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OVING EIGENVALUES AND EIGENVECTORS

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IMPROVING APPROXIMATE EIGENVALUES AND EIGENVECTORS

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An estimate of the solution to an eigenvalue problem is often available either as the final answer in some approximate method or as an intermediate result in an iterative process. This situation may arise when using either a discrete or a continuous model. Two examples are considered herein.

1. It is common practice when computing the dynamic response of rigid frames to modify or condense the structural stiffness matrix by algebraic elimination of the joint rotations and member extensions (see, e.g., Ref. 8). The reduced stiffness matrix refers explicitly only to lateral motion of the floor levels. This makes the so-called consistent mass matrix (1) more difficult to derive so that a lumped mass approach is generally used as in Ref. 8. The process outlined reduces the size of the eigenvalue problem considerably; the number of unknowns may be reduced by as much a factor of five as compared to the problem where joint rotations and member extensions are given explicit consideration. Certain of the solutions of this reduced problem may be of sufficient interest that an improved solution is sought in which the distributed nature of the mass of the members is taken into consideration, resulting in generalized mass moments of inertia and generalized masses corresponding to member end rotations and extensions. The solution of the reduced problem is then considered only a first approximation to the expanded problem.

2. Frequently the Holzer method (2) is used to bracket eigenvalues in both vibration and buckling problems. A determinant is formed which vanishes if the (homogeneous) boundary conditions are satisfied. It is then tested for change of sign as the trial approximation to the eigenvalue is incremented. A

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change of sign of this determinant between two successive values of the trial eigenvalue indicates an eigenvalue falling between the two approximate values.

In the two types of problems just described, it would be useful to have a procedure for improving an eigenvalue and the corresponding eigenvector when either the model is refined by the inclusion of more degrees of freedom (in the stiffness and mass matrix of example 1) or where only an approximate eigenvalue has been found for a fairly complicated model. A procedure, actually a modification of the Newton-Raphson method (5), is presented herein which accomplishes this objective of improving eigenvalues and eigenvectors in an extremely effective fashion.

The procedure has been tested successfully on discrete and continuous systems. It has also been applied to a problem where multiple (nonsimple) eigenvalues are known to exist. The procedure is illustrated by three sample problems which show the generality of the method. It will be demonstrated in some detail on linear eigenvalue problems by the three problems. A description is also given of the extension required to solve certain other important eigenvalue problems associated with nonlinear structural response.

LINEAR EIGENVALUE PROBLEMS

The problems to be solved herein may be described by

$$AX - \lambda BX = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

and appropriate boundary conditions where necessary. In Eq. 1 A and B are operators which may be matrices, differential or other linear operators. Operators A and B are assumed to be self adjoint and B positive definite (3). Quantity λ is an eigenvalue and X is the corresponding eigenvector or eigenfunction. If $\lambda^{(i)}$ and $X^{(i)}$ are approximate solutions of Eq. 1 then

$$AX^{(i)} - \lambda^{(i)} BX^{(i)} = R^{(i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

in which R is a residual and superscript (i) refers to either the initial estimate of some known reference configuration.

The object is to remove the residual in Eq. 2; the Newton-Raphson technique is introduced for this purpose. Eq. 1 is interpreted as a nonlinear equation in X and λ where the nonlinearity arises from the term λBX. Eq. 1 is linearized about the configuration corresponding to superscript (i), giving

$$A\delta X^{(i)} - \lambda^{(i)} B\delta X^{(i)} - \delta\lambda^{(i)} B X^{(i)} = - R^{(i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

in which $\delta X^{(i)}$ and $\delta\lambda^{(i)}$ are the linear parts of the incremental change in those quantities about the reference state specified by $X^{(i)}$ and $\lambda^{(i)}$. The residual, $R^{(i)}$, is available from Eq. 2. The unknowns in Eq. 3 are only $\delta X^{(i)}$ and $\delta\lambda^{(i)}$, because the approximate eigenvalue $\lambda^{(i)}$ and eigenvector $X^{(i)}$ are known. Because $\delta\lambda^{(i)}$ is an extra unknown, a side condition must be introduced which, together with Eq. 3, determines $\delta X^{(i)}$ and $\delta\lambda^{(i)}$. The nature of this side condition may be best explained in the context of an actual problem. In each of the succeeding sections a convenient choice for the side condition is given.

There is a formal relation between the present work and that presented by Rall (14) for the discrete case. However, Rall's method does not treat the

eigenvalue explicitly as an extra unknown, and an unfortunate choice of coordinates can lead to failure of the procedure.

SAMPLE PROBLEM 1

For purposes of illustration, the lowest mode of vibration of a one-story frame will be determined by use of a discrete model (see Fig. 1). The equations of motion corresponding to free vibrations of the frame are derived using Lagrange's equations. In order to reduce the system to one having a finite number of degrees-of-freedom, the displacements of the individual members are restricted to a cubic function in the transverse direction and a linear function along the axis of an individual member. The distributed nature of the mass of the individual members is considered, so that the formulation leads to the so-called consistent mass method.

