10
29A
69
Copy 2

ENGINEERING STUDIES

RUCTURAL RESEARCH SERIES NO. 69

1

~~A REVIEW OF~~ NUMERICAL INTEGRATION METHODS FOR DYNAMIC RESPONSE OF STRUCTURES

Metz Reference Room
Civil Engineering Department
B106 C. E. Building
University of Illinois
Urbana, Illinois 61801.

By

T. P. TUNG

and

N. M. NEWMARK

Technical Report
to
OFFICE OF NAVAL RESEARCH
Contract N6ori-071(06), Task Order VI
Project NR-064-183

Mar 54

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

A REVIEW OF NUMERICAL INTEGRATION METHODS FOR DYNAMIC RESPONSE OF STRUCTURES

by

T. P. Tung

and

N. M. Newmark

Technical Report

to

Office of Naval Research

Contract N6ori-071(06), Task Order VI

Project NR-064-183

Department of Civil Engineering
University of Illinois
Urbana, Illinois
March 1954

CONTENTS

LIST OF FIGURES

I. INTRODUCTION

1. Summary

The purpose of this study is to investigate the applicability various methods of numerical integration to problems of dynamic response of structures. The scope of the study is, therefore, limited to differe tial equations of order not higher than two, but the method of analysis be applied equally well to differential equations of higher orders.

Problems of dynamics of structures are usually considered to b initial-value problems. Unlike the boundary-value problem which has its given prescribed values along a closed boundary, the initial value probl has its given conditions prescribed at the beginning of the time coordinate. Analytical treatment for both types of problems are different, ar so are the numerical approaches. For boundary-value problems a typical numerical approach is the method of relaxation which solves a system of algebraic equations, approximating the given set of differential equatic in the sense of finite differences, at a finite number of points. For initial-value problems the numerical solution is usually obtained throug a marching process (a step-by-step integrating procedure). The range of interest of the time coordinate is divided into a finite number of small intervals. Since the necessary quantities have been given at the beginr of the interval, the values at the end of the interval can be obtained b simple formulas of integration; these values will serve as the initial values for the next interval. The accuracy of the numerical solution is affected by the method of numerical integration and by the fineness of t subdivision of the time coordinate.

A simple procedure would leave large truncational errors, and, therefore, needs a finely divided interval to reduce the error; while an accurate method of integration could tolerate a coarse time interval, bu here the computations for each step are more complicated. What is neede in design is a simple procedure with moderate accuracy which does not entail laborious computational work.

In the present report, various methods of numerical integratio are reviewed. Comparisons between the numerical solution and exact solu tions are made for a system with a single degree of freedom, with consta spring modulus. Since it is well known that motion of multi-degree-of-freedom systems can be split into eigenmodes with each vibrating with it own frequency, the analytical comparison can be used equally well in the case of multi-degree-of-freedom systems, without loss of generality. Systems with viscous damping or negative spring constant are also considered.

In a step-by-step procedure it is usually known that any error either truncational or round-off, committed in one step will continuousl affect the results of subsequent computation; under certain circumstance the error may grow without bound and eventually destroy the numerical si nificance of the answer. This instability of error is discussed in deta herein for problems of structural dynamics. The instability criteria of various methods are studied, and means of suppressing the unstable solution are suggested.

Although the present analysis is made for differential equatio with constant coefficients, the result can give some qualitative indicat in cases where the coefficients become time-dependent. If the time inte is taken small enough, within the interval it is reasonable to assume th the coefficients are constant. But this point-by-point checking does no in general, give reliable information.[1]

However, it must be noted that the application of step-by-step methods of integration is not merely limited to problems of initial-valu type; nor is the relaxation method limited only to boundary value proble For instance, attempts have been made to solve a transient heat-flow pro blem by the relaxation technique,[2] and there are also possibilities of obtaining solutions to a boundary value problem by means of step-by-step integration.[3] The choice of the methods mainly depends upon the compu tational facilities available.

(1) S. H. Crandall, "Stability Criterion of Difference Equation Selectic of a Partial Differential Equation" June, 1953, pp. 80-81. J. Phy. Math.

(2) D. N. de G. Allen and R. T. Steven, "The Application of Relaxation Methods to the Solution of Non-Elliptic Partial Differential Equations". Quart. J. Mech. and Applied Math. 4, (1951) 209-22; 5 (1952) 447-454.

(3) M. Hyman, "A Non-Iterative Solution of Boundary Value Problem", Applied Scientific Research. Section B, 2, No. 5, pp. 325-351, 1952

II. GENERAL DESCRIPTION OF VARIOUS METHODS OF INTEGRATION

2. Classification of Methods of Numerical Integration

Methods of numerical integration have been a subject of intere since Euler's day. Most of them can be found in the standard text books of numerical analysis; however only some of the most celebrated methods are discussed here with respect to their application to problems of dyna of structure. For the sake of simplifying the discussion, all available methods are tentatively classified into three groups based on the operations used in the method.

(1) Methods using derivatives. - Methods in this group are usually constructed on the basis of ordinary Taylor expansion with highe order terms truncated. Typical examples are several integration formula devised by W. E. Milne.(4)

(2) Methods of subdivision in each step. - Typical examples c this group are the famous Runge-Kutta methods in which the integrands ar more accurately defined by subdividing the interval.

(3) Methods of using higher order differences. - By extensive introducing higher order differences into the expression the method can produce results as accurate as is needed. Methods of this type are numerous, for instance, Adams' extrapolation procedure and interpolation procedures.

This classification is by no means rigid; there are procedures which possess principal features of two groups.(5)

(4) Milne, W.G., "A Note on the Numerical Integration of Differential Equations", J. Res. Nat. Bur. Stand., 43, pp. 537-542 (1949).

(5) Zürmühl, R., "Runge-Kutta Method unter Verwendung höherer Ableitung ZaMM, 32, pp. 153-154, 1952.

In the following, the discussion of various methods of numeric integration is limited to integrating differential equations of first or second order, since differential equations of second order can be split into two equations of first order, i.e.,

$$y'' = f\ (t,v,y) \tag{2}$$

is equivalent to

$$\begin{aligned} v' &= f\ (t,v,y) \\ y' &= v \end{aligned} \tag{2}$$

where y = dynamic displacement of the mass

v = velocity of the mass.

3. Methods Using Derivatives - Milne's Methods.

In one of his papers published in 1949[(6)], Professor Milne lis a series of integration formulas constructed on the basis of Taylor expa sion of the integrand for differential equations of first and second ord The higher the order of derivatives included in the expression the bette is the approximation in general, but this further complicates the numeri operations of the procedure. The most accurate formula in the list is o with a residual of order of magnitude of h^7, where h denotes the length interval. This formula was discovered independently by Lotkin in 1952.

(6) Milne, W.G., "The Remainder in Linear Methods of Approximation", J. Res. Nat. Bur. Stand., Rp. 2401, 43, pp. 501-511 (1949).

Lotkin, M., "A New Integrating Procedure of High Accuracy", J. Math. Phy., 31, pp. 29-34, 1952.

The derivation of the formula, for integrating differential equations of the first order, is briefly reproduced as follows. Based on the Taylor expansion of f in terms of f_o, f_o', $f_o''\ldots f_o^{(n)}$ it is clear that the $\int_{t_o}^{t_o+h} d\tau\, f(\tau)$ can be expressed in a power series,

$$\int_{t_o}^{t_o+h} f(\tau)\, d\tau = hf_o + \frac{h^2}{2!} f_o' + \frac{h^3}{3!} f_o'' + \ldots + \frac{h^n}{n!} f_o^{(n)} \qquad (3$$

where f_o, f_o', f_o'', ... $f_o^{(n)}$ represent the value of f, $\frac{df}{dt}$, $\frac{d^2f}{dt^2}\ldots\frac{d^nf}{dt^n}$ at $t=t_o$ respectively.

Similarly, f can be expressed in terms of f_1, f_1', f_1'',....$f_1^{(n)}$ where f_1, f_1', $f_1''\ldots f_1^{(n)}$ represent values of f, $\frac{df}{dt}$, $\frac{d^2f}{dt^2}$,...$\frac{d^nf}{dt^n}$ at $t=t_o+h$ Thus,

$$\int_{t_o}^{t_o+h} d\tau\, f(\tau) = hf_1 - \frac{h^2}{2!} f_1' + \frac{h^3}{3!} f_1'' - \ldots \qquad (3$$

Now sum eqs. (3-1) and (3-2) and average the sum; the following result i obtained

$$\int_{t_o}^{t_o+h} d\tau\, f(2) = \frac{h}{2}\,(f_o+f_1) + \frac{h^2}{2.2!}\,(f_o'-f_1') + \frac{h^3}{2.3!}\,(f_o''+f_1'') +\ldots \quad ($$

In truncating the unwanted terms the following two Taylor expansions are found very useful,

$$f_1^{(n)} = f_o^{(n)} + h\, f_o^{(n+1)} + \frac{h^2}{2!}\, f_o^{(n+2)} + \ldots \qquad ($$

$$f_o^{(n)} = f_1^{(n)} - h\, f_1^{(n+1)} + \frac{h^2}{2!}\, f_1^{(n+2)} + \ldots \qquad ($$

The difference of the two expressions is

$$f_1^{(n)} - f_o^{(n)} = \frac{1}{2}\Big[h\,(f_o^{(n+1)} + f_1^{(n+1)}) + \frac{h^2}{2}(f_o^{(n+2)} - f_1^{(n+2)}) + \ldots$$

(3·

or more specifically for n = 1, 3, and 5,

$$f_1^{(1)} - f_o^{(1)} = \frac{1}{2}\left[h\ (f_o^{(2)} + f_1^{(2)}) + \frac{h^2}{2}\ (f_o^{(3)} - f_1^{(3)}) + \ldots\right] \qquad (3$$

$$f_1^{(3)} - f_o^{(3)} = \frac{1}{2}\left[h\ (f_o^{(4)} + f_1^{(4)}) + \frac{h^2}{2}(f_o^{(5)} - f_1^{(5)}) + \ldots\right] \qquad (3$$

$$f_1^{(5)} - f_o^{(5)} = \frac{1}{2}\left[h\ (f_o^{(6)} + f_1^{(6)}) + \frac{h^2}{2}\ (f_o^{(7)} - f_1^{(7)}) + \ldots\right] \qquad (3$$

these three equations can be used to eliminate the terms involving h^4, h^5 and h^6, in eq. (3-3), the final form of the Milne-Lotkin formula is then

$$\int_{t_o}^{t_o+h} d\tau\ f(\tau) = \frac{h}{2}\ (f_1 + f_o) + \frac{h^2}{10}\ (f_o' - f_1') + \frac{h^3}{120}\ (f_o'' + f_1'') + 0(h^7 \qquad (3-$$

It is not surprising to find that the residual is of the order of h^7, sin six parameters f_o, f_1, f_o', f_1'', f_o'', f_1'' have been incorporated into the formula to define the integrand within the interval.

Milne also devised an approximate but simple expression for the residual, indicating the order of magnitude of the residual at each step. This can serve as a guide to the number of significant figures of the numerical solution.

Based on the same technique one can construct a formula for second differential equation

$$y'' = f(t,v,y)$$

Integrating Eq. (3-3) once more with respect to t, one finds

$$\int_{t_o}^{t_o+h}\int d\tau^2\ f(\tau) = \frac{1}{2}\left[\frac{h^2}{2}(f_o + f_1) + \frac{h^3}{3!}\ (f_o' - f_1') + \frac{h''}{4!}\ (f_o'' + f_1'') + \ldots\right] \qquad (3-1$$

By using Eqs. (3-7), (3-8) and (3-9) to eliminate terms involving h^5, h^6 and h^7, in Eq. (3-11), one arrives at the integration formula for second order equations with the residual of the order of h^8.

$$\int_{t_o}^{t_o+h}\int d\tau^2\, f(\tau) = \frac{h^2}{4}(f_o+f_1) + \frac{h^3}{20}(f'_o-f'_1) + \frac{h^4}{240}(f''_o+f''_1) + O(h^8) \qquad (3$$

The complete scheme of computation is then based on two different integration formulas with the order of magnitudes of the residuals differed by a factor h,

$$y_1 = y_o + v_o h + \frac{h^2}{4}(f_o+f_1) + \frac{h^3}{20}(f'_o-f'_1) + \frac{h^4}{240}(f''_o+f''_1) - \frac{509\, h^{(8)} f^{(8)}(s)}{806,400} \qquad (3-$$

$$v_1 = v_o + \frac{h}{2}(f_o+f_1) + \frac{h^2}{10}(f'_o-f'_1) + \frac{h^3}{120}(f''_o+f''_1) - \frac{h^7 f^{(7)}(s)}{100,800} \qquad (3-$$

$$t_o \leq s \leq t_o+h$$

The merits claimed for this type of integration procedure can be clearly seen from the facts that

(1) the residual, or truncational error, can be easily comput[e] or estimated from the derived expression and a knowledge of the derivati[ves] of f;

(2) the length of the interval, h, can be changed at any time so long as the error of the result is within the allowable.

However the procedure is subject to the following objections:

(1) since f_1 and its derivatives appear on the right hand sid[e] of the equations, normally an iterative procedure would have to be used;

(2) in a complex situation such as a multi-degree of-freedom system where f is a function containing $y_1 \cdots\cdots y_n$, $v_1 \cdots\cdots v_n$ in a not necessarily linear manner, the computation of derivatives of f presents a tedious task which would become prohibitive if n becomes large. For instance, if $\quad f_i = f_i\,(y_1 \cdots y_n,\ v_1 \cdots v_n,\ t)$

then
$$\frac{df_i}{dt} = \frac{\partial f_i}{\partial t} + \sum_{j=1}^{n} v_j \frac{\partial f_i}{\partial y_j} + \sum_{j=1}^{n} f_j \frac{\partial f_i}{\partial v_j}$$

4. Methods of Subdivision- Runge-Kutta and Nystrom Methods.

This celebrated method has its merit in improving the result without introducing the derivatives into the operation, in contrast to the methods of Taylor series duscussed in the foregoing section, but the construction of the procedure is still based on the Taylor expansion of the integrand, which is the foundation of practically all numerical methods. The basic idea was first used by Runge and later modified by Kutta. Through continuous elaboration and generalization, mostly by students of Runge, the method is now among the most widely used ones, especially in Europe. The construction of a Runge-Kutta fourth order method for first order differential equations is briefly described below.

Let
$$y' = f(t_1 y) \tag{4-1}$$

be the given differential equation with initial value y_o at t_o. It is required to compute y at $t = t_o + h$. By substituting $y = y_o$ in (4-1), one obtains the slope of the curve $\frac{dy}{dt}$ at t_o. Then define k as follows:

$$k_o = h\, f(t_o, y_o). \tag{4-2}$$

Now proceed along this tangent line a distance mh on the t-axis and k_1 is defined as

$$k_1 = h\, f(t_o+mh,\ y_o+mk_o) \qquad (4\text{-}$$

Having computed k_o and k_o, one define a third element

$$k_2 = h\, f(t_o+nh,\ y_o+(n-r)k_o+rk_1) \qquad (4\text{-}$$

and then a fourth element

$$k_3 = h\, f(t_o+ph,\ y_o+(p-s-q)k_o+ sk_1 + qk_2) \qquad (4\text{-}$$

where n, r, p, s, and q are parameters of the same kind as m.

The increment of y, Δ y, is defined as

$$\Delta y = ak_o + bk_1 + ck_2 + dk_3 \qquad (4\text{-}$$

It must be noted that the points, mh, nh, and ph, measured from the initi point t_o are only used as additional points to obtain an improved definition of f; they are not the end points of the broken lines along which the procedure of numerical integration is made. The choice of the ten parameters, a, b, c, d, m, n, r, p, s, and q, is based on a comparison between the Taylor expansion of Δ y and a power series in h for Δy obtained through successive elimination of k_o, k_1, k_2, and k_3, defined in equations (4-2) to (4-5). The latter can be obtained by expanding k's into power series in h by means of Taylor expansions. Having equated the coefficients of the first four terms of both expressions, one obtaines the following conditions for a result whose first error term is of the order h^5.

$$\begin{aligned} & a + b + c + d = 1 \\ & bm^2 + cn^2 + dp^2 = \frac{1}{3} \\ & Cmr + d\ (nt + ms) = \frac{1}{6} \\ & Cm^2r + d\ (n^2t + m^2s) = \frac{1}{12} \end{aligned} \qquad (4\text{-}7$$

$$bm + cn + dp = \frac{1}{2}$$

$$bm^3 + cn^3 + dp^3 = \frac{1}{4} \qquad (4\text{-}7)$$

$$cmnr + dp\ (\ nt + ms) = \frac{1}{8}$$

$$dmrt = \frac{1}{24}$$

This system of eight equations leaves a doubly infinite number of choices for the ten parameters. For an efficient scheme the parameters should take on simple numerical values, a commonly used Runge-Kutta method is

$$y = h(k_o + 2k_1 + 2k_2 + k_3)/6 \qquad (4\text{-}8$$

where

$$k_o = h\ f(t_o, y_o)$$
$$k_1 = h\ f(t_o + \tfrac{1}{2}h,\ y_o + \tfrac{1}{2}k_o)$$
$$k_2 = h\ f(t_o + \tfrac{1}{2}h,\ y_o + \tfrac{1}{2}k_1) \qquad (4\text{-}9$$
$$k_3 = h\ f(t_o + h,\ y_o + k_2)$$

Gill[7] chose the parameters on the basis that the number of registers ca be reduced when the procedure is used to solve problems on a digital computing machine, this leads to irrational numbers for some of the paramete

Using the similar technique, Nystrom[8] developed a Runge-Kutta method for differential equations of second order. With four k functions

---

(7) Gill, S. "A Process for the Step-by-Step Integration of Differential Equations in an Automatic Digital Computing Machine", Proc. Camb. Philos. Soc. 47, 46-108 (1951).

(8) Nyström, E.J., "Uber die Numericshe Integration von Differentialgleichungen," Aeta Soc. Sci. Fennicae, 50, no. 13, 56 pp. (1926).

the method can yield a solution in which both the displacement, y, and the velocity, dy/dt, are correct to h^5 provided that no velocity terms is involved in f, otherwise the solution is correct only to the h^4 term. Th following is one of his schemes for

$$y'' = f(t, y, v)$$

$$\Delta y = h\left[r_o + \frac{23\,k_o + 75k_1 - 27k_2 + 25k_3}{192}\right]$$

$$\Delta v = \frac{23k_o + 125k_1 - 81k_2 + 125k_3}{192}$$

With

$$k_o = f(t_o, y_o)h \qquad r_1 = h(v_o + \frac{k_o}{5})$$

$$k_1 = f(t_o + \frac{2h}{5}, y_o + \frac{2r_1}{5})h \qquad r_2 = h(v_o + \frac{k_1}{3}) \tag{4-1}$$

$$k_2 = f(t_o + \frac{2}{3}h, y_o + \frac{2}{3}r_1)\,h \qquad r_3 = h(v_o + \frac{k_1 + k_o}{5})$$

$$k_3 = f(t_o + \frac{4h}{5}, y_o + \frac{4r_3}{5})h$$

By the use of similar reasoning Blaess[9] devised a numerical method and later on it was improved and generalized by Bukovics[10]. The scheme was simplified by letting the parameters taking on same number, but the accuracy was preserved at the expense of more number of substitutions.

---

(9) Blaess, "On the Approximate Solutions of Ordinary Differential Equations", Z. Ver. Dtsch. Ing. 81, pp. 587-596 (1937).

(10) Bukovics, E. "An Improvement and Generalization of the Blaess Method for Numerical Integration of Differential Equations", Ost. Ing. Arch. 4, pp. 338-349 (1950).

The advantages of the Runge-Kutta method are

(1) No iteration is needed.

(2) No derivatives are involved in the computation.

(3) The length of the interval can be changed according to expediency.

(4) The operation is simple if the parameters are chosen as simple figures.

However, objections to the use of the Runge-Kutta method can be found in the literatures; there is chiefly that a highly accurate Runge-Kutta met of order m needs more than m substitutions when m is greater than 4. As far as error is concerned, Bieberbach(11) has found an expression indicating an upper bound for the error within a given step of the Runge-Kutta process, but the bound is always found to be too overcautious to give any practical significance. Lotkin(12) improved the expression yet the bound is still far on the safe side.

