Math. J. Okayama Univ. **64** (2022), 75–107

CRITERIA FOR GOOD REDUCTION OF HYPERBOLIC POLYCURVES

Ippei Nagamachi

Abstract. We give good reduction criteria for hyperbolic polycurves, i.e., successive extensions of families of curves, under some assumptions. These criteria are higher dimensional versions of the good reduction criterion for hyperbolic curves given by Oda and Tamagawa.

Contents

1. Introduction

Let K be a discrete valuation field, O_K the valuation ring of K, $p\,(\geq 0)$ the residual characteristic of K, K^{sep} a separable closure of K, G_K the absolute Galois group $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ of K, and I_K an inertia subgroup of G_K. (Note that I_K, as a subgroup of G_K, depends on the choice of a prime ideal in the integral closure of O_K in K^{sep} over the maximal ideal of O_K, but it is independent of this choice up to conjugation.) Let X be a proper smooth variety over K. X is said to have good reduction if there

Mathematics Subject Classification. Primary 14H30; Secondary 14G20.

Key words and phrases. good reduction, hyperbolic curve, polyucurve, étale fundamental group.

The author thanks his supervisor Atsushi Shiho for useful discussions and helpful advice. During the research, he was supported by the Program for Leading Graduate Course for Frontier of Mathematical Sciences and Physics. A large percentage of the work was done during his stay at Researches Institute for Mathematical Sciences, Kyoto University. He thanks them for the hospitality and encouragement, especially Akio Tamagawa and Yuichiro Hoshi.

exists a proper smooth scheme $\mathfrak{X}$ over O_K whose generic fiber is isomorphic to X over O_K. (Such a scheme $\mathfrak{X}$ is called a smooth model of X.) In arithmetic geometry, it is important to know criteria to determine whether X has good reduction. Various criteria for good reduction in terms of Galois representations have been established for certain classes of varieties. Néron, Ogg, and Shafarevich established a criterion in the case of elliptic curves, and Serre and Tate generalized this criterion to the case of abelian varieties [17]. Their criterion claims that an abelian variety has good reduction if and only if the action of I_K on the first l-adic étale cohomology group of $X \otimes_K K^{\mathrm{sep}}$ is trivial for some prime number $l \neq p$.

As a non-abelian version of the above result, Oda showed that a proper hyperbolic curve has good reduction if and only if the outer action of I_K on the pro-l fundamental group of $X \otimes_K K^{\mathrm{sep}}$ is trivial (cf. [12] and [13]). To state Oda's result precisely, we fix some notations. For a profinite group G and p as above (resp. a prime number l), we denote the pro-p' (resp. pro-l) completion of G, which is defined to be the limit of the projective system of quotient groups of G with finite order prime to p (resp. with finite l-power order) by $G^{p'}$ (resp. G^l). Here, if $p = 0$, the order of every finite group is considered to be prime to 0.

Suppose that X is a proper hyperbolic curve (i.e., a geometrically connected proper smooth curve of genus ≥ 2) over K. Then the pro-l completion $\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l$ of the étale fundamental group $\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})$ (with a base point $\bar{t}$) admits a continuous homomorphism

$$(1.1)\qquad \begin{aligned} \rho : G_K \to& \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l) \\ :=& \mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l)/\mathrm{Inn}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l). \end{aligned}$$

We refer to this outer representation as the outer Galois representation associated with X. Here, $\mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l)$ (resp. $\mathrm{Inn}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l)$) is the group of continuous automorphisms of the profinite group $\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l$ (resp. the group of inner automorphisms of the profinite group $\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l$). Oda proved that X has good reduction if and only if the restriction of ρ to I_K is trivial. Tamagawa generalized this criterion to (not necessarily proper) hyperbolic curves [19] (cf. Definition 2). Further research has been done using the weight filtration on $\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^l$) (see, for example, [2], [21, §17]).