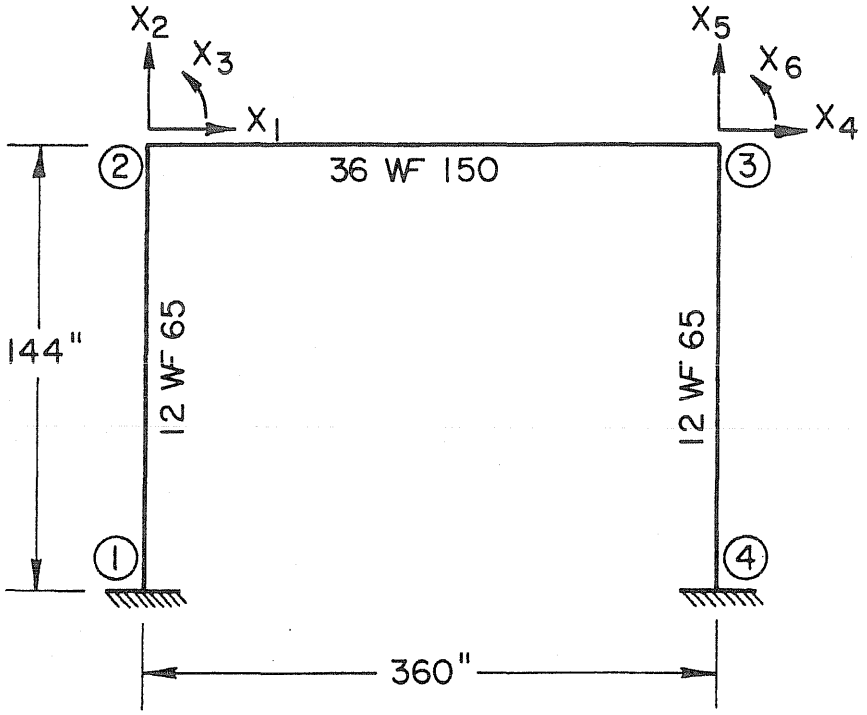

FIG. 1.—RIGID FRAME—SAMPLE PROBLEMS 1 AND 2

The motion of the frame is assumed to be specified by the generalized coordinates $x_1, x_2, \ldots x_6$ as functions of time t (Fig. 1). The generalized coordinates are written in matrix form as $\{X\}$. The matrix form of the equations of motion for free vibration is (7)

$$[M]\{\ddot{X}\} + [K]\{X\} = \{0\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

in which $[M]$ is the generalized mass matrix; $[K]$ is the stiffness matrix and a dot indicates differentiation with respect to time. For free vibrations, vector $\{X\}$ is assumed to vary sinusoidally with time, $\{X\} = \{\bar{X}\} \sin \omega t$ in which ω is the circular frequency, in radians per second. From Eq. 4

$$-\lambda[M]\{\bar{X}\} + [K]\{\bar{X}\} = \{0\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

in which $\lambda = \omega^2$ is an eigenvalue for the free vibration problem and $\{\bar{X}\}$ is

the corresponding eigenvector. Matrices $[M]$ and $[K]$ are given in Appendix III.

If an approximate eigenvalue and eigenvector are known for this problem, Eqs. 2 and 3 become

$$-\lambda^{(i)}[M]\{\bar{X}^{(i)}\} + [K]\{\bar{X}^{(i)}\} = \{R^{(i)}\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

$$-\delta\lambda^{(i)}[M]\{\bar{X}^{(i)}\} - \lambda^{(i)}[M]\{\delta\bar{X}^{(i)}\} + [K]\{\delta\bar{X}^{(i)}\} = -\{R^{(i)}\} \quad (7)$$

Eqs. 6 and 7, if supplemented by a scalar side condition, should be solvable for the $\delta\bar{X}^{(i)}$ and $\delta\lambda^{(i)}$.

In this particular case, the side condition is taken to be

$$\{\bar{X}^{(i)}\}^T[M]\{\delta\bar{X}^{(i)}\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

This is equivalent to saying that the allowable changes in the eigenvector are orthogonal to the latest eigenvector with respect to the mass matrix. This prevents unlimited drift in the eigenvector which is, after all, not determined in magnitude.

TABLE 1.—EIGENVALUES, λ, AND RESIDUALS, $R^{(i)}$, FOR SUCCESSIVE ITERATIONS—SAMPLE PROBLEM 1

Approximation Number			
0	1	2	3
λ, in radians squared per second squared			
9,138.3	9,354.4	9,335.2	9,335.2
-0.90642×10^{2}	-0.108×10^{0}	0.203×10^{-7}	0.0
-0.10641×10^{5}	-0.243×10^{2}	0.481×10^{-5}	-0.182×10^{-11}
-0.43659×10^{5}	-0.101×10^{2}	0.206×10^{-5}	-0.131×10^{-9}
0.86416×10^{2}	-0.982×10^{-1}	0.177×10^{-7}	0.291×10^{-9}
-0.66446×10^{2}	-0.144×10^{0}	0.290×10^{-7}	-0.369×10^{-11}
0.12215×10^{6}	0.381×10^{3}	-0.920×10^{-4}	-0.175×10^{-9}

The resulting set of simultaneous linear algebraic equation may be written in partitioned form as

$$\begin{bmatrix} [K] - \lambda^{(i)}[M] & -[M]\{\bar{X}^{(i)}\} \\ -\{\bar{X}^{(i)}\}^T[M] & 0 \end{bmatrix} \begin{Bmatrix} \delta\bar{X}^{(i)} \\ \delta\lambda^{(i)} \end{Bmatrix} = -\begin{Bmatrix} R^{(i)} \\ 0 \end{Bmatrix} \quad \ldots\ldots\ldots (9)$$

Note that the coefficient matrix of the incremental quantities is symmetric. This is a significant advantage of the particular side condition expressed by Eq. 8. The explanation just given makes it reasonable to expect that the coefficient matrix in Eq. 9 is nonsingular. A formal proof of this fact is given in Appendix I. Eqs. 9 may then be solved by Gauss elimination, or by any other suitable technique, to yield $\delta\bar{X}^{(i)}$ and $\delta\lambda^{(i)}$. The $(i+1)$th approximation becomes

$$\left.\begin{aligned} \{\bar{X}^{(i+1)}\} &= \{\bar{X}^{(i)}\} + \{\delta\bar{X}^{(i)}\} \\ \lambda^{(i+1)} &= \lambda^{(i)} + \delta\lambda^{(i)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

The $\{\bar{X}^{(i+1)}\}$ and $\lambda^{(i+1)}$ may be substituted into Eq. 6 again to yield a new residual vector $\{R^{(i+1)}\}$ which may be checked against some acceptable tolerance. If $\{R^{(i+1)}\}$ is not satisfactory, the process is repeated until the residual vector is within the allowable tolerance.

The initial estimate for the eigenvalue and eigenvector may be obtained by considering the frame to be a single degree-of-freedom system corresponding to horizontal motion of the girder. Mass $\bar{M}$ of this simplified system may be estimated as the mass of the girder plus one-third the mass of the columns. The stiffness, $\bar{K}$, is found by computing the force necessary to give the girder a unit horizontal displacement, and the eigenvector is estimated by computing the joint displacements corresponding to the unit displacement. The initial eigenvalue is then taken as $\lambda^{(0)} = \bar{K}/\bar{M}$.

