UILU-ENG-85-2004

CIVIL ENGINEERING STUDIES

STRUCTURAL RESEARCH SERIES NO. 519

ISSN: 0069-4274

THREE-DIMENSIONAL NON-AXISYMMETRIC ANISOTROPIC STRESS CONCENTRATIONS

By

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Technical Report of Research
Sponsored by
THE AIR FORCE OFFICE OF SCIENTIFIC RESEARCH
(Grant No. AFOSR 82-0047)

UNIVERSITY OF ILLINOIS
at URBANA-CHAMPAIGN
URBANA, ILLINOIS
MAY 1985

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50272-101

REPORT DOCUMENTATION PAGE	1. REPORT NO. UILU-ENG-85-2004	2.	3. Recipient's Accession No.
4. Title and Subtitle Three-Dimensional Non-Axisymmetric Anisotropic Stress Concentrations			5. Report Date Published May 1985 6.
7. Author(s) Abdul Hamid Zureick and Robert A. Eubanks			8. Performing Organization Rept. No. SRS No. 519
9. Performing Organization Name and Address Department of Civil Engineering University of Illinois 208 N. Romine St. Urbana, Illinois 61801			10. Project/Task/Work Unit No. 2307/B1 11. Contract(C) or Grant(G) No. (C) (G) AFOSR 82-0047
12. Sponsoring Organization Name and Address Air Force Office of Scientific Research Structural Mechanics Division Bolling Air Force Base Washington, D.C. 20332			13. Type of Report & Period Covered Technical Report 14.

15. Supplementary Notes

16. Abstract (Limit: 200 words)

Unified explicit analytical solutions for the (non-axisymmetric) first and second boundary value problems of elasticity theory for a spheroidal cavity embedded in a transversely isotropic medium are presented. The analysis is based upon solutions of the homogeneous displacement equations of equilibrium in terms of three quasi-harmonic potential functions, each of which is harmonic in a space different from the physical space. Thus, three spheroidal coordinate systems with different metric scales (one for each potential) are introduced such that the three coordinate systems coincide on the surface of the spheroidal cavity. These potential functions are taken in a unique combination of the associated Legendre functions of the first and second kind.

Extensive numerical data are obtained for the stress concentration factors associated with axisymmetric and non-axisymmetric problems for a variety of materials. The effect of anisotropy on the stress concentration factor is discussed in much greater detail than has been previously available in the literature.

17. Document Analysis a. Descriptors

Anisotropy	Non-Axisymmetric	Stress Concentration
Boundary Value Problems	Potential Functions	Three-Dimensional Linear Elasticity
Composite Materials	Spheroidal Cavity	Transverse Isotropy
Legendre Associated Functions	Stratified Materials	

b. Identifiers/Open-Ended Terms

c. COSATI Field/Group

18. Availability Statement	19. Security Class (This Report)	21. No. of Pages
Approved for public release -- Distribution unlimited	UNCLASSIFIED	128
	20. Security Class (This Page) UNCLASSIFIED	22. Price

ACKNOWLEDGMENTS

The research reported herein was conducted by Dr. Abdul Hamid Zureick under the supervision of Dr. Robert A. Eubanks, Professor of Civil Engineering and of Theoretical and Applied Mechanics at the University of Illinois at Urbana-Champaign.

This investigation was supported by the Air Force Office of Scientific Research under Grant No. AFOSR 82-0047.

Numerical results were obtained with the use of the CDC Cyber 175 computing facilities, supported by the Office of Computer Services, University of Illinois at Urbana-Champaign.

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CHAPTER 1

INTRODUCTION

1.1 Objective and Scope

The availability, diversity, and utilization of anisotropic materials have increased at a remarkable rate during the past few decades. When attempts were made to predict the behavior of these materials, it was evident that the available theoretical methods were far from satisfactory. As a result, much renewed interest has been directed to the theory of anistropic elasticity. In particular, the problem of stress concentration around cavities in a medium occupying the entire space has long been the subject of numerous investigations, not only because of its important role in fundamental elasticity problems but also because of modern engineering design concerns.

The objective of this study is to develop an explicit analytical solution for the non-axisymmetric problem of three-dimensional stress concentration in a transversely isotropic medium containing a spheroidal or spherical cavity.

1.2 History and Literature Survey

Stress concentration can be described as the local intensification of stress as the result of nearby changes in geometry or discontinuities in material or load. When a cavity exists inside the material, it gives rise to stress concentration and often leads to structural failure. Since the first investigations of the stress concentration problem during the second half of the 19th century, considerable research has been done on the problems of determining the stress concentration produced in an elastic body by a spherical or spheroidal cavity. A general review of stress concentration in linear elasticity has been given in a 1958 survey paper by Sternberg [40]. Eight years later, Neuber and Hahn [31] again reviewed stress concentration in scientific research and engineering. Because of the enormous number of articles concerning this subject, we will only briefly present the work relevant to the three-dimensional stress concentration problems in an infinite medium containing a spheroidal or spherical cavity or inclusion.

As early as 1866, Lamé [22] investigated the problem of the spherical shell under uniform external pressure. A few decades later, Larmor [24] studied the effect of a spherical cavity on a field of pure shear. A closed form solution for a spherical cavity in an infinite isotropic medium under uniaxial tension was presented, without derivation, by Southwell and Gough [39]. Goodier [14] investigated the concentration of stress around spherical and cylindrical inclusions and obtained in a general form the Southwell and Gough solution. The spheroidal cavity under uniform axial tension, pure shear, and torsion was first considered by Neuber [30]. An investigation of the stress concentration around an ellipsoidal cavity in an isotropic medium under arbitrary plane stress perpendicular to the axis of revolution of the cavity was undertaken in 1947 by Sadowsky and Sternberg [35]. They further presented an analysis of the triaxial ellipsoidal cavity in an infinite medium under a uniform stress field at infinity [36]. In 1951, Edwards [9] obtained a solution for the stress concentration around a spheroidal inclusion in a medium subjected to a uniform stress field at infinity. A general representation of the stress concentration problems in isotropic materials appeared in 1965 when Podil'chuk [32] studied the deformation of an axisymmetrically loaded elastic spheroidal cavity. Later, he extended his study to include the non-axisymmetric deformation of spheroidal cavities [33].

Although the stress concentration problem has been extended to anisotropic materials, very few three-dimensional anisotropic elasticity problems have been solved. Chen [5] solved the problem of uniaxial axisymmetric tension applied at large distances from a cavity in a transversely isotropic medium. In 1971, he also presented the general solution for an infinite elastic transversely isotropic medium containing a spheroidal inclusion [7] with the restriction that the prescribed stress field is axisymmetric and torsionless. In addition, he solved the spheroidal inclusion problem under pure shear in and out of the plane of isotropy and modified Bose's solution [3] for the torsion of a transversely isotropic medium containing an isotropic inclusion, to the case of a spheroidal inclusion in which the material may be transversely isotropic, provided that the axes of anisotropy of both the medium and the inclusion be the same [4]. To the the best of our knowledge, Chen's solution for the stress concentration around spheroidal inclusions and cavities in a transversely isotropic material under pure shear are the only non-axyisymmetric problems whose solutions are available. The present work extends the work by Chen [4-7] whose solutions can be deduced from this approach.

1.3 Organization of the Study

After the aforementioned historical review of the problem of stress concentration, the basic formulae and the potential functions approach in the three-dimensional elasticity theory of transversely isotropic materials are presented in Chapters 2 and 3, respectively.

In Chapter 4, we introduce the special coordinate systems used to solve the cavity problem and the potential functions in terms of the associated Legendre functions of the first and second kind as well as some important identities that the potential functions satisfy. The formulation and the analytical approach of solving the first and second boundary value problems of elasticity theory for a spheroidal cavity embedded in a transversely isotropic medium are presented in Chapters 5 and 6. Problems for cavities and inclusions subjected to a general constant stress field at infinity are solved at the end of these two chapters.

In Chapter 7, numerical investigation of the stress concentration factors associated with axisymmetric and non-axisymmetric problems for a variety of materials is carried out. The effect of anisotropy on the stress concentration factor is discussed.

Chapter 8 summarizes the developments of this study and makes recommendations for further study.

CHAPTER 2

TRANSVERSELY ISOTROPIC MATERIALS

2.1 Definition

A transversely isotropic material is an anisotropic one for which the Hookean matrix at a point remains invariant under an arbitrary rotation about an axis (the "axis of elastic symmetry" of the material).

There are some crystalline materials recognized as transversely isotropic. These include ice, cadmium, cobalt, magnesium, and zinc. Other examples of transverse isotropy are provided by materials having hexagonal structures such as fiber-reinforced composites [2] as shown in Fig. 1 and the honeycomb structures of Fig. 2, as well as laminated media (Fig. 3). Stratified rocks and soils can also be modeled as a homogeneous transversely isotropic medium [23,37,38,42].

2.2 Fundamental Formulae

Consider a homogeneous transversely isotropic elastic medium occupying a region of the three-dimensional Euclidian space referred to a fixed cylindrical coordinates system (r, θ, z) in which the z-axis coincides with material axis of symmetry. Let (u_r, u_θ, u_z) denote the cylindrical scalar components of the displacement vector and $(\varepsilon_{rr}, \varepsilon_{\theta\theta}, \gamma_{zz}, \gamma_{\theta z}, \gamma_{rz}, \gamma_{r\theta})$ denote engineering strains by

$$
\begin{aligned}
\varepsilon_{rr} &= \frac{\partial u_r}{\partial r} \\
\varepsilon_{\theta\theta} &= \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \\
\varepsilon_{zz} &= \frac{\partial u_z}{\partial z} \\
\gamma_{\theta z} &= \frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \\
\gamma_{rz} &= \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \\
\gamma_{r\theta} &= \frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}
\end{aligned}
\tag{2.2.1}
$$

In the absence of electrical and thermal effects, the generalized Hooke's law [16,26] can be expressed in the form:

$$
\begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{bmatrix}
=
\begin{bmatrix}
c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\
c_{12} & c_{11} & c_{13} & 0 & 0 & 0 \\
c_{13} & c_{13} & c_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & c_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & c_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2}(c_{11} - c_{12})
\end{bmatrix}
\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix}
\tag{2.2.2}
$$

where c_{11}, c_{12}, c_{13}, c_{33} and c_{44} are the five independent elastic constants characterizing the medium and (σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, $\sigma_{\theta z}$, σ_{rz}, $\sigma_{r\theta}$) are the components of the stress tensor. We may also write for convenience $c_{66} = (c_{11} - c_{12})$.

In certain applications of the theory of elasticity of transversely isotropic materials, Hooke's law may be written in the form

$$\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\ a_{12} & a_{11} & a_{13} & 0 & 0 & 0 \\ a_{13} & a_{13} & a_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(a_{11} - a_{12}) \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sigma_{\theta z} \\ \sigma_{rz} \\ \sigma_{r\theta} \end{bmatrix} \qquad (2.2.3)$$

In the literature, the coefficients a_{11}, a_{12}, a_{13}, a_{33}, and a_{44} are called "compliances" whereas the elastic constants are called "stiffnesses". We may also write, for convenience, $a_{66} = 2(a_{11} - a_{12})$.

It is evident from Eqs. (2.2.2) and (2.2.3) that the elastic constants (stiffnesses) can be converted to the compliances and vice versa by the standard determinant procedure for solving simultaneous equations. By doing so, the following relationships are found:

$$a_{11} = \frac{(c_{11}c_{33} - c_{13}^2)}{(c_{11} - c_{12})(c_{11}c_{33} + c_{12}c_{33} - 2c_{13}^2)}$$

$$a_{12} = -\frac{(c_{12}c_{33} - c_{13}^2)}{(c_{11} - c_{12})(c_{11}c_{33} + c_{12}c_{33} - 2c_{13}^2)}$$

$$a_{13} = -\frac{c_{13}}{(c_{11}c_{33} + c_{12}c_{33} - 2c_{13}^2)}$$

$$a_{33} = \frac{(c_{11} + c_{12})}{(c_{11}c_{33} + c_{12}c_{33} - 2c_{13}^2)}$$

$$a_{44} = \frac{1}{c_{44}} \tag{2.2.4}$$

$$a_{66} = 2(a_{11} - a_{12}) = \frac{1}{c_{66}} = \frac{2}{c_{11} - c_{12}}$$

and

$$c_{11} = \frac{(a_{11}a_{33} - a_{13}^2)}{(a_{11} - a_{12})(a_{11}a_{33} + a_{12}a_{33} - 2a_{13}^2)}$$

$$c_{12} = - \frac{(a_{12}a_{33} - a_{13}^2)}{(a_{11} - a_{12})(a_{11}a_{33} + a_{12}a_{33} - 2a_{13}^2)}$$

$$c_{13} = - \frac{a_{13}}{(a_{11}a_{33} + a_{12}a_{33} - 2a_{13}^2)} \tag{2.2.5}$$

$$c_{33} = \frac{(a_{11} + a_{12})}{(a_{11}a_{33} + a_{12}a_{33} - 2a_{13}^2)}$$

$$c_{44} = \frac{1}{a_{44}}$$

$$c_{66} = \frac{1}{2}(c_{11} - c_{12}) = \frac{1}{a_{66}} = \frac{1}{2(a_{11} - a_{12})}$$

2.3 The Engineering Constants

Some authors replace the elastic constants and the compliances by the so-called engineering constants which can be interpreted to be Young's moduli, the shear moduli, and Poisson's ratios associated with various directions (see [25]). If we introduce these constants which are related to the compliances by:

$$a_{11} = \frac{1}{E} \quad , \quad a_{12} = -\frac{\nu}{E} \quad , \quad a_{13} = -\frac{\bar{\nu}}{\bar{E}}$$
$$a_{33} = \frac{1}{\bar{E}} \quad , \quad a_{44} = \frac{1}{\bar{G}} \quad , \quad \left(a_{66} = \frac{1}{G} = \frac{2(1+\nu)}{E}\right) \tag{2.3.1}$$

Equations (2.2.3) take the form:

$$\begin{bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\bar{\nu}}{\bar{E}} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\bar{\nu}}{\bar{E}} & 0 & 0 & 0 \\ -\frac{\bar{\nu}}{\bar{E}} & -\frac{\bar{\nu}}{\bar{E}} & \frac{1}{\bar{E}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\bar{G}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\bar{G}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1+\nu)}{E} \end{bmatrix} \begin{bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \gamma_{\theta z} \\ \gamma_{rz} \\ \gamma_{r\theta} \end{bmatrix} \tag{2.3.2}$$

where:

E and $\bar{E}$ are Young's moduli in the plane of isotropy and perpendicular to it respectively.

ν is Poisson's ratio characterizing transverse contraction in the plane of isotropy when tension is applied in this plane.

$\bar{\nu}$ is Poisson's ratio characterizing transverse contraction in the plane of isotropy when tension is applied in a direction normal to the plane of isotropy.

G and $\bar{G}$ are shear moduli for the plane of isotropy and any plane perpendicular to it respectively.

The engineering constants and the elastic constants can be related to each other by means of Eqs. (2.2.5) and (2.3.1). These relations are given by:

$$E = 4\,c_{66}\left(1 - \frac{c_{33}c_{66}}{c_{11}c_{33} - c_{13}^2}\right)$$

$$\bar{E} = c_{33} - \frac{c_{13}^2}{c_{11} - c_{66}}$$

$$\nu = 1 - \frac{2\,c_{33}c_{66}}{c_{11}c_{33} - c_{13}^2} \tag{2.3.3}$$

$$\bar{\nu} = \frac{c_{13}}{c_{11} + c_{12}} = \frac{c_{13}}{2(c_{11} - c_{66})}$$

$$\bar{G} = c_{44}$$

$$G = \frac{E}{2(1+\nu)} = c_{66}$$

and, inversely,

$$c_{11} = \frac{2G(\frac{\bar{E}}{E} - \bar{\nu}^2)}{[\frac{\bar{E}}{E}(1-\nu) - 2\bar{\nu}^2]}$$

$$c_{12} = \frac{2G(\frac{\bar{E}}{E}\nu + \bar{\nu}^2)}{[\frac{\bar{E}}{E}(1-\nu) - 2\bar{\nu}^2]}$$

$$c_{13} = \frac{\bar{E}\,\bar{\nu}}{[\frac{\bar{E}}{E}(1-\nu) - 2\bar{\nu}^2]} \qquad (2.3.4)$$

$$c_{33} = \frac{\frac{\bar{E}^2}{E}(1-\nu)}{[\frac{\bar{E}}{E}(1-\nu) - 2\bar{\nu}^2]}$$

$$c_{44} = \bar{G}$$

$$c_{66} = G$$

For isotropic materials, the engineering constants and the elastic constants are reduced to

$$\bar{E} = E \ , \quad \bar{\nu} = \nu \ , \quad \bar{G} = G = \frac{E}{2(1+\nu)} \qquad (2.3.5)$$

and

$$c_{11} = c_{33} = \lambda + 2\mu = \frac{2(1-\nu)\mu}{1-2\nu} \qquad (2.3.6)$$

$$c_{12} = c_{13} = \lambda = \frac{2\nu\mu}{1-2\nu}$$

$$c_{44} = c_{66} = \mu = \frac{E}{2(1+\nu)}$$

where λ and μ are the well known Lame's constants, ν is Poisson's ratio, and E is Young's moduli.

2.4 Restriction on the Elastic Constants

Eubanks and Sternberg [15] show that the necessary and sufficient conditions for positive definiteness of the strain energy density are that

$$c_{11} > |c_{12}| > 0 \quad , \quad c_{33} > 0 \quad , \quad c_{44} > 0 \quad , \quad c_{66} > 0$$

$$c_{33}(c_{11} + c_{12}) - 2c_{13}^2 = c_{11}c_{33} - c_{13}^2 - c_{13}c_{66} > 0 \tag{2.4.1}$$

or

$$a_{11} > |a_{12}| > 0 \quad , \quad a_{33} > 0 \quad , \quad a_{44} > 0 \quad , \quad a_{66} > 0$$

$$a_{33}(a_{11} + a_{12}) - 2a_{13}^2 > 0 \tag{2.4.2}$$

or, equivalently,

$$E > 0 \quad , \quad \bar{E} > 0 \quad , \quad \bar{G} > 0 \quad , \quad G = \frac{E}{2(1+\nu)} > 0$$

$$(1-\nu) - 2\bar{\nu}^2 \frac{E}{\bar{E}} > 0 \quad , \quad -1 < \nu < 1 \tag{2.4.3}$$

Negative values of Poisson's ratio have been reported indirectly by Hearmon in Reference [19] for zinc and cadmium sulfide (CdS). This data is questionable, since virtually all real materials have positive values of Poisson's ratio. The negative values of Poisson's ratio are, therefore, omitted from the discussion in Chapter 7.

Inequalities (2.4.3) become for materials having positive Poisson's ratio:

$$E > 0 \quad , \quad \bar{E} > 0 \quad , \quad \bar{G} > 0 \quad , \quad 0 < \nu < 1 - 2\bar{\nu}^2 \frac{E}{\bar{E}} \qquad (2.4.4)$$

the last inequality of (2.4.4) is plotted in Fig. 4. The interior area bounded by a parabola corresponding to a specific value of $\frac{\bar{E}}{E}$, represents the valid domain of variation of Poisson's ratio. Line OI represents the isotropic materials. It is noteworthy that for values of $\frac{\bar{E}}{E} > 2$, Poisson's ratio $\bar{\nu}$ might exceed unity.

2.5 Values of the Elastic Constants

The numerical values of the elastic constants for a variety of different hexagonal crystalline and non-crystalline materials are reported by Hearmon in the revised edition of Landolt-Börnstein [19]. Some of these materials are listed in Tables 1 and 2. In both tables, the unit used for the elastic constants c_{ij} is the gigapascal (GPa = 10^{10} dyne/cm^2) and the unit used for the compliances a_{ij} is the reciprocal of terapascal (TPa = 10^3 GPa).

Moreover, the elastic constants of fiber reinforced materials can be calculated by utilizing the technique developed by Hlavacek [17,18] or by Hashin and Rosen [15]. Achenbach [1] has presented a procedure for the computation of the effective moduli of laminated media while Salamon [37] derived expressions for the elastic moduli of a stratified rock mass and Wardle and Gerrard [42] discussed the restrictions on the ranges of some of the five elastic constants.

Table 1. Elastic Constants for Hexagonal Crystals

Material	Values of the Elastic Constants (GPa)					Values of the Compliances (TPa)				
	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	a_{11}	a_{12}	a_{13}	a_{33}	a_{44}
Beryllium, Be	292	24	6	349	163	3.45	-0.28	-0.05	2.87	6.16
Cadmium, Cd	116	42	41	50.9	19.6	12.2	-1.2	-8.9	33.8	51.1
Cobalt, Co	295	159	111	335	71.0	5.11	-2.37	-0.94	3.69	14.1
Graphite, C	1060	180	15	36.5	4	0.98	-0.16	-0.33	27.5	250
Hafnium, Hf	181	77	66	197	55.7	7.16	-2.48	-1.57	6.13	18.0
Ice, H_2O at (-16°C)	13.5	6.5	5.9	14.9	3.09	105	-40	-25	87	325
Magnesium, Mg	59.3	25.7	21.4	61.5	16.4	22.0	-7.8	-5.0	19.7	60.9
Quartz, β-SiO	117	16	33	110	36.0	9.41	-0.6	-2.6	10.6	27.7
Rhenium, Re	616	273	206	683	161	2.11	-0.80	-0.40	1.70	6.21
Silver Aluminum, Ag Aℓ	142	85	75	168	34.1	11.9	-5.7	-2.8	8.35	29.3
Titanium, Ti	160	90	66	181	46.5	9.69	-4.71	-1.82	6.86	21.5
Zinc, Zn	165	31.1	50.0	61.8	39.6	8.22	0.60	-7.0	27.7	25.3
Zinc Oxide (Zincite), ZnO	209	120	104	218	44.1	7.82	-3.45	-2.10	6.64	22.4
Zirconium, α-Zr	144	74	67	166	33.4	10.1	-4.0	--2.4	8.0	30.1

Table 2. Elastic Constants for Hexagonal Systems
Non-Crystalline Materials

Material		Values of the elastic constants (GPa)					Values of the compliances (TPa)				
		c_{11}	c_{12}	c_{13}	c_{33}	c_{44}	a_{11}	a_{12}	a_{13}	a_{33}	a_{44}
$Al-CuAl_2$ composite		123	60	60	123	29.7	11.9	-3.9	-3.9	11.9	33.7
Bone (dried phalanx)		21.2	9.5	10.2	37.4	7.5	63	-23	-11	33	133
Bone (fresh phalanx)		19.7	12.1	12.6	32.0	5.4	89	-43	-18	45	185
Bone (dried femur)		23.8	10.2	11.2	33.4	8.2	56	-18	-13	38	122
Ceramics $BaTiO_3$ (Barium-Titanate)		166	77	78	162	43	8.55	-2.61	-2.85	8.93	23.3
Fiber Reinforced Resins											
Carbon fiber/epoxy resin											
Type of Fiber	Fiber Fraction										
Graphite Thornel 50	0.55	10.0	4.8	5.6	186	5.9	130	-62	-2.08	5.49	169
Graphite Thornel 75	0.56	9.2	5.0	5.2	309	6.3	154	-84	-1.20	3.27	159
Rocks											
Micha schist		85.6	18.6	20.1	68.6	25.2	12.9	-2.0	-3.2	16.4	39.7
Eclogite		171	60	59	208	58.5	7.02	-1.99	1.43	5.61	17.1

CHAPTER 3

POTENTIAL SOLUTIONS FOR TRANSVERSELY ISOTROPIC MATERIALS

3.1 Introduction

At the beginning of the twentieth century, Michell [28] and Fredholm [13] generated important three-dimensional solutions in transversely isotropic materials. Lekhnitskii [25] presented a solution procedure for axisymmetric torsionless problems in terms of a single stress function satisfying a fourth-order partial differential equation. Elliott [10-11] presented a comparable solution in terms of two stress functions each satisfying a second-order partial differential equation. Shortly after Elliot's presentation, Eubanks and Sternberg [12] proved the completeness of the Lekhnitskii and Elliot representations for the case of axisymmetry. Hu [21] introduced a third potential function suitable for non-axisymmetric stress field. This solution was implied in Fredholm's work [13]. The completeness of the three-function approach for general non-axisymmetric problems was proven in Reference [41].

3.2 Fundamental Formulae

Adopting the notations in Section 2.2 and combining Eqs. (2.2.1) and (2.2.2), we obtain the stress-displacement relations that take the form

$$\sigma_{rr} = c_{11} \frac{\partial u_r}{\partial r} + c_{12}\left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + c_{13} \frac{\partial u_z}{\partial z}$$

$$\sigma_{\theta\theta} = c_{12} \frac{\partial u_r}{\partial r} + c_{11}\left(\frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + c_{13} \frac{\partial u_z}{\partial z}$$

$$\sigma_{zz} = c_{13}\left(\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} \right) + c_{33} \frac{\partial u_z}{\partial z}$$

$$\sigma_{\theta z} = c_{44}\left(\frac{1}{r} \frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z} \right) \qquad (3.2.1)$$

$$\sigma_{rz} = c_{44}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right)$$

$$\sigma_{r\theta} = \frac{1}{2} (c_{11} - c_{12})\left(\frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right)$$

It has been shown [41] that in the absence of body forces, the following representation is complete

$$u_r = \frac{\partial}{\partial r} (\phi_1 + \phi_2) + \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$u_\theta = \frac{1}{r} \frac{\partial}{\partial \theta} (\phi_1 + \phi_2) - \frac{\partial \psi}{\partial r} \qquad (3.2.2)$$

$$u_z = k_1 \frac{\partial \phi_1}{\partial z} + k_2 \frac{\partial \phi_2}{\partial z}$$

where $\phi_1(r,\theta,z)$, $\phi_2(r,\theta,z)$ and $\psi(r,\theta,z)$ satisfy the equations

$$\nabla_1^2 \phi_1 = \nabla_2^2 \phi_2 = \nabla_3^2 \psi = 0 \qquad (3.2.3)$$

in which

$$\nabla_j^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \nu_j^2 \frac{\partial^2}{\partial z^2} \qquad (3.2.4)$$

In these equations, k_1 and k_2 are the roots of the equation

$$c_{44}(c_{13}+c_{44})k^2 + [(c_{13}+c_{44})^2 + c_{44}^2 - c_{11}c_{33}]k + c_{44}(c_{44}+c_{13}) = 0 \qquad (3.2.5)$$

and ν_1^2 and ν_2^2 are the roots of the equation

$$c_{11}c_{44}\,\nu^4 + [c_{13}(2c_{44}+c_{13}) - c_{11}c_{13}]\nu^2 + c_{33}c_{44} = 0 \qquad (3.2.6)$$

whereas ν_3^2 is defined by

$$\nu_3^2 = \frac{2c_{44}}{c_{11} - c_{12}} = \frac{c_{44}}{c_{66}} \qquad (3.2.7)$$

The constants ν_1^2 and ν_2^2 are either real or complex conjugate (with a real part different from zero) depending upon the elastic constants, but the constant ν_3^2 is always real and positive. We also specify that ν_1, ν_2, and ν_3 always have positive real parts.

