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# QUANTUM SYLVESTER-FRANKE THEOREM

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Abstract. A quantum version of classical Sylvester-Franke theorem is presented. After reviewing some representation theory of the quantum group $GL_q(n,\mathbb{C})$, the commutation relations of the matrix elements are verified. Once quantum determinant of the representation matrix is defined, the theorem follows naturally.

## 1. Introduction

It is a fundamental fact of invariants of the general linear group that a one-dimensional rational representation of $GL(n,\mathbb{C})$ is of the form $(\det)^k$ with $k \in \mathbb{Z}$. Given an irreducible (polynomial) representation $\rho_\lambda$ of $GL(n,\mathbb{C})$ corresponding to a partition $\lambda$ with $\ell(\lambda) \le n$, the determinant of the representation matrix $\rho_\lambda(g)$ $(g \in GL(n,\mathbb{C}))$ gives a one-dimensional representation. By counting the degree of the polynomials, one has $\det \rho_\lambda(g) = (\det g)^{\frac{|\lambda|}{n}\dim \rho_\lambda}$. This result is called the Sylvester-Franke theorem (cf. [1] and [4]).

One may expect that there exists a $q$-analogue of this theorem in the framework of quantum groups. In this note we prove the quantum Sylvester-Franke theorem in the simplest case $\lambda = (1^{n-1})$ for the quantum $GL(n,\mathbb{C})$. The point is that the representation matrix of $\lambda = (1^{n-1})$ is a quantum matrix in the sense that the entries satisfy the same commutation relations as those of quantum $GL(n,\mathbb{C})$. The general commutation relations of the quantum minor determinants are fully described by Goodearl in [2]. We use a portion of his results to prove our quantum determinant formula.

In this paper, $[n]$ denotes the set $\{1,\cdots,n\}$.

## 2. Classical case

Let $A_n = \mathbb{C}\left[x_{ij} | 1 \le i,j \le n\right]$ be the polynomial algebra of $n^2$ variables $x_{ij}$ $(1 \le i,j \le n)$. This is regarded as the coordinate ring of the matrix space $M_n = Mat\,(n,\mathbb{C})$, namely, $x_{ij}$ is the coordinate function $x_{ij}(a) = a_{ij}$ for $a = (a_{ij})_{i,j} \in M_n$. Let $X = (x_{ij})_{i,j}$ denote the matrix of these coordinate functions.

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Since $M_n$ is a $\mathbb{C}$-algebra, $A_n$ has a coalgebra structure with coproduct $\Delta$ and counit $\varepsilon$ defined as follows:

$$\Delta\left(x_{ij}\right)=\sum_{k=1}^{n} x_{ik}\otimes x_{kj},\quad \varepsilon\left(x_{ij}\right)=\delta_{ij}\quad \left(1\leq i,j\leq n\right). \tag{2.1}$$

Let

$$\det=\det X:=\sum_{\sigma\in\mathfrak{S}_n}\left(-1\right)^{\ell(\sigma)} x_{\sigma(1)1}\cdots x_{\sigma(n)n}\in A_n. \tag{2.2}$$

Appending the inverse $\det^{-1}$ to $A_n$, one has the coordinate ring $\mathscr{A}_n$ of $G_n = GL\left(n,\mathbb{C}\right)$:

$$\mathscr{A}_n:=\mathbb{C}\left[x_{ij},\det^{-1}\middle|1\leq i,j\leq n\right]=A_n\left[\det^{-1}\right] \tag{2.3}$$

with

$$\Delta\left(\det^{\pm1}\right)=\det^{\pm1}\otimes\det^{\pm1},\quad \varepsilon\left(\det^{\pm1}\right)=1. \tag{2.4}$$

Moreover $\mathscr{A}_n$ has a Hopf algebra structure with antipode $S$ defined by

$$S\left(x_{ij}\right)=\widetilde{x}_{ji}\cdot\det^{-1}\quad\left(1\leq i,j\leq n\right), \tag{2.5}$$

where $\widetilde{x}_{ji}=\left(-1\right)^{i-j}\xi_{\hat{i}}^{\hat{j}}$ is the $(j,i)$-cofactor. Here $\xi_{\hat{i}}^{\hat{j}}$ is the minor determinant of a submatrix of $X$ consisting of rows $\hat{j}=[n]\setminus\{j\}$ and columns $\hat{i}=[n]\setminus\{i\}$.

