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A NONLINEAR PROGRAMMING APPROACH TO THE MINIMUM WEIGHT ELASTIC DESIGN OF STEEL STRUCTURES

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NOTATION

A = the cross-sectional area of a member

A_{net} = the area of the net section of a tension member

B = the boundary of the feasible region, R

$\underline{b}_i$ = a lower bound for the ith design variable

$\overline{b}_i$ = an upper bound for the ith design variable

C_c = the effective slenderness ratio, kL/r, above which a column buckles elastically

C_m = a multiplier

E_n = Euclidian n-dimensional space

$F(\overline{x})$ = $-W(\overline{x})/W_o$ = the current value of the objective function

F.S. = the factor of safety for axial compression

F_a = the allowable stress in axial compression

F_b = the allowable stress in bending

F'_e = the elastic buckling stress for an axially loaded column

F_{max} = the optimum value of the objective function

F_t = the allowable stress in tension on the net section

F_y = the yield stress

f_a = the computed axial compressive stress

f_b = the computed bending stress

G = the intersection of the hypersurfaces $G_1, \ldots, G_q$

G_i = the hypersurface in E_n defined by $\varphi_i(\overline{x}) = 0$

$\overline{g}(\overline{x})$ = $\nabla F(\overline{x})$ = the gradient of the objective function at $\overline{x}$

$H_i(\overline{x}_o)$ = the supporting hyperplane to G_i at $\overline{x}_o$

I = a moment of inertia; also, the identity matrix

i = a positive integer

j = a positive integer

$K(\epsilon)$ = an upper bound on the number of steps required to reach $\bar{x}_{max}$

k = a positive integer; also, a multiplier

kL = the effective length of a compression member

L = the maximum length of R; also, the length of a member; also, the unbraced length of a compression member in a pin-jointed structure

ℓ = a positive integer

M = a bending moment

M_L, M_R = the clockwise bending moments at the left and right ends, respectively, of a beam

M_T, M_B = the clockwise bending moments at the top and bottom ends, respectively, of a column

M_{max} = the maximum moment in a beam for design purposes

M^+_{max} = the maximum positive moment in a beam

m = the number of constraints; also, a positive integer

N = an upper bound on the number of iterations required to correct back to the feasible region

n = the number of variables; also, a positive integer

P = an externally applied load; also, the axial force in a member of a pin-jointed frame; also, the compressive force in a column

$P_q(\bar{x})$ = $I - U_q(\bar{x})V_q(\bar{x})U_q^T(\bar{x})$ = the symmetric n x n projection matrix

$Q(\bar{x}_o)$ = the intersection of the supporting hyperplanes $H_1(\bar{x}_o)$, ..., $H_q(\bar{x}_o)$ at $\bar{x}_o$

q = the current number of active constraints

R = the feasible region, a subset of E_n

R' = a region in E_n which contains R

r = the radius of gyration in the plane of bending of a member in a rigid frame; also, the minimum radius of gyration of a member of a pin-jointed truss

r_{min} = the minimum radius of gyration of a member

$\bar{r}(\bar{x}_\nu)$ = $V_q(\bar{x}_\nu)U_q^T(\bar{x}_\nu)\bar{g}$ = a vector of order q

S = a section modulus

$U_q(\bar{x})$ = $\{\bar{u}_1(\bar{x}),\ldots,\bar{u}_q(\bar{x})\}$ = a qxq matrix

$\bar{u}_i(\bar{x})$ = $\nabla\varphi_i(\bar{x})$ = the gradient of $\varphi_i(\bar{x})$ at $\bar{x}$, a vector of order n

$V_q(\bar{x})$ = $[U_q^T(\bar{x})U_q(\bar{x})]^{-1}$ = a symmetric qxq matrix

v_{ii} = the ith diagonal element of $V_q(\bar{x})$

$W(\bar{x})$ = the current total weight of a structure

W_o = a reference value for defining the non-dimensional objective function

w = the weight of a member per unit length

$\bar{w}_q(\bar{x}_\nu)$ = $\{\varphi_1(\bar{x}_\nu),\ldots,\varphi_q(\bar{x}_\nu)\}$ = the error vector of order q

$|\bar{w}_q(\bar{x}_\nu)|$ = the length of the error vector, a measure of the nearness of the point $\bar{x}_\nu$ to the intersection G

$\bar{x}$ = $\{x_1,\ldots,x_n\}$ = the current design, a vector of order n, a point in E_n

x_i = the current value of the ith design variable

$\bar{x}_{max}$ = the optimum solution

$\bar{x}_o$ = a reference value for defining the non-dimensional design variables

$\bar{x}_\nu$ = a point which lies near the intersection, G, of q constraints, $1 \leq q \leq m$

$\bar{z}(\bar{x}_\nu)$ = $P_q(\bar{x}_\nu)\bar{g}/|P_q(\bar{x}_\nu)\bar{g}|$ = a unit vector of order n which points in the direction of the projected gradient

α = a measure of the linear independence of the vectors $\bar{u}_1(\bar{x}),\ldots,$ $u_q(x)$; also, the angle of inclination of an external load on a three-bar truss

$\beta(\bar{x}_\nu)$ = $\max\,\{|P_q(\bar{x}_\nu)\bar{g}|,\beta_\ell\}$

β_i = the angle of inclination of the ith member in a three-bar truss

$\beta_i(\bar{x}_\nu)$ = $r_i(\bar{x}_\nu)/2\sqrt{v_{ii}}$

$\beta_\ell(\bar{x}_\nu)$ = $\max_i\,\{\beta_i(\bar{x}_\nu)\}$

δ = a tolerance which controls the nearness of the point x_ν to the intersection G

ϵ = a tolerance which controls the nearness of the point $\bar{x}_{max}$ to the true optimum point

λ = a measure of the curvature of the intersection of the active constraints

ξ = a measure of the distance from the end of a uniformly loaded beam to the point of maximum positive moment

σ' = a measure of the optimum step length

τ = a step length

$\varphi_i(\bar{x})$ = the current value of the ith constraint

$\nabla\varphi_i(\bar{x})$ = $\bar{u}_i(\bar{x})$ = the gradient of $\varphi_i(\bar{x})$ at $\bar{x}$, a vector of order n

$\nabla F(\bar{x})$ = $\bar{g}(\bar{x})$ = the gradient of the objective function at $\bar{x}$, a vector of order n

I. INTRODUCTION

1.1 Object and Scope of Study

The object of this investigation is to demonstrate the feasibility of using programming techniques for obtaining minimum-weight design of elastic steel structures, and to develop a method by which this can be accomplished with the aid of high-speed digital computers.

The present study is limited to relatively simple statically indeterminate rigid frames and trusses which can be fabricated from standard rolled steel wide-flange (WF) sections. Structures which are subjected to more than one loading condition are included. Stresses due to bending moment, axial force, and combined stresses are considered; the possibilities of buckling of columns and excessive deflections of the structures are included as constraints in the design considerations. Local buckling, lateral buckling of beams, shear stresses, and other secondary effects are not considered; however, they can be included if necessary.

The study includes only structures for which the overall configuration is known a priori, and for which the selection of a combination of member sizes leading to the least weight is desired. The method may be extended to include optimization problems in which it is required to select the member sizes as well as certain overall dimensions.

For convenience, the weight of a structure has been selected as the criterion for optimization. Although the weight of a structure is not necessarily the best measure of optimality of civil engineering structures, it may nevertheless be one of the considerations in design, and if the remaining phases of the design process[(3)*] have been performed efficiently

* Superscripts in parentheses refer to entries in the Bibliography.

then a savings in weight can be of importance. For aero-space structures, least-weight design is the most important consideration.

1.2 Formulation of Problem

In designing a statically determinate structure of a given overall configuration it is necessary merely to analyze the structure under the prescribed loads and to select member sizes such that stresses and deflections are less than prescribed allowable values.

However, in the case of a statically indeterminate structure the results of the analysis are dependent upon the relative sizes of the individual members. Thus in the usual process of design, a set of member sizes is assumed, the structure is analyzed, the member sizes are revised, and the cycle is repeated until each member is fully stressed under at least one loading condition. This process does not always lead to the least weight design, especially when the structure is subjected to more than one loading condition, and when limitations on deflections are prescribed. Thus there is a need for a method by which the least weight design of a statically indeterminate structure can be determined with reasonable mathematical certainty. Nonlinear programming provides a formal basis for the optimization problems of structural design, and the related computational algorithms provide feasible numerical techniques for the determination of an optimum design.

A programming problem may be defined as follows:

Given the objective function

$$W(x) = W(x_1, x_2, \ldots, x_n) \qquad (1)$$

and the m constraint functions

$$\varphi_i(\overline{x}) = \varphi_i(x_1, x_2, \ldots, x_n), \; i = 1, \; \ldots, \; m \tag{2}$$

determine the variables

$$\overline{x} = (x_1, x_2, \ldots, x_n), \tag{3}$$

so as to

$$\text{minimize } W(\overline{x})$$

under the constraints

$$\varphi_i(\overline{x}) \geq 0 \;, \; i = 1, \ldots, m; \text{ where } m \geq n \tag{4}$$

In the context of Eqs. (1) through (4), the minimum-weight design problem is as follows:

The weight, W, of the structure may be expressed as a function of n geometric and/or structural variables $x_1, x_2, \ldots, x_n$, which are the design variables. The variables $\overline{x} = (x_1, x_2, \ldots, x_n)$ could be the areas of an n-member trussed structure, or the n moments of inertia that define a framed structure.

In any design, the variables $\overline{x}$ must be determined such that they satisfy the requirements or limitations of specifications. For instance, the computed stresses in the members and deflections at certain points in the structure should not exceed some prescribed allowable values. These limitations are also functions of the design variables $\overline{x}$, and hence constitute the constraints given by Eq. (4).

The design variables, $\overline{x}$, of the structures considered herein are assumed to be the sizes of the n members that make up the structure. Furthermore, it is assumed that the structures are fabricated from available wide-flange sections. Thus in reality the functions $W(\overline{x})$ and $\varphi_i(\overline{x})$ are defined only for discrete values of the member sizes, $\overline{x}$; this means that formally the problem is an integer programming problem. At present (1965) the available nonlinear programming techniques require that the functions W and φ_i be continuous functions of the variables $\bar{x}$.

1.3 Organization of Report

Chapter 2 includes a general discussion of programming problems and various programming techniques, and a detailed presentation of the gradient projection method of nonlinear programming, including modifications and revisions which were found to be necessary.

In Chapter 3 it is shown how the design problems can be formulated as a programming problem for structures composed of rolled steel wide-flange shapes.

Several illustrative examples are presented in Chapter 4, and in Chapter 5, the results of the study are summarized and several conclusions are presented.

1.4 Previous Applications of Programming Techniques to Optimum Design Problems

In the recent past several authors have investigated the possibility of using mathematical programming techniques to obtain minimum weight designs of structures.

Based on the work of Prager, Heyman, Livesley, and others Bigelow[4] developed a method for obtaining the minimum weight of plastically designed steel frames. A problem in minimum weight plastic design leads to a programming problem in which the constraints are linear and the objective function is nonlinear. Using a linear approximation to the objective function and the simplex algorithm of linear programming Bigelow obtained a first approximation to the minimum weight design. On the basis of available WF members a minimum weight plastic design is determined. The design method presented by Bigelow enables one to obtain the least-weight combination of WF members for a given indeterminate frame in an efficient manner and should have a wide application to a variety of practical design problems.

Schmit(8) formulated the minimum weight design of a three-bar planar truss as a nonlinear programming problem. The structure considered is shown in Fig. 1. The predetermined overall configuration is defined by β_1, β_2, β_3, and N. The constraints consist of upper and lower limits on the stresses in the members and limits on the deflections at the point S. The structure may be subjected to several loading conditions as defined by P_n and α_n, and may also be subjected to temperature changes. The objective function is a linear function of the member sizes, or areas. The resulting nonlinear programming problem was solved by a "method of alternate steps" by which if the current design, (the set of values of A_1,A_2, and A_3) is not constrained (i.e. none of the computed stresses or deflections are equal to the allowable values) the member sizes are modified in the direction of maximum change in weight; if the current design is constrained the member sizes are modified in one of three alternate directions of zero weight change. The method is continued until a step in the direction of maximum weight change cannot be taken. Schmit and Kicher(9) extended the three-bar truss problem to include material properties and overall configuration as design variables. This was accomplished by obtaining minimum weight designs for several discrete combinations of material properties and geometric configuration, and selecting the minimum weight design from among these answers. Schmit and Morrow(11) further extended the three-bar truss problem to include buckling constraints based on the assumption of annular members, and Schmit and Mallett(10) studied the three-bar truss problem to include material properties and overall configuration as continuous rather than discrete design variables. This results in a problem of nine variables and necessitates a revision in the method of alternate steps to the use of a random direction of travel when modifying the members in a direction of zero weight change.

Moses[5] used the cutting plane method which involves solving a succession of linear programming problems to obtain the solution to a three-bar truss problem and also to obtain the solution to a more practical problem consisting of a single-story single-bay rigid frame.

The papers by Schmit and others have demonstrated several important points concerning minimum-weight structural design, among which are the following:

(1) Minimum weight structural design problems can be formulated as nonlinear programming problems,

(2) The minimum-weight design is not necessarily one in which each member is fully stressed under some loading condition,

(3) A proof that a nonlinear programming solution leads to an absolute optimum rather than a local optimum would be desirable.

It is hoped that the present investigation will demonstrate the advantages and promise of programming techniques as a practical approach to the formulation of the problem of minimum-weight structural design, and to the solution of a broad range of practical design problems.

II. SOLUTION OF THE PROGRAMMING PROBLEM

2.1 Nonlinear Programming Techniques

A concise discussion of developments in nonlinear programming up to 1962 is presented by Wolfe[12]; it is concluded that while the subject of linear programming is well in hand both from the theoretical and computational standpoints, the same is not true of nonlinear programming. The bulk of the report is devoted to describing several of the algorithms proposed for the solution of various special types of nonlinear programming problems.

The programming problem as defined by Wolfe is presented in Section 1.2 above. The problem is illustrated in two dimensions in Fig. 2. The shaded area is called the feasible region, i.e. the set of all $\bar{x}$ satisfying the constraints, Eq. (4). The boundary of the feasible region consists of all $\bar{x}$ such that some constraint function vanishes. The vertices of the feasible region consist of points at which enough constraint functions vanish to define a unique point. A contour of the objective function W is a set of points on which the function has a constant value. The programming problem consists of finding the highest (or lowest) valued contour of W having some point in common with the feasible region. The gradient of W at $\bar{x}$, denoted by $\nabla W(\bar{x})$, is defined by

$$\nabla W(\bar{x}) = \left\{ \frac{\partial W(\bar{x})}{\partial x_1} , \ldots , \frac{\partial W(\bar{x})}{\partial x_n} \right\} . \qquad (5)$$

The gradient at $\bar{x}$ is normal to the surface (or curve) defined by $W(\bar{x})$ = constant which passes through $\bar{x}$, and the gradient points in the direction of steepest ascent (or descent), i.e. the direction of maximum rate of change of W. (It should be noted that the problem of minimizing a function W is the same as the problem of maximizing -W.) It is convenient to assume that W is a concave function, i.e. the graph of the function lies entirely below any tangent plane. In addition the functions φ_i are assumed to be convex, i.e. the graph of the function lies above any tangent plane. Consequently, the feasible region is a convex set. Under these assumptions about W and φ_i it can be shown that any local minimum is also a global minimum. A local minimum is a point $\bar{x}$ such that there is no point in the immediate neighborhood of $\bar{x}$ which gives a smaller value of W. A global minimum is a point $\bar{x}$ such that there is no other point in the feasible region which gives an improved value of W.

Among the programming techniques discussed by Wolfe the cutting-plane method best seems to fit the requirements of the minimum weight design problem. By this method the nonlinear constraints are approximated at the feasible point $\bar{x}_o$ by linear first order Taylor series expansions. The resulting linear programming problem is solved to yield a solution which does not necessarily satisfy the original nonlinear constraints. This solution is revised so that the original constraints are satisfied and a new feasible point $\bar{x}_o'$ and the process is repeated until the linear programming solution satisfies the nonlinear constraints within some acceptable tolerance.

Rosen(6),(7) presented a gradient projection method for nonlinear programming, and concluded that the gradient projection method has some advantages over the cutting-plane method. In the cutting-plane method it is necessary to solve a linear programming problem each time a linear approximation to the constraints is constructed. In the gradient projection method only a single inverse matrix based on local linearization is required at each step. Also, in the gradient projection method if a feasible starting point is known, then all points obtained in the optimization procedure are also feasible points; hence, the procedure may be stopped at any time with an improved feasible solution. In addition, with gradient projection it is only necessary to approximate those nonlinear constraints which actually constrain at a particular point.

Because of these advantages and because it is felt that the gradient projection method provides a better insight into the basic nature of the minimum weight design problem it was decided to use the gradient projection method for the solution of the minimum weight problem of structural design.

2.2 The Gradient Projection Method of Calculation

The following summary of the gradient projection method of nonlinear programming has been condensed and presented here as a convenience for the reader. For a more detailed presentation of the method including proofs of the results, the papers of Rosen[(6),(7)] should be consulted.

A Euclidian n-dimensional space E_n is considered, a point in the space being represented by the vector

$$\bar{x} = \left\{x_1,\ x_2, \ldots, x_n\right\} \tag{6}$$

It is assumed that a convex and bounded region R within E_n is defined by a set of m constraints, at least one of which is a nonlinear function of the variables $x_1, x_2, \ldots, x_n$. These constraints are assumed to be in the form

$$\varphi_i(\bar{x}) \geq 0\ ,\ i = 1,\ 2,\ \ldots,\ m, \tag{7}$$

where the m functions, $\varphi_i(\bar{x})$, are concave and satisfy certain conditions noted below. The convexity of R follows from the assumption that the $\varphi_i(\bar{x})$ are concave functions. A geometric interpretation of convexity can be presented as follows: The region R is convex if all points $\bar{x}$ on the line joining any two points $\bar{x}_1$ and $\bar{x}_2$ in R are also in R.

Corresponding to each function $\varphi_i(\bar{x})$ is a hypersurface G_i defined by

$$G_i = \left\{ \bar{x} \middle| \; \varphi_i(\bar{x}) = 0 \right\}, \quad i = 1,2,\ldots,m \quad . \tag{8}$$

If a point lies in the q hypersurfaces G_i, $i = 1,\ldots,q$, it is said to lie in the intersection of the q hypersurfaces, denoted by G.

The closed and bounded convex feasible region R and its boundary B are defined as follows:

$$R = \left\{ \bar{x} \;\middle|\; \varphi_i(\bar{x}) \geq 0, \; i = 1,\ldots,m \right\}, \tag{9}$$

$$B = \left\{ \bar{x} \;\middle|\; \bar{x} \text{ in } R, \; \varphi_i(\bar{x}) = 0 \text{ for at least one } i \right\} . \tag{10}$$

Every point in R is called a feasible point. An interior point $\bar{x}$ is one for which $\varphi_i(\bar{x}) > 0$, $i = 1,\ldots,m$. $\bar{x}$ is said to lie in a δ-neighborhood of the intersection G if $\sum_{i=1}^{q} \varphi_i^2(\bar{x}) \leq \delta^2$. The bounded region R can be enclosed in a region R' determined by an upper and lower bound on each variable,

$$R' = \left\{ \bar{x} \;\middle|\; -\underline{b}_i \leq x_i \leq \bar{b}_i, \; i = 1,\ldots,n \right\} . \tag{11}$$

The maximum length of the region R' is defined by L, where

$$L^2 = \sum_{i=1}^{n} (\underline{b}_i + \bar{b}_i)^2 . \tag{12}$$

It is assumed that the m functions $\varphi_i(\bar{x})$ have continuous and bounded second partial derivatives for all $\bar{x}$ in R'. At each point $\bar{x}$ in R' the gradient of each function $\varphi_i(\bar{x})$ is a vector,

$$\bar{u}_i(\bar{x}) = \nabla\varphi_i(\bar{x}) = \left\{ \frac{\partial\varphi_i(\bar{x})}{\partial x_1}, \ldots, \frac{\partial\varphi_i(\bar{x})}{\partial x_n} \right\}, \quad i = 1,\ldots,m . \tag{13}$$

It is assumed that the $\varphi_i(\bar{x})$ have been normalized so that $|\bar{u}_i(\bar{x})| \leq 1$ for $\bar{x}$

in R' and i = 1,...,m. Let $\bar{x}_o$ be a point which lies in a δ-neighborhood of the intersection, G, of q hypersurfaces G_i, i = 1,...q, $1 \leq q \leq m$. It is assumed that $\bar{u}_i(\bar{x}_o) \neq 0$, i = 1,...,q, and that the q vectors $\bar{u}_i(\bar{x}_o)$ are linearly independent.

Because the gradient vector of any function points in the direction of the most rapid increase of that function, and because of the inequalities (7) which determine the feasible region R, it is convenient to think of the vector $\bar{u}_i(\bar{x}) = \nabla\varphi_i(\bar{x})$ as pointing into R if $\bar{x}$ is in B and toward R if $\bar{x}$ is outside of B. For any point $\bar{x}_o$ in the hypersurface G_i the vector $\bar{u}_i(\bar{x}_o)$ is orthogonal to G_i. A hyperplane containing $\bar{x}_o$ which is also orthogonal to $\bar{u}_i(\bar{x}_o)$ is given by

$$H_i(\bar{x}_o) = \left\{ \bar{x} \mid \bar{x}^T \cdot \bar{u}_i(\bar{x}_o) - x_o^T \cdot \bar{u}_i(\bar{x}_o) = 0 \right\} \tag{14}$$

$H_i(\bar{x}_o)$ is the <u>supporting hyperplane</u> to G_i at $\bar{x}_o$. Suppose that a point $\bar{x}_o$ lies in the intersection G of q hypersurfaces G_i, i = 1,...,q. Corresponding to each G_i is a supporting hyperplane at $\bar{x}_o$, $H_i(\bar{x}_o)$, which contains $\bar{x}_o$. Then these q supporting hyperplanes form an intersection $Q(\bar{x}_o)$ which contains $\bar{x}_o$.

Only the simplest linear objective function $F(\bar{x}) = x_n$ need be considered since a problem with any other linear or nonlinear objective function can be reformulated to have this objective function by adding one variable and one constraint. In particular, if the original objective function is $F(x_1,\ldots,x_{n-1})$ the variable x_n is added along with the additional constraint

$$\varphi(\bar{x}) = F(x_1,\ldots,x_{n-1}) - x_n \geq 0 \ . \tag{15}$$

This new constraint will satisfy the conditions given above if $F(x_1,\ldots,x_{n-1})$ already satisfies these same conditions.

The nonlinear programming problem can now be stated as follows:

$$\text{maximize} \left\{ x_n \mid x_n \text{ in } R \right\} . \tag{16}$$

Note that since we are considering a linear objective function the desired solution $\bar{x} = \bar{x}_{max}$ will always occur on the boundary, B. The gradient of the linear function $F(\bar{x}) = x_n$ is the constant unit vector $\bar{g}$, where

$$\bar{g} = \left\{0,\ldots,0,1\right\} \tag{17}$$

A matrix $U_q(\bar{x})$ is defined in terms of the normals $\bar{u}_i(\bar{x})$ to $H_i(\bar{x})$, $i = 1,\ldots,q \leq m$,

$$U_q(\bar{x}) = [\bar{u}_1(\bar{x}),\ \bar{u}_2(\bar{x}),\ldots,\bar{u}_q(\bar{x})], \qquad (\bar{x} \text{ in } R). \tag{18}$$

Because of the assumed linear independence of the $\bar{u}_i(\bar{x})$ at each boundary point $\bar{x}$ in the boundary B of R the qxq symmetric matrix $U_q^T(\bar{x})U_q(\bar{x})$ is nonsingular for all $\bar{x}$ in B. Let

$$V_q(\bar{x}) = [U_q^T(\bar{x})U_q(\bar{x})]^{-1} . \tag{19}$$

A symmetric nxn projection matrix $P_q(\bar{x})$ is defined for each $\bar{x}$ in B by

$$P_q(\bar{x}) = I - U_q(\bar{x})V_q(\bar{x})U_q^T(\bar{x}) . \tag{20}$$

This matrix takes any vector in E_n into the intersection $Q(\bar{x})$ of the supporting linear hyperplanes to the hypersurfaces, G_i, $i = 1,\ldots,q$, at $\bar{x}$.

In the course of an optimization calculation using gradient projection it is necessary to obtain the projection of the gradient vector on various intersections Q. For a particular set of q vectors $\bar{u}_i(\bar{x})$, it is always possible to form the matrix $U_q^T(\bar{x})U_q(\bar{x})$ and invert it to obtain $V_q(\bar{x})$. Alternatively the inverse can be obtained by means of recursion relations presented in part I of Rosen's paper. For the purposes of this study $V_q(\bar{x})$ is determined by direct inversion.

The stepwise optimization procedure can be described as follows. It is assumed that a feasible starting point is known. If the starting point is an interior point of R the constant gradient $\bar{g}$ is followed until a boundary point is reached. At a boundary point the supporting hyperplanes to the constraint hypersurfaces in which the point lies are determined. The gradient is projected on the intersection of these supporting hyperplanes. A step is taken in the direction of the projected gradient to a new point with an increased value of F. Since the constraints are nonlinear the new point will generally not be a feasible point; it is then necessary to correct back to the feasible region in such a way that F remains greater than its value prior to taking the step. A correction procedure is presented which uses the inverse matrix V_q, which has already been generated, to determine a vector through the new non-feasible point which is normal to the intersection, G, of the tangent hyperplanes. The new feasible point is then determined by the vector normal to the intersection and containing the new non-feasible point, and the elements of the error vector.

The basic algorithm for the gradient projection method is given below. Several of the quantities referred to are defined in the paragraphs preceding Theorem 1. It is assumed that a tolerance $\epsilon > 0$ has been specified and that another tolerance δ is given by $\delta = \epsilon^2/3\lambda(8nL\alpha)^2$, and that an initial feasible point is known. Here α is a constant which is a measure of the linear independence of the vectors $\bar{u}_i(\bar{x})$. λ is a constant which depends on the dimensionality and geometry of the region R. For almost linear constraints $\lambda = 1$.

ALGORITHM

The following algorithm is extracted from Ref. 6:

1. If $\bar{x}_o$ is an interior point of R let $\bar{x}_1 = \bar{x}_o + \tau\bar{g}$ where τ is chosen so that $\bar{x}$ is in B. This is a one-dimensional interpolation problem to determine τ such that $\varphi_i(\bar{x}_o + \tau\bar{g}) \geq 0$, $i = 1,\ldots,m$, where the equality holds for at least one i.

2. Consider a point x_ν which lies in a δ-neighborhood of the intersection G of q constraints, $1 \leq q \leq m$. For convenience let these be G_i, $i = 1,\ldots,q$, so that $|\bar{w}_q(\bar{x}_\nu)| \leq \delta$ where $\bar{w}_q(\bar{x})$ is given by Eq. (21). The point $\bar{x}_\nu$ may be either the initial point $\bar{x}_o$ or a subsequent point. Compute $V_q(\bar{x}_\nu)$ by inversion of $U_q^T(\bar{x}_\nu)U_q(\bar{x}_\nu)$, $r(\bar{x}_\nu) = V_q(\bar{x}_\nu)U_q^T(\bar{x}_\nu)\bar{g}$, $P_q(\bar{x}_\nu)\bar{g} = \bar{g} - U_q(\bar{x}_\nu)\bar{r}(\bar{x}_\nu)$ and $\beta = \beta(\bar{x}_\nu)$ as given by Eqs. (24) and (25). If $\beta \leq \epsilon/2nL\alpha$ then $F(\bar{x}_\nu)$ differs from the global maximum by at most ϵ (Theorem 2).

3. If $\beta > \epsilon/2nL\alpha$ and $|P_q\bar{g}| \geq \beta_\ell$, compute the unit vector $\bar{z}(\bar{x}_\nu)$ according to Eq. (26). This is illustrated by Fig. 3. A sequence of n points is computed according to Eqs. (28) and (29) with $\tau = \alpha\beta/6\lambda$, and n the smallest integer such that $|\bar{w}_q(\bar{x}^{(n)}(\tau))| \leq \delta$. If $\varphi_i(\bar{x}^{(n)}(\tau)) \geq 0$, $i = q+1,\ldots k$, let $\bar{x}_{\nu+1} = \bar{x}^{(n)}(\tau)$. Then $F(\bar{x}_{\nu+1}) - F(\bar{x}_\nu) \geq \alpha\beta^2/12\lambda$ by theorem 1. The point $\bar{x}_{\nu+1}$ lies in a δ-neighborhood of G. If $\varphi_i(\bar{x}^{(n)}(\tau)) < 0$ for at least one i, $i = q + 1, \ldots, m$, we interpolate for the value $\tau = \tau' < \alpha\beta/6\lambda$ such that $\varphi_i(\bar{x}^{(n)}(\tau')) \geq 0$, $i = q + 1,\ldots,m$, with the equality holding for at least one value of i. This is shown in Fig. 4. We let $\bar{x}_{\nu+1} = \bar{x}^{(n)}(\tau')$, giving $F(\bar{x}_{\nu+1}) - F(\bar{x}_\nu) \geq \tau'\beta/2$ by theorem 1. The point $\bar{x}_{\nu+1}$ lies in a δ-neighborhood of a new intersection consisting of G and at least one additional constraint.

4. If $\beta > \epsilon/2nL\alpha$ and $|P_q\bar{g}| < \beta_\ell$, drop $\bar{u}_\ell$ from U_q and obtain U_{q-1} and the corresponding inverse matrix V_{q-1}. Compute $\bar{r} = V_{q-1}U_{q-1}^T\bar{g}$,

$P_{q-1}\bar{g} = \bar{g} - U_{q-1}\bar{r}$, and $\bar{z} = P_{q-1}\bar{g}/|P_{q-1}\bar{g}|$. A sequence of n points is then computed according to Eqs. (28) and (29) with U_{q-1}, V_{q-1}, and $\bar{w}_{q-1}(\bar{x}_j)$ replacing the corresponding quantities U_g, etc. The value $\tau = \alpha\beta/6\lambda$ is used, and n is taken as the smallest integer such that $|\bar{w}_{q-1}(\bar{x}^{(n)}(\tau))| \leq \delta$. This is illustrated in Fig. 5. By theorem 1 $\varphi_2(\bar{x}^{(n)}(\tau)) > 0$. If in addition $\varphi_i(\bar{x}^{(n)}(\tau)) \geq 0$, $i = q+1,\ldots,m$, let $\bar{x}_{\nu+1} = \bar{x}^{(n)}(\tau)$ which gives $F(\bar{x}_{\nu+1}) - F(\bar{x}_\nu) \geq \alpha\beta^2/12\lambda$. The point $\bar{x}_{\nu+1}$ lies in a δ-neighborhood of the intersection of the q-1 constraints G_i, $i = 1,\ldots,q$, $i \neq \ell$. If $\varphi_i(\bar{x}^{(n)}(\tau)) < 0$ for at least one i, $i = q + 1,\ldots,m$, interpolate for τ' such that $\varphi_i(\bar{x}^{(n)}(\tau')) \geq 0$, $i = q + 1,\ldots,m$, with an equality for at least one i. Let $\bar{x}_{\nu+1} = \bar{x}^{(n)}(\tau')$, giving $F(\bar{x}_{\nu+1}) - F(\bar{x}_\nu) \geq \tau'\beta/2$. The point $\bar{x}_{\nu+1}$ lies in a δ-neighborhood of the intersection of the q-1 constraints G_i, $i = 1,\ldots,q$, $i \neq \ell$, and at least one other constraint.

This completes the algorithm for a typical step. The convergence of a sequence of steps taken according to this algorithm is shown in Theorem 3.

It is shown in theorem 1 how a step can be taken from a feasible point to a new feasible point with an increased value of F by showing that given a feasible point $\bar{x}_\nu$ for which the quantity $\beta(\bar{x}_\nu)$ is positive, another feasible point can be formed with an increase in F which is proportional to $\beta(\bar{x}_\nu)$ and the step length.

Consider the intersection G of q hypersurfaces G_i, $i = 1,\ldots,q \leq m$. A measure of the "distance" of a point $\bar{x}$ from G is given by the length of the error vector

$$\left|\bar{w}_q(\bar{x})\right| = \left|\left\{\varphi_1(\bar{x}),\ldots,\varphi_q(\bar{x})\right\}\right| . \tag{21}$$

Consider a point $\bar{x}_\nu$ and the corresponding matrices $U_q(\bar{x}_\nu)$, $V_q(\bar{x}_\nu)$ and $P_q(\bar{x}_\nu)$ as given by Eqs. (18), (19), and (20). A q-dimensional vector with components $\bar{r}_i(\bar{x}_\nu)$ is defined by

$$\bar{r}(\bar{x}_\nu) = \left\{\bar{r}_1(\bar{x}_\nu), \ldots, \bar{r}_q(\bar{x}_\nu)\right\} = V_q(\bar{x}_\nu)U_q^T(\bar{x}_\nu)\bar{g} , \tag{22}$$

and q quantities $\beta_i(\bar{x}_\nu)$ are given by

$$\beta_i(\bar{x}_\nu) = r_i(\bar{x}_\nu)/2\sqrt{v_{ii}} , \tag{23}$$

where the v_{ii} are the diagonal elements of $V_q(\bar{x}_\nu)$. Let

$$\beta_\ell = \max_i \left\{\beta_i(\bar{x}_\nu)\right\} , \tag{24}$$

$$\beta(\bar{x}_\nu) = \max \left\{ |P_q(\bar{x}_\nu)\bar{g}|, \ \beta_\ell(\bar{x}_\nu)\right\} . \tag{25}$$

We define the unit vector

$$\bar{z}(\bar{x}_\nu) = P_q(\bar{x}_\nu)\bar{g}/|P_q(\bar{x}_\nu)\bar{g}| . \tag{26}$$

Then the following theorem can be proved:[7]

<u>Theorem 1</u>. Let the point $\bar{x}_\nu$ and the constant δ be such that

$$|\bar{w}_q(\bar{x}_\nu)| \leq \delta \leq \beta^2/48\lambda , \tag{27}$$

where $\beta = \beta(\bar{x}_\nu) > 0$ is given by Eq. (25). For $\beta = |P_q(\bar{x}_\nu)\bar{g}| \geq \beta_\ell$, a sequence of points is given by

$$\bar{x}^{(0)}(\tau) = \bar{x}_\nu + \tau\bar{z}(\bar{x}_\nu) , \tag{28}$$

$$\bar{x}^{(j+1)}(\tau) = \bar{x}^{(j)}(\tau) - U_q(\bar{x}_\nu)V_q(\bar{x}_\nu)\bar{w}_q(\bar{x}^{(j)}), \ j = 0,1,\ldots . \tag{29}$$

For $\beta = \beta_2 > |P_q(\bar{x}_\nu)\bar{g}|$ a sequence of points is given by Eqs. (28) and (29) with P_{q-1} replacing P_q in Eq. (26) and U_{q-1}, V_{q-1}, and $\bar{w}_{q-1}$ replacing the corresponding quantities in Eqs. (28) and (29). Then for any τ such that

$$\frac{8\alpha\delta}{\beta} \leq \tau \leq \frac{\alpha\beta}{6\lambda} , \tag{30}$$

and every integer n such that

$$n \geq N(\delta) = 1.443 \log \left(\frac{1}{24\lambda\delta}\right) , \tag{31}$$

then

$$F(\overline{x}^{(n)}(\tau)) - F(\overline{x}_\nu) \geq \tau\beta/2 \geq 4\alpha\delta. \tag{32}$$

Furthermore, for $|P_q(\overline{x}_\nu)\overline{g}| \geq \beta_\ell$,

$$|\overline{w}_q(\overline{x}^{(n)}(\tau))| \leq \delta \quad . \tag{33}$$

And if $\beta = \beta_\ell \leq \frac{1}{4\alpha}$, then

$$\varphi_2(\overline{x}^{(n)}(\tau)) \geq \frac{\tau\beta}{2} \geq 4\alpha\delta. \tag{34}$$

Theorem 1 states that a step can be taken from a feasible point to a new feasible point with an improved value of F. That is, given a feasible (within a specified tolerance) point $\overline{x}_\nu$ for which $\beta(\overline{x}_\nu)$ is positive, another feasible point can be found with an increase in the value of F which is proportional to the product of $\beta(\overline{x}_\nu)$ and the step length τ.