5. Methods of Finite Differences.

Methods under this heading consist of a great variety of procedures of numerical integration which are perfected by introducing high order differences extensively. Typical examples in this category are th Adams' extrapolation procedures, interpolation procedures, and methods o

---

11. Bieberbach, L. "On the Remainder of the Runge-Kutta Formula in the Theory of Ordinary Differential Equation". ZaMP, 2, No. 4, 1951, pp. 233-248.

12. Lotkin, M., "On the Accuracy of Runge-Kutta Method", Math Tables an Other Aids to Computations", 5, No. 35, pp. 128-133, July, 1951.

central difference. The formulation of the expressions can be found in most books on numerical analysis. For a first order differential equatic

$$y' = f\ (t, y) \qquad (5\cdot$$

The extrapolation procedure is to compute y from

$$y_{n+1} = y_n + h\ (f_n + \frac{\nabla f_n}{2} + \frac{5\nabla^2}{12} f_n + \frac{3\nabla^3 f_n}{8} + \cdots) \qquad (5\cdot$$

where $\nabla f_n = f_n - f_{n-1}$, the backward difference.

The interpolation procedure gives

$$y_{n+1} = y_n + h\ (f_{n+1} - \frac{\nabla f_{n+1}}{2} - \frac{\nabla^2 f_{n+1}}{12} - \frac{\nabla^3 f_{n+1}}{24} - \ldots) \qquad (5\cdot$$

and the method of central difference gives

$$y_{n+1} = y_n + h\ (2f_n + \frac{\nabla^2}{3} f_{n+1} - \frac{\nabla^4 f_{n+2}}{90} + \frac{1}{756} \nabla^6 f_{n+3} - \ldots) \qquad (5\cdot$$

The last two procedures needed to be carried out on an iterative basis since the unknown y_{n+1} is involved in the right-hand side.

In view of the finite differences equations an obvious questior is how to start the problem. Besides the given initial values there is not enough information leading to the computation of the various differences needed in the formula. A usual technique is to compute y's at firs few steps by a simple formula such as Taylor series method, Runge-Kutta methods; these values will give a series of backward differences of y with reliable accuracy, then new y's are computed by Eqs. (5-1), (5-2), or (5-3). This process may need several repetitions until the y's reach stable values, afterwards the procedure will be carried on as usual.

This starting of the process is certainly a great disadvantage of the methods, especially in the case of analyzing dynamic behavior of a structure, the elastic property of which may change as certain parts of the member yields; and, as a consequence, the equation of equilibrium may take a new, different form and the problem has to start afresh, Moreover as it will be shown later on, the introduction of higher order difference is not always beneficial, sometimes extraneous solutions are introduced into the result due to the presence of higher order differences.

In order to get the beneficial result of using higher order differences the following procedures have been suggested. They are carried out on an iterative basis and the numerical result can be obtaine as accurately as it is needed. For a first order differential equation $y' = f(t,y)$ it is always possible to write the finite difference equivalence as

$$y_n = A_n y_{n-1} + B_n + E_n \qquad (5\cdot$$

By first neglecting the residual, or truncational error, E_n, y's can be obtained from the basic equation

$$y_n = A_n y_{n-1} + B_n \qquad (5\cdot$$

as a first approximation. Then the first approximation of E_n can be computed and used to find the second approximation for y_n in the Eq. (5-5). This resembles the starting process for an extrapolation or interpolatio procedure except that the process will be used throughout the whole probl

The efficiency of the procedure depends upon a balanced choice of A and E. If A is chosen to contain all terms in Eq. (5-2) that are needed for the desired accuracy, E will consist of the remaining higher order terms. In this case, the y's computed for the first trial will be as accurate as required. Few or no cycle of iterations would be needed. But if A is chosen this way, so it would be an involved expression which would be time-consuming to evaluate at each step. It can be seen that a simple expression for A, at the expense of larger residual error, would constitute a better basis for computations. The following example,

$$y_n = y_{n-1} + h\,(y'_{n-1} + y'_n) + E_n \tag{5-7}$$

with
$$E_n = \left(-\frac{\delta^3}{12} + \frac{\delta^5}{120} - \frac{\delta^7}{840} + \dots\right) f_{\frac{1}{2}+n}$$

where δ is the central difference operator, applied to a differential equation of first order

$$y' = f\,y + g$$

The recurring relation is shown by the following difference equation

$$(1 - f_n h)\, y_n = (1 + f_{n-1} h)\, y_{n-1} + h\,(g_n + g_{n-1}) + E_n \tag{5-8}$$

Several similar procedures can be found in a paper by Fox[13] and Goodwin.

The same technique can be used for second order differential equations. For second order differential equations without terms involving the first derivative, (any second order differential equation can be put in this form by a suitable transformation), a simple and highly accurate method is recommended by Fox. In the later discussion it will be referred to as Fox's method, but the basic formula has been discussed

---

13. Fox, L., Goodwin, E.T., "Some New Methods for the Numerical Integration of Ordinary Differential Equations", Proc. Camb. Philos. Soc. 45, pp. 373-388, (1949).

and used in several publications.[14] Let the differential equation be

$$y'' = y f + g$$

The presence of g does not cause extra difficulty as has been seen in the last example, thus g may be neglected. If y' is replaced by its central difference equivalent and then difference operator $(1 + \frac{\sigma^2}{12})$ is applied throughout the equation, a recurrence relation is obtained

$$(1 - \frac{h^2 f_{n+1}}{12}) y_{n+1} = (2 + \frac{5h^2 f_n}{6}) y_n - (1 - \frac{h^2 f_{n-1}}{12}) y_{n-1} + E_n \qquad (5$$

and the truncation error $E_n = (- \frac{\delta^6}{240} + \frac{13\delta^8}{15120} - \ldots) y_n$ where δ is the central difference operator. The recurrence relation is quite simple and the truncational error is of order of magnitude of δ^6, which is very small in most normal cases.

If f is a function of y as well as t, then the differential equation is no longer linear. Trial values for f have to be assumed and the process becomes a doubly iterative procedure. If the time interval, h, is kept small enough, convergence will not be a serious problem.

In all the methods of integration by iterative procedures, the truncational errors are computed in terms of the differences of the function values previously established; central differences are usually to be recommended. However, it generally happens that in the first and the last few steps the given information is not sufficient to determine

---

14. Feinstein, L., and Scharzshild, M., "Automatic Integration of Linear Second Order Differential Equations by Means of Punch Card Machines," Rev. Sci. Inst. 12, 1941, pp. 405-8.
Lindberg, N. A., "Integration of Second Order Differential Equation on the Type 602 Calculating Punch", Proc. Sci. Computation Forum, p. 23, 1948, IBM.
Collatz, L., "Numerische Behandling von Differentialgleichungen," Springer, 1951, S 80 Gl. (5.40).

all the central differences that are needed. In such a case these differences are usually supplied by extrapolation or guess. In order to utilize all the given information available, a mixed central difference and backward difference scheme has been suggested. Lowdin[15] in his recent paper succeeded in supplying the additional central differences by means of the given backward differences in the last backward line.

Lowdin's method has features similar to those of the step-by-step iterative integration procedure, namely, a basic formula upon which the first approximation is constructed, and a difference correction that takes into account truncational error in terms of differences. To start the problem, an iterative procedure is used to establish a few values at the beginning, but is not continued. After this iterative procedure has been used up to a certain number of steps, the process is transformed into a "marching process", with all the needed central differences supplied by the backward difference formulas. When the marching process has been completed for the whole range of integration it is possible to improve the accuracy of "y" by utilizing the actual values of central differences formed in the later stages of calculation. An correction may be added if it is significant. Using a similar technique Lowdin[16] published a procedure for differential equations of second order.

---

15. Lowdin, P.O., "On the Numerical Integration of Ordinary Differential Equations of the First Order", Quart, Appl. Math. 10, No. 2, July, 1952, pp. 97-111.

16. Lowdin, P. O., and Sjolander, A., "A Note on the Numerical Calculation of Asyenptotic Phases with a Numerical Study of Hulthin's Variational Principle", Archiv for Fysik, 3, No. 11, pp. 155-159 (1951

The essential points of Lowdin's procedure are:

(1) A mixed use of central backward differences to utilize all available information.

(2) An iterative process is needed only to start the problem, then a marching process continues on to get all the rest of the solution.

(3) To correct the solution on the basis of central difference computed from the values of the solution obtained, a correction can be added; the correction is usually small.

As a concluding remark, these methods have the advantage that a clear indication of truncational error can be easily obtained for the classical method of finite differences (namely Adams' method). Since the iterative procedures of numerical integration will yield the same solutio but with a truncational error which is practically zero, they differ only in the operating scheme.

However, there are some annoying features of the methods:

(1) The same time interval has to be used throughout the whole problem.

(2) The methods needed a starting device to establish an approximate solution from which the differences can be obtained.

(3) The operating scheme is complicated and time-consuming.

(4) It is not convenient to use the methods for problems where the differential equations change their functional forms when critical limits of certain parameters are reached, since the solution is not supposed to reach its final form until the last iteration. These methods are less objectionable for non-linear differential equations, since iteration has to be used anyway.

III. INSTABILITY OF NUMERICAL SOLUTION

6. General Description

In the method of using an equivalent difference scheme to replace the given, partial or ordinary, differential equations, it has been observed that under certain circumstances the method may lead to a soluti(that is radically different from what is anticipated. For most of the initial value problems this erroneous solution usually takes the form of an oscillation with an ever-increasing amplitude. Its existence has been observed and reported by many investigators and suitable remedies have been suggested. It is generally agreed that the instability of error is an intrinsic property of the method of solution with respect to the associated problem.

For partial differential equations of hyperbolic and parabolic type, Brien, Hyman and Kaplan(17) have found, by a method used by von Neumann in his unpublished study of unstable solution of difference equations, that certain relationships between the meth sizes of the net must be maintained in order to keep down the growth of errors. Similar efforts have been made by Eddy(18) in studying unstable solution in heat

---

17. Brien, G.O., Hyman, M., and Kaplan, S. "A Study of the Numerical Solutions of Partial Differential Equations", Jour. Math. and Physics 29, pp. 223-251, 1951.

18. Eddy, R.P. "Stability in the Numerical Solution of Initial Value Problems in Partial Differential Equations", Technical Memo. 10232, Naval Ordnance Laboratory.

conduction problem and by Collatz[19] in deriving the instability criteric for differential equations in the problem of the vibration of beams, and also for the use of the method of central difference.[20]

Recently Rutishauser[21] made instability tests for several methc of numerical integration of ordinary differential equations of simple type He attributes the unstable solution to the presence of "extra" solutions arising from the use of higher order differences, which result in a higher order characteristic algebraic equation. A generalization of the method of analysis was also reported.

In all the methods of analysis a test differential equation was replaced by its equivalent difference equation, obtained with the aid of some numerical method. Then the solution of the differential equation is compared with that of the difference equation. The trucational error, due to the replacement, is expected that the two solutions are of the same order of magnitude and can have similar physical interpretation. If they cannot be compared on this basis, the truncational error of the numerical method is certainly serious enough that the numerical solution has to be rejected.

---

19. Collatz, L. "Zur Stabilität des Differenzenverfahrens bei der Stabschwingsgleichung", Z. Angew. Math. Mech. 31, S. 392-393, 1951.

20. Collatz, L. "Über die Instabilität beim Verfahren der zentralen Differenzen fur Differential-gleichungen zweiter Ordung", ZaMP 4, 1953, seite 153-154.

21. Rutishauser, H. "Über die Instabilität von Methoden zur Integration gewöhnlicher Differentialgleichungen", ZaMP, 3, seite 65-74, 1952.

7. Criterion of Instability

In the present study the test differential equation is

$$y'' + p^2 y = 0 \tag{7-1}$$

the solution of which is well known as

$$y = y_o \cos pt + \frac{v_o}{p} \sin pt \tag{7-2}$$

where y_o and v_o are the initial displacement and initial velocity respectively. Since a second order differential equation can always be split into two first order differential equations, Eq. (7-1) can be formulated into

$$y' = v, \; v' = -p^2 y \tag{7-3}$$

then the corresponding difference equation of Eq. (7-1) will take two different forms depending on the type of integration formula to be used.

If the Runge-Kutta Fourth Order method, Eq. (4-8), is used to solve the system of two first order differential equations, after successive substitutions of k's the following difference equations are obtained.

$$y_n = (1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}) y_{n-1} + (1 - \frac{\theta^2}{6}) h v_{n-1}$$

$$v_n h = \theta^2 (-1 + \frac{\theta^2}{6}) y_{n-1} + (1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}) h v_{n-1} \tag{7-4}$$

The solution to the above equations can be assumed to take the form

$$\begin{pmatrix} y_n \\ v_n h \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \lambda^n \tag{7-5}$$

After direct substitution for y_n and v_n, it can be proved that the following determinate has to be zero if the two constants, a and b, will take non-zero values.

$$\begin{vmatrix} 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \lambda & 1 - \frac{\theta^2}{6} \\ - \theta^2(1 - \frac{\theta^2}{6}) & 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \lambda \end{vmatrix} = 0 \qquad (7\text{-}$$

where θ = ph.

The characteristic equation is

$$(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \lambda)^2 + \theta^2 \, (1 - \frac{\theta^2}{6})^2 = 0$$

from which the roots for λ can be obtained.

It is obvious that the solution to the difference equations has to be sinusoidal in view of the exact answer, Eq. (7-2). To achieve this, the λ's must be complex numbers, nevertheless, the absolute value of the complex roots must be unity in order to assure a sinusoidal solution with constant amplitude. In the present case λ is always complex but unfortunately the amplitude is not conserved, since

$$1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \lambda \pm i \, \theta \, (1 - \frac{\theta^2}{6}) = 0 \qquad (7\text{-}7$$

and

$$|\lambda|^2 = 1 - \frac{\theta^6}{72} + \frac{\theta^8}{578}$$

Apparently the amplitude can be conserved if the interval h is chosen small enough to make $\theta \ll 1$,

In the next example the method of using derivatives is tested, a double integration formula is chosen to get y_n and a simple integration formula for v_n, namely Eqs. (3-13) and (3-14).

Since $$f = -p^2 y$$

therefore
$$f' = -p^2 v \tag{7-8}$$
$$f'' = p^4 y$$

After eliminating f, f', etc., and f'' from Eqs. (3-13) and (3-14) by the above expressions the resulting difference equations are obtained.

$$y_n = y_{n-1} + v_{n-1}h - \frac{\theta^2}{4}(y_n + y_{n-1}) + \frac{\theta^3}{20}(v_n - v_{n-1})h + \frac{\theta^4}{240}(y_n + y_{n-1})$$

(7-

$$v_n h = v_{n-1}h - \frac{\theta^2}{2}(y_n + y_{n-1}) + \frac{\theta^3}{10}(v_n - v_{n-1})h + \frac{\theta^4}{120}(y_n + y_{n-1})$$

which is equivalent to the following system after a few algebraic simplifications

$$y_n \left(1 + \frac{3\theta^2}{20} - \frac{\theta^4}{240}\right) = y_{n-1}\left(1 - \frac{7\theta^2}{20} + \frac{\theta^4}{240}\right) + v_{n-1}h\left(1 - \frac{\theta^2}{10}\right)$$

(7-1

$$v_n h \left(1 + \frac{3\theta^2}{20} - \frac{\theta^4}{240}\right) = -y_{n-1}\,\theta^2\left(1 - \frac{\theta^2}{60}\right) + v_{n-1}h\left(1 - \frac{7\theta^2}{20} + \frac{\theta^4}{240}\right)$$

The characteristic equation of above system is

$$\left[1 - \frac{7\theta^2}{20} + \frac{\theta^4}{240} - \lambda\left(1 + \frac{3\theta^2}{20} - \frac{\theta^4}{240}\right)\right]^2 + \theta^2\left(1 - \frac{\theta^2}{10}\right)\left(1 - \frac{\theta^2}{60}\right) = 0$$

from which the roots for λ are given as

$$\lambda = \frac{1 - \frac{7\theta^2}{20} + \frac{\theta^4}{240} \pm i\,\theta\sqrt{\left(1 - \frac{\theta^2}{10}\right)\left(1 - \frac{\theta^2}{60}\right)}}{1 + \frac{3\theta^2}{20} - \frac{\theta^4}{240}}$$

In this case the roots become complex conditionally; therefore, the soluti is conditionally stable but will not behave sinusoidal when

$$10 < \theta^2 < 60$$

But this time the amplitude is always conserved because of the fact that $|\lambda|^2 = 1$ no matter what value for h is taken so long as the solution is kept stable.

From the examples given above, the stability of the numerical solution is mainly dependent on the nature of the roots of the characteristic equation. If the matrix of the coefficients in the right hand side of Eq. (7-3) has complex eigen-values of unit absolute values stability i assured. It is unnecessary to test every numerical method for stability, since it can be shown that the eigen-value of the matrix can always be made complex if the same integration formula is used for the system of tw first order differential equations, Eq. (7-2).

No matter what method of integration is used, it is always possible to write down the formula as

$$\int_{t_o}^{t_o+h} f(\tau)\, d\tau = h \left[A\, f_1 + Bf_o + Cf_{-1} + \ldots.\right]$$

if higher order differences are not included the formula could be even simpler

$$\int_{t_o}^{t_o+h} f(\tau)\, d\tau = h\ (A\, f_1 + B\, f_o) \tag{7-11}$$

where A, B, are functions of θ . Then Eq. (7-2) can be transformed into the corresponding difference equations

$$(v_n - v_{n-1}) = -\, p^2 \int y d_2 = -\, p^2 h\ (A\, y_n + B\, y_{n-1})$$

$$y_n - y_{n-1} = \int v d_2 = h\ (A\, y_n + B\, v_{n-1})$$

transposing,

$$y_n - A\, V_n h = y_{n-1} + B\, V_{n-1} h$$

(7-12

$$A\theta^2 y_n + V_n h = - B\theta^2 y_{n-1} + V_{n-1} h$$

λ , in this case, must satisfy the following condition

$$\begin{vmatrix} 1 - \lambda & B + A\lambda \\ -\theta^2(B + A\lambda) & 1 - \lambda \end{vmatrix} = 0$$

$$\therefore \quad 1 - \lambda = i\,(B + A\lambda)$$

Therefore, λ will become complex independent of the values taken by A and B. Moreover, the inclusion of higher order terms merely complicates the elements in the determinant and assigns more roots to λ . Since elements along the principal diagonal are always the same and the elements along the other diagonal are always differed by a factor $-\theta^2$, λ will never become real. This observation confirms the result in the first example in which the Runge-Kutta Fourth Order method is used and the numerical solution is sinusoidal; while in the second example two different integration formulas are used, one for simple integration and one for double integration, the numerical solution is not always sinusoidal despite the fact that amplitude can be conserved.

8. Conservation of Amplitude

However, if λ is complex, it doesn't necessarily follow that amplitude is conserved. There are separate criteria to find whether or not amplitude is conserved. From theory of matrices, it can be proved that if the complex eigen-value of the matrix in Eq. (7-4) has unit absolute value then the determinant of the matrix is unity, (see Appendix A). The numerical value of the determinant of matrix in Eq. (7-10) can certainly confirm this statement.

