_U, Name, x_0, ...
   [],[], ...
   A, b_L, b_U, c, dc, d2c, ConsPattern, c_L, c_U);

% Brand 8 does not take part in competition

mpec=zeros(20,6);
for(i=1:7)
    mpec(i,1)=i;
    mpec(i,6)=i;
end
for(i=8:20)
    mpec(i,1)=i+1;
    mpec(i,6)=i+1;
end
Prob = BuildMPEC(Prob,mpec);
Prob.prilevopt=3;
Result = tomRun('knitro',Prob,1);
x = Result.x_k
```

constraint.m

```
function c = constraint(x,Prob)
% No of manufacturers f1
f1=15;
c=zeros(21,1);
% m refers to the total number of customers
% Cost refers to the cost vector
% Ca refers to the weighing coefficient of each consumer towards the attributes
% Cp refers to the weighing coefficient with respect to price
% A1,A2,A3,A4 refer to the set of atribute values for m customers
% Af1,Af2,Af3,Af4 refer to the set of attribute values for 14 manufacturers
% Let us take an example of five consumers
m=30;

AA=[0 0.0835 0 0.0331
0 0.543 0.0075 0.0204
0 0.4889 0.0055 0
0.479 0.0568 0 0.0725
0.3202 0 0.0013 0
0 0.1395 0 0
0 0.4805 0 0
0.0649 0.3759 0.0022 0
0 0.3834 0 0
0.3431 0.0908 0 0.0695
0.0484 0.3229 0.0351 0
0.2696 0.0741 0.0005 0.111
0.4348 0.0276 0.0013 0.0605
0.2634 0 0.0022 0
0.3163 0.0581 0 0
0.0859 0.0488 0 0.1355
0.3197 0.032 0.0424 0.063
0.1872 0.7724 0 0.0186
0.4398 0.0235 0.023 0.0765
0 0.196 0 0.0604
0.0242 0.5938 0.0016 0.0002
0.0016 0.5157 0.0399 0.0079
0.2584 0.0761 0.0024 0.0065
```

```
0 0.5171 0 0
0.1094 0.1291 0 0.0934
0.0153 0.2855 0 0
0.1851 0.0874 0.0322 0.0903
0.1289 0.262 0.1226 0
0.0472 0.2513 0.0059 0
0.2752 0.0199 0.0003 0.0224
];
A1=AA(:,1);
A2=AA(:,2);
A3=AA(:,3);
A4=AA(:,4);

% Subject's importance

CC=[15.1354 3.8655 -4.4286
4.6278 1 -2.0476
2.2123 1 -1.8206
0 4.0706 -3.2257
0 2.9537 -2.1314
10.5894 1.5244 -2.758
5.0178 1 -1.9722
3.5191 3.0352 -2.7977
9.101 3.0648 -3.1728
0 2.6051 -2.228
10.5342 7.6762 -5.1675
0 7.5246 -4.4067
0 5.3952 -3.0808
0 5.7735 -3.4689
0 3.2881 -2.6675
7.4649 4.944 -4.1138
0.6457 2.0779 -1.8347
4.8654 1 -3.5624
0.5351 3.9169 -2.3135
5.3182 1.9882 -2.2817
6.8606 5.2027 -4.387
5.6944 1 -1.8547
0 4.7539 -2.755
5.986 2.3496 -2.6193
14.4747 1 -2.6596
13.5548 1 -2.9508
13.0129 1 -2.5012
22.7317 1.9678 -3.6522
5.1373 3.4133 -2.8745
0.0755 5.1061 -2.7871];

Ca=CC(:,1);
Cp=CC(:,2);

%Error term

e=CC(:,3);

% Af defines the attribute values of the manufacturer

Af=[0 0.5 0 0
0.4 0 0.032 0
0 0.5 0 0
0.325 0 0 0.15
0.325 0 0 0
0.324 0 0 0.1
0.421 0 0.032 0.075
0.5 0 0 0.1
0 0.5 0 0
```

```
0.25 0.25 0.065 0
0 0.5 0 0
0 0.5 0 0
0 0.325 0 0
0.227 0.194 0 0.075
0.1239 0.2011 0 0
];
Af1=Af(:,1);
Af2=Af(:,2);
Af3=Af(:,3);
Af4=Af(:,4);