Next the following sufficient condition is given for a global maximum:

<u>Theorem 2</u>: Let $\overline{x}_\nu$ be a point in the intersection G as described prior to theorem 1. Let a tolerance $\epsilon > 0$ be specified and $\beta(\overline{x}_\nu)$ be the quantity given by Eq. (25). Then if

$$\beta(\overline{x}_\nu) \leq \frac{\epsilon}{2nL\alpha} , \tag{35}$$

we have for any $\overline{x}$ in R.

$$F(\overline{x}) - F(\overline{x}_\nu) \leq \epsilon . \tag{36}$$

Theorem 2 states that if the quantity $\beta(\overline{x}_\nu)$ is sufficiently small, then the value of the objective function, $F(\bar{x}_\nu)$, is as close as desired to the global optimum value. The proof of theorem 2 depends on the fact that the constraints determine a convex feasible region.

The convergence of the algorithm to the global maximum is based on the following theorem:

Theorem 3: Let $\epsilon > 0$ be specified and

$$\delta = \epsilon^2/3\lambda(8nL\alpha)^2 \quad . \tag{37}$$

Starting at any feasible point $\bar{x}_o$ in R the algorithm gives a sequence of points $\bar{x}_j$, $j = 1,2,\ldots$, such that

$$F(\bar{x}_j) \geq F(\bar{x}_{j-1}) \quad , \tag{38}$$

$$\varphi_i(\bar{x}_j) \geq -\delta, \; i = 1,\ldots,m \quad . \tag{39}$$

Furthermore, if

$$F_{max} = F(\bar{x}_{max}) = \max \left\{ F(\bar{x}) \mid \bar{x} \text{ in } R \right\} \tag{40}$$

there exists a finite member $k(\epsilon)$ such that

$$F_{max} - F(\bar{x}_j) \leq \epsilon \tag{41}$$

for every $j \geq K(\epsilon)$.

As part of the proof it is shown that at most $K(\epsilon)$ steps are required to reach $\bar{x}_{max}$, where

$$K(\epsilon) = \frac{3\lambda L}{2} \left(\frac{8nL\alpha}{\epsilon}\right)^3 \quad , \tag{42}$$

and where the step length $\tau = \alpha\beta/6\lambda$ is used whenever possible. In practice the actual number of steps required will be far less than $K(\epsilon)$. Also, more importantly, the number of steps required will depend directly on the closeness of the starting point $\bar{x}_o$ to the optimum point $\bar{x}_{max}$.

The proofs of the above three theorems are omitted but are given in Ref. 7.

In connection with the iteration Eqs. (28) and (29), in the usual case only a few iterations are required per step to get within the δ-neighborhood of the intersection G, and if the active constraings are nearly linear, the number of iterations required approaches zero.

To determine the optimum step length, the quantity $\tau = \alpha\beta/6\lambda$ can be estimated by applying a single iteration, Eqs. (28) and (29) with a small value of $\tau = \tau'$ to get $\bar{x}^{(1)}(\tau')$. It then can be shown that for $\sigma' = |\bar{w}^{(1)}|/|\bar{w}^{(o)}|$

$$\frac{\sigma'}{2\tau'} \simeq \frac{\lambda}{\alpha} . \qquad (43)$$

The algorithm can also be applied to the problem of maximizing a linear function in a nonconvex region. This is true because Theorem 1 does not require the functions $\varphi_i(\bar{x})$ to be concave. Hence, even for a nonconvex region R, if $\beta(\bar{x}_j) > \epsilon/2nL\alpha$, a point $\bar{x}_{j+1}$ is obtained by the algorithm such that $F(\bar{x}_{j+1}) - F(\bar{x}_j) \geq \tau\beta/2$. Since Theorem 2 depends on the convexity of R, it can be shown that for a nonconvex region the algorithm will converge to a constrained stationary point. Because gradient projection is a steepest ascent procedure the point obtained for a nonconvex region will almost always be a constrained maximum point, which may be only a local maximum.

This completes the summary of the gradient projection method presented by Rosen.[7] In the following section some modification and simplifications to the method are suggested for the purpose of obtaining a workable computer program based on the method. In Chapter 4, the various steps of the algorithm are illustrated in detail in connection with a two-dimensional example.

2.3 Computational Aspects of the Problem and Modifications to Rosen's Gradient Projection Algorithm

In order to apply the gradient projection method to minimum weight design problems, it is necessary to violate certain of the requirements on F and φ_i as given in Section 2.2. Also, in order to obtain a workable computer program it was necessary to modify certain portions of the algorithm.

It is assumed in Section 2.2 that the feasible region R is convex and bounded. It can be shown (see Example 4.1 of Chapter 5) that the feasible region for a minimum weight design problem is not necessarily convex. Since Theorem 1 is applicable to a nonconvex feasible region the gradient projection method is still valid. But since Theorem 2 applies strictly only for convex feasible regions one cannot be very sure that a solution necessarily leads to a minimum. This difficulty can be overcome by starting the algorithm at two or more points which are significantly different and comparing the final solutions. Furthermore, in a minimum weight design problem it is not necessary to provide constraints in the form of upper bounds on each member size, since from the physical significance of the objective function it is observed that no member size can be expected to increase beyond a reasonable bound.

It is assumed in Section 2.2 that the functions φ_i have continuous and bounded second partial derivatives. These derivatives are required for estimating the constant λ which is a measure of the curvature of the constraints. The value of λ is used for calculating the maximum step length and in testing for convergence. In order to be able to complete the algorithm in a reasonable length of computation time it was found advantageous to use an arbitrary step length of L/100, where L is defined in Eq. (12). Also it will be seen that the tolerance δ need not depend on λ as in Eq. (37). The vectors $\overline{u}_i(\overline{x})$ as given by Eq. (13) can be conveniently calculated by a forward finite difference technique. Therefore the only requirement necessary for the functions φ_i is that they be continuous for all $\bar{x}$ in R.

It is also assumed in Section 2.2 that $\bar{u}_i(\bar{x}_o) \neq 0$, $i = 1,\ldots,q$, and that the vectors $\bar{u}_i(\bar{x}_o)$ are linearly independent, where $\bar{x}_o$ is a point in a δ-neighborhood of an intersection G. That this requirement must be met should be pointed out. The vanishing of $\bar{u}_i(\bar{x}_o)$, or $\bar{u}_i(\bar{x}_o) = 0$, is possible for a constraint which does not depend on at least one variable x_i. Linearly dependent $\bar{u}_i$ may result when two or more constraints are redundant. This can happen, for example, if constraints are written for two corresponding members of a symmetrical structure subjected to a symmetrical loading.

In Section 2.2 it is suggested that only the simplest objective function $F(\bar{x}) = x_n$ need be considered. This leads to no great savings in computation time or simplification in the computer program. Therefore it was decided to compute $F(\bar{x}) = -\overline{\nabla w}(\bar{x})$ at each step in order to gain a better understanding of the behavior of the problem. Furthermore, it is assumed that a tolerance $\epsilon > 0$ has been specified and that another tolerance δ is given by $\delta = \epsilon^2/3\lambda(8n\bar{L}\alpha)^2$. The difficulty in calculating λ has already been noted. The constant α is a measure of the linear independence of the vectors $\bar{u}_i$, and is a property of the matrix $V_q = [U_q^T U_q]^{-1}$.

The tolerance δ determines the closeness of each constrained point $\bar{x}_o$ to the corresponding intersection G. Because the IBM 7094 digital computer provides single-precision accuracy of 8 decimal digits, it was found that for $\delta \lesssim 1 \times 10^{-7}$ the sequence of points given by Eqs. (28) and (29) does not always converge to an acceptable point. Also, the sequence given by Eqs. (28) and (29) converges faster for larger values of δ. In the course of this study values of δ between 1×10^{-3} and 1×10^{-5} were found to give satisfactory results.

The tolerance ϵ determines the closeness of the indicated solution to the true optimum. In Section 2.2 ϵ and δ are formulated so that for the step length as determined by $\tau = \alpha\beta/6\lambda$ the objective function will never decrease for a given step. Using the arbitrary step length $\tau = L/100$ and the tolerance δ in the range 1×10^{-3} to 1×10^{-5} it is no longer possible to say that a given step will not decrease the value of the objective function. Under these conditions it is necessary to stop the algorithm whenever a decrease in the objective function is noticed. The desired solution $\bar{x}_{max}$ will then lie somewhere between the point at which computation was stopped and the point immediately preceding. By redefining the step length to be less than L/100 and restarting the algorithm at the last feasible point, $\bar{x}_{max}$ can be obtained with greater accuracy. Alternatively an interpolation procedure could be applied which should converge to the point $\bar{x}_{max}$.

The above difficulty only occurs when the solution lies in the intersection of $q < m$ constraints. When the solution lies in the intersection of $q = m$ constraints, such an intersection defines a unique point (at least within a δ-neighborhood), and in such cases at $\bar{x}_{max}$, β as given by Eq. (25) was found to be of the order of magnitude of 1×10^{-11}.

For problems in which $\bar{x}_{max}$ is obtained at the intersection of $q = m$ constraints, values of ϵ in the range of 1×10^{-8} to 1×10^{-10} were found to be satisfactory if computation is stopped when $\beta \leq \epsilon/2nL$. For problems in which $\bar{x}_{max}$ is obtained at the intersection of $q < m$ constraints, the same value of β was found satisfactory provided the algorithm is stopped when a decrease in F occurs.

In Section 2.2. it is implicitly assumed that the feasible region R contains only points $\bar{x}$ for which $\bar{x}_i > 0$, $i = 1,\ldots,n$. It is possible in the course of the algorithm that one or more variables may approach zero. This can be prevented, if necessary, by defining n constraints in the form of lower bounds on each variable. Also, it is possible that during certain steps the values of some x_i may become negative. In such a case it may not be possible to analyze the structure or evaluate the constraints. Therefore provision is made in the computer program to prevent obtaining negative values for the variables x_i.

III. STRUCTURAL SYSTEMS OF WIDE-FLANGE STEEL MEMBERS

3.1 Rigid Frames

As was noted in Chapter I, the weight of the structure and the constraints on stresses and deflections are expressed as functions of n geometric and/or structural variables $x_1, x_2, \ldots, x_n$, which are the design variables of a structure. A typical wide-flange section has several properties, known as section properties, which are necessary to define the elastic behavior of a structure. These properties include moment of inertia, section modulus, area, weight, and two radii of gyration. Rather than include each section property as a design variable it is convenient when possible, to use only one of these section properties as the design variable for a particular member, and to obtain the other section properties as functions of the designated design variable.

For an indeterminate rigid frame, the analysis of the structure depends on the relative moments of inertia of the members. Therefore for framed structures the member sizes, or design variables x_i, $i = 1, 2, \ldots, n$, will be taken to be proportional to the moments of inertia of the members, I_i. In formulating the constraint functions, it may be observed that stresses due to bending depend on the section moduli, S_i; and stresses resulting from combined bending moment and axial force depend on the section moduli and the areas, A_i. For frames braced against lateral buckling, the compressive stresses depend on the area and the in-plane radii of gyration, r_i. Deflections also are functions of moments of inertia. For minimum weight design, the objective function is the total weight of a struoture and can be formulated as a function of the weight of each member per foot of length, w_i.

Thus for problems involving rigid frames it is necessary to determine approximate relationships giving section modulus, S, area, A, radius of gyration, r, and weight per foot, w, as continuous functions of the moment of inertia, I. Approximate curves representing the above relationships have been constructed for the so-called economy sections which provide a given section modulus for the least weight. These sections are tabulated in the AISC handbook.[1] In table 1, values of I, S, w, and r are listed for these economy sections, and these data are plotted in Figs. 6, 7, and 8.

Figure 6 shows the variation of section modulus with moment of inertia. The approximate curves were obtained by passing a parabola through the points

$$I = 0 \quad , \quad S = 0$$

$$I = 5350 \quad , \quad S = 354.6$$

$$I = 9000 \quad , \quad S = 502.9$$

and a straight line through the points

$$I = 9{,}000 \quad , \quad S = 502.9$$

$$I = 20{,}300 \quad , \quad S = 1105.1$$

Thus the following relationship is assumed to represent the continuous variation of section modulus with moment of inertia:

$$S = \begin{cases} \sqrt{60.6I + 84{,}100} - 290, & 0 \leq I \leq 9{,}000 \\ (I - 8056.3)/1.876, & 9000 \leq I \leq 20{,}300 \end{cases} \qquad (44)$$

Figure 7 shows the variation of weight per linear foot of member with moment of inertia. The approximate curves were obtained by passing a parabola through the points

$$I = 0 \quad , \quad w = 0$$

$$I = 9{,}000 \quad , \quad w = 150,$$

where the parabola has a vertical slope at the origin; and a straight line through the points

$$I = 9{,}000 \ , \quad w = 150$$

$$I = 20{,}300 \ , \quad w = 300$$

Thus the following relationship is assumed to represent the continuous variation of unit weight with moment of inertia:

$$w = \begin{cases} 1.58\sqrt{I}\,, & 0 \le I \le 9000 \\ (I + 2300)/75.3, & 9000 \le I \le 20{,}300 \ . \end{cases} \qquad (45)$$

Since steel weighs about 490 pounds per cubic foot, the area of a section and the unit weight are directly related by the equation

$$A = \frac{144}{490}\, w \ . \qquad (46)$$

Hence, from Eq. (45)

$$A = \begin{cases} 0.465\sqrt{I}\,, & 0 \le I \le 9000 \\ (I + 2300)/25.6 & 9000 \le I \le 20{,}300 \end{cases} \qquad (47)$$

Figure 8 shows the variation of in-plane radius of gyration with moment of inertia. The approximate curves for $0 \le I \le 9000$ were obtained by using

$$r = \sqrt{\frac{I}{A}}$$

and Eq. (47). A linear variation was assumed for $9000 \le I \le 20{,}300$. Thus

$$r = \begin{cases} \sqrt[4]{\dfrac{I}{.21583}}\,, & 0 \le I \le 9000 \\[2ex] \dfrac{I + 174{,}498}{12{,}841} & 9000 \le I \le 20{,}300 \end{cases} \qquad (48)$$

3.2 Trusses

In the case of an indeterminate pin-pointed frame, or truss, the analysis of the structure depends on the relative areas of the members. Therefore for a trussed structure, the member sizes, or design variables, x_i, $i = 1,2,\ldots,n$, will be taken to be proportional to the areas of the members, A_i. In computing the constraint functions buckling stresses depend on the least radii of gyration of the members, r_{min_i}. Thus it is necessary to determine an approximate relationship giving r_{min} as a continuous function of the area, A.

In computing the objective function the unit weight of each member is a linear function of the area as given by Eq. (46), and hence the total weight is a linear function of the member sizes A_i.

Figure 9 shows the variation of r_{min} with A for all the WF sections. Figure 10 shows the variation of r_{min} with A for only the 14 WF sections. Since the 14 WF sections provide nearly the same range in A and r_{min} as do all the WF sections, and since it is convenient to specify the same depth for the members of a truss, the illustrative study made herein is further restricted to trusses made of 14 WF sections.

The approximate curves giving r_{min} in terms of A were obtained by passing straight lines through the points

$$A = 9.4, \quad r_{min} = 1.4,$$
$$A = 34.7, \quad r_{min} = 3.94,$$

and through the points

$$A = 34.7, \quad r_{min} = 3.94,$$
$$A = 125.0, \quad r_{min} = 4.34.$$

Thus the following relationship is assumed to represent the continuous variation of minimum radius of gyration with area:

$$r_{min} = \begin{cases} 0.01131A - 0.60122, & 9.4 \le A \le 34.7 \\ 0.000382A + 3.786718, & 34.7 \le A \le 125 \end{cases} \quad (49)$$

It is noted that Eqs. (44) through (49) are only crude approximations to the actual variations of the section properties of selected WF members. However, it will be seen that any such approximation is sufficient for the purpose of finding the least weight design from among commercially available sections.

3.3 Least Weight Design from Available Sections

METHOD 1. Using continuous approximations to represent the variations among the properties of the available sections (Eqs. (44),(45), (46),(47),(48),(49)), the minimum-weight design problem can be formulated as a nonlinear programming problem. The gradient projection method can then be used to solve the resulting programming problem and obtain what may be called the "continuous solution." However, in general, there may not be any combination of commercially available sections that corresponds to the continuous solution. It is therefore necessary to select the combination of available sections which gives an available minimum-weight design. This combination of available sections will be called the "available solution."

It seems reasonable to assume that the available solution will occur somewhere in the neighborhood of the continuous solution. That is, for each member size to be selected, only the few available sections whose sizes are slightly less than or greater than the size indicated by the continuous solution need be considered. Having selected such a range of available sizes for each member, all possible combinations of these available members are considered. Beginning with the lowest indicated size for each member, the structure is analyzed and the values of the constraints are computed, based on the actual section properties. The member sizes are then increased one at a time to the next available size until an available design is found which does not violate any of the constraints. Thereafter the structure is analyzed only for those combinations of available sections which lead to a lighter structure. By this method the available solution is obtained from all possible combinations of available sections in the neighborhood of the continuous solution.

METHOD 2. The above method was found to be quite satisfactory for problems in which the number of design variables was small ($n \leq 5$). However, in the case of problems in which the number of design variables was large ($n \geq 10$) the number of possible combinations became too large for this method to be practical. In such a case the following alternative method was used.

An initial available solution is determined by selecting for each member the available section giving a member size not less than that indicated by the continuous solution. If any constraint is violated for this initial available solution, the next greater available size is chosen for each member, until a satisfactory initial available solution is obtained. The size of each member is then successively changed to the next lower available section; the member giving the greatest weight decrease is changed first. The procedure is terminated when no member size can be decreased without violating some constraint. The resulting combination of available sections is then the minimum weight design.

The second procedure was applied to a 10-dimensional problem (see Sec. 5.4) and was found to work quite satisfactorily as regards computation time. It was also applied to a 4-dimensional problem (see Sec. 5.2) and was found to give the same available solution as the first procedure.

IV. ILLUSTRATIVE MINIMUM-WEIGHT DESIGNS

4.1 General Remarks

In this chapter, the gradient projection method is applied to several typical Civil Engineering structures.

In order to obtain a computer program which would be applicable to any general type of structure the computation was divided among a main program and several auxiliary subroutines. The function of the main program is to carry out the steps of the gradient projection algorithm as set forth in Section 2.2 and modified in Section 2.3. The functions of the auxiliary subroutines are to read in and write out variables which do not directly concern the main program, to analyze the structure, to calculate the values of the constraint functions, to calculate the values of the objection function and gradient vector, and to select an available design based on the continuous solution. The main program is applicable to any design problem which has been formulated in terms which correspond to the requirement of the gradient projection method. The auxiliary subroutines may need to be revised for each individual problem, depending on the type of structure, the method of analysis, the constraints, the objective function, the relationships among the properties of the members, and the type of output desired

A description of the computer program for minimum-weight design, a machine-independent flow diagram for the main program, short descriptions of the auxiliary subroutines, and input data for a typical rigid frame and a typical truss are presented in Appendices A.1, A.2, A.3, and A.4, respectively.

In the examples to follow, the Argyris matrix formulation[2] of structural analysis was used throughout. The displacement method was used in the cases of rigid frames, and the force method was used in the case of the indeterminate pin-jointed truss. It should be pointed out that the main computer program is independent of the method of structural analysis used.

4.2 Symmetrical Portal Frame

As a first example the structure shown in Fig. 11 is considered. The structure is a rigid frame of 30-ft. span, and 15-ft. height; the supports are rigidly fixed. The structure is loaded by 30-kip loads applied at the quarter points of the horizontal member.

The moment of inertia, section modulus, area, and unit weight of the columns are denoted by I_1, S_1, A_1, and w_1, respectively; and the moment of inertia, section modulus, and weight per foot of the beam are denoted by I_2, S_2, and w_2 respectively. In accordance with the matrix displacement method of formulation the external joints A and B and the internal joints 1, 2,...,6 are denoted as shown. The bending moment diagram is defined by the moments at the internal joints, $M = \{M_1, M_2, \ldots, M_6\}$.

The non-dimensional design variables x_i, i = 1,2, are taken to be $x_i = I_i/x_o$, where the reference value, x_o, is taken to be 9000 in^4.

The non-dimensional constraints, φ_i, i = 1, 2, 3, are defined so that the stresses due to bending moment, or bending moment and axial force, are not greater than 20 ksi. Thus,

$$\left.\begin{aligned} \varphi_1 &= 1 - \frac{|M_2|}{20S_1} - \frac{30}{20A_1} \geq 0 \\ \varphi_2 &= 1 - \frac{|M_2|}{20S_2} \geq 0 \end{aligned}\right\} \qquad (50)$$

$$\varphi_3 = 1 - \frac{2700 - |M_3|}{20 S_2} \geq 0$$

$\varphi_1 \geq 0$ requires that the stress in the column due to bending moment and axial load is not greater than 20 ksi. $\varphi_2 \geq 0$ and $\varphi_3 \geq 0$ require that the bending stresses in the beam resulting from the maximum negative moment and the maximum positive moment, respectively, are not greater than 20 ksi.

The non-dimensional objective function is defined by $F = -W/W_o$, where W is the total weight of members 1 and 2, and the reference value, W_o, is taken to be 10,000 lb. Letting L_i denote the total length, in feet, of members of size I_i, i = 1, 2,

$$W = \sum_{i=1}^{2} L_i w_i ,$$

and

$$F = \frac{-1}{W_o} \sum_{i=1}^{2} L_i w_i .$$

Thus the gradient of the objective function, $\bar{g}$, is given by

$$\bar{g} = \nabla F = \frac{-1}{W_o} \left\{ L_1 \frac{\partial w_1}{\partial x_1} , L_2 \frac{\partial w_2}{\partial x_2} \right\}$$

Differentiating Eqs. (45),

$$g_i = \left\{ \begin{array}{ll} \frac{-0.79}{W_o} \frac{L_i}{\sqrt{x_i x_o}} & , \text{ if } x_i x_o \leq 9000 \text{ in.}^4 \\ \frac{-1.0}{W_o} \frac{L_i}{75.3} & , \text{ if } x_i x_o \geq 9000 \text{ in.}^4 \end{array} \right\} ; \ i = 1,2 \qquad (51)$$

It is instructive to consider the steps in the solution of this rather simple problem in detail. If the gradient projection algorithm is initiated at the point $\overline{x}_o = \{1.0, 1.0\}$ (see Fig. 12) the gradient, $\overline{g}$, is followed until the point $\overline{x} = \{0.1109, 0.1109\}$ is reached, at which point $\varphi_1 \leq \delta^2$. Following the constraint surface $\varphi_1 \geq 0$, (which, in this case, is a curve) the solution $\overline{x}_{max} = \{0.1136, 0.1007\}$ is obtained. In order to illustrate the boundary of the feasible region R, the algorithm was restarted at $\overline{x}_o' = \{0.7, 0.2\}$, the gradient was followed to the point $\overline{x} = \{0.6573, 0.1202\}$, and the constraint curve $\varphi_2 = 0$ was followed to the point $\overline{x}_{max}$. By plotting successive values of $\{x_1, x_2\}$ during the travel along the curve $\varphi_2 = 0$, a plot of the portion of the boundary of R defined by $\varphi_2 = 0$ was obtained, and is shown in Fig. 12. Similarly, by starting the algorithm at $\overline{x}_o'' = \{0.1, 0.7\}$, the portion of the boundary of R defined by $\varphi_1 = 0$ was obtained and is also shown in Fig. 12. It can be seen from this figure that the boundary is indeed nonlinear. Also it can be seen that R is not a convex region, but that this fact does not render the gradient projection method ineffective in the present case.

In order to visualize the geometrical meanings of the various steps of the gradient projection method a typical series of calculations, in connection with this two-dimensional problem are illustrated. For this purpose the algorithm was started at the point $\overline{x}_o = \{0.165, 0.130\}$. The results are shown on Fig. 13, which is an enlarged region of the solution space in the neighborhood of $\overline{x}_{max}$.

In accordance with the flow diagram given in the appendix, the structure is analyzed, the constraints are calculated, and it is determined that the point $\overline{x}_o$ is feasible (i.e. $\varphi_i(\overline{x}_o) \geq 0$, $i = 1, 2, 3$), and that no constraint is active (i.e. $(\varphi_i(\overline{x}_o))^2 > \delta^2$, $i = 1, 2, 3$). Then by means of

Eqs. (51) the gradient of the objective function at $\bar{x}_o$, $\bar{g}(\bar{x}_o) = \{-0.615, -0.693\} \times 10^{-4}$, is determined. In accordance with part 1 of the algorithm, movement occurs in the direction of the gradient to the point $\bar{x}_1 = \{0.1423, 0.1044\}$, at which point $(\varphi_2(\bar{x}_1))^2 \leq \delta^2$, $\varphi_1 > 0$, $\varphi_3 > 0$. Thus the point $\bar{x}_1$ lies in a δ-neighborhood of the graph of the function $\varphi_2(\bar{x}) = 0$. By means of the forward finite difference technique described in the flow diagram the gradient of φ_2 at $\bar{x}_1$, $\bar{u}_2(\bar{x}_1) = \left\{ \frac{\partial\varphi_2(\bar{x}_1)}{\partial x_1}, \frac{\partial\varphi_2(\bar{x}_1)}{\partial x_2} \right\} = \{-1.0696, 9.7759\}$ is calculated using Eq. (13). The vector $\bar{u}_2(\bar{x}_1)$ is normal to $\varphi_2(\bar{x}) = 0$ at the point $\bar{x}_1$. The line $H_2(\bar{x}_1)$, which is tangent to the graph of $\varphi_2(\bar{x}) = 0$ at $\bar{x}_1$, is constructed which is normal also to $\bar{u}_2(\bar{x}_1)$ at $\bar{x}_1$. The matrix $V_q(\bar{x}_1) = (U_q^T(\bar{x}_1)U_q(\bar{x}_1))^{-1} = (0.01034)$ is determined, and the gradient of the objective function at $\bar{x}_1, \bar{g}(\bar{x}_1) = \{-0.6623, -0.7732\} \times 10^{-4}$, is calculated according to Eqs. (51). $\bar{r}(\bar{x}_1) = V_q(\bar{x}_1)U_q^T(\bar{x}_1)\bar{g}(\bar{x}_1) = (-0.7083) \times 10^{-5}$ is calculated by Eq. (22), and $U_q(\bar{x}_1)\bar{r}(\bar{x}_1) = \{0.0758, -0.6924\} \times 10^{-4}$ is determined. $U_q\bar{r}$ is the component of the gradient, $\bar{g}(\bar{x}_1)$, normal to the curve $\varphi_2(\bar{x}) = 0$ at $\bar{x}_1$, i.e. the component of $\bar{g}(\bar{x}_1)$ in the direction of $\bar{u}_2(\bar{x}_1)$. Subtracting this component $U_q\bar{r}$ from $\bar{g}$, the component of $\bar{g}$ which lies in the tangent of the graph of $\varphi_2(\bar{x}) = 0$ at x_1 is obtained. Thus $P_q(\bar{x}_1)\bar{g}(\bar{x}_1) = \bar{g}(\bar{x}_1) - U_q(\bar{x}_1)\bar{r}(\bar{x}_1) = \{-0.7381, -0.0808\} \times 10^{-4}$.

The quantity $\beta_1 = -0.3483 \times 10^{-4}$ is computed according to Eq. (23), and $\beta_2 = \beta_1 = -0.3483 \times 10^{-4}$ is computed according to Eq. (24), $|P_q\bar{g}| = \sqrt{g_1^2 + g_2^2} = 0.7425 \times 10^{-4}$ is determined, and $\beta(\bar{x}_1) = |P_q\bar{g}| = 0.7425 \times 10^{-4}$ is computed according to Eq. (25). Since $\beta > \epsilon/2nL = 0.1768 \times 10^{-9}$ and $|P_q\bar{g}| \geq \beta_2$, the unit vector $\bar{z}(\bar{x}_1)$ is computed according to Eq. (26). $\bar{z}(\bar{x}_1)$ is a unit vector in the direction of the projected gradient and lies along the tangent to $\varphi_2(\bar{x}) = 0$ at $\bar{x}_1$. Using the step length $\tau = L/25$, a movement is taken in the direction of $\bar{z}$ to the point $\bar{x}^{(0)}(\tau) = \{0.0860, 0.0983\}$. Due to the curvature of the graph of $\varphi_2(\bar{x}) = 0$, $(\varphi_2(\bar{x}^{(0)}(\tau)))^2 > \delta^2$; it is then necessary to apply the correction procedure prescribed by Eqs. (28) and (29) until the point $\bar{x}_2(\tau) = \bar{x}^{(5)}(\tau) = \{0.0863, 0.0958\}$ is obtained. At point $\bar{x}_2(\tau_3)$, $\varphi_1(\bar{x}_2(\tau)) < 0$. One-dimensional interpolation is used to find τ' such that $0 \leq \tau' \leq \tau$, which yields $\bar{x}_2(\tau') = \bar{x}_2 = \{0.1136, 0.1007\}$. At $\bar{x}_2$, $(\varphi_1(\bar{x}_2))^2 + (\varphi_2(\bar{x}_2))^2 \leq \delta^2$. Thus the point $\bar{x}_2$ lies in a δ-neighborhood of the intersection of the two constraint surfaces $\varphi_1(\bar{x}) = 0$ and $\varphi_2(\bar{x}) = 0$.

The gradients of φ_1 and φ_2 at $\bar{x}_2$ are calculated according to Eq. (13) giving $U_q(\bar{x}_2) = [\bar{u}_1, \bar{u}_2]$, where $\bar{u}_1 = \{5.8601, 1.6000\}$, and $\bar{u}_2 = \{-1.5677, 10.4151\}$. $\bar{u}_1(\bar{x}_2)$ and $\bar{u}_2(\bar{x}_2)$ are normal to the graphs of $\varphi_1(x_2) = 0$ and $\varphi_2(\bar{x}) = 0$, respectively, at the point $\bar{x}_2$. The lines $H_1(\bar{x}_2)$ and $H_2(\bar{x}_2)$, which are tangent at $\bar{x}_2$ to the graphs of $\varphi_1 = 0$ and $\varphi_2 = 0$ respectively, are then constructed. The curvature of the graph of $\varphi_2(\bar{x}) = 0$ may be seen by considering the angle between the tangent lines $H_2(\bar{x}_1)$ and $H_2(\bar{x}_2)$. The gradient of the objective function at $\bar{x}_2$, $\bar{g}(\bar{x}_2) = \{-0.74107460\pm, -0.78711168\pm\} \times 10^{-4}$ is calculated from Eq. (51). $\bar{r}(\bar{x}_2) = V_q(\bar{x}_2)U_q^T(\bar{x}_2)\bar{g}(\bar{x}_2) = \{-0.1409, -0.0539\} \times 10^{-4}$ is calculated using Eq. (22), and $U_q(\bar{x}_2)\bar{r}(\bar{x}_2) = \{-0.74107461 \pm, -0.78711167 \pm\} \times 10^{-4}$ is determined. Since, at $\bar{x}_2$, there are two active constraints, $U_q\bar{r}$ is the component of the gradient which is normal to the intersection of the lines $H_1(\bar{x}_2)$ and $H_2(\bar{x}_2)$. As before we obtain $P_q(\bar{x}_2)\bar{g}(\bar{x}_2) = \bar{g}(\bar{x}_2) - U_q(\bar{x}_2)\bar{r}(\bar{x}_2) = \{0.0909, -0.1819\} \times 10^{-11}$, which is the projection of the gradient on the intersection of the lines $H_1(\bar{x}_2)$ and $H_2(\bar{x}_2)$. Since the intersection of two lines is a point, this latter component is zero. The quantities $\beta_1 = -0.4250 \times 10^{-4}$, $\beta_2 = -0.2821 \times 10^{-4}$, and $\beta = 0.2034 \times 10^{-11}$ are computed according to Eqs. (23), (24), and (25). Since $\beta \leq \epsilon/2nL = 0.1768 \times 10^{-9}$, the point $\bar{x}_2$ is the desired optimum point $\bar{x}_{max}$.

The algorithm was restarted at the point $\bar{x}_3 = \{0.120, 0.146\}$, and similar results were obtained; the steps are sketched in Fig. 13.

In the manner described above, the gradient projection method yields a continuous solution to the problem of minimizing the weight of the structure shown in Fig. 11, subject to the prescribed constraints on certain stresses in the structure. It remains to choose available WF sections which provide a realizable optimum structure. Taking $i_{band} = 3$, 7 possibilities for each member are considered. The tolerance $\epsilon_a = 0.01$ is chosen so that a computed stress which is in excess of the allowable stress by less than 1% of the allowable stress is considered acceptable. The available sections considered for each member are shown in Table 2.