For an irreducible polynomial representation $(\rho, V)$ of $G_n$, the alternating tensor representation $Alt^d\left(\rho\right): G_n\to GL\left(Alt^d\left(V\right)\right)$ is also irreducible. It is well-known that an irreducible polynomial representation of $G_n$ appears as a constituent of the tensor product of some alternating tensor representation, and irreducible polynomial representations are in one-to-one correspondence with Young diagrams of length less than or equal to $n$. We denote the representation by $\lambda$ if it corresponds to the Young diagram $\lambda$. As explained in Introduction we know the following theorem:

**Theorem 2.1** (Sylvester-Franke theorem)**.**

$$\det\lambda\left(g\right)=\left(\det g\right)^{\frac{|\lambda|}{n}\dim\lambda}\quad\left(g\in G_n\right) \tag{2.6}$$

The original version of Sylvester-Franke theorem is for the case $\lambda=\left(1^d\right)$ $(1\leq d\leq n)$. The representation matrix $\Xi_d$ consisits of minor determinants $\xi_J^I$ of rows $I=\{i_1<\cdots<i_d\}$ and columns $J=\{j_1<\cdots<j_d\}$:

**Theorem 2.2** (Original Sylvester-Franke theorem)**.**

$$\det\Xi_d=\left(\det X\right)^{\binom{n-1}{d-1}}. \tag{2.7}$$

Let $(\varphi, V)$ be a $d$-dimensional rational representation of $G_n$. Taking a basis $\{v_i\}_{i=1}^{d}$ for $V$, one has a matrix representation $(\varphi_{ij})_{i,j=1,\cdots,d}$ defined by

$$\varphi(g)(v_j) = \sum_{i=1}^{m} \varphi_{ij}(g)\, v_i \quad (1 \le j \le m)\,. \tag{2.8}$$

This gives $V$ an $\mathscr{A}_n$-comodule structure as follows. There exists a comodule map $\omega\colon V \to V \otimes \mathscr{A}_n$ defined by

$$\omega(v_j) = \sum_{i=1}^{m} v_i \otimes \varphi_{ij} \quad (1 \le j \le m)\,. \tag{2.9}$$

Conversely, a finite dimensional $\mathscr{A}_n$- comodule $V$ affords a rational representation of $G_n$.

## 3. Quantum case

Let $q \in \mathbb{C}^{\times}$ be generic, *i.e.*, not a root of unity. A $q$-analogue $A_n(q)$ of the algebra $A_n$ is a $\mathbb{C}$-algebra generated by $x_{ij}$ $(1 \le i, j \le n)$ subject to the fundamental relations:

$$\begin{array}{llll} \text{(i)} & x_{ij}x_{i\ell} = q x_{i\ell}x_{ij} & (j < \ell), \\ \text{(ii)} & x_{ij}x_{kj} = q x_{kj}x_{ij} & (i < k), \\ \text{(iii)} & x_{i\ell}x_{kj} = x_{kj}x_{i\ell} & (i < k,\ j < \ell), \\ \text{(iv)} & x_{ij}x_{k\ell} - x_{k\ell}x_{ij} = \left(q - q^{-1}\right) x_{i\ell}x_{kj} & (i < k,\ j < \ell). \end{array} \tag{3.1}$$

We set $X(q) = (x_{ij})_{i,j}$.

We regard $A_n(q)$ as the coordinate ring of the "quantum space" $M_n(q) = Mat_q(n, \mathbb{C})$. The algebra structure of $M_n(q)$ reflects to the coalgebra structure of $A_n(q)$:

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik} \otimes x_{kj}, \quad \varepsilon(x_{ij}) = \delta_{ij} \quad (1 \le i, j \le n)\,. \tag{3.2}$$

The quantum determinant is defined as

$$\det_q = \det_q X(q) := \sum_{\sigma \in \mathfrak{S}_n} (-q)^{\ell(\sigma)}\, x_{\sigma(1)1} \cdots x_{\sigma(n)n} \in A_n(q)\,. \tag{3.3}$$

The following result due to Reshetikhin-Takhtajan-Faddeev [5] is very important for our purpose:

**Lemma 3.1** ([5], see also [3])**.** *The quantum determinant* $\det_q$ *belongs the center* $ZA_n(q)$ *of* $A_n(q)$*. Furthermore* $ZA_n(q)$ *is the polynomial ring* $\mathbb{C}[\det_q]$.