% Costs

Cost=[ 0.4
0.1328
0.4
0.1275
0.0975
0.1172
0.1541
0.17
0.4
0.301
0.4
0.4
0.26
0.2383
];


Dhoc=[0.3 0.8 0.4 0.2];
Cost(15)=x(16)*0.3+x(17)*0.8+x(18)*0.4+x(19)*0.2;

% L-function of error-\chi in the paper
L=3;
Du=ones(m,f1);

% i refers to the customer
% j refers to the firm
% Con(i,j) refers to the sum of the attribute terms in the dis utility
% function

for(i=1:m)
    for(j=1:f1-1)
        Du(i,j)=Ca(i)*(Af1(j)-A1(i))^2+Ca(i)*(Af2(j)-A2(i))^2+Ca(i)*(Af3(j)-A3(i))^2+Ca(i)*(Af4(j)-A4(i))^2+Cp(i)*x(j)+e(i);
    end
Du(i,15)=Ca(i)*(x(16)-A1(i))^2+Ca(i)*(x(17)-A2(i))^2+Ca(i)*(x(18)-A3(i))^2+Ca(i)*(x(19)-A4(i))^2+Cp(i)*x(15)+e(i);
end

% Summing up the values of the disutility function for the denominator of
% firm j's optimization problem for consumer i ( The process excludes
% disutility of i from the picture). F refers to the summation

F=zeros(m,1);
for(i=1:m)
    for(j=1:f1)
            F(i)=F(i)+exp(-L*Du(i,j));
    end
end
for(i=1:m)
    for(j=1:f1)
        Pr(i,j)=(exp(-L*(Du(i,j)))/F(i));
    end
```

```
end

% Sum 1 represents the sum of Prij values for all i.
% Sum 2 represents the sum of all wi*(1+Prij) values for all i
Sum1=zeros(f1,1);
Sum2=zeros(f1,1);
for(j=1:f1)
    for(i=1:m)
        Sum1(j)=Sum1(j)+Pr(i,j);
        Sum2(j)=Sum2(j)+Cp(i)*Pr(i,j)*(1-Pr(i,j));
    end
end

% Declaration of the constraint vector
for(j=1:f1)
    if(j==8)
        continue
    else
    c(j)=-f1/m*(Sum1(j)-((x(j)-Cost(j))*L*Sum2(j)));
    end
end
% complementarity cons w.r.t. new prod attributes
% Sum3 - for attr 1 of new prod
% Sum4 - for attr 2 of new prod
% Sum5 - for attr 3 of new prod
% Sum6 - for attr 4 of new prod
Sum3=0;
Sum4=0;
Sum5=0;
Sum6=0;
for(i=1:m)
   Sum3=Sum3+Ca(i)*(x(16)-A1(i))*Pr(i,15)*(1-Pr(i,15));
   Sum4=Sum4+Ca(i)*(x(17)-A2(i))*Pr(i,15)*(1-Pr(i,15));
   Sum5=Sum5+Ca(i)*(x(18)-A3(i))*Pr(i,15)*(1-Pr(i,15));
   Sum6=Sum6+Ca(i)*(x(19)-A4(i))*Pr(i,15)*(1-Pr(i,15));
end
c(16)=2*L*(15/m)*(x(15)-Cost(15))*Sum3+(Sum1(15)*(15/m)*0.3)-x(20)+x(21);
c(17)=2*L*(15/m)*(x(15)-Cost(15))*Sum4+(Sum1(15)*(15/m)*0.8)-x(20)+x(21);
c(18)=2*L*(15/m)*(x(15)-Cost(15))*Sum5+(Sum1(15)*(15/m)*0.4);
c(19)=2*L*(15/m)*(x(15)-Cost(15))*Sum6+(Sum1(15)*(15/m)*0.2);
c(20)=x(16)+x(17)-0.325;
c(21)=-x(16)-x(17)+0.5;
```

min.m

```
function f = min(x,Prob)
f=0;
```

A.2.7 Code-pain killer-new product equilibrium-decomposition

This solves the problem in a decomposed fashion. price1.m, constraint.m and min.m represent the codes for solving the existing agents' problem at every step. newp.m, attr.m and Ffun.m represent the codes for solving the new product's problem at every step. It is to be noted that the starting point has to be keyed in manually at every iteration based on the output at the previous iteration. This process is to be repeated till convergence is obtained.

price1.m

```
% Pain killer problem with the new product
% Decomposition
% 14 complementarity constraints
% 1-14 - Price FOC'S(KKTs)


% No (explicitly) linear constraints.