The constants k_1 and k_2 are related to ν_1^2 and ν_2^2 respectively by

$$k_j = \frac{c_{11}\,\nu_j^2 - c_{44}}{c_{13} + c_{44}} = \frac{\nu_j^2(c_{13} + c_{44})}{c_{33} - c_{44}\,\nu_j^2} \qquad (3.2.8)$$

or, equivalently

$$\nu_j^2 = \frac{k_j(c_{13} + c_{44}) + c_{44}}{c_{11}} = \frac{k_j\,c_{33}}{c_{44}(1 + k_j) + c_{13}} \qquad (3.2.9)$$

By using Eqs. (3.2.5), (3.2.6) and (3.2.8), it is not difficult to obtain the following identities

$$\nu_1^2 \nu_2^2 = \frac{c_{33}}{c_{11}}$$

$$\nu_1^2 + \nu_2^2 = \frac{c_{11}c_{33} - c_{13}^2 - 2c_{13}c_{44}}{c_{11}c_{44}} \tag{3.2.10}$$

$$k_1 k_2 = 1$$

$$k_1 + k_2 = \frac{c_{11}c_{33} + c_{44}^2 + (c_{13} + c_{44})^2}{c_{44}(c_{13} + c_{44})}$$

Sometimes, it is convenient to employ three new variables defined by

$$z_j = \frac{z}{\nu_j} \quad , \qquad j = 1,\ 2,\ 3 \tag{3.2.11}$$

These new three variables generates three distinct spaces (r,θ,z_1), (r,θ,z_2) and (r,θ,z_3) different from the physical space. The differential operations ∇_j^2 defined by Eq. (3.2.4) become, in the spaces (r,θ,z_j), the Laplacian defined by

$$\nabla_j^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z_j^2} \tag{3.2.12}$$

the displacement components in Eq. (3.2.2) can be, alternately, expressed in the following form

$$u_r = \frac{\partial}{\partial r}(\phi_1 + \phi_2) + \frac{1}{r}\frac{\partial \psi}{\partial \theta}$$

$$u_\theta = \frac{1}{r}\frac{\partial}{\partial \theta}(\phi_1 + \phi_2) - \frac{\partial \psi}{\partial r} \qquad (3.2.13)$$

$$u_z = \frac{k_1}{\nu_1}\frac{\partial \phi_1}{\partial z_1} + \frac{k_2}{\nu_2}\frac{\partial \phi_2}{\partial z_2}$$

and, consequently, the stress components can be written in the form

$$\sigma_{rr} = -c_{44}\left[\frac{(1+k_1)}{\nu_1^2}\frac{\partial^2 \phi_1}{\partial z_1^2} + \frac{(1+k_2)}{\nu_2^2}\frac{\partial^2 \phi_2}{\partial z_2^2}\right]$$

$$-2c_{66}\left[\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)(\phi_1 + \phi_2) - \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\right]$$

$$\sigma_{\theta\theta} = -c_{44}\left[\frac{(1+k_1)}{\nu_1^2}\frac{\partial^2 \phi_1}{\partial z_1^2} + \frac{(1+k_2)}{\nu_2^2}\frac{\partial^2 \phi_2}{\partial z_2^2}\right]$$

$$-2c_{66}\left[\frac{\partial^2}{\partial r^2}(\phi_1 + \phi_2) + \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial \theta}\right)\right] \qquad (3.2.14)$$

$$\sigma_{zz} = c_{44}\left[(1+k_1)\frac{\partial^2 \phi_1}{\partial z_1^2} + (1+k_2)\frac{\partial^2 \phi_2}{\partial z_2^2}\right]$$

$$\sigma_{\theta z} = c_{44}\left[\frac{(1+k_1)}{\nu_1}\frac{1}{r}\frac{\partial^2 \phi_1}{\partial \theta \partial z_1} + \frac{(1+k_2)}{\nu_2}\frac{1}{r}\frac{\partial^2 \phi_2}{\partial \theta \partial z_2} - \frac{1}{\nu_3}\frac{\partial^2 \psi}{\partial r \partial z_3}\right]$$

$$\sigma_{rz} = c_{44}\left[\frac{(1+k_1)}{\nu_1}\frac{\partial^2 \phi_1}{\partial r \partial z_1} + \frac{(1+k_2)}{\nu_2}\frac{\partial^2 \phi_2}{\partial r \partial z_2} + \frac{1}{\nu_3}\frac{1}{r}\frac{\partial^2 \psi}{\partial \theta \partial z_3}\right]$$

$$\sigma_{r\theta} = 2c_{66}\left(\frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial}{\partial \theta}\right)(\phi_1 + \phi_2) + c_{66}\left(\frac{2}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z_3^2} + \frac{2}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}\right)$$

Numerical values of ν_j^2 and k_j are calculated for some materials and listed in Tables 3 and 4 for real and complex k_j respectively.

Table 3. Values of ν_j^2 and k_j for Selected Materials

Material	ν_1^2	ν_2^2	ν_3^2	k_1	k_2
Cobalt	2.999	0.379	1.044	4.471	0.2237
Graphite	9.040	0.004	0.009	504.1	0.0020
Hafnium	1.755	0.620	1.071	2.153	0.4644
Ice	2.706	0.408	0.883	3.719	0.2689
Magnesium	2.052	0.505	0.976	2.785	0.3590
Quartz	1.670	0.563	0.713	2.310	0.4329
Titanium	1.880	0.602	1.329	2.261	0.4423
Zirconium	2.675	0.431	0.954	3.504	0.2854

Table 4. Values of ν_j^2 and k_j for Selected Materials*

Material	ν_3^2	ν_1^2	k_1
Berylium	1.216	1.050 + 0.3058 i	0.8491 + 0.5283 i
Cadmium	0.5297	0.575 + 0.3283 i	0.7779 + 0.6284 i
Zinc	0.5915	0.286 + 0.5411 i	0.0847 + 0.9964 i

*Values of ν_2^2 and k_2 are the complex conjugate of ν_1^2 and k_1 respectively.

CHAPTER 4

THE SPHEROIDAL CAVITY PROBLEM

4.1 Statement of Problem

Consider an infinite elastic transversely isotropic medium, as shown in Fig. 5, containing a spheroidal cavity whose surface is defined by

$$\frac{z^2}{a^2} + \frac{r^2}{b^2} = 1 \tag{4.1.1}$$

where a and b are the two semi-axes of the spheroid.

In the absence of body forces, we wish to develop displacement fields whose stresses vanish at infinity and which satisfy given boundary conditions on the cavity surface. In this chapter, we introduce applicable coordinate systems and potential functions. General solutions to problems in which either the displacement vector or the surface traction vector is prescribed on the surface of the cavity will be considered in the following chapters.

4.2 The Coordinate System

We have to determine the potential functions $\phi_1(r,\theta,z_1)$, $\phi_2(r,\theta,z_2)$ and $\psi(r,\theta,z_1)$ which are the solution of Laplace's equation in the spaces (r,θ,z_1), (r,θ,z_2) and (r,θ,z_3) respectively. Since we have a problem in which the natural boundaries are spheroids with center at the origin of coordinates, it is suitable to employ [5-9] the spheroidal coordinate systems defined by

$$x = \alpha_j(q_j^2 - 1)^{\frac{1}{2}}\ (1 - p_j^2)^{\frac{1}{2}} \cos\theta$$

$$y = \alpha_j(q_j^2 - 1)^{\frac{1}{2}}\ (1 - p_j^2)^{\frac{1}{2}} \sin\theta \qquad (4.2.1)$$

$$z_j = \alpha_j\ q_j\ p_j$$

or equivalently,

$$r = \alpha_j(q_j^2 - 1)^{\frac{1}{2}}\ (1 - p_j^2)^{\frac{1}{2}}$$

$$z_j = \alpha_j\ q_j\ p_j \qquad (4.2.2)$$

where q_j and p_j are parameters which can be determined for any point whose coordinates (r,z) are known. The α_j are constants to be determined later.

Let ρ_j denote the value q_j on the spheroidal surface, then the three coordinate systems coincide on the surface of spheroid if the following equalities are satisfied

$$\alpha_1^2\ \nu_1^2\ \rho_1^2 = \alpha_2^2\ \nu_2^2\ \rho_2^2 = \alpha_3^2\ \nu_3^2\ \rho_3^2 = a^2$$

$$\alpha_1^2\ (\rho_1^2 - 1) = \alpha_2^2\ (\rho_2^2 - 1) = \alpha_3^2\ (\rho_3^2 - 1) = b^2 \qquad (4.2.3)$$

from which we obtain

$$\alpha_j^2 = \frac{a^2}{\nu_j^2} - b^2 \qquad (4.2.4)$$

$$\rho_j^2 = \frac{a^2}{a^2 - b^2\nu_j^2} \qquad (4.2.5)$$

where j = 1, 2 and 3.

It is interesting to note that when $q_j = \rho_j$, Eq. (4.2.2) reduces to the equation of the spheroidal surface described by Eq. (4.1.1) and the parameters p_j become independent of ν_j^2 and equal $\frac{z}{a}$. The coordinate systems defined by Eq. (4.2.2) can be represented graphically in the physical space for real values of ν_j^2. For example, when the cavity is spherical and ν_j^2 = 0.5 and 2.0, Eq. (4.2.2) generates two types of spheroidal and hyperboloidal surfaces that have a common spherical surface corresponding to the same value ν_j. These coordinate systems are shown in Figs. 5 and 6.

4.3 The Potential Functions

As indicated in Section 3.2, the potential functions ϕ_1, ϕ_2, and ψ must satisfy Laplace's equation in the spaces (r,θ,z_1), (r,θ,z_2) and (r,θ,z_3) respectively. It is well known [20, 29, 34] that the products $P_n^{-m}(p_j)\, Q_n^m(q_j)$ [cos mθ, sin mθ] are harmonic in the spaces (r,θ,z_j) and regular in the region exterior to a spheroid, where $P_n^{-m}(p_j)$ and $Q_n^m(q_j)$ are the associated Legendre functions of the first and second kind respectively and will be defined explicitly later. For the problem at hand, the potential functions are constructed by the following combination of the Legendre associated functions of first and second kind:

$$\phi_j = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \frac{2\alpha_j}{(2n+1)} [P_{n+1}^{-m}(p_j)\, Q_{n+1}^{m}(q_j) - P_{n-1}^{-m}(p_j)\, Q_{n-1}^{m}(q_j)]$$

$$\cdot\, [A_{jnm} \cos m\theta + B_{jnm} \sin m\theta]$$

$$\psi = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \frac{2\alpha_3}{(2n+1)} [P_{n+1}^{-m}(p_3)\ Q_{n+1}^{m}(q_3) - P_{n-1}^{-m}(p_3)\ Q_{n-1}^{m}(q_3)] \cdot [A_{3nm}\sin m\theta + B_{3nm}\cos m\theta] \qquad (4.3.1)$$

$j = 1,2$

because they are harmonic in (r,θ,z_j) and their first and second derivatives vanish at large distances from the spheroid. Furthermore, as will be shown later, the first derivative of the chosen potential functions with respect to r or z_j can be written in simple forms which lead to possible solutions of the first and second boundary value problems.

In these potential functions, $P_n^{-m}(p_j)$ and $Q_n^m(q_j)$ are the associated Legendre functions of degree n and order m, of the first and second kind respectively. They are defined [20,27,34] by

$$P_n^{-m}(p_j) = (1-p_j^2)^{-m/2} \int_{p_j}^{1} \int_{p_j}^{1} \cdots\cdots\cdots \int_{p_j}^{1} P_n(p_j)(dP_j)^m \qquad (4.3.2)$$

$$Q_n^{-m}(q_j) = \frac{(n-m)!}{(n+m)!}\ Q_n^m \qquad (4.3.3)$$

$$Q_n^m\ (q_j) = (q_j^2 - 1)^{m/2}\ \frac{d^m Q_n(q_j)}{dq_j^m} \qquad (4.3.4)$$

in which

$$P_n(p_j) = \frac{1}{2^n\ n!} \frac{d^n(p_j^2 - 1)^n}{dp_j^n} \qquad (4.3.5)$$

$$Q_n(q_j) = \frac{1}{2} P_n(q_j) \, \mathrm{Ln} \frac{q_j + 1}{q_j - 1} - T_n(q_j) \tag{4.3.6}$$

$$T_n(q_j) = \frac{(2n-1)}{1 \cdot n} P_{n-1}(q_j) + \frac{(2n-5)}{3(n-1)} P_{n-3}(q_j) + \\ + \frac{(2n-9)}{5(n-2)} P_{n-5}(q_j) + \; . \; . \; . \; . \tag{4.3.7}$$

Legendre associated functions defined by Eqs. (4.3.2) and (4.3.6) are listed in Appendix A for different values of n and m.

The Legendre associated functions defined by Eqs. (4.3.2) and (4.3.7) satisfy the following identities

$$-Q_n^{m+1}(q) + (n+m)(n-m+1)\, Q_n^{m-1}(q) = \frac{2\,m\,q}{(q^2-1)^{\frac{1}{2}}}\, Q_n^m(q)$$

$$-Q_n^{m+1}(q) + (n-m)(n-m+1)\, Q_n^{m-1}(q) = \frac{2\,m}{(q^2-1)^{\frac{1}{2}}}\, Q_{n-1}^m(q) \tag{4.3.8}$$

$$-Q_n^{m+1}(a) + (n+m)(n+m+1)\, Q_n^{m-1}(q) = \frac{2\,m}{(q^2-1)^{\frac{1}{2}}}\, Q_{n+1}^m(q)$$

and

$$(n-m)(n+m+1)\, P_n^{-(m+1)}(p) + P_n^{-(m-1)}(p) = \frac{2\,m\,p}{(1-p^2)^{\frac{1}{2}}}\, P_n^{-m}(p)$$

$$(n+m)(n+m+1)\, P_n^{-(m+1)}(p) + P_n^{-(m-1)}(p) = \frac{2\,m}{(1-p^2)^{\frac{1}{2}}}\, P_{n-1}^{-m}(p) \tag{4.3.9}$$

$$(n-m)(n-m+1)\, P_n^{-(m+1)}(p) + P_n^{-(m-1)}(p) = \frac{2\,m}{(1-p^2)^{\frac{1}{2}}}\, P_{n+1}^{-m}\;(p)$$

The first derivatives of $P_n^{-m}(p)$ and $Q_n^{-m}(q)$ with respect to the argument are given by

$$(1-p^2)\frac{d\,P_n^{-m}(p)}{d\,p} = -(1-p^2)^{\frac{1}{2}}\,P_n^{-(m-1)}(p) + m\,p\,P_n^{-m}(p)$$

$$(q^2-1)\frac{d\,Q_n^{m}(q)}{d\,q} = (q^2-1)^{\frac{1}{2}}\,Q_n^{(m+1)}(q) + m\,q\,Q_n^{m}(q) \qquad (4.3.10)$$

The representation of the potential functions in terms of the associated Legendre function as shown in Eq. (4.3.1) enables us to write the partial derivative with respect to r and z_j in very simple forms. Recalling Eq. (4.2.2), one can easily obtain:

$$\frac{\partial p_j}{\partial r} = -\frac{p_j\,r}{\alpha_j^2(q_j^2 - p_j^2)}$$

$$\frac{\partial p_j}{\partial z_j} = \frac{(1-p_j^2)\,z_j}{\alpha_j^2\,p_j(q_j^2-p_j^2)}$$

$$\frac{\partial q_j}{\partial r} = \frac{q_j\,r}{\alpha_j^2\,(q_j^2 - p_j^2)} \qquad (4.3.11)$$

$$\frac{\partial q_j}{\partial z_j} = \frac{(q_j^2 - 1)\,z_j}{\alpha_j^2\,q_j\,(q_j^2-p_j^2)}$$

By means of Eqs. (4.3.8)-(4.3.10), it can be readily shown that

$$\frac{\partial \phi_j}{\partial r} = \sum_{n=0}^{\infty}\sum_{m=0}^{n+1} -[P_n^{-(m+1)}(p_j)\,Q_n^{(m+1)}(q_j) - P_n^{-(m-1)}(p_j)\,Q_n^{(m-1)}(q_j)]$$

$$\cdot[A_{jnm}\cos m\theta + B_{jnm}\sin m\theta] \qquad (4.3.12)$$

$$\frac{\partial \phi_j}{\partial z_j} = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} 2\, P_n^{-m}(p_j)\, Q_n^m(q_j)\, [A_{jnm} \cos m\theta + B_{jnm} \sin m\theta] \tag{4.3.13}$$

and Eq. (4.3.1) can also be written in the form

$$\phi_j = \frac{r}{m\alpha_j} [P_n^{-(m+1)}(p_j)\, Q_n^{(m+1)}(q_j) + P_n^{-(m-1)}(p_j)\, Q_n^{(m-1)}(q_j)] \cdot [A_{jnm} \cos m\theta + B_{jnm} \sin m\theta] \tag{4.3.14}$$

4.4 Some Remarks on the Potential Functions

(i) The Legendre associated function of the first kind used in the expressions of the potential functions in Eq. (4.3.2) is equivalent to the following definition

$$P_n^{-m}(p) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(p) \tag{4.4.1}$$

where

$$P_n^m(p) = (-1)^m (1-p^2)^{m/2} \frac{d^m P_n(p)}{dp} \tag{4.4.2}$$

for all values of $m \leq n$.

(ii) For values of $m > n$, Eq. (4.3.2) generates a function of p singular at $p = 1$ or $p = -1$. This situation is encountered in our potential functions when $m = n$ and $m = n+1$. However, it will be shown later that for these two particular cases, singularities inside the medium are always removable.

(iii) The sum with respect to m in the expression of the potential function is truncated after the value $m = n+1$.

CHAPTER 5

THE FIRST BOUNDARY VALUE PROBLEM

5.1 Introduction

The first boundary value problem consists of finding stresses and displacements of an elastic body in equilibrium when the body forces are known and the displacements of the body are prescribed. In this chapter, the first boundary value problem is solved for an unbounded elastic transversely isotropic medium containing a spheroidal cavity with zero body forces and vanishing stress and displacement fields at infinity. Explicit solutions are presented for two problems involving a rigid spheroidal inclusion.

5.2 Fundamental Formulae

The components of the displacement vector in Eqs. (3.2.13) can be written as

$$\begin{aligned} u_r &= u_{r1} + u_{r2} + u_{r3} \\ u_\theta &= u_{\theta 1} + u_{\theta 2} + u_{\theta 3} \qquad (5.2.1) \\ u_z &= u_{z1} + u_{z2} \end{aligned}$$

where

$$u_{rj} = \frac{\partial \phi_j}{\partial r} \quad , \quad u_{\theta j} = \frac{1}{r}\frac{\partial \phi_j}{\partial \theta} \quad , \quad u_{zj} = \frac{k_j}{\nu_j}\frac{\partial \phi_j}{\partial z_j}$$

$$u_{r3} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad , \quad u_{\theta 3} = -\frac{\partial \psi}{\partial r} \tag{5.2.2}$$

Substituting the expressions of ϕ_j and ψ defined by Eqs. (4.3.1) into Eqs. (5.2.2) we obtain:

$$u_{rj} = \sum_{n=0}^{\infty}\sum_{m=0}^{n+1} -[P_n^{-(m+1)}(p_j)Q_n^{(m+1)}(q_j) - P_n^{-(m-1)}(p_j)\,Q_n^{(m-1)}(q_j)] \cdot [A_{jnm}\cos m\theta + B_{jnm}\sin m\theta]$$

$$u_{r3} = \sum_{n=0}^{\infty}\sum_{m=0}^{n+1} [P_n^{-(m+1)}(p_3)\;Q_n^{(m+1)}(q_3) + P_n^{-(m-1)}(p_3)\;Q_n^{(m-1)}(q_3)] \cdot [A_{3nm}\cos m\theta - B_{3nm}\sin m\theta]$$

$$u_{\theta j} = \sum_{n=0}^{\infty}\sum_{m=0}^{n+1} [P_n^{-(m+1)}(p_j)\;Q_n^{(m+1)}(q_j) + P_n^{-(m-1)}(p_j)\;Q_n^{(m-1)}(q_j)] \cdot [B_{jnm}\cos m\theta - A_{jnm}\sin m\theta] \tag{5.2.3}$$

$$u_{\theta 3} = \sum_{n=0}^{\infty}\sum_{m=0}^{n+1} [P_n^{-(m+1)}(p_3)\;Q_n^{(m+1)}(q_3) - P_n^{-(m-1)}(p_3)\;Q_n^{(m-1)}(q_3)] \cdot [B_{3nm}\cos m\theta + A_{3nm}\sin m\theta]$$

$$u_{zj} = \sum_{n=0}^{\infty}\sum_{m=0}^{n+1} \frac{2k_j}{\nu_j} P_n^{-m}(p_j)\;Q_n^{m}(q_j)\;[A_{jnm}\cos m\theta + B_{jnm}\sin m\theta]$$

On the surface of a spheroidal cavity ($q_j = \rho_j$ and $p_j = p$), the components of the displacement vector become:

$$u_r = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \{[\alpha_{nm}\cos m\theta - \bar{\alpha}_{nm}\sin m\theta]\ P_n^{-(m+1)}(p)$$

$$+ [\beta_{nm}\cos m\theta + \bar{\beta}_{nm}\sin m\theta]\ P_n^{-(m-1)}(p)\}$$

$$u_\theta = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \{[\alpha_{nm}\sin m\theta + \bar{\alpha}_{nm}\cos m\theta]\ P_n^{-(m+1)}(p) \qquad (5.2.4)$$

$$+ [-\beta_{nm}\sin m\theta + \bar{\beta}_{nm}\cos m\theta]\ P_n^{-(m-1)}(p)\}$$

$$u_z = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} [\gamma_{nm}\cos m\theta + \bar{\gamma}_{nm}\sin m\theta]\ P_n^{-m}(p)$$

where

$$\alpha_{nm} = -Q_n^{(m+1)}(\rho_1)\ A_{1nm} - Q_n^{(m+1)}(\rho_2)\ A_{2nm} + Q_n^{(m+1)}(\rho_3)\ A_{3nm}$$

$$\beta_{nm} = Q_n^{(m-1)}(\rho_1)\ A_{1nm} + Q_n^{(m-1)}(\rho_2)\ A_{2nm} + Q_n^{(m-1)}(\rho_3)\ A_{3nm}$$

$$\gamma_{nm} = \frac{2k_1}{\nu_1}\ Q_n^m(\rho_1)\ A_{1nm} + \frac{2k_2}{\nu_2}\ Q_n^m(\rho_2)\ A_{2nm}$$

$$(5.2.5)$$

$$\bar{\alpha}_{nm} = Q_n^{(m+1)}(\rho_1)\ B_{1nm} + Q_n^{(m+1)}(\rho_2)\ B_{2nm} + Q_n^{(m+1)}(\rho_3)\ B_{3nm}$$

$$\bar{\beta}_{nm} = Q_n^{(m-1)}(\rho_1)\ B_{1nm} + Q_n^{(m-1)}(\rho_2)\ B_{2nm} - Q_n^{(m-1)}(\rho_3)\ B_{3nm}$$

$$\bar{\gamma}_{nm} = \frac{2k_1}{\nu_1}\ Q_n^m(\rho_1)\ B_{1nm} + Q_n^m(\rho_2)\ B_{2nm}$$

With respect to rectangular Cartisian coordinates, the components of the displacement vector (u_x, u_y, u_z) can be written in terms of (u_r, u_θ, u_z) as follows:

$$u_x = u_r \cos\theta - u_\theta \sin\theta$$

$$u_y = u_r \sin\theta + u_\theta \cos\theta \qquad (5.2.6)$$

$$u_z = u_z$$

Substituting Eqs. (5.2.4) into Eqs. (5.2.6) we obtain:

$$u_x = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \{[\alpha_{nm} \cos(m+1)\theta - \bar{\alpha}_{nm} \sin(m+1)\theta]\, P_n^{-(m+1)}(p) + [\beta_{nm} \cos(m-1)\theta + \bar{\beta}_{nm} \sin(m-1)\theta]\, P_n^{-(m-1)}(p)\}$$

$$u_y = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \{[\alpha_{nm} \sin(m+1)\theta + \bar{\alpha}_{nm} \cos(m+1)\theta]\, P_n^{-(m+1)}(p) + [-\beta_{nm} \sin(m-1)\theta + \bar{\beta}_{nm} \cos(m-1)\theta]\, P_n^{-(m-1)}(p)\} \qquad (5.2.7)$$

$$u_z = \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} [\gamma_{nm} \cos m\theta + \bar{\gamma}_{nm} \sin m\theta]\, P_n^{-m}(p)$$

If a displacement vector on the surface of the spheriodal cavity is prescribed, then the components of that vector, in Cartesian coordinates, u, v, and w can be represented by a series expansion of spherical harmonics as follows:

$$u = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (u_{nm} \cos m\theta + \bar{u}_{nm} \sin m\theta)\, P_n^{-m}(p)$$

$$v = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (v_{nm} \cos m\theta + \bar{v}_{nm} \sin m\theta)\, P_n^{-m}(p) \qquad (5.2.8)$$

$$w = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (w_{nm} \cos m\theta + \bar{w}_{nm} \sin m\theta)\, P_n^{-m}(p)$$

where $P_n^{-m}(p)$ is the Legendre associated function, defined by Eq. (4.3.2), satisfying the following orthogonality condition

$$\int_{-1}^{1} P_i^{-m}(p)\, P_j^{-m}(p)\, dp = \frac{\delta_{ij}}{(i+\frac{1}{2})} \frac{(i-m)!}{(i+m)!} \tag{5.2.9}$$

Using this orthogonality property and that of the trigonometric functions, the coefficients in Eqs. (5.2.8) can be evaluated by:

$$u_{nm} = \frac{(n+m)!}{(n-m)!} \frac{(2n+1)}{2\pi\lambda_m} \int_0^{2\pi} d\theta \int_0^1 u(p,\theta)\, P_n^{-m}(p) \cos m\theta\, dp$$

$$\bar{u}_{nm} = \frac{(n+m)!}{(n-m)!} \frac{(2n+1)}{2\pi\lambda_m} \int_0^{2\pi} d\theta \int_0^1 u(p,\theta)\, P_n^{-m}(p) \sin m\theta\, dp \tag{5.2.10}$$

in which $\lambda_o = 2$ and $\lambda_m = 1$ for $m \neq 0$. The coefficients v_{nm}, $\bar{v}_{nm}$, w_{nm}, and $\bar{w}_{nm}$ can be evaluated in a similar manner. Equating coefficients in the identical trigonometric and spherical functions in (5.2.7) to those in (5.2.8) yields

$$\alpha_{nm} = \frac{\pi_{nm}}{2} (u_{nm+1} + \bar{v}_{nm+1})$$

$$\bar{\alpha}_{nm} = \frac{\pi_{nm}}{2} (v_{nm+1} - \bar{u}_{nm+1})$$

$$\pi_{nm} = \begin{cases} \frac{1}{2} & \text{for } m = 0 \\ 1 & \text{for } m < n \\ 0 & \text{for } m = n, n+1 \end{cases}$$

$$\beta_{n0} = -\frac{1}{n(n+1)} \alpha_{n0}$$

$$\beta_{n1} = u_{n0}$$

$$\beta_{nm} = \frac{1}{2} (u_{nm-1} - \bar{v}_{nm-1}) \qquad \text{for} \quad 2 \leq m \leq n+1$$

$$(5.2.11)$$

$$\bar{\beta}_{n1} = v_{n0}$$

$$\bar{\beta}_{nm} = \frac{1}{2} (v_{nm-1} + \bar{u}_{nm-1}) \qquad \text{for} \quad 2 \leq m \leq n+1$$

$$\gamma_{nm} = w_{nm} \quad \text{for} \quad m \leq n \quad \text{and} \quad \gamma_{nm} = 0 \quad \text{for} \quad m = n+1$$

$$\bar{\gamma}_{nm} = \bar{w}_{nm} \quad \text{for} \quad m \leq n \quad \text{and} \quad \bar{\gamma}_{nm} = 0 \quad \text{for} \quad m = n+1$$