The coordinate ring $\mathscr{A}_n(q)$ of the "quantum group" $G_n(q) = GL_q(n, \mathbb{C})$ is defined by

$$\mathscr{A}_n(q) := A_n(q)\left[\det_q^{-1}\right] \tag{3.4}$$

with

$$\Delta\left(\det_q^{\pm 1}\right) = \det_q^{\pm 1} \otimes \det_q^{\pm 1}, \quad \varepsilon\left(\det_q^{\pm 1}\right) = 1. \tag{3.5}$$

A detailed account of the structure of $\mathscr{A}_n(q)$ and the finite dimensional comodules over it is found in [3]. An important family of elements in $\mathscr{A}_n(q)$ is quantum minor determinants: for $d$-sets $I = \{i_1 < i_2 < \cdots < i_d\}$, $J = \{j_1 < j_2 < \cdots < j_d\}$ in $[n]$, we define

$$\xi^I_J(q) := \sum_{\sigma \in \mathfrak{S}_d} (-q)^{\ell(\sigma)} x_{i_{\sigma(1)} j_1} \cdots x_{i_{\sigma(d)} j_d} \in A_n(q). \tag{3.6}$$

The bialgebra $\mathscr{A}_n(q)$ has a Hopf algebra structure with the antipode

$$S(x_{ij}) = \widetilde{x}_{ji} \cdot \det_q^{-1} \quad (1 \le i, j \le n), \tag{3.7}$$

where $\widetilde{x}_{ji} = (-q)^{i-j} \xi^{\widehat{j}}_{\widehat{i}}(q)$ is the $(j,i)$-cofactor of the matrix $X(q)$. If we put $\widetilde{X}(q) := (\widetilde{x}_{ji})_{1 \le i,j \le n}$, then we have

$$\widetilde{X}(q) X(q) = \det_q \cdot I_n = X(q) \widetilde{X}(q). \tag{3.8}$$

A finite dimensional rational representation of $G_n(q)$ is, by definition, an $\mathscr{A}_n(q)$-comodule. The alternating tensor representation is realized as follows. Let $E$ be a $\mathbb{C}$-algebra generated by $n$ letters $y_1, \cdots, y_n$ subject to the relations:

$$y_j y_i = -q y_i y_j \quad (1 \le i < j \le n) \ \textit{and} \ y_i^2 = 0 \quad (1 \le i \le n). \tag{3.9}$$

It is a graded algebra $E = \bigoplus_{d=0}^n E_d$, where $E_d$ is the space of all homogeneous elements of degree $d$. The space $E_d$ is an irreducible $A_n(q)$-comodule through the algebra homomorphism

$$\omega_E(y_j) = \sum_{i=1}^n y_i \otimes x_{ij}. \tag{3.10}$$

For $J = \{j_1 < \cdots < j_d\} \subseteq [n]$, put $y_J = y_{j_1} \cdots y_{j_d}$. It is verified that

$$\omega_E(y_J) = \sum_{|I|=d} y_I \otimes \xi^I_J. \tag{3.11}$$

Namely the representation matrix for $E_d$ is

$$\Xi_d(q) = \left(\xi^I_J(q)\right)_{\substack{I,J \subseteq [n] \\ |I|=|J|=d}}. \tag{3.12}$$

We will show the quantum version of Theorem 2.2 for the case $d = n-1$. To this end we first show that the quantum determinant of the representation matrix $\Xi_{n-1}(q)$ makes sense, that is, the following commutation relations hold:

**Proposition 3.2.**

$$\begin{array}{llll} & \text{(I)} & \xi_{ij}\xi_{i\ell} = q\xi_{i\ell}\xi_{ij} & (j<\ell) \\ & \text{(II)} & \xi_{ij}\xi_{kj} = q\xi_{kj}\xi_{ij} & (i<k) \\ (3.13) & \text{(III)} & \xi_{i\ell}\xi_{kj} = \xi_{kj}\xi_{i\ell} & (i<k,\ j<\ell) \\ & \text{(IV)} & \xi_{ij}\xi_{k\ell} - \xi_{k\ell}\xi_{ij} = \left(q-q^{-1}\right)\xi_{i\ell}\xi_{kj} & (i<k,\ j<\ell) \end{array}$$

Here, we put $\xi_{ij} = \xi^{\widehat{n-i+1}}_{\widehat{n-j+1}}(q)$. Note that $\Xi_{n-1}(q) = (\xi_{ij})_{1\le i,j\le n}$.