By considering all possible combinations of these available sections, it was found that an 18WF55 column section and an 18WF50 beam section provide the optimum available solution.

The moment diagram corresponding to the continuous solution is shown in Fig. 14, and the moment diagram corresponding to the available solution is shown in Fig. 15.

4.3 Symmetrical Two-Story Single-Bay Frame

The second example considered is shown in Fig. 16. It is a two-story single-bay frame having the overall dimensions shown. Members of A7 steel are used, and the frame is to be designed according to current (1965) AISC Specification.[1] For design purposes, it is assumed that a 1 klf uniformly distributed load acts on the roof beams, a 2 klf uniformly distributed load acts on the floor beam, and that wind forces of 3.57^k and 8.92^k act at the roof and floor levels, respectively. Since the wind may act from either direction, it is assumed that the final design will by symmetrical.

This structure was also considered by Bigelow,[4] who obtained the minimum-weight plastic design according to AISC specifications. The results of the elastic and plastic designs will be compared later.

Two cases are considered: In Case I, deflections will not be constrained; in Case II, an allowable lateral deflection at the roof line is specified to be 0.3 in.

As in the first example, the non-dimensional design variables are $x_i = I_i/x_o$, i = 1, 2, 3, 4, where x_o = 9000 in^4; where I_1 corresponds to the roof beam, I_2 the floor beam, and so on, as shown in Fig. 16.

The constraints are defined so that stresses due to bending moment, and combined bending moment and axial load are within the allowable stresses of the AISC Specification. The effects of shear and local buckling are not considered; also, the frame is assumed to be adequately braced against out-of-plane deformations. If necessary, these secondary effects could be considered in the form of additional constraints.

To obtain the constraint functions, consider the typical members shown in Fig. 17. A typical beam of length L is subjected to a total load W = wL, where w is the intensity of the uniform load, and to the end moments, M_L and M_R, where positive moments produce clockwise rotation of the ends of the member in accordance with the matrix formulation of the analysis. The distance from the left and of the beam to the point of maximum positive moment is

$$\xi = 0.5 - \frac{(M_L + M_R)}{WL/2} \quad . \tag{52}$$

The maximum positive moment M^+_{max} is then given by

$$M^+_{max} = \frac{WL}{2}(\xi - \xi^2) + M_L(1-\xi) - M_R\xi \quad . \tag{53}$$

A typical column of length L is subjected to an axial load P and to end moments M_T and M_B.

In formulating the constraints it is preferable to define a single constraint for each member in each loading condition, rather than a separate constraint for each maximum stress in each member. To illustrate this, consider the gravity load stresses in a roof beam of the two-story two bay frame shown in Fig. 24. Neglecting shear and other secondary effects it is necessary to limit the bending stresses due to the end moments, M_L and M_R, and due to the maximum positive moment, M^+_{max}. Suppose that a separate constraint is formulated for each of these moments, and suppose further that for some combination of member sizes $M_L = M_R$. If for this combination of member sizes the constraint pertaining to M_L were active, then the constraint pertaining to M_R would also be active. Since we can only drop one constraint at a time, these two constraints would remain active during the course of the algorithm, and computation would proceed, always maintaining the equality between the end moments. Therefore, since this possibility would lead to an unreasonably restricted answer, it is preferable to write a single constraint which pertains to the controlling stress in a given member for a given loading condition.

In formulating the constraints it is necessary to avoid the possibility of obtaining active constraints which have linearly dependent gradient vectors. For the example of Fig. 16, under gravity loads, the stresses in column AB are identical to the stresses in column CD because of symmetry. Thus the constraint $\varphi_3 \geq 0$ (see Fig. 15) applies both to column AB and to column CD, and a separate constraint for column CD should not be included.

In view of the preceding comments, there are eleven independent constraints as shown in Fig. 18. This figure indicates the correspondence between the constraint functions and the members and loading conditions. $\varphi_1 \geq 0$, for example, requires that the controlling bending stress in the roof beam due to gravity loads be not greater than the allowable stress prescribed by the AISC specifications. $\varphi_{11} \geq 0$ requires that the lateral deflection be not greater than a prescribed value. To illustrate the formulation of these constraints, the following typical cases are considered:

1) typical beam - gravity loads,

2) typical beam - wind and gravity loads,

3) typical column - gravity loads,

4) typical column - wind and gravity loads.

In the following, it is assumed that the forces shown correspond to the loading condition being considered.

1) <u>Typical beam - gravity loads</u>. According to section 1.5.1.4 of the AISC specification, the allowable bending stress is limited to 0.66 F_y, where F_y is the yield stress. For A7 steel, F_y = 33 ksi. The member may be proportioned for 9/10 of the negative moments produced by gravity loading if the maximum position moment is increased by 1/10 of the average of the negative moments. Thus, for gravity loads, the k^{th} constraint corresponding to beam j is given by,

$$\varphi_k = 1 - \frac{M_{max}}{0.66 F_y S_j} , \tag{54}$$

where

$$M_{max} = \max\left\{ |0.9M_L|, |0.9M_R|, |M^+_{max} + .05(|M_L| + |M_R|)\right\} \tag{55}$$

and S_j is the section modulus of the j^{th} beam.

2) Typical beam-wind and gravity loads. According to section 1.5.6 of the AISC Specification, allowable bending stresses for this case may be increased by 1/3. Thus, for wind and gravity loads, the k^{th} constraint corresponding to beam j is,

$$\varphi_k = 1 - \frac{M_{max}}{(4/3)0.66F_y S_j} \quad . \tag{56}$$

Here $M_{max} = \max\,(|M_L|, \; |M_R|, \; |M^+_{max}|)$

3) Typical column gravity loads. According to section 1.5.1.5 and 1.6.1 of AISC Specifications, columns must be proportioned such that

$$\frac{f_a}{F_a} + \frac{C_m f_b}{(1 - \frac{f_a}{F'_e})F_b} \leq 1.0$$

and

$$\frac{f_a}{0.6F_y} + \frac{f_b}{F_b} \leq 1.0$$

where,

$$F_a = \frac{\left[1 - \frac{(kL/r)^2}{2C_c^2}\right] F_y}{F.S.}$$

kL = effective length of the column,

r = radius of gyration in the plane of bending,

C_c = 131.7 for A-7 steel,

F.S. = $\frac{5}{3} + \frac{3}{8}\frac{(kL/r)}{C_c} - \frac{(kL/r)^3}{8\,C_c^3}$,

F_y = yield stress = 33 ksi for A-7 steel,

F_b = 0.66 F_y ,

$$F'_e = \frac{149\ 000\ 000}{(kL/r)^2} ,$$

f_a = computed axial stress = P/A_j ,

f_b = computed bending stress = 0.9 max $\{|M_T|, |M_B|\}/S_j$,

C_m = 0.85 except when $f_a/F_a \leq 0.15$;

For this case, require $\frac{f_a}{F_a} + \frac{f_b}{F_b} \leq 1.0$.

Thus, for gravity loads, the k^{th} constraint corresponding to column j is formulated as follows:

if $P/A_j F_y \leq 0.15$,

$$\varphi_k = \min \left\{ 1 - \frac{f_a}{F_a} - \frac{f_b}{F_b} ,\ 1 - \frac{f_a}{0.6F_y} - \frac{f_b}{F_b} \right\} ; \quad (57a)$$

if $P/A_j F_y > 0.15$,

$$\varphi_k = \min \left\{ 1 - \frac{f_a}{F_a} - \frac{C_m f_b}{(1 - \frac{f_a}{F'_e})F_b} ,\ 1 - \frac{f_a}{0.6F_y} - \frac{f_b}{F_b} \right\} . \quad (57b)$$

4) <u>Typical column-wind and gravity loads</u>. In this case the stresses F_b, F_a, and F'_e may be increased by 1/3 over the values given above. Thus, for wind and gravity loads, the k^{th} constraint corresponding to column j is as follows:

if $P/Aj\ F_y \leq 0.15$,

$$\varphi_k = \min \left\{ 1 - \frac{f_a}{(4/3)F_a} - \frac{f_b}{(4/3)F_b} ,\ 1 - \frac{f_a}{(4/3)0.6F_y} - \frac{f_b}{(4/3)F_b} \right\} ; \quad (58a)$$

if $P/Aj\ F_y > 0.15$

$$\varphi_k = \min \left\{ 1 - \frac{f_a}{(\frac{4}{3})F_a} - \frac{C_m f_b}{(1 - \frac{f_a}{(\frac{4}{3})F'_e})(\frac{4}{3})F_b} ,\ 1 - \frac{f_a}{(\frac{4}{3})0.6F_y} - \frac{f_b}{(\frac{4}{3})F_b} \right\} . \quad (58b)$$

The objective function is defined as in the previous example by using $F = -W/W_o$, where W is the total weight of the structure, and the reference value W_o is taken to be 10,000 lb. The gradient vector $\bar{g}$ is defined by Eqs. (51), for values of i = 1, 2, 3, 4.

Starting the algorithm at the point $\bar{x}_o$ = {0.5, 0.1, 0.5, 0.1}, the gradient projection procedure leads to the continuous solution $\bar{x}_{max}$ = {0.0747, 0.1748, 0.1156, 0.0937} for Case I, in which deflections are not limited. The active constraints at the point $\bar{x}_{max}$ are φ_1, φ_3, φ_6, and φ_{10}. The continuous solution for Case I is shown in Fig. 19.

Considering all combinations of seven available sections for each member, the following available solution was obtained for Case I:

$I_1 = 515$ in^4	,	16 WF40
$I_2 = 1479$	,	21 WF68
$I_3 = 889$	,	18 WF55
$I_4 = 583$	,	16 WF45

This available solution is shown in Fig. 20. The minimum-weight plastic design of the frame as obtained by Bigelow[4] is compared with the present optimum elastic design in Table 3.

For this frame, plastic design leads to a weight savings of 690 lb, or 10.2 percent.

For Case II, in which the lateral deflection at the roof line was limited to 0.3 in., the gradient projection procedure was started at $\bar{x}_o$ = {0.1, 0.5, 0.1, 0.5}, and led to the continuous solution $\bar{x}^{II}_{max+1}$ = {0.07037, 0.57943, 0.07766, 0.41884}. The active constraints at this point are φ_1, φ_3, and φ_{11}.

In Case I the solution $\bar{x}^{I}_{max}$ corresponds to the intersection of four constraint hypersurfaces, which, in a four-dimensional problem, defines a unique point which is a vertex of the feasible region. However, in Case II the solution $\bar{x}^{II}_{max+1}$ corresponds to the intersection of only three constraint hypersurfaces, which are sufficient only to define a curve in a four-dimensional problem. This situation is illustrated in Fig. 21. In this figure, G denotes the curve which is the intersection of $q = n - 1$ constraint hypersurfaces, where n is the number of variables. The algorithm proceeds from the point $\bar{x}_{max-1}$ in the direction of $\bar{z}\ (\bar{x}_{max-1})$ to the point $\bar{x}_{max+1}$, and since the weight of the structure at $\bar{x}_{max+1}$, $W(\bar{x}_{max+1})$, is greater than $W\ (\bar{x}_{max-1})$, $\bar{x}_{max+1}$ is indicated as the solution point according to the modifications to the gradient projection procedure described in Section 3.3 above. Actually, the solution $\bar{x}_{max}$ lies somewhere between the points $\bar{x}_{max+1}$ and $\bar{x}_{max-1}$ along the curve G. Since it was necessary to use a small step length $\tau = L/100$ in traveling along the boundary of the feasible region, the points $\bar{x}_{max-1}$ and $\bar{x}_{max+1}$ are both reasonably close to the actual solution point.

In the present example, the algorithm was restarted at the point $\bar{x}_o' = \{1.0, 1.0, 1.0, 1.0\}$ and led to the continuous solution $\bar{x}^{II}_{max+1}$ = 0.07032, 0.55852, 0.07761, 0.44308 , at which point the active constraints are φ_1, φ_3, and φ_{11} as before. The optimum weights corresponding to the two solution points are 99000 lb and 98992 lb.

The continuous solution corresponding to $\bar{x}_{max+1}$ = {0.07032, 0.55852, 0.07761, 0.44308} is shown in Fig. 22. The available solution was obtained by considering all combinations of seven sections for each member, and is shown in Fig. 23. The same available solution is obtained when the algorithm is started at the alternate point $\bar{x}_o'$.

Beginning at the point $\bar{x}'_o = \{1.0, 1.0, 1.0, 1.0\}$, the path that leads to the continuous solution can be described in the following manner:

begin with feasible point,

$\bar{x} = \{1.0, 1.0, 1.0, 1.0\}$;

move along gradient and interpolate to find first active constraint,

$\bar{x} = \{0.3146, 0.3146, 0.7009, 0.5256\}$

φ_{11} is active;

move along hypersurface defined by $\varphi_{11} = 0$ until next constraint is encountered,

$\bar{x} = \{0.0768, 0.4894, 0.5424, 0.5043\}$

φ_1 and φ_{11} are active;

move along hypersurface defined by

$\varphi_1 = 0$ and $\varphi_{11} = 0$ until next constraint is encountered,

$\bar{x} = \{0.0703, 0.5585, 0.0776, 0.4431\}$

φ_1, φ_3, and φ_{11} are active;

move along hypersurface (curve) defined by $\varphi_1 = 0$, $\varphi_3 = 0$, and $\varphi_{11} = 0$ until weight increases,

$\bar{x}_{max+1} = \{0.0703, 0.5585, 0.0776, 0.4431\}$.

The computer time required for this path was approximately 50 seconds on the IBM 7094.

4.4 Unsymmetrical Two-Story Two-Bay Frame

The third illustrative design is shown in Fig. 24. It is a two-story two-bay frame having the overall dimensions shown in Fig. 24. Members of A36 steel are to be used and the frame is to be designed according to the current (1965) AISC Specification. The structure is subjected to the same loads as in the second example (Section 5.3 above) except, in this case, it is assumed that the wind can act from either direction.

The non-dimensional design variables $x_i = I_i/x_o$, $i = 1,2,\ldots,10$ are defined as before, with $x_o = 9000$ in^4. The members are numbered as shown in Fig. 24.

The constraints are formulated in accordance with the procedure outlined in the second example, the only difference being that higher allowable stresses are specified for A-36 steel. The constraints are numbered as shown in Fig. 25. The constraints $\varphi_{31} \geq 0$, $\varphi_{32} \geq 0$, and $\varphi_{33} \geq 0$ require that the lateral deflection at the roof line be not greater than a specified limit. This limit was taken to be 30 in. so that the deflection constraints would not be come active. In this manner, the solution can be obtained without deflection constraints. The deflections corresponding to this minimum weight design can then be examined, and if they are two large, the problem can be solved again with appropriate limitations on deflections. In this example the maximum lateral deflection was 0.54 inches and was considered acceptable. The effective column lengths are assumed to be 1.2 and 2.0 times the actual lengths for the upper and lower stories, respectively.

The objective function is defined as in the first example (Section 5.2) by $F = -W/W_o$, where W is the total weight and W_o is taken to be 10000 lb. The gradient vector, $\overline{g}$, is as defined by Eqs. (51), for i = 1, 2, ..., 10.

Beginning the gradient projection algorithm at the point $\overline{x}_o$ = {1.0, 1.0,...,1.0}, the solution $\overline{x}_{max}$ = {0.07837, 0.01837, 0.38405, 0.27490, 0.41602, 0.22726, 0.14805, 0.36987, 0.22785, 0.97379} is obtained. The active constraints at the point $\overline{x}_{max}$ are φ_1, φ_2, φ_3, φ_5, φ_6, φ_7, φ_8, φ_9, φ_{10}, and φ_{24}. Since ten constraints are active, and since the problem is ten-dimensional, the point $\bar{x}_{max}$ corresponds to a unique point which is a vertex of the feasible region. The continuous solution is shown in Fig. 26. The path taken according to the gradient projection procedure is obtaining this solution can be summarized as follows:

Begin at feasible point $\bar{x}_o = \{1.0, 1.0, \ldots, 1.0\}$;

follow gradient to first active constraint,

$q = 1$, $\underline{\varphi_{10}}$ is active;

follow projected gradient to next active constraint,

$q = 2$, $\underline{\varphi_8}$, φ_{10} are active;

follow projected gradient to next active constraint,

$q = 3$, φ_8, $\underline{\varphi_9}$, φ_{10} are active;

follow projected gradient to next active constraint,

$q = 4$, $\underline{\varphi_7}$ φ_8, φ_9, φ_{10} are active;

follow projected gradient to next active constraint,

$q = 5$, $\underline{\varphi_4}$, φ_7, φ_8, φ_9, φ_{10} are active;

follow projected gradient to next active constraint,

$q = 6$, $\underline{\varphi_3}$, φ_4, φ_7, φ_8, φ_9, φ_{10} are active;

follow projected gradient to next active constraint,

$q = 7$, φ_3, φ_4, $\underline{\varphi_6}$, φ_7, φ_8, φ_9, φ_{10} are active;

follow projected gradient to next active constraint,

$q = 8$, φ_3, φ_4, φ_6, φ_7, φ_8, φ_9, φ_{10}, $\underline{\varphi_{24}}$ are active;

follow projected gradient to next active constraint,

$q = 9$, φ_3, φ_4, $\underline{\varphi_5}$, φ_6, φ_7, φ_8, φ_9, φ_{10}, φ_{24} are active;

drop φ_4 from intersection of active constraints,

$q = 8$, φ_3, φ_5, φ_6, φ_7, φ_8, φ_9, φ_{10}, φ_{24} are active;

follow projected gradient to next active constraint,

$q = 9$, $\underline{\varphi_2}$, φ_3, φ_5, φ_6, φ_7, φ_8, φ_9, φ_{10}, φ_{24} are active;

follow projected gradient to next active constraint,

$q = 10,\ \underline{\varphi_1},\ \varphi_2,\ \varphi_3,\ \varphi_5,\ \varphi_6,\ \varphi_7,\ \varphi_8,\ \varphi_9,\ \varphi_{10},\ \varphi_{24}$ are active;

at this point, $\beta \leq \epsilon/2nL$; hence $\bar{x} = \bar{x}_{max}$.

Here, projected gradient means the projection of the gradient along the intersection of the active constraints, and q denotes the number of active constraints. Also, the most recently added constraint is underlined in the above summary of steps.

The computer time required for this solution was approximately 6 minutes 40 seconds on the IBM 7094.

Using the second method of obtaining an available solution (Section 4.2), the following available solution is obtained:

$I_1 = 447\ in^4$	,	16WF36
$I_2 = 147$	,	14B17.2
$I_3 = 3610$	,	27WF102
$I_4 = 2370$	,	24WF84
$I_5 = 3610$	,	27WF102
$I_6 = 1600$	,	21WF73
$I_7 = 1328$	,	21WF62
$I_8 = 3260$	,	27WF94
$I_9 = 2093$	,	24WF76
$I_{10} = 9000$	,	36WF150

This available solution leads to a frame weighing 30,053 lb., and is shown in Fig. 27. To obtain the available solution from a known continuous solution, using the second method, required approximately 36 seconds computation time.

4.5 Pin-Jointed Truss

The fourth and last example considered in the present study is shown in Fig. 28. It is a pin-jointed truss, which is 20 ft. high by 10 ft. wide, loaded by 1000k vertical loads and 300k lateral loads as shown. It is assumed that the lateral loads can act from either side. Since the lateral loads can act from either side, the vertical members are all taken to be of the same size; for the same reason, the two diagonal members in a given panel are taken to be identical. Thus it is required to select 5 unknown member sizes. The members are numbered as shown in Fig. 28. The cross-sectional area and least radius of gyration of the members are denoted, respectively, by A_i and r_{min_i}, $i = 1,\ldots,5$.

The non-dimensional variables, x_i, $i = 1,\ldots,5$, are taken to be $x_i = A_i/x_o$, where the reference value, x_o, is taken to be 34.7 in^2.

The non-dimensional constraints are defined so that tensile stresses are limited by the equation

$$1 - \frac{P}{0.8A\ F_t} \geq 0\ , \tag{59}$$

and compressive stresses by

$$1 + \frac{P}{A\ F_a} \geq 0\ , \tag{60}$$

where

P = axial force acting on member being considered (tension positive),

A = cross-sectional area of member being considered (for tension members the net section is assumed to be 80% of the gross section),

F_t = allowable tensile stress on net section

= $0.6\ F_y$,

F_y = yield stress = 36 ksi,

F_a = allowable compressive stress

$$= \frac{\left[1 - 0.5\left(\frac{L/r}{C_c}\right)^2\right] Fy}{F.S.}, \text{ for } L/r \leq C_c,$$

$$= \frac{149000}{(L/r)^2} \text{ for } L/r > C_c,$$

L/r = slenderness ration of member being considered,

L = unbraced length of member,

r = r_{min} = minimum radius of gyration of member,

C_c = 126.1 for A-36 Steel,

$$F.S. = \frac{5}{3} + \frac{3}{8}\left(\frac{L/r}{C_c}\right) - \frac{1}{8}\left(\frac{L/r}{C_c}\right)^3$$

Additional constraints are given requiring that the member sizes should not be smaller than a specified minimum value. In this example, the minimum member size was taken to be 8.81 in^2, which is the cross-sectional area of the smallest available 14WF section. The size constraints are written in the form

$$\frac{x_i x_o}{8.81} - 1 \geq 0; \; i = 1,\ldots,5 \qquad (61)$$

Thus, the problem involves 15 constraints as shown on Fig. 29; 10 of these constraints control the stresses in the members, and the other 5 constraints limit the sizes of the members.

The non-dimensional objective function is defined by $F = -W/W_o$ where W is the total weight of the members and the reference value, W_o, is taken to be 10,000 lb. Letting L_i denote the total length, in ft., of members of size A_i, and w_i the weight per linear foot of members of size A_i,

$$W = \sum_{i=1}^{5} L_i w_i$$

and
$$F = -\frac{1}{W_o}\sum_{i=1}^{5} L_i w_i \ .$$

Since $w_i = \frac{490}{144}$, $A_i = \frac{490}{144} x_i x_o$,

$$F = -\frac{1}{W_o}\sum_{i=1}^{5}\frac{490}{144} L_i x_i x_o = \frac{490 x_o}{144 W_o}\sum_{i=1}^{5} L_i x_i \ .$$

Hence, the gradient of the objective function, $\bar{g}$, is given by

$$\left.\begin{aligned} g_i &= \frac{\partial F}{\partial x_i} = -\frac{490}{144}\frac{x_o}{W_o} L_i \quad , \quad i = 1,\ldots,5, \\ \bar{g} &= \{g_1,\ldots,g_5\} \ . \end{aligned}\right\} \qquad (62)$$

Using the gradient projection technique to obtain the continuous solution and the first method to obtain the available solution, the design shown in Fig. 30 is obtained.

Since, in this design, the size of members 2 and 3 are equal to the lower limit of 8.81 in^2, it was considered possible that a weight savings might result by eliminating one or both of these members, or by selecting a statically determinate design obtained by eliminating one diagonal from each panel. The resulting alternative indeterminate minimum weight designs are shown in Figs. 30 through 32. These designs, along with the determinate designs, are compared in Table 4.

From Table 4, it is evident that the least weight design corresponds to configuration II, in which the lower horizontal bar has been omitted, and the upper horizontal bar has been retained at the minimum size.

It is interesting to note here that a minimum weight solution in which one or more design variables are equal to the lower limit does not necessarily imply that these members should be eliminated. The elimination of one or more members leads to a structure of a different topology, and thus to a new programming problem having fewer variables and no direct relationship to the original programming problem. In order to solve cases II and III, it was necessary in each case to reformulate the constraints and to redefine the flexibility matrix for analyzing the structure. To solve cases IV, V, and VI, no programming was involved, since these structures are statically determinate.

In comparing the designs shown in Table 4 it is obvious that the lightest configuration (Case II) may not necessarily be the cheapest one. In order to choose the most economical design it would be necessary to study the costs of fabricating and erecting each of the alternative designs as well as any other variables that might affect cost. These and other considerations which bear on the choice of actual designs are beyond the scope of this investigation.

In none of the cases considered was any member size controlled by tensile stress. Thus, the assumption $A_{net} = 0.8\ A$ is immaterial as long as the tension connections can be designed so that the stresses on the net sections of the members indicated in Figs. 30 through 35 are acceptable.

V. SUMMARY AND CONCLUSIONS

5.1 Summary

A flexible technique has been developed for obtaining the minimum weight design of indeterminate rigid frames and trusses based on the assumption of elastic behavior. This has been accomplished by assuming that the structures are to be fabricated from commercially available steel sections. By obtaining approximate relationships among the necessary section properties, the number of constraints on stresses and deflections are reduced to a relatively small number; the objective function is formulated in terms of the weight of the structure. The resulting constraints and objective function define a nonlinear programming problem which is an approximation to the original design problem, and which can be solved by means of the gradient projection method of nonlinear programming. The solution of the nonlinear programming problem, called the "continuous solution," may be thought of as a first approximation to the solution of the original design problem. On the basis of the continuous solution an "available solution" is obtained from the combination of available sections that gives the least weight structure.

5.2 Conclusions

The following conclusions can be presented as a result of this investigation:

1. Problems of minimum weight design of statically indeterminate rigid frames and trusses can be formulated as nonlinear programming problems.

2. In order to formulate the design problem as a programming problem, it is necessary to obtain approximate relationships among the section

properties of the structural members. It is possible to do this for the economy WF sections for rigid frames, and for the 14WF sections for trusses.

3. The gradient projection method of nonlinear programming provides an efficient numerical technique for solving nonlinear programming problems in connection with structural design problems. The method is suitable for use on large digital computers, and provides a sound mathematical basis on which to develop a rational design method.

4. It is possible to select a set of commercially available member sizes on the basis of the solution to the programming problem. This can be done with reasonable certainty that the available solution is in fact the lightest available design.

5. The method which has been developed has proven to be flexible. At the present stage, instability constraints, size constraints, and deflection constraints have been formulated in addition to constraints on stresses. In order to accommodate other types of constraints it is necessary merely to obtain the relationships among the required section properties and the design variables. Also, overall dimensions can be included as design variables.

6. A minimum-weight design problem does not necessarily lead to a programming problem with a convex feasible region. Thus, it is not mathematically certain that the programming technique will always lead to a global minimum rather than a local minimum. This difficulty can be overcome by starting the gradient projection algorithm at more than one initial feasible point and comparing the final answers.

7. A minimum weight design is not necessarily a fully-stressed design, especially if the structure is subjected to more than one loading condition and if deflections are constrained. In other words, the solution

to a programming problem does not necessarily occur at a vertex of the feasible region. This occurs when the number of active constraints corresponding to the optimum solution is less than the dimensionality of the feasible region.

8. It is not always possible to obtain the minimum-weight design merely by removing members which tend to become small during the solution of the programming problem. Rather it is necessary to reformulate the programming problem because of changes in the dimensionality of the programming problem and changes in the topology of the structure.

9. The minimum weight design is not necessarily the most economical design. In order to obtain the most economical design the problem must be formulated with the cost of the structure as the objective function of the design variables. The technique proposed herein, however, is applicable to minimum-cost designs.

BIBLIOGRAPHY

1. American Institute of Steel Construction, Manual of Steel Construction, Sixth Edition, New York, 1964.

2. Argyris, J. H. and Kelsey, S., Energy Theorems and Structural Analysis, Butterworth, London, 1960.

3. Asimow, Morris, Introduction to Design, Prentice Hall, Englewood Cliffs, New Jersey.

4. Bigelow, Richard Hugh, Minimum Weight of Plastically Designed Steel Frames, Ph.D. Thesis, University of Illinois, Urbana, Illinois, 1963.

5. Moses, Fred, "Optimum Structural Design Using Linear Programming," Proceedings of the American Society of Civil Engineers, Vol. 90, No. ST6, December, 1964.

6. Rosen, J. B., "The Gradient Projection Method for Nonlinear Programming. Part I: Linear Constraints," Journal of the Society for Industrial and Applied Mathematics, Vol. 8, No. 1, March, 1960.

7. Rosen, J. B., "The Gradient Projection Method for Nonlinear Programming, Part II: Nonlinear Constraints," Journal of the Society for Industrial and Applied Mathematics, Vol. 9, No. 4, December, 1961.

8. Schmit, Lucien A., Jr., "Structural Design by Systematic Synthesis," Second Conference on Electronic Computation, American Society of Civil Engineers, New York, September, 1960.

9. Schmit, Lucien A., Jr., and Kicher, Thomas P., "Synthesis and Material and Configuration Selection," Proceedings of the American Society of Civil Engineers, Vol. 88, No. ST3, June, 1962.

10. Schmit, Lucien A., Jr., and Mallet, Robert H., "Structural Synthesis and Design Parameter Heirarchy," Proceedings of the American Society of Civil Engineers, Vol. 89, No. ST4, August, 1963.

11. Schmit, Lucien A., Jr., and Morrow, William M., "Structural Synthesis with Buckling Constraints," Proceedings of the American Society of Civil Engineers, Vol. 89, No. ST2, April, 1963.

12. Wolfe, Philip, Recent Developments in Nonlinear Programming, Report No. R-401-PR, The Rand Corporation, Santa Monica, California, May, 1962.

APPENDICES

A.1 Computer Program for Minimum-Weight Design

The complete computer program for obtaining the minimum weight elastic design of a structure consists of the following parts:

1) main program;

2) auxiliary subroutines;

3) input data.

The function of the main program is to perform the gradient projection calculations according to the method presented in Sections 2.2 and 2.3 of the text. It is advantageous for this main program to be in such a form that it can be used as a tool to obtain the design of any type of structure. For this reason the main program was designed to be independent of the type of structure being considered. A machine-independent flow diagram of the main program is presented in Appendix A.2.

The auxiliary subroutines perform the following functions:

1) read input variables other than those which pertain to the main program;

2) write the appropriate output quantities whenever this is required by the main program;

3) calculate the section properties;

4) analyze the structure;

5) calculate the values of the constraints;

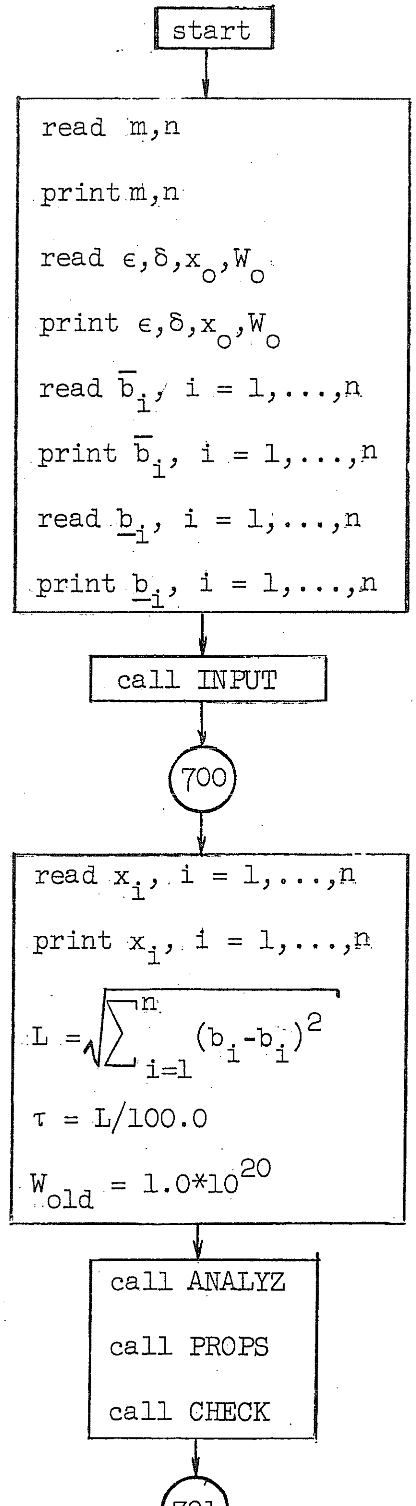
6) calculate the values of the gradient vector and the objection function;

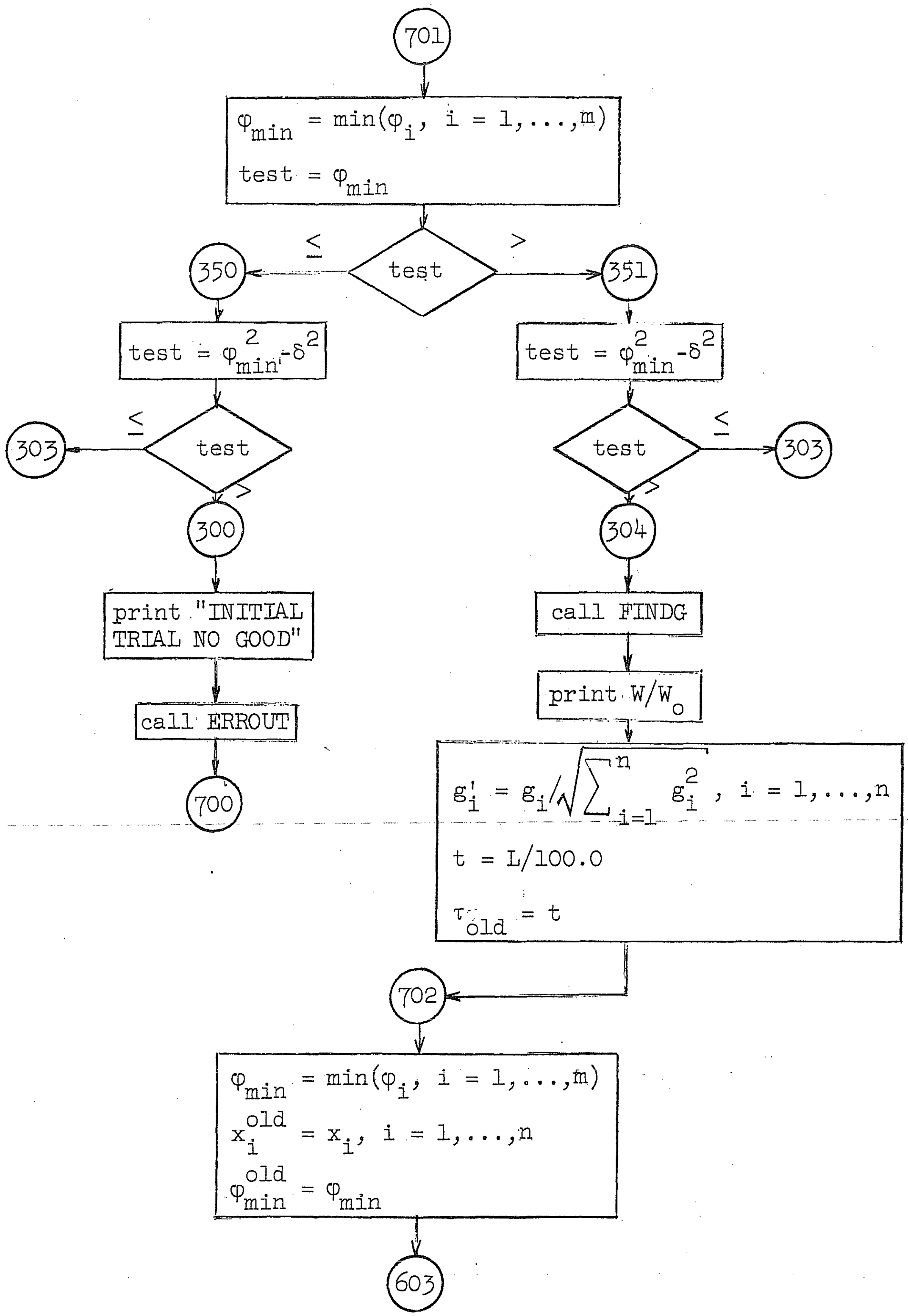
7) select an available design on the basis of the continuous solution.