In this case, the initial approximation to the eigenvalue is 9,138.3 rad^2/sec^2. After two cycles of iteration, the eigenvalue stabilizes at 9,335.2 rad^2/sec^2 (see Table 1). Note also (Table 1) that the residuals decrease rapidly. This extremely rapid convergence is consistent with the theoretical character of the process in the neighborhood of an eigenvalue, which is developed in Appendix II.

SAMPLE PROBLEM 2

The procedure used to find eigenvalues and eigenvectors of continuous systems is illustrated by determining the lowest mode of vibration of the frame in Fig. 1. This also permits a comparison between the discrete and continuous methods. The differential equations of motion are taken as (7)

$$\left.\begin{aligned} EI\,\frac{\partial^4 w}{\partial s^4} + m\,\frac{\partial^2 w}{\partial t^2} &= 0 \\ EA\,\frac{\partial^2 u}{\partial s^2} - m\,\frac{\partial^2 u}{\partial t^2} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (11)$$

in which EI is the flexural rigidity; m is the mass per unit length; EA is the extensional rigidity; w is the transverse displacement; u is the axial displacement; and s is the distance along the center line of the frame measured as positive from point 1 on the frame clockwise to point 4. With $u = U \sin \omega t$ and $w = W \sin \omega t$, Eqs. 11 become

$$\left.\begin{aligned} W^{\mathrm{IV}} - \lambda\,\frac{m}{EI}\,W &= 0 \\ U'' + \lambda\,\frac{m}{EA}\,U &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (12)$$

in which $\lambda = \omega^2$ and a prime denotes differentiation with respect to s.

If an approximate eigenvalue and eigenvector are substituted into Eqs. 11, the residuals are

$$\left.\begin{aligned} {W^{(i)}}^{\mathrm{IV}} - \lambda^{(i)}\,\frac{m}{EI}\,W^{(i)} &= R_W^{(i)} \\ {U^{(i)}}'' + \lambda^{(i)}\,\frac{m}{EA}\,U^{(i)} &= R_U^{(i)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (13)$$

As in the discrete method, the residuals are removed by use of the Newton-Raphson method. The nonlinearity in Eqs. 12 arises from the terms $\lambda(m/EI)\,W$, $\lambda(m/EA)\,U$. Eqs. 12 are linearized about the reference configuration resulting in

$$\left.\begin{aligned} \delta W^{(i)^{IV}} - \lambda^{(i)}\frac{m}{EI}\,\delta W^{(i)} - \delta\lambda^{(i)}\frac{m}{EI}\,W^{(i)} &= -R_W^{(i)} \\ \delta U^{(i)^{''}} + \lambda^{(i)}\frac{m}{EA}\,\delta U^{(i)} + \delta\lambda^{(i)}\frac{m}{EA}\,U^{(i)} &= -R_U^{(i)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (14)$$

in which the $\delta W^{(i)}$, $\delta U^{(i)}$, and $\delta\lambda^{(i)}$ are unknown.

This system of linear differential equations may be solved for $\delta\lambda^{(i)}$ and the incremental changes in the eigenfunction. The side condition is taken analogous to that for the discrete problem, i.e.

$$\int_{S_1}^{S_4} m\left[W^{(i)}\,\delta W^{(i)} + U^{(i)}\,\delta U^{(i)}\right] ds = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Eqs. 14 are uncoupled within an individual member of the frame but coupling takes place because of the compatibility and equilibrium requirements at the intersection of the girder and columns. These requirements are:

$$(EAU')_+ = -(EIW''')_-$$

$$(EIW'')_+ = (EIW'')_-$$

$$(EIW''')_+ = (EAU')_-$$

$$(U)_+ = (W)_-$$

$$(W')_+ = (W')_-$$

$$(W)_+ = -(U)_-$$

TABLE 2.—INITIAL VALUES OF DIFFERENTIAL EQUATIONS SAMPLE PROBLEM 2

Quantity	Homogeneous Solutions				Particular Solution
	$i = 1$	$i = 2$	$i = 3$	$i = 4$	$i = 5$
$\delta W_i(0)$	0	0	0	0	0
$\delta W_i'(0)$	0	0	0	0	0
$\delta W_i''(0)$	1	0	0	0	0
$\delta W_i'''(0)$	0	1	0	0	0
$\delta U_i(0)$	0	0	0	0	0
$\delta U_i'(0)$	0	0	1	0	0
$\delta\lambda$	0	0	0	1	0
Right-hand side Eq. 14	0	0	0	0	$-R_W$
Right-hand side Eq. 14	0	0	0	0	$-R_U$

(The direction from - to + is in the direction s_1 to s_4.)

The modified boundary value problem defined by Eqs. 14 and 15 and appropriate boundary conditions is converted to an initial value problem (see e.g., Ref. 12) which is then solved by numerical integration using a generalized trapezoidal method (3). (In the present case rapidly growing solutions are absent and the process is straightforward.) The integral in Eq. 15 is evaluated by Simpson's rule. Several initial value problems are propagated from point 1 around the frame to point 4 (see Fig. 1). A linear combination of these initial value problems is constructed to satisfy the boundary conditions at point 4 in addition to the side condition defined by Eq. 15.

The initial value problems are combined to satisfy the following boundary conditions at point 4:

$$\left.\begin{aligned} 0 &= \delta W_5(s_4) + \sum_{j=1}^{4} \alpha_j \, \delta W_j(s_4) \qquad (W = 0 \text{ at } s_4) \\ 0 &= \delta W_5'(s_4) + \sum_{j=1}^{4} \alpha_j \, \delta W_j'(s_4) \qquad (W' = 0 \text{ at } s_4) \\ 0 &= \delta U_5(s_4) + \sum_{j=1}^{4} \alpha_j \, \delta U_j(s_4) \qquad (U = 0 \text{ at } s_4) \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (16)$$

in which subject j refers to the initial value solution number and the α_j = the amplitudes of the initial value solutions.

In addition, the side condition, Eq. 15, is expressed as

$$0 = \int_{s_1}^{s_4} m\left[U^{(i)} \, \delta U_5 + W^{(i)} \, \delta W_5\right] ds + \sum_{j=1}^{4} \alpha_j \int_{s_1}^{s_4} m\left[U^{(i)} \, \delta U_j + W^{(i)} \, \delta W_j\right] ds \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (17)$$

Eqs. 16 and 17 may then be solved for α_j. The initial values for each solution are given in Table 2. Once the values of α_j are computed from Eqs. 16*a* and 16*b* the correct incremental solution may be determined, i.e.

$$\delta W^{(k)} = \delta W_5^{(k)} + \sum_{i=1}^{4} \alpha_i \, \delta W_i^{(k)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (18)$$

with a similar relationship for $\delta U^{(k)}$ and all higher derivatives.