More generally, there are cases in which both the y_n and v_n are expressed only implicitly such as in Eq. (7-9). Then the criterion for conserving amplitude is that values of determinants of the two matrices on the left and right hand sides must be equal. This is a direct consequence of the foregoing statement and its proof is trivial. Eq. (7-9) can serve as an example; after transposing, it is found

$$\left(1 + \frac{\theta^2}{4} - \frac{\theta^4}{240}\right) y_n + \frac{\theta^2}{20} v_n h = \left(1 - \frac{\theta^2}{4} + \frac{\theta^4}{240}\right) y_{n-1} + \left(1 + \frac{\theta^2}{20}\right) v_{n-1} h$$

$$\left(\frac{\theta^2}{2} - \frac{\theta^4}{120}\right) y_n + \left(1 + \frac{\theta^2}{10}\right) v_n h = \left(-\frac{\theta^2}{2} + \frac{\theta^4}{120}\right) y_{n-1} + \left(1 + \frac{\theta^2}{10}\right) v_{n-1} h$$

and

$$\begin{vmatrix} 1 + \frac{\theta^2}{4} - \frac{\theta^4}{240} & \frac{\theta^2}{20} \\ \frac{\theta^2}{2} - \frac{\theta^4}{120} & 1 + \frac{\theta^2}{10} \end{vmatrix} = 1 + \frac{7\theta^2}{20} - \frac{\theta^4}{240}$$

$$\begin{vmatrix} 1 - \frac{\theta^2}{4} + \frac{\theta^4}{240} & 1 + \frac{\theta^2}{20} \\ -\frac{\theta^2}{2} + \frac{\theta^4}{120} & 1 + \frac{\theta^2}{10} \end{vmatrix} = 1 + \frac{7\theta^2}{20} - \frac{\theta^4}{240}$$

To formulate a general principle upon which numerical methods can be constructed with the aim of conserving amplitude is quite involved. However, it can be shown that the Milne's methods always conserve amplituc

If the simple integration formula is used symmetrically for the system of two first order differential equations, Eq. (7-2), then

$$\int_{t_o}^{t_o+h} d\tau\, y(\tau) = \sum a_n \left[y_o^{(n)} + (-)^n y_1^{(n)} \right]$$

and (8.

$$\int_{t_o}^{t_o+h} d\tau\, v(\tau) = \sum a_n \left[v_o^{(n)} + (-)^n v_1^{(n)} \right]$$

where a_n are functions of h, can be simplified by making use of the recurring property of the original differential equation; namely,

$$y'' + p^2 y = 0$$

$$y''' + p^2 y = 0$$

$$y^{iv} - p^2 y = 0$$

$$\cdots\cdot \qquad \cdots\cdot$$

Therefore, $$y_n - y_{n-1} = A_1 h\,(v_n + v_{n-1}) + A_2\,(y_{n-1} - y_n)$$

$$h\,(v_n - v_{n-1}) = -p^2 \left[B_1\,(y_n + y_{n-1}) + B_2 h\,(v_{n-1} - v_n) \right]$$

where A and B are functions of p and h, transposing,

$$y_n\,(1 + A_2) + V_n h\,(-A_1) = y_{n-1}(1 + A_2) + V_{n-1} h\, A_1$$

(8-2

$$y_n\, B_1\, p^2 + V_n h\,(1 - B_2 p^2) = y_{n-1}(-p^2 B_1) + V_{n-1} h\,(1 - B_2 p^2)$$

It is obvious that the determinants of the two matrices on both sides are equal; the statement is then proved.

To prove conservation of amplitude, in the case of the use of one simple integration and one double integration formula of the Milne's methods additional property of the methods of this group is required. Examination made of Eqs. (3-13) and (3-14), will show that the two formuls are in the ratio, h/2. But it will be proved that this relation always holds true as long as both keep the same order of terms no matter how many terms are involved. The proof is in Appendix B.

This additional property is found useful in the proof of the property of amplitude-conserving of the numerical solution. Instead of using the two original formulas,

$$y_n = y_{n-1} + V_{n-1}h + \iint f\, d\tau^2$$

$$V_n = V_{n-1} + \int f\, d\tau \qquad (8\text{-}$$

either one of them can be replaced by the additional property just found, namely

$$y_n = y_{n-1} + v_{n-1}h + \frac{h}{2}\left[\int f\, d\tau\right]$$

$$= y_{n-1} + v_{n-1}h + \frac{h}{2}(v_n - v_{n-1}) \qquad (8\text{-}$$

$$= y_{n-1} + \frac{h}{2}(v_n + v_{n-1})$$

Now let the second of Eq. (8-2) be written as see Eq. (7-11)

$$h\ (V_n - V_{n-1}) = -p^2 \left[B_1 (y_n + y_{n-1}) + B_2 h\ (V_{n-1} - V_n) \right] \qquad (8\text{-}4$$

After this equation is combined with Eq. (8-3), it becomes clear that the amplitude of the numerical solution computed by the Milne's methods will be conserved.

9. Influence of Higher Order Differences

Theoretically the inclusion of higher order difference in formu for numerical integration will reduce truncational error; it is logical t surmise that the growth of error could be suppressed. But from the resul of several investigators this seems not the case.

Inclusion of higher order differences in the formulas (see Adam procedures) usually would raise the order of the characteristic equation the difference equations which replaces the original differential equatio Hence extraneous roots will be produced as a consequence. Houbolt[22] uses a third order backward differences for the acceleration; this produc a cubic characteristic equation of which one of the roots must be real. Fortunately the real root is always less than one. It corresponds to a damped component in the numerical solution. Similar results were found i the "parabolic acceleration method"[23], in which the acceleration is assumed varying parabolically within the small time interval. The result

---

22. Houbolt, J.C., "A Recurrence-Matrix Solution for the Dynamic Respons of Elastic Aircraft" NACA TN 2060, March, 1950.

23. Newmark, N.M. and Chan, S.P., "A Comparison of Numerical Methods for Analyzing the Dynamic Response of Structures", Civil Engineering Studies, Structural Research Series No. 36, University of Illinois, pp. 16-19.

characteristic equation is again cubic, but the real root is not always less than one, and the complex roots alone do not produce an amplitude-conserving sinusoidal solution as they should.

The foregoing example may suggest that higher order differences might be included only up to an even order. This will produce a characte istic equation of even order. Here, then, is the possibility of suppress all the real roots and leaving the roots complex conjugates with the furt requirements that the absolute value of the roots should be equal to unit Even if all these could be achieved the numerical solution would be a mix- ture of several sinusoidal waves each having a particular frequency, instead of a single wave.

It seems that influence of using higher order differences is no always beneficial, despite the theoretical support which claims to reduce truncational error by inclusion of the differences. However, it can be shown that the theoretical reasoning is correct if a proper process is used. One of the processes is the method of iterative numerical integration discussed in section 5. The truncational error, computed in term of higher order differences of values in previous trial, is considered as known and put at the right-hand side of the difference equations. This leaves the homogeneous difference equations of the difference equations always of second order; consequently, removing the possibility of appearance of extraneous roots corresponding to unwanted solutions. This may be called a Picard process; the particular solution of the difference equation is improved each trial.

In the following are two examples, in which the same problem is worked out by two different procedures, the time interval, h, is so chosen that one of these procedures leads to an unstable solution. But later on the instability is suppressed; after correction the unstable scheme shows a better result than the stable one.

The first procedure is the Timoshenko's modified method,[24] for a given problem

$$y'' + p^2 y = 0 \qquad (9\text{-}$$

the equivalent difference equation is

$$(1 + \frac{\theta^2}{4})\, y_{n+1} - (2 - \frac{\theta^2}{2})\, y_n + (1 + \frac{\theta^2}{4})\, y_{n-1} + E_n \qquad (9\text{-}$$

with the truncational error $E_n = (- \frac{\delta^4}{6} + \frac{\delta^6}{720} - \frac{\delta^8}{1008} + \ldots)\, y_n$

The second scheme is the famous Fox method[25]; the equivalent difference equation is

$$(1 + \frac{\theta^2}{12})\, y_{n+1} - (2 - \frac{5}{6}\theta^2)\, y_n + (1 + \frac{\theta^2}{12})\, y_{n-1} = E_n \qquad (9\text{-}$$

with $E_n = (- \frac{\delta^6}{240} + \frac{13\delta^8}{15120} - \ldots.)\, y_n$

Now take ph = 2.5, and p = 1. This will make the second solution unstabl to begin with, while Timoshenko's method will always give a stable solution. The initial conditions are given with $y_o = 0$ and $y_1 = \cos h$, which corresponds to unit initial velocity. Since the differential equati

---

24. Timoshenko, S., "Vibration Problems in Engineering", 2nd Edition D. Van Nostrand Co., Inc., New York, 1937, pp. 126-128.

25. Fox, L., and Goodwin, E.T., "Some New Methods for the Numerical Integration of Ordinary Differential Equations", Proc. Camb. Phil. Soc., 45, p. 381, 1949.

consists of members of even order, the central differences can be assigned without guessing. After the first trial, (see Table I and II) the result shows that the unstable solution, computed by Fox's method, comes closer to the exact solution. But the divergent oscillation of higher order differences, with ever increasing magnitude, constitutes a serious drawback to the method, and limits its application. In particular, if the process is going to be carried out on digital computing machinery, the divergence in the values of differences will eventually run the number out of bound. Even if the scale of numbers is changed, the round-off error will cause a serious concern. The examples merely show that a procedure with smaller truncational error could yield a better answer despite the fact that it has to start with an unstable solution.

In view of the oscillatory property of the unstable solution it may be inferred that a method of averaging the successive value might bring down the magnitude of the figure without impairing the accuracy. (This idea is suggested in Lowdin's paper[15]).

It would be helpful to view the method by studying the following difference equation

$$y_{n+1} - (2-\alpha^2)y_n + y_{n-1} = 0 \tag{9-}$$

where $\alpha^2 = \dfrac{12p^2h^2}{12 + p^2h^2}$. This equation replaces Eq. (9-1) after the Fox's method is used, see Eq. (9-3). The instability criterion for the method is $p^2h^2 < 6$. After y_{n-1}, and y_n are given, $\bar{y}_{n+1}$ and $\bar{y}_{n+2}$ can be computed by the recurrence formulas, Eq. (9-4), now instead of carrying out the routine, the required y_{n+1} is computed by averaging y_n, $\bar{y}_{n+1}$, and $\bar{y}_{n+2}$, i.e.

$$y_{n+1} = \frac{1}{4} (\bar{y}_{n+2} + 2\,\bar{y}_{n+1} + y_n) \tag{9-}$$

Algebraically this is equivalent to a modification of Eq. (9-4) to the following difference equation

$$y_{n+1} - \frac{(4 - \alpha^2)(2 - \alpha^2)}{4} y_n + \frac{4 - \alpha^2}{4} y_{n-1} = 0 \qquad (9-6$$

In the case of a multi-degree-of-freedom system the time interval, h, may be fine enough to give accurate answer for lower modes but still not meet the instability requirement for higher modes, then the average procedure will not impair the accuracy for the lower modes but can postpone the starting of instability, as can be seen in the following derivation. The characteristic equation in this case is

$$\lambda^2 - \frac{(4 - \alpha^2)(2 - \alpha^2)}{4} \lambda + \frac{4 - \alpha^2}{4} = 0 \qquad (9-$$

If the λ's which cause the instability before average and after average are plotted against θ^2 in Fig. 1, it is apparent that the root which causes instability has its magnitude reduced; this implies a delay in starting instability. But for $\theta^2 > 20$ the root for average procedure becomes greater than that for Fox's method. Moreover, another unfortunate feature of this procedure is the fact that the amplitude is not conserved since

$$|\lambda|^2 = 1 - \frac{\alpha^2}{4}$$

IV ANALYSIS OF ERROR

10. Interpretation of Errors

It is not difficult to formulate a numerical method of integrati but to analyze the error involved in the method and its sequential effect is not a simple task. The present discussion of error is limited only to truncational error and its cumulative influences. The investigation of round-off error will not be considered because of its complicated and random nature, and its dependence on every detail of the particular computing process by which the method is carried out. A rigorous mathematica treatment of round-off is not impossible but usually it leads to an overestimated bound of error which is far on the safe side; yet, a statistical analysis of the error based on the random character of round-off, sometime would underestimate the actual error.

However, to analyze the influence of truncational error of a numerical method in a generalized way is by no means easy. Quite often the results are not reliable and devoid of physical significance. Numerou analyses of errors have been made, most of them are subject to two serious defects shown from the following points:

(1) analysis always tend to cover a great variety of integrand functions but lack the means of depicting its behavior, thus leave the class of functions described by few parameters or the bounds of the parameters. For instance, two problems with the same bound for the Lifschitz constant for their integrand functions may have radically different solutions even if the same numerical method is used to get the solutions.

Sometimes two different numerical procedures may have their performance compared with respect to a certain class of integrand functions, but the indications would be completely reversed if a problem, the integrand function of which belongs to the class, is solved numerically by the two methods.

(2) On account of the complicated form of the resulting difference equations the study of propagation of error is often reduced to a study of bound of error in each step. After simplifying a few undesirable terms of the expressions, this oversimplification may lead to unreliable and controversial results. In the course of this simplification it may happen that an accurate procedure gets a larger bound for its error than a less accurate method.

To avoid incorrect estimates of error the present investigation is made on a realistic basis. Instead of estimating error or its bound the intention is to find out how truncational error manifests itself in a different part of the numerical solution. Since the nature of the differential equations of problems of dynamic structural response is fairly well known, the numerical method is applied to such a standard problem that generalization of the characteristics of the numerical solution may be possible.

11. Manifestation of Truncational Error in Various Methods

Without loss of generality the standard problem is taken as

$$y'' + p^2 y = 0 \tag{11-1}$$

with initial condition y_o and v_o. Separate discussion will be given in th next chapter for cases involving viscous damping.

The Milne-Lotkin 6 order method, Eq. (3-10) is applied to the system

$$v' = -p^2 y$$
$$y' = v \qquad (11\text{-}2)$$

The replacement difference equations are

$$v_1 = v_o + \frac{h}{3}(-p^2 y_o - p^2 y_1) + \frac{h^2}{10}(-p^2 v_o + p^2 v_1) + \frac{h^3}{120}(p^4 y_o + p^4 y_1)$$

$$y_1 = y_o + \frac{h}{2}(v_o + v_1) + \frac{h^2}{10}(-p^2 y_o + p^2 y_1) + \frac{h^3}{120}(-p^4 v_o - p^4 y_1)$$

after the derivatives are eliminated by the following equations, obtained by differentiating Eqs. (11-1),

$$v'' = -p^2 y^1 = -p^2 v,$$
$$v''' = -p^2 v^1 = p^4 y,$$
$$y'' = v' = p^2 y,$$
$$y''' = -p^2 y^1 = -p^2 v, \quad \text{etc.}$$

Now let $ph = \theta$ and simplify the expression, then

$$y_1\left(1 - \frac{\theta^2}{10}\right) - v_1 h\left(\frac{1}{2} - \frac{\theta^2}{120}\right) = y_o\left(1 - \frac{\theta^2}{10}\right) + v_o h\left(\frac{1}{2} - \frac{\theta^2}{120}\right)$$

(11-3

$$y_1\theta^2\left(\frac{1}{2} - \frac{\theta^2}{120}\right) + v_1 h\left(1 - \frac{\theta^2}{10}\right) = -y_o\theta^2\left(\frac{1}{2} - \frac{\theta^2}{120}\right) + v_o h\left(1 - \frac{\theta^2}{10}\right)$$

From the results in the previous analysis it can be inferred directly that the solution of the difference equations is always sinusoidal and conserves the amplitude, the λ in the present case is

$$D\lambda = (1 - \frac{\theta^2}{10})^2 - \theta^2(\frac{1}{2} - \frac{\theta^2}{120}) \pm i\theta\ (1 - \frac{\theta^2}{10})(1 - \frac{\theta^2}{60}) \qquad \text{(11-}$$

where $D = (1 - \frac{\theta^2}{10})^2 + \theta^2\ (\frac{1}{2} - \frac{\theta^2}{120})^2$

After the initial conditions are incorporated into the solution, it is found that:

$$y_n = y_o \cos n\mu + \frac{v_o}{p} \sin n\mu$$

$$v_n = - y_o\ p \sin n\mu + v_o \cos n\mu \qquad \text{(11-}$$

where μ/h is defined as the pseudo-circular frequency of the response an

$$\mu = \tan^{-1} \frac{\theta(1 - \frac{\theta^2}{10})(1 - \frac{\theta^2}{60})}{(1 - \frac{\theta^2}{10})^2 - \theta^2(1 - \frac{\theta^2}{60})^2} \qquad \text{(11-}$$

Then the pseudo period of the solution can be computed, $T_s = 2\pi h/\mu$ and compared with the actual period, $T = 2\pi/p$. In Fig. 2. T_s/T is plotted against θ, the error is only 10 percent when an interval as long as the period of the system is used.

In the following a 6-order formula for double integration and a 6-order simple integration formula Eqs. (15) and (16) are applied to Eq. (11-1); the replacement difference equations are

$$y_1 = y_o + v_o h + \frac{h^2}{4}(-p^2 y_o - p^2 y_1) - \frac{h^3}{20}(p^2 v_o - p^2 v_1) + \frac{h^4}{240}(p^4 y_o + p^4 y_1)$$

$$v_1 = v_o + \frac{h}{2}(-p^2y_o - p^2y_1) - \frac{h^2}{10}(p^2v_o - p^2v_1) + \frac{h^3}{120}(p^4y_o + p^4y_1) \qquad (11-$$

From previous analysis (see page 24) the numerical solution is conditionally stable with a conserved amplitude. The λ's are given as

$$\lambda = \frac{1 - \frac{7\theta^2}{20} + \frac{\theta^4}{240} \pm i\theta\sqrt{(1 - \frac{\theta^2}{10})(1 - \frac{\theta^2}{60})}}{1 + \frac{3\theta^2}{20} - \frac{\theta^4}{240}} \qquad (11-$$

and $\mu = \tan^{-1}\dfrac{\theta\sqrt{(1 - \frac{\theta^2}{10})(1 - \frac{\theta^2}{60}}}{1 - \frac{7\theta^2}{20} + \frac{\theta^4}{240}}$

The complete solution is

$$y_n = y_o \cos n\mu + \frac{v_o}{p}\sqrt{\frac{1 - \frac{\theta^2}{10}}{1 - \frac{\theta^2}{60}}}\sin n\mu$$

$$v_n = y_o p\sqrt{\frac{1 - \frac{\theta^2}{60}}{1 - \frac{\theta^2}{10}}}\sin n\mu + v_o \cos n\mu \qquad (11-$$

In the plot of T_s/T, Fig. 2, this method gives consideratle error in freqnency, sometimes as high as 15 percent. Moreover, another annoying feature is the presence of factors $\sqrt{\frac{1 - \frac{\theta^2}{10}}{1 - \frac{\theta^2}{60}}}$ in the coefficient of v_o in expression for y_n; $\sqrt{\frac{1 - \frac{\theta^2}{60}}{1 - \frac{\theta^2}{10}}}$ in the coefficient of y_o in expression for v_n. Both factors should be unity but this can only be achieved by using $\theta \sim 0$.

By using a similar technique, the 4th order formulas for double and simple integration can be obtained, namely

$$\iint f\, d\tau^2 = \frac{h^2}{4}(f_o + f_1) + \frac{h^3}{24}(f'_o - f'_1) \qquad (11\text{-}$$

$$\int f\, d\tau = \frac{h}{2}(f_o + f_1) + \frac{h^2}{12}(f'_o - f'_1) \qquad (11\text{-}$$

This formulas have the advantage of saving the trouble of computing f'' but at the expense of introducing higher truncational error. If Eq. (11-: is used for the system of first order differential equations, Eq. (7-2), tl replacement difference equations are

$$y_1\left(1 - \frac{\theta^2}{12}\right) - \frac{v_1 h}{2} = y_o\left(1 - \frac{\theta^2}{12}\right) + \frac{v_o h}{2}$$

$$\frac{\theta^2}{2} y_1 + v_1 h\left(1 - \frac{\theta^2}{12}\right) = -\frac{\theta^2}{2} y_o + v_o h\left(1 - \frac{\theta^2}{12}\right) \qquad (11\text{-}$$

and the pseudo frequency μ/h is given as

$$\tan\mu = \frac{\theta\left(1 - \frac{\theta^2}{12}\right)}{1 - \frac{5\theta^2}{12} + \frac{\theta^4}{144}} \qquad (11\text{-}$$

the numerical solution is similar to that of the 6th order method except for a larger error in frequency, as can be seen in Fig. 2, where the error is about 40 percent if θ is taken as 6.

$$y_n = y_o \cos n\mu + \frac{v_o}{p}\sin n\mu$$

$$v_n = -y_o p \sin n\mu + v_o \cos n\mu \qquad (11\text{-}$$

If both Eqs. (11-10) and (11-11) are used to integrate Eq. (11-1

$$y'' = -p^2 y$$

to obtain y_1 and v_1, it is found the replacement difference equations take the following form

$$y_1 = y_o + v_o h - \frac{\theta^2}{4}(y_o + y_1) \, , \frac{\theta^2}{24}(-v_o h + v_1 h)$$

(11-1

$$v_1 h = v_o h - \frac{\theta^2}{2}(y_o + y_1) + \frac{\theta^2}{12}(-v_o h + v_1 h)$$

and the numerical solution is

$$y_n = y_o \cos n\mu + \frac{v_o}{p} \cdot \frac{\sin n\mu}{\sqrt{1 - \frac{\theta^2}{12}}}$$

(11-16

$$v_n = -y_o p \sqrt{1 - \frac{\theta^2}{12}} \sin n\mu + v_o \cos n\mu$$

with $\tan\mu = \dfrac{\theta\sqrt{1 - \frac{\theta^2}{12}}}{1 - \frac{\theta^2}{3}}$ if $\theta^2 < 12.$ (11-17

Again this time, erroneous factors $(1 - \frac{\theta^2}{12})^{-\frac{1}{2}}$ and $(1 - \frac{\theta^2}{12})^{\frac{1}{2}}$ are present in the equations.