Equations (5.2.11) can be written in matrix forms as follows

$$\begin{bmatrix} -Q_n^{(m+1)}(\rho_1) & -Q_n^{(m+1)}(\rho_2) & Q_n^{(m+1)}(\rho_3) \\ Q_n^{(m-1)}(\rho_1) & Q_n^{(m-1)}(\rho_2) & Q_n^{(m-1)}(\rho_3) \\ \frac{2k_1}{\nu_1} Q_n^m(\rho_1) & \frac{2k_2}{\nu_2} Q_n^m(\rho_2) & 0 \end{bmatrix} \begin{bmatrix} A_{1nm} \\ A_{2nm} \\ A_{3nm} \end{bmatrix} = [R_{nm}] \qquad (5.2.12a)$$

$$\begin{bmatrix} Q_n^{(m+1)}(\rho_1) & Q_n^{(m+1)}(\rho_2) & Q_n^{(m+1)}(\rho_3) \\ Q_n^{(m-1)}(\rho_1) & Q_n^{(m-1)}(\rho_2) & -Q_n^{(m-1)}(\rho_3) \\ \frac{2k_1}{\nu_1} Q_n^m(\rho_1) & \frac{2k_2}{\nu_2} Q_n^m(\rho_2) & 0 \end{bmatrix} \begin{bmatrix} B_{1nm} \\ B_{2nm} \\ B_{3nm} \end{bmatrix} = [\bar{R}_{nm}] \qquad (5.2.12b)$$

in which $[R_{nm}]$ and $[\bar{R}_{nm}]$ are 3 x 1 matrices defined, for different values of m, as follows:

for $m = 1$

$$[R_{n1}] = \begin{bmatrix} \frac{1}{2}(u_{n2} + \bar{v}_{n2}) \\ u_{n0} \\ w_{n1} \end{bmatrix}, \qquad [\bar{R}_{n1}] = \begin{bmatrix} \frac{1}{2}(v_{n2} - \bar{u}_{n2}) \\ v_{n0} \\ \bar{w}_{n1} \end{bmatrix} \qquad (5.2.13a)$$

for $2 \leq m \leq n-1$

$$[R_{nm}] = \begin{bmatrix} \frac{1}{2}(u_{nm+1} + \bar{v}_{nm+1}) \\ \frac{1}{2}(u_{nm-1} - \bar{v}_{nm-1}) \\ w_{nm} \end{bmatrix}, \qquad [\bar{R}_{nm}] = \begin{bmatrix} \frac{1}{2}(v_{nm+1} - \bar{u}_{nm+1}) \\ \frac{1}{2}(v_{nm-1} + \bar{u}_{nm-1}) \\ \bar{w}_{nm} \end{bmatrix} \qquad (5.2.13b)$$

for $m = n \neq 1$

$$[R_{nn}] = \begin{bmatrix} 0 \\ \frac{1}{2}(u_{nn-1} - \bar{v}_{nn-1}) \\ w_{nn} \end{bmatrix}, \qquad [\bar{R}_{nn}] = \begin{bmatrix} 0 \\ \frac{1}{2}(v_{nn-1} + \bar{u}_{nn-1}) \\ \bar{w}_{nn} \end{bmatrix} \qquad (5.2.13c)$$

for m = n + 1

$$[R_{nn+1}] = \begin{bmatrix} 0 \\ \frac{1}{2}(u_{nn} - \bar{v}_{nn}) \\ 0 \end{bmatrix} \quad , \quad [\bar{R}_{nn+1}] = \begin{bmatrix} 0 \\ \frac{1}{2}(v_{nn} + \bar{u}_{nn}) \\ 0 \end{bmatrix} \quad (5.2.13.d)$$

If $m = 0$, the stress field becomes independent of θ. This is the axisymmetric case which will be discussed in the next section. Equations (5.2.12a) and (5.2.12b) have a unique solution if the determinant Δ is not zero:

$$\begin{aligned} \Delta = \frac{2k_2}{\nu_2} Q_n^m(\rho_2) & [Q_n^{(m+1)}(\rho_1) Q_n^{(m-1)}(\rho_3) + Q_n^{(m-1)}(\rho_1) Q_n^{(m+1)}(\rho_3) \\ - \frac{2k_1}{\nu_1} Q_n^m(\rho_1) & [Q_n^{(m+1)}(\rho_2) Q_n^{(m-1)}(\rho_3) + Q_n^{(m-1)}(\rho_2) Q_n] \end{aligned} \quad (5.2.14)$$

As mentioned in the previous chapter, the Legendre associated function of the first kind defined by (4.3.2) is singular at $p = 1$ and -1 for the two cases corresponding to $m = n$ and $m = n+1$. However, the equations $\alpha_{nn} = \alpha_{nn+1} = \bar{\alpha}_{nn} = \alpha_{nn+1} = \gamma_{nn+1} = \bar{\gamma}_{nn+1} = 0$ automatically remove these singularities inside the medium.

5.3 The Axisymmetric Problems

Consider the class of problems in which the potentials do not depend upon θ, therefore the first boundary value problem can be deduced directly from the preceeding section by taking $m = 0$ for which the potential functions become:

$$\Phi_j = \sum_{n=0}^{\infty} \frac{2\alpha_j}{(2n+1)} A_{jn0} \left[P_{n+1}(p_j)\, Q_{n+1}(q_j) - P_{n-1}(p_j)\, Q_{n-1}(q_j)\right] \qquad (5.3.1)$$

$$\psi = 0$$

$$j = 1,2$$

and the displacements in cylindrical coordinates take the form:

$$u_{rj} = \sum_{n=0}^{\infty} -2\, A_{jn0}\, P_n^{-1}(p_j)\, Q_n^{1}(q_j)$$

$$u_{zj} = \sum_{n=0}^{\infty} \frac{2k_j}{\nu_j} A_{jn0}\, P_n(p_j)\, Q_n(q_j) \qquad (5.3.2)$$

$$u_{\theta j} = 0$$

The components of the prescribed displacement vector on the surface of the spheroidal cavity, in the axisymmetric case, can be represented in the forms:

$$u = \sum_{n=0}^{\infty} u_{n1}\, P_n^{-1}(p)$$

$$w = \sum_{n=0}^{\infty} w_{n0}\, P_n(p) \qquad (5.3.3)$$

Here u and w are the components of the displacement vector in the r- and z-direction. Expressing Eqs. (5.3.2) on the surface of the spheroid and equating their coefficients to those in Eqs. (5.3.3) we obtain:

$$Q_n^1(\rho_1)\ A_{1n0} + Q_n^1(\rho_2)\ A_{2n0} = -\frac{1}{2}\,u_{n1}$$

$$\frac{k_1}{\nu_1}\,Q_n(\rho_1)\ A_{1n0} + Q_n(\rho_2)\ A_{2n0} = \frac{1}{2}\,w_{n0} \tag{5.3.4}$$

By solving (5.3.4) we determine the unknowns A_{1n0} and A_{2n0} which enable us to determine the stress field without difficulty.

To illustrate the method developed in this chapter, we will consider the problem of a rigid spheroidal inclusion embedded in a transversely isotropic medium when the medium is subjected to the following loads at large distances from the inclusion:

(i) Uniaxial tension in the direction of the axis of symmetry of the medium.

(ii) Pure shearing stress in the plane perpendicular to the axis of symmetry of the medium.

5.4 Rigid Spheroidal Inclusion Under Axisymmetric Uniaxial Tension

The boundary conditions in this case are:

$$\sigma_{zz} = T_0\ ,\quad \sigma_{rr} = \sigma_{\theta\theta} = \sigma_{rz} = 0 \quad \text{at infinity} \tag{5.4.1}$$

$$u_r = u_\theta = u_z = 0 \quad \text{on the surface of spheroid} \tag{5.4.2}$$

In the absence of the inclusion, the uniform stress field (Eqs. 5.4.1) can be extended throughout the space, thereby violating the boundary conditions in Eqs. (5.4.2). The stress and displacement field are found to be:

$$\sigma_{zz} = T_0\ ,\quad \sigma_{rr} = \sigma_{\theta\theta} = \sigma_{rz} = 0 \tag{5.4.3}$$

$$u_r = - \frac{c_{13}\ T_0\ r}{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]} \quad , \quad u_z = \frac{(c_{11} + c_{12})\ T_0\ z}{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]} \qquad (5.4.4)$$

We now seek a solution which, upon superposition on the uniform field, removes the residual displacements, given by Eqs. (5.4.4), on the surface of the spheroid. Therefore, consider the problem where the negative of the displacements given by Eqs. (5.4.4) are specified on the boundary of the spheroidal inclusion. These displacements are:

$$u_r = \frac{c_{13}\ T_0}{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]}\ r = \frac{2c_{13}\ T_0\ b}{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]}\ P_1^{-1}(p)$$

$$w = \frac{-(c_{11} + c_{12})\ T_0}{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]}\ z = \frac{-(c_{11} + c_{12})\ T_0\ a}{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]}\ P_1(p) \qquad (5.4.5)$$

in which a and b are the semiaxes of the spheroid. From Eqs. (5.3.3) it is easily found that the only non-zero coefficients are those corresponding to $n = 1$. Thus,

$$u_{11} = \frac{2c_{13}\ T_0\ b}{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]}$$

$$w_{10} = \frac{-(c_{11} + c_{12})\ T_0\ a}{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]} \qquad (5.4.6)$$

Equations (5.3.4) become in this case:

$$\begin{bmatrix} Q_1^1(\rho_1) & Q_1^1(\rho_2) \\ \\ \frac{k_1}{\nu_1} Q_1(\rho_1) & \frac{k_2}{\nu_2} Q_1(\rho_2) \end{bmatrix} \begin{bmatrix} A_{110} \\ \\ A_{210} \end{bmatrix} = \begin{bmatrix} - \dfrac{c_{13}\ T_0\ b}{[c_{33}(c_{11}+c_{12}) - 2c_{13}^2]} \\ \\ - \dfrac{(c_{11}+c_{12})\ T_0\ a}{2[c_{33}(c_{11}+c_{12}) - 2c_{13}^2]} \end{bmatrix} \quad (5.4.7)$$

Solving the simultaneous equations we obtain:

$$A_{110} = \frac{T_0}{2\,\Delta\,[c_{33}(c_{11}+c_{12}) - 2c_{13}^2]}\ [-\frac{2c_{13}k_2}{\nu_2}\ b\ Q_1(\rho_2) + (c_{11}+c_{12})\ a\ Q_1^1(\rho_2)]$$

$$A_{210} = \frac{-T_0}{2\,\Delta\,[c_{33}(c_{11}+c_{12}) - 2c_{13}^2]}\ [-\frac{2c_{13}k_1}{\nu_1}\ b\ Q_1(\rho_1) + (c_{11}+c_{12})\ a\ Q_1^1(\rho_1)] \quad (5.4.8)$$

where

$$\Delta = [\frac{k_2}{\nu_2} Q_1(\rho_2)\ Q_1^1(\rho_1) - \frac{k_1}{\nu_1} Q_1(\rho_1)\ Q_1^1(\rho_2)]$$

The potential functions, therefore, are

$$\phi_j = \frac{2\alpha_j}{3} A_{j10}[P_2(p_j)\ Q_2(q_j) - Q_0(q_j)] \qquad (j=1,2) \qquad (5.4.9)$$

Substituting Eqs. (5.4.9) into Eqs. (3.2.13) and (3.2.14) and adding the solution obtained in the absence of the inclusion, we obtain the desired solution for which the displacements and stresses become:

$$u_r = -\left[\frac{c_{13}\ T_0}{[c_{33}(c_{11}+c_{12}) - 2c_{13}^2} + \frac{A_{110}}{\alpha_1} \frac{Q_1^1(q_1)}{(q_1^2-1)^{1/2}} + \frac{A_{210}}{\alpha_2} \frac{Q_1^1(q_2)}{(q_2^2-1)^{1/2}} \right] r$$

$$u_z = \left[\frac{(c_{11}+c_{12})T_0}{[c_{33}(c_{11}+c_{12}) - 2c_{13}^2]} + \frac{2k_1\ A_{110}}{\alpha_1\ \nu_1^2} \frac{Q_1(q_1)}{q_1} + \frac{2k_2\ A_{210}}{\alpha_2\ \nu_2^2} \frac{Q_1(q_2)}{q_2} \right] z$$

$$u_\theta = 0$$

$$\sigma_{rr} = \sum_{j=1}^{2} 2c_{44} \frac{A_{j10}}{\alpha_j} \left\{ -\frac{(1+k_j)}{\nu_j^2} \left[\frac{1}{2}\ \mathrm{Ln}\ \frac{q_j+1}{q_j-1} - \frac{q_j}{(q_j^2-p_j^2)} \right] + \frac{1}{\nu_3^2} \frac{Q_1^1(q_j)}{(q_j^2-1)^{1/2}} \right\} \tag{5.4.10}$$

$$\sigma_{\theta\theta} = \sum_{j=1}^{2} 2c_{44} \frac{A_{j10}}{\alpha_j} \left\{ -\frac{(1+k_j)}{\nu_j^2} \left[\frac{1}{2}\ \mathrm{Ln}\ \frac{q_j+1}{q_j-1} - \frac{q_j}{(q_j^2-p_j^2)} \right] + \frac{1}{\nu_3^2} \left[\frac{Q_1^1(q_j)}{(q_j^2-1)^{1/2}} + \frac{2\,q_j(1-p_j^2)}{(q_j^2-1)(q_j^2-p_j^2)} \right] \right\}$$

$$\sigma_{zz} = T_0 + \sum_{j=1}^{2} 2c_{44} \frac{A_{j10}(1+k_j)}{\alpha_j\ q_j} \left[\frac{q_j\ Q_1^1\ (q_j)}{(q_j^2-1)^{1/2}} + \frac{q_j^2(1-p_j^2)}{(q_j^2-1)(q_j^2-p_j^2)} \right]$$

$$\sigma_{rz} = \sum_{j=1}^{2} - 2c_{44} \frac{A_{j10}(1+k_j)}{\alpha_j\ \nu_j} \frac{p_j(1-p_j^2)^{1/2}}{(q_j^2-1)^{1/2}(q_j^2-p_j^2)}$$

5.5 Rigid Spheroidal Inclusion Under Pure Shearing Stress in the Plane of Isotropy

The boundary conditions in this case are:

$$\sigma_{xy} = \tau_0 \quad , \quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \quad \text{at infinity} \tag{5.5.1}$$

$$u_x = u_y = u_z = 0 \quad \text{on the surface of the spheroid} \tag{5.5.2}$$

the loading conditions (5.5.1) is equivalent, in cylindrical coordinates, to

$$\sigma_{rr} = -\sigma_{\theta\theta} = \tau_0 \sin 2\theta \; , \quad \sigma_{r\theta} = \tau_0 \cos 2\theta \; , \quad \sigma_{zz} = \sigma_{rz} = \sigma_{z\theta} = 0 \tag{5.5.3}$$

In the absence of the inclusion, Eqs. (5.5.1) or (5.5.3) represent the stress field in the medium. The displacement field, accordingly, is found to be:

$$u_x = \frac{\tau_0}{2c_{66}}\, y \quad , \quad u_y = \frac{\tau_0}{2c_{66}}\, x \quad , \quad u_z = 0 \tag{5.5.4}$$

Consider the problem where the negative of these displacements are specified on the boundary of the spheroid. Thus,

$$\begin{aligned} u &= -\frac{\tau_0}{2c_{66}}\, y = -\frac{\tau_0}{2c_{66}}\, b(1-p^2)^{\frac{1}{2}} \sin\theta = -\frac{\tau_0 b}{c_{66}}\, P_1^{-1}(p)\sin\theta \\ v &= -\frac{\tau_0}{2c_{66}}\, x = -\frac{\tau_0}{2c_{66}}\, b(1-p^2)^{\frac{1}{2}} \cos\theta = -\frac{\tau_0 b}{c_{66}}\, P_1^{-1}(p)\cos\theta \\ w &= 0 \end{aligned} \tag{5.5.5}$$

Comparing Eqs. (5.5.5) to Eqs. (5.2.8), we find that the only non-zero coefficients are:

$$\bar{u}_{11} = -\frac{\tau_0 b}{c_{66}} \quad , \quad v_{11} = -\frac{\tau_0 b}{c_{66}} \tag{5.5.6}$$

By inspecting Eqs. (5.2.11) and (5.2.13) we conclude that the desired problem is corresponding to n = 1, m = 2, $A_{j12} = 0$, and B_{j12} to be determined. Therefore, Eqs. (5.2.12B) become:

$$\begin{bmatrix} Q_1^3(\rho_1) & Q_1^3(\rho_1) & Q_1^3(\rho_3) \\ Q_1^1(\rho_1) & Q_1^1(\rho_2) & -Q_1^1(\rho_3) \\ \frac{2k_1}{\nu_1} Q_1^2(\rho_1) & \frac{2k_2}{\nu_2} Q_1^2(\rho_2) & 0 \end{bmatrix} \begin{bmatrix} B_{112} \\ B_{212} \\ B_{312} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{\tau_0 b}{c_{66}} \\ 0 \end{bmatrix} \tag{5.5.7}$$

which have the solution:

$$B_{112} = -\frac{k_2 \nu_1}{(k_1 - k_2)D} \frac{\tau_0 b}{c_{66}}$$

$$B_{212} = \frac{k_1 \nu_2}{(k_1 - k_2)D} \frac{\tau_0 b}{c_{66}} \tag{5.5.8}$$

$$B_{312} = -\frac{1}{\alpha_3^2 \nu_3} \frac{\tau_0 b}{c_{66}}$$

where

$$D = \frac{k_2 \nu_1 Q_1^1(\rho_1)}{\alpha_1^2 \nu_3^2 (k_1 - k_2)} - \frac{k_1 \nu_2 Q_1^1(\rho_2)}{\alpha_2^2 \nu_3^2 (k_1 - k_2)} - \frac{Q_1^1(\rho_3)}{\alpha_3^2 \nu_3}$$

The potential functions corresponding to the problem at hand are:

$$\phi_j = \frac{2\alpha_j}{3} B_{j12}[P_2^{-2}(p_j)\, Q_2^2(q_j) - P_0^{-2}(p_j)\, Q_0^2(q_j)] \sin 2\theta$$
$$\psi = \frac{2\alpha_3}{3} B_{312}[P_2^{-2}(p_3)\, Q_2^2(q_3) - P_0^{-2}(p_3)\, Q_0^2(q_3)] \cos 2\theta \qquad (5.5.9)$$

Substituting into Eqs. (3.2.2) and adding the solution of that in the absence of the cavity, we obtain the following displacements and stresses:

$$u_r = \{ \frac{\tau_0 r}{c_{66}} + \frac{r}{2} [- \frac{B_{312}}{\alpha_3} \frac{Q_1^1(q_3)}{(q_3^2 - 1)^{\frac{1}{2}}} + \sum_{j=1}^{2} \frac{B_{j12}}{\alpha_j} \frac{Q_1^1(q_j)}{(q_j^2 - 1)^{\frac{1}{2}}}]$$
$$- \frac{1}{3r^3} \sum_{j=1}^{3} B_{j12}\, \alpha_j^2\, q_j (p_j^4 - 6p_j^2 + 8p_j - 3) \}\sin 2\theta$$

$$u_\theta = \{- \frac{\tau_0 r}{c_{66}} + \frac{r}{2} [- \frac{B_{312}}{\alpha_3} \frac{Q_1^1(q_3)}{(q_3^2 - 1)^{\frac{1}{2}}} + \sum_{j=1}^{2} \frac{B_{j12}}{\alpha_j} \frac{Q_1^1(q_j)}{(q_j^2 - 1)^{\frac{1}{2}}}]$$
$$+ \frac{1}{3r^3} \sum_{j=1}^{3} B_{j12}\, \alpha_j^2\, q_j (p_j^4 - 6p_j^2 + 8p_j - 3) \}\cos 2\theta$$

$$u_z = \frac{2}{3r^2} \sum_{j=1}^{2} B_{j12} \frac{k_j\, \alpha_j^2}{\nu_j} (p_j^3 - 3p_j + 2)\sin 2\theta$$

$$\sigma_{rr} = \{\tau_0 + 2c_{44}\, r^2 \sum_{j=1}^{2} \frac{B_{j12}(1+k_j)}{\alpha_j^3\, \nu_j^2} \frac{q_j}{(q_j^2-1)^2(q_j^2-p_j^2)}$$

$$+ c_{66}[- \frac{B_{312}}{\alpha_3} \frac{Q_1^1(q_3)}{(q_3^2-1)^{\frac{1}{2}}} + \sum_{j=1}^{2} \frac{B_{j12}}{\alpha_j} \frac{Q_1^1(q_j)}{(q_j^2-1)^{\frac{1}{2}}}]$$

$$+ \frac{2c_{66}}{r^4} \sum_{j=1}^{3} B_{j12}\, \alpha_j^2\, q_j (p_j^4 - 6p_j^2 + 8p_j - 3)\}\sin 2\theta$$

$$\sigma_{\theta\theta} = \{-\tau_0 + 2c_{44}\, r^2 \sum_{j=1}^{2} \frac{B_{j12}}{\alpha_j^3} (\frac{1+k_j}{\nu_j^2} - \frac{2}{\nu_3^2}) \frac{q_j}{(q_j^2-1)^2(q_j^2-p_j^2)}$$

$$- c_{66}[- \frac{B_{312}}{\alpha_3} \frac{Q_1^1(q_3)}{(q_3^2-1)^{\frac{1}{2}}} + \sum_{j=1}^{2} \frac{B_{j12}}{\alpha_j} \frac{Q_1^1(q_j)}{(q_j^2-1)^{\frac{1}{2}}}$$

$$- \frac{2c_{66}}{r^4} \sum_{j=1}^{3} B_{j12}\, \alpha_j^2\, q_j (p_j^4 - 6p_j^2 + 8p_j - 3)\}\sin 2\theta$$

$$\sigma_{zz} = -2c_{44}\, r^2 \sum_{j=1}^{2} \frac{B_{j12}(1+k_j)}{\alpha_j^3} \frac{q_j}{(q_j^2-1)^2(q_j^2-p_j^2)} \sin 2\theta$$

$$\sigma_{rz} = - \frac{4}{3} c_{44} \{ \frac{1}{r^3} \sum_{j=1}^{3} \frac{B_{j12}\, \alpha_j^2(1+k_j)}{\nu_j} (p_j^3 - 3p_j + 2)$$

$$- \frac{3}{2} r \sum_{j=1}^{2} \frac{B_{j12}\, (1+k_j)}{\alpha_j^2\, \nu_j} \frac{p_j}{(q_j^2-1)(q_j^2-p_j^2)} \} \sin 2\theta$$

$$\sigma_{\theta z} = 2c_{44} \{-r \frac{B_{312}}{\alpha_3^2 \nu_3} \frac{p_3}{(q_3^2 - 1)\ (q_3^2 - p_3^2)}$$

$$+ \frac{2}{3r^3} \sum_{j=1}^{3} \frac{B_{j12}\ \alpha_j^2 (1 + k_j)}{\nu_j} (p_j^3 - 3p_j + 2)\} \cos 2\theta$$

$$\sigma_{r\theta} = \{\tau_0 - 2c_{66}\ r^2 \frac{B_{312}}{\alpha_3^2} \frac{q_3}{(q_3^2 - 1)^2 (q_3^2 - p_3^2)} - c_{66} \frac{B_{312}}{\alpha_3} \frac{Q_1^1(q_3)}{(q_3^2 - 1)^{\frac{1}{2}}} \tag{5.5.10}$$

$$+ c_{66} \sum_{j=1}^{2} \frac{B_{j12}}{\alpha_j} \frac{Q_1^1(q_j)}{(q_j^2 - 1)^{\frac{1}{2}}}$$

$$+ \frac{2c_{66}}{r^4} \sum_{j=1}^{3} B_{j12}\ \alpha_j^3\ q_j\ (p_j^4 - 6p_j^2 + 8p_j - 3)\} \cos 2\theta$$

At the spheroid interface, these equations may be reduced to:

$$u_r = u_\theta = u_z = 0$$

$$\sigma_{rr} = 2c_{44} \frac{ar^2}{b^4} \sum_{j=1}^{2} \frac{B_{j12}(1 + k_j)}{\nu_j^3\ (\rho_j^2 - p^2)} \sin 2\theta$$

$$\sigma_{\theta\theta} = 2c_{44} \frac{ar^2}{b^4} \sum_{j=1}^{2} \frac{B_{j12}}{\nu_j} \left(\frac{1 + k_j}{\nu_j^2} - \frac{2}{\nu_3^2} \right) \frac{1}{(\rho_j^2 - p^2)} \sin 2\theta$$

$$\sigma_{zz} = -2c_{44} \frac{ar^2}{b^4} \sum_{j=1}^{2} \frac{B_{j12}(1 + k_j)}{\nu_j(\rho_j^2 - p^2)} \sin 2\theta$$

$$\sigma_{rz} = 2c_{44} \frac{r}{b^2} \sum_{j=1}^{2} \frac{B_{j12}(1 + k_j)}{\nu_j} \frac{p}{(\rho_j^2 - p^2)} \sin 2\theta$$

$$\sigma_{\theta z} = -2c_{44} \frac{r}{b^2} \frac{B_{312}}{\nu_3} \frac{p}{(\rho_3^2 - p^2)} \cos 2\theta$$

$$\sigma_{r\theta} = -2c_{66}\,\frac{ar^2}{b^4}\,\frac{B_{312}}{\nu_3}\,\frac{1}{(\rho_3^2 - p^2)}\cos 2\theta \qquad (5.5.11)$$

If we obtain a transformation of the stress components from cylindrical coordinates to spheroidal coordinates by means of the equations in Appendix B, we obtain

$$\sigma_{\eta\eta} = \frac{2}{a\,b\,D}\,\frac{r^2}{r^2 + s^4 z^2}\,\tau_0 \sin 2\theta$$

$$\sigma_{\phi\phi} = \frac{2}{a\,b\,D}\,\frac{r^2}{(k_1 - k_2)}\left[\frac{(1+k_1)(1-\nu_2^2)}{(r^2 + \nu_2^2 s^4 z^2)} - \frac{(1+k_2)(1-\nu_1^2)}{(r^2 + \nu_1^2 s^4 z^2)} - \frac{(k_1 - k_2)}{(r^2 + s^4 z^2)}\right]\tau_0 \sin 2\theta \qquad (5.5.12)$$

$$\sigma_{\theta\theta} = \frac{2}{a\,b\,D}\,\frac{r^2}{(k_1 - k_2)}\left[\frac{k_1[\nu_3^2(1+k_2) - 2\nu_2^2]}{(r^2 + \nu_2^2 s^4 z^2)} - \frac{k_2[\nu_3^2(1+k_1) - 2\nu_1^2]}{(r^2 + \nu_1^2 s^4 z^2)}\right]\tau_0 \sin 2\theta$$

$$\sigma_{\eta\phi} = \frac{2s}{a^2 D}\,\frac{rz}{r^2 + s^4 z^2}\,\tau_0 \sin 2\theta$$

$$\sigma_{\eta\theta} = \frac{2}{a\,b\,D}\,\frac{r}{\sqrt{r^2 + s^4 z^2}}\,\tau_0 \cos 2\theta$$

$$\sigma_{\phi\theta} = \frac{2S}{a^2 D}(1 - \nu_3^2)\,\frac{zr^2}{(r^2 + \nu_3^2 s^4 z^2)\sqrt{r^2 + s^4 z^2}}\,\tau_0 \cos 2\theta$$

where D is defined by Eq. (5.5.8) and $s = \frac{b}{a}$.