The incremental solution is added to the total solution to yield

$$\left.\begin{aligned} W^{(k+1)} &= W^{(k)} + \delta W^{(k)} \\ U^{(k+1)} &= U^{(k)} + \delta U^{(k)} \\ \lambda^{(k+1)} &= \lambda^{(k)} + \delta\lambda^{(k)} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (19)$$

The $(k + 1)$th approximation becomes the new reference position which is then substituted into Eq. 13 to compute new residuals that are examined for accuracy. If the residuals are small enough, the process is terminated. If not,

the entire procedure is repeated for the new reference position given by Eq. 19, until the residuals are acceptable.

In this particular case, the starting eigenvalue for the continuous process is the one obtained from the discrete process. The initial eigenvector may be derived by the following technique. A series of three initial value solutions corresponding to $\delta W_3^{\prime\prime\prime}(0) = \beta$, $\delta W_2^{\prime\prime\prime}(0) = 1$, $\delta U_1^{\prime} = 1$, in which β is set at some convenient magnitude (in this case 0.01), are propagated to point 4. The amplitudes of solutions 2 and 3 are determined so that two of the three boundary conditions at point 4 are satisfied. All subsequent corrections to this approximate eigenvector are determined in such a way that the final eigenvector does satisfy all boundary conditions. The choice of the boundary condition which is not satisfied in the initial approximation to the eigenvector can sometimes be important. Experience indicates that the softest constraint should be relaxed.

The eigenvalue for the first mode as determined by this process is 9,118.4 rad^2/sec^2 which as expected is lower than the 9,335.2 rad^2/sec^2 determined by the discrete process.

Table 3 shows the starting eigenvalue and the eigenvalue for the next few iterations. It stabilizes rapidly.

SAMPLE PROBLEM 3

As an example of a problem having a multiple eigenvalue, a simply supported strut with a spring at its midpoint is considered (see Fig. 2).

When $K = 16\,\pi^2 EI/L^3$, there are two independent eigenfunctions corre-

TABLE 3.—EIGENVALUES AND MAXIMUM RESIDUALS FOR SUCCESSIVE ITERATES—SAMPLE PROBLEM 2

Quantity	Approximation Number			
	0	1	2	3
λ, in radians squared per second squared	9,335.20	9,118.44	9,118.44	9,118.44
[a] $R^{(i)}_{W\max}$, in inches^{-3}	0	0.110×10^{-9}	0.186×10^{-15}	0.132×10^{-22}
[a] $R^{(i)}_{U\max}$, in inches^{-1}	0	0.113×10^{-8}	0.515×10^{-14}	0.256×10^{-16}

[a] Maximum value of residual over entire range of S, the arc length.

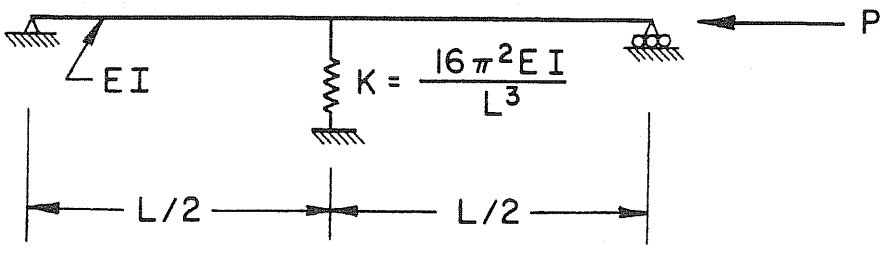

FIG. 2.—SPRING-SUPPORTED STRUT—SAMPLE PROBLEM 3

sponding to $\lambda = P/EI = 4\pi^2/L^2$ (11). The differential equation of equilibrium is (11)

$$W^{IV} + \lambda W'' = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (20)$$

The solution process is similar to that used in sample problem 2. There is, however, one significant difference. If a single side condition is introduced, e.g.

$$\int_0^L (W' \, \delta \, W') ds = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

and the problem is solved by the initial value approach, the equations which determine the initial values are singular. This is expected because there are two independent eigenvectors possible for the same eigenvalue, which means that a single side condition is not sufficient to determine the initial values uniquely. Because of numerical round-off, the equations for the initial values are not exactly singular. For this reason, the method still isolates the eigenvalue fairly well, but the resulting eigenvectors never stabilize, i.e., as further iterations are performed, changes in the eigenvectors do not become small.

In order to avoid this problem, two eigenvectors are generated simultaneously (called U and V). Another side condition is introduced for each of these eigenvectors, resulting in two side conditions for each eigenvector. In computing δV, the side conditions are

$$\left.\begin{aligned} \int_0^L V' \delta V' \, ds &= 0 \\ \int_0^L U' \delta V' \, ds &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (22)$$

and in computing δU

$$\left.\begin{aligned} \int_0^L U' \, \delta U' \, ds &= 0 \\ \int_0^L V' \, \delta U' \, ds &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (23)$$

in which U and V are two independent eigenvectors and δU and δV their increments.

It will be recalled that the function of the side condition (see sample problem 1) is to insure that changes in the eigenvector are orthogonal to the latest approximation. The side condition guarantees that large changes parallel to the eigenvector are eliminated from $\delta X^{(i)}$. In the case of a multiple eigenvalue, unlimited drift would be possible in the entire subspace spanned by the independent eigenvectors. In the present example, the changes in $\delta U^{(i)}$ and $\delta V^{(i)}$ will be small provided that each of them is orthogonal to the latest $U^{(i)}$ and $V^{(i)}$. (It is advisable to take the initial $U^{(i)}$ and $V^{(i)}$ as orthogonal.)

The extra side condition for each eigenvector would seem to over-determine the amplitudes of the initial value solutions, i.e., there are more equations to be satisfied than unknowns. However, this is only an apparent over-determination because if δU and δV are to be small, they should be restricted to have no components in the two dimensional subspace spanned by U and V.