The truncational error in a Runge-Kutta Fourth order method can be easily seen from the result of a stability analysis made in the last chapter, Eq. (7-7). It must be remarked that although there are several Runge-Kutta 4th order rules, depending upon the choice of numerical values for the parameters, that are satisfying equations (4-7), the numerical coefficients in the replacement difference equations, Eqs. (7-4), are independent of the numerical values of the parameters. It can be shown that the parameters appearing in the coefficients are so grouped that they take precisely the same functional forms as those in Eqs. (4-7). This leaves no choice of numerical values for the parameters on the basis of minimizing truncational errors.

$$y_n = |\lambda|^n \left(y_o \cos n\mu + \frac{v_o}{p} \sin n\mu\right)$$

(11-1

$$v_n = |\lambda|^n \left(-y_o p \sin n\mu + v_o \cos n\,\mu\right)$$

with
$$\mu = \tan^{-1} \frac{\theta\sqrt{1 - \frac{\theta^2}{6}}}{1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}}$$
(11-1

$$|\lambda|^2 = 1 - \frac{\theta^6}{72} + \frac{\theta^8}{578}$$

The numerical solution of Runge-Kutta method is sinusoidal; nevertheless, it does not conserve amplitude. In Fig. 3 the amplification factor is plotted against ph. It seems that in the usable range for θ , is $\theta < 2.8$. Beyond that, an undesirable divergent oscillation will result. The pseudo period, T_s/T, of Runge-Kutta solution is also plotted in Fig. 2 for comparison.

In the following, one of the Nystrom method Eq. (4-10) is demonstrated for

$$y'' = -p^2 y$$

After the k-functions have been successively eliminated the difference equations take a simple form

$$y_1 = y_o \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{600}\right) + v_o h \left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}\right) .$$

$$v_1 h = - \theta^2 y_o \left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}\right) + v_o h \left(1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}\right) \tag{11-20}$$

The eigen-value of the matrix, λ, in this case, is given as

$$\lambda = 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24} - \frac{\theta^6}{1200} \pm i\theta \sqrt{\left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}\right)^2 - \frac{\theta^{10}}{1200^2}} \tag{11-21}$$

This time the solution can simusoidal only when

$$1 - \frac{\theta^2}{6} + \frac{\theta^4}{120} > \frac{\theta^5}{1200} , \text{ i.e. } \theta < 2.9394 \tag{11-22}$$

The solution of the difference equations takes a complicated form

$$y_n = \lambda^n \left[y_o \left\{ \cos n\mu - \frac{\theta^5}{1200} \cdot \frac{\sin n\mu}{\sqrt{\left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}\right)^2 - \frac{\theta^{10}}{1200^2}}} \right\} + \frac{v_o}{p} \cdot \frac{1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}}{\sqrt{\left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}\right)^2 - \frac{\theta^{10}}{1200^2}}} \sin n\mu \right]$$

$$v_n = \lambda^n \left[- \frac{1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}}{\sqrt{\left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}\right)^2 - \frac{\theta^{10}}{1200^2}}} y_o p \sin n\mu + v_o \left\{ \cos n\mu - \frac{\theta^5}{1200} \cdot \frac{\sin n\mu}{\sqrt{\left(1 - \frac{\theta^2}{6} + \frac{\theta^4}{120}\right)^2 - \frac{\theta^{10}}{1200^2}}} \right\} \right] \tag{11-23}$$

The amplitude is longer conserved, besides, there are errors in the phase and the coefficients, which can only be minimized by using a small interval. So far as the errors in the amplitude and in the period are concerned the Nystrom method shows better performance than the Runge-Kutta Fourth Order Rule if the interval is kept within the stability limit.

From the foregoing analysis it can be seen that the truncational error can conceal itself in the frequency, or in the amplitude, or in both and it can also introduce an erroneous phase angle as well as an erroneous coefficient modifying the amplitude to some extent. However, it will be verified in the following that the coefficients can be kept free from errors by the using of the same formulas of simple integration for the system of two first order differential equations Eqs. (11-2).

$$\text{Let} \int_{t_o}^{t_o + h} f \, d\tau = A f_o + B f_1 \qquad (11\text{-}2$$

After applying the formula to Eq. (11-2) one obtains a system of two first difference equations, namely

$$\begin{aligned} y_1 - y_o &= A v_o + B v_1 \\ v_1 - v_o &= -p^2 (A y_o + B y_1) \end{aligned} \qquad (11\text{-}2$$

If y_n and $v_n h$ are assumed as

$$\begin{pmatrix} y_n \\ v_n h \end{pmatrix} = \lambda^n \begin{pmatrix} a \\ b \end{pmatrix} \qquad (11\text{-}2$$

then

$$\lambda_{1,2} = \frac{1 \pm i\, p\, A}{1 \mp i\, p\, B} \qquad (11\text{-}2$$

therefore the final solution is

$$\begin{aligned} y_n &= \lambda^n \left[y_o \cos n\mu + \frac{v_o}{p} \sin n\mu \right] \\ v_n &= \lambda^n \left[-y_o\, p \sin n\mu + v_o \cos n\mu \right] \end{aligned} \qquad (11\text{-}2$$

where

$$\tan \mu = \frac{p\,(A + B)}{1 - p^2 AB}$$

V. β METHOD

12. General Description

One of the most versatile methods of numerical integration is the β method, in which the value of β can be changed so that the method will give the best possible result under the particular condition. The formulas are simple to use and the numerical operations and sequence of operations are convenient for most computing facilities, namely

$$y_1 = y_o + v_o h + (\frac{1}{2} - \beta)\, f_o h^2 + \beta\, f_1 h^2$$

$$v_1 = v_o + \frac{h}{2}(f_1 + f_o) \tag{12-1}$$

where the differential equation to be integrated is of the type

$$y'' = f(f, v, y)$$

The properties and practicability of the method have been extensively explored [26,27]. When β is taken as 0, the method reduces to Levy's procedure; β = 1/4, it is the Timoshenko's modified method; β = 1/6, it implies the assumption that the variation of acceleration is linear within the interval; for β = 1/12, the method is equivalent to the Fox's method (or the Collatz method of Central Difference).

---

26. N.M. Newmark, "Computation of Dynamic Structural Response in the Range Approaching Failure", Proc. Symposium on Earthquake and Blast Effects on Structures, University of California, Los Angles, Cal., June, 1952. pp. 116-126.

27. N. M. Newmark and S. P. Chan "Comparison of Numerical Methods for Analyzing the Dynamic Response of Structures" Civil Engineering Studies, Structural Research Group, No. 36, 1952.

When the function f is taken as $-p^2y$, the replacement difference equation is found to be

$$y_{n+1} - (2 - \frac{\theta^2}{1 + \beta\theta^2})\, y_n + y_{n-1} = 0 \qquad \text{(12-}$$

and the numerical solution is

$$y_n = y_o \cos n\mu + \frac{v_o}{p} \frac{\sin n\mu}{\sqrt{1 - (\frac{1}{4} - \beta)\theta^2}} \qquad \text{(12-}$$

$$v_n = - y_o p \sqrt{1 - (\frac{1}{4} - \beta)\theta^2} \sin n\mu + v_o \cos n\mu$$

where

$$\tan \mu = \frac{\sqrt{\frac{\theta^2}{1 + \beta\theta^2} (4 - \frac{\theta^2}{1 + \beta\theta^2})}}{2 - \frac{\theta^2}{1 + \beta\theta^2}}$$

Therefrom it is claimed that the solution conserves amplitude but is only conditionally sinusoidal and subjected to the restriction

$$\theta^2 < \frac{4}{1 - 4\beta} \qquad \text{(12-}$$

Furthermore the velocity-response and the velocity, due to the initial displacement, y_o, are both in error by factors, $\left[1 - (\frac{1}{4} - \beta)\theta^2\right]^{-\frac{1}{2}}$ and $\left[1 - (\frac{1}{4} - \beta)\theta^2\right]^{\frac{1}{2}}$ respectively. If $\beta = 1/4$, these factors reduce to unity and the choice of time interval is no longer restricted by the criterion for instability. In this case, where instability is no longer to be feared, this leaves the truncational error in the frequency of the response. The $\beta = 1/4$ method has the largest error of all the β methods as can be seen in Fig. 4. Perhaps this is less objectionable in comparison to the errors in other parts of the solution, but in a multi-degree-of-freedom system if the solution is the super-position of several participating modes, then, the error in phase will yield an erroneous peak value.

13. A Modified $\beta = 1/4$ Method

It would be of practical value if the $\beta = 1/4$ method could be modified to have the error in frequency reduced without changing other parts of the solution. To explore this possibility it is suggested that the solution produced by the $\beta = 1/4$ procedure by reviewed, i.e.

$$y_n = y_o \cos n\mu + \frac{v_o}{p} \sin n\mu$$

$$v_n = -y_o \sin n\mu + v_o \cos n\mu \qquad (13-$$

with $\tan \mu = \dfrac{\theta}{1 - \dfrac{\theta^2}{2}}$

or $\tan \frac{\mu}{2} = \frac{\theta}{2}$ (13-

Since $\tan \frac{\theta}{2} = \frac{\theta}{2} + \frac{1}{3}(\frac{\theta}{2})^3 + \ldots \cong \frac{1}{2}(\theta + \frac{\theta^3}{11})$ for $\theta \sim 1$ (13-

it is inferred that, instead of using p in the given differential equatio $q = p\ (1+\frac{\theta^2}{11})$ is used in all numerical operations. This reduces the truncational error in the frequency and the modified frequency φ/h, is given as

$$\tan \frac{\varphi}{2} = \frac{qh}{2} \qquad (13-4$$

Numerical values for T_s/T are plotted against ph in Fig. 4. Considerable improvement can be seen within the stable zone for other β procedures whi higher modes are kept from growing wild. In doing so, an error is introduced into terms involving velocity since q is in the place where p was, i

$$y_n = y_o \cos n\varphi + \frac{\bar{v}_o}{q} \sin n\varphi$$

$$\bar{v}_n = -y_o\, q \sin n\varphi + \bar{v}_o \cos n\varphi \qquad (13-5$$

However this can be remedied by modifying the velocity by a factor q/p, in other words v_o has to be multiplied by q/p before it enters in the numerical operations, the relation between the pseudo velocity, $\bar{v}$, and the actual velocity, v, is

$$\bar{v} = v\ (q/p) \tag{13-6}$$

Upon substituting the above expression for $\bar{v}$ in eqs. (13-5) one can obtain the right result.

A further improved result can be obtained if higher order terms in the series expansion of $\tan \frac{\theta}{2}$ are added.

All of this seems quite obvious, but this can also be done for a multi-degree of freedom system without knowing the actual values of frequencies of system. Without loss of generality the equations of equilibrium can be assumed as

$$\ddot{y}_i + \sum_j P_{ij}\, y_j = 0 \qquad i,j = 1, 2 .. N. \tag{13-7}$$

From theory of matrices, if P_{ij} has p_n^2, $n = 1 \ldots N$, as its eigen-values, then, Q_{ij}

$$Q_{ij} = P_{ij} + \frac{2h^2}{11} P_{ij}^2 + \frac{h^4}{121} P_{ij}^3 \tag{13-8}$$

will have q_n, $n = 1 \ldots N$, as the eigen-values, since

$$q^2 = p^2\ (1 + \frac{p^2h^2}{11})^2 \tag{13-9}$$

Therefore P_{ij} is replaced by Q_{ij} in actual numerical operations. Similarly the factor q/p is equivalent to a matrix A_{ij} with

$$A_{ij} = \delta_{ij} + \frac{h^2}{11} P_{ij}$$

where δ_{ij} is the unit matrix, or the Kronecker's delta. The pseudo velocity vector $\bar{v}_i$ is them obtained by

$$\bar{v}_i = \sum_j A_{ij}\, v_j \qquad i,j = 1 \ldots N \tag{13-10}$$

This modified $\beta = 1/4$ procedure is certainly not convenient to use in case of time-dependent or non-linear elements in the matrix because of the necessary modifications to be made in the matrix, in each step of integration. For problems with insignificant higher modes this modified method may lead to a correct solution, without introducing serious error in peak values.

14. β Method as Applied to System with Viscous Damping

In an actual vibration system damping is always present but only to some extent. Therefore, it is imperative to study different integration methods for a damped system and to examine the respective merits of the various methods. In the following, a viscous damping force is added to the test differential equation and only the β methods are used to study the result.

Let the test equation be

$$y'' + 2rpy' + p^2 y = 0 \tag{14-1}$$

the exact solution is

$$y\ (t) = e^{-rpt}(y_o \cos qt + \frac{v_o + p\ y_o}{q} \sin qt) \tag{14-2}$$

where $q^2 = p^2(1-r^2)$

The replacement difference equation is

$$y_{n+1}\ (1 + c) - y_n(2 - \alpha^2) + y_{n-1}(1 - c) = 0 \tag{14-3}$$

with $\quad c = \dfrac{rph}{1 + \beta p^2 h^2} \qquad$ and $\qquad \alpha^2 = \dfrac{p^2 h^2}{1 + \beta p^2 h^2}$

The characteristic equation is therefore

$$\lambda^2 (1 + c) - (2 - \alpha^2) \lambda + (1 - c) = 0 \tag{14-4}$$

and

$$\lambda = \frac{2 - \alpha^2 \pm \sqrt{(2 - \alpha^2)^2 - 4 (1 - c)^2}}{2 (1 + c)}$$

The truncational error of the β method creeps into the criterion for critical damping of the system, which is

$$4 c^2 - 4 \alpha^2 + \alpha^4 = 0$$

more precisely, $4 r^2 p^2 h^2 - 4 p^2 h^2 (1 + \beta p^2 h^2) + p^4 h^4 = 0$ (14-5

The system is supposed to be in a state of critical damping if $r = 1$, but now this criterion is dependent upon the value of β and the time interval used in the analysis. It can be seen that the criterion coincides with the exact one by taking $\beta = 1/4$. Again this suggests a speculation that if the same simple integration formula is applied to

$$v' = -\delta v - p^2 y \tag{14-6}$$

$$y' = v$$

the truncational error of the formula will not enter into the condition for critical camping.

Let the integration formula be

$$\int_{t_o}^{t_o + h} f \, d\tau = A f_o + B f_1 \tag{14-7}$$

Having carried out the substitution one obtains the following replacement difference equations

$$v_1 - v_o = - \delta (A v_1 + B v_o) - p^2 (A y_1 + B y_o) \tag{14-8}$$

$$y_1 - y_o = A v_1 + B v_o$$

and λ-determinant

$$\begin{vmatrix} 1 - \lambda & B + A\lambda \\ -p^2(B + A\lambda) & 1 - \lambda - \delta\,(B + A\,\lambda) \end{vmatrix} = 0 \qquad (14-$$

Therefore the characteristic equation is

$$(1 - \lambda)^2 - \delta\,(B + A\lambda)(1 - \lambda) + p^2(B + A\lambda)^2 = 0$$

hence

$$\frac{1 - \lambda}{B + A\lambda} = \frac{\delta \pm \sqrt{\delta^2 - 4p^2}}{2} \qquad (14-1$$

The solution begins to behave aperiodically when

$$\delta^2 - 4\,p^2 > 0 \qquad (14-1$$

Hence the critical damping of the system is 2p.

Now if

$$(2 - \alpha^2) - 4\,(1 - c^2) < 0$$

The numerical periodic solution of the difference equation then can be compared with the exact one in two respects, namely the pseudo frequency and pseudo attentuation factor. The exact solution is

$$y = e^{-rpt}(y_o \cos qt + \frac{v_o + rp\,y_o}{q} \sin q\,t)$$

$$(14-1$$

$$v = e^{-rpt}(v_o \cos qt - \frac{v_o rp + p^2 y_o}{q} \sin qt)$$

while the numerical solution takes following form

$$y_n = R^n \left[y_o \cos n\varphi + \frac{v_o\{1 - r^2\theta^2(1 - 4\beta)\} + y_o rp\{1 - \frac{\theta^2}{2}(1-4\beta)\}}{p\sqrt{1 - (\frac{1}{4} - \beta)\theta^2 - r^2}} \sin n\varphi \right]$$

(14-13)

$$v_n = R^n \left[v_o \cos n\varphi - \frac{v_o rp\{1-(1-4\beta)\frac{\theta^2}{2}\} + y_o p^2\{1- \frac{\theta^2}{4}(1-4\beta)\}}{p\sqrt{1 - (\frac{1}{4} - \beta)\theta^2 - r^2}} \sin n\varphi \right]$$

with $\tan \varphi = \dfrac{\sqrt{1 - c^2 - (1 - \frac{\alpha^2}{2})^2}}{1 - \frac{\alpha^2}{2}}$

and $R = \sqrt{\dfrac{1 - c}{1 + c}}$.

It is of interest to note that the truncational error of the method enters into different parts of the numerical solution, besides the errors in frequency and attenuation factor. Errors also appear in the coefficients as happened in the case of undamped system. However if β is taken as 1/4, these errors disappear, and the numerical solution takes exactly the same form as the exact solution except the errors in frequency and attenuation factor.

To compare the pseudo frequency of the response with respect to the exact value, values of T_s/T $(=qh/\varphi)$ are plotted against the different values of ph in Fig. 5, for β = 0, 1/12, 1/8, 1/6, and 1/4. Except for β = 1/4, the validity of the curves is limited by the aperiodic boundary beyond which aperiodic response will result despite the fact that physically the response should behave periodically.

In Fig. 6, the ratio between the actual attenuation factor and that of the numerical solution is plotted against different values of ph, for $\beta = 1/4$, the curves show the numerical solutions are all under-damped; for $\beta = 0$, the numerical solutions are all over-damped; for the rest value of β, solution for $r = 3/4$ show over-damped trend; $r = 1/2$, $1/4$, under-damped.

So far as the aperiodic response is concerned the performance of β method can not be seen clearly just by comparing a few quantities. Since the exact solution behaves monotonically decreasing as time is increasing it is imperative to have λ taking positive values less than one, where

$$\lambda = \frac{2 - \alpha^2 \pm \sqrt{(2 - \alpha^2)^2 - 4\,(1 - c^2)}}{a\,(1 + c)} \tag{14-14}$$

To achieve that $|\lambda| < 1$, it is found that

$$\left| 2 - \alpha^2 \pm \sqrt{(2 - \alpha^2)^2 - 4\,(1 - c^2)} \right| < 2\,(1 + c)$$

It is sufficient to take the negative sign, thus

$$\sqrt{(2 - \alpha^2)^2 - 4\,(1 - c^2)} + \alpha^2 - 2 < 2\,(1 + c)$$

It follows immediately that $\alpha^2 < 4$, after a few algebraic operations. If the positive sign is taken it is found that $\alpha^2 > 0$ which is automatically satisfied for any choice of ph and β.

The above condition limits the numerical aperiodic solution from growing divergently either monotonically or in an oscillatory manner. To make sure that λ be positive it can be proved that $\alpha^2 < 2$ is a sufficien criterion, that is

$$\theta^2 < \frac{2}{1 - z\beta}$$

Thus it can be concluded that if β and ph are so chosen that α^2 is larger than 2, the solution may become oscillatory but convergent if α^2 is larger than 4, the oscillation becomes divergent.

The performance of the β method is best illustrated in the following examples in which different values for β, ph, and r are chosen to cover a wide range, for the sake of comparison. The differential equation is taken as

$$y'' + 4ry' + 4y = 0$$

with initial conditions: $y_o = 1$, $v_o = 1$. The damping coefficient, r, takes on values 1, 1.5, and 2; β takes on 0, 1/12, 1/8, 1/6, and 1/4. Numerical analysis is made by using h taking on 1/4, 1/2, and 1. Numerical results are plotted in Fig. 7, and compared with the exact solutions. It seems the comparison gives favorable consideration to $\beta = 1/4$, and 1/6 procedures provided the criterion $\alpha^2 < 2$ is satisfied. Otherwise undesirab oscillation would result.