Oda and Tamagawa's criterion can be regarded as a result in anabelian geometry. Indeed, a hyperbolic curve is a typical example of an anabelian variety, i.e., a variety which is determined by its outer Galois representation $G_K \to \mathrm{Out}\, \pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})$ (under a suitable assumption on K) (cf. [19] and [7]). Therefore, it would be natural to expect that we can obtain information on the reduction of X from the outer Galois representation associated with X.

The class of hyperbolic polycurves, that is, varieties X which admit a structure of successive smooth fibrations (called a sequence of parameterizing morphisms (cf. Definition 3))

$$X = X_n \overset{f_n}{\to} X_{n-1} \overset{f_{n-1}}{\to} \cdots \overset{f_2}{\to} X_1 \overset{f_1}{\to} \operatorname{Spec} K \tag{1.2}$$

whose fibers are hyperbolic curves, is considered to be anabelian. Indeed, the Grothendieck conjecture for hyperbolic polycurves of dimension up to 4 holds under suitable assumptions on K [7] [5]. Moreover, in the case where X is a strongly hyperbolic Artin neighborhood (of any dimension) (cf. [16, Definition 6.1]) and K is finitely generated over $\mathbb{Q}$, the Grothendieck conjecture for such a variety holds [16]. Thus, we can expect that there exists a good reduction criterion for hyperbolic polycurves analogous to that of Oda and Tamagawa.

In [9], we studied a good reduction criterion for proper hyperbolic polycurves under some assumptions. In this paper, we improve the main theorem of [9] and discuss not necessarily proper cases. The main results of this paper are as follows:

Theorem 1.1. *Let $K, O_K, p, K^{\mathrm{sep}}, G_K$, and I_K be as above. Let X be a proper hyperbolic polycurve over K and g_X the maximum genus of X (cf. Definition 3.3). Consider the following conditions:*

- : *(A) X has good reduction.*
- : *(B) The outer Galois representation $I_K \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{t})^{p'})$ is trivial.*

Then we have the following:

1. *(A) implies (B).*
2. *If $p = 0$, (B) implies (A).*
3. *If $p > 2g_X + 1$ and the dimension of X is 2, (B) implies (A).*
4. *Suppose that $p > 2g_X + 1$, X has a K-rational point x, and the Galois representation $I_{K(x)} \to \mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \bar{x})^{p'})$ defined as (2.4) in Section 2 is trivial. Then (A) holds.*

Theorem 1.2. *Let K, O_K, p, K^{sep}, and I_K be as in Theorem 1.1. Let X be a hyperbolic polycurve over K with a sequence of parameterizing morphisms*

$$\mathcal{S} : X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = \operatorname{Spec} K. \tag{1.3}$$

Write $b_{\mathcal{S}}$ for the maximal first Betti number of $\mathcal{S}$ (cf. Definition 3.3). Consider the following conditions:

: *(A) There exists a hyperbolic polycurve $\mathfrak{X} \to \operatorname{Spec} O_K$ with a sequence of parameterizing morphisms*

$$\mathfrak{X} = \mathfrak{X}_n \to \mathfrak{X}_{n-1} \to \ldots \to \mathfrak{X}_1 \to \mathfrak{X}_0 = \operatorname{Spec} O_K \tag{1.4}$$

whose generic fiber is isomorphic to $(X, \mathcal{S})$ (cf. Definition 3.1).

: *(B) The outer Galois representation $I_K \to \operatorname{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{x})^{p'})$ is trivial.*

Then we have the following:

(1) (A) implies (B).
(2) If $p = 0$, (B) implies (A).
(3) Suppose that $p > b_{\mathcal{S}} + 1$ and the dimension of X is 2. Then (B) implies (A).

Remark. At the time of writing, the author does not know whether or not the lower bound $p > 2g_X + 1$ in Theorem 1.1.3 and Theorem 1.1.4 (resp. $p > b_{\mathcal{S}} + 1$ in Theorem 1.2.3) is best possible.