TABLE 4.—INCREMENTS OF MIDSPAN DEFLECTION AND EIGENVALUES FOR SUCCESSIVE ITERATIONS—SAMPLE PROBLEM 3

Iteration Number	Numerical Values Obtained by Procedure for Multiple Eigenvalues (Double Precision Arithmetic = 15 Figures)				Numerical Values Obtained by Procedure for Single Eigenvalues (Double Precision Arithmetic)		Numerical Values Obtained by Procedure for Single Eigenvalues (Single Precision Arithmetic)	
	$\delta U\left(\frac{L}{2}\right)$	$\delta V\left(\frac{L}{2}\right)$	$\lambda_U L^2$	$\lambda_V L^2$	$\delta U\left(\frac{L}{2}\right)$	$\delta V\left(\frac{L}{2}\right)$	$\delta U\left(\frac{L}{2}\right)$	$\delta V\left(\frac{L}{2}\right)$
(1)	(2)[a]	(3)[a]	(4)	(5)	(6)[a]	(7)[a]	(8)[a]	(9)[a]
0	0.500×10^{3}	0.13236×10^{8}	36.0000	36.0000	0.500×10^{3}	0.132×10^{8}	0.500×10^{3}	0.132×10^{8}
1	0.279×10^{2}	-0.324×10^{6}	39.5433	39.6719	0.268×10^{2}	-0.20×10^{6}	0.268×10^{2}	-0.201×10^{6}
2	0.129×10^{0}	0.161×10^{5}	39.4817	39.4817	0.280×10^{0}	0.312×10^{5}	0.283×10^{0}	0.313×10^{5}
3	-0.146×10^{-3}	-0.520×10^{2}	39.4817	39.4817	-0.586×10^{2}	-0.148×10^{7}	-0.587×10^{2}	0.309×10^{5}
4	0.312×10^{-9}	-0.305×10^{-4}	39.4817	39.4817	-0.526×10^{2}	-0.224×10^{7}	-0.525×10^{2}	-0.125×10^{6}
5	-0.417×10^{-16}	-0.809×10^{-9}	39.4817	39.4817	-0.628×10^{2}	-0.221×10^{7}	-0.628×10^{2}	0.171×10^{5}

[a] The reason δU and δV are of different orders of magnitude is that the initial estimates for U and V were of different orders of magnitude.

This system of $n + 1$ equations in n unknowns may be conveniently solved by a least squares technique (6). If the equations specifying satisfaction of the boundary conditions and the two side conditions are written as $[\bar{C}]\{y\} = \{D\}$, in which $[\bar{C}]$ has one more row than column, the least squares approximation is computed from

$$[\bar{C}]^T [\bar{C}] \{y\} = [\bar{C}]^T \{D\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (24)$$

A demonstration of the nonsingular character of the square matrix $[\bar{C}]^T [\bar{C}]$ is given in Appendix I for a discrete system.

In this way, two independent eigenvectors are insured which do stabilize rapidly. Table 4 shows the results for a few cycles of the process both with and without the introduction of the extra side condition.

EIGENVALUE PROBLEMS ASSOCIATED WITH NONLINEAR RESPONSE

The form of Eq. 1 is also applicable to a nonlinear problem if A and B are taken to be functions of the eigenvalue λ. This dependence on λ is often indirect; e.g., in buckling problems A and B may depend on the prebuckling configuration (PBC), which in turn depends on the applied load λ. With this interpretation of Eq. 1, the linearization results in

$$A^{(i)}\,\delta X^{(i)} - \lambda^{(i)} B^{(i)}\,\delta X^{(i)} = \delta A^{(i)}\,X^{(i)} + \lambda^{(i)}\,\delta B^{(i)}\,X^{(i)}$$
$$+ \delta\lambda^{(i)}\,B^{(i)}\,X^{(i)} - R^{(i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (25)$$

Examination of Eq. 25 reveals two types of incremental quantities, i.e., an increment of eigenvector $X^{(i)}$ and incremental quantities corresponding to changes of the fundamental state (in the case of buckling problems, the PBC). In the context of an arch buckling problem the $\delta A^{(i)}$ and $\delta B^{(i)}$ correspond to changes in the stiffness of the PBC (symmetrical in this case). The $\delta X^{(i)}$ and $\delta\lambda^{(i)}$ are interpreted as the incremental changes in the eigenvector and eigenvalue, respectively (in this case corresponding to sidesway in the plane of the arch).

If $A^{(i)}$, $B^{(i)}$, and $\lambda^{(i)}$ correspond to the PBC just at the onset of buckling, then, of course, $\delta\lambda^{(i)} = 0$ and the problem is simply one of determining X, the eigenvector. In general, however, $A^{(i)}$, $B^{(i)}$, and $\lambda^{(i)}$ do not correspond to the PBC at the onset of buckling, so that $\delta\lambda^{(i)} \neq 0$. Given the $\delta\lambda^{(i)}$ computed by the use of Eq. 25, it is a routine matter using the Newton-Raphson method (12) to modify $A^{(i)}$, $B^{(i)}$, and $\lambda^{(i)}$, i.e., to find A and B. However, the general procedure fails exactly at the configuration of interest, i.e., the PBC just at the onset of buckling where the operator $A - \lambda B$ in the equations for the PBC becomes singular (10).

A technique analogous to that used in the linear problem with a double eigenvalue (sample problem 3) is presented herein which disposes of the difficulty associated with the singularity of the operator. All that is required is that the change in the PBC be orthogonal to the latest estimate of the eigenvector of the system. This may be expressed as

$$X^{(i)^T} C^{(i)}\,\delta\bar{X}^{(i)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (26)$$

in which $\delta\bar{X}$ as used herein refers to the incremental changes in the PBC and

$C^{(i)}$ is a suitable self-adjoint positive-definite operator. It should be emphasized that this device is employed only for the determination of accurate changes in the PBC near the onset of buckling.

The introduction of Eq. 26 results in an apparent over-determination of the system, but at the onset of buckling, Eq. 26 is necessarily satisfied (9). Therefore, the correct $\delta\bar{X}^{(i)}$ may be determined as the best one by a least squares technique as was done in sample problem 3.

If it is desired to trace the post-buckling behavior of the arch, some multiple of the eigenvector is added to the PBC, and the Newton-Raphson method used until a new equilibrium state is found (with sidesway in the case of an arch).