A   = [];
b_L = [];
b_U = [];

% Provide a pattern for the nonlinear constraints
ConsPattern = [];
HessPattern = [];

% Bounds

c_L = zeros(14,1);
c_U = inf*ones(14,1);
x_L = zeros(14,1);
x_L(8,1)=0.199;
x_U = inf*ones(14,1);
x_U(8,1)=0.199;

% Initial point

Name = 'pr';
x_0=[];

% Functions for calculating the nonlinear function and derivative values.
f   = 'min';
c   = 'constraint';
g=[];
H=[];
dc=[];
d2c=[];
Prob = conAssign(f, g, H, HessPattern, x_L, x_U, Name, x_0, ...
   [],[], ...
   A, b_L, b_U, c, dc, d2c, ConsPattern, c_L, c_U);

mpec=zeros(13,6);
for(i=1:7)
    mpec(i,1)=i;
    mpec(i,6)=i;
end
for(i=8:13)
    mpec(i,1)=i+1;
    mpec(i,6)=i+1;
end
Prob = BuildMPEC(Prob,mpec);

Result = tomRun('knitro',Prob);

x = Result.x_k
```

constraint.m

```
function c = constraint(x,Prob)
% No of manufacturers f1
f1=15;
c=zeros(14,1);
% m refers to the total number of customers
% Cost refers to the cost vector
```

```
% Ca refers to the weighing coefficient of each consumer towards the attributes
% Cp refers to the weighing coefficient with respect to price
% A1,A2,A3,A4 refer to the set of atribute values for m customers
% Af1,Af2,Af3,Af4 refer to the set of attribute values for 14 manufacturers
% Let us take an example of five consumers
m=30;
AA=[0 0.0835 0 0.0331
0 0.543 0.0075 0.0204
0 0.4889 0.0055 0
0.479 0.0568 0 0.0725
0.3202 0 0.0013 0
0 0.1395 0 0
0 0.4805 0 0
0.0649 0.3759 0.0022 0
0 0.3834 0 0
0.3431 0.0908 0 0.0695
0.0484 0.3229 0.0351 0
0.2696 0.0741 0.0005 0.111
0.4348 0.0276 0.0013 0.0605
0.2634 0 0.0022 0
0.3163 0.0581 0 0
0.0859 0.0488 0 0.1355
0.3197 0.032 0.0424 0.063
0.1872 0.7724 0 0.0186
0.4398 0.0235 0.023 0.0765
0 0.196 0 0.0604
0.0242 0.5938 0.0016 0.0002
0.0016 0.5157 0.0399 0.0079
0.2584 0.0761 0.0024 0.0065
0 0.5171 0 0
0.1094 0.1291 0 0.0934
0.0153 0.2855 0 0
0.1851 0.0874 0.0322 0.0903
0.1289 0.262 0.1226 0
0.0472 0.2513 0.0059 0
0.2752 0.0199 0.0003 0.0224
];
A1=AA(:,1);
A2=AA(:,2);
A3=AA(:,3);
A4=AA(:,4);

% Change prices at every step..depending on price alone or New product
% decomp..and run this code again (price1)

% Initial Price values
p=[ 0.6268
0.2259
0.6268
0.2276
0.197
0.2184
0.2472
0.199
0.6268
0.4762
0.6268
0.6268
0.4261
0.3926
0.3853
];
```