In a special case where the medium is isotropic, Eqs. (5.5.12) become after taking the proper limit:

$$\sigma_{\eta\eta} = \frac{8(1-\nu)}{D_0} \frac{r^2}{r^2 + s^4 z^2} \tau_0 \sin 2\theta$$

$$\sigma_{\phi\phi} = \frac{8\nu}{D_0} \frac{r^2}{r^2 + s^4 z^2} \tau_0 \sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{8\nu}{D_0} \frac{r^2}{r^2 + s^4 z^2} \tau_0 \sin 2\theta \qquad (5.5.13)$$

$$\sigma_{\eta\phi} = \frac{8(1-\nu)s^2}{D_0} \frac{rz}{r^2 + s^4 z^2} \tau_0 \sin 2\theta$$

$$\sigma_{\eta\theta} = \frac{8(1-\nu)}{D_0} \frac{r}{\sqrt{r^2 + s^4 z^2}} \tau_0 \cos 2\theta \, .$$

$$\sigma_{\phi\theta} = 0$$

where

$$D_0 = (\rho_0^2 - 1)(3\rho_0^2 - 7 + 8\nu)\, Q_1(\rho_0) - (\rho_0^2 + 8\nu - 7)$$

$$\rho_0 = \frac{1}{\sqrt{1 - s^2}} \quad , \quad s = \frac{b}{a} \qquad (5.5.14)$$

For a rigid spherical inclusion we must evaluate the limit of D_0 as $\rho_0 \to \infty$, therefore:

$$\lim_{\rho_0 \to \infty} D_0 = \frac{16}{15}(4 - 5\nu) \qquad (5.5.15)$$

Substituting the limiting value of D_0 into Eqs. (5.5.13), we obtain the stresses in the matrix at the spherical inclusion interface.

CHAPTER 6

THE SECOND BOUNDARY VALUE PROBLEM

6.1 Introduction

The second boundary value problem consists of finding stresses and displacements of an elastic body in equilibrium when the body forces are known and the surface forces are prescribed. In succeeding sections of this chapter the second boundary value problem is solved for equilibrium of an elastic transversely isotropic medium containing a spheroidal cavity. Examples are given for a variety of constant loadings applied at sufficiently large distances from the unloaded cavity in the absence of body forces.

6.2 Fundamental Formulae

Let n_r and n_z denote the components of the unit normal (direction cosines) to the spheroidal surface for which $q_j = \rho_j$. The n's may be evaluated from the expressions

$$n_r = \frac{a}{b\lambda} r \quad , \qquad n_z = \frac{b}{a\lambda} z \tag{6.2.1}$$

in which

$$\lambda = \sqrt{\frac{a^2}{b^2} r^2 + \frac{b^2}{a^2} z^2}$$

The projections t_r, t_z, and t_θ of the traction vector acting upon an area to which the normal is given by the direction cosines n_r and n_z can be expressed as follows

$$\begin{aligned} t_r &= \sigma_{rr}\, n_r + \sigma_{rz}\, n_z \\ t_z &= \sigma_{zr}\, n_r + \sigma_{zz}\, n_z \qquad (6.2.2) \\ t_\theta &= \sigma_{\theta r}\, n_r + \sigma_{\theta z}\, n_z \end{aligned}$$

We shall express these components in terms of the potential functions defined by Eqs. (4.3.1). First, however, we will obtain an expression that will be used frequently in the analysis:

$$\frac{\partial}{\partial q_j} = \frac{\partial}{\partial r}\frac{\partial r}{\partial q_j} + \frac{\partial}{\partial z_j}\frac{\partial z_j}{\partial q_j} = \frac{\alpha_j\, q_j (1-p_j^2)^{\frac{1}{2}}}{(q_j^2-1)^{\frac{1}{2}}}\frac{\partial}{\partial r} + \alpha_j\, p_j \frac{\partial}{\partial z_j} \qquad (6.2.3)$$

On the surface of the spheroidal cavity we have $q_j = \rho_j$ and $p_j = p$. Thus,

$$\left[\frac{\partial}{\partial q_j}\right]_{q_j=\rho_j} = \left[\frac{\alpha_j\, \rho_j (1-p^2)^{\frac{1}{2}}}{(\rho_j^2 - 1)^{\frac{1}{2}}}\frac{\partial}{\partial r} + \alpha_j\, p \frac{\partial}{\partial z_j}\right]_{q_j=\rho_j} \qquad (6.2.4)$$

Using Eqs. (4.2.3) and (6.2.1) we obtain

$$\left[\frac{\partial}{\partial q_j}\right]_{q_j=\rho_j} = \frac{\alpha_j \lambda}{\nu_j\, b}\left[n_r \frac{\partial}{\partial r} + \nu_j\, n_z \frac{\partial}{\partial z_j}\right]_{q_j=\rho_j} \qquad (6.2.5)$$

therefore,

$$[n_r \frac{\partial}{\partial r} + \nu_j n_z \frac{\partial}{\partial z_j}]_{q_j=\rho_j} = \frac{\nu_j b}{\alpha_j \lambda} [\frac{\partial}{\partial q_j}]_{q_j=\rho_j} \tag{6.2.6}$$

in which j = 1, 2, and 3.

6.3 The Components of the Traction Vector Due to ϕ_1 and ϕ_2

Consider now the gradient solution which is based on two potentials ϕ_1 and ϕ_2. The components of the displacement vector due to either part, say ϕ_j, where j = 1, 2, are:

$$u_{rj} = \frac{\partial \phi_j}{\partial r} \quad , \quad u_{\theta j} = \frac{1}{r}\frac{\partial \phi_j}{\partial \theta} \quad , \quad u_{zj} = \frac{k_j}{\nu_j}\frac{\partial \phi_j}{\partial z_j} \tag{6.3.1}$$

Substituting Eqs. (3.2.1) into Eqs. (6.2.2), the components of the traction vector due to the potential ϕ_j become:

$$t_{rj} = [c_{11} \frac{\partial u_{rj}}{\partial r} + c_{12}(\frac{1}{r}\frac{\partial u_{\theta j}}{\partial \theta} + \frac{u_{rj}}{r}) + \frac{c_{13}}{\nu_j}\frac{\partial u_{zj}}{\partial z_j}]_{q_j=\rho_j} \cdot n_r$$
$$+ c_{44}[\frac{1}{\nu_j}\frac{\partial u_{rj}}{\partial z_j} + \frac{\partial u_{zj}}{\partial r}]_{q_j=\rho_j} \cdot n_z$$

$$t_{\theta j} = c_{66}[\frac{1}{r}\frac{\partial u_{rj}}{\partial \theta} + \frac{\partial u_{\theta j}}{\partial r} - \frac{u_{\theta j}}{r}]_{q_j=\rho_j} \cdot n_r$$
$$+ c_{44}[\frac{1}{\nu_j}\frac{\partial u_{\theta j}}{\partial z_j} + \frac{1}{r}\frac{\partial u_{zj}}{\partial \theta}]_{q_j=\rho_j} \cdot n_z \tag{6.3.2}$$

$$t_{zj} = c_{44}[\frac{1}{\nu_j}\frac{\partial u_{rj}}{\partial z_j} + \frac{\partial u_{zj}}{\partial r}]_{q_j=\rho_j} \cdot n_r$$
$$+ [c_{13}\frac{\partial u_{rj}}{\partial r} + c_{13}(\frac{1}{r}\frac{\partial u_{\theta j}}{\partial \theta} + \frac{u_{rj}}{r}) + \frac{c_{33}}{\nu_j}\frac{\partial u_{zj}}{\partial z_j}]_{q_j=\rho_j} \cdot n_z$$

It is not difficult to show that by means of Eqs. (3.2.8), (6.2.6), and (6.3.1), Eqs. (6.3.2) take the form:

$$t_{rj} = \frac{1}{\lambda} \{ c_{44}\, b \frac{(1+k_j)}{\alpha_j \nu_j} [\frac{\partial u_{rj}}{\partial q_j}]_{q_j=\rho_j} + \frac{a}{b} (c_{12} - \frac{c_{13}\, k_j}{\nu_j^2}) + [u_{rj} + \frac{\partial u_{\theta j}}{\partial \theta}]_{q_j=\rho_j} \}$$

$$t_{\theta j} = \frac{1}{\lambda} \{ c_{44}\, b \frac{(1+k_j)}{\alpha_j \nu_j} [\frac{\partial u_{\theta j}}{\partial q_j}]_{q_j=\rho_j} - \frac{a}{b} (c_{12} - \frac{c_{13}\, k_j}{\nu_j^2}) [\frac{\partial u_{rj}}{\partial \theta} - u_{\theta j}]_{q_j=\rho_j} \} \tag{6.3.3}$$

$$t_{zj} = \frac{1}{\lambda} c_{44}\, b \frac{(1+k_j)\nu_j}{\alpha_j k_j} [\frac{\partial u_{zj}}{\partial q_j}]_{q_j=\rho_j}$$

6.4 The Components of the Traction Vector Due to ψ

The components of the displacement vector due to the potential function ψ (the "curl" portion) can be written as:

$$u_{r3} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} , \quad u_{\theta 3} = - \frac{\partial \psi}{\partial r} , \quad u_{z3} = 0 \tag{6.4.1}$$

It can be shown that Eqs. (6.4.1) satisfy the following equation:

$$\frac{1}{r}\frac{\partial u_{\theta 3}}{\partial \theta} + \frac{u_{r3}}{r} = - \frac{\partial u_{r3}}{\partial r} \tag{6.4.2}$$

Substituting Eqs. (3.2.1) into Eqs. (6.2.2) and setting $u_{z3} = 0$, the components of the traction vector due to the potential function ψ become:

$$t_{r3} = [c_{11} \frac{\partial u_{r3}}{\partial r} + \frac{c_{12}}{r} (\frac{\partial u_{\theta 3}}{\partial \theta} + u_{r3})]_{q_3=\rho_3} \cdot n_r$$

$$+ \frac{c_{44}}{\nu_3} [\frac{\partial u_{r3}}{\partial z_3}]_{q_3=\rho_3} \cdot n_z$$

$$t_{\theta 3} = c_{66} [\frac{1}{r} \frac{\partial u_{r3}}{\partial \theta} + \frac{\partial u_{\theta 3}}{\partial r} - \frac{u_{\theta 3}}{r}]_{q_3=\rho_3} n_r + \frac{c_{44}}{\nu_3} [\frac{\partial u_{\theta 3}}{\partial z_3}]_{q_3=\rho_3} \cdot n_z \qquad (6.4.3)$$

$$t_{z3} = \frac{c_{44}}{\nu_3} [\frac{\partial u_{r3}}{\partial z_3}]_{q_3=\rho_3} n_r + c_{13} [\frac{\partial u_{r3}}{\partial r} + \frac{1}{r} \frac{\partial u_{\theta 3}}{\partial \theta} + \frac{u_{r3}}{r}]_{q_3=\rho_3} \cdot n_z$$

By using Eqs. (3.2.8), (6.2.6), (6.4.1), and (6.4.2), it can be shown that Eqs. (6.4.3) have the form:

$$t_{r3} = \frac{1}{\lambda} \{ c_{66} b \frac{\nu_3}{\alpha_3} [\frac{\partial u_{r3}}{\partial q_3}]_{q_3=\rho_3} - c_{66} \frac{a}{b} [u_{r3} + \frac{\partial u_{\theta 3}}{\partial \theta}]_{q_3=\rho_3} \}$$

$$t_{\theta 3} = \frac{c_{66}}{\lambda} \{ b \frac{\nu_3}{\alpha_3} [\frac{\partial u_{\theta 3}}{\partial q_3}]_{q_3=\rho_3} + \frac{a}{b} [\frac{\partial u_{r3}}{\partial \theta} - u_{\theta 3}]_{q_3=\rho_3} \} \qquad (6.4.4)$$

$$t_{z3} = \frac{c_{44}}{\lambda} \frac{a}{b \nu_3} [r \frac{\partial u_{r3}}{\partial z_3}]_{q_3=\rho_3}$$

6.5 The Components of the Traction Vector in Terms of the Legendre Associated Functions

Expressions of the displacement components are given by Eqs. (5.2.3). Substituting these expressions into Eqs. (6.3.3) and (6.4.4) we obtain, after some algebraic manipulation, the following expressions for the components of the traction vector on the surface of the spheroidal cavity:

$$t_{rj} = \frac{1}{\lambda}[M_{jnm} P_n^{-(m+1)}(p) + N_{jnm} P_n^{-(m-1)}(p)][A_{jnm} \cos m\theta + B_{jnm} \sin m\theta]$$

$$t_{r3} = \frac{1}{\lambda}[M_{3nm} P_n^{-(m+1)}(p) + N_{3nm} P_n^{-(m-1)}(p)][A_{3nm} \cos m\theta - B_{3nm} \sin m\theta]$$

$$t_{\theta j} = \frac{1}{\lambda}[-M_{jnm} P_n^{-(m+1)}(p) + N_{jnm} P_n^{-(m-1)}(p)][B_{jnm} \cos m\theta - A_{jnm} \sin m\theta]$$

$$(6.5.1)$$

$$t_{\theta 3} = \frac{1}{\lambda}[M_{3nm} P_n^{-(m+1)}(p) - N_{3nm} P_n^{-(m-1)}(p)][B_{3nm} \cos m\theta + A_{3nm} \sin m\theta]$$

$$t_{zj} = \frac{1}{\lambda} S_{jnm} P_n^{-m}(p) [A_{jnm} \cos m\theta + B_{jnm} \sin m\theta]$$

$$t_{z3} = \frac{1}{\lambda} S_{3nm} P_n^{-m}(p) [A_{3nm} \cos m\theta - B_{3nm} \sin m\theta]$$

where:

$$M_{jnm} = -c_{44}\, b \frac{(1+k_j)}{\alpha_j \nu_j} \left[\frac{\partial Q_n^{(m+1)}(q_j)}{\partial q_j} \right]_{q_j=\rho_j} - \frac{a}{b}\left(c_{12} - \frac{c_{13} k_j}{\nu_j^2}\right)(m+1)\, Q_n^{(m+1)}(\rho_j)$$

$$M_{3nm} = c_{66}\, b \frac{\nu_3}{\alpha_3} \left[\frac{\partial Q_n^{(m+1)}(q_3)}{\partial q_3} \right]_{q_3=\rho_3} - c_{66} \frac{a}{b} (m+1)\, Q_n^{(m+1)}(\rho_3)$$

$$N_{jnm} = c_{44}\, b \frac{(1+k_j)}{\alpha_j \nu_j} \left[\frac{\partial Q_n^{m-1}(q_j)}{\partial q_j} \right]_{q_j=\rho_j}$$

$$- \frac{a}{b} \left(c_{12} - \frac{c_{13} k_j}{\nu_j^2}\right)(m-1)\, Q_n^{(m-1)}(\rho_j)$$

$$N_{3nm} = c_{66}\, b \frac{\nu_3}{\alpha_3} \left[\frac{\partial Q_n^{(m-1)}(q_3)}{\partial q_3} \right]_{q_3=\rho_3} + \frac{a}{b} c_{66} (m-1)\, Q_n^{(m-1)}(\rho_3) \qquad (6.5.2)$$

$$S_{jnm} = 2c_{44}\, b \frac{(1+k_j)}{\alpha_j} \left[\frac{\partial Q_n^{m}(q_j)}{\partial q_j} \right]_{q_j=\rho_j}$$

$$S_{3nm} = 2c_{44} \frac{a}{b\nu_3}\, m\, Q_n^{-m}(\rho_3)$$

By setting

$$t_r = t_{r1} + t_{r2} + t_{r3}$$

$$t_\theta = t_{\theta 1} + t_{\theta 2} + t_{\theta 3} \qquad (6.5.3)$$

$$t_z = t_{z1} + t_{z2} + t_{z3}$$

and using the following transformation:

$$t_x = t_r \cos\theta - t_\theta \sin\theta$$

$$t_y = t_r \sin\theta + t_\theta \cos\theta \qquad (6.5.4)$$

$$t_z = t_z$$

we obtain:

$$t_x = \frac{1}{\lambda} \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \{[\beta_{nm} \cos(m+1)\theta + \bar{\beta}_{nm} \sin(m+1)\theta]\ P_n^{-(m+1)}$$
$$+ [\gamma_{nm} \cos(m-1)\theta + \bar{\gamma}_{nm} \sin(m-1)\theta]\ P_n^{-(m-1)}(p)\}$$

$$t_y = \frac{1}{\lambda} \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} \{[\beta_{nm} \sin(m+1)\theta - \beta_{nm} \cos(m+1)\theta]\ P_n^{-(m+1)}(p) \qquad (6.5.5)$$
$$+ [-\gamma_{nm} \sin(m-1)\theta + \bar{\gamma}_{nm} \cos(m-1)\theta]\ P_n^{-(m-1)}(p)\}$$

$$t_z = \frac{1}{\lambda} \sum_{n=0}^{\infty} \sum_{m=0}^{n+1} [\zeta_{nm} \cos m\theta + \bar{\zeta}_{nm} \sin m\theta]\ P_n^{-m}(p)$$

where:

$$\beta_{nm} = M_{1nm} A_{1nm} + M_{2nm} A_{2nm} + M_{3nm} A_{3nm}$$

$$\gamma_{nm} = N_{1nm} A_{1nm} + N_{2nm} A_{2nm} + N_{3nm} A_{3nm}$$

$$\zeta_{nm} = S_{1nm} A_{1nm} + S_{2nm} A_{2nm} + S_{3nm} A_{3nm}$$

(6.5.6)

$$\bar{\beta}_{nm} = M_{1nm} B_{1nm} + M_{2nm} B_{2nm} - M_{3nm} B_{3nm}$$

$$\bar{\gamma}_{nm} = N_{1nm} B_{1nm} + N_{2nm} B_{2nm} - N_{3nm} B_{3nm}$$

$$\bar{\zeta}_{nm} = S_{1nm} B_{1nm} + S_{2nm} B_{2nm} - S_{3nm} B_{3nm}$$

If a surface force on the spheroidal cavity is prescribed, then the components of that vector, in Cartesian coordinates, t_x^0, t_y^0, and t_z^0 can be represented by a series expansion of spherical harmonics as follows:

$$t_x^0 = \frac{1}{\lambda} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (g_{nm} \cos m\theta + \bar{g}_{nm} \sin m\theta)\ P_n^{-m}(p)$$

$$t_y^0 = \frac{1}{\lambda} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (h_{nm} \cos m\theta + \bar{h}_{nm} \sin m\theta)\ P_n^{-m}(p) \qquad (6.5.7)$$

$$t_z^0 = \frac{1}{\lambda} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (\ell_{nm} \cos m\theta + \bar{\ell}_{nm} \sin {}_{m}\theta)\ P_n^{-m}(p)$$

in which the coefficients g_{nm}, $\bar{g}_{nm}$, h_{nm}, $\bar{h}_{nm}$, ℓ_{nm}, and $\bar{\ell}_{nm}$ can be evaluated by using the orthogonality property which leads to equations similar to those of Eqs. (5.2.10). By equating the coefficients in the identical trigonometric and spherical functions in (6.5.5) to those in (6.5.7) we obtain the following equations:

$$\begin{bmatrix} M_{1nm} & M_{2nm} & M_{3nm} \\ N_{1nm} & N_{2nm} & N_{3nm} \\ S_{1nm} & S_{2nm} & S_{3nm} \end{bmatrix} \begin{bmatrix} A_{1nm} \\ A_{2nm} \\ A_{3nm} \end{bmatrix} = [V_{nm}]$$

$$(6.5.8)$$

$$\begin{bmatrix} M_{1nm} & M_{2nm} & -M_{3nm} \\ N_{1nm} & N_{2nm} & -N_{3nm} \\ S_{1nm} & S_{2nm} & -S_{3nm} \end{bmatrix} \begin{bmatrix} B_{1nm} \\ B_{2nm} \\ B_{3nm} \end{bmatrix} = [\bar{V}_{nm}]$$

where $[v_{nm}]$ and $[\bar{v}_{nm}]$ are defined as follows:

for m = 1

$$[V_{n1}] = \begin{bmatrix} \frac{1}{2}(g_{n2} + \bar{h}_{n2}) \\ g_{n0} \\ \ell_{n1} \end{bmatrix} , \qquad [\bar{V}_{n1}] = \begin{bmatrix} \frac{1}{2}(h_{n2} - \bar{g}_{n2}) \\ h_{n0} \\ \ell_{n1} \end{bmatrix}$$

for $2 \leq m \leq n - 1$

$$[V_{nm}] = \begin{bmatrix} \frac{1}{2}(g_{nm+1} + \bar{h}_{nm+1}) \\ \frac{1}{2}(g_{nm-1} - \bar{h}_{nm-1}) \\ \ell_{mn} \end{bmatrix} , \qquad [\bar{V}_{nm}] = \begin{bmatrix} \frac{1}{2}(h_{nm+1} - \bar{g}_{nm+1}) \\ \frac{1}{2}(h_{nm-1} + \bar{g}_{nm-1}) \\ \bar{\ell}_{nm} \end{bmatrix}$$

for $m = n > 1$ (6.5.9)

$$[V_{nn}] = \begin{bmatrix} 0 \\ \frac{1}{2}(g_{nn-1} - \bar{h}_{nn-1}) \\ \ell_{nn} \end{bmatrix} \qquad [\bar{V}_{nn}] = \begin{bmatrix} 0 \\ \frac{1}{2}(h_{nn-1} + \bar{g}_{nn-1}) \\ \bar{\ell}_{nm} \end{bmatrix}$$

for $m = n + 1$

$$[V_{nn+1}] = \begin{bmatrix} 0 \\ \frac{1}{2}(g_{nn} - \bar{h}_{nn}) \\ 0 \end{bmatrix} \qquad [\bar{V}_{nn+1}] = \begin{bmatrix} 0 \\ \frac{1}{2}(h_{nn} + \bar{g}_{nn}) \\ 0 \end{bmatrix}$$

For values of $m = n$ and $m = n + 1$, singularities at $p = 1, -1$ are removed by the equations $\beta_{nm} = \bar{\beta}_{nm} = \gamma_{nn+1} = \bar{\gamma}_{nn+1} = \zeta_{nn+1} = \bar{\zeta}_{nn+1} = 0$.

6.6 The Axisymmetric Problem

For the axisymmetric problem in which the stress field is independent of θ, the potential functions take the form

$$\phi_j = \sum_{n=0}^{\infty} \frac{2\alpha_j}{(2n+1)} A_{jn0}[P_{n+1}(p_j)Q_{n+1}(q_j) - P_{n-1}(p_j)Q_{n-1}(q_j)] \qquad (6.6.1)$$

$$\psi = 0$$

$$j = 1, 2$$

If a surface force on the spheroidal cavity is prescribed, the components of this vector can be represented by:

$$t_r^0 = \frac{1}{\lambda} \sum_{n=0}^{\infty} g_{n1} P_n^{-1}(p)$$

$$t_z^0 = \frac{1}{\lambda} \sum_{n=0}^{\infty} \ell_{n0} P_n(p) \qquad (6.6.2)$$

Therefore, Eqs. (6.5.6) which determine the coefficient A_{1n0} and A_{2n0} are reduced to:

$$\begin{bmatrix} M_{1n0} & M_{2n0} \\ S_{1n0} & S_{2n0} \end{bmatrix} \begin{bmatrix} A_{1n0} \\ A_{2n0} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} g_{n1} \\ \ell_{n0} \end{bmatrix} \qquad (6.6.3)$$

By solving (6.6.3), the stress and displacement field can be readily obtained

As an illustration, we will consider in the next sections the problem of transversely isotropic medium containing a spheroidal cavity when the medium is subjected, at large distances from the cavity, to

(i) Uniaxial tension in the direction of axis of symmetry of the medium (z-direction).

(ii) Hydrostatic tension in the plane of isotropy (xy-plane).

(iii) Pure shear stress in the plane of isotropy (xy-plane).

(iv) Uniaxial tension in the direction perpendicular to the axis of symmetry of the medium (x-direction).

Numerical evaluation for case (i) and (iv) are presented in the next chapter.

6.7 Uniaxial Tension in the z-Direction

This problem is an axisymmetric one in which $\sigma_{r\theta} = \sigma_{\theta z} = 0$. Therefore, the boundary conditions in this case are

$$\sigma_{zz} = T_0 \quad , \quad \sigma_{rr} = \sigma_{\theta\theta} = \sigma_{rz} = 0 \quad \text{at infinity} \tag{6.7.1}$$

$$t_r = t_z = 0 \quad \text{on the spheroidal surface} \tag{6.7.2}$$

In the absence of the cavity, the state of stress is:

$$\sigma^0_{zz} = T_0 \quad , \quad \sigma^0_{rr} = \sigma^0_{\theta\theta} = \sigma^0_{rz} = 0 \tag{6.7.3}$$

The components of the residual traction vector on the spheroid are:

$$\begin{aligned} t_r &= \sigma^0_{rr}\, n_r + \sigma^0_{rz}\, n_z = 0 \\ t_z &= \sigma^0_{zr}\, n_r + \sigma^0_{zz}\, n_z = T_0 \frac{b}{\lambda}\, p = \frac{T_0 b}{\lambda} P_1(p) \end{aligned} \tag{6.7.4}$$

Now, consider the problem in which the negative of these traction components are specified on the spheroidal cavity, therefore:

$$t_r^0 = 0$$

$$t_z^0 = -\frac{1}{\lambda} T_0 b \; P_1(p) \qquad (6.7.5)$$

Comparing Eqs. (6.7.5) to Eqs. (6.6.2), we find that the only non-zero coefficient is $\ell_{10} = -T_0 b$. Therefore,

$$M_{110} = -2c_{44} \frac{(1 + k_1)}{\nu_1} Q_1(\rho_1) + \frac{2a}{b} c_{66} Q_1^1(\rho_1)$$

$$M_{210} = -2c_{44} \frac{(1 + k_2)}{\nu_2} Q_1(\rho_2) + \frac{2a}{b} c_{66} Q_1^1(\rho_2)$$

$$S_{110} = 2c_{44}(1 + k_1) Q_1^1(\rho_1) \qquad (6.7.6)$$

$$S_{210} = 2c_{44}(1 + k_2) Q_1^1 (\rho_2)$$

and Eqs. (6.6.3) which determine A_{110} and A_{210} become:

$$\begin{bmatrix} M_{110} & M_{210} \\ S_{110} & S_{210} \end{bmatrix} \begin{bmatrix} A_{110} \\ A_{210} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{2} T_0 b \end{bmatrix} \qquad (6.7.7)$$

from which we obtain

$$A_{110} = \frac{T_0 b}{2(k_2 - k_1)\Delta} \left[-\nu_3^2 \frac{(1+k_2)}{\nu_2} Q_1(\rho_2) + \frac{a}{b} Q_1^1(\rho_2) \right]$$

(6.7.8)

$$A_{210} = \frac{-T_0 b}{2(k_2 - k_1)\Delta} \left[-\nu_3^2 \frac{(1+k_1)}{\nu_1} Q_1(\rho_1) + \frac{a}{b} Q_1^1(\rho_1) \right]$$

where

$$\Delta = c_{44} \frac{a}{b} Q_1^1(\rho_1) \, Q_1^1(\rho_2) \, F_1$$

(6.7.9)

$$F_1 = 1 + \frac{b\nu_3^2(1+k_1)(1+k_2)}{a(k_2 - k_1)} \left[\frac{Q_1(\rho_2)}{\nu_2 \, Q_1^1(\rho_2)} - \frac{Q_1(\rho_1)}{\nu_1 \, Q_1^1(\rho_1)} \right]$$

The potential functions in this case are:

$$\phi_j = \frac{2\alpha_j}{3} A_{j10} [P_2(p_j) \, Q_2(q_j) - Q_0(q_j)] \qquad (j = 1,2) \qquad (6.7.10)$$