15. β Method as Applied to System with Negative Spring Constant

In studying dynamic response of structures realistically, certai secondary influences which were usually neglected in the classical approac because of their insignificance and their hindrance to elegant mathematica methods, may have to be included, especially when the structure is to be analyzed in the range approaching failure. For example, besides non-linea behavior of structural members and joints, a secondary effect of excessive deflection and axial thrust may gain practical significance in the analysi This raises the possibility that certain components of the structure might behave as if the restoring force of the component were not tending to obstruct the motion but to aggravate the motion. This situation may be idealized as one in which a negative spring constant has been inserted for the component.

To investigate the performance of β method for problems of this nature the test differential equation is simply set up as

$$y'' - k^2 y = 0 \tag{15-1}$$

its exact solution is

$$\begin{aligned} y &= y_o \cosh kt + \frac{v_o}{k} \sinh kt \\ v &= v_o \cosh kt + y_o k \sinh kt \end{aligned} \qquad (15\text{-}2$$

The replacement difference equation is then

$$y_{n+1} - (2 + \gamma)\, y_n + y_{n-1} = 0 \qquad (15\text{-}3$$

with $\gamma = \dfrac{\overline{kh}^2}{1 - \beta \overline{kh}^2}$

Since a monotone solution is expected the roots should be real and positive, where λ is now given as

$$\lambda_{1,2} = \frac{2 + \gamma \pm \sqrt{(2 + \gamma^2) - 4}}{2} \qquad (15\text{-}4$$

This can be easily achieved by maintaining $\gamma > 0$, which means

$$(kh)^2 < \frac{1}{\beta} \qquad (15\text{-}5$$

as the criterion for a non-oscillatory numerical solution.

However this does not insure that the numerical solution is always increasing in magnitude as it should be expected; it may become monotonically decreasing if $|\lambda|$ is less than one.

The complete solution to the difference equation is

$$y_n = y_o \cosh (n \ln\lambda) + \frac{v_o}{k\sqrt{1 + (\frac{1}{4} - \beta)\,\overline{kh}^2}} \sinh (n \ln\lambda)$$

$$v_n = y_o k \sqrt{1 + (\frac{1}{4} - \beta)\overline{kh}^2}\ \sinh (n \ln\lambda) + v_o \cosh (n \ln \lambda) \qquad (15\text{-}6)$$

where $\lambda = \dfrac{2 + \gamma + \sqrt{\gamma(4 + \gamma)}}{2}$

In order to compare the over-all performance of the different solutions, values of $\ln \lambda/kh$ is plotted in Fig. 8 against different values of kh. This ratio, $\ln\lambda/kh$, is called the growth factor which gives an indicatior of how fast the numerical solution is growing in comparison with the exaci rate. It is interesting to note that $\beta = 0$ is free from the restriction for an oscillisory solution and its numerical solution grows more slowly than the exact solution. For other values of β, namely 1/4, 1/6, 1/8, and 1/12, the numerical solutions are always growing with a faster rate and erroneous solutions will result if the time interval used is approaching the critical limit, $1/k\sqrt{\beta}$.

So far as the growth ratio is concerned a best choice of β is to make $\ln \lambda = kh$, that is

$$1 + \frac{\gamma}{2} + \sqrt{(1 + \frac{\gamma}{2})^2 - 1} = e^{kh} \qquad (15\text{-}7)$$

After a few algebraic operations one arrives at the following expression for β which will preserve the growth rate for a given kh.

$$\beta = \frac{(e^{kh} - 1)^2 - (kh)^2 e^{kh}}{(kh)^2 (e^{kh} - 1)^2} \tag{15-8}$$

Fig. 9 shows the general trend of β as a function of kh. As kh takes on very small values the asymptotic value of β is 1/12.

CONCLUSIONS

As a result of the present study the β method is suggested for obtaining solutions to problems of dynamic structural response because of its simplicity and flexibility in various applications. Among them the $\beta = 1/4$ procedure is recommended for its being not restricted by instability for undamped vibration and possessing correct critical damping in damped system. This is important for analyzing a system with a large number of degree-of freedom; but for aperiodically damped systems and systems with negative spring constant the use of $\beta = 1/4$ would encounter limitations on the choice of time interval.

For a system having only a few number of degree-of-freedom bett accuracy may be obtained by introducing the derivatives in the formulatio as has been done in the Milne-Lotkin Methods. The methods share all the advantages of the $\beta = 1/4$ method and improve their accuracy by taking in the derivative if the computations of the derivatives do not entail laborious work. However, it must be noted that sometimes a cut-down of time interval would give better accuracy and need less work than the innovation made by introducing derivatives.

In case the problems are so proposed that the iterative procedu is highly undesirable, one has to resort to methods like the Runge-Kutta Method, the Nystrom Method, or the BlässMethod. All these methods have t annoying feature of not conserving amplitude of the motion; on that accou it seems that the $\beta = 0$ procedure is preferable except that the time inte val has to be chosen within the stable limit.

To perfect a numerical procedure by means of higher order differ ences is a well-known fact, but the contribution of higher order differenc can become beneficial only when the used time interval is within certain limits. Due to the presence of extraneous solutions, produced by the high degree of the characteristic polynomial by the introducing of higher order differences, the regular introducing of higher order differences is not recommended provided a compatible time interval is used.

If the functional form of the differential equations does not change during the course of analysis and if a highly accurate result is expected, the iterative integration method, namely Fox, Lowdin's methods, is suggested for using. The method can eliminate the truncational error t the desired significant figures.

The investigation of instability of numerical solution with respect to an undamped system is made under two different requirements; namely, sinuosity of the solution and conservation of amplitude. It is found that a symmetric use of the same simple integration formula for the two first order differential equations always leads to a sinusoidal soluti but the conservation of amplitude is achieved only in some methods, for instance, Milne-Lotkin methods and β methods.

For the case of damped system there exists a criterion similar to that for instability in undamped system, namely, criterion for critica damping. If the interval, h, is not properly chosen, an under-damped system may have a numerical solution showing over-damped response. If β is taken as 1/4, this criterion is automatically satisfied.

If the given system will show aperiodic response, the analysis of β method finds that the choice of time interval and the parameter β

has to comply with the following criterion

$$(ph)^2 < \frac{2}{1 - 2\beta}$$

in order to remove any oscillation that may result in the numerical solution.

If a system happens to have a negative spring constant, the choice of time interval is limited by

$$h < 1/k\sqrt{\beta}$$

It seems the $\beta = 0$ procedure would be free from this restriction, but the use of $\beta = 1/4$ procedure can eliminate some truncational errors appearing in the coefficients as usually happened in the damped and undamped cases. But in Figs. 8 and 9, it seems that a best choice of β will take value between 0 and 1/12, which is also depending on the given value on k and h.

APPENDIX A

If y_n and v_n are connected to y_{n-1} by the following relation

$$\begin{pmatrix} y_n \\ v_n h \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{matrix} y_{n-1} \\ y_{n-1} h \end{matrix} \qquad \text{(A-1}$$

and if y_n and v_n are supposed to exhibit an oscillatory motion then the motion will be of amplitude-conserving type if the determinant of the matrix is equal to unity.

Let
$$\begin{pmatrix} y_n \\ v_n h \end{pmatrix} = \lambda^n \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \qquad \text{(A-2}$$

therefore
$$\begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \qquad \text{(A-3}$$

or
$$ad - (d + a)\lambda + \lambda^2 - b\,c = 0$$

$$\lambda = \frac{a + d \pm \sqrt{(a + d)^2 - 4\,(ad - bc)}}{2}$$

$$= \frac{a + d \pm i\sqrt{4\,(ad - bc) - (a + d)^2}}{2}$$

Since y and v represent deflection and velocity of a sinusoidal motion, λ must be a complex number. Now if $|\lambda| = 1$ then $ad - bc = 1$. Therefore the statement is proved.

APPENDIX B

In this appendix it is proved that the double integration formula, $\int_{t_o}^{t_o+h}\int d\tau^2\, f(\tau)$ and the simple integration formula, $\int_{t_o}^{t_o+h} d\tau\, f(\tau)$ of function f are in a simple relation, namely,

$$\iint d\tau^2 f(\tau) = \frac{h}{2}\int d\tau\, f(\tau) \qquad \text{(B-1)}$$

where the two integration formulas are constructed by the Milne-Lotkin Method and involving the same terms. In constructing the integration formulas it has been found that the following "null" function, g_n are quite useful, in eliminating superfluous terms.

$$g_i = f_o^{(i)} - f_1^{(i)} + \frac{h}{2}\left(f_o^{(i+1)} + f_o^{(i+1)}\right) + \frac{h^2}{2\cdot 2!}\left(f_o^{(i+1)} - f_1^{(i+1)}\right) + \ldots$$

$$i = 1, 3, 5 \qquad \text{(B-2)}$$

see Eqs. (3-4) to (3-9). Therefore, to prove the fore-mentioned statement it is sufficient to prove the following equivalent arguement that

$$\int_{t_o}^{t_o+h}\int d\tau^2\, f(\tau) - \frac{h}{2}\int_{t_o}^{t_o+h} d\tau\, f(\tau) = \sum_{i=1,3,5\ldots} c_i g_i \qquad \text{(B-3)}$$

where $c_i \neq 0$ for some i, and $\int d\tau f$ and $\iint d\tau^2 f$ are in their original unrefined forms, namely,

$$\int d\tau\, f = \frac{h}{2}\left[f_o + f_1 + \frac{h^2}{2}(f_o' - f_1') + \frac{h^3}{3!}(f_o'' + f_1'') + \ldots\right]$$

and $$\iint d\tau^2 f = \frac{h}{2}\left[\frac{h^2}{2!}(f_o + f_1) + \frac{h^3}{3!}(f_o' - f_1') + \frac{h^4}{4!}(f_o'' + f_1'') + \ldots\right]$$

Now

$$\frac{h}{2}\int d\tau f - \iint d\tau^2 f = \frac{h^3}{24}(f_o' - f_1') + \frac{h^4}{48}(f_o'' + f_1'') + \frac{h^5}{160}(f_o''' - f_1''') + \ldots$$

$$\ldots + \frac{n-2}{4 \cdot n!} h^{n+1}\left[f_o^{(n-1)} + (-)^{n-1} f_1^{(n-1)}\right] \qquad \text{(B-4)}$$

After Eqs. (B-2) are substituted for g_i's in Eq. (B-3) and the coefficient of h of the same power on both right-hand and left-hand sides are equated, two sets of linear algebraic equations for c_i are obtained, depending on odd or even power of h, as follows,

$$\begin{aligned}
c_1 &= \frac{1}{4}\left(\frac{1}{2!} - \frac{2}{3!}\right) \\
\frac{1}{2}\cdot\frac{1}{2} c_1 + c_2 &= \frac{1}{4}\left(\frac{1}{4!} - \frac{2}{5!}\right) \\
\frac{1}{2}\cdot\frac{1}{4!} c_1 + \frac{1}{2}\cdot\frac{1}{2!} c_2 + c_3 &= \frac{1}{4}\left(\frac{1}{6!} - \frac{2}{7!}\right) \qquad \text{(B-} \\
\ldots \qquad \ldots \qquad &\qquad \ldots
\end{aligned}$$

and

$$\begin{aligned}
c_1 &= \frac{1}{4}\left(\frac{2}{3!} - \frac{4}{4!}\right) \\
\frac{1}{3!} c_1 + c_2 &= \frac{1}{4}\left(\frac{2}{5!} - \frac{4}{6!}\right) \\
\frac{1}{5!} c_1 + \frac{1}{3!} c_2 + c_3 &= \frac{1}{4}\left(\frac{2}{7!} - \frac{4}{8!}\right) \qquad \text{(B-6} \\
\ldots \qquad \ldots \qquad &\qquad \ldots
\end{aligned}$$

It can be seen that the first few values of c_i, obtained from the first set of equations, satisfies the second set, and vice versa. However the existence of c_i needs the proof that these two sets of equations share the same unique solution.

In order that the arguement will not be confused by the presence of the same notation in both sets of equations, it is suggested to use a_i for c_i in the first set and b_i for c_i in the second set, then proceed to prove that

$$a_i = b_i$$

for all values of i.

If the first equation of Eqs. (B-5) remains unmodified; the second one is multiplied by $-x^2$, where x is an arbitrary number; and the third one is multiplied by $+x^4$; the fourth, by $-x^6$, and so forth; the operation proceeds according to this rule, then sum all the equations, the following equation is obtained

$$(1 - \frac{1}{2}\cdot\frac{1}{2!}\ x^2 + \frac{1}{2}\cdot\frac{x^4}{4!} - \frac{1}{2}\circ\frac{x^6}{6!} + \ldots\ldots)(a_1 - a_2 x^2 + a_3 x^4 - a_4 x^6 + \ \ldots)$$

$$= \frac{1}{4}\ (\frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!} - \ldots) - \frac{1}{2}\ (\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \ldots\ldots)$$

after transposing and simplifying, one arrives at a clear expression,

$$a_1 - a_2\, x^2 + a_3\, x^4 - \ldots = \frac{2 \sin x - x\ (1 + \cos x)}{2\ x^3\ (\ + \cos x)} \qquad \text{(B-}$$

Similarly, if the first equation of Eq. (B-6) is multiplied by x; the second, by $-x^3$; the third by $+x^5$; the fourth, by $-x^7$, and so forth; the summation of all these modified equations yields,

$$(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots)\ (b_1 - b_2\, x^2 + b_3\, x^4 - b_4\, x^6 + \ldots)$$

$$= \frac{1}{4}\ (\frac{x}{3!} - \frac{x^3}{5!} + \frac{x^5}{7!} - \frac{x^7}{9!} + \ldots) - \frac{1}{2}\ (\frac{x}{4!} - \frac{x^3}{6!} + \frac{x^5}{8!} - \frac{x^7}{10!} + \ldots)$$

which reduces to

$$b_1 - b_2x^2 + b_3x^4 - b_4x^6 + \ldots = \frac{2\ (1 - \cos x) - x \sin x}{2\ x^3 \sin x} \qquad \text{(B-8}$$

Since the right hand side of both Eqs. (B-7) and (B-5) are identical, the following is, then, true for any value of x,

$$a_1 - a_2x^2 + a_3x^4 - a_4x^6 + \ldots = b_1 - b_2x^2 + b_3x^4 - b_4x^6 + \ldots$$

Hence

$$a_i = b_i$$

for all values of i.

TABLE I TIMOSHENKO MODIFIED METHOD

$$2.5625y_1 = -1.125y_o - 2.5625y_{-1} + E$$

$$E = \left(-\frac{\delta^4}{6} + \frac{7\delta^6}{720} - \frac{\delta^8}{1008}\right) y_o$$

t_0	y_0	δ	δ^2_0	δ^3	δ^4_0	δ^5	δ^6_0	δ^7	δ^8_0	E/2.5625	y
		+.5984		-1.4596		+3.5602		- 8.6834			
2.5	.5984		-1.4596		+3.5602		-8.6834		+21.1791	-.2727	.5984
		-.8612		+2.1005		-5.1232		+12.4957			
5.0	-.2627		+ .6408		-1.5630		+3.8122		- 9.2981	+.1197	-.5354
		-.2203		+ .5375		-1.3109		+ 3.1975			
7.5	-.4831		+1.1783		-2.8740		+7.0097		-17.0970	+.2201	-.2436
		+.9579		-2.3365		+5.6987		-13.8994			
10.0	+.4748		-1.1581		+2.8247		-6.8897		+16.8041	-.2163	+.8625
		-.2001		+ .4882		-1.1909		+ 2.9046			
12.5	+.2746		- .6698		+1.6338		-3.9850		+ 9.7195	-.1251	-.3513
		-.8700		+2.1221		-5.1759		+12.6242			
15.0	-.5954		+1.4522		-3.5420		+8.6392		-21.0713	+.2713	-.8334
		+.5821		-1.4199		+3.4632		- 8.4470			
17.5	-.0132		+ .0323		- .0787		+ .1921		- .4687	+.0060	+.9886
		+.6144		-1.4987		+3.6554		- 8.9157			
20.0	+.6012		-1.4664		+3.5766		-8.7236		+21.2770	-.2739	+.4054
		-.8519		+2.0779		-5.0681		+12.3612			
22.5	-.2507		+ .6114		-1.4914		+3.6376		- 8.8724	+.1142	-.1365
		-.2404		+ .5864		-1.4304		+ 3.4888			
25.0	-.4911		+1.1979		-2.9218		+7.1265				
		+.9575		-2.3354		+5.6961					
27.5	+.4663		-1.1374		+2.7742						
		-.1799		+ .4388							
30.0	+.2864		- .6986								
		-.8785									
32.5	-.5921										

TABLE II FOX'S METHOD

$$y_1 = -2.109589056y_o - y_1 + E/1.52083$$

$$E = (-\frac{\delta^6}{240} + \frac{13\delta^8}{15120})\, y_o$$

t_0	y_0	δ	δ^2_0	δ^3	δ^4_0	δ^5	δ^6_0	δ^7	δ^8_0	E	y_0	exact
		.598		- 2.460		+ 10.111		- 41.556				
2.5	.598		- 2.460		+ 10.111		- 41.556		+ 170.789	+ .210	.598	
		- 1.861		+ 7.651		- 31.445		+129.233				
5.0	- 1.263		+ 5.191		- 21.334		+ 87.676		- 360.319	- .443	-1.052	-.958
		+ 3.329		-13.682		+ 56.230		-231.086				
7.5	+ 2.066		- 8.491		+ 34.896		-143.410		+ 589.356	+ .726	+1.178	+.937
		- 5.162		+21.213		- 87.179		+358.269				
10.0	- 3.095		+12.722		- 52.282		+214.859		- 882.983	-1.087	- .707	-.544
		+ 7.559		-31.068		+127.680		-524.713				
12.5	+ 4.464		-18.346		+ 75.397		-309.853		+1273.371	+1.568	- .774	-.066
		-10.786		+44.328		-182.173		+748.658				
15.0	- 6.322		+25.982		-106.775		+438.805		-1803.310	-2.221	+3.909	+.650
		+15.195		-62.447		+256.631		-1054.651				
17.5	+ 8.873		-36.464		+149.856		-615.846		+2530.876	+3.118	-9.288	-.975
		-21.269		+87.408		-359.214		+1476.225				
20.0	-12.396		+50.943		-209.358		+860.378					
		+29.674		+121.949		+501.163						
22.5	+17.278		-71.005		+291.804							
		-41.331		+169.855								
25.0	-24.053		+98.849									
		+57.517										
27.5	+33.464											