We verify the above commutation relations by using the results of Goodearl [2]. First recall some notations. For $r \in [n]$, $\mathscr{N}_r$ denotes the set of $r$-subsets of $[n]$.

**Definition of $I \le J$.** We define the following partial order $\le$ on $\mathscr{N}_r$. For $I = \{i_1 < \cdots < i_r\}, J = \{j_1 < \cdots < j_r\} \in \mathscr{N}_r$, we denote by $I \le J$ if and only if $i_\ell \le j_\ell$ for $1 \le \ell \le r$. Furthermore, if $I \ne J$, then we write $I < J$. On the other hand, we use the notation $\prec$ for the lexicographic order on $\mathscr{N}_r$. Note that, for $i, j \in [n]$, following relation hold:

$$i > j \Leftrightarrow \widehat{i} \prec \widehat{j} \Leftrightarrow \widehat{i} < \widehat{j} \tag{3.14}$$

**Definition of $\xi_q(I;J)$.** For $d \in \mathbb{N}$, we define the $-q$-analogue of $d$ by

$$\begin{aligned} [d]_{-q} &:= \frac{(-q)^d - (-q)^{-d}}{(-q)-(-q)^{-1}} \\ &= (-q)^{1-d}\left(1+q^2+q^4+\cdots+q^{2d-2}\right). \end{aligned} \tag{3.15}$$

In addition, for $I = \{i_1 < \cdots < i_r\} \in \mathscr{N}_r$, $J \in \mathscr{N}_r$ with $I \ge J$, we set $d_\ell := |[1, i_\ell] \cap J| - \ell + 1 \in \mathbb{N}$ for $1 \le \ell \le r$, and

$$\xi_q(I;J) := \prod_{\ell=1}^{r} [d_\ell]_{-q} \tag{3.16}$$

with the convention that $\xi_q(\emptyset;\emptyset) = 1$.

**Definitions of $\{< X \parallel Y\}$, $\{> X \parallel Y\}$, $\mathscr{L}(U, X, Y)$, and $\mathscr{L}^\natural(V, X, Y)$.** For $X, Y \in \mathscr{N}_r$, we define the set $\{< X \parallel Y\}$ and $\{> X \parallel Y\}$ as follows:

$$\{< X \parallel Y\} := \{U \subseteq X \cup Y | X \cap Y \subseteq U, |X| = |U|, U < X\}, \tag{3.17}$$

$$\{> X \parallel Y\} := \{V \subseteq X \cup Y | X \cap Y \subseteq V, |X| = |V|, V > X\}. \tag{3.18}$$

Moreover, for $U, V \in \mathscr{N}_r$, the integers $\mathscr{L}(U,X,Y)$ and $\mathscr{L}^{\natural}(V,X,Y)$ are defined by

(3.19)
$$\mathscr{L}(U,X,Y) := \ell\left(\left(U \setminus U^{\natural}\right) \cup (Y \setminus X); X \setminus U\right) - \ell\left(\left(U \setminus U^{\natural}\right) \cup (Y \setminus X); U \setminus X\right),$$
(3.20)
$$\mathscr{L}^{\natural}(V,X,Y) := \ell\left(\left(V^{\natural} \setminus V\right) \cup (X \setminus Y); V \setminus X\right) - \ell\left(\left(V^{\natural} \setminus V\right) \cup (X \setminus Y); X \setminus V\right),$$

where

- $W^{\natural} := (X \cap Y) \sqcup ((X \cup Y) \setminus W)$ for $W \in \mathscr{N}_r$ with $X \cap Y \subseteq W \subseteq X \cup Y$,
- $\ell(S;T) := \sharp\{(s,t) \in S \times T | s > t\}$ for $S, T \in \mathscr{N}_r$.