The technique just described has been applied (13) to one of the arch problems solved by Huddleston (4) and the results obtained are in close agreement with those of Fig. 2A of Ref. 4.

SUMMARY AND CONCLUSIONS

A numerical method has been presented for improving eigenvalues and eigenvectors for certain classes of operators. It has been illustrated by three detailed sample problems and a presentation of certain aspects of a fourth problem. The sample problems indicate various facets of the method as applied to discrete and continuous systems. A case of a double eigenvalue is considered in sample problem 3 and the special treatment of this problem is outlined. The extension of the procedure to an eigenvalue multiplicity of any order is apparent.

The application of the method to problem involving nonlinear structural response results in a more straightforward computational process than a previously proposed technique (10).

The proposed procedure converges rapidly as indicated empirically in Tables 1, 3, and 4 and verified analytically in Appendix II.

Although the method has been illustrated by finding the lowest modes for the sample problems, it may be used to find any eigenvalue (and its corresponding eigenvector) for which an approximate eigenvalue is known. In fact, one of the promising applications of the method may well be in determining the higher eigenvalues and eigenvectors.

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APPENDIX I.—SOLVABILITY OF BASIC EQUATIONS OF METHOD

Consider the proposed method as applied to the determination of the Jth eigenvalue and eigenvector of the system $[A]\{X\} = \lambda[B]\{X\}$, in which $[A]$ and $[B]$ are self-adjoint and $[B]$ is positive definite.

The method will fail if the basic coefficient matrix used in the computation of the increments of an approximate eigenvalue and eigenvector is singular. It is to be expected that if this occurs, it will be exactly at the eigenvalue in question. Thus, in the following presentation, the exact eigenvalue is substituted into the coefficient matrix used in the solution for the unknown incremental quantities.

From Eq. 9 the symmetric coefficient matrix may be written as

$$[C] = \left[\begin{array}{c|c} [A] - \lambda_j[B] & -[B]\{X_j\} \\ \hline -\{X_j\}^T[B] & 0 \end{array}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (27)$$

The eigenvectors are normalized with respect to $[B]$. If the order of the original problem is $n \times n$, the coefficient matrix in Eq. 27 is $(n + 1) \times (n + 1)$.

Matrix $[C]$ in Eq. 27 will be shown to be nonsingular by a consideration of the eigenvalues of the system

$$[C]\{y\} = \bar{\lambda}[D]\{y\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (28)$$

in which $[D] = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & 1 \end{array}\right]$.

It may be verified by direct substitution that the eigenvectors $\{y\}_m$, $(m = 1, \ldots, n + 1)$ are $\begin{Bmatrix} X_J \\ \hline 1 \end{Bmatrix}$, $\begin{Bmatrix} X_J \\ \hline -1 \end{Bmatrix}$, $\begin{Bmatrix} X_k \\ \hline 0 \end{Bmatrix}$, $(k = 1, \ldots n, k \neq J)$, in which X_J and X_k are eigenvectors of the eigenvalue problem $[A]\{X\} - \lambda[B]\{X\}$. The corresponding eigenvalues of Eq. 28, $\bar{\lambda}$ are -1, $+1$, and $(\lambda_k - \lambda_J)$.

It is not difficult to show that the determinant of $[C]$ is equal to the product of the $\bar{\lambda}$'s multiplied by the determinant of $[D]$. Because the latter is equal to the determinant of $[B]$, which is positive, the determinant of $[C]$ is nonzero, provided that none of the $\bar{\lambda}$'s are zero. Only in the case of a multiple root λ_J can a $\bar{\lambda}$ be zero. Thus, if the eigenvalue under study is simple, the basic method proposed encounters no numerical difficulties associated with a singularity of $[C]$.

The case of a double root, say $\lambda_J = \lambda_K$, is treated in sample problem 3 by a modification of the basic method. It will now be formally shown that the matrix equation in this modified method is also nonsingular. Take X_J and X_K orthogonal with respect to $[B]$. Now Eq. 28 becomes

$$[C]\{y_K\} = \{0\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (29)$$

in which $\{y_K\} = \begin{Bmatrix} X_K \\ \hline 0 \end{Bmatrix}$.

Following the procedure used in sample problem 3, another side condition is appended, resulting in

$$[\bar{C}] = \begin{bmatrix} C \\ \hline X_K^T B \mid 0 \end{bmatrix} = \begin{bmatrix} C \\ \hline y_K^T D \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (30)$$

in which $[D] = \begin{bmatrix} B \mid 0 \\ \hline 0 \mid 1 \end{bmatrix}$. The least squares approach used in sample problem 3 yields a new coefficient matrix $[\bar{C}]^T [\bar{C}]$, which, by virtue of the form of Eq. 30, becomes

$$[\bar{C}]^T [\bar{C}] = [C]^T [C] + [D] \{y_K\} \{y_K\}^T [D] \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (31)$$

The object is to show that the matrix given in Eq. 31 is nonsingular. From Eq. 29, $\{y_K\}$ is an eigenvector of $[C]$ (and thus of $[C]^T$) corresponding to a zero eigenvalue. The remaining eigenvalues of $[C]$ are nonzero as was shown earlier in this Appendix.

Now consider matrix $[F] = [D] \{y_K\} \{y_K\}^T [D]$ in Eq. 31. It may be shown by direct substitution that

$$[F] \{y_K\} = 1 [D] \{y_K\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (32)$$

so that eigenvector $\{y_K\}$ is also an eigenvector of $[F]$ and the corresponding eigenvalue is unity. By the nature of $[F]$, the remaining eigenvalues are zero because $[F]$ is a symmetric matrix of rank one. The remaining eigenvectors of $[F]$ may therefore be taken the same as those of $[C]$. The statement that the remaining eigenvectors have zero eigenvalues is precisely the condition of orthogonality the $\{y_k\}$ with respect to $[D]$, $\{y_k\}^T [D] \{y_k\} = 0$, $k \neq K$.

It is easy to show that for two matrices having the same set of eigenvectors, the eigenvalues of the sum of the two matrices are simply the sums of the corresponding eigenvalues of the individual matrices.