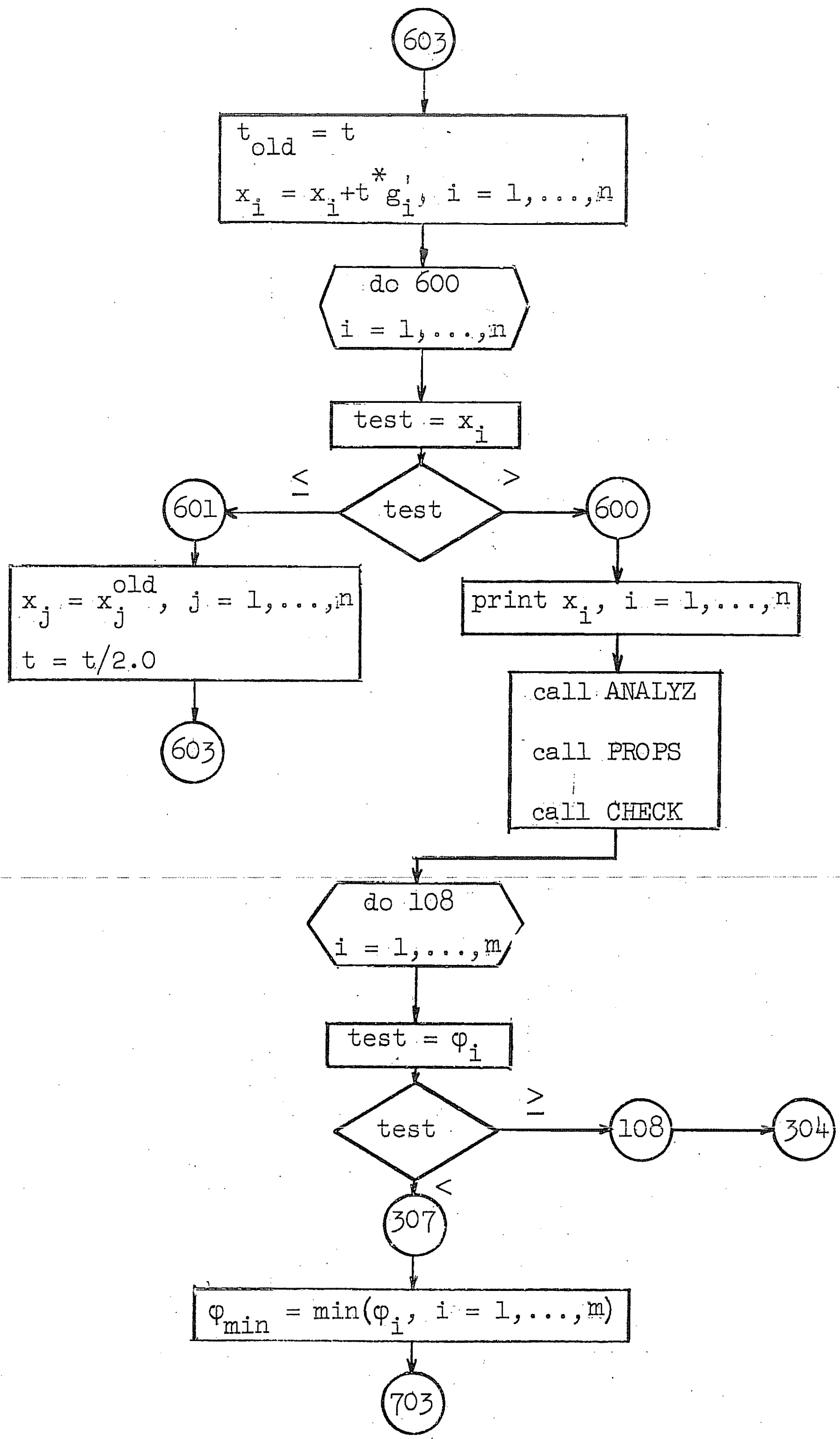
No attempt was made to obtain auxiliary subroutines which would be applicable to all structures. It was necessary to write a separate subroutine for calculating the values of the constraints for each problem. The subroutines for reading input variables and writing output variables, analyzing the structure, calculating the section properties, calculating the objective function and gradient vector, and selecting the available design were written in two forms, depending on whether the structure was a rigid frame or a truss. Short descriptions of the auxiliary subroutines written for the study are presented in Appendix A.3.

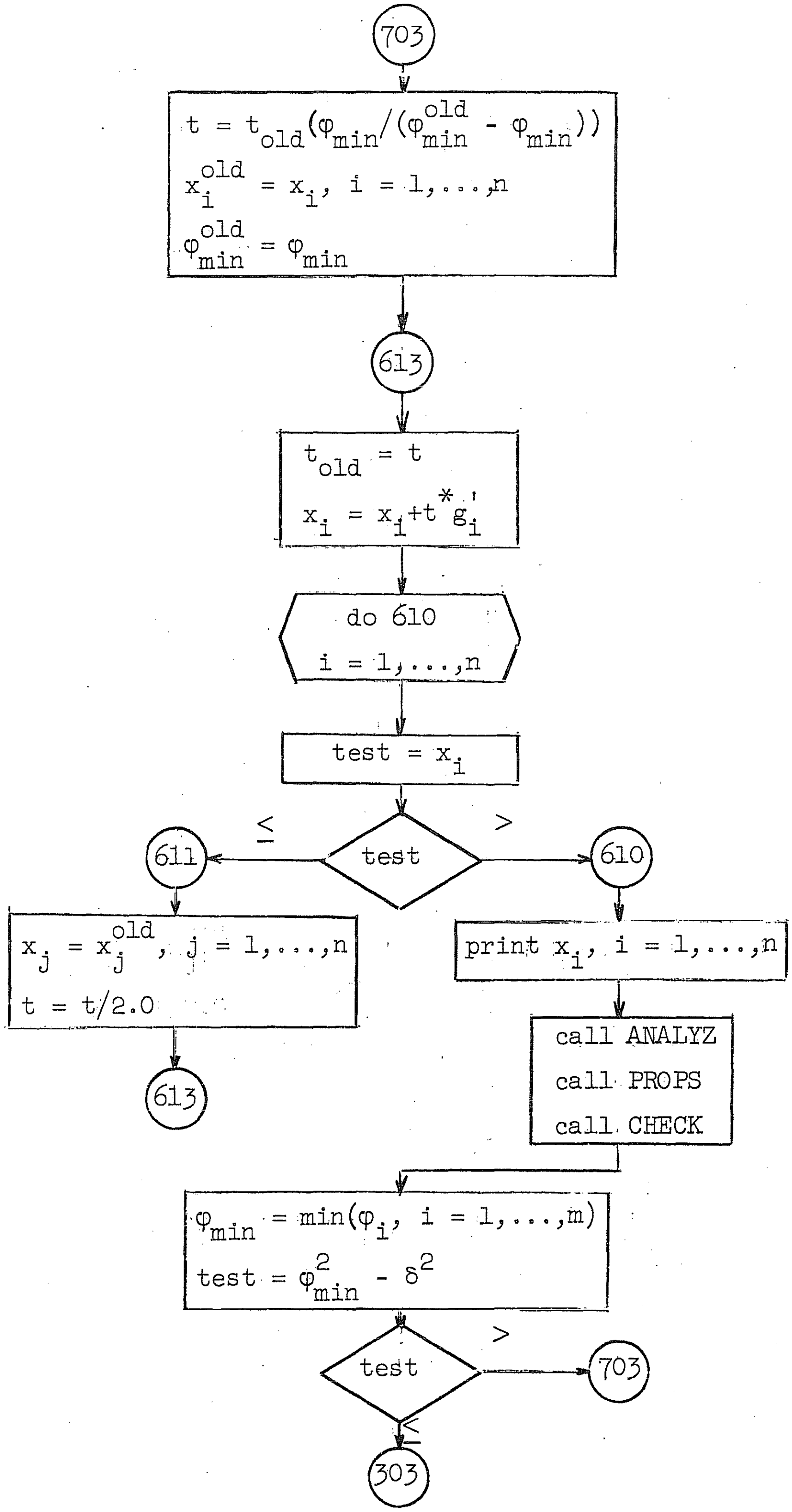
The input data was read according to one of two formats, depending on whether the structure was a rigid frame or a truss. Input data for a typical rigid frame problem and for a typical truss problem are presented and explained in Appendix A.4.

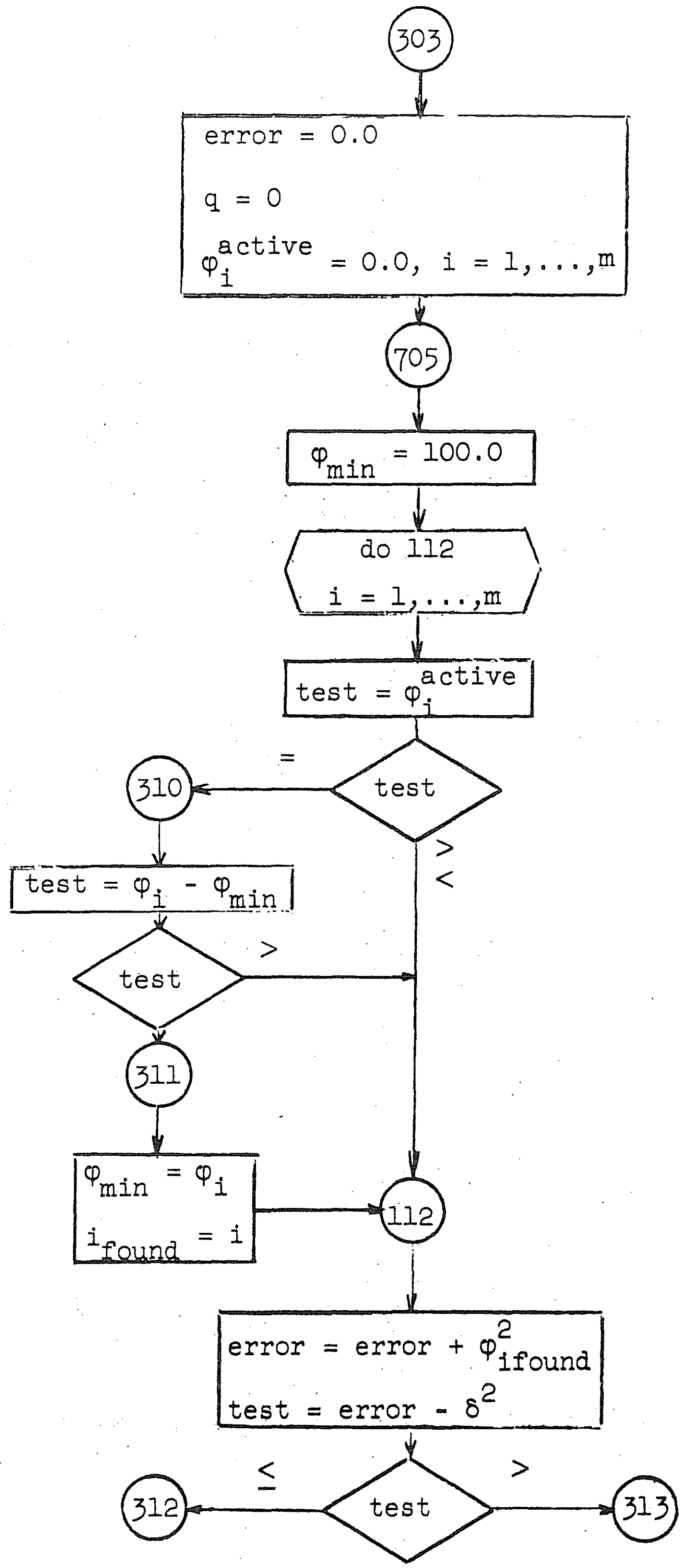
A.2. Flow Diagram--Main Program

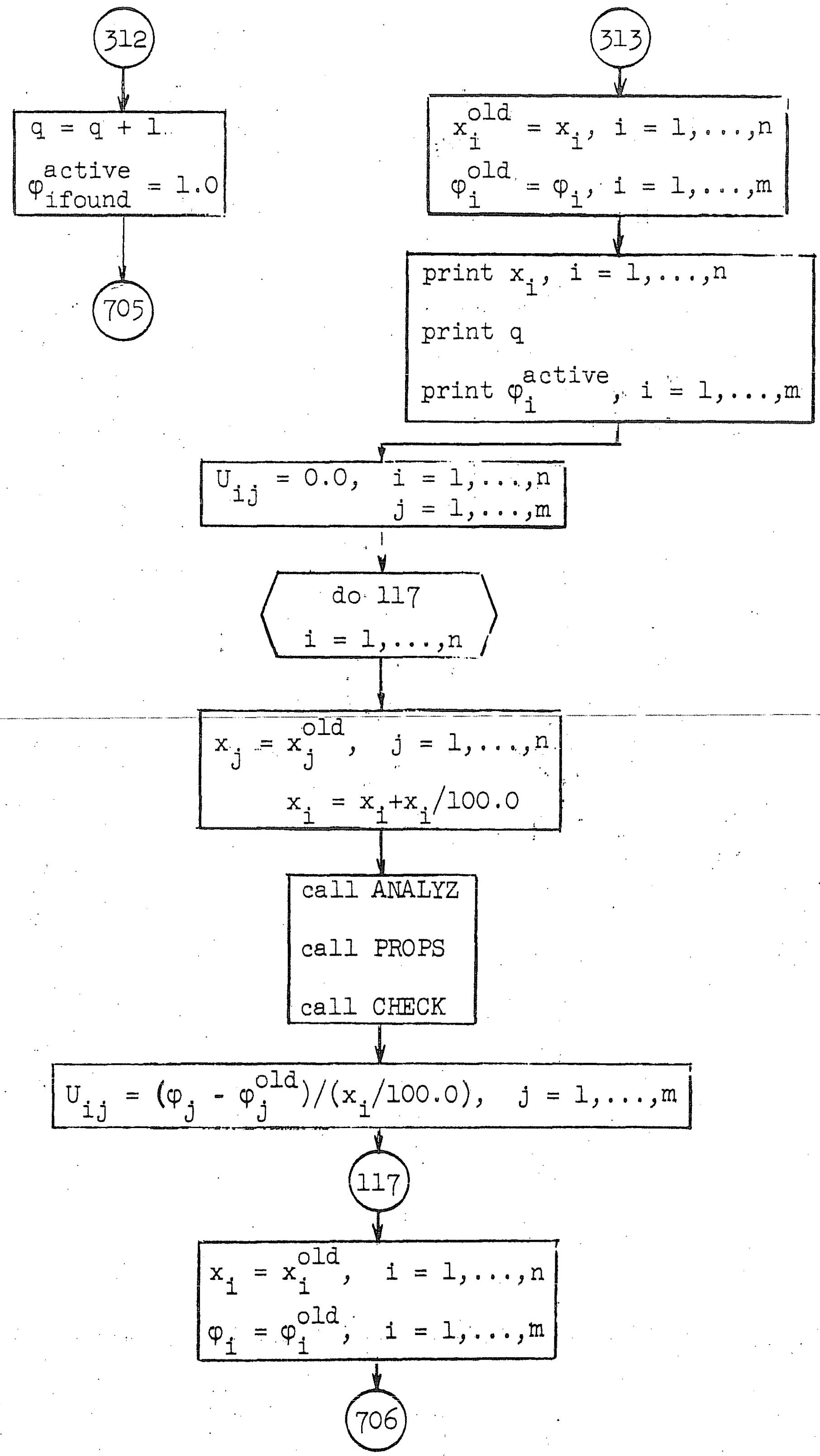


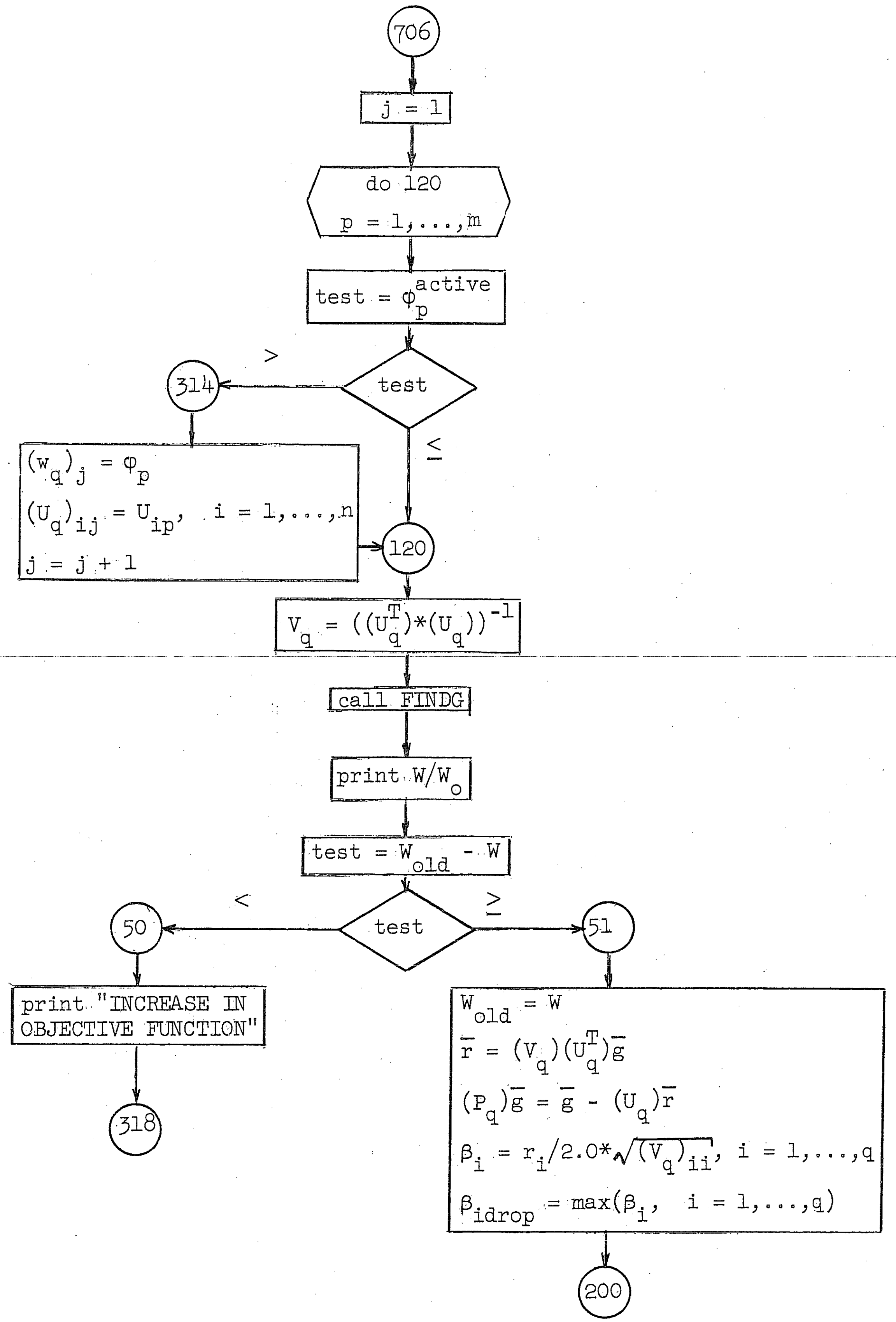

701
φmin = min(φi, i = 1,...,m)
test = φmin
test
≤
>
350
351
test = φmin^2-δ^2
test = φmin^2-δ^2
test
≤
>
303
test
≤
>
303
300
304
print "INITIAL TRIAL NO GOOD"
call ERROUT
700
call FINDG
print W/Wo
g'i = gi/√(Σ i=1 to n gi^2), i = 1,...,n
t = L/100.0
τold = t
702
φmin = min(φi, i = 1,...,m)
xi^old = xi, i = 1,...,n
φmin^old = φmin
603


603
t old = t
x i = x i +t * g' i , i = 1,...,n
do 600
i = 1,...,n
test = x i
≤
test
>
601
600
x j = x j old , j = 1,...,n
t = t/2.0
603
print x i , i = 1,...,n
call ANALYZ
call PROPS
call CHECK
do 108
i = 1,...,m
test = φ i
test
≥
108
304
<
307
φ min = min(φ i , i = 1,...,m)
703

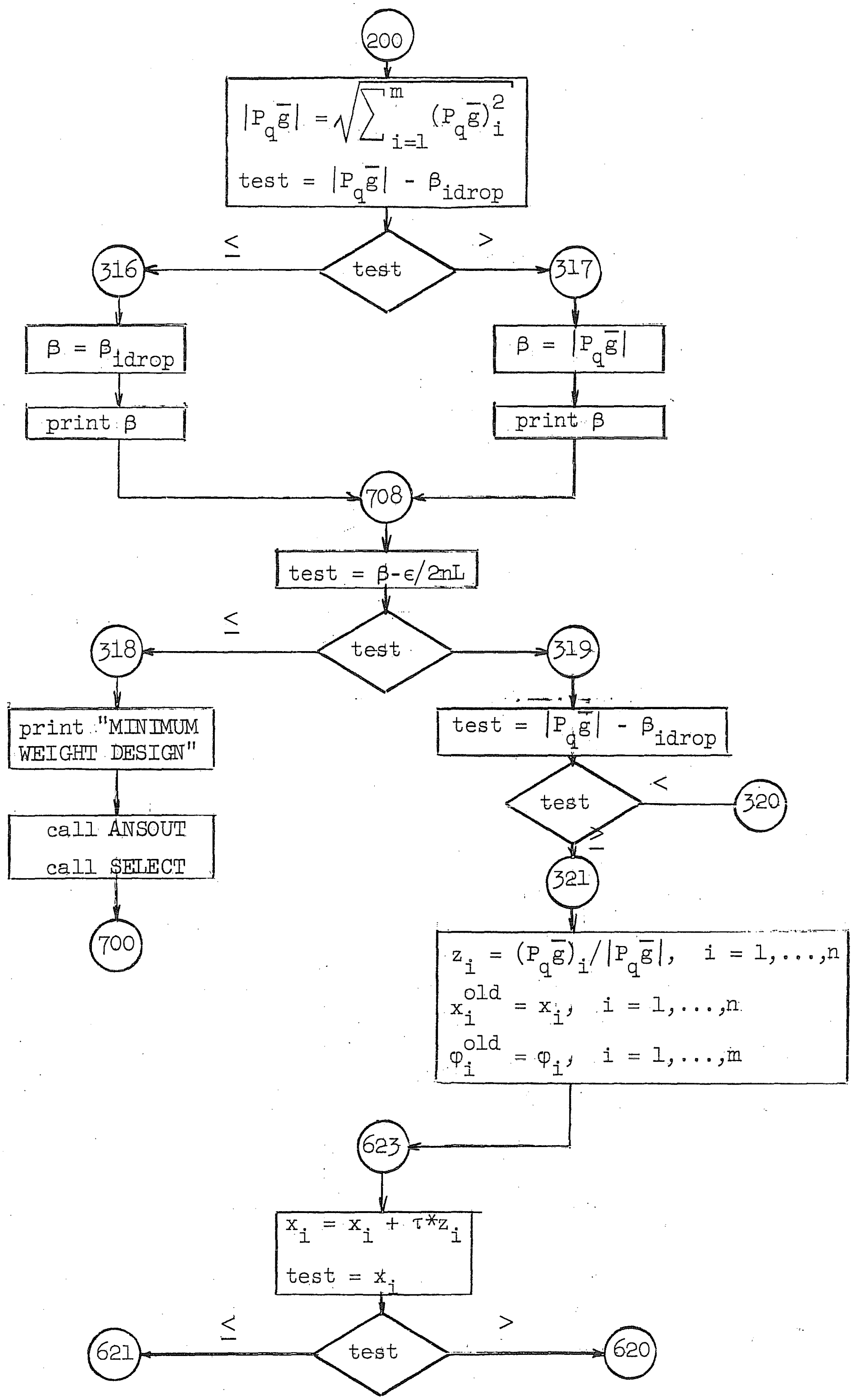

703
t = t_old(φ_min/(φ_min^old - φ_min))
x_i^old = x_i, i = 1,...,n
φ_min^old = φ_min
613
t_old = t
x_i = x_i+t* g_i'
do 610
i = 1,...,n
test = x_i
≤
test
>
611
610
x_j = x_j^old, j = 1,...,n
t = t/2.0
613
print x_i, i = 1,...,n
call ANALYZ
call PROPS
call CHECK
φ_min = min(φ_i, i = 1,...,m)
test = φ_min^2 - δ^2
test
>
703
≤
303

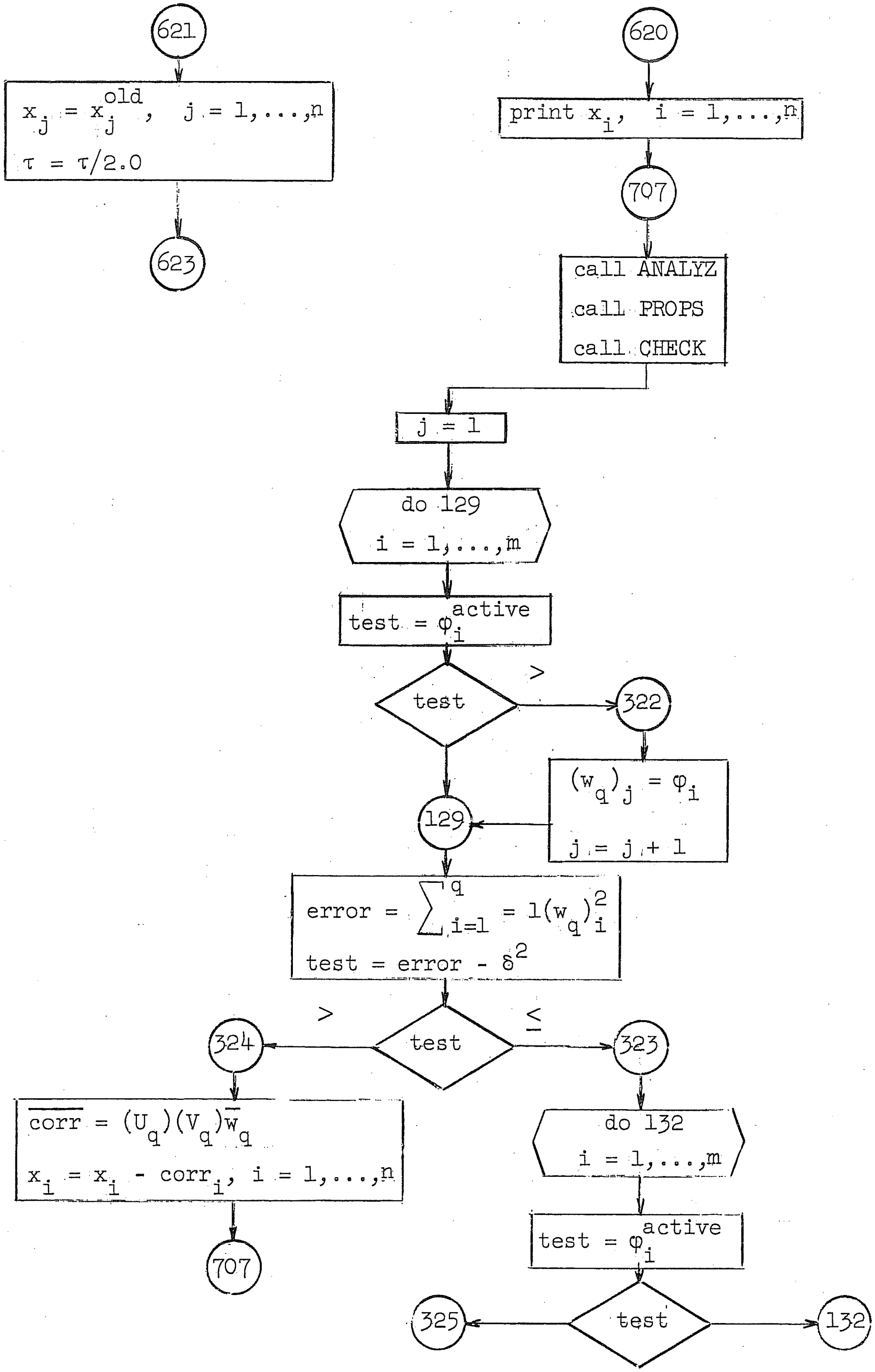

303
error = 0.0
q = 0
φ_i^active = 0.0, i = 1,...,m
705
φ_min = 100.0
do 112
i = 1,...,m
test = φ_i^active
test
=
310
>
<
test = φ_i - φ_min
test
>
311
φ_min = φ_i
i_found = i
112
error = error + φ_ifound^2
test = error - δ^2
test
≤
>
312
313

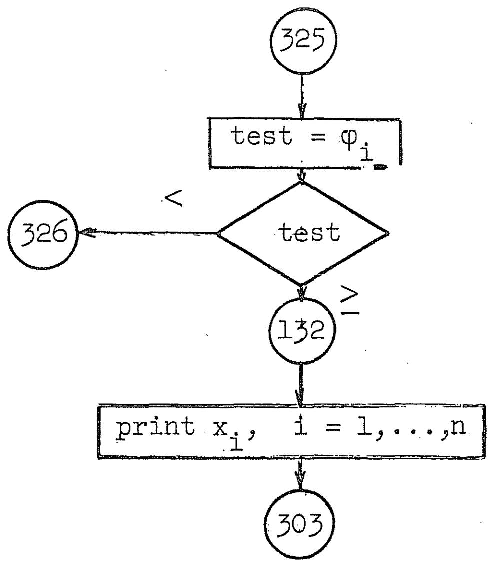

312
$q = q + 1$
$\varphi^{active}_{ifound} = 1.0$
705
313
$x^{old}_i = x_i$, $i = 1,\ldots,n$
$\varphi^{old}_i = \varphi_i$, $i = 1,\ldots,m$
print x_i, $i = 1,\ldots,n$
print q
print φ^{active}_i, $i = 1,\ldots,m$
$U_{ij} = 0.0$, $i = 1,\ldots,n$ $j = 1,\ldots,m$
do 117
$i = 1,\ldots,n$
$x_j = x^{old}_j$, $j = 1,\ldots,n$
$x_i = x_i + x_i/100.0$
call ANALYZ
call PROPS
call CHECK
$U_{ij} = (\varphi_j - \varphi^{old}_j)/(x_i/100.0)$, $j = 1,\ldots,m$
117
$x_i = x^{old}_i$, $i = 1,\ldots,n$
$\varphi_i = \varphi^{old}_i$, $i = 1,\ldots,m$
706

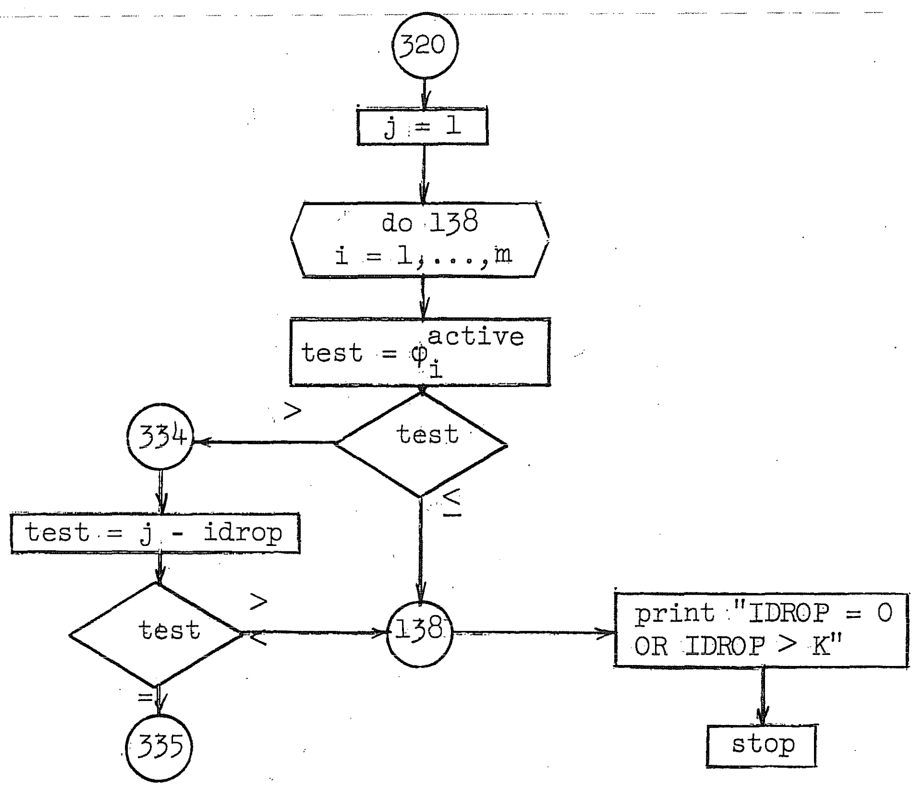


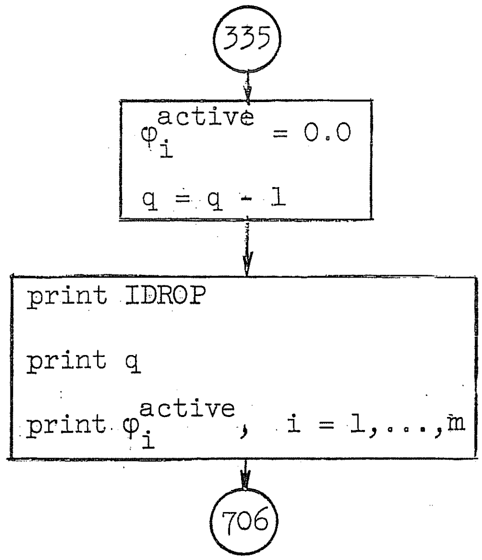
706
j = 1
do 120
p = 1,...,m
test = φ_p^active
test
>
≤
314
(w_q)_j = φ_p
(U_q)_ij = U_ip, i = 1,...,n
j = j + 1
120
V_q = ((U_q^T)*(U_q))^-1
call FINDG
print W/W_o
test = W_old - W
test
<
≥
50
print "INCREASE IN OBJECTIVE FUNCTION"
318
51
W_old = W
r̄ = (V_q)(U_q^T)ḡ
(P_q)ḡ = ḡ - (U_q)r̄
β_i = r_i/2.0*√((V_q)_ii), i = 1,...,q
β_idrop = max(β_i, i = 1,...,q)
200