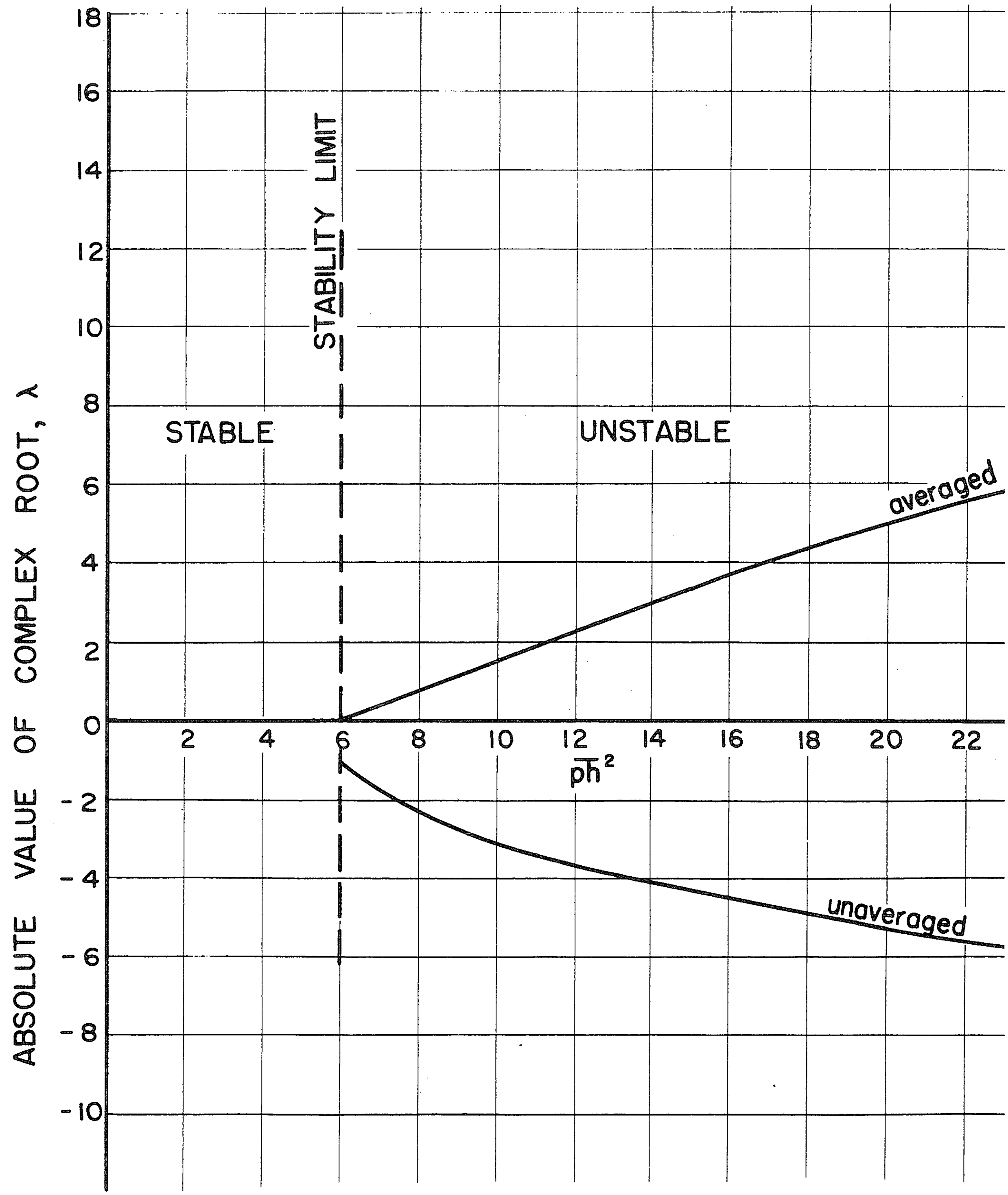

FIG. 1 COMPARISON OF $|\lambda|$ BEFORE AND AFTER AVERAGING

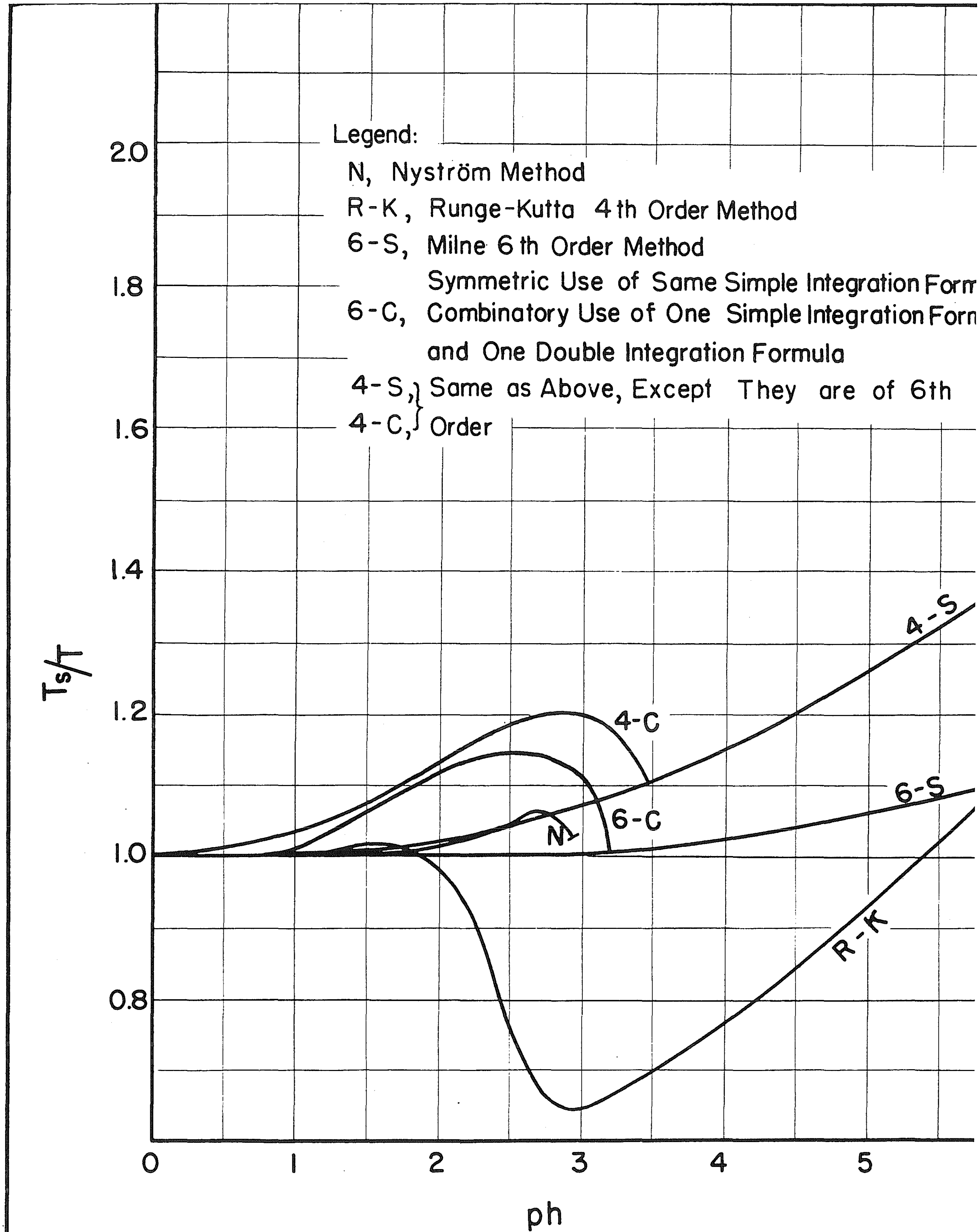

FIG. 2 COMPARISON OF PSEUDO-PERIODS IN MILNE, RUNGE-KUTTA, AND NYSTRÖM METHODS

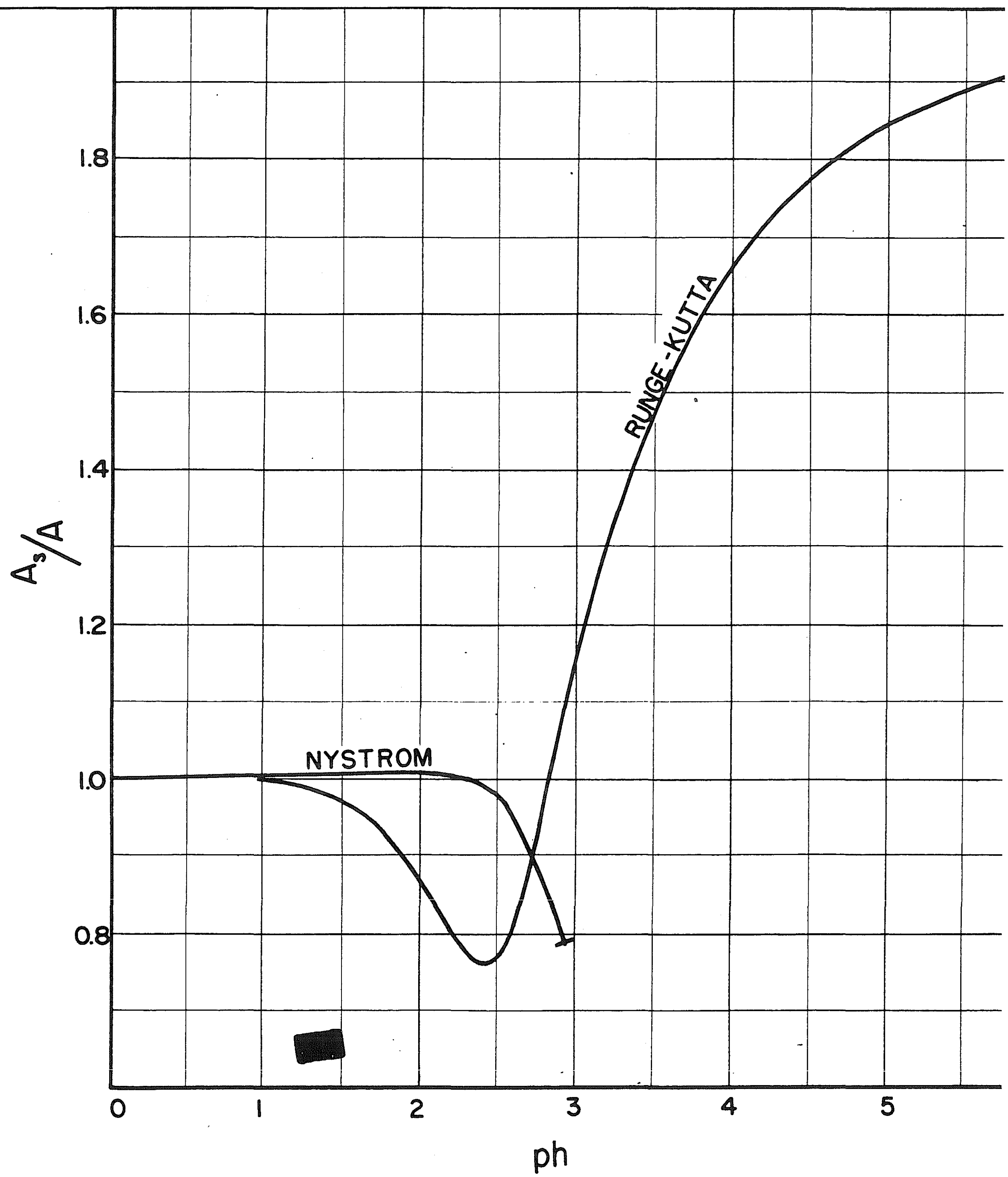

FIG. 3 COMPARISON OF AMPLIFICATION FACTORS COMPUTED BY NYSTROM AND RUNGE-KUTTA METHODS

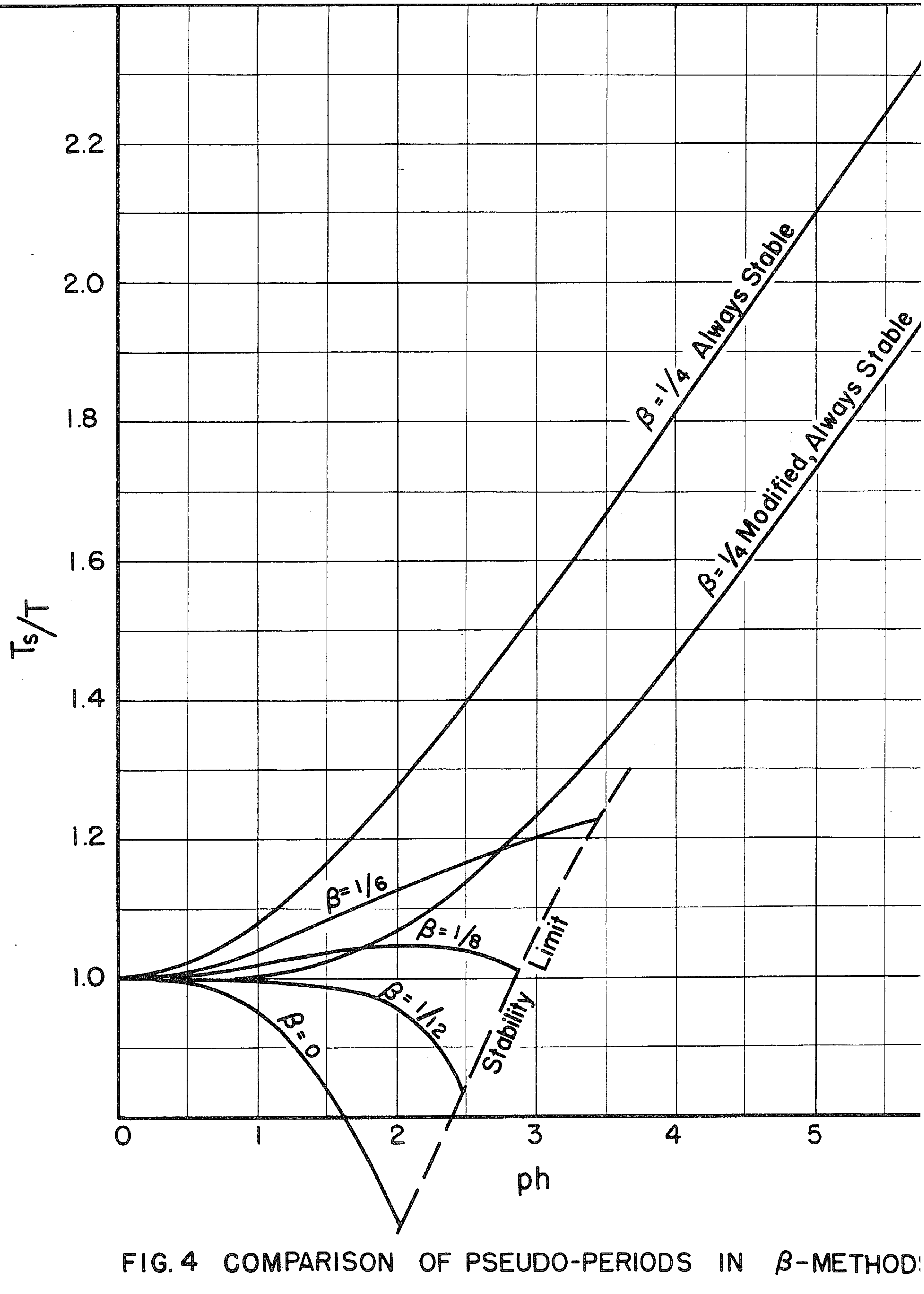

FIG. 4 COMPARISON OF PSEUDO-PERIODS IN β-METHODS

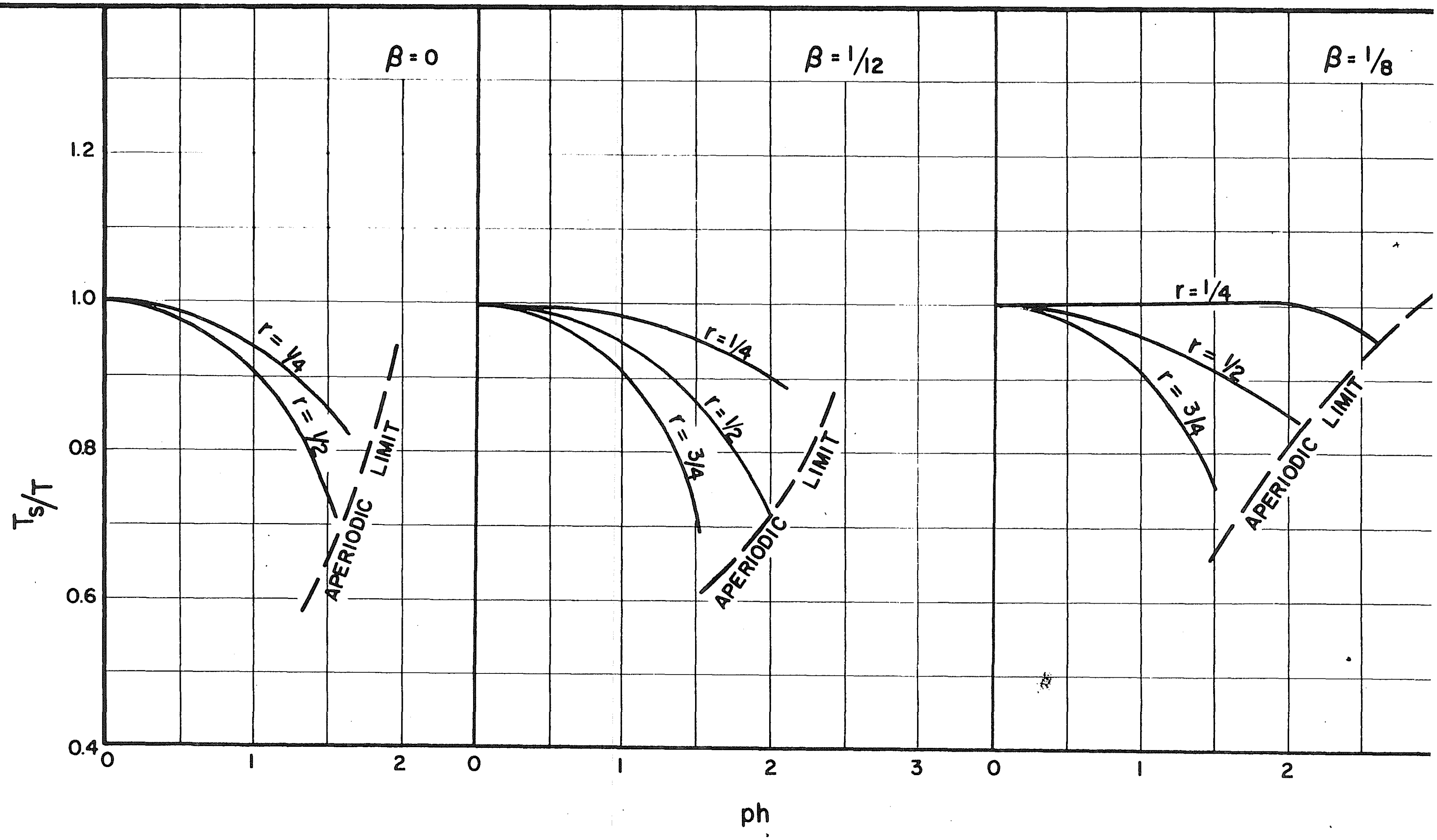

FIG. 5a COMPARISON OF PSEUDO PERIOD OF A VISCOUSLY DAMPED SYSTEM, ANALYZED BY VARIOUS β METHODS, (β = 0, 1/12, 1/8)

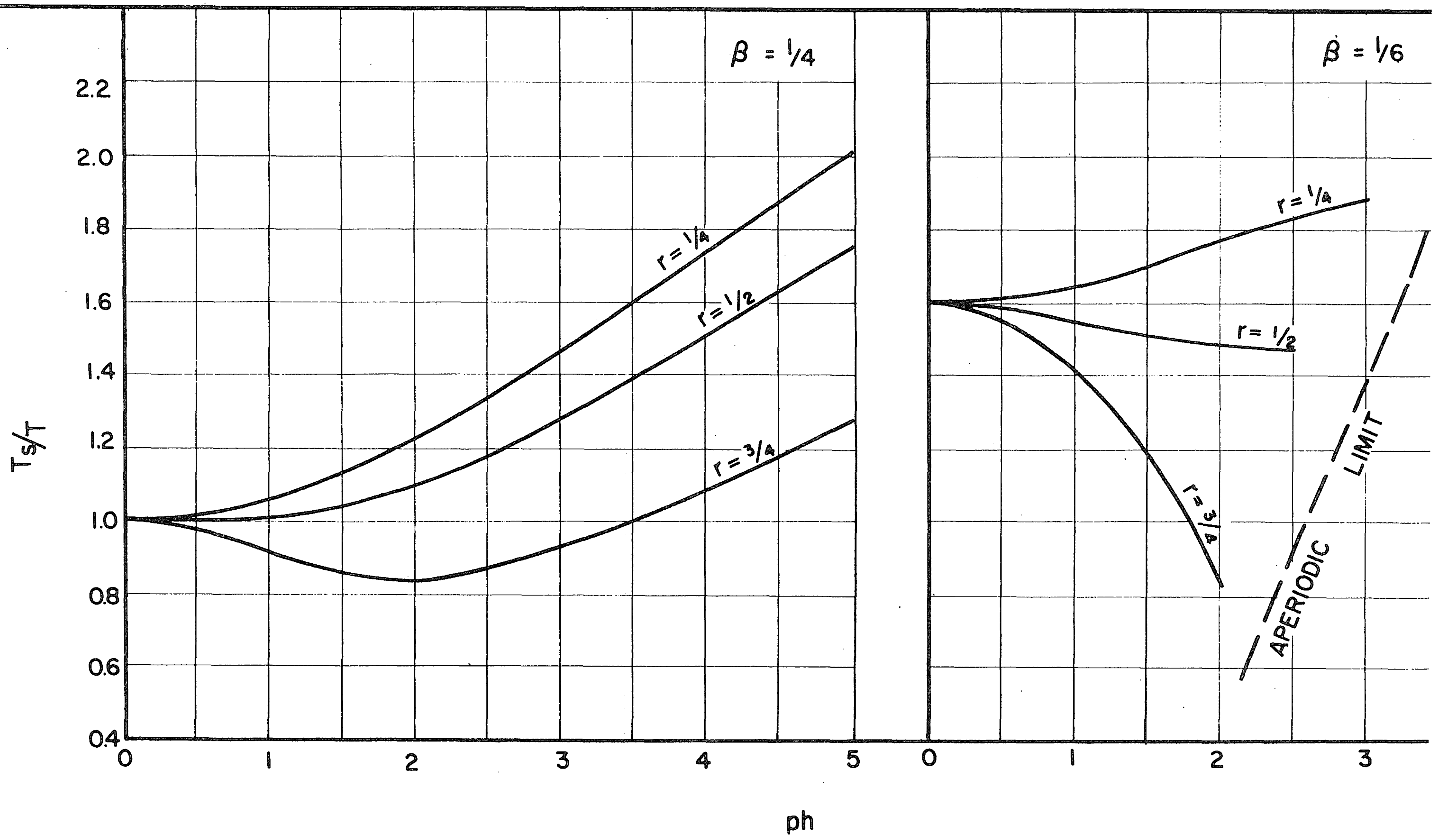

FIG. 5b COMPARISON OF PSEUDO PERIOD OF A VISCOUSLY DAMPED SYSTEM, ANALYZED BY VARIOUS β METHODS, ($\beta = 1/4$, $1/6$)