If we assume a very strong condition on $b_{\mathcal{S}}$ and p, (B) implies (A) in the case where $\dim X \geq 3$.

Theorem 1.3. *Let K, O_K, I_K, X, $\mathcal{S}$, $b_{\mathcal{S}}$, and n be as in Theorem 1.2. Suppose that $n \geq 3$. Define a function $f_{b_{\mathcal{S}}}(m)$ for $m \geq 3$ in the following way:*

- *For $m = 3$, $f_{b_{\mathcal{S}}}(3) = 2^{b_{\mathcal{S}}^2}$.*
- *For $m \geq 3$,*

$$f_{b_{\mathcal{S}}}(m+1) = (f_{b_{\mathcal{S}}}(m)) \times (2^{b_{\mathcal{S}}^2 \times f_{b_{\mathcal{S}}}(m)^2})^{f_{b_{\mathcal{S}}}(m)}.$$

Consider the conditions (A) and (B) in Theorem 1.2. If $p > 2^{b_{\mathcal{S}} \times f_{b_{\mathcal{S}}}(n)}$, (B) implies (A).

Remark. The main result of [9] is described as follows: Let K, O_K, and I_K be as in Theorem 1.1. Let X be a proper hyperbolic polycurve over K which has a sequence of parameterizing morphisms

$$X = X_n \to \ldots \to X_0 = \operatorname{Spec} K$$

such that, for each $1 \leq i \leq n$, $X_i \to X_{i-1}$ has a section. Write g_X for the minimum of the maximal genera of such sequences of parameterizing morphisms of X (cf. Definition 3). Consider the condition (A) in Theorem 1.1 and the following condition:
(B)' Let x be a closed point of X, $O_{K(x)}$ a valuation ring of the residual field $K(x)$ of x over O_K, $K(x)^{\mathrm{sep}}$ a separable closure of $K(x)$, and $I_{K(x)}$ an inertia subgroup of $O_{K(x)}$ in the absolute Galois group $\operatorname{Gal}(K(x)^{\mathrm{sep}}/K(x))$. Then the action of $I_{K(x)}$ on $\pi_1(X \otimes_K K(x)^{\mathrm{sep}}, \overline{x})^{p'}$ is trivial.

Then (A) implies (B)'. If $p = 0$ or $p > 2g_X + 1$, (B)' implies (A).

The condition (B)' is stronger than the condition (B) in Theorem 1.1 or the condition given in Theorem 1.1.4. Hence, the main result of [9] is weaker than Theorem 1.1 because we need to assume that each $X_i \to X_{i-1}$ has a section and that the condition (B)' is satisfied.

To prove the implication (B) $\Rightarrow$ (A) or (B)' $\Rightarrow$ (A) by induction on the dimension of X, we need a homotopy exact sequence of étale fundamental groups of hyperbolic curves. In the previous paper [9], we constructed homotopy exact sequences of Tannakian fundamental groups of certain categories of smooth $\mathbb{Q}_l$-sheaves by using the existence of a section of each morphism $X_i \to X_{i-1}$. Also, we needed the assumption (B)', which is stronger than (B), because we used a criterion for smoothness of $\mathbb{Q}_l$-sheaves which is due to Drinfeld [4].

In this paper, we use different arguments from those of [9] to obtain stronger results. Since the implication (A) $\Rightarrow$ (B) follows from a standard specialization argument, we explain key ingredients of the proof of the implication (B) $\Rightarrow$ (A) (assuming the condition on p, g_X, and b_S in the assertions), which enables us to improve the result of [9]. Let X be as in Theorem 1.1, Theorem 1.2, or Theorem 1.3. Take a geometric point of $X \otimes_K K^{\mathrm{sep}}$ and write Δ (resp. Π) for the étale fundamental group of the scheme $X \otimes_K K^{\mathrm{sep}}$ (resp. X) determined by this geometric point.