Note that $W^{\natural} = X^{\natural} = Y$ $\left(\textit{resp. } W^{\natural} = Y^{\natural} = X\right)$ if $W = X$ $(\textit{resp. } W = Y)$.

We are ready to state the theorem of Goodearl which we need to verify Propsition 3.2.

**Theorem 3.3** ([2],Corollary 6.8.). *For $I, J, K, L \in \mathscr{N}_r$, we have*

(3.21)
$$q^{|I\cap K|}\xi^I_J\xi^K_L + q^{|I\cap K|}\sum_{P\in\{>J\|L\}}\widetilde{\mu}_P\xi^I_P\xi^K_{P^{\natural}} = q^{|J\cap L|}\xi^K_L\xi^I_J + q^{|J\cap L|}\sum_{Q\in\{<I\|K\}}\widetilde{\lambda}_Q\xi^{Q^{\natural}}_L\xi^Q_J,$$

*where*

$$\widetilde{\mu}_P := \left(-q+q^{-1}\right)^{|P\setminus J|}(-q)^{-\mathscr{L}^{\natural}(P,J,L)}\xi_q(P\setminus J; J\setminus P), \tag{3.22}$$
$$\widetilde{\lambda}_Q := \left(-q+q^{-1}\right)^{|I\setminus Q|}(-q)^{-\mathscr{L}(Q,I,K)}\xi_q(I\setminus Q; Q\setminus I) \tag{3.23}$$

*for $P \in \{> J \parallel L\}, Q \in \{< I \parallel K\}$.*

In the following proof of Proposition 3.2, we put $i^* = n - i + 1$ for $i \in [n]$.

*Proof of (I) of Proposition 3.2.* Let $i, j$, and $\ell \in [n]$ satisfy $j < \ell$ and suppose $I = K = \widehat{i^*}$, $J = \widehat{j^*}$, and $L = \widehat{\ell^*}$ in Theorem 3.3.

The set $\{< I \parallel K\} = \{< I \parallel I\}$ is empty. In fact, if an element $Q \in \{< I \parallel I\}$ exists, then $I \cap I \subseteq Q \subseteq I \cup I$ and $|I| = |Q|$ holds by the definition of $\{< I \parallel I\}$ and $Q$ is equal to $I$. This contradicts to $Q < I$. Therefore, the summation of the right hand side of Theorem 3.3 is empty.

Suppose $P \in \{> J \parallel L\}$. Then $J \cap L \subseteq P \subseteq J \cup L = [n]$ and $|P| = |J| = n-1$ hold by the definition of $\{> J \parallel L\}$, and we see that $J \cap L = [n] \setminus \{j^*, \ell^*\}$ since $j^* > \ell^*$. Thus, since $P$ is the $n-1$-subset of $[n]$ containing $n-2$ elements of $[n]$ other than $j^*$ and $\ell^*$, we see that $P = J$ or $P = L$. However, we must have $P > J$ since $P \in \{> J \parallel L\}$. Furthermore, we have $J = \widehat{j^*} < \widehat{\ell^*} = L$ by (3.14). Hence, we see that $P = L$ and $\{> J \parallel L\} = \{L\}$. Moreover,

$$\widetilde{\mu}_P = \widetilde{\mu}_L = \left(-q + q^{-1}\right)^1 (-q)^0 \cdot 1 = q^{-1} - q. \tag{3.24}$$

By the above discussion, we see that

$$q^{n-2}\xi^I_J \xi^I_L = q^{n-1}\xi^I_L \xi^I_J, \tag{3.25}$$

that is,

$$\xi_{ij}\xi_{i\ell} = q\xi_{i\ell}\xi_{ij} \tag{3.26}$$

by Theorem 3.3. □

*Proof of (IV) of Proposition 3.2.* Let $i, j, k$, and $\ell \in [n]$ satisfy $i < k, j < \ell$, and suppose $I = \widehat{i^*}, J = \widehat{j^*}, K = \widehat{k^*}$, and $L = \widehat{\ell^*}$ in Theorem 3.3.