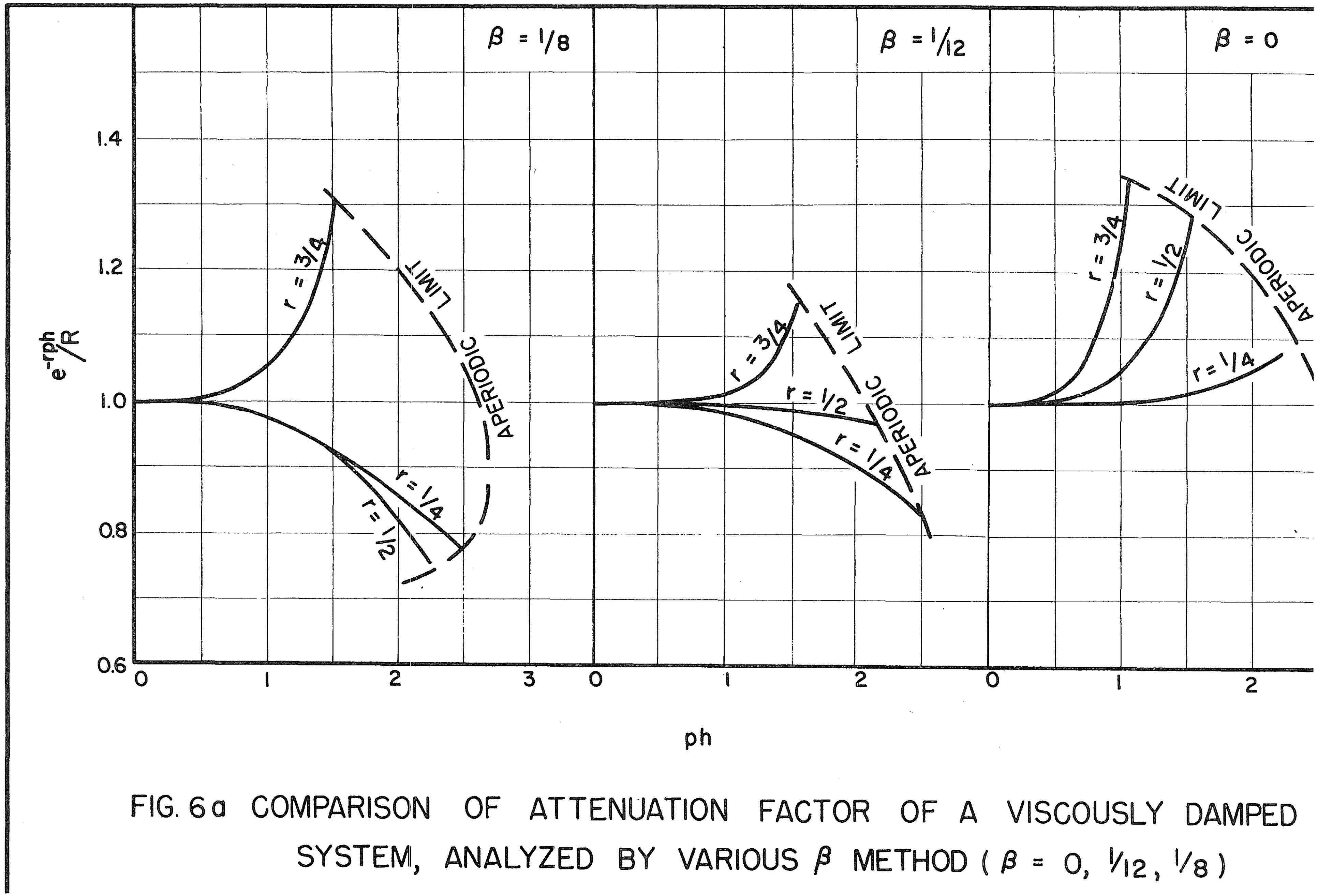

FIG. 6a COMPARISON OF ATTENUATION FACTOR OF A VISCOUSLY DAMPED SYSTEM, ANALYZED BY VARIOUS β METHOD (β = 0, 1/12, 1/8)

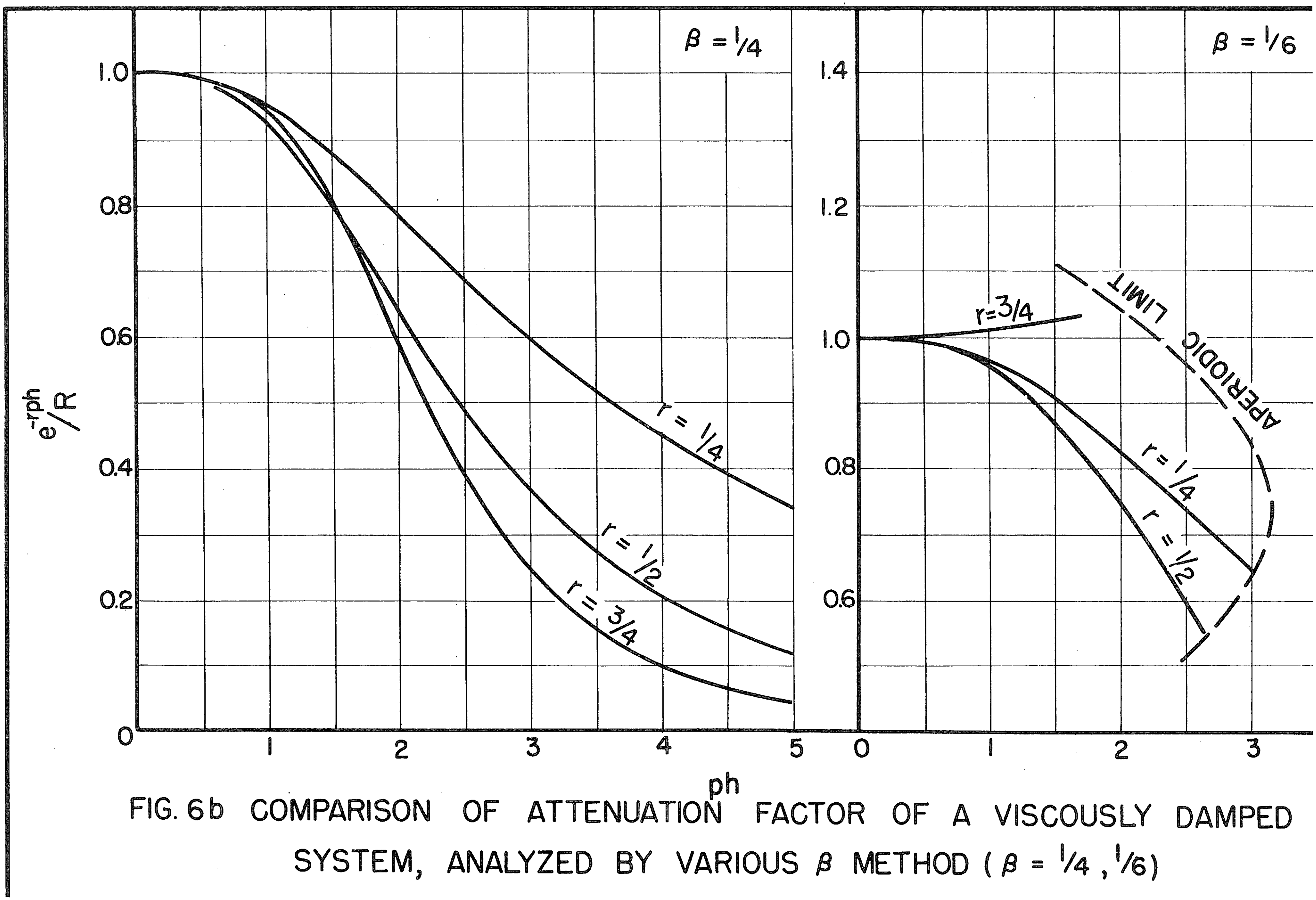

FIG. 6b COMPARISON OF ATTENUATION FACTOR OF A VISCOUSLY DAMPED SYSTEM, ANALYZED BY VARIOUS β METHOD (β = 1/4 , 1/6)