(1) <u>A decomposition of Π</u>

If $p = 0$, we can obtain a decomposition $\Pi \cong \Delta \times Z_\Pi(\Delta)$, where $Z_\Pi(\Delta)$ is the centralizer subgroup of Δ in Π by using the assumption that the outer Galois action of I_K is trivial and the homotopy exact sequences in [5, PROPOSITION 2.5] (in the case where $p = 0$). We can prove the implication (B) $\Rightarrow$ (A) by using this decomposition.

(2) <u>Intermediate quotient groups</u>

Note that we do not have appropriate homotopy exact sequences associated with the fibrations $X_i \to X_{i-1}$ $(2 \leq i \leq n)$ if $p > 0$. Indeed, the functor of taking pro-p' completion (of profinite groups) is not exact. Moreover, in the case the characteristic of K is positive, the sequence in [5, PROPOSITION 2.5] is no longer exact. In this paper, we consider an intermediate quotient group of Δ (which we will write $\Delta^{(l,p')}$ for) between $\Delta^{p'}$ and Δ^{l}, for which we can obtain a homotopy exact sequence. If the dimension of X is 2, we can show the implication (B) $\Rightarrow$ (A) by using the center-freeness of $\Delta^{(l,p')}$ and applying an argument similar to that in 1. If X admits a K-rational point x, we can use $I_{K(x)}$ instead of $Z_\Pi(\Delta)$. Then we can prove Theorem 1.1.4.

(3) <u>Further intermediate quotient groups</u>

If the dimension of X is equal to or greater than 3, we do not know whether the group $\Delta^{(l,p')}$ is center-free or not. However, if p is big enough, we can find a certain quotient $\overline{\Delta}$ of Δ which is center-free and for which there exists a homotopy exact sequence. Thus, we can prove the implication (B) $\Rightarrow$ (A) in higher-dimensional cases if p is big enough.

The content of each section is as follows: In Section 2, we give a review of outer Galois representations associated with homotopy exact sequences of étale fundamental groups of hyperbolic curves. In Section 3, we give a precise definition of a hyperbolic polycurve and a first step of the proof of Theorem 1.1 and Theorem 1.2. In Section 4, we give a proof of Theorem 1.1 and Theorem 1.2 in the case of residual characteristic 0. In Section 5, we give a proof of Theorem 1.1 and 1.2 in the case of residual characteristic $p > 0$. In Section 6, we give a proof of Theorem 1.3. In Section 7, we review the property of an extension of a family of proper hyperbolic curves proved in [8] and prove a non-proper version of it. In Section 8, we give an example of a hyperbolic polycurve for which the naive analogue of the criterion of Oda and Tamagawa does not hold.

2. Good reduction criterion for hyperbolic curves

In this section, we recall the good reduction criterion for hyperbolic curves proven by Oda and Tamagawa. Let $K, O_K, p, K^{\mathrm{sep}}, G_K$, and I_K be as in Section 1.

Definition 1. Let S be a scheme, $\overline{X}$ a scheme over S, and D an effective divisor on $\overline{X}$. We shall say that the pair $(\overline{X}, D)$ is a *hyperbolic curve* over S if the following three conditions are satisfied:

- The morphism $\overline{X} \to S$ is proper, smooth, with geometrically connected fibers of dimension one and genus g.
- The morphism $D \to S$ is finite étale of degree n.
- $2g + n - 2 > 0$.

If $n = 0$ (resp. $n > 0$), we call the number $2g$ (resp. $2g + n - 1$) the first Betti number of the curve.

Definition 2. (1) Let X be a proper smooth scheme geometrically connected over K. We say that X has good reduction if there exists a proper smooth O_K-scheme $\mathfrak{X}$ whose generic fiber $\mathfrak{X} \otimes_{O_K} K$ is isomorphic to X over K. We refer to $\mathfrak{X}$ as a *smooth model* of X.