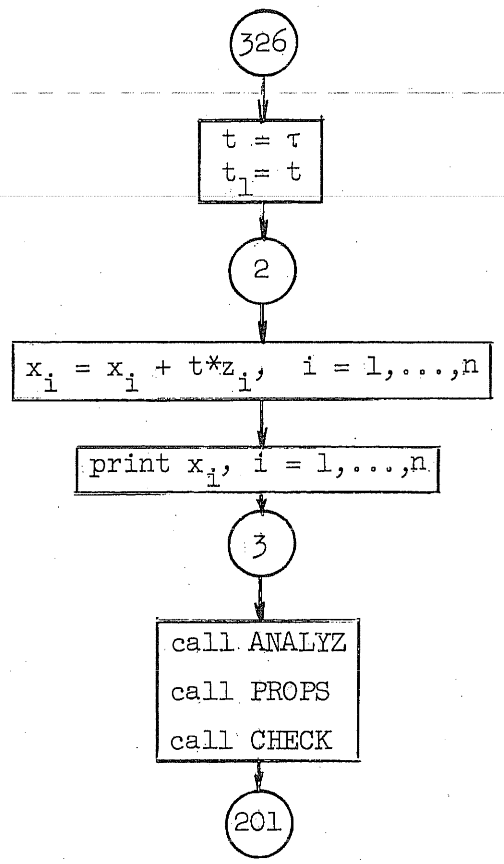

200
|Pqḡ| = √(Σ_{i=1}^{m} (Pqḡ)_i^2)
test = |Pqḡ| - β_idrop
≤
test
>
316
317
β = β_idrop
β = |Pqḡ|
print β
print β
708
test = β-ε/2nL
≤
test
318
319
print "MINIMUM WEIGHT DESIGN"
test = |Pqḡ| - β_idrop
test
<
320
≥
call ANSOUT
call SELECT
321
700
z_i = (Pqḡ)_i/|Pqḡ|, i = 1,...,n
x_i^old = x_i, i = 1,...,n
φ_i^old = φ_i, i = 1,...,m
623
x_i = x_i + τ*z_i
test = x_i
≤
test
>
621
620

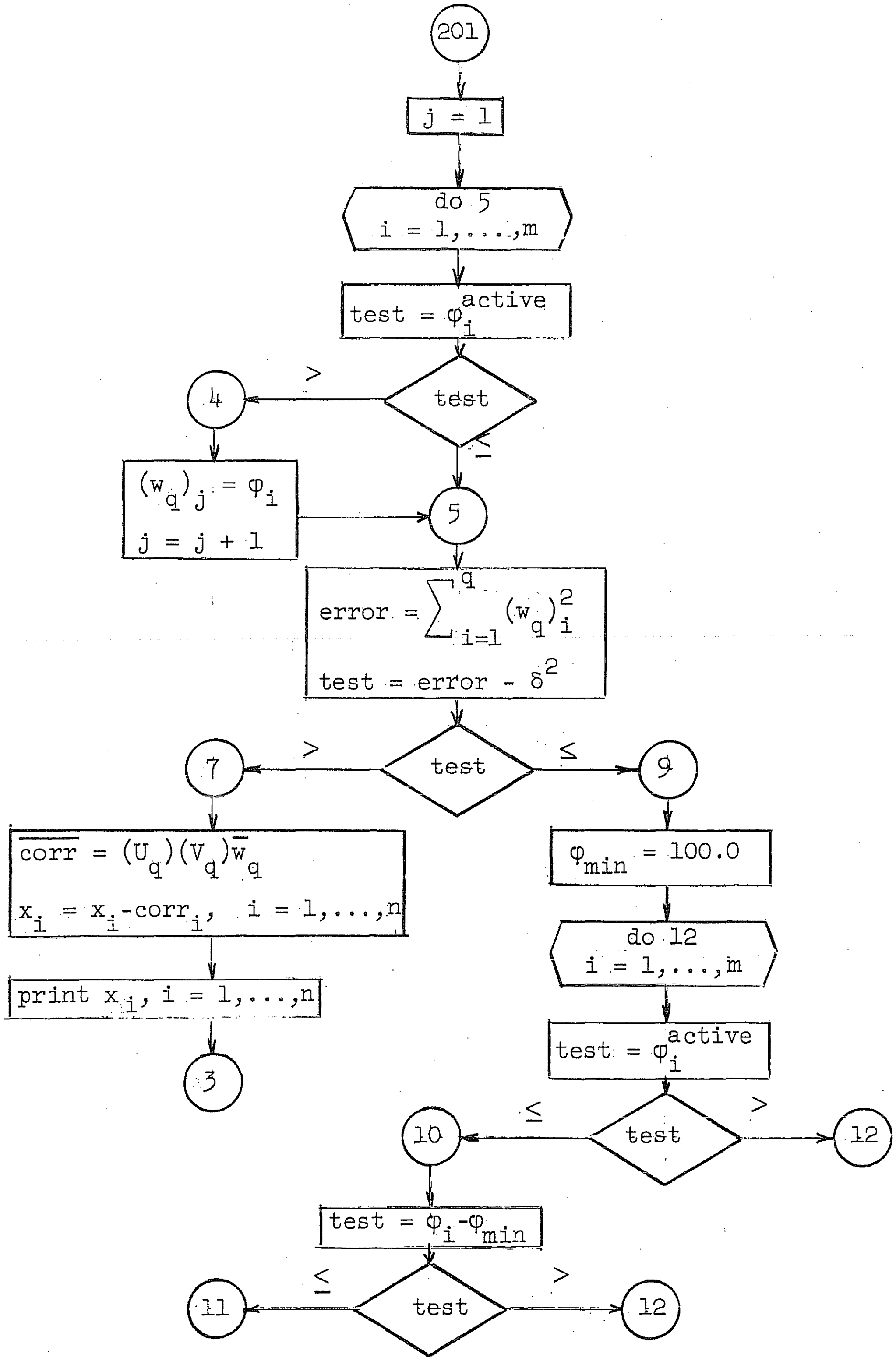

621
x_j = x_j^old, j = 1,...,n
τ = τ/2.0
623
620
print x_i, i = 1,...,n
707
call ANALYZ
call PROPS
call CHECK
j = 1
do 129
i = 1,...,m
test = φ_i^active
test
>
322
(w_q)_j = φ_i
j = j + 1
129
error = Σ_{i=1}^{q} = 1(w_q)_i^2
test = error - δ^2
>
test
≤
324
323
corr = (U_q)(V_q)w_q
x_i = x_i - corr_i, i = 1,...,n
707
do 132
i = 1,...,m
test = φ_i^active
325
test
132

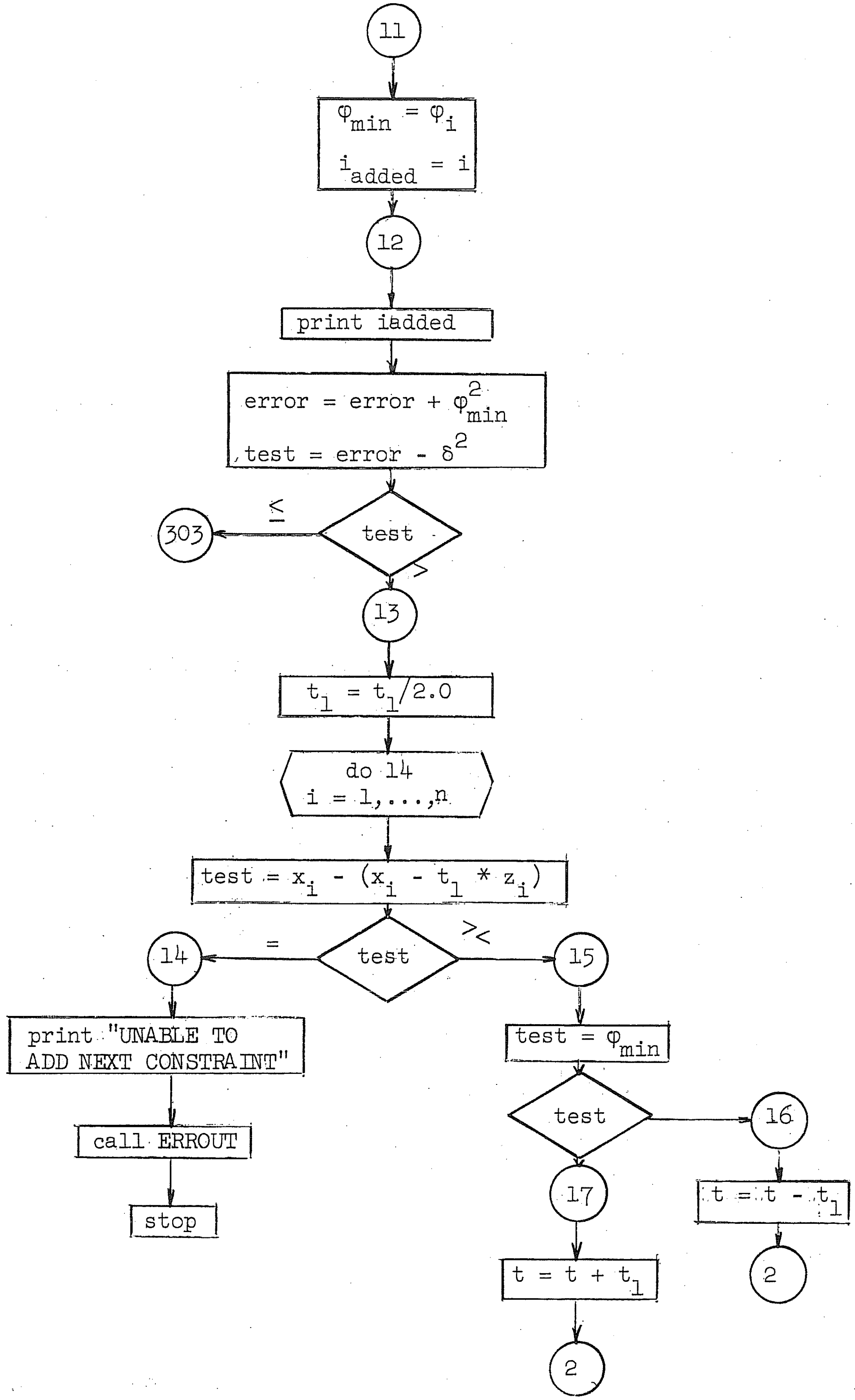

325
test = φ_i
test
<
326
≥
132
print x_i, i = 1,...,n
303


320
j = 1
do 138
i = 1,...,m
test = φ_i^{active}
test
>
334
≤
test = j - idrop
test
>
<
=
335
138
print "IDROP = 0
OR IDROP > K"
stop


335
$\varphi_i^{active} = 0.0$
q = q - 1
print IDROP
print q
print φ_i^{active}, i = 1,...,m
706


326
t = τ
t_1 = t
2
$x_i = x_i + t*z_i$, i = 1,...,n
print x_i, i = 1,...,n
3
call ANALYZ
call PROPS
call CHECK
201


201
j = 1
do 5
i = 1,....,m
test = φ_i^active
test
>
≤
4
(w_q)_j = φ_i
j = j + 1
5
error = Σ_{i=1}^{q} (w_q)_i^2
test = error - δ^2
test
>
≤
7
9
corr = (U_q)(V_q)w_q
x_i = x_i-corr_i, i = 1,...,n
print x_i, i = 1,...,n
3
φ_min = 100.0
do 12
i = 1,...,m
test = φ_i^active
test
≤
>
10
12
test = φ_i-φ_min
test
≤
>
11
12



11
φmin = φi
iadded = i
12
print iadded
error = error + φ²min
test = error - δ²
≤
303
test
>
13
t1 = t1/2.0
do 14
i = 1,...,n
test = xi - (xi - t1 * zi)
=
><
14
test
15
print "UNABLE TO ADD NEXT CONSTRAINT"
test = φmin
call ERROUT
test
16
stop
17
t = t - t1
t = t + t1
2
2

A.3. Descriptions of Auxiliary Subroutines

INPUT: A subroutine which reads input variables required by subroutines other than the main program.

ANALYZ: A subroutine which analyzes the structure for unknown forces and displacements on the basis of the current design and variables supplied by INPUT.

PROPS: A subroutine which calculates the necessary section properties on the basis of the current design using the approximate relationships developed in Chapter 2.

CHECK: A subroutine which calculates the values of the constraint functions φ_i, $i = 1,\ldots,m$, on the basis of the current design and the results of ANALYZ and PROPS.

ERROUT: A subroutine which prints current values of the design variables and other quantities of interest when the main program fails.

FINDG: A subroutine which computes the values of the objective function and the gradient vector on the basis of the current design and the variables read by INPUT.

ANSOUT: A subroutine which prints the continuous solution $\bar{x}$max and other information of interest when the main program has obtained a solution.

SELECT: A subroutine which selects an available minimum-weight design on the basis of the continuous solution and the available sections properties.

A.4. Typical Input Data

Input data for a typical rigid frame problem are shown in Table 5. This data corresponds to the symmetrical braced portal frame considered in

Section 4.2 of the text. In Table 5, m and n denote the number of constraints and the number of variables respectively. ϵ and δ denote the tolerances defined in Chapter 2. x_o and W_o denote the reference values, $x_o = 9000$ in^4 and $W_o = 10{,}000$ lb. $\bar{b}_i$ and $\underline{b}_i$ denote the upper and lower bounds defined in Chapter 2. n_{ls}, n_{mf}, n_{eal}, and n_{imd} are variables pertaining to the Argyris formulation of structural analysis and are respectively, the number of loading conditions considered, the number of member forces to be determined, the number of loads applied; and the number of internal member displacements to be calculated. F_e is a $n_{eal} \times n_{ls}$ matrix of externally applied loads, F_{fem} is a $n_{mf} \times n_{ls}$ matrix of fixed-end moments, and a_e is a $n_{imd} \times n_{eal}$ matrix giving internal joint displacements corresponding to unit displacements of the external joints. L_i denotes the member lengths. n_{sects} denotes the total number of available WF economy sections. i_{band} is a measure of the number of these available sections to be considered as possibilities for each member to be selected. For each member the available section next larger in size than indicated by the continuous solution plus i_{band} sections above and i_{band} sections below that one are considered. Thus $2\, i_{band} + 1$ available sections are considered for each number in the determination of an available solution. ϵ_a is a tolerance specifying an acceptable error by which a constraint may be violated in an available design; that is, a given combination of available sections is considered acceptable if $\varphi_i \geq -\epsilon_a$, $i = 1,\ldots,m$. P_{sects} is a table of section properties giving the moment of inertia, section modulus, unit weight, maximum radius of gyration, and cross-sectional area of each available WF economy section. The cards in the table P_{sects} are sorted in order of increasing moment of inertia. The last card (or cards) gives the values of the variables, x_i, for the initial feasible solution. This input would be typical for a rigid frame analyzed by the Argyris displacement method. All matrices except the table P_{sects} are punched by columns; P_{sects} is punched by rows.

Input data for a typical truss problem are shown in Table 6. These data correspond to the twice indeterminate pin-jointed truss considered in Section 4.5 of the text. In Table 6, m and n denote the number of constraints and number of variables respectively. ϵ and δ denote the tolerances defined in Chapter 3. x_o and W_o denote the reference values $x_o = 34.7$ in^2 and $W_o = 10000$ lb. $\overline{b}_i$ and $\underline{b}_i$ denote upper and lower bounds defined in Chapter 2. n_{if}, n_{eal}, n_{rif}, and n_{ls} are variables pertaining to the Argyris formulation of structural analysis, and are, respectively, the number of internal member forces, the number of externally applied loads, the number of redundant internal forces, and the number of loading conditions considered. Z is a 4x1 matrix of external loads, b_o is a n_{if} x n_{eal} matrix of internal member forces caused by unit external forces acting on the structure without the redundant members, and b_1, is a n_{if} x n_{rif} matrix of internal member forces caused by unit values of the redundant member forces. n_{sects} denotes the number of sections available, ϵ_a is a tolerance specifying an acceptable error by which a constraint may be violated in an acceptable available design. P_{sects} is a table of the properties of the 14-inch wide flange sections and gives the minimum radius of gyration and cross-sectional area for each section. The cards in the table P_{sects} are sorted in order of increasing area. $\overline{x}$ is the initial solution. This input would be typical for a pin-jointed truss analyzed by the Argyris force method. All matrices except the table P_{sects} are punched by columns; P_{sects} is punched by rows.

TABLE 1. PROPERTIES OF THE ECONOMY SECTIONS

Section	Moment of Inertia (inches⁴)	Section Modulus (inches³)	Weight Per Foot (lb per ft)	In-Plane Radius of Gyration (inches)	Area (inches²)
3⊔ 4.1	1.65	1.1	4.1	1.17	1.19
3⊔ 5.0	1.80	1.2	5.0	1.12	1.46
6Jr 4.4	7.20	2.4	4.4	2.37	1.30
7Jr 5.5	12.20	3.5	5.5	2.74	1.61
8Jr 6.5	18.80	4.7	6.5	3.12	1.92
10Jr⊔ 8.4	32.50	6.5	8.4	3.61	2.47
10Jr 9.0	39.00	7.8	9.0	3.85	2.64
12Jr⊔10.6	55.80	9.3	10.6	4.23	3.12
10B 11.5	32.50	10.5	11.5	3.92	3.39
12Jr 11.8	72.00	12.0	11.8	4.57	3.45
12B 14.0	88.80	14.8	14.0	4.61	4.14
12B 16.5	105.00	17.5	16.5	4.65	4.86
14B 17.2	147.00	21.0	17.2	5.40	5.05
12B 19.0	128.40	21.4	19.0	4.81	5.62
10WF 21	106.40	21.5	21.0	4.14	6.19
12B 22.0	151.80	25.3	22.0	4.91	6.47
~~10WF 25~~	~~133.00~~	~~26.4~~	~~25.0~~	~~4.26~~	~~7.35~~
12WF 27	204.00	34.1	27.0	5.06	7.97
14WF 30	290.00	41.8	30.0	5.73	8.81
14WF 34	340.00	48.5	34.0	5.83	10.00
16WF 36	447.00	56.3	36.0	6.49	10.59
16WF 40	515.00	64.4	70.0	6.62	11.77
16WF 45	583.00	72.4	45.0	6.64	13.24
16WF 50	657.00	80.7	50.0	6.68	14.70
18WF 50	802.00	89.0	50.0	7.38	14.71
18WF 55	889.00	98.2	55.0	7.41	16.19
18WF 60	985.00	107.8	60.0	7.47	17.64
21WF 62	1328.00	126.4	62.0	8.53	18.23
21WF 68	1479.00	139.9	68.0	8.59	20.02
21WF 73	1600.00	150.7	73.0	8.64	21.46
24WF 76	2093.00	175.4	76.0	9.68	22.37
24WF 84	2370.00	196.3	84.0	9.78	24.71
24WF 94	2680.00	220.1	94.0	9.85	27.63
27WF 94	3260.00	242.8	94.0	10.87	27.65
24WF 100	2990.00	248.9	100.0	10.08	29.43
27WF 102	3610.00	266.3	102.0	10.96	30.01
30WF 108	4460.00	299.2	108.0	11.85	31.77
30WF 116	4920.00	327.9	116.0	12.00	34.13
30WF 124	5350.00	354.6	124.0	12.11	36.45
33WF 130	6700.00	404.8	130.0	13.23	38.26
33WF 141	7440.00	446.8	141.0	13.39	41.51
36WF 150	9000.00	502.9	150.0	14.29	44.16

(Continued)

TABLE 1. (Continued)

Section	Moment of Inertia (inches^4)	Section Modulus (inches^3)	Weight Per Foot (lb per ft)	In-Plane Radius of Gyration (inches)	Area (inches^2)
36WF 160	9730.00	541.0	160.0	14.38	47.09
36WF 170	10470.00	579.1	170.0	14.47	49.98
36WF 182	11280.00	621.2	182.0	14.52	53.54
36WF 194	12100.00	663.6	194.0	14.56	57.11
33WF 200	11050.00	669.6	200.0	13.71	58.79
33WF 220	12320.00	740.6	220.0	13.79	64.73
36WF 230	15000.00	835.5	230.0	14.88	67.73
36WF 245	16080.00	892.5	245.0	14.95	72.03
36WF 260	17240.00	951.1	260.0	15.00	76.56
36WF 280	18820.0	1031.2	280.0	15.12	82.32
36WF 300	20300.00	1105.1	300.0	15.17	88.17

TABLE 2. AVAILABLE SECTIONS CONSIDERED
TWO-DIMENSIONAL EXAMPLE

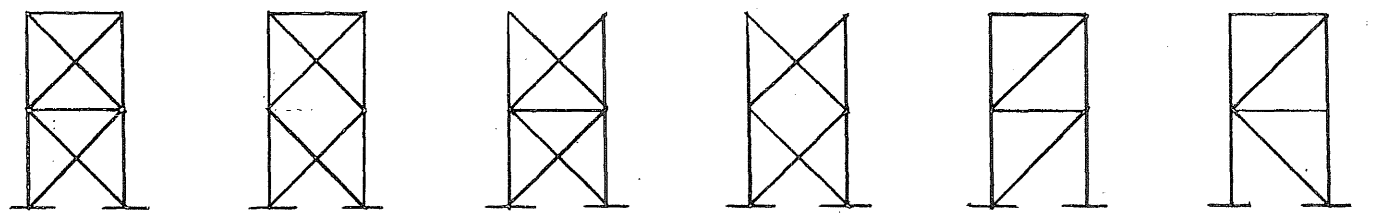
Column	Beam
18WF50	16WF50
18WF55	18WF50
18WF60	18WF55
21WF62	18WF60
21WF68	21WF62
21WF73	21WF68
24WF76	21WF 73

TABLE 3. COMPARISON OF PLASTIC AND ELASTIC DESIGNS
OF TWO-STORY SINGLE-BAY FRAME

Member	Plastic Design	Elastic Design
1	16WF40	16WF40
2	21WF62	21WF68
3	16WF45	18WF55
4	16WF40	16WF45
Weight	6080#	6770#

TABLE 4. COMPARISON OF ALTERNATE DESIGNS FOR PIN-JOINTED TRUSS MEMBERS SELECTED FOR AVAILABLE SOLUTIONS

Configuration



	I	II	III	IV	V	VI
Verticals	14WF264	14WF264	14WF264	14WF287	14WF342	14WF342
Upper Horizontal	14WF30	14WF30	-	-	14WF61	14WF61
Lower Horizontal	14WF30	-	14WF30	-	14WF103	14WF61
Upper Diagonals	2-14WF76	2-14WF68	2-14WF84	2-14WF103	1-14WF84	1-14WF84
Lower Diagonals	2-14WF103	2-14WF95	2-14WF103	2-14WF103	1-14WF142	1-14WF142
Weight of Continuous Solution	15 725 lb	15 348	15 787	16 420	17 920	17 450
Weight of Available Solution	16 172 lb	15 479	16 156	17 330	18 520	17 910

TABLE 5. INPUT DATA FOR SYMMETRICAL BRACED PORTAL FRAME

Cols. 1-10	Cols. 11-20	Cols. 21-30	Cols. 31-40	Cols. 41-50	Cols. 51-60	Identification
3	2					k,m
.000000001	.00001	9000.0	10000.0			ϵ,δ,x_o,W_o
1.0	1.0					b_i
0.0	0.0					$\underline{b}_i$
1	6	2	6			$n_{ls},n_{mf},n_{eal},n_{imd}$
+2025.0	-2025.0					F_e (2x1)
0.0	0.0	-2025.0	2025.0	0.0	0.0	F_{fem} (6x1)
0.0	1.0	1.0	0.0	0.0	0.0	a_e (6x2)
0.0	0.0	0.0	1.0	1.0	0.0	
30.0	30.0					L_i
54	4	0.01				$n_{sects},i_{band},\epsilon_a$
3 4.1	1.65	1.1	4.1	1.17	1.19	P_{sects} (54x6)
3 5.0	1.80	1.2	5.0	1.12	1.46	
6Jr 4.4	7.20	2.4	4.4	2.37	1.30	
7Jr 5.5	12.20	3.5	5.5	2.74	1.61	
8Jr 6.5	18.80	4.7	6.5	3.12	1.92	
10Jr 8.4	32.50	6.5	8.4	3.61	2.47	
10B 11.5	32.50	10.5	11.5	3.92	3.39	
10Jr 9.0	39.00	7.8	9.0	3.85	2.64	
12Jr 10.6	55.80	9.3	10.6	4.23	3.12	
12Jr 11.8	72.00	12.0	11.8	4.57	3.45	
12B 14.0	88.80	14.8	14.0	4.61	4.14	
12B 16.5	105.00	17.5	16.5	4.65	4.86	
10WF 21	106.40	21.5	21.0	4.14	6.19	
12B 19.0	128.40	21.4	19.0	4.81	5.62	
10WF 25	133.00	26.4	25.0	4.26	7.35	
14B 17.2	147.00	21.0	17.2	5.40	5.05	
12B 22.0	151.80	25.3	22.0	4.91	6.47	
12WF 27	204.00	34.1	27.0	5.06	7.97	
14WF 30	290.00	41.8	30.0	5.73	8.81	
14WF 34	340.00	48.5	34.0	5.83	10.00	
16WF 36	447.00	56.3	36.0	6.49	10.59	

(Continued)

TABLE 5. (Continued)

Cols. 1-10	Cols. 11-2-	Cols. 21-30	Cols. 31-40	Cols. 41-50	Cols. 51-60	Identification
16WF 40	515.00	64.4	40.0	6.62	11.77	
16WF 45	583.00	72.4	45.0	6.64	13.24	
16WF 50	657.00	80.7	50.0	6.68	14.70	
18WF 50	802.00	89.0	50.0	7.38	14.71	
18WF 55	889.00	98.2	55.0	7.41	16.19	
18WF 60	985.00	107.8	60.0	7.47	17.64	
21WF 62	1328.00	126.4	62.0	8.53	18.23	
21WF 68	1479.00	139.9	68.0	8.59	20.02	
21WF 73	1600.00	150.7	73.0	8.64	21.46	
24WF 76	2093.00	175.4	76.0	9.68	22.37	
24WF 84	2370.00	196.3	84.0	9.78	24.71	
24WF 94	2680.00	220.1	94.0	9.85	27.63	
24WF 100	2990.00	248.9	100.00	10.08	29.43	
27WF 94	3260.00	242.8	94.0	10.87	27.65	
27WF 102	3610.00	266.3	102.0	10.96	30.01	
30WF 108	4460.00	299.2	108.0	11.85	31.77	
30WF 116	4920.00	327.9	116.0	12.00	34.13	
30WF 124	5350.00	354.6	124.0	12.11	36.45	
33WF 130	6700.00	404.8	130.0	13.23	38.26	
33WF 141	7440.00	446.8	141.0	13.39	41.51	
36WF 150	9000.00	502.9	150.0	14.29	44.16	
36WF 160	9730.00	541.0	160.0	14.38	47.09	
36WF 170	10470.00	579.1	170.0	14.47	49.98	
33WF 200	11050.00	669.6	200.0	13.71	58.79	
36WF 182	11280.00	621.2	182.0	14.52	53.54	
36WF 194	12100.00	663.6	194.0	14.56	57.11	
33WF 220	12320.00	740.6	220.0	13.79	64.73	
36WF 230	15000.00	835.5	230.0	14.88	67.73	
36WF 245	16080.00	892.5	245.0	14.95	72.03	
36WF 260	17240.00	951.1	260.0	15.00	76.56	
36WF 280	18820.00	1031.2	280.0	15.12	82.32	
36WF 300	20300.00	1105.1	300.0	15.17	88.17	
1.0	1.0					x
0.7	0.2					x
0.1	0.7					x
0.165	0.130					x
0.120	0.142					x

TABLE 6. INPUT DATA FOR PIN-JOINTED TRUSS

Cols. 1-10	Cols. 11-20	Cols. 21-30	Cols. 31-40	Cols. 41-50	Cols. 51-60	Identification
15	5					k,m
.00000001	.00001	34.7	10000.0			$\epsilon, \delta, x_o, W_o$
3.61	3.61	3.61	3.61	3.61		$\bar{b}_i$
0.25	0.25	0.25	0.25	0.25		$\underline{b}_i$
10	4	2	1			$n_{if}, n_{eal}, n_{rif}, n_{ls}$
1000.0	1000.0	300.0	300.0			Z (4x1)
-1.0	-1.0	0.0	0.0	0.0	0.0	b_o (10x4)
0.0	0.0	0.0	0.0	0.0	0.0	
-1.0	-1.0	0.0	0.0	0.0	0.0	
0.0	0.0	0.0	1.0	-1.0	-2.0	
-1.0	0.0	1.414	-1.0	0.0	1.414	
0.0	0.0	0.0	-1.0	0.0	0.0	
0.0	-1.0	0.0	1.4.4			
-0.707	0.0	-0.707	0.0	-0.707	1.0	b_1 (10x2)
1.0	-0.707	0.0	0.0	0.0	-0.707	
0.0	-0.707	0.0	0.0	0.0	-0.707	
1.0	1.0					
40.0	10.0	10.0	28.28	28.28		L_i
39	0.01	3				$n_{sects}, \epsilon_a, i_{band}$
14WF 30	1.41	8.81				P_{sects}
14WF 34	1.46	10.00				
14WF 38	1.49	11.17				
14WF 43	1.89	12.65				
14WF 48	1.91	14.11				
14WF 53	1.92	15.59				
14WF 61	2.45	17.94				
14WF 68	2.46	20.00				
14WF 74	2.48	21.76				
14WF 78	3.00	22.94				
14WF 84	3.02	24.71				
14WF 87	3.70	25.56				
14WF 95	3.71	27.94				
14WF 103	3.72	30.26				
14WF 111	3.73	32.65				
14WF 119	3.75	34.99				
14WF 127	3.76	37.33				

(Continued)

TABLE 6. (Continued)

Cols. 1-10	Cols. 11-20	Cols. 21-30	Cols. 31-40	Cols. 41-50	Cols. 51-60	Identification
14WF 136	3.77	39.98				
14WF 142	3.97	41.85				
14WF 150	3.99	44.08				
14WF 158	4.00	46.47				
14WF 167	4.01	49.09				
14WF 176	4.02	51.73				
14WF 184	4.04	54.07				
14WF 193	4.05	56.73				
14WF 202	4.06	59.39				
14WF 211	4.07	62.07				
14WF 219	4.08	64.36				
14WF 228	4.10	67.06				
14WF 237	4.11	69.69				
14WF 246	4.12	72.33				
14WF 264	4.14	77.63				
14WF 287	4.17	84.87				
14WF 320	4.17	94.12				
14WF 314	4.20	92.30				
14WF 342	4.24	100.59				
14WF 370	4.27	108.78				
14WF 398	4.31	116.98				
14WF 426	4.34	125.25				
3.0	3.0	3.0	3.0	3.0		$\bar{x}$

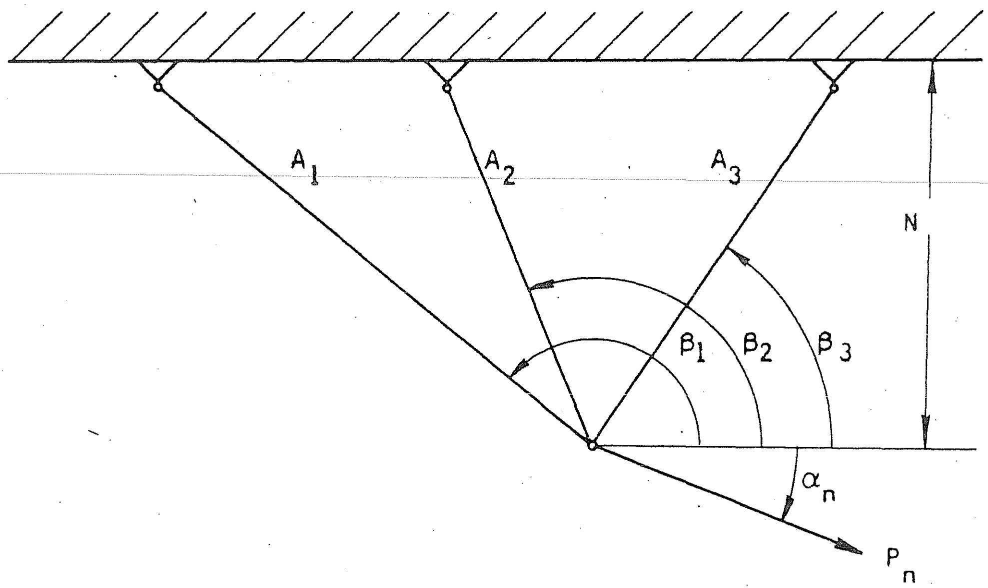


FIG. 1. THREE BAR TRUSS (AFTER SCHMIT)