```
% Subject's importance

CC=[15.1354 3.8655 -4.4286
4.6278 1 -2.0476
2.2123 1 -1.8206
0 4.0706 -3.2257
0 2.9537 -2.1314
10.5894 1.5244 -2.758
5.0178 1 -1.9722
3.5191 3.0352 -2.7977
9.101 3.0648 -3.1728
0 2.6051 -2.228
10.5342 7.6762 -5.1675
0 7.5246 -4.4067
0 5.3952 -3.0808
0 5.7735 -3.4689
0 3.2881 -2.6675
7.4649 4.944 -4.1138
0.6457 2.0779 -1.8347
4.8654 1 -3.5624
0.5351 3.9169 -2.3135
5.3182 1.9882 -2.2817
6.8606 5.2027 -4.387
5.6944 1 -1.8547
0 4.7539 -2.755
5.986 2.3496 -2.6193
14.4747 1 -2.6596
13.5548 1 -2.9508
13.0129 1 -2.5012
22.7317 1.9678 -3.6522
5.1373 3.4133 -2.8745
0.0755 5.1061 -2.7871];

Ca=CC(:,1);
Cp=CC(:,2);

%Error term

e=CC(:,3);

% For New product problem- Change Af1(15)...Af4(15) at every step and run
% this code again (price1)

% Af defines the attribute values of the manufacturer

Af=[0 0.5 0 0
0.4 0 0.032 0
0 0.5 0 0
0.325 0 0 0.15
0.325 0 0 0
0.324 0 0 0.1
0.421 0 0.032 0.075
0.5 0 0 0.1
0 0.5 0 0
0.25 0.25 0.065 0
0 0.5 0 0
0 0.5 0 0
0 0.325 0 0
0.227 0.194 0 0.075
0.1026 0.2224 0 0
];
Af1=Af(:,1);
Af2=Af(:,2);
Af3=Af(:,3);
```

```
Af4=Af(:,4);

% Costs

Cost=[ 0.4
0.1328
0.4
0.1275
0.0975
0.1172
0.1541
0.17
0.4
0.301
0.4
0.4
0.26
0.2383
];


% L-function of error

L=3;
Du=ones(m,f1);

% i refers to the customer
% j refers to the firm
% Con(i,j) refers to the sum of the attribute terms in the dis utility
% function

for(i=1:m)
    for(j=1:f1)
        Con(i,j)=Ca(i)*(Af1(j)-A1(i))^2+Ca(i)*(Af2(j)-A2(i))^2+Ca(i)*(Af3(j)-A3(i))^2+Ca(i)*(Af4(j)-A4(i))^2;
        Du(i,j)=Ca(i)*(Af1(j)-A1(i))^2+Ca(i)*(Af2(j)-A2(i))^2+Ca(i)*(Af3(j)-A3(i))^2+Ca(i)*(Af4(j)-A4(i))^2+Cp(i)*p(j)+e(i);
    end
end
% Summing up the values of the disutility function for the denominator of
% firm j's optimization problem for consumer i ( The process excludes
% disutility of i from the picture). F refers to the summation

F=zeros(m,f1);
for(i=1:m)
    for(j=1:f1)
        for(k=1:f1)
            if(k>j|k<j)
            F(i,j)=F(i,j)+exp(-L*Du(i,k));
            else
                continue
            end
        end
    end
end
for(i=1:m)
    for(j=1:(f1-1))
        Pr(i,j)=exp(-L*(Con(i,j)+Cp(i)*x(j)+e(i)))/(exp(-L*(Con(i,j)+Cp(i)*x(j)+e(i)))+F(i,j));
    end
end
% Sum 1 represents the sum of Prij values for all i.
% Sum 2 represents the sum of all wi*(1+Prij) values for all i
Sum1=zeros((f1-1),1);
Sum2=zeros((f1-1),1);
for(j=1:(f1-1))
    for(i=1:m)
```

```
        Sum1(j)=Sum1(j)+Pr(i,j);
        Sum2(j)=Sum2(j)+Cp(i)*Pr(i,j)*(1-Pr(i,j));
    end
end
% Declaration of the constraint vector
for(j=1:(f1-1))
    if(j==8)
        continue
    else
    c(j)=-f1/m*(Sum1(j)-((x(j)-Cost(j))*L*Sum2(j)));
    end
end
```

min.m

```
function f=min(x,Prob)
f=0;
```

newp.m

```
% Pain killer problem with the new product
% Decomp for the new prod.-7 constraints
% 1-5 - Price and attribute FOC'S(KKTs)
% Multipliers - 6,7 constraints for the new product


% No (explicitly) linear constraints.