Coefficient matrix $[\bar{C}]^T [\bar{C}]$ has the same eigenvectors as $[C]$, thus the eigenvalues are the squares of those of $[C]$ except for the zero eigenvalue which becomes + 1. Since all the eigenvalues of $[\bar{C}]^T [\bar{C}]$ are nonzero, it is nonsingular and the method proceeds without difficulty.

When nonlinear structural response is considered, the equations which determine the behavior of the prebuckling configuration are progressively more ill-conditioned as bifurcation points are approached, becoming singular at a bifurcation point itself. The ill-conditioning may be removed by specifying an additional side condition (see presentation in the main text). A new coefficient matrix is then generated using the least squares approach. The new coefficient matrix may be shown to be nonsingular by a procedure similar to, and somewhat simpler than, that used in the multiple eigenvalue problem described in the last paragraphs.

APPENDIX II.—STUDY OF CONVERGENCE OF ITERATIVE PROCESS IN THE NEIGHBORHOOD OF AN EIGENVECTOR

An eigenvalue problem defined by the matrix equation

$$[A]\{X\} = \lambda[B]\{X\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (33)$$

is considered, in which $[A]$ and $[B]$ are self-adjoint and $[B]$ is positive definite. Eigenvectors $\{\bar{X}_m\}$ $(m = 1, 2, \ldots n)$ are normalized with respect to $[B]$; the corresponding eigenvalues are λ_m, n is the order of the system.

Substitution of an approximate eigenvalue $\lambda^{(i)}$ and eigenvector $\{X^{(i)}\}$ into Eq. 33 yields

$$[A]\{X^{(i)}\} - \lambda^{(i)}[B]\{X^{(i)}\} = \{R^{(i)}\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (34)$$

in which $\{R\}$ is a residual and superscript (i) denotes the ith approximation. Eq. 33 is now linearized which results in

$$[A]\{\delta X^{(i)}\} - \lambda^{(i)}[B]\{\delta X^{(i)}\} = \delta\lambda^{(i)}[B]\{X^{(i)}\} - \{R^{(i)}\} \quad \ldots\ldots \quad (35)$$

The approximate eigenvector $\{X^{(i)}\}$ may be expanded in a series of the true eigenvectors yielding

$$\{X^{(i)}\} = \sum_{m=1}^{n} \alpha_m^{(i)} \{\bar{X}_m\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (36)$$

in which the $\alpha_m^{(i)}$ are scalar coefficients. Eq. 36, together with Eq. 35, implies

$$[A] \sum_{m=1}^{n} \delta\alpha_m^{(i)} \{\bar{X}_m\} - \lambda^{(i)}[B] \sum_{m=1}^{n} \delta\alpha^{(i)} \{\bar{X}_m\}$$

$$= \delta\lambda^{(i)}[B] \sum_{m=1}^{n} \alpha_m^{(i)} \{\bar{X}_m\} - \{R^{(i)}\} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (37)$$

Premultiplication of Eq. 37 by the eigenvector $\{\bar{X}_k\}^T$ and substitution of $\{R^{(i)}\}$ from Eq. 37 results in

$$\delta\alpha_k^{(i)} = -\frac{\lambda_k - \lambda^{(i)} - \delta\lambda^{(i)}}{\lambda_k - \lambda^{(i)}} \alpha_k^{(i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (38)$$

Now consider that $X^{(i)}$ and $\lambda^{(i)}$ are in the vicinity of $\bar{X}_J$ and λ_J. A side condition is introduced analogous to the one used in sample problem 1, i.e.

$$\{X^{(i)}\}^T [B]\{X^{(i)}\} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (39)$$

Eq. 35 is then substituted into Eq. 39 to give

$$\sum_{k=1}^{n} \alpha_k^{(i)} \delta\alpha_k^{(i)} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (40)$$

Eq. 38 may be substituted into Eq. 40 resulting in

$$[\alpha_j^{(i)}]^2 \left\{1 - \frac{\delta\lambda^{(i)}}{\lambda_j - \lambda^{(i)}}\right\} + \sum_{\substack{m=1 \\ m \neq j}}^{n} [\alpha_m^{(i)}]^2 \left\{1 - \frac{\delta\lambda^{(i)}}{\lambda_m - \lambda^{(i)}}\right\} = 0 \quad \ldots\ldots \quad (41)$$

As the process is supposed to be isolating the Jth mode

$$\left.\begin{aligned} &\max_{m \neq J} \left| \frac{\alpha_m^{(i)}}{\alpha_j^{(i)}} \right| = \epsilon^{(i)} << 1 \\ &\text{and } \frac{\delta\lambda^{(i)}}{\lambda_m - \lambda^{(i)}} << 1 \qquad m \neq J \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (42)$$

If Eq. 41 is divided by $[\alpha_j^{(i)}]^2$, the last term becomes $K[\epsilon^{(i)}]^2$ (K a positive number $< n$). Eq. 41 becomes

$$\delta\lambda^{(i)} = [\lambda_j - \lambda^{(i)}]\{1 + K[\epsilon^{(i)}]^2\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (43)$$

Substitution of Eq. 43 into Eq. 38 yields

$$\delta\alpha_m^{(i)} = \left\{ -1 + \frac{[\lambda_J - \lambda^{(i)}]\{1 + K[\epsilon^{(i)}]^2\}}{\lambda_m - \lambda^{(i)}} \right\} \alpha_m^{(i)} \quad \ldots\ldots\ldots\ldots \quad (44)$$

It now follows that

$$\alpha_m^{(i+1)} = \alpha_m^{(i)} + \delta\alpha_m^{(i)} = \frac{\lambda_J - \lambda^{(i)}}{\lambda_m - \lambda^{(i)}} \{1 + K[\epsilon^{(i)}]^2\} \alpha_m^{(i)} \quad \ldots\ldots\ldots \quad (45)$$

The error in the eigenvalue at the ith iteration is denoted by

$$\Delta^{(i)} = \lambda_J - \lambda^{(i)} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (46)$$

Eqs. 42, 45, and 46 yield

$$\epsilon^{(i+1)} = \max_{m \neq J} \left| \frac{\alpha_m^{(i+1)}}{\alpha_J^{(i+1)}} \right| = \left| \frac{\Delta^{(i)}}{\lambda_{m_1} - \lambda^{(i)}} \right| \{1 + K[\epsilon^{(i)}]^2\} \epsilon^{(i)} \quad \ldots\ldots \quad (47)$$

in which m_1 is selected to make the quantity $\left|\lambda_{m_1} - \lambda^{(i)}\right|$ a minimum. Eq. 47 gives (neglecting terms of order $[\epsilon^{(i)}]^2$ as compared with 1)

$$\frac{\epsilon^{(i+1)}}{\epsilon^{(i)}} \cong \left| \frac{\Delta^{(i)}}{\lambda_{m_1} - \lambda^{(i)}} \right| \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (48)$$

Also, from Eqs. 43 and 46

$$\frac{\Delta^{(i+1)}}{\Delta^{(i)}} = K_1[\epsilon^{(i)}]^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (49)$$

in which K_1 is bounded.