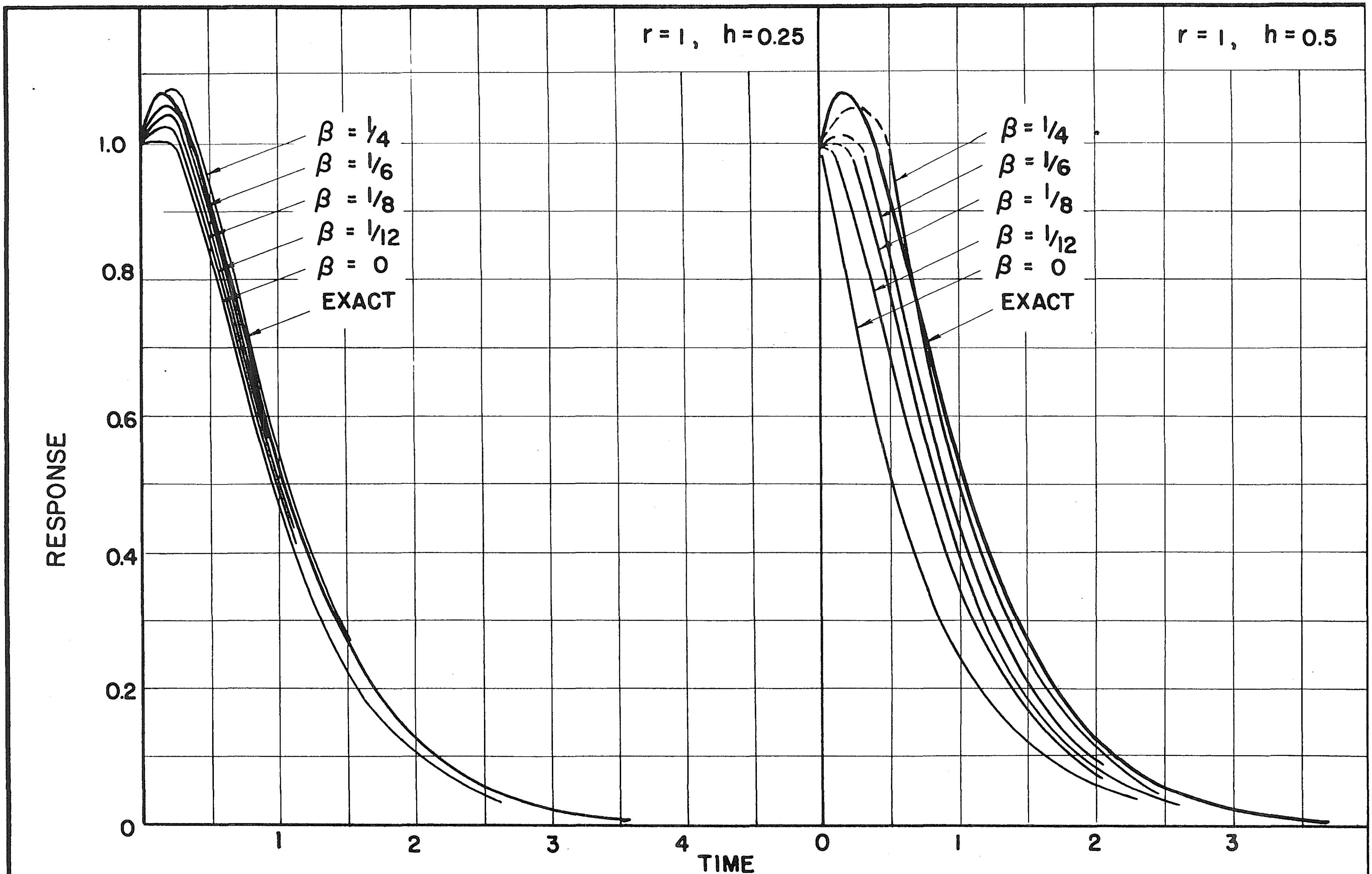

FIG. 7a. COMPARISON OF RESPONSES OF AN OVER-DAMPED SYSTEM, ANALYZED BY β METHODS

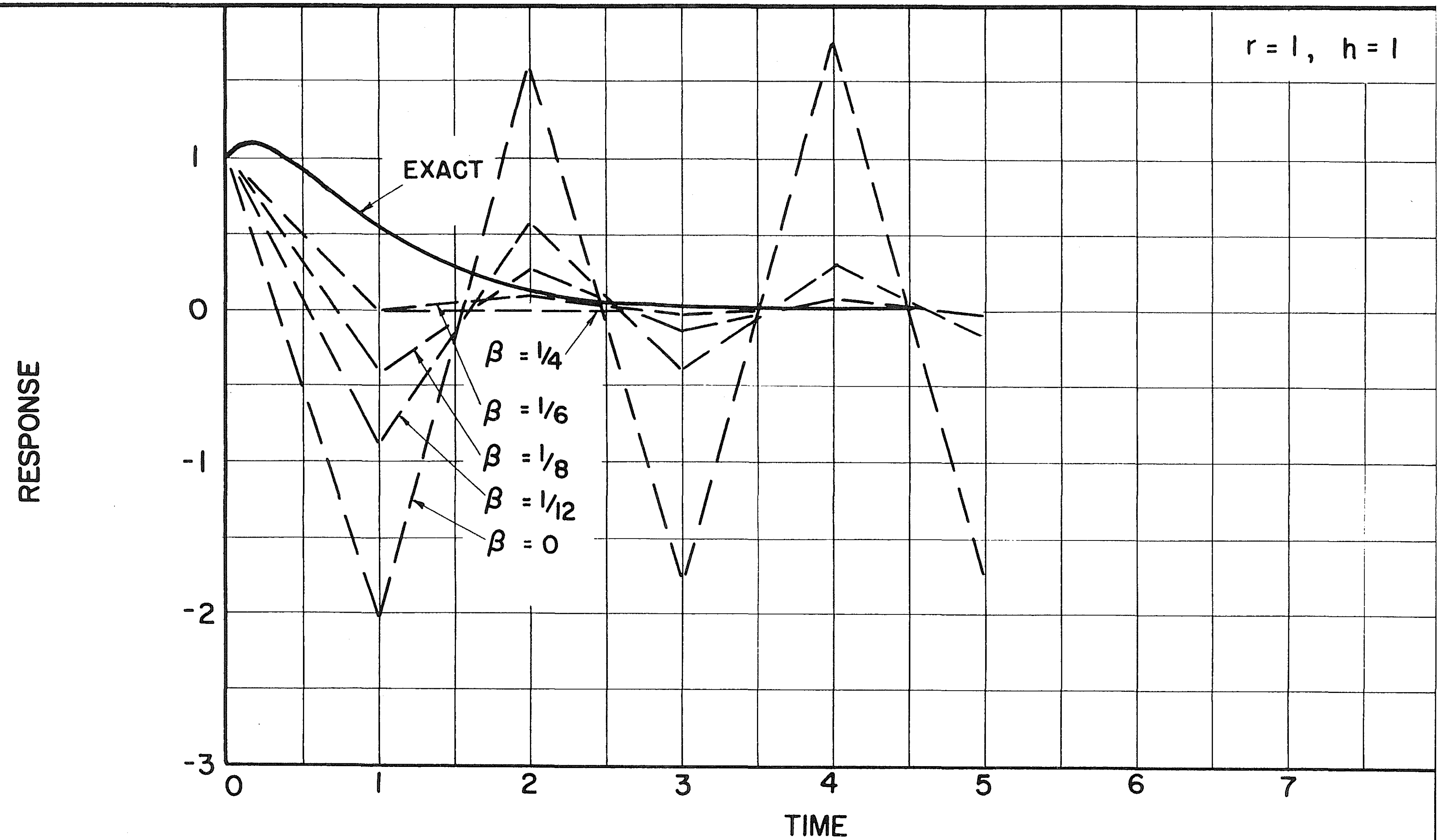

FIG.7b. COMPARISON OF RESPONSES OF AN OVER-DAMPED SYSTEM, ANALYZED BY β METHODS

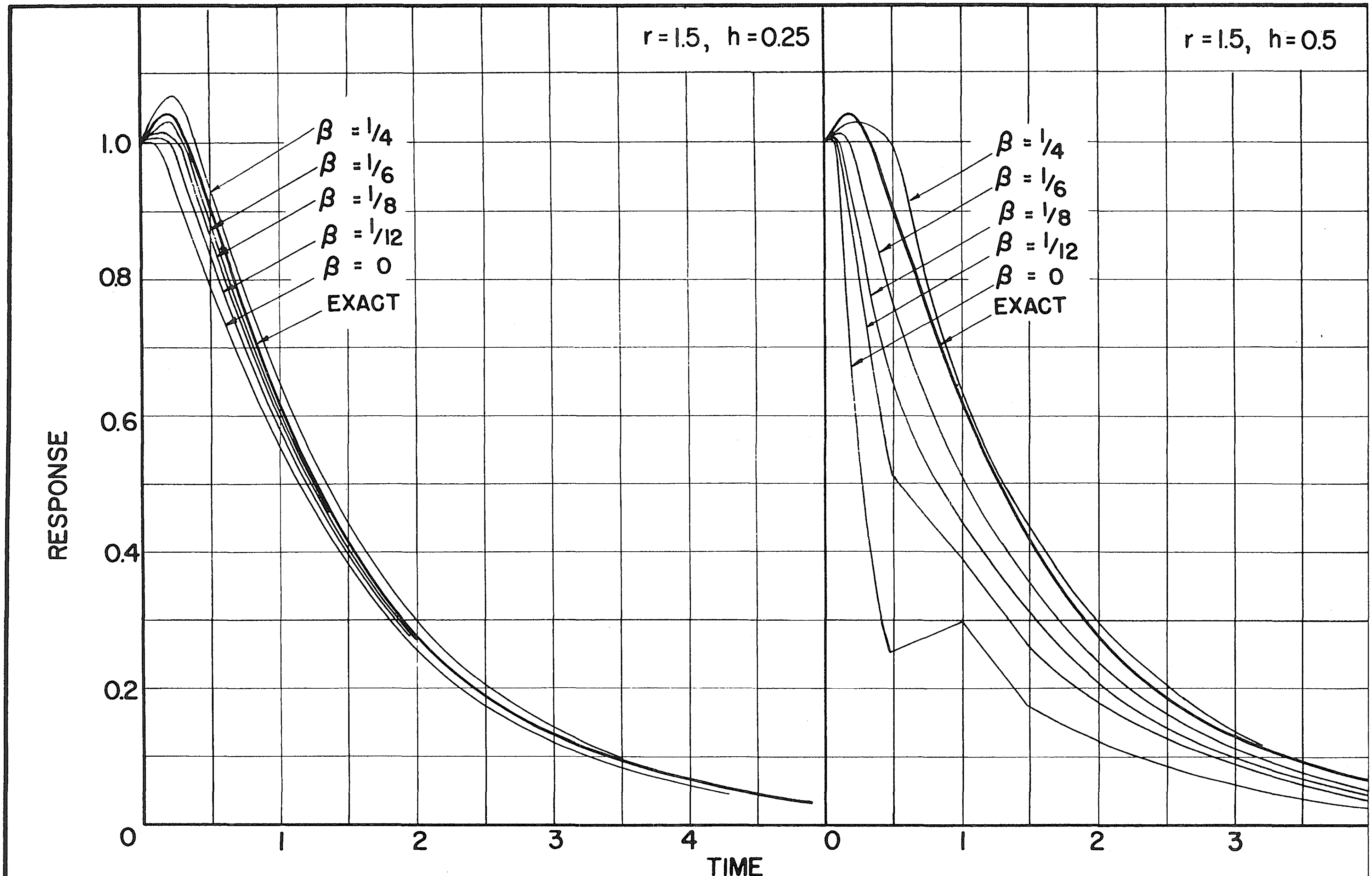

FIG. 7c. COMPARISON OF RESPONSES OF AN OVER-DAMPED SYSTEM, ANALYZED BY β METHODS

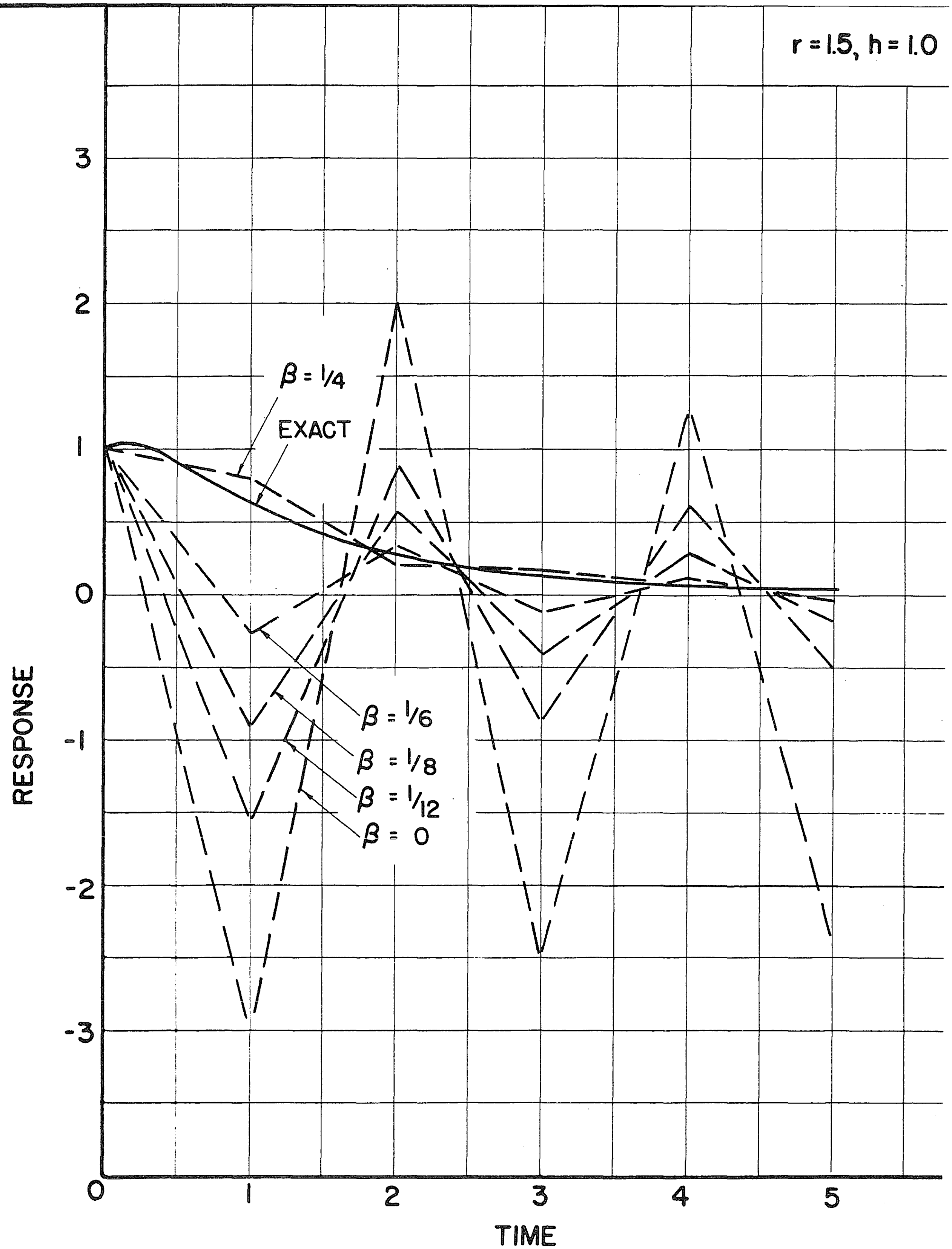

FIG.7d. COMPARISON OF RESPONSES OF AN OVER-DAMPED SYSTEM, ANALYZED BY β METHODS

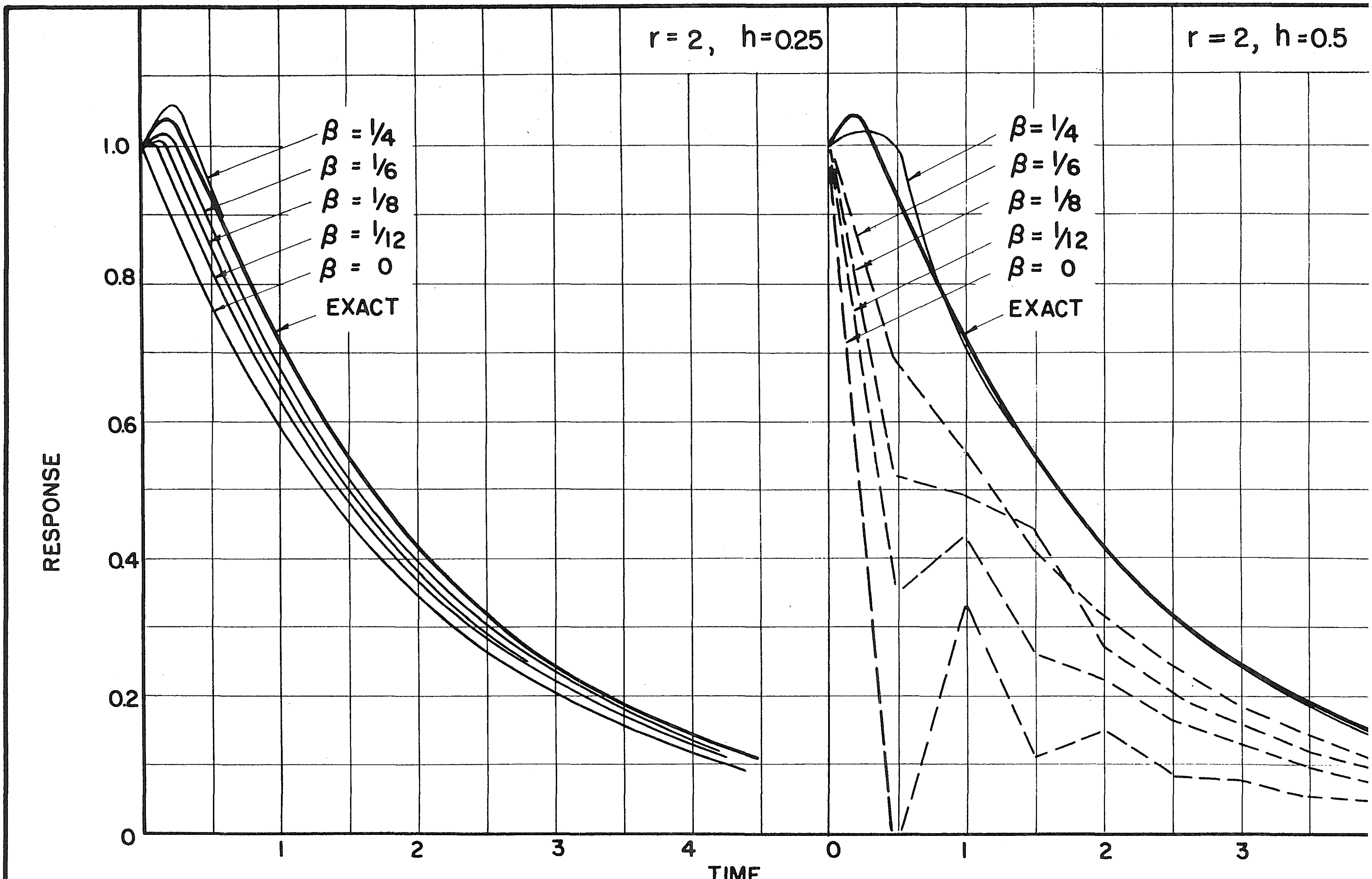

FIG. 7e. COMPARISON OF RESPONSES OF AN OVER-DAMPED SYSTEM, ANALYZED BY β METHODS

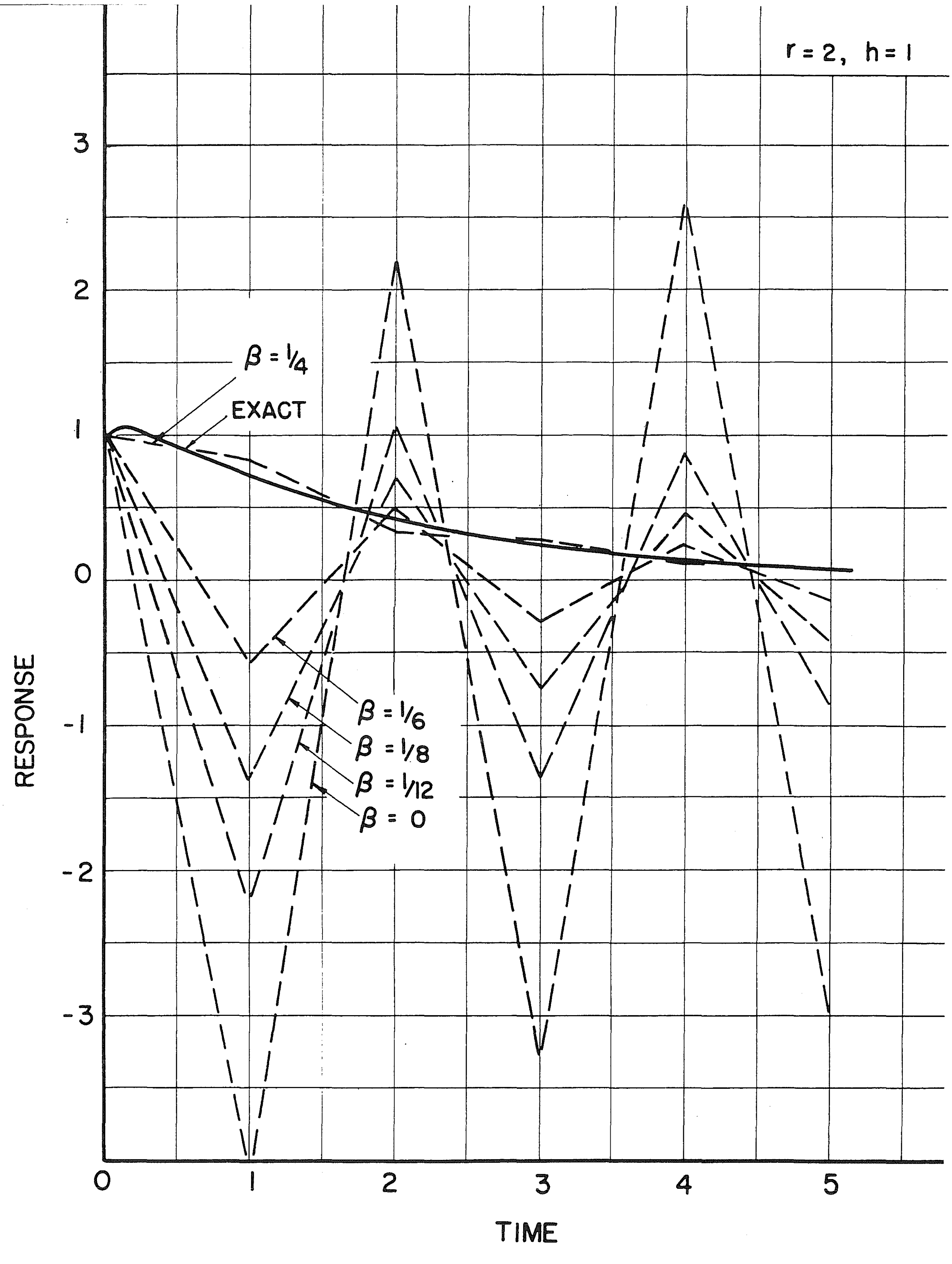

FIG. 7f. COMPARISON OF RESPONSES OF AN OVER-DAMPED SYSTEM, ANALYZED BY β METHODS

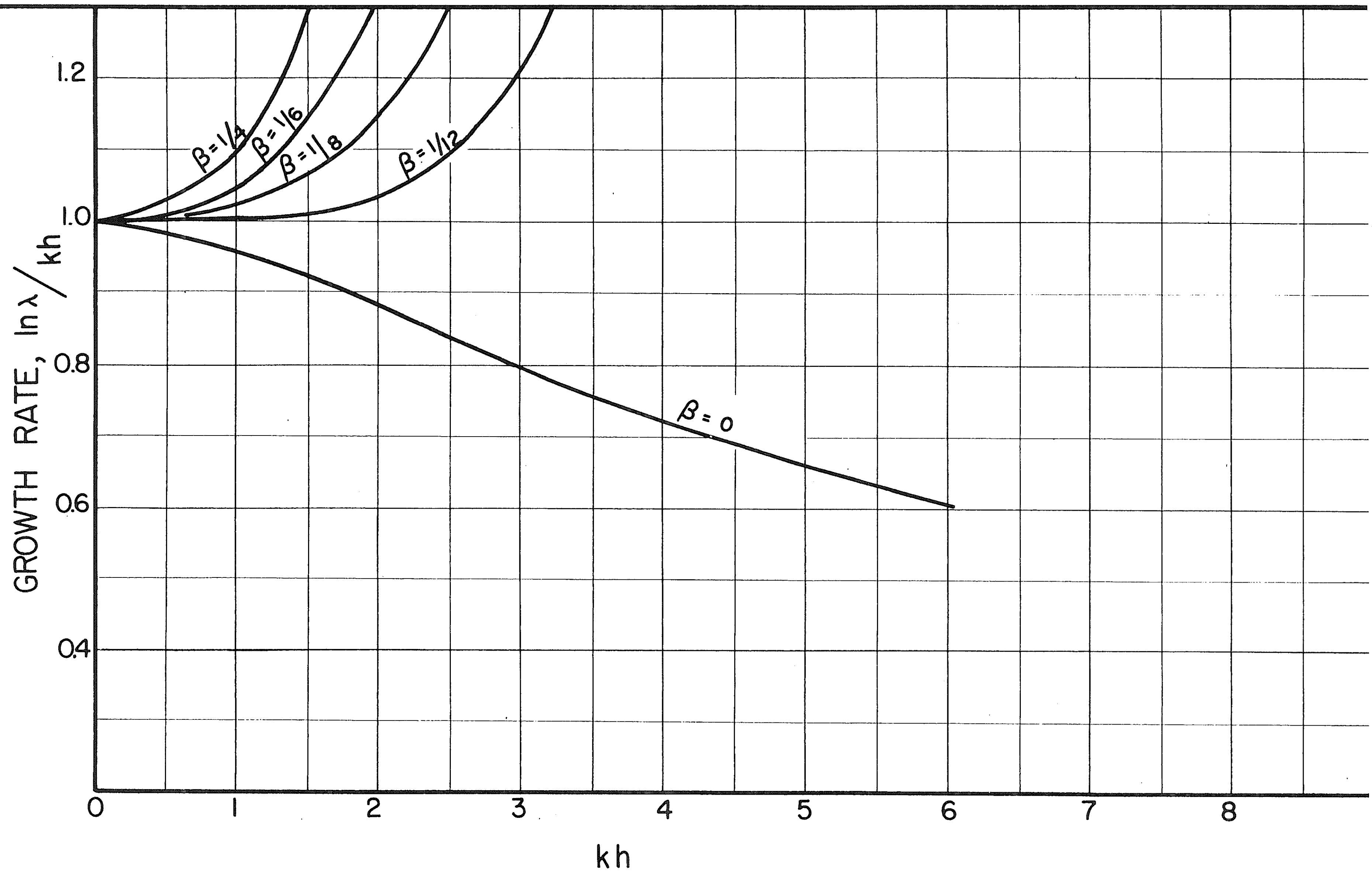

FIG. 8 COMPARISON OF GROWTH RATE OF A SYSTEM WITH NEGATIVE SPRING CONSTANT, ANALYZED BY β METHOD

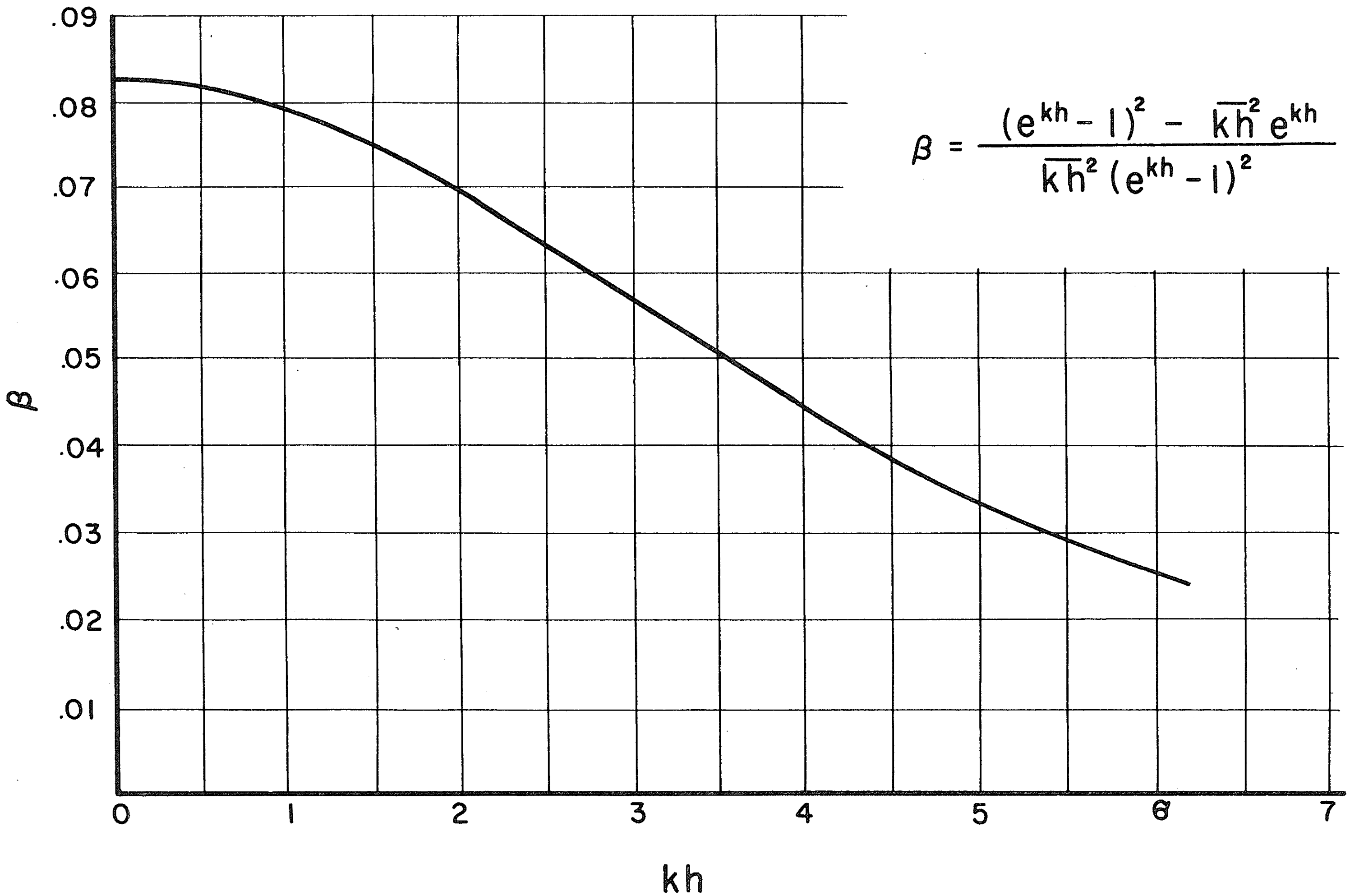

FIG. 9 β FOR CONSERVING GROWTH RATE