A   = [];
b_L = [];
b_U = [];

% Provide a pattern for the nonlinear constraints

ConsPattern = [];
HessPattern = [];

% Upper and lower bounds on constraints

c_L = zeros(7,1);
c_U = inf*ones(7,1);
x_L = zeros(7,1);
x_U = inf*ones(7,1);
Name = 'np';
% Initial vecotr
x_0=[0.1 0.225 0.3 0.3 0.4 1.82 0];

% Functions for calculating the nonlinear function and derivative values.
f   = 'Ffun';
c   = 'attr';
g=[];
H=[];
dc=[];
d2c=[];
Prob = conAssign(f, g, H, HessPattern, x_L, x_U, Name, x_0, ...
   [],[], ...
   A, b_L, b_U, c, dc, d2c, ConsPattern, c_L, c_U);

mpec=zeros(7,6);
for(i=1:7)
    mpec(i,1)=i;
    mpec(i,6)=i;
end
Prob = BuildMPEC(Prob,mpec);
```

```
Prob.PrilevOpt=3;
Result = tomRun('knitro',Prob,1);

x = Result.x_k
```

attr.m

```
function c = attr(x,Prob)
% No of manufacturers f1
%x=[ 0.1018 0.2232 0 0 3.86]
c=zeros(7,1);
f1=14;
% m refers to the total number of customers
% Cost refers to the cost vector
% Ca refers to the weighing coefficient of each consumer towards the attributes
% Cp refers to the weighing coefficient with respect to price
% A1,A2,A3,A4 refer to the set of attribute values for m customers
% Af1,Af2,Af3,Af4 refer to the set of attribute values for 14 manufacturers
% Let us take an example of five consumers
m=30;
A=[0 0.0835 0 0.0331
0 0.543 0.0075 0.0204
0 0.4889 0.0055 0
0.479 0.0568 0 0.0725
0.3202 0 0.0013 0
0 0.1395 0 0
0 0.4805 0 0
0.0649 0.3759 0.0022 0
0 0.3834 0 0
0.3431 0.0908 0 0.0695
0.0484 0.3229 0.0351 0
0.2696 0.0741 0.0005 0.111
0.4348 0.0276 0.0013 0.0605
0.2634 0 0.0022 0
0.3163 0.0581 0 0
0.0859 0.0488 0 0.1355
0.3197 0.032 0.0424 0.063
0.1872 0.7724 0 0.0186
0.4398 0.0235 0.023 0.0765
0 0.196 0 0.0604
0.0242 0.5938 0.0016 0.0002
0.0016 0.5157 0.0399 0.0079
0.2584 0.0761 0.0024 0.0065
0 0.5171 0 0
0.1094 0.1291 0 0.0934
0.0153 0.2855 0 0
0.1851 0.0874 0.0322 0.0903
0.1289 0.262 0.1226 0
0.0472 0.2513 0.0059 0
0.2752 0.0199 0.0003 0.0224
];

% DU coefficients
Coeff=[15.1354 3.8655 -4.4286
4.6278 1 -2.0476
2.2123 1 -1.8206
0 4.0706 -3.2257
0 2.9537 -2.1314
10.5894 1.5244 -2.758
5.0178 1 -1.9722
3.5191 3.0352 -2.7977
9.101 3.0648 -3.1728
0 2.6051 -2.228
```

```
10.5342 7.6762 -5.1675
0 7.5246 -4.4067
0 5.3952 -3.0808
0 5.7735 -3.4689
0 3.2881 -2.6675
7.4649 4.944 -4.1138
0.6457 2.0779 -1.8347
4.8654 1 -3.5624
0.5351 3.9169 -2.3135
5.3182 1.9882 -2.2817
6.8606 5.2027 -4.387
5.6944 1 -1.8547
0 4.7539 -2.755
5.986 2.3496 -2.6193
14.4747 1 -2.6596
13.5548 1 -2.9508
13.0129 1 -2.5012
22.7317 1.9678 -3.6522
5.1373 3.4133 -2.8745
0.0755 5.1061 -2.7871
];