Substituting (6.7.10) into (3.2.1) and adding the solution of the problem in the absence of the cavity, the displacement and stress field for the problem at hand become:

$$u_r = -r \left[\frac{c_{13}}{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]} T_0 + \sum_{j=1}^{2} \frac{A_{j10} \, Q_1^1(q_j)}{\alpha_j (q_j^2 - 1)^{\frac{1}{2}}} \right]$$

$$u_z = z \left[\frac{(c_{11} + c_{12})}{[c_{33}(c_{11} + c_{12}) - 2c_{13}^2]} T_0 + \sum_{j=1}^{2} \frac{2k_j A_{j10} \, Q_1^1(q_j)}{\alpha_j \nu_j^2 \, (q_j^2 - 1)^{\frac{1}{2}}} \right]$$

$$\sigma_{rr} = \sum_{j=1}^{2} \frac{2A_{j10}}{\alpha_j} \{[-c_{44} \frac{(1+k_j)}{\nu_j^2 q_j} Q_1(q_j) + c_{66} \frac{Q_1^1(q_j)}{(q_j^2 - 1)^{\frac{1}{2}}}]$$

$$+ c_{44} \frac{(1+k_j)\, p_j^2}{\nu_j^2\, q_j (q_j^2 - p_j^2)} \}$$

$$\sigma_{\theta\theta} = \sum_{j=1}^{2} \frac{2A_{j10}}{\alpha_j} \{[-c_{44} \frac{(1+k_j)}{\nu_j^2 q_j} Q_1(q_j) + c_{66} \frac{Q_1^1(q_j)}{(q_j^2 - 1)^{\frac{1}{2}}}]$$

$$+ [\frac{c_{44}(1+k_j) p_j^2}{\nu_j^2 q_j (q_j^2 - p_j^2)} + 2 \frac{c_{66} q_j (1 - p_j^2)}{(q_j^2 - 1)(q_j^2 - p_j^2)}]\} \qquad (6.7.11)$$

$$\sigma_{zz} = T_0 + \sum_{j=1}^{2} 2c_{44} \frac{A_{j10}(1+k_j)}{\alpha_j} [\frac{Q_1^1(q_j)}{(q_j^2 - 1)^{\frac{1}{2}}} + \frac{q_j (1 - p_j^2)}{(q_j^2 - 1)(q_j^2 - p_j^2)}]$$

$$\sigma_{rz} = \sum_{j=1}^{2} - 2c_{44} \frac{A_{j10}(1+k_j)}{\alpha_j \nu_j} \frac{p_j (1 - p_j^2)^{\frac{1}{2}}}{(q_j^2 - 1)^{\frac{1}{2}} (q_j^2 - p_j^2)}]$$

At the spheroid interface, the equations of the stresses in the medium take the form:

$$\sigma_{rr} = \frac{2c_{44} s^2}{a^3} \sum_{j=1}^{2} A_{j10}\, \alpha_j^2\, \nu_j (1 + k_j) \frac{z^2}{r^2 + \nu_j^2 s^4 z^2}$$

$$\sigma_{\theta\theta} = \frac{2c_{44} s^2}{a^3} \sum_{j=1}^{2} A_{j10}\, \alpha_j^2\, \nu_j [\frac{(1 + k_j) z^2}{(r^2 + \nu_j^2 s^4 z^2)} + \frac{2}{\nu_3^2 s^4} \frac{r^2}{r^2 + {}_j^2 s^4 z^2}]$$

(6.7.12)

$$\sigma_{zz} = \frac{2c_{44}}{a\, b^2} \sum_{j=1}^{2} A_{j10}\, \alpha_j^2\, \nu_j (1 + k_j) \frac{r^2}{r^2 + \nu_j^2 s^4 z^2}$$

$$\sigma_{rz} = -\frac{2c_{44}s}{a\,b^2}\sum_{j=1}^{2} A_{j10}\,\alpha_j^2\,\nu_j(1+k_j)\,\frac{rz}{r^2+\nu_j^2 s^4 z^2}$$

By transforming Eqs. (6.7.12) from cylindrical coordinates to spheroidal coordinates by means of Eqs. (B.3) in Appendix B we obtain:

$$\sigma_{\phi\phi} = \frac{2c_{44}}{a\,b^2}(r^2+s^4z^2)\left[\frac{A_{110}(1+k_1)\alpha_1^2\nu_1}{r^2+\nu_1^2 s^4 z^2} + \frac{A_{210}(1+k_2)\alpha_2^2\nu_2}{r^2+\nu_2^2 s^4 z^2}\right]$$

$$\sigma_{\theta\theta} = \frac{2c_{66}}{a\,b^2}\left\{\frac{A_{110}\,\alpha_1^2\,\nu_1[\nu_3^2 s^4(1+k_1)z^2+2r^2]}{(r^2+\nu_1^2 s^4 z^2)} + \frac{A_{210}\,\alpha_2^2\,\nu_2[\nu_3^2 s^4(1+k_2)z^2+2r^2}{r^2+\nu_2^2 s^4 z^2}\right\} \tag{6.7.13}$$

For an isotropic medium containing a spheroidal cavity, Eqs. (6.7.13) are reduced by the limiting process to:

$$\sigma_{\gamma\gamma} = \frac{3\,T_0}{2(7-5\nu)}[-(1+5\nu)+10\sin^2\gamma]$$

$$\sigma_{\theta\theta} = \frac{3\,T_0}{2(7-5\nu)}[-(1+5\nu)+10\nu\sin^2\gamma] \tag{6.7.14}$$

where γ is the meridional angle shown in Fig. 8.

6.8 Hydrostatic Tension in the Plane of Isotropy

The boundary conditions of this problem are

$$\sigma_{rr} = \sigma_{\theta\theta} = T_0 \quad , \quad \sigma_{zz} = \sigma_{rz} = 0 \tag{6.8.1}$$

$$t_r = t_z = 0 \quad \text{on the spheroidal surface} \tag{6.8.2}$$

In the absence of the cavity, the state of stress is represented by Eqs. (6.8.1). Therefore, the tractions that must be removed to satisfy the boundary conditions of (6.8.2) are:

$$\begin{aligned} t_r &= \sigma_{rr}\, n_r + \sigma_{rz}\, n_z = T_0 \frac{a}{\lambda} (1 - p^2)^{\frac{1}{2}} = \frac{2\, T_0\, a}{\lambda}\, P_1^{-1}(p) \\ t_z &= \sigma_{zr}\, n_r + \sigma_{zz}\, n_z = 0 \end{aligned} \tag{6.8.3}$$

We now consider the problem in which the negative of these tractions are applied on the surface of the spheroidal cavity. Thus,

$$t_r^0 = - \frac{2\, T_0\, a}{\lambda}\, P_1^{-1}(p) \quad , \quad t_z^0 = 0 \tag{6.8.4}$$

It follows that the only non-zero term in Eqs. (6.6.2) is $g_{11} = -2T_0 a$. This term is corresponding to n = 1. The coefficients A_{110} and A_{210} can be determined by solving the equations

$$\begin{bmatrix} M_{110} & M_{210} \\ S_{110} & S_{210} \end{bmatrix} \begin{bmatrix} A_{110} \\ A_{210} \end{bmatrix} = \begin{bmatrix} -T_0\, a \\ 0 \end{bmatrix} \tag{6.8.5}$$

where M_{110}, M_{210}, S_{110}, and S_{210} are given by Eqs. (6.7.6). By solving Eqs. (6.8.5) we obtain:

$$A_{110} = -\frac{b}{2c_{66}} \frac{(1+k_2)}{(k_2-k_1)Q_1^1(\rho_1)} \frac{T_0}{F_1}$$

$$A_{210} = -\frac{b}{2c_{66}} \frac{(1+k_1)}{(k_2-k_1)Q_1^1(\rho_2)} \frac{T_0}{F_1} \qquad (6.8.6)$$

$$F_1 = 1 + \frac{b\nu_3^2}{a} \frac{(1+k_1)(1+k_2)}{k_2-k_1} \left[\frac{Q_1(\rho_2)}{\nu_2 Q_1^1(\rho_2)} - \frac{Q_1(\rho_1)}{\nu_1 Q_1^1(\rho_1)} \right]$$

The potential functions for this case take the form:

$$\phi_j = \frac{2\alpha_j}{3} A_{j10}[P_2(p_j)\, Q_2(q_j) - Q_0(q_j)] \qquad (6.8.7)$$

The stress field becomes:

$$\sigma_{rr} = T_0 + \sum_{j=1}^{2} \frac{2A_{j10}}{\alpha_j} \{[-c_{44} \frac{(1+k_j)}{\nu_j^2 q_j} Q_1(q_j) + c_{66} \frac{Q_1^1(q_j)}{(q_j^2-1)^{\frac{1}{2}}}] + c_{44} \frac{(1+k_j)\, p_j^2}{\nu_j^2 q_j (q_j^2 - p_j^2)}\}$$

$$(6.8.8)$$

$$\sigma_{\theta\theta} = T_0 + \sum_{j=1}^{2} \frac{2A_{j10}}{\alpha_j} \{[-c_{44} \frac{(1+k_j)}{\nu_j^2 q_j} Q_1(q_j) + c_{66} \frac{Q_1^1(q_j)}{(q_j^2-1)^{\frac{1}{2}}}] + [\frac{c_{44}(1+k_j)p_j^2}{\nu_j^2 q_j (q_j^2 - p_j^2)} + \frac{2c_{66}\, q_j (1-p_j^2)}{(q_j^2-1)(q_j^2-p_j^2)}]\}$$

$$\sigma_{zz} = \sum_{j=1}^{2} 2c_{44} \frac{A_{j10}(1+k_j)}{\alpha_j} \left[\frac{Q_1^1(q_j)}{(q_j^2 - 1)^{1/2}} + \frac{q_j(1-p_j^2)}{(q_j^2 - 1)(q_j^2 - p_j^2)} \right]$$

$$\sigma_{rz} = \sum_{j=1}^{2} -2c_{44} \frac{A_{j10}(1+k_j)}{\alpha_j \nu_j} \frac{p_j(1-p_j^2)^{1/2}}{(q_j^2 - 1)^{1/2}(q_j^2 - p_j^2)}$$

On the spheroidal surface, these equations take the form:

$$\sigma_{rr} = \frac{2c_{44}}{a} \sum_{j=1}^{2} \frac{A_{j10}(1+k_j)}{\nu_j} \frac{p^2}{(\rho_j^2 - p^2)}$$

$$\sigma_{\theta\theta} = \frac{2c_{44}}{a} \sum_{j=1}^{2} A_{j10} \left[\frac{(1+k_j)}{\nu_j} \frac{p^2}{(\rho_j^2 - p^2)} + \frac{2a^2}{b^2\nu_3^2} \frac{(1-p^2)}{\nu_j(\rho_j^2 - p^2)} \right]$$

(6.8.9)

$$\sigma_{zz} = 2c_{44} \frac{a}{b^2} \sum_{j=1}^{2} \frac{A_{j10}(1+k_j)}{\nu_j} \frac{(1-p^2)}{(\rho_j^2 - p^2)}$$

$$\sigma_{rz} = -\frac{2c_{44}}{b} \sum_{j=1}^{2} \frac{A_{j10}(1+k_j)}{\nu_j} \frac{p(1-p^2)^{1/2}}{(\rho_j^2 - p^2)}$$

By transforming Eqs. (6.8.9) from cylindrical coordinates to spheroidal coordinates by means of Eqs. (B.3) in Appendix B we obtain:

$$\sigma_{\phi\phi} = T_0 \frac{\nu_3^2}{SF_1} \frac{(1+k_1)(1+k_2)}{(k_2-k_1)} (r^2+s^4 z^2)$$

$$\left[\frac{(1-s^2\nu_2^2)}{\nu_2(r^2+\nu_2^2 s^4 z^2)Q_1^1(\rho_2)} - \frac{(1-s^2\nu_1^2)}{\nu_1(r^2+\nu_1^2 s^4 z^2)Q_1^1(\rho_1)} \right] \quad (6.8.10)$$

$$\sigma_{\theta\theta} = T_0 \frac{1}{S\, F_1\, (k_2-k_1)} \left\{ \frac{(1+k_1)(1-s^2\nu_2^2)}{\nu_2\; Q_1^1(\rho_2)} \frac{[\nu_3^2(1+k_2)s^4 z^2+2r^2]}{(r^2+\nu_2^2 s^4 z^2)} - \frac{(1+k_2)(1-s^2\nu_1^2)}{\nu_1\; Q_1^1(\rho_1)} \frac{[\nu_3^2(1+k_1)s^4 z^2+2r^2]}{(r^2+\nu_1^2 s^4 z^2)} \right\}$$

For isotropy, by taking the proper limit we find

$$(\lim F_1)_{\text{isotropy}} = \frac{d_1}{(1-\nu)\rho^2(\rho^2-1)[Q_1^1(\rho)]^2} \quad (6.8.11)$$

where

$$d_1 = [-(1+\nu)(\rho^2-1)^2[Q_1(\rho)]^2 + (\rho^2-1)(2\nu+1-3\rho^2)Q_1(\rho) + \rho^2 - \nu] \quad (6.8.12)$$

and, accordingly,

$$\sigma_{\phi\phi} = -\frac{T_0}{d_1}\{(\rho^2-1)[(3\rho^2-1)Q_1(\rho)-1+2[(\rho^2-1)Q_1(\rho)-1]\frac{s^4z^2}{r^2+s^4z^2}\}$$

$$\sigma_{\theta\theta} = -\frac{T_0}{d_1}\{(3\rho^2-2\nu+1)(\rho^2-1)Q_1(\rho)+2\nu-1-\rho^2 \qquad (6.8.13)$$

$$+2\nu\,[(\rho^2-1)Q_1(\rho)-1]\frac{s^4z^2}{r^2+s^4z^2}\}$$

For an isotropic medium containing a spherical cavity, Eqs. (6.8.13) become

$$\sigma_{\gamma\gamma} = -\frac{3T_0}{(7-5\nu)}(1-5\cos^2\gamma)$$

$$\sigma_{\theta\theta} = -\frac{3T_0}{(7-5\nu)}(-4+5\nu\sin^2\gamma) \qquad (6.8.14)$$

where γ is the meridional angle shown in Fig. 8.

6.9 Pure Shear Stress in the Plane of Isotropy

The boundary conditions are

$$\sigma_{xy} = \tau_0 \quad , \quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = \sigma_{yz} = 0 \text{ at infinity} \qquad (6.9.1)$$

$$t_x = t_y = t_z \text{ on the spherical surface.} \qquad (6.9.2)$$

In the absence of the cavity, the state of stress in the medium can be represented by:

$$\sigma_{xy} = \tau_0 \quad , \quad \sigma_{xx} = \sigma_{yy} = \sigma_{zz} = \sigma_{xz} = \sigma_{zy} = 0 \qquad (6.9.3)$$

or, equivalently in cylindrical coordinates:

$$\sigma_{rr} = -\sigma_{\theta\theta} = -\tau_0 \sin 2\theta \ , \quad \sigma_{r\theta} = \tau_0 \cos 2\theta \quad , \ \sigma_{zz} = \sigma_{rz} = \sigma_{z\theta} = 0 \tag{6.9.4}$$

and the displacement components are:

$$u_r = \frac{\tau_0}{2c_{66}} r \sin 2\theta \ , \quad u_\theta = -\frac{\tau_0}{2c_{66}} r \cos 2\theta \quad , \ u_z = 0 \tag{6.9.5}$$

The components of the traction vector generated by the solution of Eq. (6.9.3) are:

$$t_x = \sigma_{xx}n_x + \sigma_{xy}n_y + \sigma_{xz}n_z = \tau_0 \frac{a}{\lambda} (1-p^2)^{\frac{1}{2}} \sin\theta = \frac{2a}{\lambda} \tau_0 P_1^{-1}(p) \sin\theta$$

$$t_y = \sigma_{yx}n_x + \sigma_{yy}n_y + \sigma_{yz}n_z = \tau_0 \frac{a}{\lambda} (1-p^2)^{\frac{1}{2}} \cos\theta = \frac{2a}{\lambda} \tau_0 P_1^{-1}(p) \cos\theta \tag{6.9.6}$$

$$t_y = \sigma_{zx}n_x + \sigma_{zy}n_y + \sigma_{zz}n_z = 0$$

Now, we consider the problem in which the negative of these traction components are specified on the surface of the spheroid. Thus,

$$t_x^0 = -\frac{2a}{\lambda} \tau_0 \ P_1^{-1}(p) \sin\theta$$

$$t_y^0 = -\frac{2a}{\lambda} \tau_0 \ P_1^{-1}(p) \cos\theta \tag{6.9.7}$$

$$t_z^0 = 0$$

From Eqs. (6.5.7) and (6.9.7), we find that the only non-zero coefficients in Eqs. (6.5.7) are:

$$\bar{g}_{11} = -2a\tau_0 \quad , \quad \bar{h}_{11} = -2a\tau_0 \tag{6.9.8}$$

therefore, the problem at hand is corresponding to the case in which $m = 2$, $n = 1$, and $A_{j12} = 0$. The coefficients B_{j12} are determined from:

$$\begin{bmatrix} M_{112} & M_{212} & -M_{312} \\ N_{112} & N_{212} & -N_{312} \\ S_{112} & S_{212} & -S_{312} \end{bmatrix} \begin{bmatrix} B_{112} \\ B_{212} \\ B_{312} \end{bmatrix} = \begin{bmatrix} 0 \\ -2a\tau_0 \\ 0 \end{bmatrix} \qquad (6.9.9)$$

where:

$$M_{112} = \frac{8\alpha_1^2}{b^2\nu_1}\,[c_{44}(1+k_1) - 6\,\frac{a^2}{b^2}\,c_{66}]$$

$$M_{212} = \frac{8\alpha_2^2}{b^2\nu_2}\,[c_{44}(1+k_2) - 6\,\frac{a^2}{b^2}\,c_{66}]$$

$$M_{312} = -8c_{66}\,\frac{\alpha_3^2}{b^2\nu_3}\,(\nu_3^2 - 6\,\frac{a^2}{b^2})$$

$$N_{112} = \frac{2c_{44}}{b^2}\,\frac{\alpha_1^2(1+k_1)}{\nu_1} + 2c_{66}\,\frac{a}{b}\,Q_1^1(\rho_1)$$

$$N_{212} = \frac{2c_{44}}{b^2}\,\frac{\alpha_2^2(1+k_2)}{\nu_2} + 2c_{66}\,\frac{a}{b}\,Q_1^1(\rho_2)$$

$$N_{312} = 2c_{66}\,[\,\frac{\alpha_3^2\nu_3}{b^2} + \frac{a}{b}\,Q_1^1(\rho_3)]$$

$$S_{112} = -8c_{44} \frac{a}{b^3} \frac{\alpha_1^2(1+k_1)}{\nu_1}$$

$$S_{212} = -8c_{44} \frac{a}{b^3} \frac{\alpha_2^2(1+k_2)}{\nu_2} \qquad (6.9.10)$$

$$S_{312} = 2c_{44} \frac{a}{b\nu_3} Q_1^{-2}(\rho_3) = 8c_{44} \frac{a}{b^3} \frac{\alpha_3^2}{\nu_3}$$

The solution of Eqs. (6.9.9), after some algebric manipulation, is given by:

$$B_{112} = -\frac{k_2 \nu_1 a b^2}{\alpha_1^2(k_1 - k_2) F_3} \frac{\tau_0}{c_{44}}$$

$$B_{212} = \frac{k_1 \nu_2 a b^2}{\alpha_2^2(k_1 - k_2) F_3} \frac{\tau_0}{c_{44}} \qquad (6.9.11)$$

$$B_{312} = -\frac{\nu_3 a b^2}{\alpha_3^2 F_3} \frac{\tau_0}{c_{44}}$$

where:

$$F_3 = -2 - \frac{ab}{\nu_3} \frac{Q_1^1(\rho_3)}{\alpha_3^2} - \frac{ab}{\nu_3^2} \frac{k_1 \nu_2 Q_1^1(\rho_2)}{\alpha_2^2(k_1 - k_2)} + \frac{ab}{\nu_3^2} \frac{k_2 \nu_1 Q_1^1(\rho_1)}{\alpha_1^2(k_1 - k_2)} \qquad (6.9.12)$$

The potential functions for this case are:

$$\phi_j = \frac{2\alpha_j}{3} B_{j12} [P_2^{-2}(p_j)\ Q_2^2(q_j) - P_0^{-2}(p_j)\ Q_0^2(q_j)] \sin 2\theta$$

$$\psi_3 = \frac{2\alpha_3}{3} B_{312} [P_2^{-2}(p_3)\ Q_2^2(q_3) - P_0^{-2}(p_3)\ Q_0^2(q_3)] \cos 2\theta \qquad (6.9.13)$$

$j = 1, 2$

Substituting Eqs. (6.9.13) into Eqs. (3.2.1) and adding the solution of the problem in the absence of the cavity we find,

$$u_r = \{ \frac{\tau_0}{2c_{66}} r + B_{112}[\frac{\alpha_1^3 q_1}{3r^3} (1-p_1)^3(3+p_1) + \frac{r}{2\alpha_1} \frac{Q_1^1(q_1)}{(q_1^2-1)^{\frac{1}{2}}}]$$

$$+ B_{212}[\frac{\alpha_2^3 q_2}{3r^3} (1-p_2)^3(3+p_2) + \frac{r}{2\alpha_2} \frac{Q_1^1(q_2)}{(q_2^2-1)^{\frac{1}{2}}}]$$

$$+ B_{312}[\frac{\alpha_3^3 q_2}{3r^3} (1-p_3)^3(3+p_2) - \frac{r}{2\alpha_3} \frac{Q_1^1(q_3)}{(q_3^2-1)^{\frac{1}{2}}}]\} \sin 2\theta$$

$$u_\theta = \{ -\frac{\tau_0}{2c_{66}} r + B_{112}[- \frac{\alpha_1^3 q_1}{3r^3} (1-p_1)^3(3+p_1) + \frac{r}{2\alpha_1} \frac{Q_1^1(q_1)}{(q_1^2-1)^{\frac{1}{2}}}]$$

$$+ B_{212}[- \frac{\alpha_2^3 q_2}{3r^3} (1-p_2)^3(3+p_2) + \frac{r}{2\alpha_2} \frac{Q_1^1(q_2)}{(q_2^2-1)^{\frac{1}{2}}}]$$

$$+ B_{312}[- \frac{\alpha_3^3 q_3}{3r^3} (1-p_3)^3(3+p_3) - \frac{r}{2\alpha_3} \frac{Q_1^1(q_3)}{(q_3^2-1)^{\frac{1}{2}}}]\} \cos 2\theta$$

$$u_z = \frac{2}{3r^2} [\frac{B_{112} k_1 \alpha_1^2}{\nu_1} (1-p_1)^2(2+p_1) + \frac{B_{212} k_2 \alpha_2^2}{\nu_2} (1-p_2)^2(2+p_2)] \sin 2\theta$$

$$\sigma_{rr} = \{\tau_0 + 2c_{44} r^2 [\frac{B_{112}(1+k_1)}{\alpha_1^3 \, \nu_1^2} \frac{q_1}{(q_1^2-1)^2(q_1^2-p_1^2)} + \frac{B_{212}(1+k_2)}{\alpha_2^3 \, \nu_2^2} \frac{q_2}{(q_2^2-1)^2(q_2^2-p_2^2)}$$

$$+ 2c_{66} \, B_{112} [- \frac{\alpha_1^3 q_1}{r^4} (1-p_1)^3(3+p_1) + \frac{Q_1^1(q_1)}{2\alpha_1 (q_1^2-1)^{\frac{1}{2}}}]$$

$$+ 2c_{66} \, B_{212} [- \frac{\alpha_2^3 q_2}{r^4} (1-p_2)^3(3+p_2) + \frac{Q_1^1(q_2)}{2\alpha_2 (q_2^2-1)^{\frac{1}{2}}}]$$

$$+ 2c_{66} \, B_{312} [- \frac{\alpha_3^3 q_3}{r^4} (1-p_3)^3(3+p_3) - \frac{Q_1^1(q_3)}{2\alpha_3 (q_3^2-1)^{\frac{1}{2}}}]\} \sin 2\theta$$

$$\sigma_{\theta\theta} = \{-\tau_0 + 2c_{44} r^2 [\frac{B_{112}}{\alpha_1^3} (\frac{1+k_1}{\nu_1^2} - \frac{2}{\nu_3^2}) \frac{q_1}{(q_1^2-1)^2(q_1^2-p_2^2)}$$

$$+ \frac{B_{112}}{\alpha_2^3} (\frac{1+k_2}{\nu_2^2} - \frac{2}{\nu_3^2}) \frac{q_2}{(q_2^2-1)^2(q_2^2-p_2^2)}]$$

$$- 2c_{66} B_{112} [- \frac{\alpha_1^3 q_1}{r^4} (1-p_1)^3(3+p_1) + \frac{Q_1^1(q_1)}{2\alpha_1 (q_1^2-1)^{\frac{1}{2}}}]$$

$$- 2c_{66} B_{212} [- \frac{\alpha_2^3 q_2}{r^4} (1-p_2)^3(3+p_2) + \frac{Q_1^1(q_2)}{2\alpha_2 (q_2^2-1)^{\frac{1}{2}}}]$$

$$- 2c_{66} B_{312} [- \frac{\alpha_3^3 q_3}{r^4} (1-p_3)^3(3+p_3) - \frac{Q_1^1(q_3)}{2\alpha_3 (q_3^2-1)^{\frac{1}{2}}}] \sin 2\theta$$

$$\sigma_{zz} = -2c_{44} r^2 [\frac{B_{112}(1+k_1)}{\alpha_1^3} \frac{q_1}{(q_1^2-1)^2(q_1^2-p_1^2)}$$

$$+ \frac{B_{212}(1+k_2)}{\alpha_2^3} \frac{q_2}{(q_2^2-1)^2(q_2^2-p_2^2)}] \sin 2\theta$$

$$\sigma_{rz} = c_{44}\{-\frac{4}{3r^3}\sum_{j=1}^{3}\frac{B_{j12}\,\alpha_j^2(1+k_j)}{\nu_j}(1-p_j)^2(2+p_j)$$

$$+\,2r\sum_{j=1}^{2}\frac{B_{j12}(1+k_j)}{\alpha_j^2\,\nu_j}\,\frac{p_j}{(q_j^2-1)(q_j^2-p_j^2)}\}\sin 2\theta$$

$$\sigma_{\theta z} = c_{44}\{\frac{4}{3r^3}\sum_{j=1}^{3}\frac{B_{j12}\,\alpha_j^2(1+k_j)}{\nu_j}(1-p_j)^2(2+p_j)$$

$$-\,2r\,\frac{B_{312}}{\alpha_3^2\nu_3}\,\frac{p_3}{(q_3^2-1)(q_3^2-p_3^2)}\}\cos 2\theta \qquad (6.9.14)$$

$$\sigma_{r\theta} = 2c_{66}\{\frac{\tau_0}{2c_{66}}-\sum_{j=1}^{2}B_{j12}[\frac{\alpha_j^3 q_j}{r^4}(1-p_j)^3(3+p_j)-\frac{Q_1^1(q_j)}{2\alpha_j(q_j^2-1)^{\frac{1}{2}}}]$$

$$-\,B_{312}[\frac{\alpha_3^2 q_3}{r^4}(1-p_3)^3(3+p_3)+\frac{Q_1^1(q_3)}{2\alpha_3(q_3^2-1)^{\frac{1}{2}}}]$$

$$-\,r^2\,\frac{B_{312}}{\alpha_3^2}\,\frac{q_3}{(q_3^2-1)^2(q_3^2-p_3^2)}\}\cos 2\theta$$