Eqs. 48 and 49 determine the character of the convergence of the process. To examine this character more conveniently, the $\Delta^{(i)}$, $\Delta^{(i+1)}$ are eliminated from Eqs. 48 and 49 as follows:

$$\frac{\epsilon^{(i+2)}}{\epsilon^{(i+1)}} = \max_{m \neq J} \left| \frac{\Delta^{(i+1)}}{\lambda_m - \lambda^{(i+1)}} \right| = \left| \frac{\Delta^{(i)}}{\lambda_{m_1} - \lambda^{(i)}} \right| K_2[\epsilon^{(i)}]^2 \quad \ldots\ldots\ldots \quad (50)$$

in which K_2 is a bounded constant. Eq. 48 is then substituted into Eq. 50 to yield

$$\frac{\epsilon^{(i+2)}\,\epsilon^{(i)}}{\epsilon^{(i+1)}\,\epsilon^{(i+1)}} = K'[\epsilon^{(i)}]^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (51)$$

A process will be said to be of order γ if

$$\epsilon^{(i+1)} = C[\epsilon^{(i)}]^\gamma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (52)$$

Thus, a first order method merely multiplies the error by a constant (less than unity for convergence), and a second order method squares the error. The order γ and the constant c are obtained by substituting Eq. 52 into Eq. 51. It is found that $\gamma = 1 + \sqrt{2} \simeq 2.41$ and $c = (K')^{\sqrt{2}/2}$. Thus the process removes the unwanted components of the modes other than the Jth more rapidly than a second order process.

In the same way, elimination of the $\epsilon^{(i)}$ terms from Eqs. 48 and 49 leads to

$$\frac{\Delta^{(i+2)}\ \Delta^{(i)}}{\Delta^{(i+1)}\ \Delta^{(i+1)}} = K''[\Delta^{(i)}]^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (53)$$

Therefore, the convergence of the eigenvalues is also a process of order $\doteq$ 2.41.

The extremely rapid convergence indicated in Tables 1, 3, and 4 is thus to be expected.

APPENDIX III.—STIFFNESS AND MASS MATRICES FOR SAMPLE PROBLEM 1

The stiffness and mass matrices for sample problem 1 are given in Table 5.

TABLE 5.—STIFFNESS AND MASS MATRICES FOR SAMPLE PROBLEM 1

X_1	X_2	X_3	X_4	X_5	X_6
		(*a*) Stiffness Matrix			
0.374×10^7	0.000×10^0	0.463×10^7	-0.368×10^7	0.000×10^0	0.000×10^0
0.000×10^0	0.405×10^7	0.125×10^8	0.000×10^0	-0.695×10^5	0.125×10^8
0.463×10^7	0.125×10^8	0.344×10^{10}	0.000×10^0	-0.125×10^8	0.150×10^{10}
-0.368×10^7	0.000×10^0	0.000×10^0	0.374×10^7	0.000×10^0	0.463×10^7
0.000×10^0	-0.695×10^5	-0.125×10^8	0.000×10^0	0.405×10^7	-0.125×10^8
0.000×10^0	0.125×10^8	0.150×10^{10}	0.463×10^7	-0.125×10^8	0.344×10^{10}
		(*b*) Mass Matrix			
0.463×10^1	0.000×10^0	0.152×10^2	0.194×10^1	0.000×10^0	0.000×10^0
0.000×10^0	0.499×10^1	0.219×10^3	0.000×10^0	0.149×10^1	-0.467×10^5
0.152×10^2	0.219×10^3	0.147×10^5	0.000×10^0	0.129×10^3	-0.107×10^5
0.194×10^1	0.000×10^0	0.000×10^0	0.463×10^1	0.000×10^0	0.152×10^2
0.000×10^0	0.149×10^1	0.129×10^3	0.000×10^0	0.499×10^1	-0.219×10^3
0.000×10^0	-0.467×10^5	-0.107×10^5	0.152×10^2	-0.219×10^3	0.147×10^5

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APPENDIX V.—NOTATION

The following symbols are used in this paper:

A, B, C = generalized linear operators, may be matrices, differential or integral operators;
$\mathrm{C}, \overline{\mathrm{C}}, \mathrm{D}$ = augmented matrices (Appendix I);
EA = extensional stiffness;
EI = flexural stiffness;
F = matrix in Eq. 32;
i = superscript indicating ith approximation;
$[K]$ = stiffness matrix;
$\overline{K}$ = frame stiffness;
K, K_1, K_2, K', K'' = positive constants (Appendix II);
$[M]$ = mass matrix;
$\overline{M}$ = equivalent mass;
m = mass per unit length;
$R^{(i)}$ = ith residual (vector or function);
s = arc length;

t = time;
$U(s)$ = modal displacement;
$U(s)$ = eigenfunction (in problem 3);
$u(s, t)$ = axial displacement;
$V(s)$ = eigenfunction (problem 3);
$W(s)$ = modal transverse displacement;
$w(s, t)$ = transverse displacement;
X = eigenvector or eigenfunction;
x_j = jth generalized coordinate;
y_r = rth eigenvector (Appendix A);
α_j = contribution of jth initial value solution;
$\alpha_m^{(i)}$ = contribution of mth true eigenvector to $X^{(i)}$ (Appendix II);
γ = order of convergent process (Appendix II);
δ = variational symbol (used to denote incremental quantity);
$\Delta^{(i)}$ = error in ith approximation to eigenvalue (Appendix II);
$\epsilon^{(i)}$ = measure of error in ith approximation to eigenvector (Appendix II);
λ = eigenvalue; and
ω = natural circular frequency.