% Initial Price values
p=[0.6268
0.2259
0.6268
0.2276
0.197
0.2184
0.2472
0.199
0.6268
0.4762
0.6268
0.6268
0.4261
0.3926
];
% Subject's importance values for price

Cost=[ 0.4
0.1328
0.4
0.1275
0.0975
0.1172
0.1541
0.17
0.4
0.301
0.4
0.4
0.26
0.2383
];
% Af defines the attribute values of the manufacturer
Af=[0 0.5 0 0 0.4
0.4 0 0.032 0 0.1328
0 0.5 0 0 0.4
0.325 0 0 0.15 0.1275
0.325 0 0 0 0.0975
0.324 0 0 0.1 0.1172
0.421 0 0.032 0.075 0.1541
0.5 0 0 0.1 0.17
```

```
0 0.5 0 0 0.4
0.25 0.25 0.065 0 0.301
0 0.5 0 0 0.4
0 0.5 0 0 0.4
0 0.325 0 0 0.26
0.227 0.194 0 0.075 0.2383
];

% L-function of error
L=3;
Du=ones(m,f1);
% i refers to the customer
% j refers to the firm
% Con(i,j) refers to the sum of the attribute terms in the dis utility
% function
Con=zeros(m,f1);
for(i=1:m)
    for(j=1:f1)
        for(n=1:4)
        Con(i,j)=Con(i,j)+Coeff(i,1)*(Af(j,n)-A(i,n))^2;
        end
        Du(i,j)=Con(i,j)+Coeff(i,2)*p(j)+Coeff(i,3);
    end
end

% Summing up the values of the disutility function for the denominator of
% firm j's optimization problem for consumer i ( The process excludes
% disutility of i from the picture). F refers to the summation
F=zeros(m);
for(i=1:m)
        for(k=1:f1)
            F(i)=F(i)+exp(-L*Du(i,k));
        end
end
Pr=zeros(m,1);
for(i=1:m)
    DTem(i)=-L*(Coeff(i,1)*((x(1)-A(i,1))^2+(x(2)-A(i,2))^2+(x(3)-A(i,3))^2+(x(4)-A(i,4))^2)+Coeff(i,2)*x(5)+Coeff(i,3));
    Pr(i)=exp(DTem)/(F(i)+exp(DTem));
end
Sum=zeros(4,1);
Sum5=0;
Sum6=0;
for(t=1:4)
for(i=1:m)
   Sum(t)=Sum(t)+Coeff(i,1)*(x(t)-A(i,t))*Pr(i)*(1-Pr(i));
end
end
for(i=1:m)
Sum5=Sum5+Pr(i);
Sum6=Sum6+Coeff(i,2)*Pr(i)*(1-Pr(i));
end
dhoc=[0.3 0.8 0.4  0.2];
%dhoc=[0 0 0 0];
cst=0;
for(t=1:4)
    cst=cst+x(t)*dhoc(t);
end
%cst=0.2;
for(t=1:4)
c(t)=2*L*(15/m)*(x(5)-cst)*Sum(t)+(Sum5*(15/m)*dhoc(t));
end
c(1)=c(1)-x(6)+x(7);
c(2)=c(2)-x(6)+x(7);
c(5)=-(15/m)*(Sum5-((x(5)-cst)*L*Sum6));
```

```
c(6)=x(1)+x(2)-0.325;
c(7)=-x(1)-x(2)+0.5;
```

Ffun.m

```
function f=Ffun(x,Prob)
f=0;
```

A.2.8 Code-pain killer-price competition-decomposition

This solves the problem in a decomposed fashion. The codes price1.m, constraint.m and min.m stated in the previous subsection are the set of codes to be used for this problem. The number of firms has to be however changed to 14 from 15 (Follow comments in the codes). It is to be noted that the starting point has to be keyed in manually at every iteration based on the output at the previous iteration. This process is to be repeated till convergence is obtained.

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