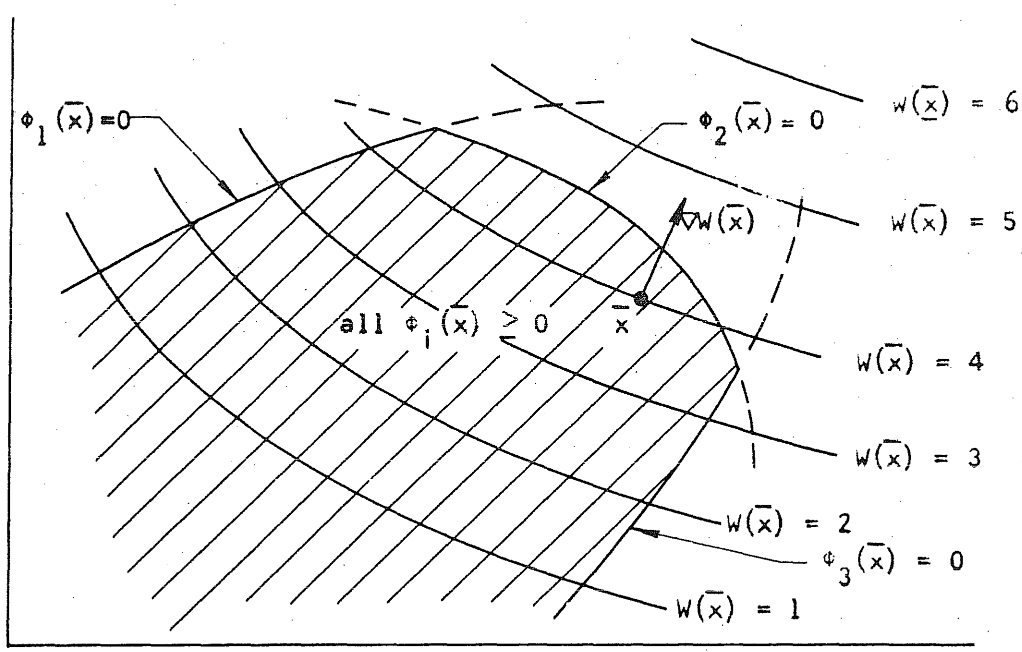


FIG. 2. TWO-DIMENSIONAL NONLINEAR PROGRAMMING PROBLEM (AFTER WOLFE)

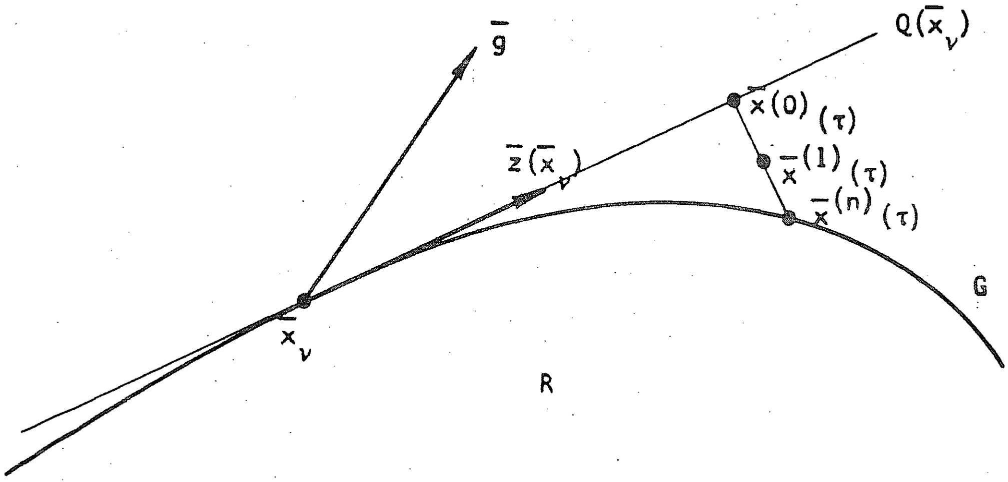


FIG. 3. ITERATION TO OBTAIN FEASIBLE POINT $\bar{x}_{\nu+1}$

IN INTERSECTION G. (AFTER ROSEN)

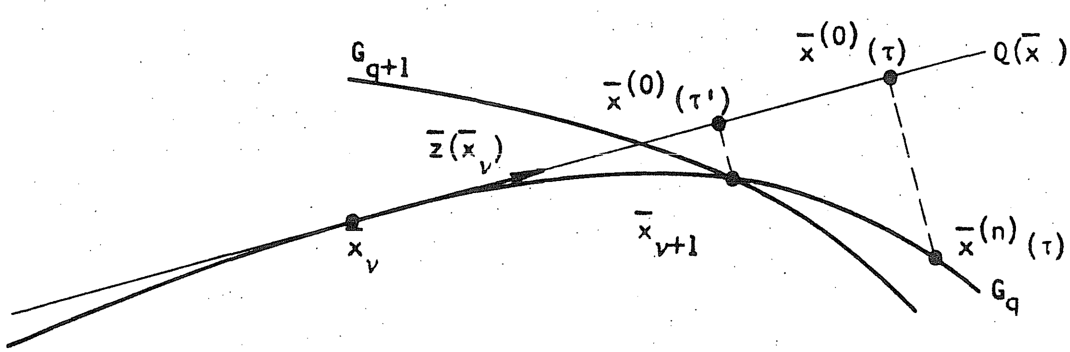


FIG. 4. INTERPOLATION TO OBTAIN FEASIBLE POINT $\bar{x}_{\nu+1}$. (AFTER ROSEN)

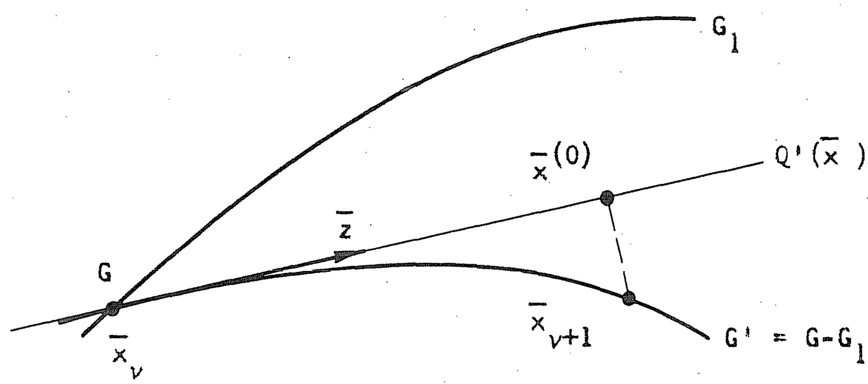


FIG. 5. DETERMINATION OF POINT $\bar{x}_{\nu+1}$ IN INTERSECTION G' BUT NOT IN G_1. (AFTER ROSEN)