On the spheroidal surface, the stress components in the medium take the form:

$$\sigma_{rr} = \frac{2\tau_0}{F_3}\{-1+\frac{r^2}{(k_1-k_2)}[\frac{(1+k_1)}{r^2+\nu_2^2 s^4 z^2}-\frac{(1+k_2)}{r^2+\nu_1^2 s^4 z^2}]\}\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{2\tau_0}{F_3}\{1+\frac{r^2}{(k_1-k_2)}[\frac{(1+k_1-\frac{2\nu_2^2 k_1}{\nu_3^2})}{r^2+\nu_2^2 s^4 z^2}-\frac{(1+k_2-\frac{2\nu_2^2 k_2}{\nu_3^2})}{r^2+\nu_1^2 s^4 z^2}]\}\sin 2\theta$$

$$(6.9.15)$$

$$\sigma_{zz} = -\frac{2\tau_0}{F_3}\frac{r^2}{(k_1-k_2)}[\frac{(1+k_1)\,\nu_2^2}{r^2+\nu_2^2 s^4 z^2}-\frac{(1+k_2)\,\nu_1^2}{r^2+\nu_1^2 s^4 z^2}]\sin 2\theta$$

$$\sigma_{rz} = 2\,\frac{\tau_0}{F_3}\,\frac{s^2}{(k_1 - k_2)}\,rz\,\left[\frac{(1+k_1)\,\nu_2^2}{r^2+\nu_2^2 s^4 z^2} - \frac{(1+k_2)\,\nu_1^2}{r^2+\nu_1^2 s^4 z^2}\right] \sin 2\theta$$

$$\sigma_{\theta z} = \frac{2\tau_0}{F_3}\,\nu_3 s^2\,\frac{zr}{r^2+\nu_3^2 s^4 z^2}\,\cos 2\theta$$

$$\sigma_{r\theta} = \frac{2\tau_0}{F_3}\left[\frac{r^2}{r^2+\nu_3^2 s^4 z^2} - 1\right] \cos 2\theta$$

The stress components in spheroidal coordinates are obtained by using the transformation in Appendix B and take the form:

$$\sigma_{\phi\phi} = \frac{2\tau_0}{F_3}\left[-1 + \frac{(1+k_1)(1-\nu_2^2)}{(k_1 - k_2)}\,\frac{r^2}{r^2+\nu_2^2 s^4 z^2} - \frac{(1+k_2)(1-\nu_1^2)}{(k_1 - k_2)}\,\frac{r^2}{r^2+\nu_1^2 s^4 z^2}\right] \sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{2\tau_0}{F_3}\left\{1 + \frac{r^2}{\nu_3^2(k_1 - k_2)}\left[\frac{(1+k_1)\nu_3^2 - 2k_1\nu_2^2}{r^2+\nu_2^2 s^4 z^2} - \frac{(1+k_2)\nu_3^2 - 2k_2\,\nu_1^2}{r^2+\nu_1^2 s^4 z^2}\right]\right\} \sin 2\theta \qquad (6.9.16)$$

$$\sigma_{\theta\phi} = -\frac{2\tau_0}{F_3}\,\nu_3^2\,s^2\,\frac{z\sqrt{r^2+s^4 z^2}}{r^2+\nu_3^2 s^4 z^2}\,\cos 2\theta$$

$$\sigma_{\eta\eta} = \sigma_{\eta\phi} = \sigma_{\eta\theta} = 0$$

For an isotropic medium, Eqs. (6.9.16) may be reduced to:

$$\sigma_{\phi\phi} = \frac{8\tau_0}{d_3}\left[-(1-\nu) + \frac{r^2}{r^2+s^4z^2}\right]\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{8\tau_0}{d_3}\left[(1-\nu) + \nu\frac{r^2}{r^2+s^4z^2}\right]\sin 2\theta \qquad (6.9.17)$$

$$\sigma_{\phi\theta} = -\frac{8\tau_0}{d_3}(1-\nu)\frac{s^2z}{\sqrt{r^2+s^4z^2}}\cos 2\theta$$

$$\sigma_{\eta\eta} = \sigma_{\eta\phi} = \sigma_{\eta\theta} = 0$$

where

$$d_3 = (\rho^2-1)(3\rho^2-7+8\nu)\,Q_1(\rho) - (\rho^2+1) \qquad (6.9.18)$$

Furthermore, for an isotropic medium containing a spherical cavity Eqs. (6.9.17) are reduced to:

$$\sigma_{\gamma\gamma} = \frac{15\tau_0}{(7-5\nu)}(1-\nu-\sin^2\gamma)\sin 2\theta$$

$$\sigma_{\theta\theta} = \frac{15\tau_0}{(7-5\nu)}(-1+\nu\cos^2\gamma)\cos 2\theta \qquad (6.9.19)$$

$$\sigma_{\gamma\theta} = \frac{15(1-\nu)}{(7-5\nu)}\tau_0\cos\gamma\sin 2\theta$$

$$\sigma_{\eta\eta} = \sigma_{\eta\gamma} = \sigma_{\eta\theta} = 0$$

6.10 Uniaxial Tension in the x-Direction

The distribution of the stress field throughout a medium subjected to uniaxial tension in the x-direction T_o may be achieved by superposition of the solutions for the following two cases:

(i) $\sigma_x = \sigma_y = \frac{1}{2} T_o$

(ii) $\sigma_x = -\sigma_y = \frac{1}{2} T_o$

The solution of case (i) may be obtained directly from Section (6.8) after replacing T_o by $\frac{1}{2} T_o$. On the other hand, the solution of case (ii) can be obtained from the solution of Section (6.9) after replacing θ by $(\theta + \frac{\pi}{4})$ and τ_o by $\frac{1}{2} T_o$. Therefore, the stress components at the spheroidal cavity surface for the desired problem are:

$$\sigma_{\phi\phi} = T_0 \frac{\nu_3^2}{2sF_1} \frac{(1+k_1)(1+k_2)}{(k_2-k_1)} (r^2 + s^4 z^2) \left[\frac{(1 - s^2\nu_2^2)}{\nu_2(r^2 + \nu_2^2 s^4 z^2) Q_1^1(\rho_2)} - \frac{(1 - s^2\nu_1^2)}{\nu_1(r^2 + \nu_1^2 s^4 z^2) Q_1^1(\rho_1)} \right] + \frac{T_0}{F_3} \left[-1 + \frac{(1+k_1)(1-\nu_2^2)}{(k_1-k_2)} \frac{r^2}{r^2 + \nu_2^2 s^4 z^2} - \frac{(1+k_2)(1-\nu_1^2)}{(k_1-k_2)} \frac{r^2}{r^2 + \nu_1^2 s^4 z^2} \right] \cos 2\theta \tag{6.10.1}$$

$$\sigma_{\theta\theta} = \frac{T_0}{2sF_1(k_2-k_1)} \left\{ \frac{(1+k_1)(1-s^2\nu_2^2)}{\nu_2\, Q_1^1(\rho_2)} \frac{[\nu_3^2(1+k_2)\, s^4 z^2 + 2r^2]}{(r^2 + \nu_2^2 s^4 z^2)} - \frac{(1+k_2)(1-s^2\nu_1^2)}{\nu_1\, Q_1^1(\rho_1)} \frac{[\nu_3^2(1+k_1)\, s^4 z^2 + 2r^2]}{(r^2 + \nu_1^2 s^4 z^2)} \right\} + \frac{T_0}{F_3} \left\{ 1 + \frac{r^2}{\nu_3^2(k_1-k_2)} \left[\frac{(1+k_1)\nu_3^2 - 2k_1\nu_2^2}{r^2 + \nu_2^2 s^4 z^2} - \frac{(1+k_2)\nu_3^2 - 2k_2\nu_1^2}{r^2 + \nu_1^2 s^4 z^2} \right] \right\} \cos 2\theta$$

$$\sigma_{\phi\theta} = \frac{T_0}{F_3} \quad \nu_3^2 \, s^2 \, \frac{z\sqrt{r^2 + s^4 z^2}}{r^2 + \nu_3^2 s^4 z^2} \quad \sin 2\theta$$

$$\sigma_{\eta\eta} = \sigma_{\eta\phi} = \sigma_{\eta\theta} = 0$$

CHAPTER 7

THE EFFECT OF ANISOTROPY ON THE STRESS CONCENTRATION FACTORS

7.1 Introduction

In this chapter, numerical results will be presented for a transversely isotropic medium containing a spherical cavity subjected to the following load cases, applied at large distances from the cavity.

(i) Uniaxial tension in the direction of the axis of symmetry of the material (z-direction).

(ii) Uniaxial tension in the direction perpendicular to the axis of symmetry of the material (x-direction).

Elementary dimensional considerations show that the stress concentration factor must depend upon four dimensionless ratios of either the five elastic constants (stiffnesses), or the compliances, or the engineering constants. Adopting the engineering constants and noting that Poisson's ratios $\bar{\nu}$ and ν are dimensionless, the remaining two dimensionless ratios among $\bar{E}$, E, and $\bar{G}$ can be taken as $\frac{\bar{E}}{E}$ and $\frac{\bar{G}}{E}$ although other ratios are possible. To show the effect of anisotropy on the stress concentration factor in the vicinity of a spherical cavity, we consider hypothetical materials for which one of the four chosen ratios can vary while the remaining three ratios are kept fixed at values corresponding to an isotropic material whose Poisson's ratio equals 0.25. Therefore, the ratios have, at isotropy with $\nu = 0.25$, the following values

$$\bar{\nu} = \nu = 0.25 \ , \ \frac{\bar{E}}{E} = 1.0 \ , \ \frac{\bar{G}}{E} = 0.4 \qquad (7.1.1)$$

Throughout this chapter, we define the stress concentration factor as the ratio of the maximum principal stress on the surface of the cavity to the magnitude of the uniform stress field applied at infinity. Positive and negative signs are used to denote tensile and compressive stresses respectively.

Numerical results for the stress concentration factor are plotted on a coordinate system in which the abscissa represents the varying ratio and the ordinate represents the stress concentration factor denoted by K_t and K_c for tension and compression respectively. Negative values of Poisson's ratios do theoretically exist but they may not be physically attainable. Therefore, they are omitted from the plots and discussion. The asterisk (*) on the plots indicates the upper or lower limit of any of the four ratios for positive definiteness of the strain energy function (Eq. 2.4.3) when the other three ratios are fixed. For each value of the varying ratio, the locations of K_t and K_c are calculated and shown on a plot in terms of two angles γ and θ shown in Fig. 8.

7.2 Uniaxial Tension in the z-Direction

This problem was first investigated by Chen [5] who evaluated the stress concentration factor for a few transversely isotropic materials. In this section, we will present more numerical data than has been reported previously in the literature.

It is well known that the tensile and compressive stress concentration factors for a spherical cavity in an isotropic medium equal $\frac{3(9-5\nu)}{2(7-5\nu)}$ and $-\frac{3(1+5\nu)}{2(7-5\nu)}$ respectively. The former occurs on the equatorial line and changes slightly from 1.929 to 2.167 as Poisson's ratio ν traverses the range 0 to 0.5 and the latter occurs at the pole and changes from -0.214 to -1.116 as ν traverses the range 0 to 0.5.

We have calculated the tensile and compressive stress concentration factors for a wide range of crystalline and non-crystalline transversely isotropic materials including some composite materials possessing high Young's modulus in the direction of the fibers. Results of the computations are presented in Table 5 which shows a substantial increase in the maximum tensile stress for highly anisotropic materials (i.e., Graphite Thornel). Furthermore, the highest tensile and compressive stresses for all of the anisotropic materials, listed in Table 5, occur at the equatorial line and pole respectively. This is a well recognized situation in isotropic materials but this observation should not lead to a general conclusion applicable to other anisotropic materials. To provide further insight into the influence of anisotropy on the stress concentration factor, sensitivity analyses for the four ratios mentioned in the preceeding section have been made and will be presented next.

7.2.1 Effect of $\frac{\bar{E}}{E}$

Figure 9 shows the variation of the highest tensile and compressive stress factors K_t and K_c against the variation of the ratio $\frac{\bar{E}}{E}$. It is seen that for values of $\frac{1}{6} < \frac{\bar{E}}{E} \leq 1$, K_t changes slightly and remains close to 2.0 but for $\frac{\bar{E}}{E} > 1$, K_t increases appreciably as $\frac{\bar{E}}{E}$ increases (i.e., for $\frac{\bar{E}}{E} = 10$ and 100, $K_t = 4.214$ and 11.11). Conversely, the factor K_c decreases as $\frac{\bar{E}}{E}$ increases and becomes negligible for very large values of (i.e., $\frac{\bar{E}}{E} = 10$ and 100, Kc = -0.140 and -0.041).

In Fig. 10, it is shown that for $\frac{\bar{E}}{E} = 0.167$ the highest tensile stress is calculated at $\gamma = 49.8°$ while the highest compressive stress is calculated at $\gamma = 19.2°$. As $\frac{\bar{E}}{E}$ deviates and becomes larger than 0.167, the location of the highest tensile stress starts shifting toward the equatorial line ($\gamma = 90°$) and remains there for $\frac{\bar{E}}{E} > 0.5$, whereas the location of highest compressive stress starts shifting toward the pole and remains at the pole for values of $\frac{\bar{E}}{E} > 0.3$.

7.2.2 Effect of $\frac{\bar{G}}{E}$

The variation of the tensile and compressive stress concentration factors with $\frac{\bar{G}}{E}$ are shown in Fig. 11. It can be observed that

(i) As the ratio $\frac{\bar{G}}{E}$ rises from 0.01 to 1, K_t decreases from 7.400 to 1.749.

(ii) Neither K_t, for $\frac{\bar{G}}{E} > 1$, nor K_c, for $\frac{\bar{G}}{E} > 0.01$, seems to be sensitive to the variation of $\frac{\bar{G}}{E}$.

(iii) For a sufficiently large value of $\frac{\bar{G}}{E}$, K_t and K_c approach the values 1.913 and −0.633 respectively.

Figure 12 shows that the location of the highest tensile stress is found to be on the equatorial line ($\gamma = 90°$) for values of $\frac{\bar{G}}{E} \leq 0.8$. This location starts shifting toward the pole as increases and deviates from 0.8. When $\frac{\bar{G}}{E}$ becomes sufficiently large, the highest tensile stress occurs, approximately, at $\gamma = 57.3°$. It should also be noted that for all values of $\frac{\bar{G}}{E}$, the highest compressive stress occurs at the pole.

7.2.3 Effect of Poisson's Ratios $\bar{\nu}$ and ν

Figures 13 and 14 show clearly that Poisson's ratios $\bar{\nu}$ and ν have little effect on the stress concentration factors K_t and K_c. When $\bar{\nu}$ transverses the range $0 < \bar{\nu} < \sqrt{\frac{3}{8}} = 0.612$, K_t changes slightly between 2.028 and 2.160 and K_c increases from -0.333 to -0.903. For values of $0 < \nu < \frac{7}{8}$ plotted in Fig. 14, K_t appears to be insensitive to the variation of ν and remains close to 2.0 whereas K_c increases from -0.45 to -2.80. The locations of the highest tensile and compressive stress are found to be at the equator and pole respectively for all values of $\bar{\nu}$ and ν.

7.3 Uniaxial Tension in the Direction Perpendicular to the Axis of Elastic Symmetry (x-Direction)

Obviously, for the case of an isotropic body with a spherical cavity, the solutions corresponding to tensions in the x-and z-direction are equivalent since these two directions are elastically and geometrically identical.

For transversely isotropic materials, the magnitude of the highest principal tensile and compressive stresses, K_t and K_c, are calculated for a variety of materials. Results of computation are given in Table 6. It is found that for the majority of these materials, the highest principal tensile stress occurs at the pole ($\gamma = 0°$). However, for some materials indicated by an asterisk in Table 6, the highest principal tensile stress occurs away from the pole. On the other hand, the highest principal compressive stress for all the materials in Table 6 occurs at $\theta = 0°$ and

$\gamma = 90°$. The effect of anisotropy on on the stress concentration factor on the stress concentration factor of the problem at hand is discussed next.

7.3.1 Effect of $\frac{\bar{E}}{E}$

Figure 15 presents the variation of the maximum principal tensile and compressive stress factors, K_t and K_c, on the cavity surface against the variation of the ratio $\frac{\bar{E}}{E}$. It is seen that the factor K_t is insensitive to the variation of $\frac{\bar{E}}{E}$, changes from a value of slightly above 2.5 to slightly below 2.0, and approaches a value of 1.946 for fairly large values of $\frac{\bar{E}}{E}$. In contrast, the factor K_c is considerably affected by the variation of $\frac{\bar{E}}{E}$, attains its absolute minimum (Kc = 2.022) at $\frac{\bar{E}}{E} = 1$, and increases monotonically as $\frac{\bar{E}}{E}$ increases. The variation of the location of the maximum principal tensile stress when $\frac{\bar{E}}{E}$ varies is shown in Fig. 16 while the maximum principal compressive stress occurs at $\theta = 0°$ and $\gamma = 90°$.

7.3.2 Effect of $\frac{\bar{G}}{E}$

In Fig. 17, K_t is plotted as a function of the ratio $\frac{\bar{G}}{E}$. It is seen that if $\frac{\bar{G}}{E}$ rises from 0.01 to 0.4, K_t decreases from 2.923 to 2.022. When $\frac{\bar{G}}{E}$ changes from 0.4 to 9, K_t changes slightly from 2.022 to 1.893. For values of $\frac{\bar{G}}{E}$ greater than 9, as $\frac{\bar{G}}{E}$ increases, the magnitude of K_t increases. The location at which K_t is evaluated is shown in the same plot. A similar plot is made for K_c and shown in Fig. 18. It appears that K_c is insensitive to the variation of $\frac{\bar{G}}{E}$ when $\frac{\bar{G}}{E} < 10$ but it increases as the ratio $\frac{\bar{G}}{E}$ increases (i.e., $\frac{\bar{G}}{E} = 100$, $K_t = 5.003$ at $\theta = 50°$ and $\gamma = 86°$, and $K_c = -4.726$ at $\theta = 40°$ and $\gamma = 86°$).

7.3.3 Effect of $\bar{\nu}$ and ν

Fig. 19 shows that neither K_t nor K_c is sensitive to the variation of $\bar{\nu}$. The location at which K_t is evaluated is shown in Fig. 20 whereas K_c is found to be at $\theta = 0°$ and $\gamma = 90°$ for all plotted values of ν. In Fig. 21 K_t and K_c are plotted as a function of ν. It is seen that when ν changes from 0 to 0.875, K_t rises from 2.002 to 4.017. The location at which K_t is evaluated is $\theta = 90°$ and $\gamma = 90°$ for $\nu \leq 0.25$ and is $\theta = 90°$ and $\gamma = 0°$ for $\nu \geq 0.25$. In the same plot, K_c changes from -0.525 to -1.115, when ν changes from 0 to 0.875, and occur at $\theta = 0°$ and $\gamma = 90°$.

Table 5. Stress Concentration Factor on the Surface of Spherical Cavity Under Uniaxial Tension in the Direction of the Axis of Elastic Symmetry

Material	Stress Concentration Factor	
	Tension (K_t)	Compression (K_c)
Beryllium	1.980	-0.248
Bone (fresh phalanx)	2.328	-0.654
Cadmium	1.786	-1.033
Ceramics ($BaTiO_3$)	2.048	-0.729
Cobalt	2.285	-0.690
Ecologite	2.130	-0.532
Graphite	2.753	-0.669
Graphite Thornel 50	4.683	-0.146
Graphite Thornel 75	5.637	-0.117
Hafnium	2.107	-0.616
Ice	2.231	-0.664
Magnesium	2.129	-0.633
Micha Schist	1.980	-0.538
Quartz	2.025	-0.491
Rhenium	2.220	-0.596
Silver Aluminum	2.210	-0.743
Titanium	2.155	-0.704
Zinc	1.621	-1.000
Zinc Oxide	2.196	-0.769
Zirconium	2.243	-0.677
Isotropic medium ($\nu = 0.25$)	2.022	-0.587

Table 6. Stress Concentration Factor on the Surface of Spherical Cavity Under Uniaxial Tension Perpendicular to the Axis of Elastic Symmetry

Material	Stress Concentration Factor	
	Tension (K_t)	Compression (K_c)
Beryllium	1.905	-0.278
Bone (fresh phalanx)	2.127	-0.976
Cadmium	2.316*	-0.763
Ceramics ($BaTiO_3$)	2.069*	-0.724
Cobalt	2.262	-0.770
Ecologite	2.476*	-0.630
Graphite	2.918*	-0.946
Graphite Thornel 50	1.935	-2.066
Graphite Thornel 75	1.921	-2.793
Hafnium	2.075	-0.649
Ice	2.142	-0.726
Magnesium	2.105	-0.665
Micha Schist	2.074*	-0.518
Quartz	2.021*	-0.526
Rhenium	2.146	-0.566
Silver Aluminum	2.176	-0.812
Titanium	2.177	-0.736
Zinc	2.315	-0.637
Zinc Oxide	2.186	-0.792
Zirconium	2.150	-0.745
Isotropic medium ($\nu = 0.25$)	2.022	-0.587

CHAPTER 8

SUMMARY, CONCLUSION, AND RECOMMENDATION FOR FURTHER STUDY

8.1 Summary and Conclusion

The principal result of this study has been the development of explicit analytical solutions for the (non-axisymmetric) first and second boundary value problems of elasticity theory for a spheroidal cavity embedded in a transversely isotropic medium. The analysis is based upon solutions of the homogeneous displacement equations of equilibrium in terms of three quasi-harmonic potential functions taken in a special combination of the associated Legendre functions of the first and second kind. Exact solutions have been obtained for problems involving a region containing a rigid spheroidal inclusion and a region containing a traction-free cavity subjected to constant loadings applied at sufficiently long distances from the cavity. A tractable problem which can be treated in a similar fashion is the hyperboloidal notch in a transversely isotropic material under arbitrary loadings (see Reference [8]).

8.2 Recommendation for Further Study

The applications of the present approach are by no means exhausted in this work. Further study should be made of the following:

1. Problems which can be solved directly by the present approach. These include solutions of spheroidal cavities and inclusions under loading conditions different from those we have obtained, as well as numerical investigations including the effect on the stress concentration factors of the shape ratio of the spheroidal cavity or inclusion and the effect of anisotropy on the decay of stresses with distance from the cavity or inclusion.
2. The mixed and mixed-mixed boundary value problems for a transversely isotropic medium containing a spheroidal cavity or inclusion.
3. Problems in which the elastic medium is bounded by other geometries (i.e., the hyperboloidal notch).