(2) Let $(\overline{X}, D)$ be a hyperbolic curve over K. We shall say that $(\overline{X}, D)$ has good reduction if there exists a hyperbolic curve $(\overline{\mathfrak{X}}, \mathfrak{D})$ over $\operatorname{Spec} K$ whose generic fiber $(\overline{\mathfrak{X}} \otimes_{O_K} K, \mathfrak{D} \otimes_{O_K} K)$ is isomorphic to $(\overline{X}, D)$ over K. We refer to $(\overline{\mathfrak{X}}, \mathfrak{D})$ as a *smooth model* of $(\overline{X}, D)$.

Remark. If a smooth model of a hyperbolic curve exists, it is unique up to canonical isomorphism by [3] and [6].

Let $(\overline{X}, D)$ be a hyperbolic curve K, X the open subscheme $\overline{X} \setminus D$ of $\overline{X}$, and $\overline{t}$ a geometric point of $X \otimes_K K^{\mathrm{sep}}$. We have the following exact sequence of profinite groups:

$$1 \to \pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{t}) \to \pi_1(X, \overline{t}) \to G_K \to 1. \tag{2.1}$$

This exact sequence yields an outer Galois action

$$G_K \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{t})). \tag{2.2}$$

Then, for any prime number $l \neq p$, we have natural homomorphisms

$$\begin{aligned} I_K \hookrightarrow G_K &\to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{t})) \\ &\to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{t})^{p'}) \\ &\to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{t})^{l}). \end{aligned} \tag{2.3}$$

Oda and Tamagawa gave the following criterion:

Proposition 2.1 ([12], [13], and [19, Section 5]). *The following are equivalent:*

- *(1) $(\overline{X}, D)$ has good reduction.*
- *(2) The outer action $I_K \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{t})^{p'})$ defined by (2.3) is trivial.*
- *(3) There exists a prime number $l \neq p$ such that the outer action $I_K \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{t})^{l})$ defined by (2.3) is trivial.*

Suppose that X is a proper hyperbolic curve over K and there exists a section $s : \mathrm{Spec}\, K \to X$. Consider the geometric point $\overline{s} : \mathrm{Spec}\, K^{\mathrm{sep}} \to X \otimes_K K^{\mathrm{sep}}$ induced by s. Then we have the exact sequence (2.1) with $\overline{t}$ replaced by $\overline{s}$. The section s defines a section of the homomorphism $\pi_1(X, \overline{s}) \to G_K$ in the homotopy exact sequence (2.1). This induces a homomorphism $G_K \to \mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s}))$ such that the composite homomorphism $G_K \to \mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s})) \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s}))$ coincides with outer representation $G_K \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s}))$ in (2.3). For a prime number $l \neq p$, we obtain homomorphisms

$$\begin{aligned} I_K \hookrightarrow G_K &\to \mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s})) \\ &\to \mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s})^{p'}) \\ &\to \mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s})^{l}). \end{aligned} \tag{2.4}$$

Proposition 2.2. *The following are equivalent:*

- *(1) X has good reduction.*
- *(2) The action of I_K on $\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s})^{p'}$ defined by (2.4) is trivial.*

(3) The action of I_K on $\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s})^l$ defined by (2.4) is trivial.

Proof. For the proof of the implication $1 \Rightarrow 2$, see Remark after Proposition 2.2. The implication $2 \Rightarrow 3$ is trivial. Assume that the action of I_K on $\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s})^l$ is trivial. Then the outer action $I_K \to \mathrm{Out}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{s})^l)$ is trivial. Therefore, X has good reduction by Proposition 2.1. □

Remark. In Proposition 2.2, to prove the implication $1 \Rightarrow 2$, we only need the hypothesis that the morphism $X \to \mathrm{Spec}\, K$ is proper, smooth, and geometrically connected. We show this assertion in the first part of the proof of Theorem 1.1.1 and Theorem 1.2.1.