Then the left hand side of Theorem 3.3 is equal to

$$q^{n-2}\xi^I_J \xi^K_L + q^{n-2} \sum_{P \in \{> J \parallel L\}} \widetilde{\mu}_P \xi^I_P \xi^K_{P^\natural}. \tag{3.27}$$

However, we see that the set $\{> J \parallel L\}$ is the singleton $\{L\}$ for the same reason of the third paragraph of the above proof of (I). Since

$$\widetilde{\mu}_P = \widetilde{\mu}_L = \left(-q + q^{-1}\right)^1 (-q)^0 \cdot 1 = q^{-1} - q, \tag{3.28}$$

(3.27) is equal to

$$q^{n-2}\xi^I_J \xi^K_L + q^{n-2}\left(q^{-1} - q\right)\xi^I_L \xi^K_J. \tag{3.29}$$

Furthermore, the right hand side of Theorem 3.3 is equal to

$$q^{n-2}\xi^K_L \xi^I_J + q^{n-2} \sum_{Q \in \{< I \parallel K\}} \widetilde{\lambda}_Q \xi^{Q^\natural}_L \xi^Q_J. \tag{3.30}$$

Nevertheless, we see that the set $\{< I \parallel K\}$ is empty for the same reason of the second paragraph of the above proof of (I). Thus, (3.30) is equal to

$$q^{n-2}\xi^K_L \xi^I_J. \tag{3.31}$$

Hence, we see that

$$q^{n-2}\xi^I_J \xi^K_L + q^{n-2}\left(q^{-1} - q\right)\xi^I_L \xi^K_J = q^{n-2}\xi^K_L \xi^I_J, \tag{3.32}$$

that is,

$$\xi_{ij}\xi_{k\ell} - \xi_{k\ell}\xi_{ij} = \left(q - q^{-1}\right)\xi_{i\ell}\xi_{kj}. \tag{3.33}$$

□

We omit the proof of the others of Proposition 3.2 because they are similar to the above. In the following we write $X$ and $\Xi_{n-1}$ in the place of $X(q)$ and $\Xi_{n-1}(q)$.

**Theorem 3.4.**

$$\det{}_q \Xi_{n-1} = (\det{}_q X)^{n-1} . \tag{3.34}$$

*Proof.* Let $B_n(q)$ be the subalgebra of $A_n(q)$ generated by $\{\xi_{ij} \mid 1 \leq i, j \leq n\}$. Then $D_q := \det_q \Xi_{n-1}$ belongs to the center $ZB_n(q)$ of $B_n(q)$. We have

$$X\widetilde{X}\begin{pmatrix} D_q & & \\ & \ddots & \\ & & D_q \end{pmatrix} = X\begin{pmatrix} D_q & & \\ & \ddots & \\ & & D_q \end{pmatrix}\widetilde{X}. \tag{3.35}$$

On the other hand, we see that

$$\begin{aligned} X\widetilde{X}\begin{pmatrix} D_q & & \\ & \ddots & \\ & & D_q \end{pmatrix} &= \begin{pmatrix} \det_q & & \\ & \ddots & \\ & & \det_q \end{pmatrix}\begin{pmatrix} D_q & & \\ & \ddots & \\ & & D_q \end{pmatrix} \\ &= \begin{pmatrix} D_q & & \\ & \ddots & \\ & & D_q \end{pmatrix}\begin{pmatrix} \det_q & & \\ & \ddots & \\ & & \det_q \end{pmatrix} \\ &= \begin{pmatrix} D_q & & \\ & \ddots & \\ & & D_q \end{pmatrix} X\widetilde{X}. \end{aligned} \tag{3.36}$$

By the above two equations, we obtain

$$X\begin{pmatrix} D_q & & \\ & \ddots & \\ & & D_q \end{pmatrix} = \begin{pmatrix} D_q & & \\ & \ddots & \\ & & D_q \end{pmatrix} X, \tag{3.37}$$

that is, $D_q \in ZA_n(q)$. Therefore, we see that

$$D_q = \alpha (\det{}_q X)^k \tag{3.38}$$

with some $\alpha \in \mathbb{C}$ and $k \in \mathbb{N}$. Comparing the degree and the "leading term" of both sides, we see that $\alpha = 1, k = n-1$. □


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