FIGURES

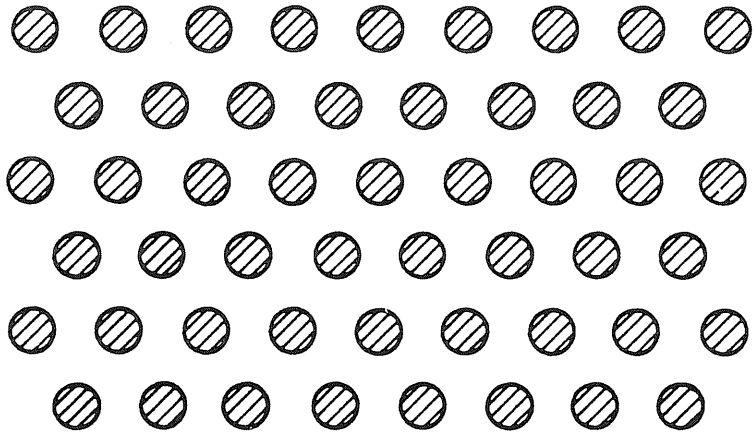

Figure 1. Hexagonal Arrays

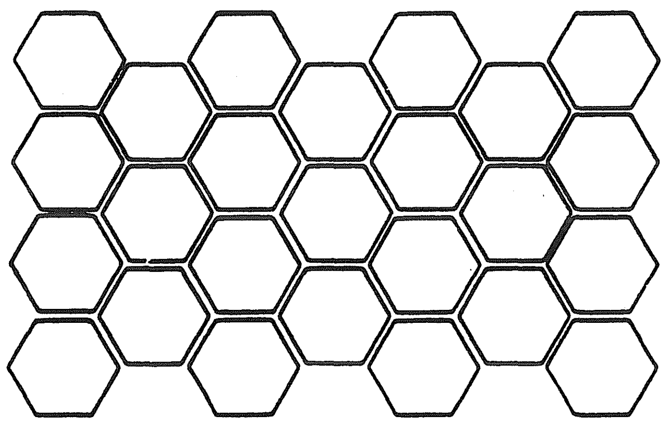

Figure 2. Honeycomb Structures

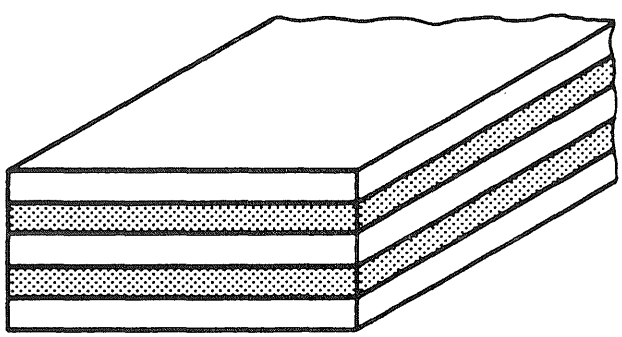

Figure 3. Stratified Materials

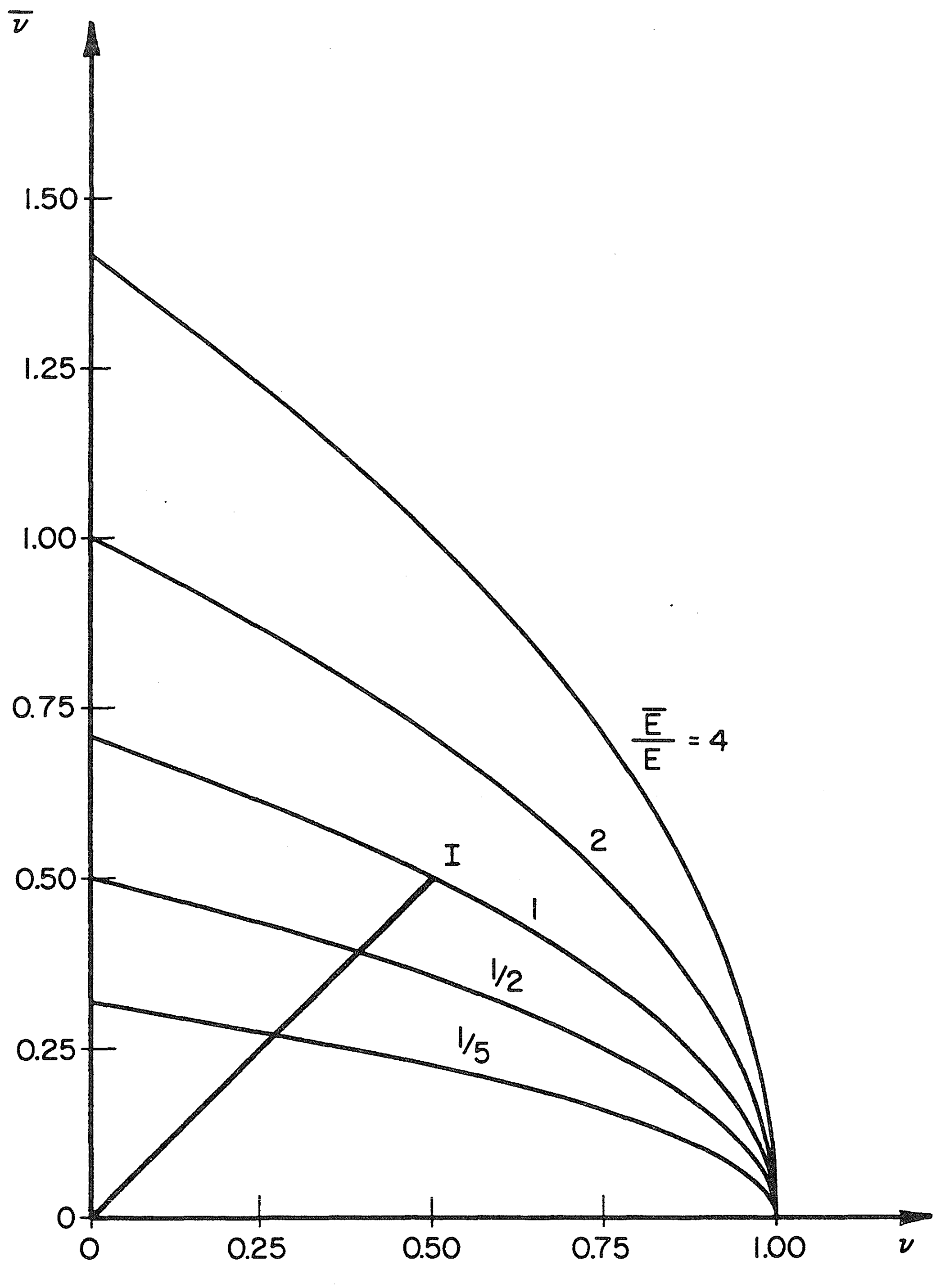

Figure 4. Variation of the Engineering Constants

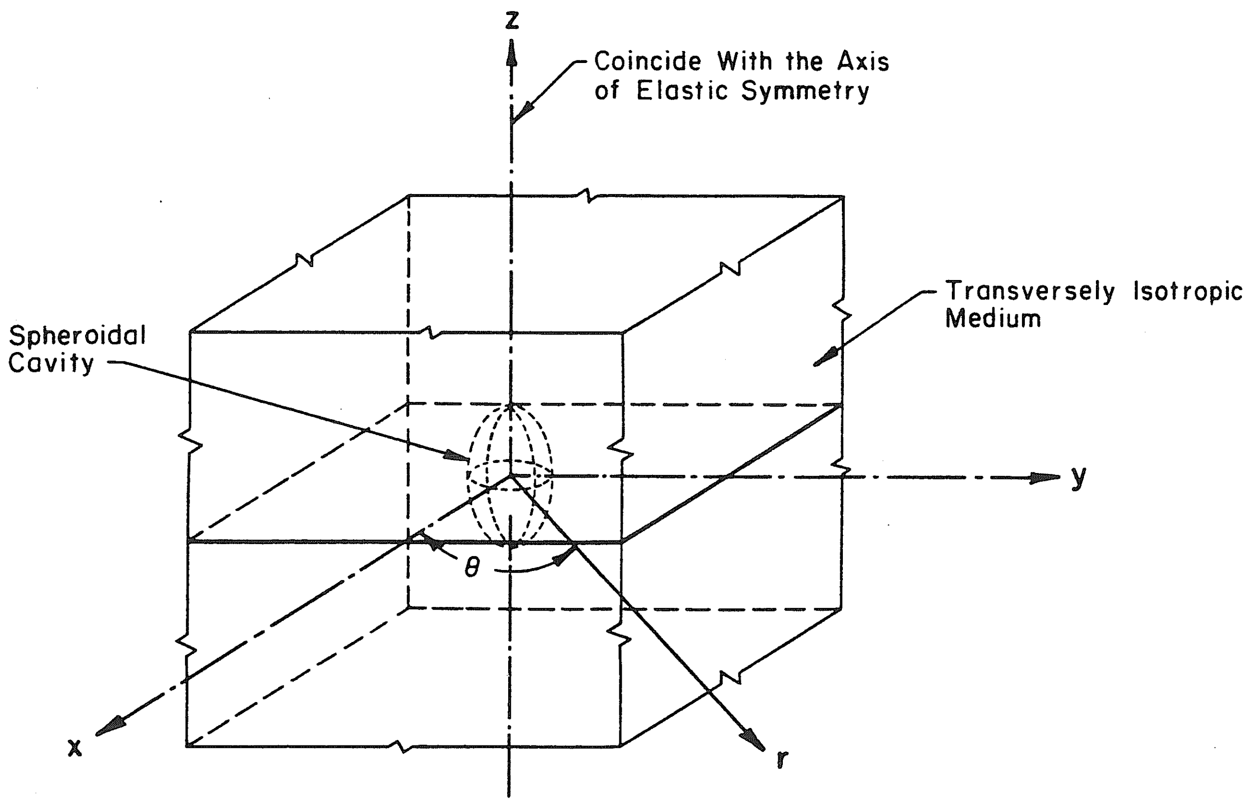

Figure 5. Transversely Isotropic Medium
Containing a Spheroidal Cavity

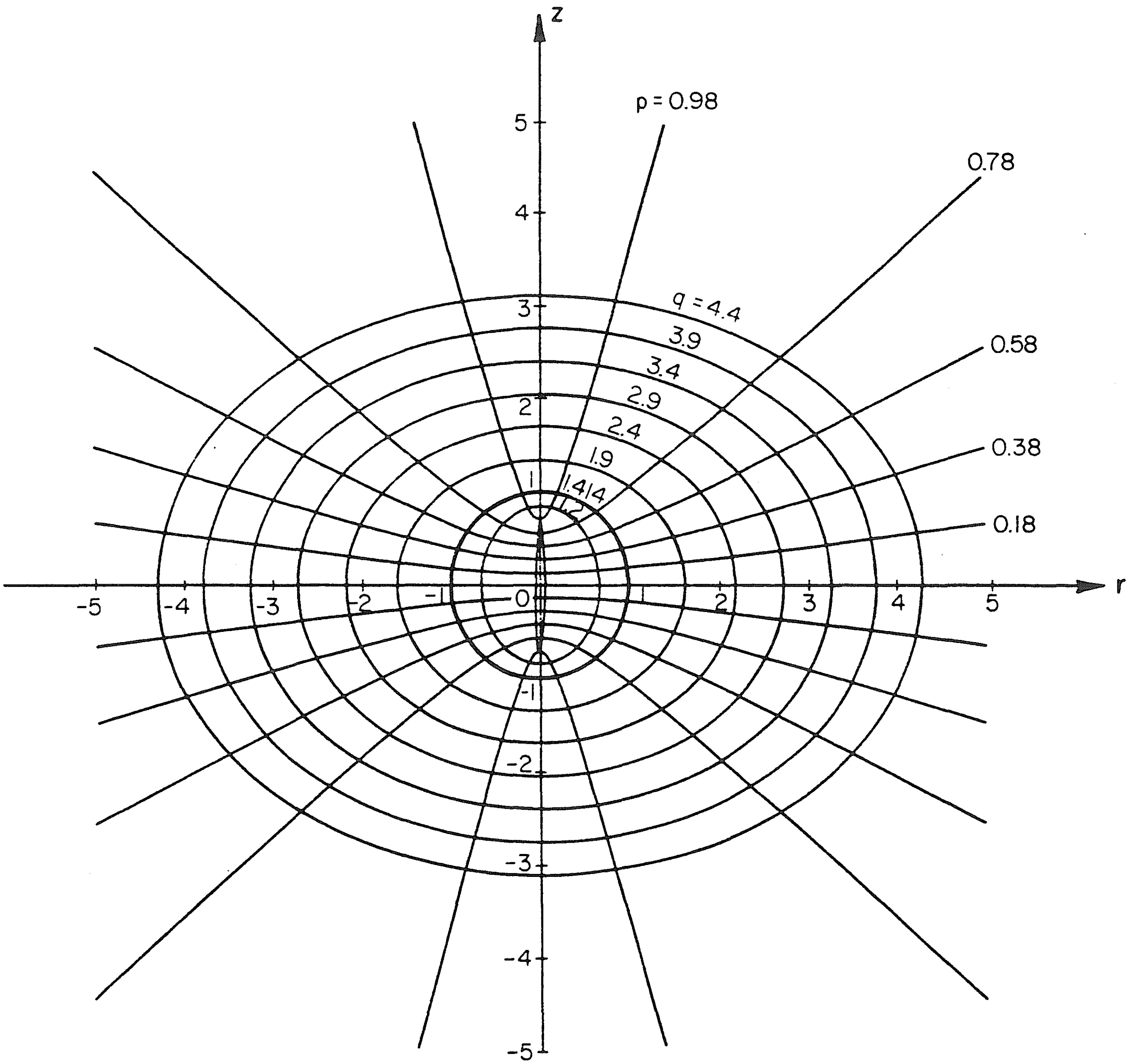

Figure 6. Example of the Coordinate System Corresponding to a Spherical Cavity with $\nu_j^2 = 0.5$

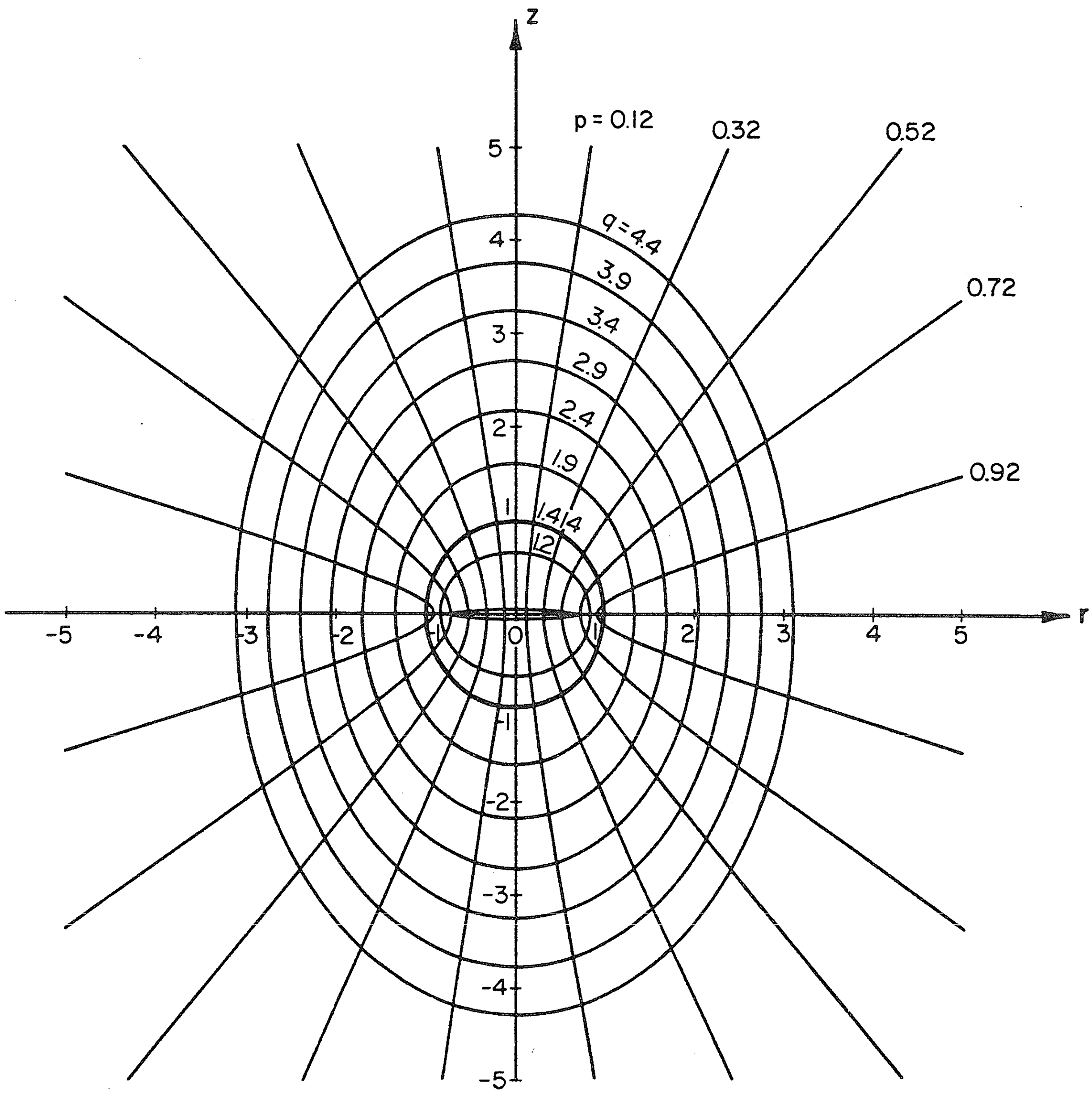

Figure 7. Example of the Coordinate System Corresponding to a Spherical Cavity with $\nu_j^2 = 2.0$

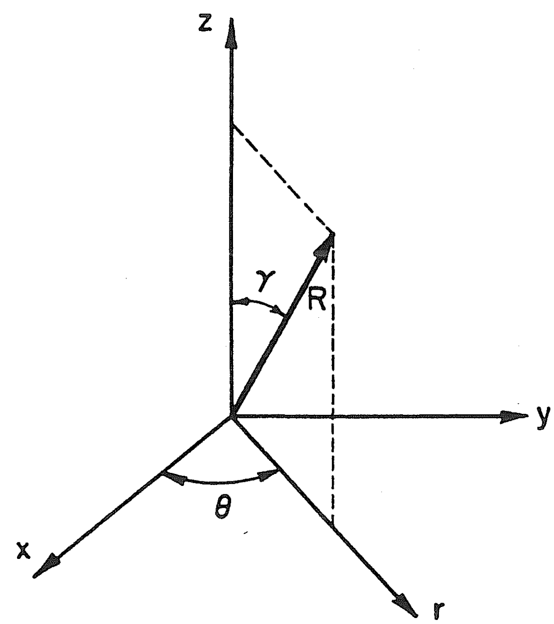

Figure 8. Spherical Coordinate System

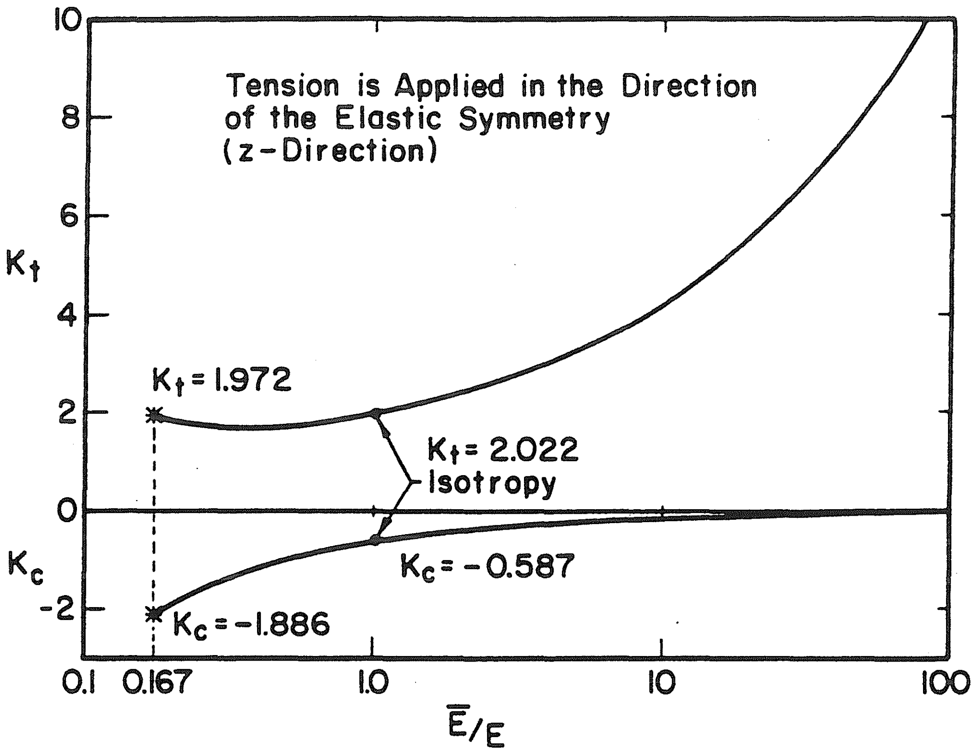

Figure 9. Variation of Stress Concentration with Modulus Ratios $\frac{\bar{E}}{E}$

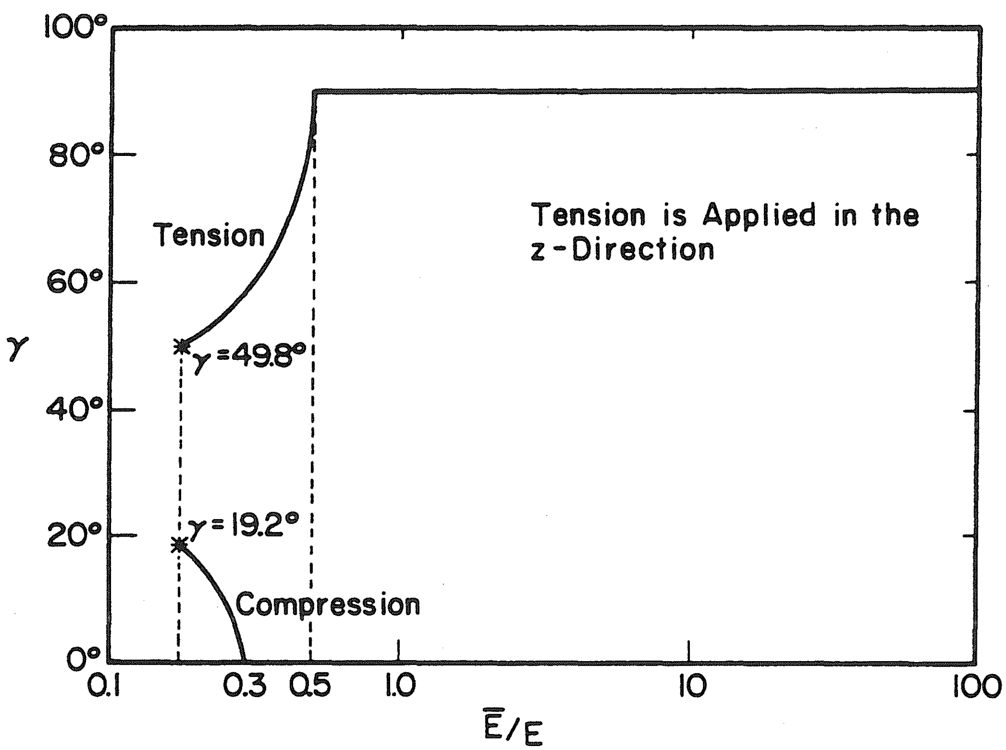

Figure 10. Meridional Position of Stress Concentration versus $\frac{\bar{E}}{E}$

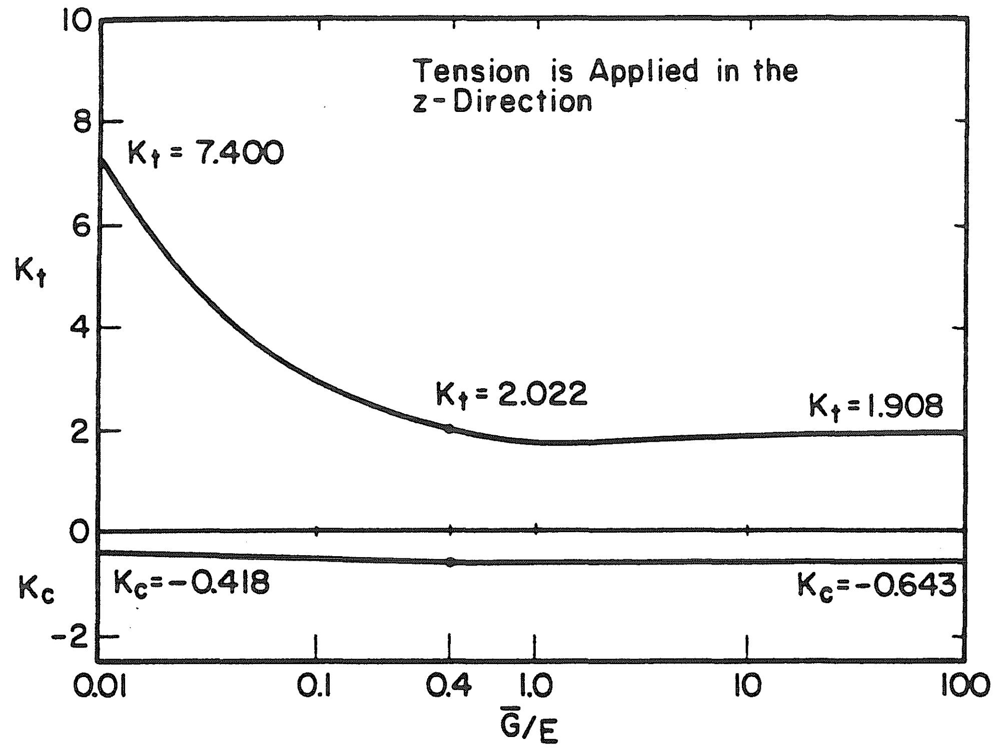

Figure 11. Variation of Stress Concentration with Modulus Ratios $\frac{\bar{G}}{E}$

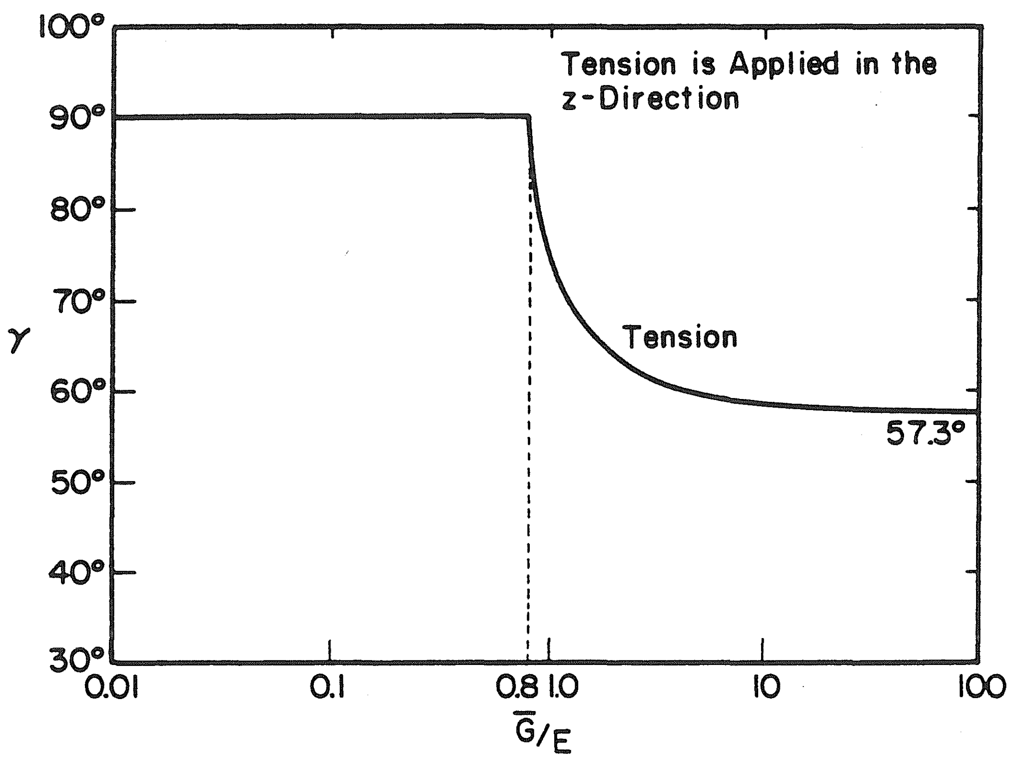

Figure 12. Meridional Position of Stress Concentration versus $\frac{\bar{G}}{E}$