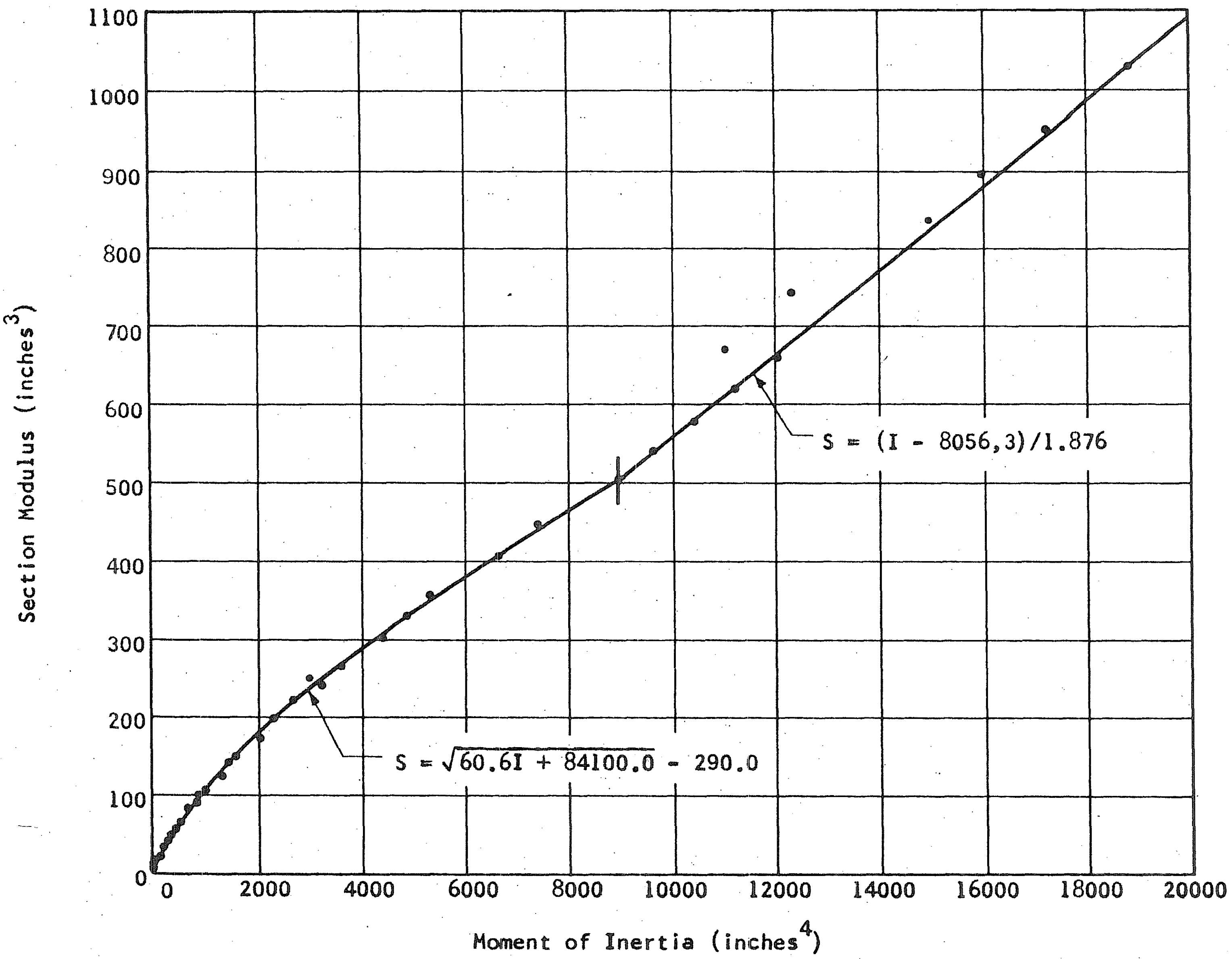


FIG. 6. SECTION MODULUS VS. MOMENT OF INERTIA FOR ECONOMY WF SECTIONS

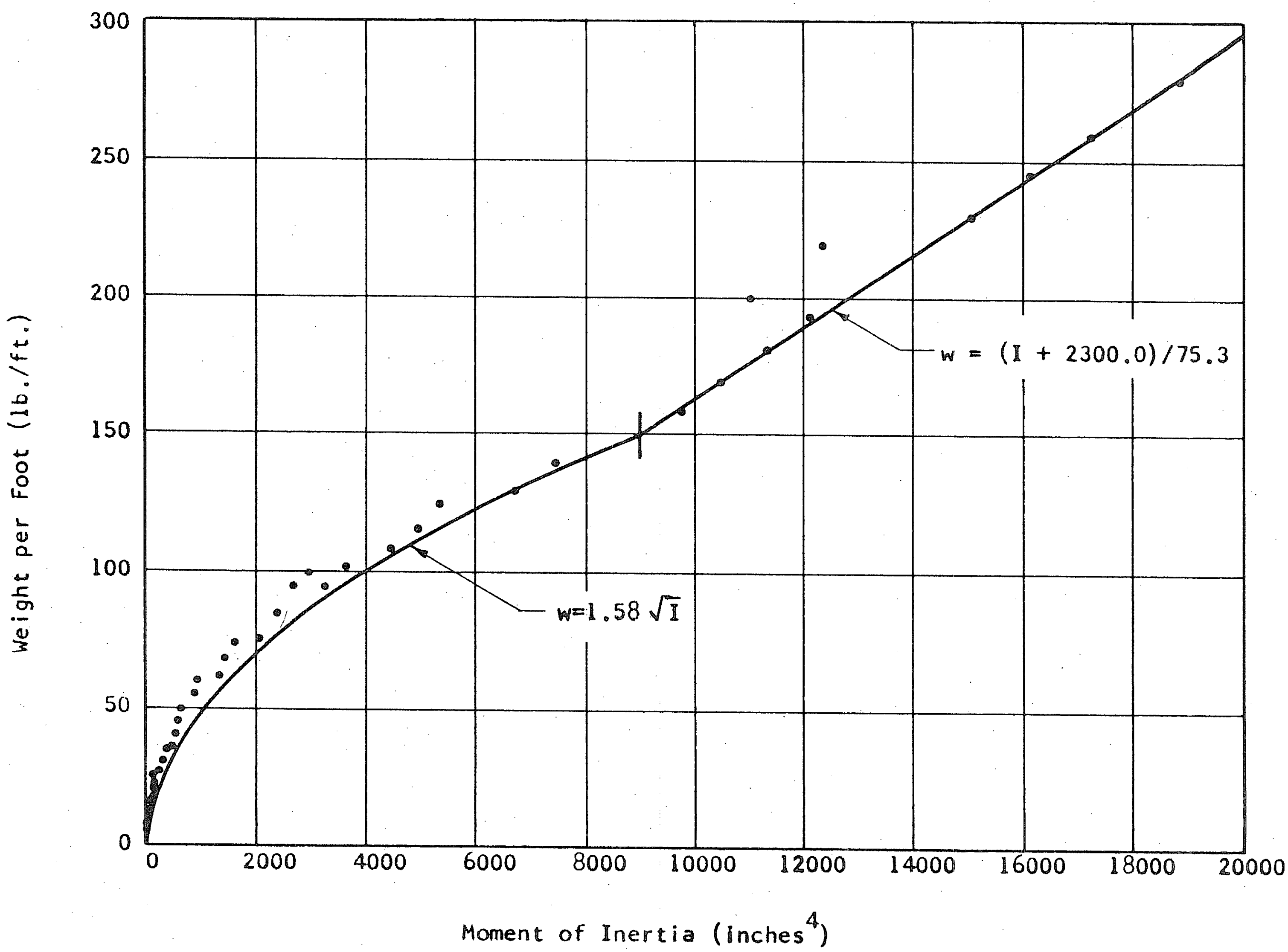


FIG. 7. WEIGHT PER FOOT VS. MOMENT OF INERTIA FOR ECONOMY WF SECTIONS

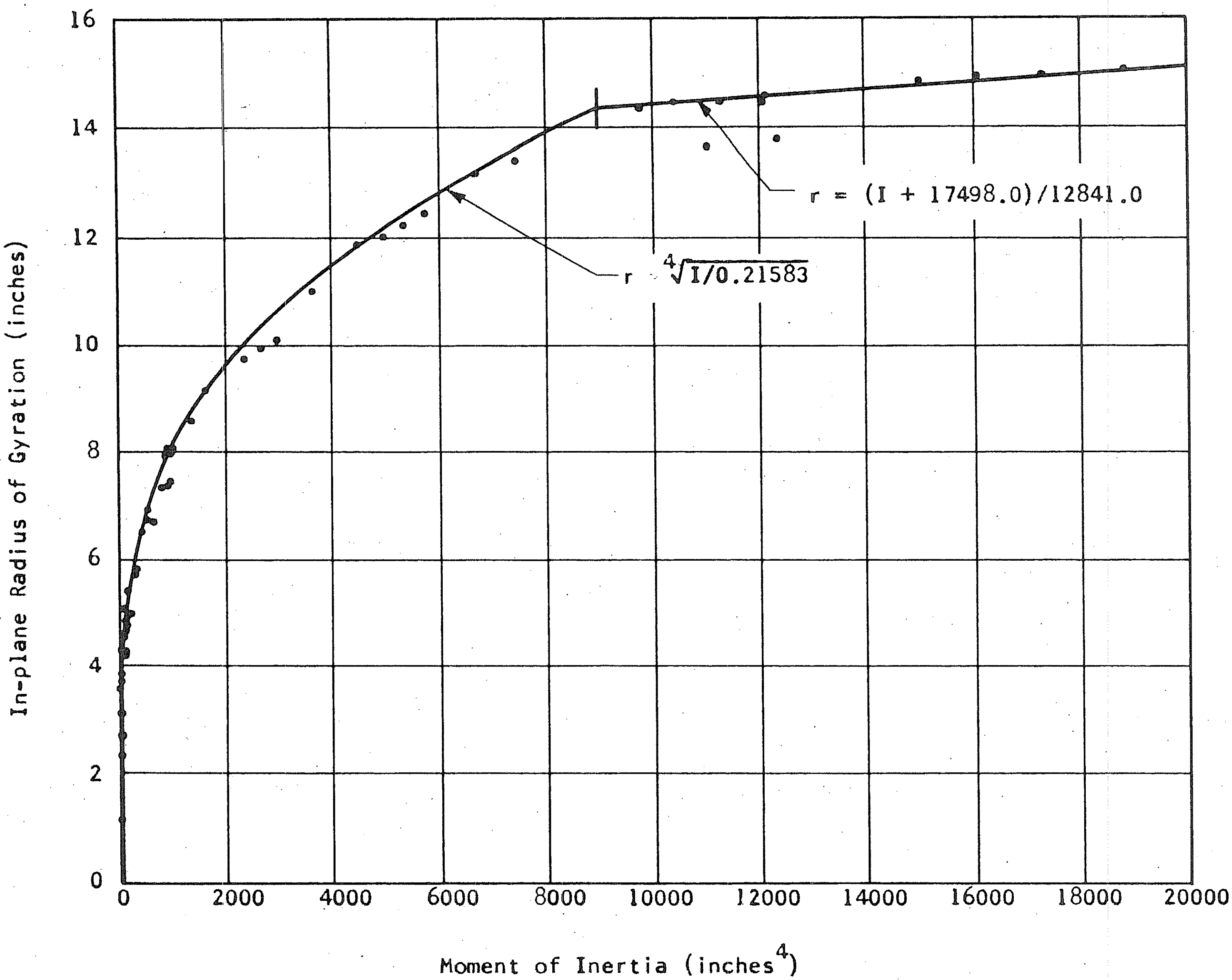


FIG. 8. IN-PLANE RADIUS OF GYRATION VS. MOMENT OF INERTIA FOR ECONOMY WF SECTIONS

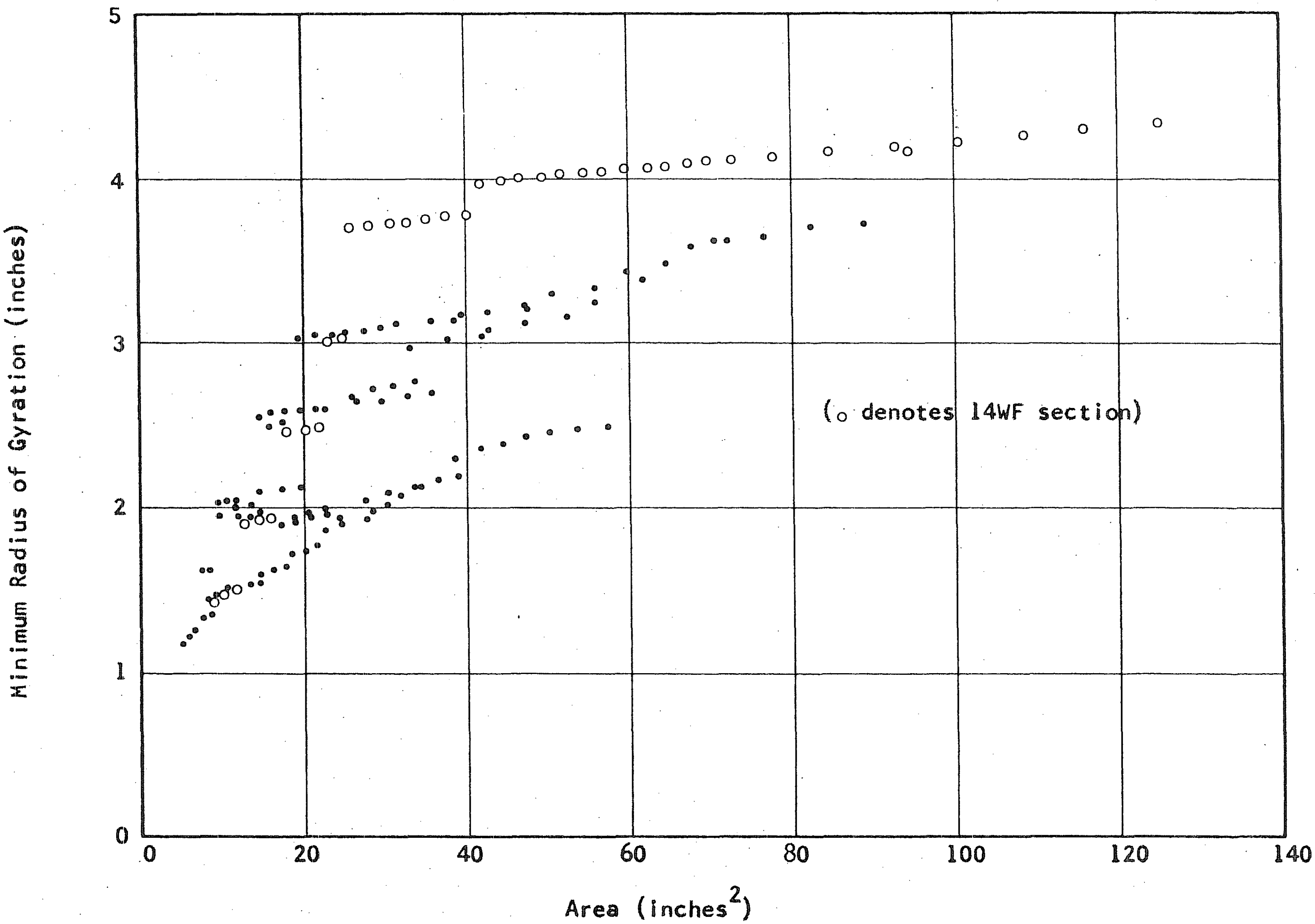


FIG. 9. MINIMUM RADIUS OF GYRATION VS. AREA FOR ALL WF SECTIONS

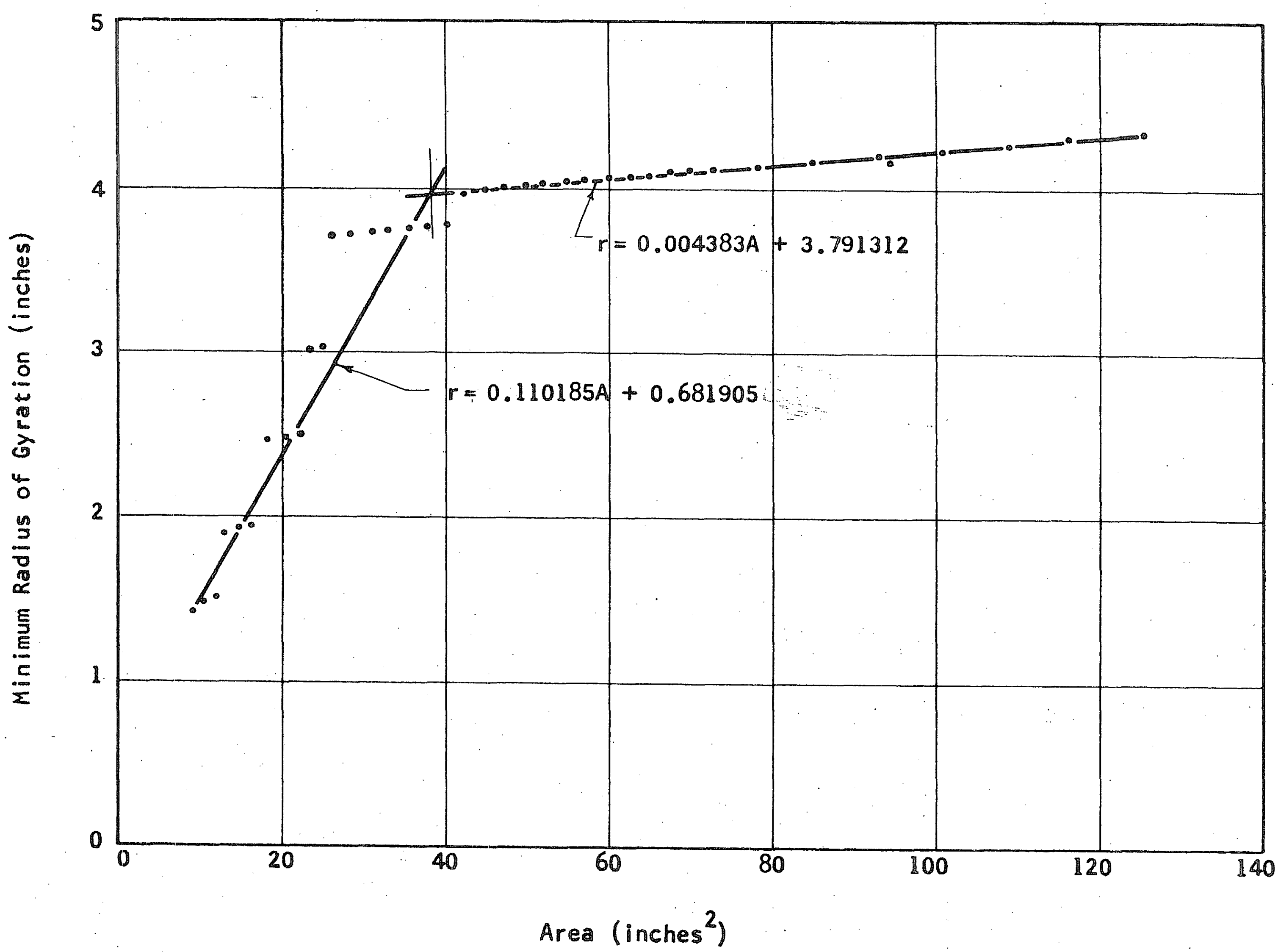


FIG. 10. MINIMUM RADIUS OF GYRATION VS. AREA FOR 14WF SECTIONS

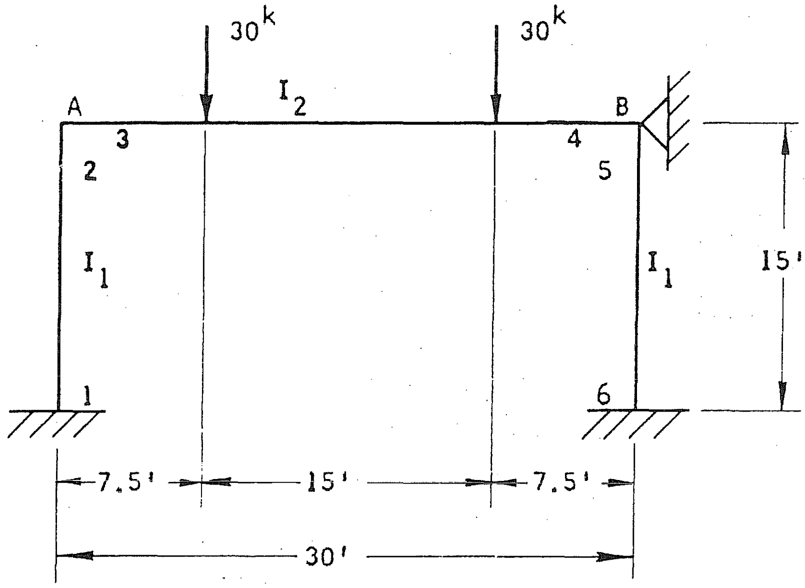


FIG. 11. SYMMETRICAL BRACED PORTAL FRAME

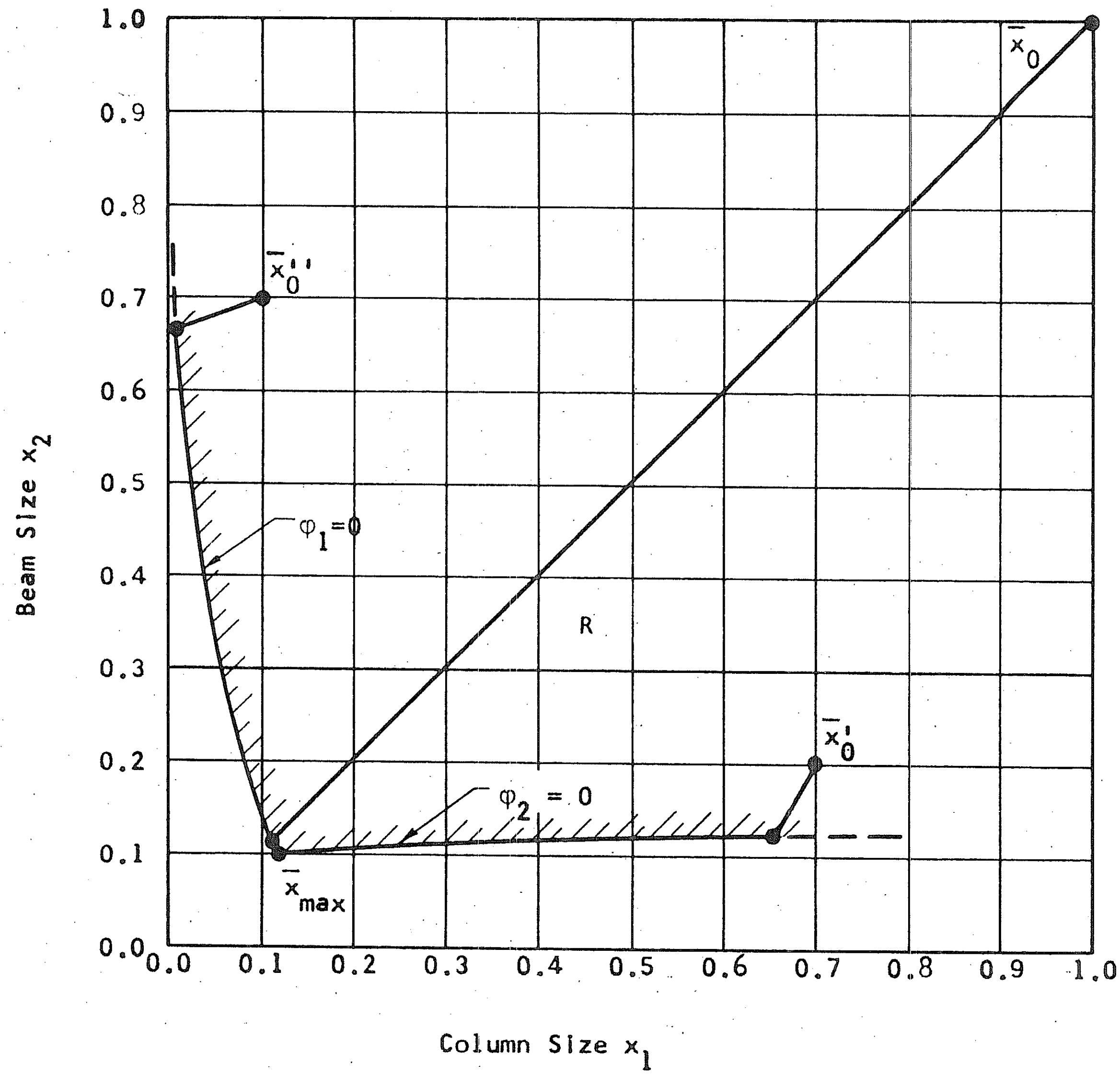


FIG. 12. SOLUTION SPACE FOR TWO-DIMENSIONAL PROBLEM

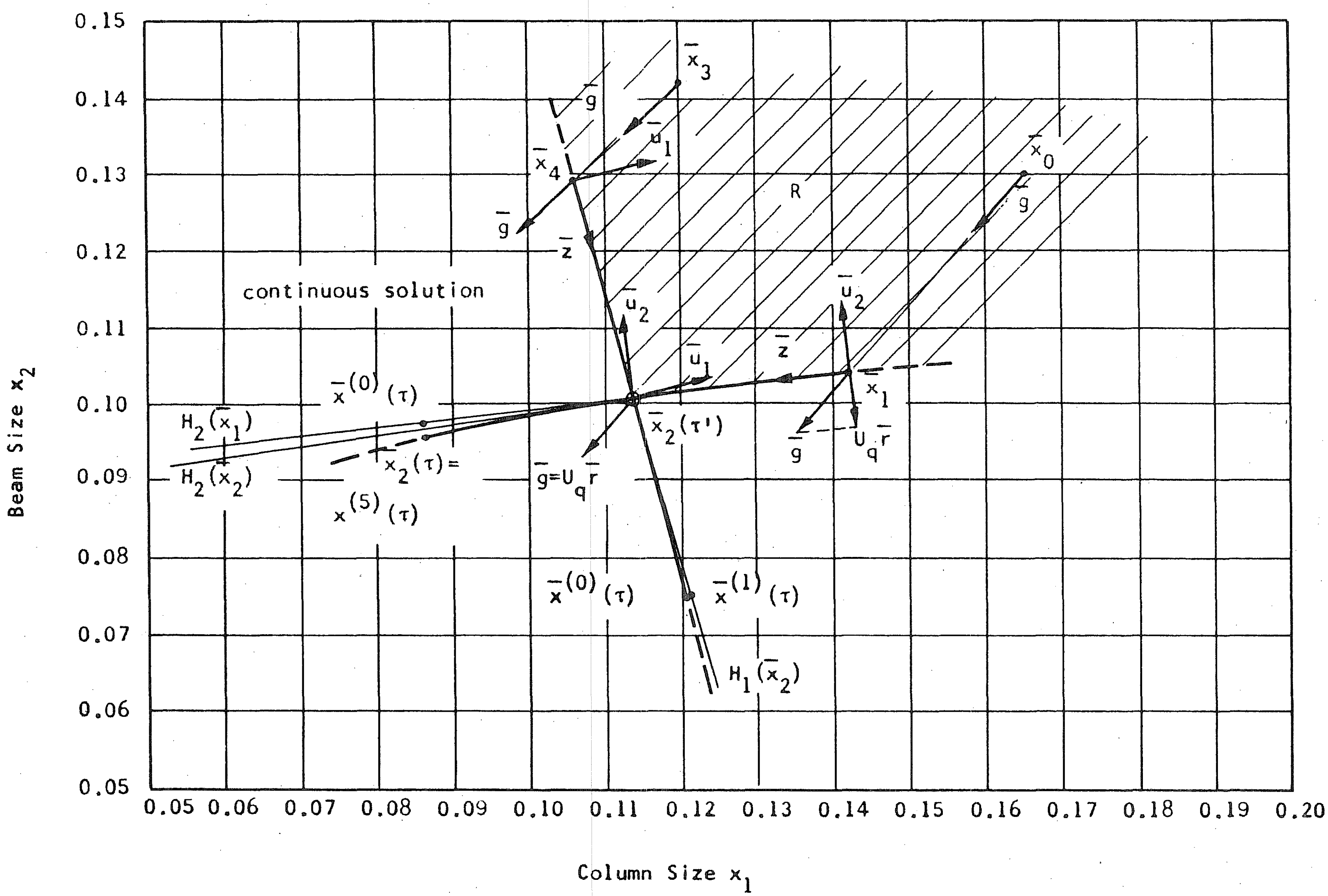


FIG. 13. TYPICAL GRADIENT PROJECTION CALCULATIONS - TWO-DIMENSIONAL EXAMPLE

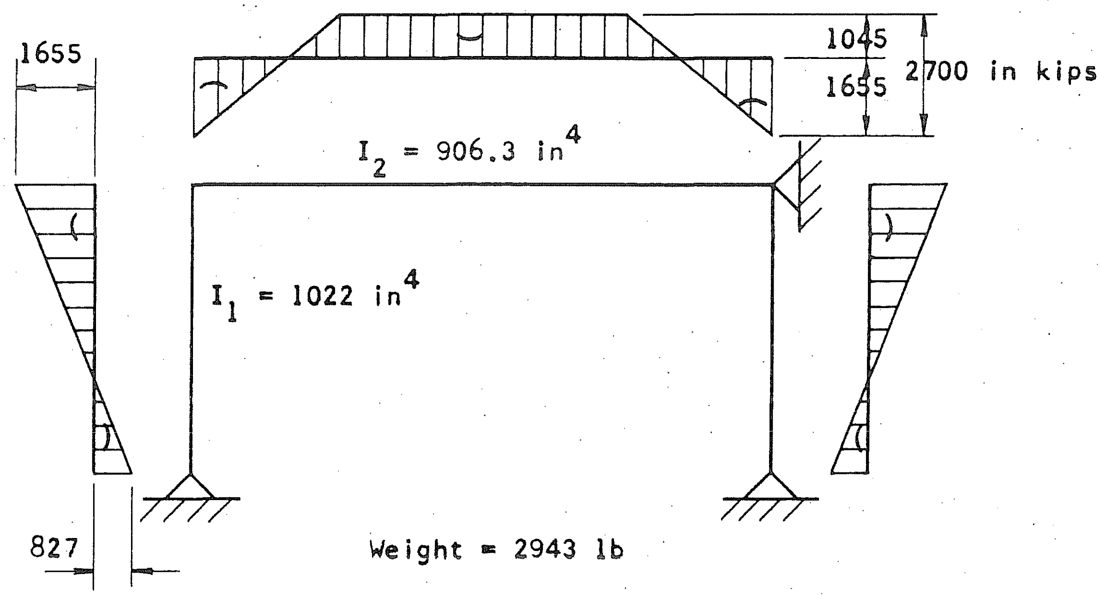


FIG. 14. CONTINUOUS SOLUTION - TWO-DIMENSIONAL EXAMPLE

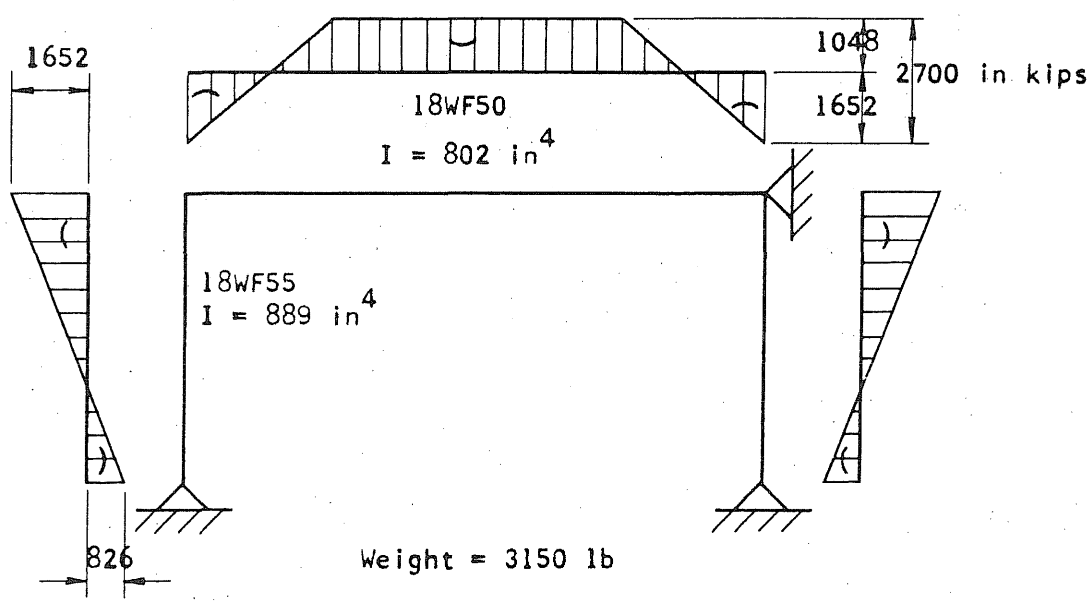
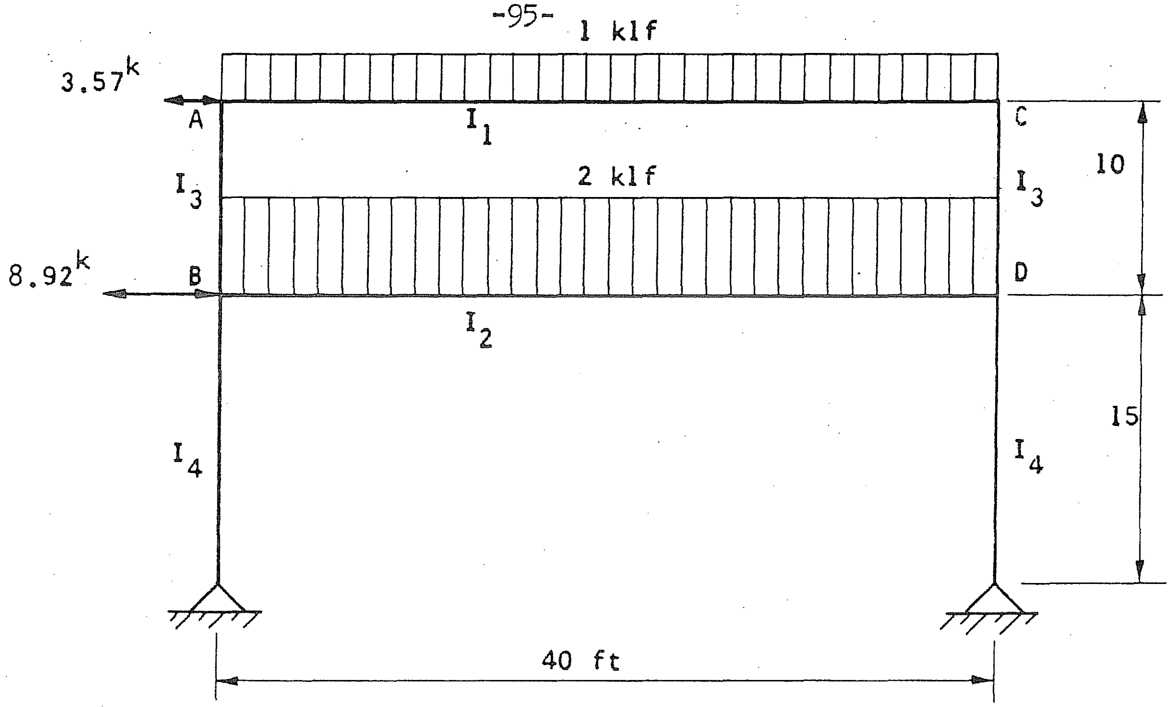


FIG. 15. AVAILABLE SOLUTION - TWO-DIMENSIONAL EXAMPLE



Frames 20 ft c to c

Gravity loads

roof: 50 psf x 20 ft / 1000 = 1.0 klf

floor: 100 psf x 20 ft / 1000 = 2.0 klf

Wind loads

wind velocity = 108 mph (assumed)

by ASCE, wind force = 37.5 psf

assume wind loads concentrated at roof and floor

roof: 35.7 x 20 x 5 / 1000 = 3.57 k

floor: 35.7 x 20 x 12.5 / 1000 = 8.92 k

AISC Specification

A-7 Steel

FIG. 16. SYMMETRICAL TWO-STORY SINGLE-BAY FRAME

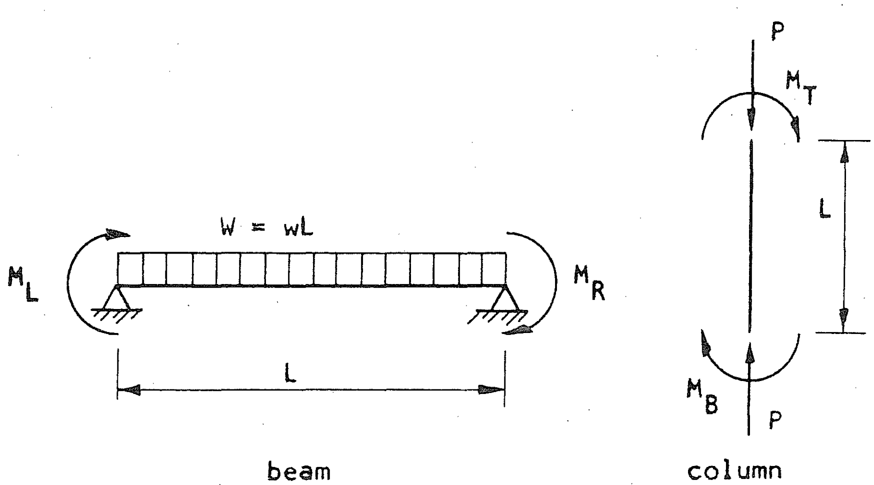
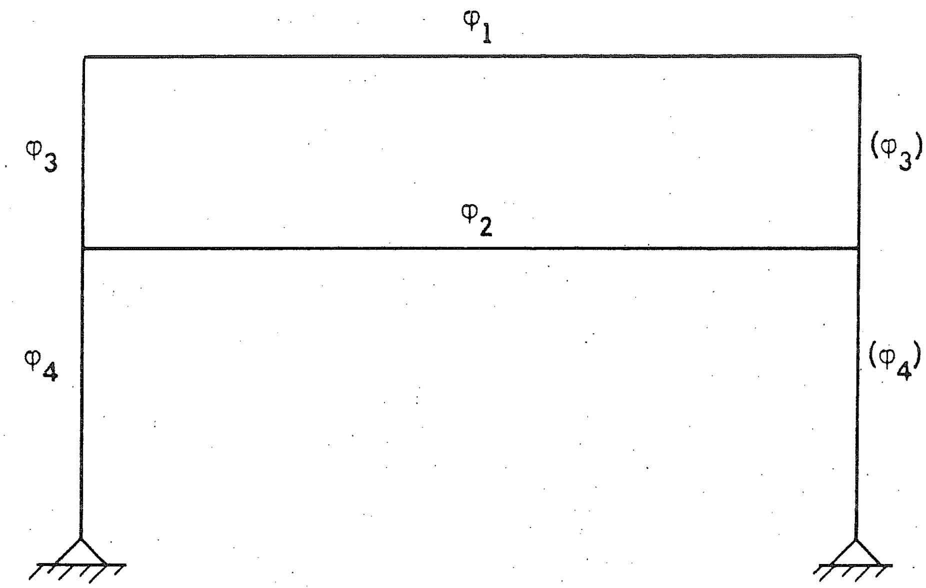
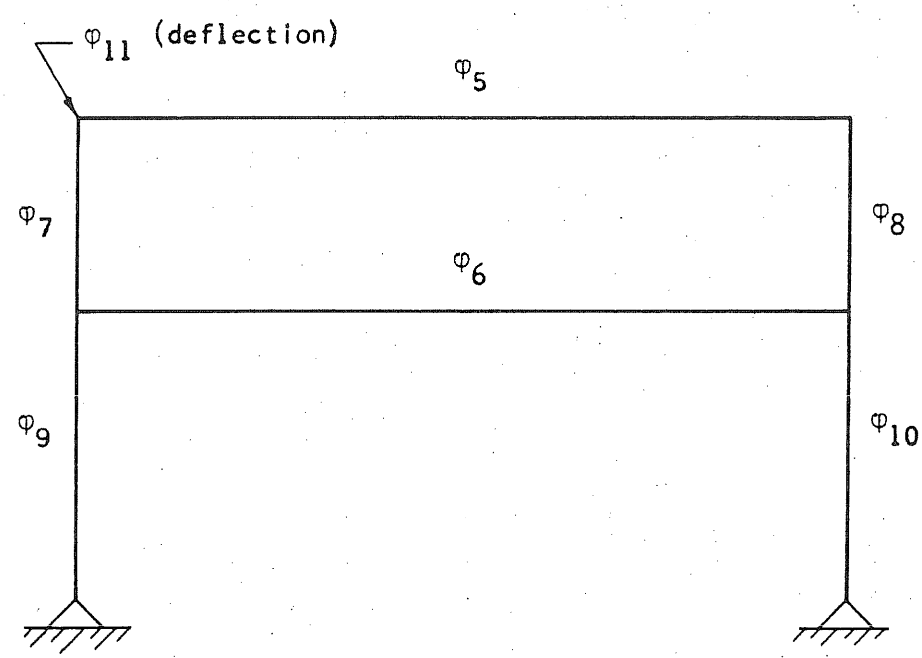


FIG. 17. FORCES ACTING ON TYPICAL MEMBERS



A. gravity loads



B. wind and gravity loads

FIG. 18. CONSTRAINTS FOR SYMMETRICAL TWO-STORY SINGLE-BAY FRAME

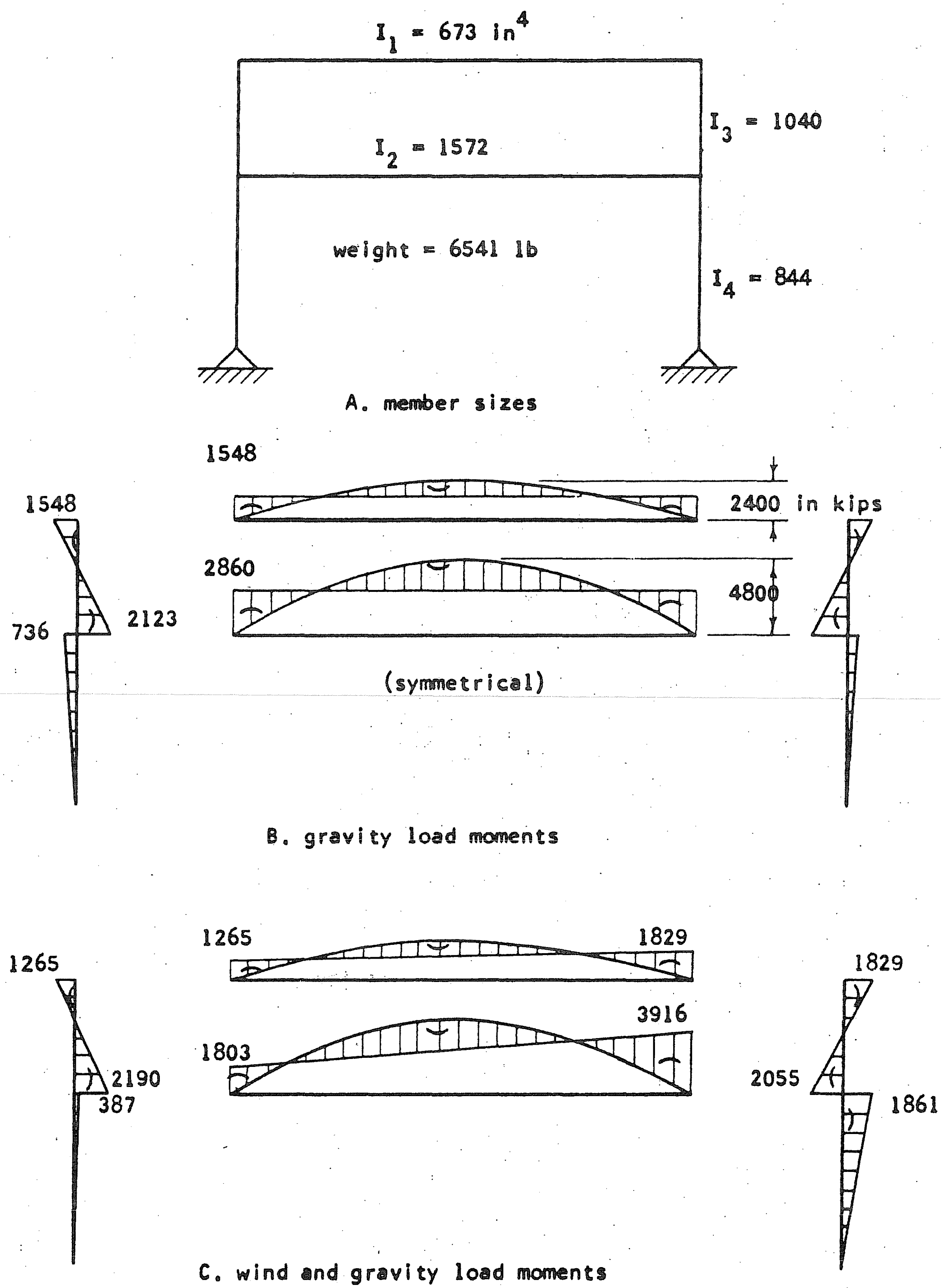
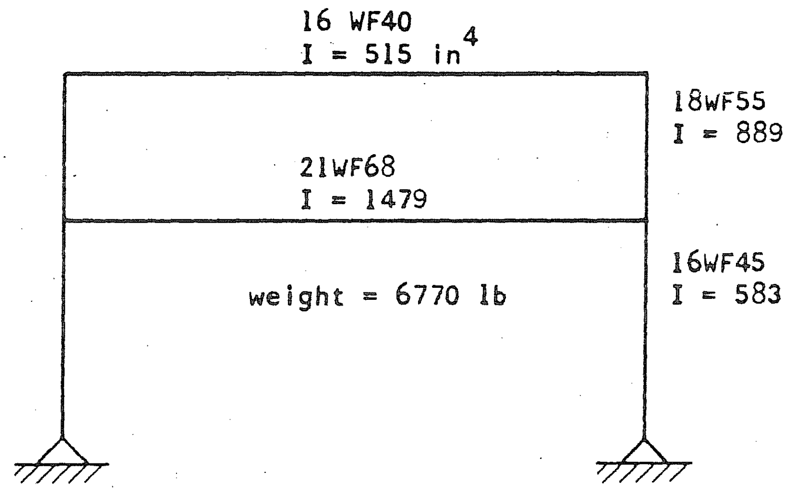
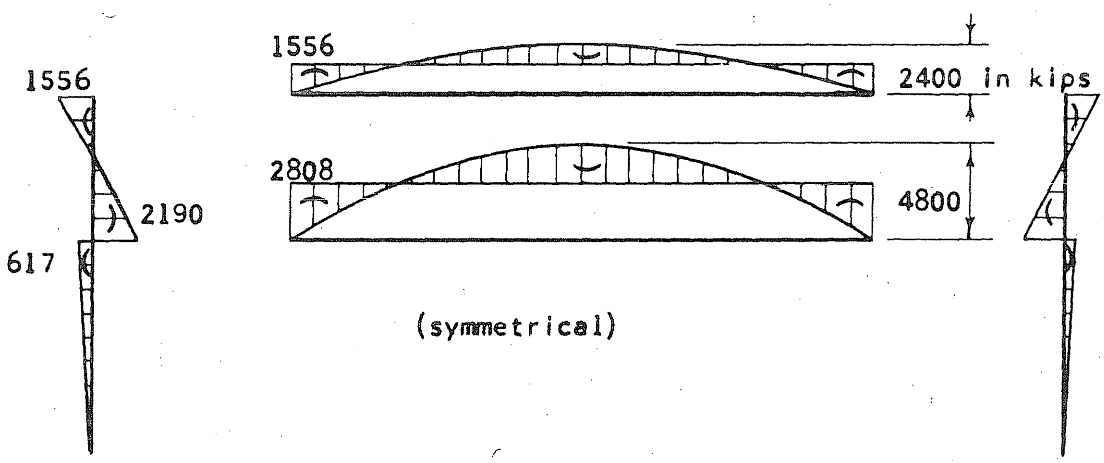


FIG. 19. CONTINUOUS SOLUTION - TWO-STORY SINGLE-BAY FRAME

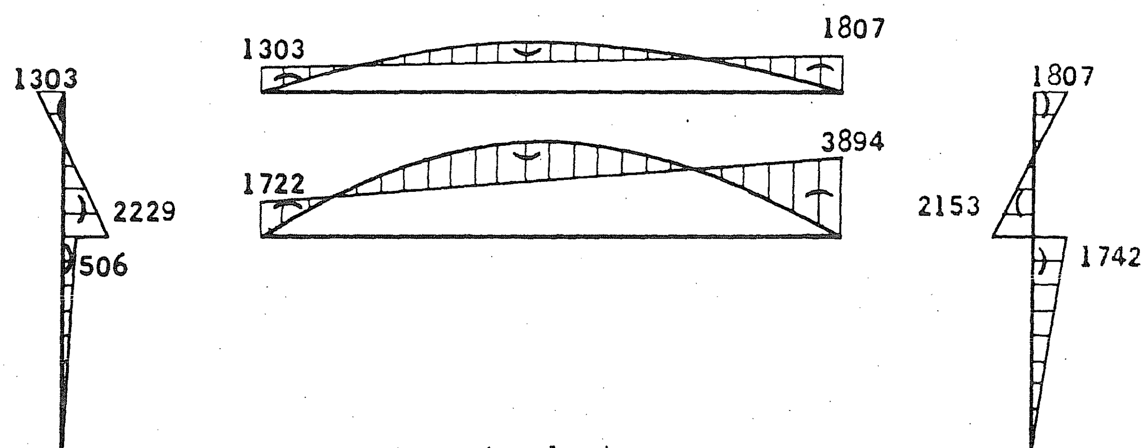
CASE I - DEFLECTIONS NOT CONSTRAINED



A. member sizes



B. gravity load moments



C. wind and gravity load moments

FIG. 20. AVAILABLE SOLUTION - TWO-STORY SINGLE-BAY FRAME

CASE I - DEFLECTIONS NOT CONSTRAINED

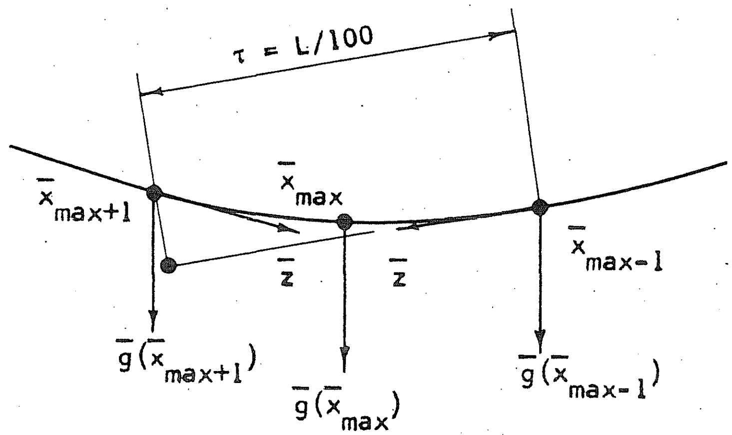


FIG. 21. SOLUTION POINT NOT AT A VERTEX OF THE FEASIBLE REGION

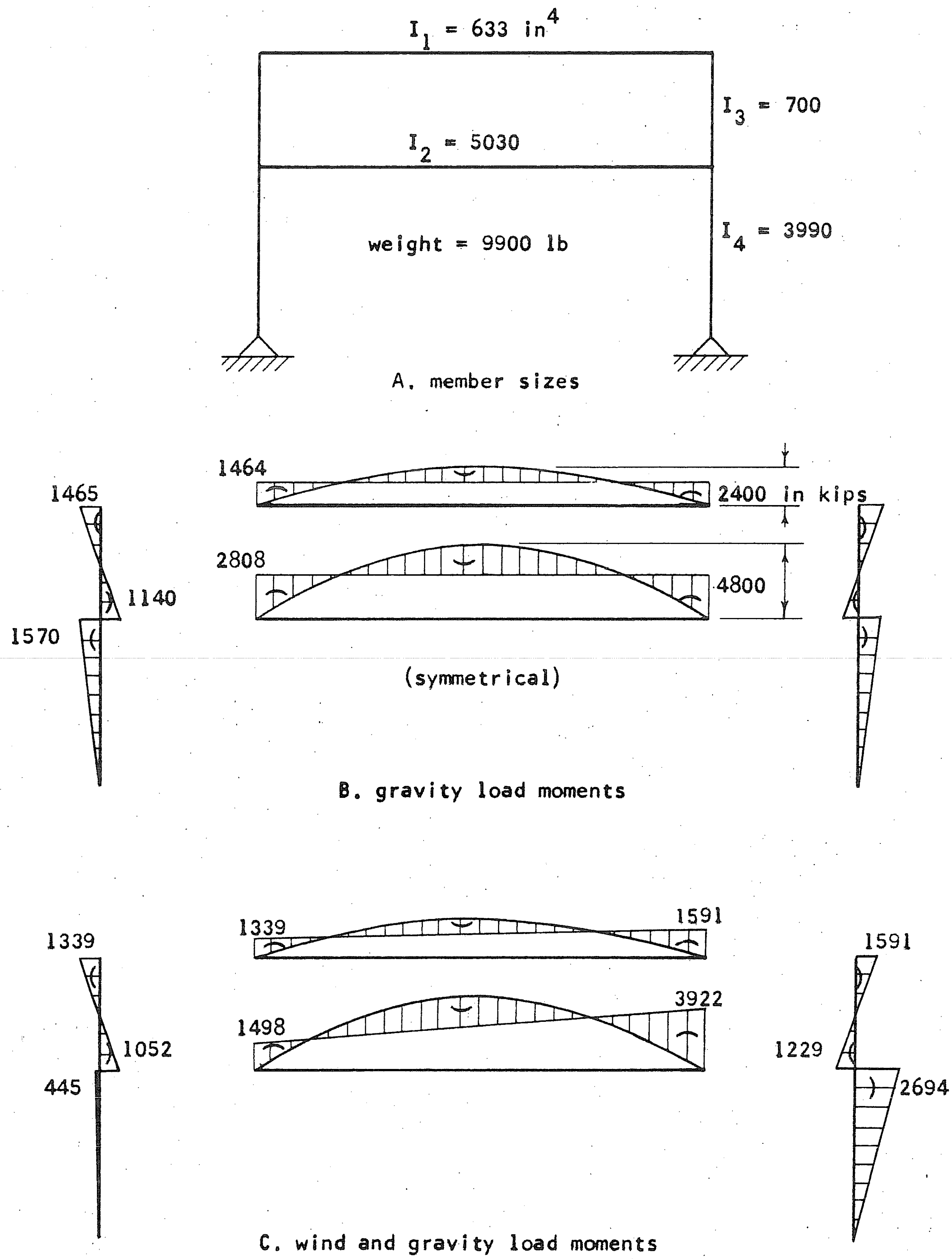


FIG. 22. CONTINUOUS SOLUTION - TWO-STORY SINGLE-BAY FRAME

CASE II - ALLOWABLE DEFLECTION = 0.3 in

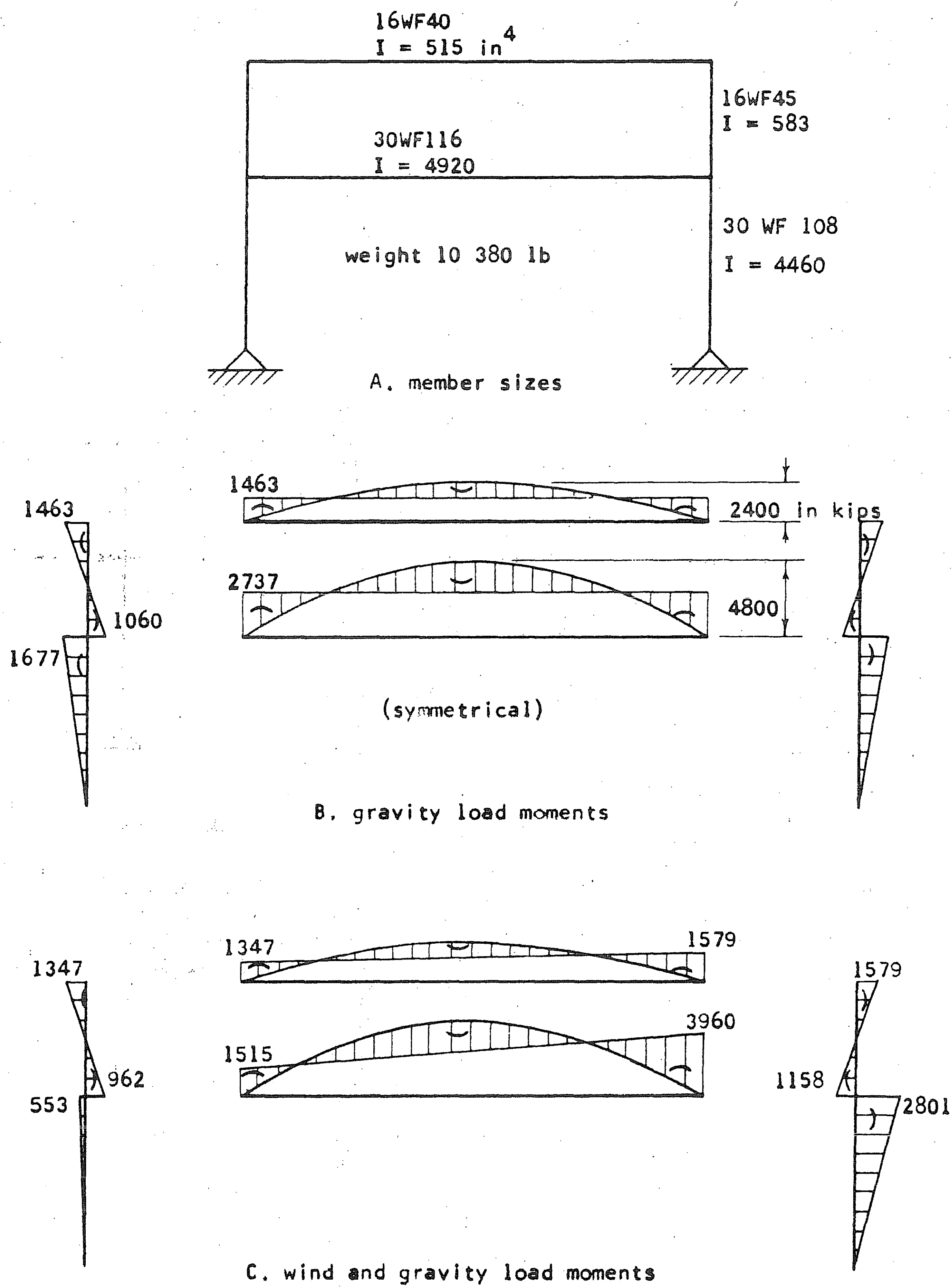
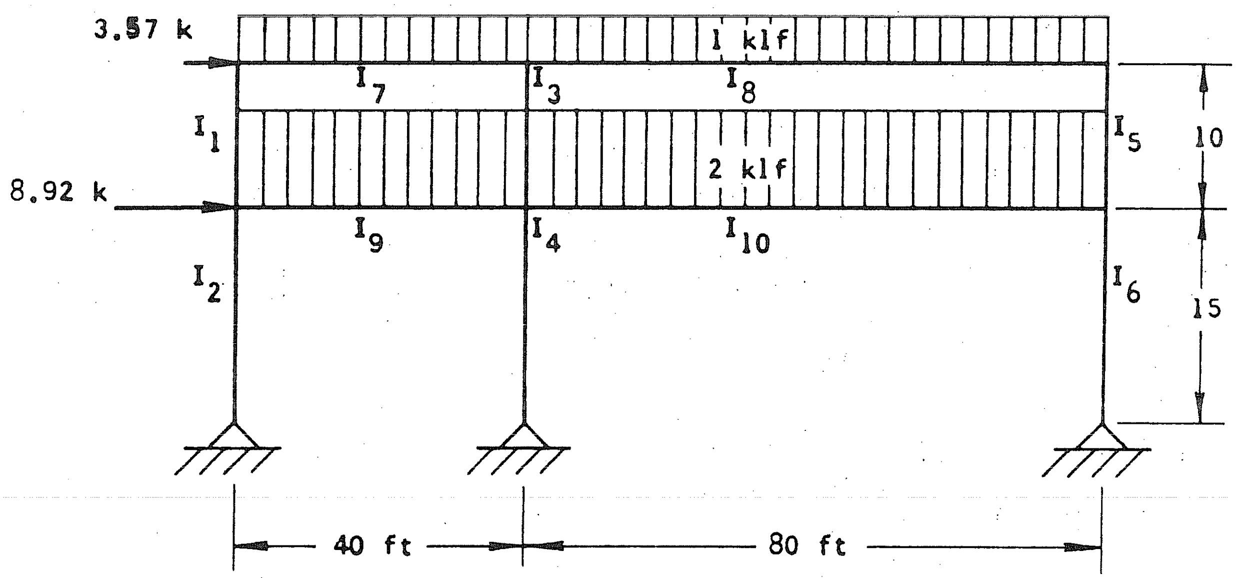