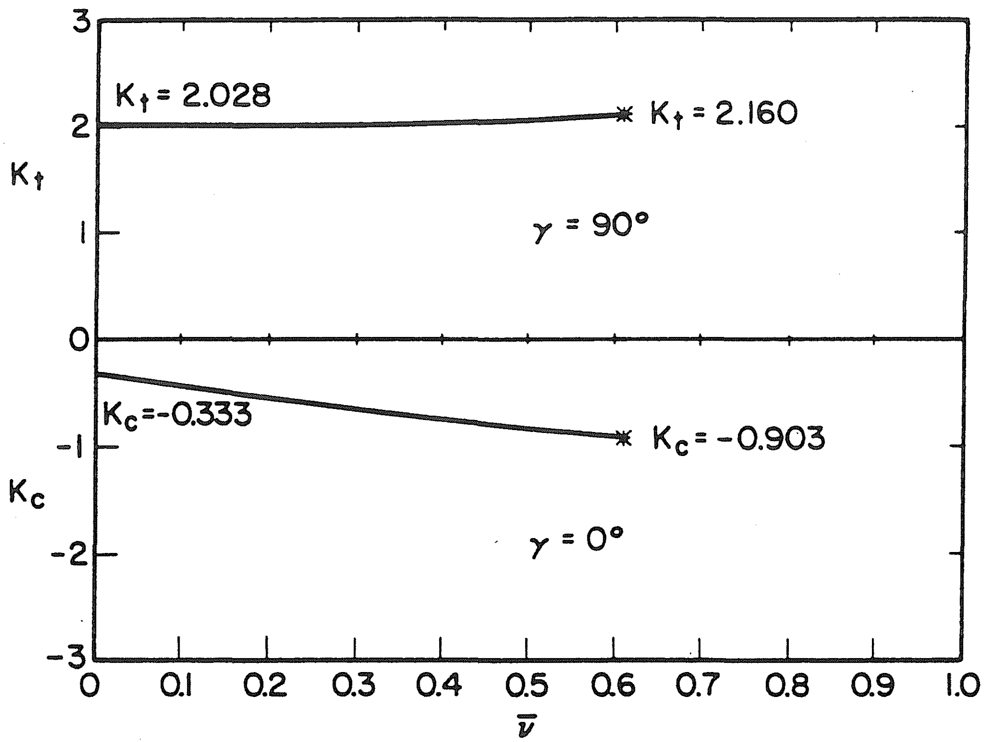

Figure 13. Variation of Stress Concentration with $\bar{\nu}$

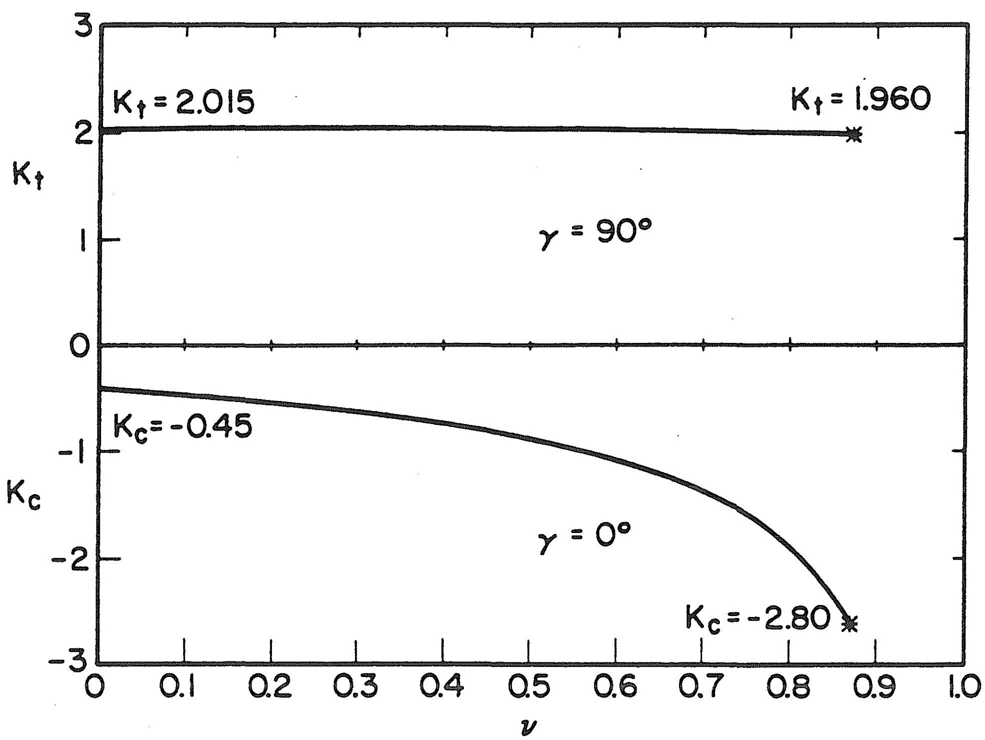

Figure 14. Variation of Stress Concentration with ν

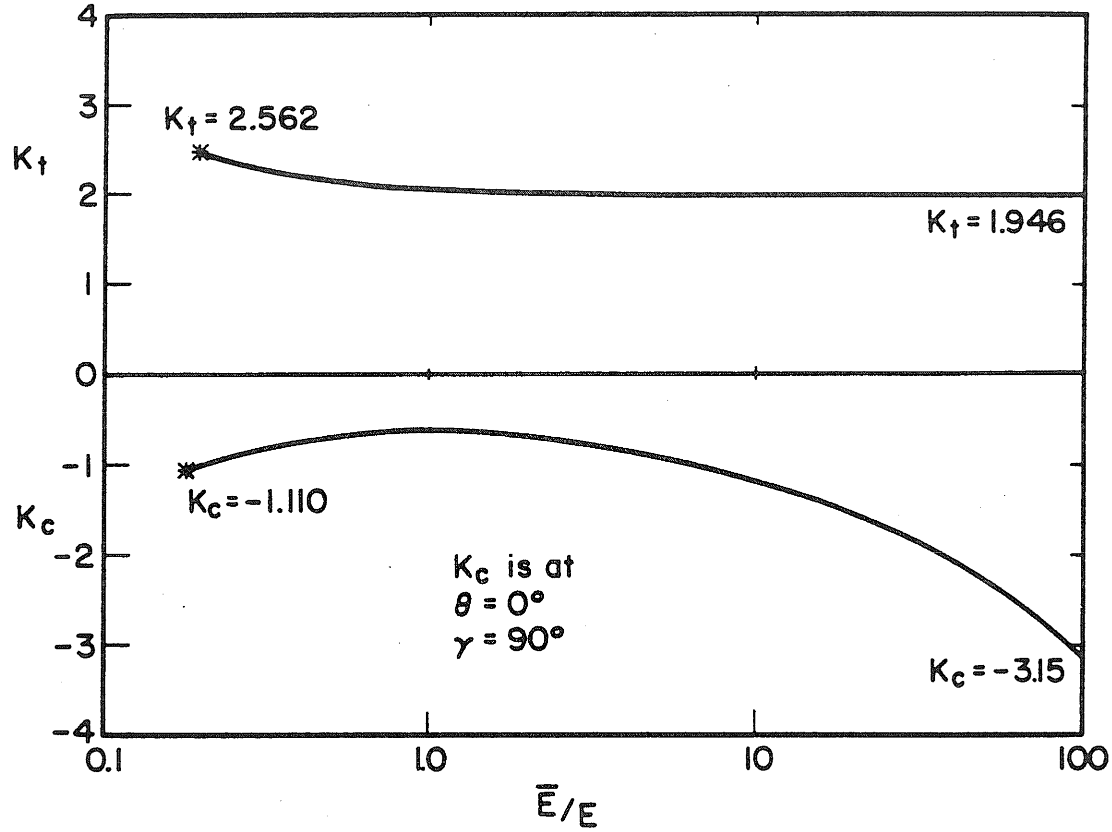

Figure 15. Variation of Stress Concentration with $\frac{\bar{E}}{E}$

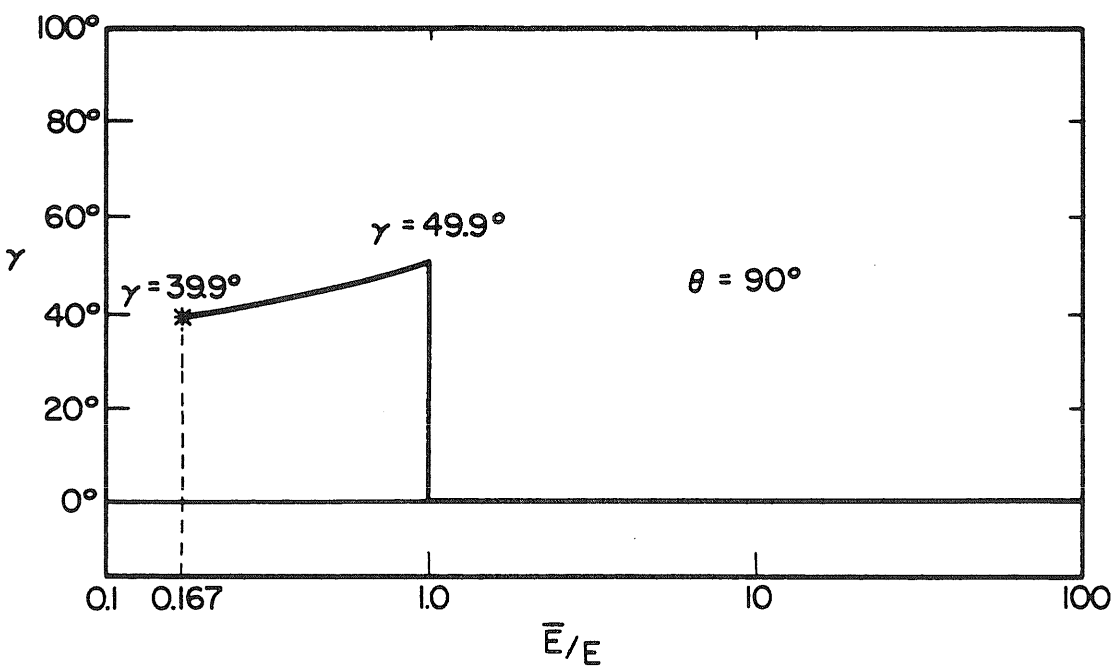

Figure 16. Meridional Position γ of the Tensile Stress Concentration Factor versus $\frac{\bar{E}}{E}$

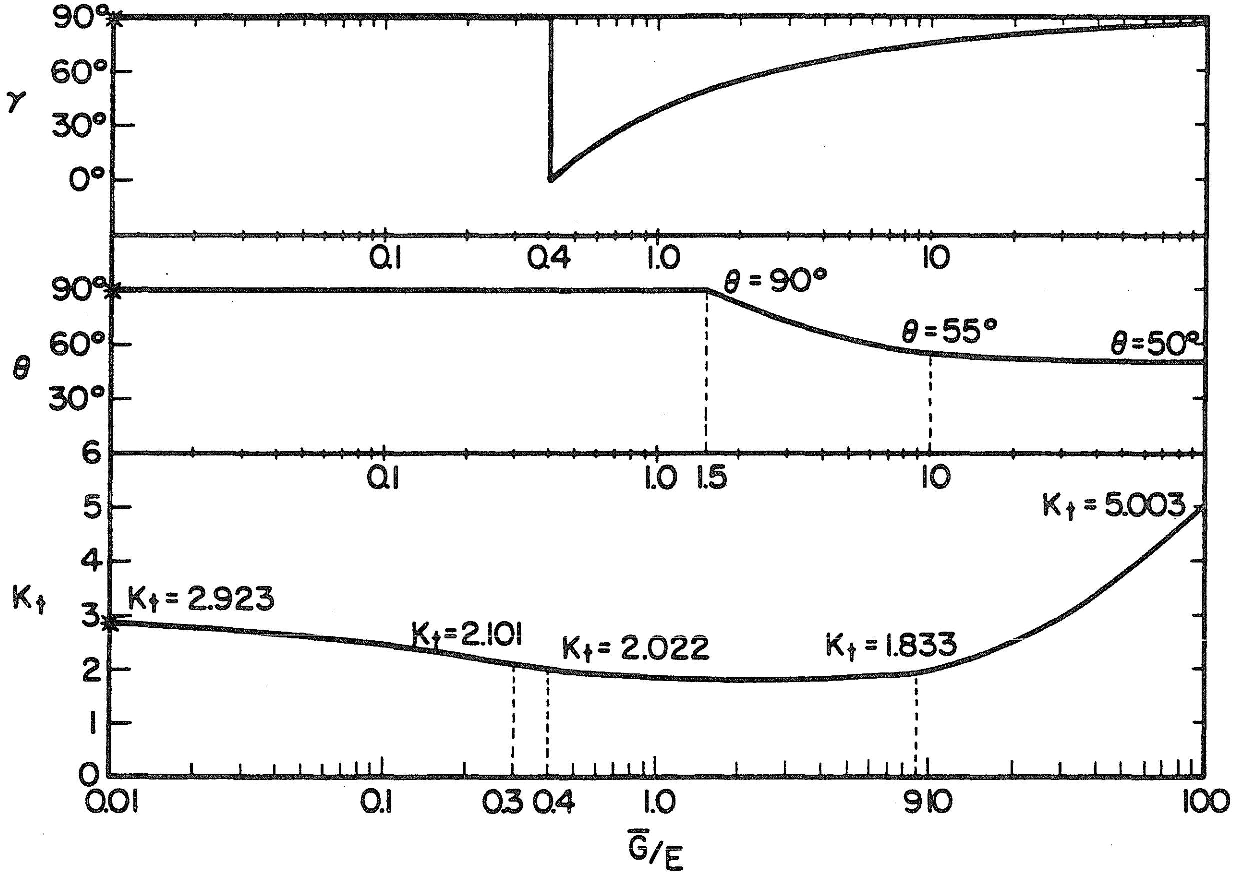

Figure 17. Tensile Stress Concentration Factor and Position versus $\frac{\bar{G}}{E}$

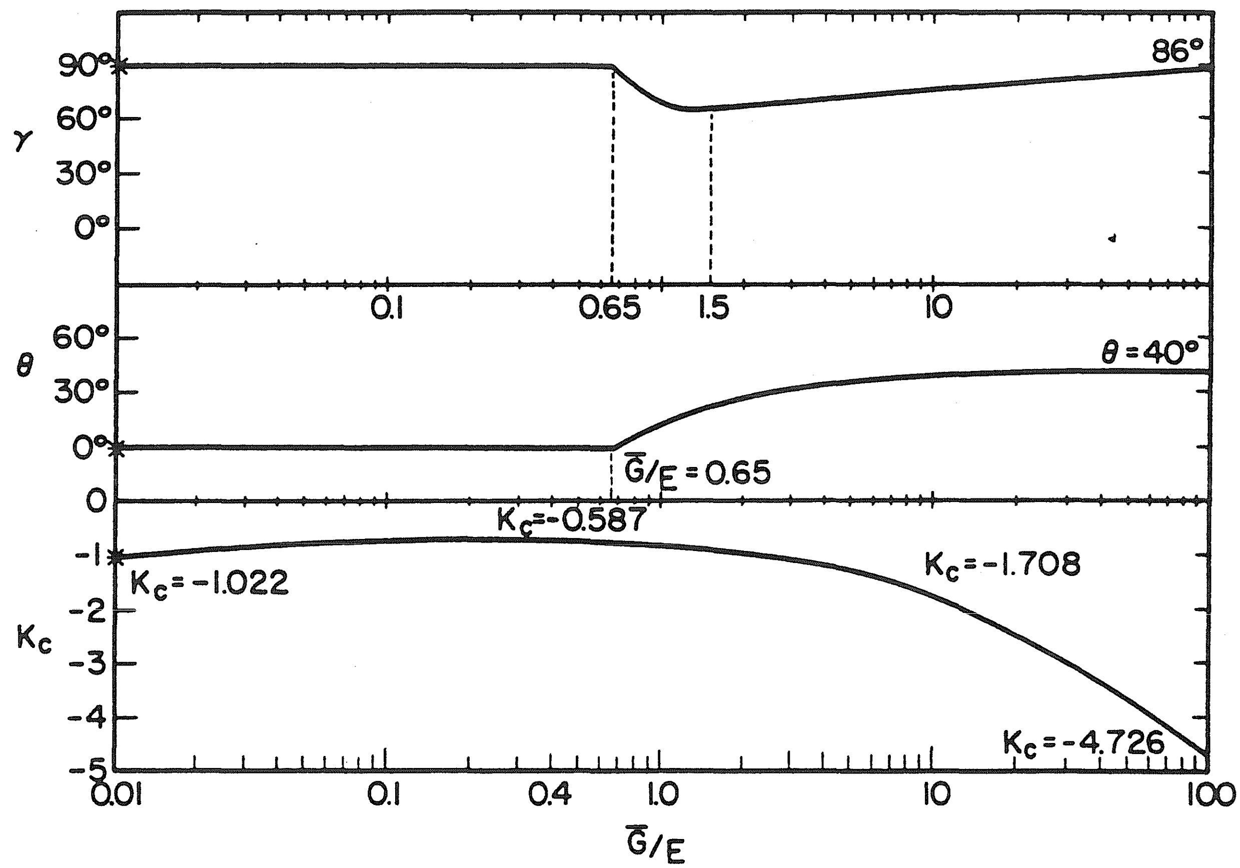

Figure 18. Compressive Stress Concentration Factor and Position versus $\frac{\bar{G}}{E}$

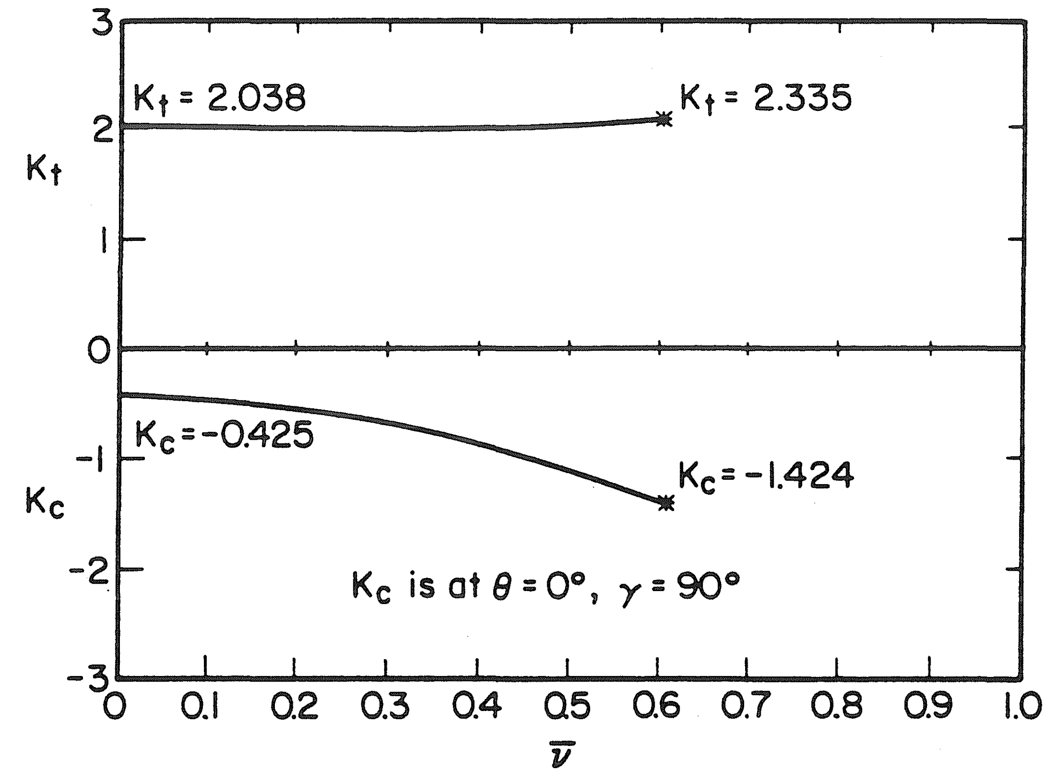

Figure 19. Variation of Stress Concentration with $\bar{\nu}$

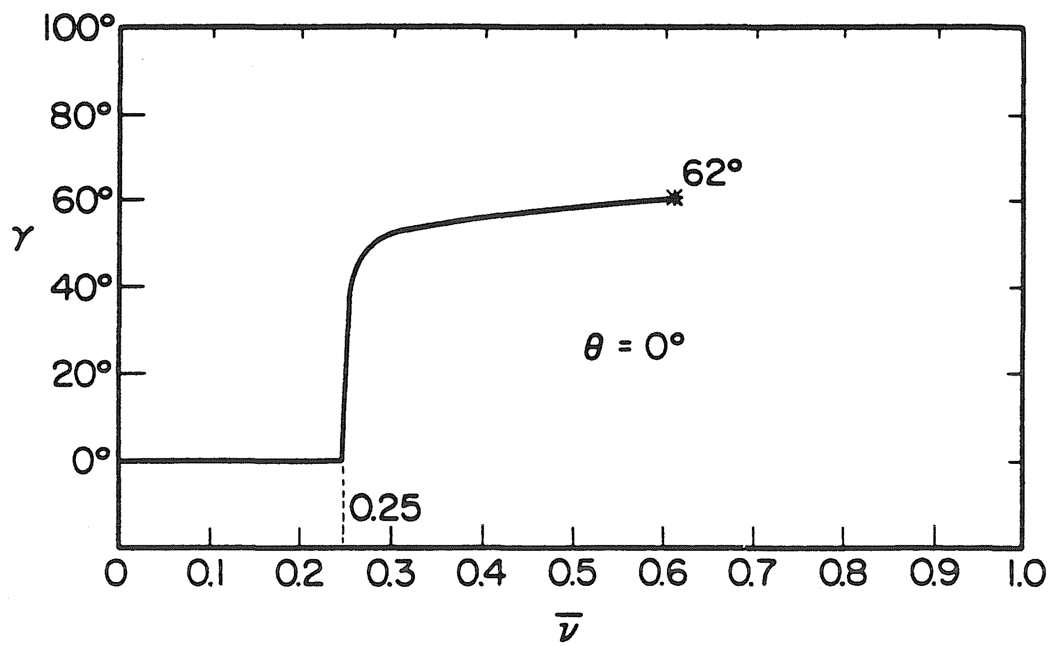

Figure 20. Meridional Position of Stress Concentration with $\bar{\nu}$

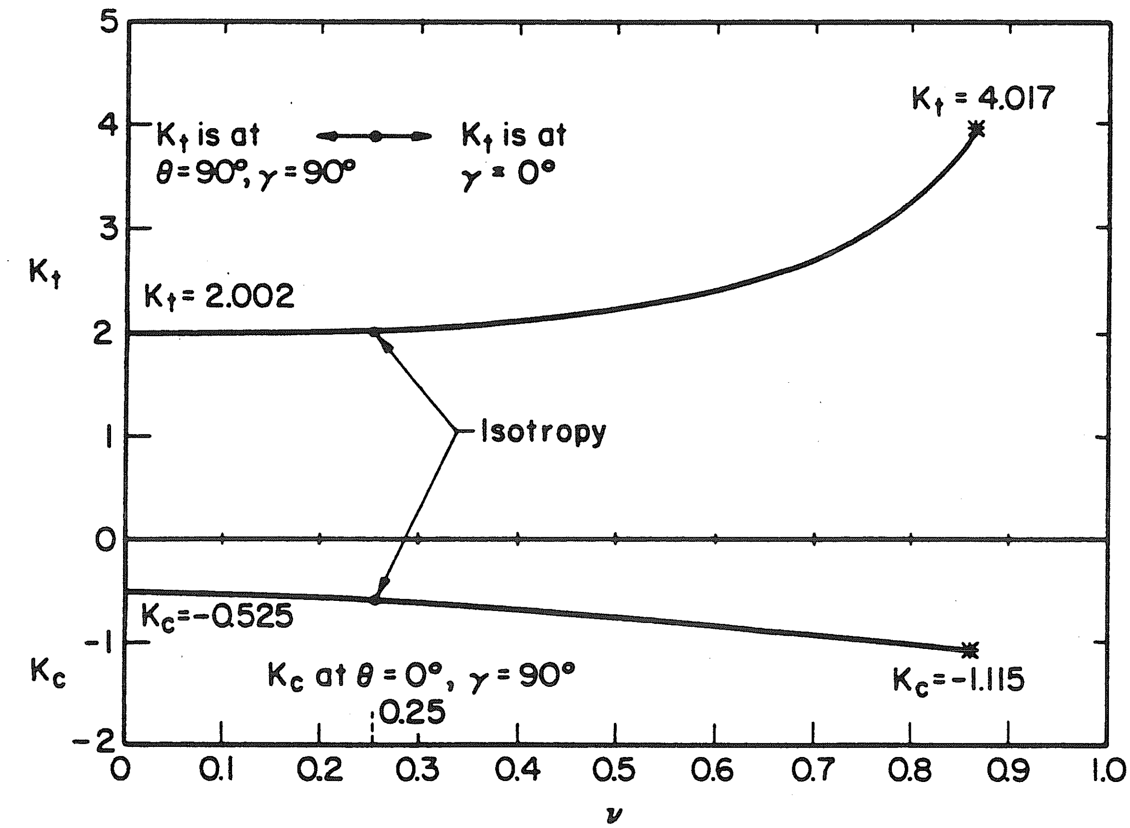

Figure 21. Variation of Stress Concentration with ν

APPENDIX A.1

LEGENDRE ASSOCIATED FUNCTIONS OF FIRST KIND

$$P_0(p) = 1$$

$$P_1(p) = p$$

$$P_2(p) = \frac{1}{2}(3p^2 - 1)$$

$$P_3(p) = \frac{p}{2}(5p^2 - 3)$$

$$P_4(p) = \frac{1}{8}(35p^4 - 30p^2 + 3)$$

$$P_0^{-1}(p) = \frac{(1-p)}{(1-p^2)^{\frac{1}{2}}}$$

$$P_1^{-1}(p) = \frac{1}{2}(1-p^2)^{\frac{1}{2}}$$

$$P_2^{-1}(p) = \frac{1}{2}\, p\, (1-p^2)^{\frac{1}{2}}$$

$$P_3^{-1}(p) = \frac{1}{8}(5p^2 - 1)(1-p^2)^{\frac{1}{2}}$$

$$P_4^{-1}(p) = \frac{1}{8}\, p\, (7p^2 - 3)(1-p^2)^{\frac{1}{2}}$$

$$P_0^{-2}(p) = \frac{(1-p)^2}{2(1-p^2)}$$

$$P_1^{-2}(p) = \frac{(p^3 - 3p + 2)}{6(1-p^2)}$$

$$P_2^{-2}(p) = \frac{1}{8}(1-p^2)$$

$$P_3^{-2}(p) = \frac{1}{8}\, p\, (1-p^2)$$

$$P_4^{-2}(p) = \frac{1}{48}(1-p^2)(7p^2 - 1)$$

$$P_1^{-3}(p) = -\frac{(p^4 - 6p^2 + 8p - 3)}{24(1-p^2)^{3/2}}$$

$$P_2^{-3}(p) = \frac{(8 - 15p + 10p^3 - 3p^2)}{120(1-p^2)^{3/2}}$$

$$P_3^{-3}(p) = \frac{1}{48}(1-p^2)^{3/2}$$

$$P_4^{-3}(p) = \frac{1}{48}\, p\, (1-p^2)^{3/2}$$

APPENDIX A.2

LEGENDRE ASSOCIATED FUNCTIONS OF SECOND KIND

$$Q_0(q) = \frac{1}{2} \operatorname{Ln} \frac{q+1}{q-1}$$

$$Q_1(q) = \frac{q}{2} \operatorname{Ln} \frac{q+1}{q-1} - 1$$

$$Q_2(q) = \frac{1}{4} (3q^2 - 1) \operatorname{Ln} \frac{q+1}{q-1} - \frac{3}{2} q$$

$$Q_3(q) = \frac{1}{4} q (5q^2 - 3) \operatorname{Ln} \frac{q+1}{q-1} - \frac{1}{6} (15q^2 - 4)$$

$$Q_4(q) = \frac{1}{16} (35q^4 - 30q^2 + 3) \operatorname{Ln} \frac{q+1}{q-1} - \frac{5}{24} q (21q^2 - 11)$$

$$Q_0^1(q) = \frac{-1}{(q^2 - 1)^{\frac{1}{2}}}$$

$$Q_1^1(q) = \frac{1}{2} (q^2 - 1)^{\frac{1}{2}} \operatorname{Ln} \frac{q+1}{q-1} - \frac{q}{(q^2 - 1)^{\frac{1}{2}}}$$

$$Q_2^1(q) = \frac{3}{2} q (q^2 - 1)^{\frac{1}{2}} \operatorname{Ln} \frac{q+1}{q-1} - \frac{(3q^2 - 2)}{(q^2 - 1)^{\frac{1}{2}}}$$

$$Q_3^1(q) = \frac{3}{4} (5q^2 - 1)(q^2 - 1)^{\frac{1}{2}} \operatorname{Ln} \frac{q+1}{q-1} - \frac{q(15q^2 - 13)}{2(q^2 - 1)^{\frac{1}{2}}}$$

$$Q_0^2(q) = \frac{2q}{q^2 - 1}$$

$$Q_1^2(q) = \frac{2}{q^2 - 1}$$

$$Q_2^2(q) = \frac{3}{2} (q^2 - 1) \operatorname{Ln} \frac{q+1}{q-1} + \frac{q(5 - 3q^2)}{q^2 - 1}$$

$$Q_3^2(q) = \frac{15}{2} q (q^2 - 1) \operatorname{Ln} \frac{q+1}{q-1} - 15q^2 + 10 + \frac{2}{q^2 - 1}$$

$$Q_0^3(q) = \frac{-2(3q^2+1)}{(q^2-1)^{3/2}}$$

$$Q_1^3(q) = -\frac{8q}{(q^2-1)^{3/2}}$$

$$Q_2^3(q) = -\frac{8}{(q^2-1)^{3/2}}$$

$$Q_3^3(q) = \frac{15}{2}(q^2-1)^{3/2}\,\mathrm{Ln}\,\frac{q+1}{q-1} - 15q(q^2-1)^{\frac{1}{2}} + \frac{10q}{(q^2-1)^{1/2}} - \frac{8q}{(q^2-1)^{3/2}}$$

APPENDIX B

TRANSFORMATION OF THE STRESS COMPONENTS FROM CYLINDRICAL TO SPHEROIDAL COORDINATES

Let the spheroidal coordinate system be defined by means of the transformation

$$r = \alpha(q^2 - 1)^{\frac{1}{2}}(1 - p^2)^{\frac{1}{2}}$$

$$z = \alpha\, q\, p \tag{B.1}$$

$$\theta = \theta$$

in which it is customary to use $q = \cosh\eta$ and $p = \cos\theta$ where the ranges of η, ϕ, and θ are

$$0 \le \eta < \infty \quad , \quad 0 \le \phi \le \pi \quad , \quad 0 \le \theta < 2\pi \tag{B.2}$$

and let η_0 be the value on the surface of a spheroid whose semi-axes are a and b. Then, $a = \alpha \cosh\eta_0$, $b = \alpha \sinh\eta_0$, and the components of the stress tensor along three orthogonal directions (η, ϕ, θ) can be written in terms of the components of the stress tensor in the cylindrical coordinate system as follows.

$$\sigma_{\eta\eta} = \frac{1}{r^2 + s^4 z^2}\,[r^2\, \sigma_{rr} + s^4 z^2 \sigma_{zz} + 2s^2\, r\, z\, \sigma_{rz}]$$

$$\sigma_{\phi\phi} = \frac{1}{r^2 + s^4 z^2}\,[s^4 z^2 \sigma_{rr} + r^2\, \sigma_{zz} - 2s^2\, r\, z\, \sigma_{rz}] \tag{B.3}$$

$$\sigma_{\theta\theta} = \sigma_{\theta\theta}$$

$$\sigma_{\phi\theta} = \frac{1}{\sqrt{r^2 + s^4 z^2}} [s^2 z \sigma_{r\theta} - r \sigma_{z\theta}]$$

$$\sigma_{\eta\theta} = \frac{1}{\sqrt{r^2 + s^4 z^2}} [r \sigma_{r\theta} + s^2 z \sigma_{z\theta}]$$

$$\sigma_{\eta\phi} = \frac{1}{r^2 + s^4 z^2} [s^2 r z(\sigma_{rr} - \sigma_{zz}) + (s^4 z^2 - r^2) \sigma_{rz}]$$

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