FIG. 23. AVAILABLE SOLUTION - TWO-STORY SINGLE-BAY FRAME

CASE II - ALLOWABLE DEFLECTION = 0.3 in



Frames 20 ft c to c
loading: see Fig. 16
AISC Specification
A-36 steel

FIG. 24. TWO-STORY TWO-BAY UNSYMMETRICAL FRAME

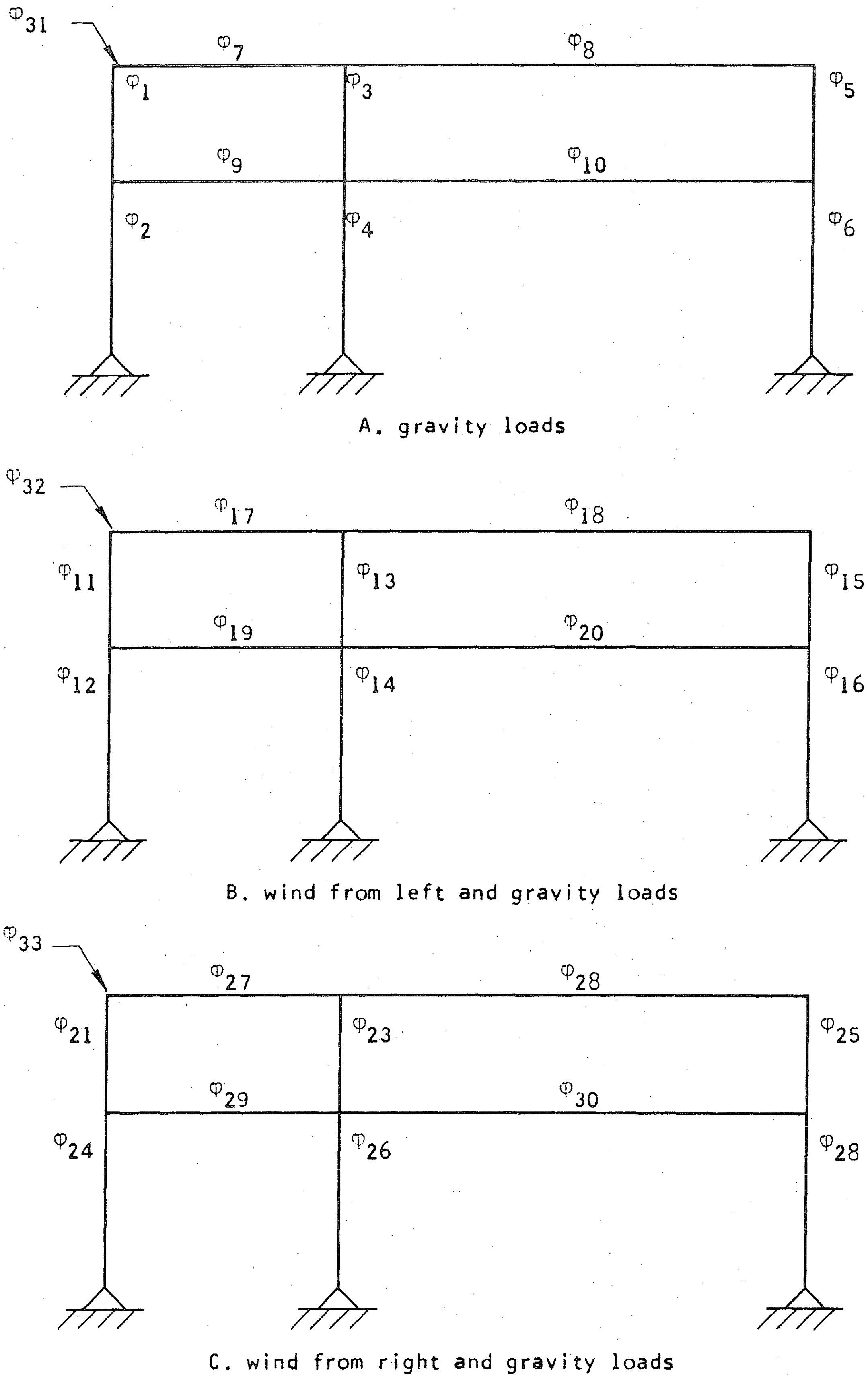


FIG. 25. CONSTRAINTS FOR TWO-STORY TWO-BAY UNSYMMETRICAL FRAME

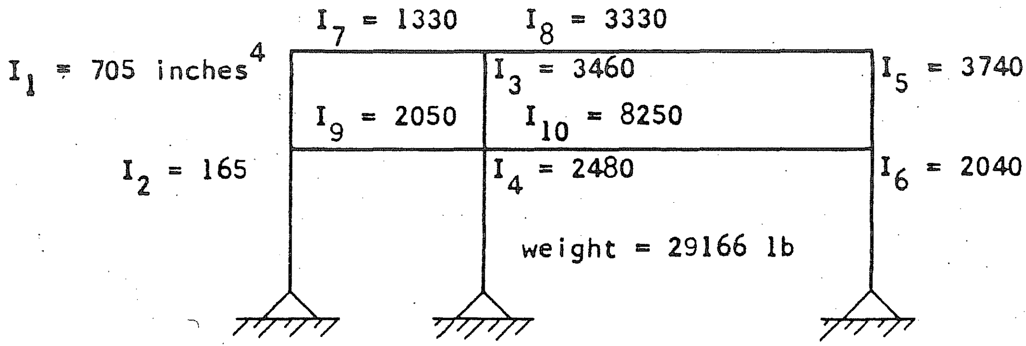
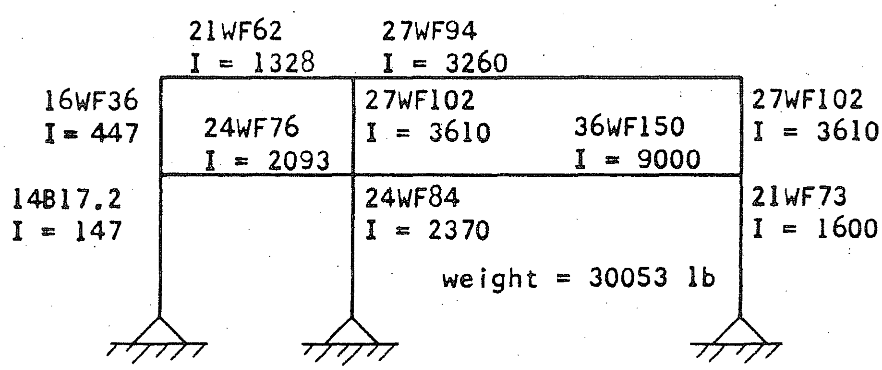
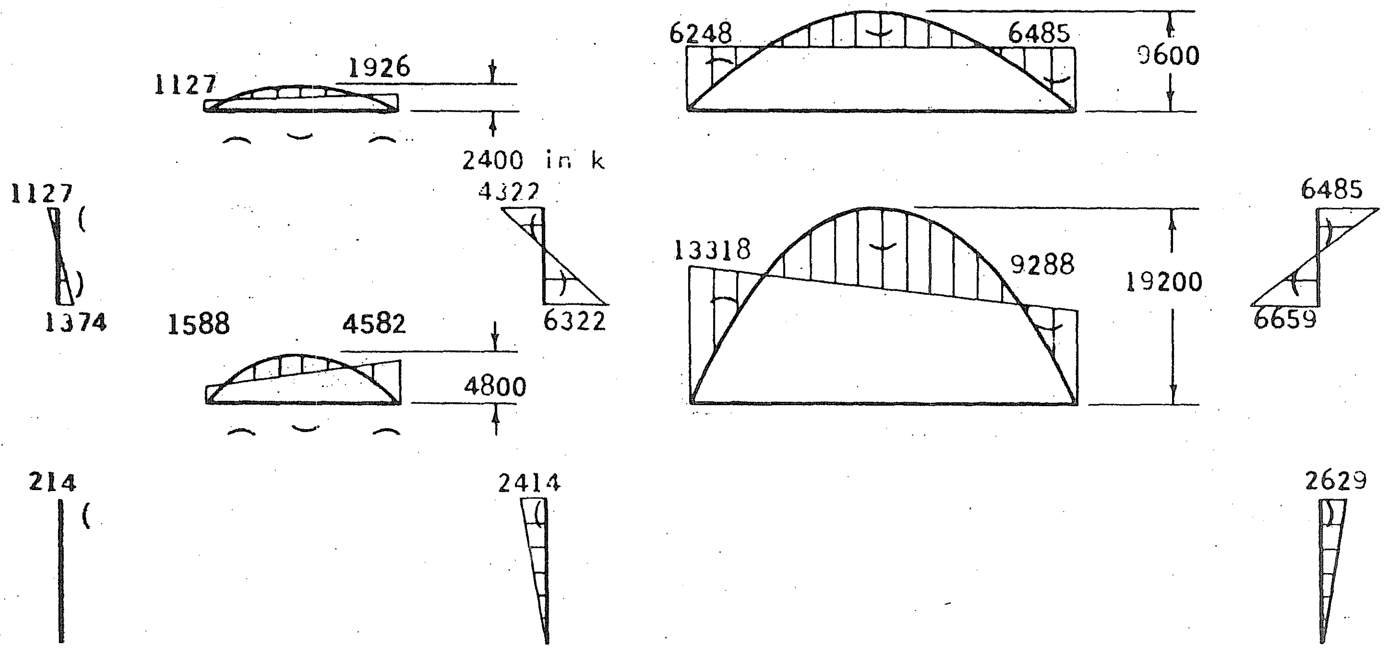


FIG. 26. CONTINUOUS SOLUTION - TWO-STORY TWO-BAY FRAME



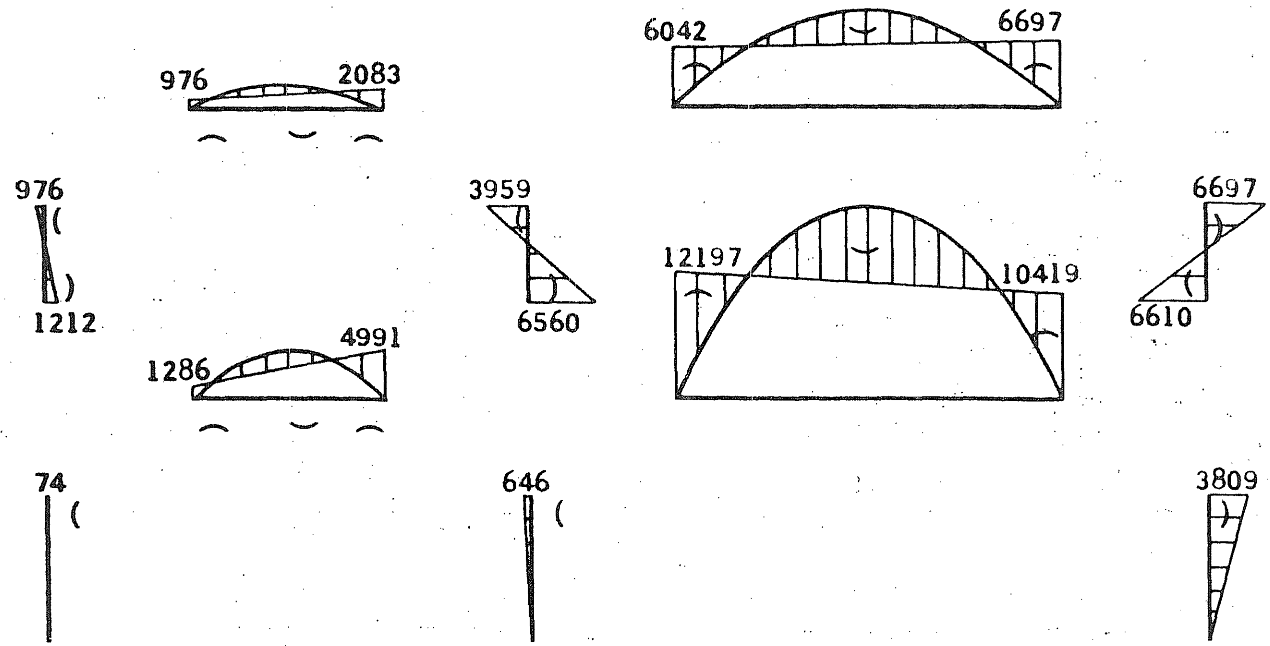
A. member sizes

FIG. 27-A. AVAILABLE SOLUTION - TWO-STORY TWO-BAY FRAME



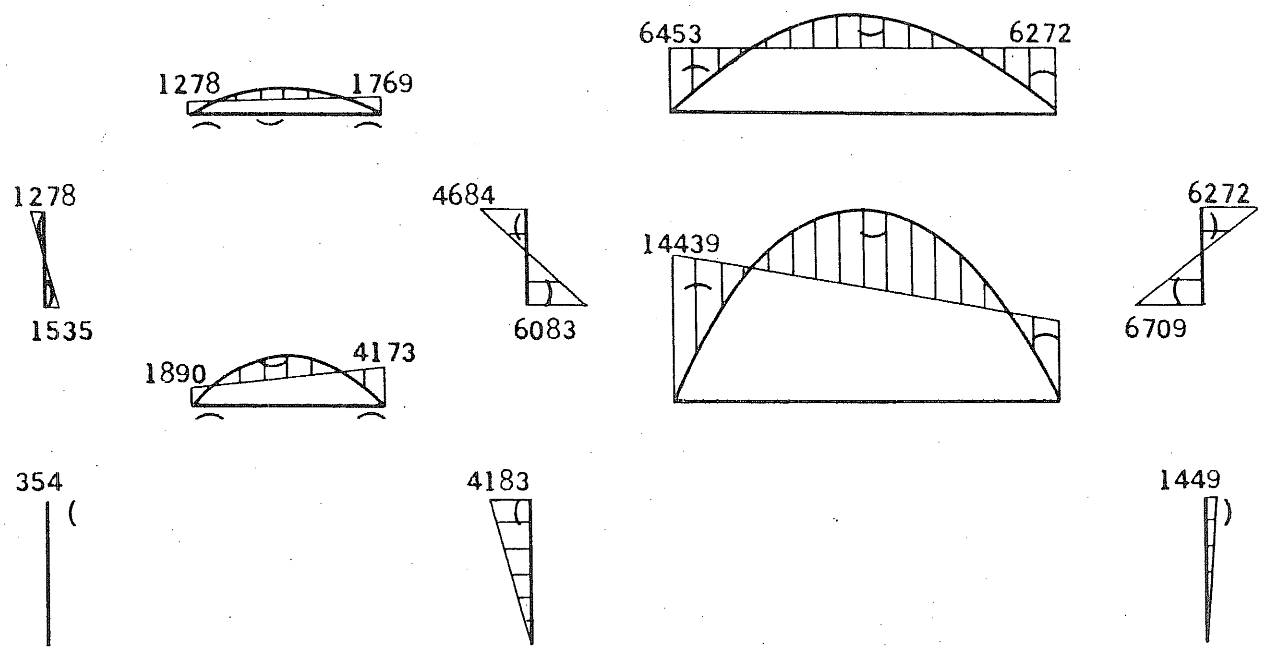
B. gravity load moments

FIG. 27-B. AVAILABLE SOLUTION - TWO-STORY TWO-BAY FRAME
(continued



C. wind from left and gravity load moments

FIG. 27-C. AVAILABLE SOLUTION - TWO-STORY TWO-BAY FRAME
(continued



D. wind from right and gravity load moments

FIG. 27-D. AVAILABLE SOLUTION - TWO-STORY TWO-BAY FRAME
(continued)

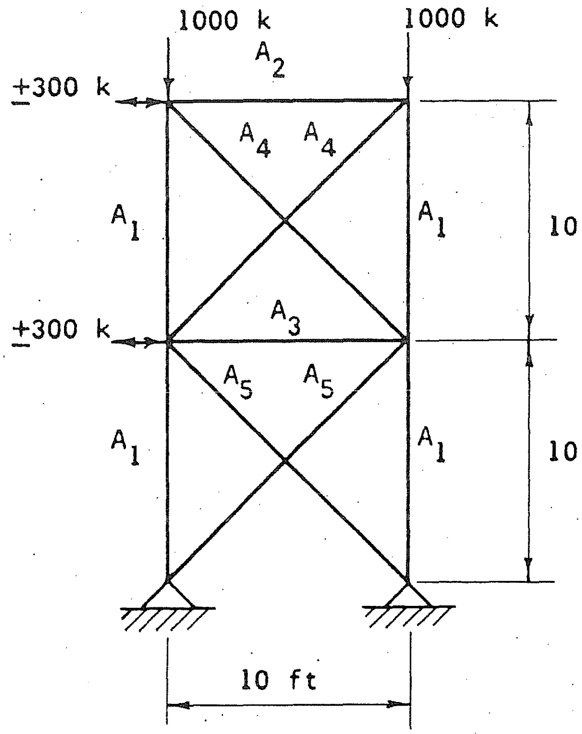


FIG. 28. PIN-JOINTED TRUSS

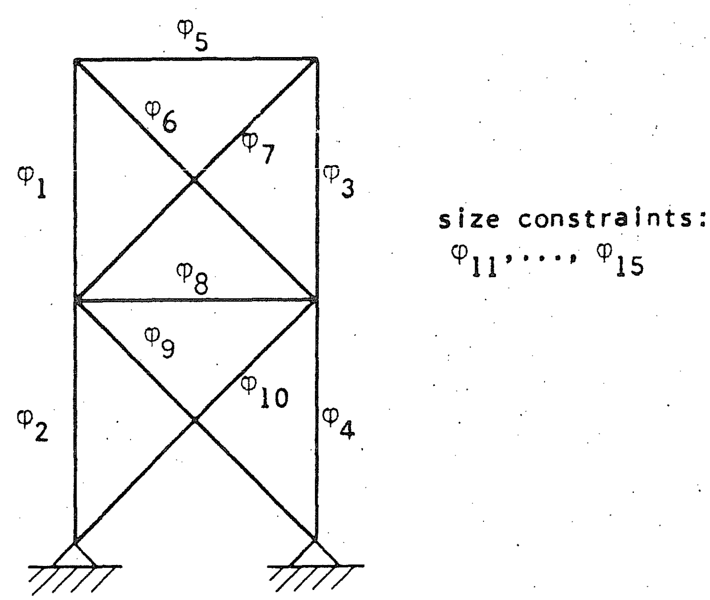
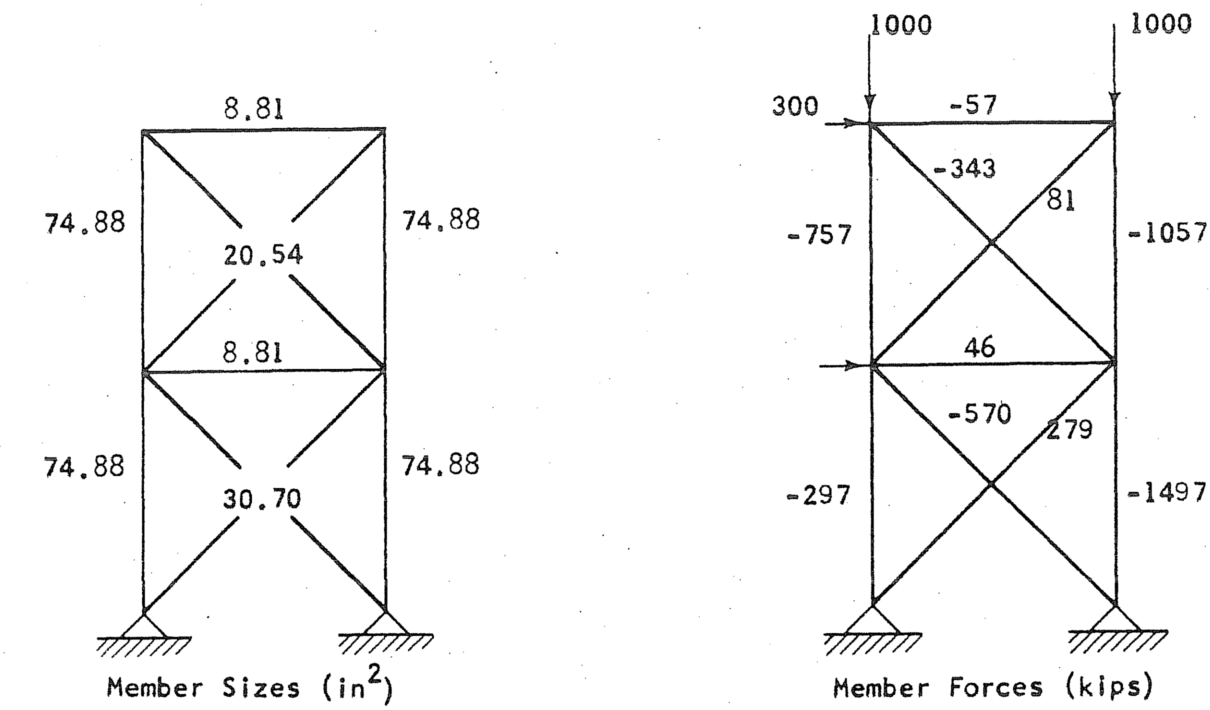
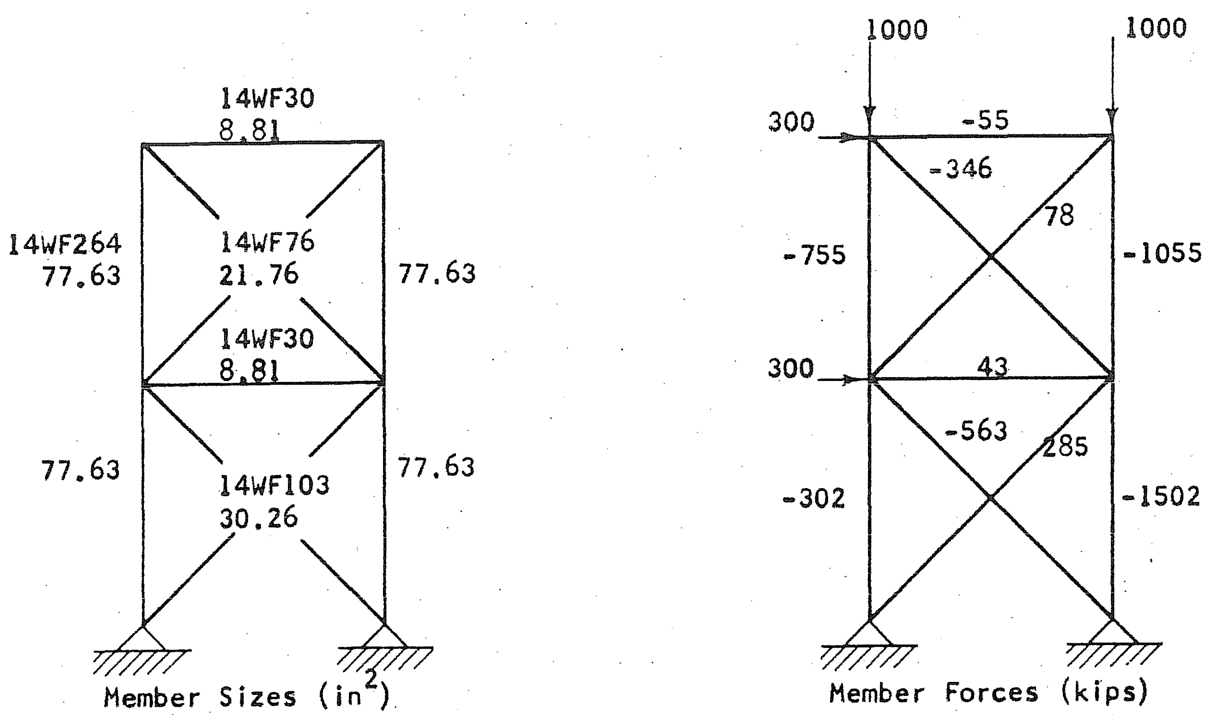


FIG. 29. CONSTRAINTS FOR PIN-JOINTED TRUSS

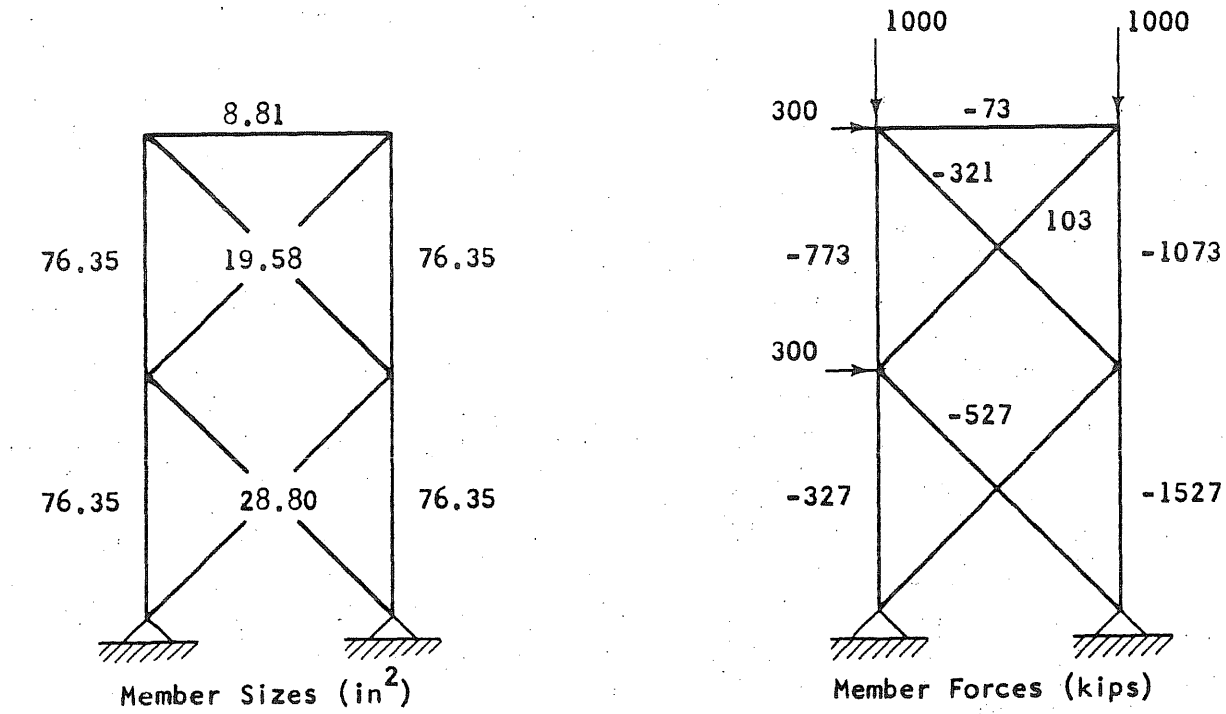


A. Continuous Solution (Weight = 15725 lb)

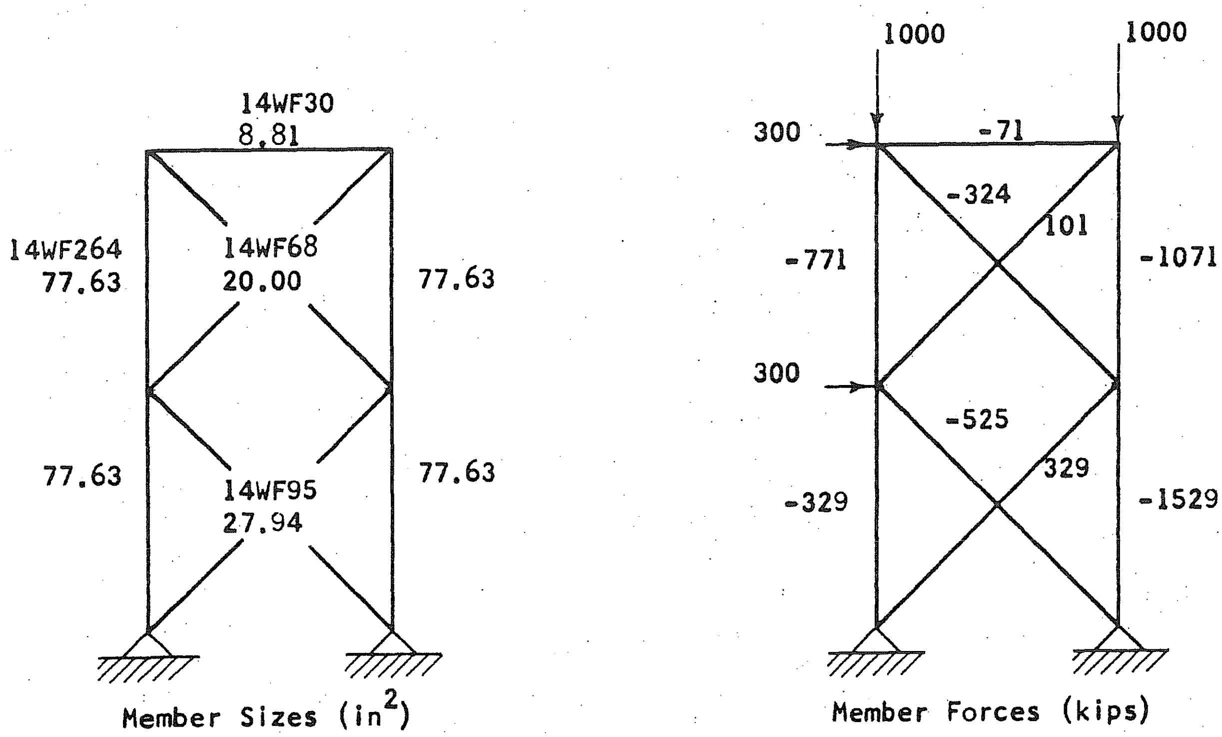


B. Available Solution (Weight = 16172 lb)

FIG. 30. MINIMUM-WEIGHT DESIGN OF PIN-JOINTED TRUSS
Case I: All members retained

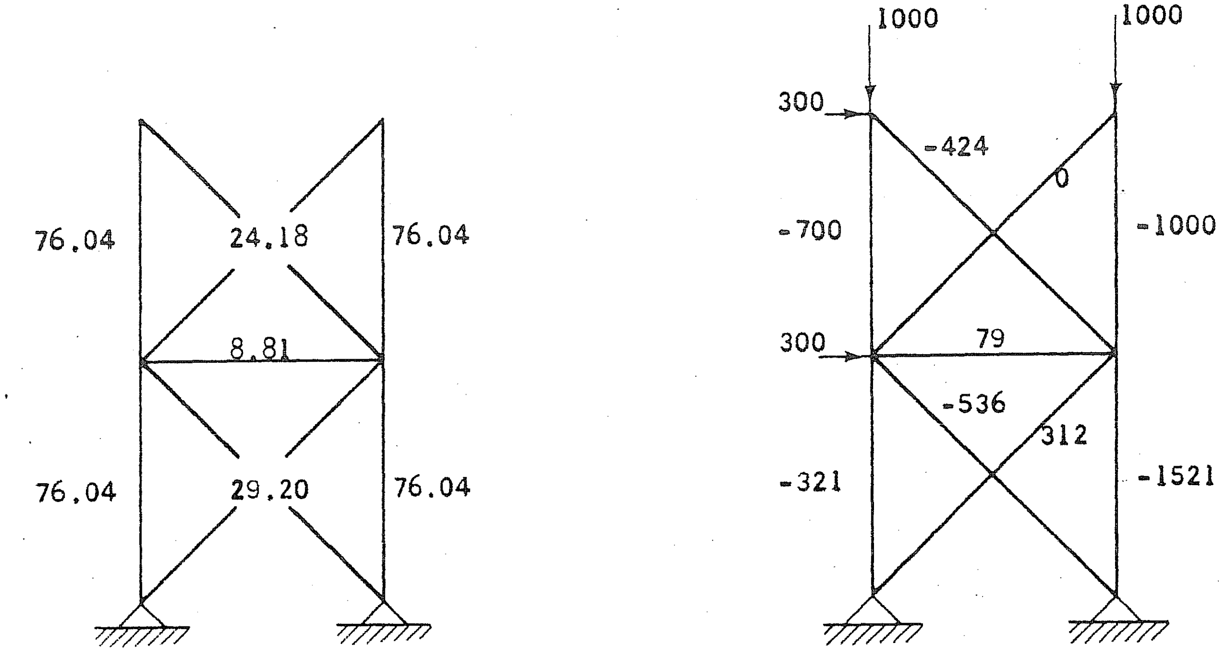


A. Continuous Solution (Weight = 15348 lb)

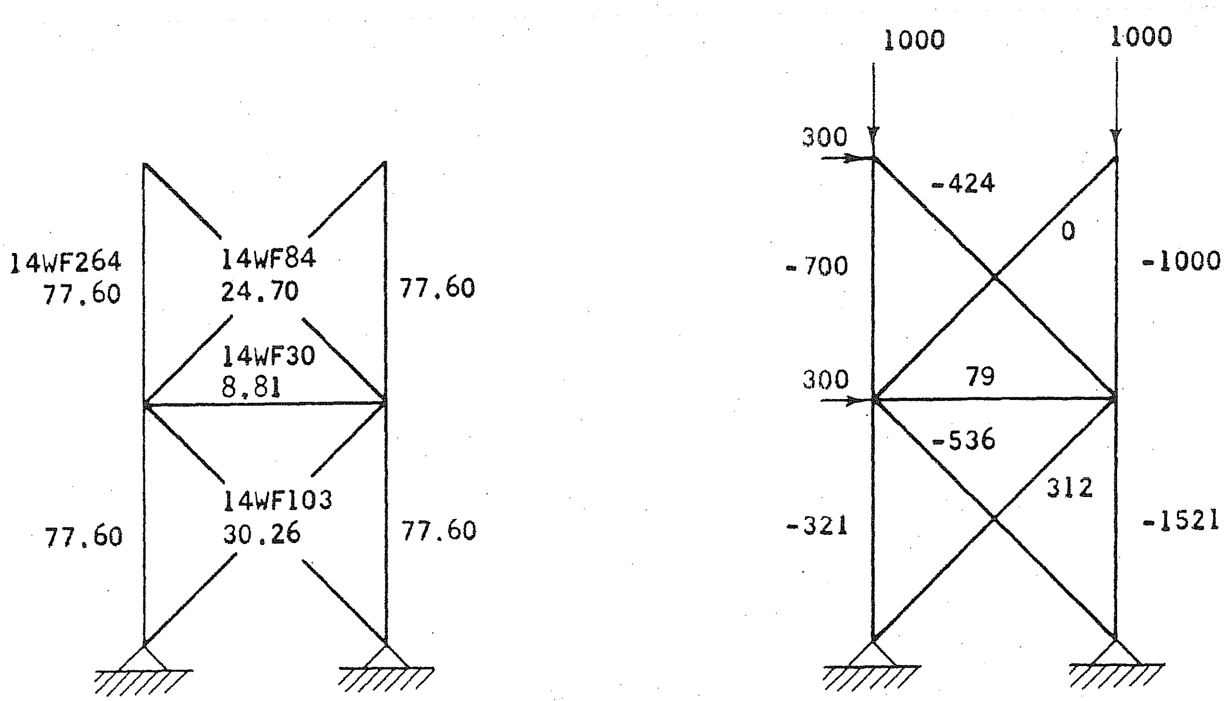


B. Available Solution (Weight = 15479 lb)

FIG. 31. MINIMUM-WEIGHT DESIGN OF PIN-JOINTED TRUSS
Case II: lower horizontal bar removed.



A. Continuous Solution (Weight 15787 lb)



B. Available Solution (Weight 16156 lb)

FIG. 32. MINIMUM-WEIGHT DESIGN OF PIN-JOINTED TRUSS
Case III: upper horizontal bar removed.