3. First reduction

In this section, we give a precise definition of a hyperbolic polycurve and the first step of the proof of Theorem 1.1, 1.2, and 1.3. Let $K, O_K, p, K^{\mathrm{sep}}, G_K$, and I_K be as in Section 1. Let O_K^{h} (resp. O_K^{sh}) be the henselization (resp. strict henselization) of O_K contained in K^{sep} defined by I_K.

Definition 3. Let S be a scheme and X a scheme over S.

(1) We shall say that X is a *hyperbolic polycurve* (of relative dimension n) over S if there exists a (not necessarily unique) factorization of the structure morphism $X \to S$

$$\mathcal{S} : X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = S \tag{3.1}$$

such that, for each $i \in \{1, \ldots, n\}$, there exists a hyperbolic curve $(\overline{X}_i, D_i)$ over X_{i-1} (cf. Definition 1) and the scheme $\overline{X}_i \setminus D_i$ is isomorphic to X_i over X_{i-1}. We refer to the above factorization of $X \to S$ as a *sequence of parameterizing morphisms*. In the case where we consider a pair of a hyperbolic polycurve X over S and a sequence of parametrizing morphisms $\mathcal{S}$ of X, we write $(X, \mathcal{S})$. We refer to such a pair as a *hyperbolic polycurve with a sequence of parametrizing morphisms*. We shall say that two hyperbolic polycurves (over S) with a sequence of parametrizing morphisms $(X, \mathcal{S})$ and $(X', \mathcal{S}')$ are isomorphic if there exists an S-isomorphism between hyperbolic polycurves of relative dimension i over S defined by $\mathcal{S}$ and $\mathcal{S}'$ for each $1 \leq i \leq n$ such that these isomorphisms are compatible with the sequence of parametrizing morphisms $\mathcal{S}$ and $\mathcal{S}'$.

(2) For a hyperbolic polycurve X over S, the following are equivalent:
  : (a) The structure morphism $X \to S$ is proper.
  : (b) For any sequence of parameterizing morphisms

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = S,$$

the morphism $X_i \to X_{i-1}$ is proper for each $1 \leq i \leq n$.

: (c) There exists a sequence of parameterizing morphisms

$$X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = S$$

such that the morphism $X_i \to X_{i-1}$ is proper for each $1 \leq i \leq n$. We call such $X \to S$ a *proper hyperbolic polycurve*.

(3) Let X be a hyperbolic polycurve (resp. proper hyperbolic polycurve) of relative dimension n over S. For a sequence of parameterizing morphisms

$$\mathcal{S} : X = X_n \to X_{n-1} \to \ldots \to X_1 \to X_0 = S, \tag{3.2}$$

we write $b_{\mathcal{S}}$ (resp. $g_{\mathcal{S}}$) for the maximum of the first Betti numbers (resp. the genera) of fibers of all the morphisms $X_i \to X_{i-1}$ and refer to $b_{\mathcal{S}}$ (resp. $g_{\mathcal{S}}$) *the maximal first Betti number* (resp. *the maximal genus*) of $\mathcal{S}$. We write for b_X (resp. g_X) the minimum of the maximal first Betti numbers (resp. the maximal genera) of sequences of parameterizing morphisms of X and refer to b_X (resp. g_X) as *the maximum first Betti number* (resp. *the maximum genus*) of X.

Notation-Propositoin 3.1. *Let* $1 \to N \to G \to Q \to 1$ *be an exact sequence of profinite groups and* $l \neq p$ *a prime number. Then* $\mathrm{Ker}\,(N \to N^{p'})$ *(resp.* $\mathrm{Ker}\,(N \to N^{l})$*) is a characteristic subgroup of* N*. We write* $G^{(p')}$ *(resp.* $G^{(l)}$*) for the quotient group* $G/\mathrm{Ker}\,(N \to N^{p'})$ *(resp.* $G/\mathrm{Ker}\,(N \to N^{l})$*). Moreover, we have the following commutative diagram with exact horizontal lines:*

$$\begin{array}{ccccccccc} 1 & \to & N & \to & G & \to & Q & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & N^{p'} & \to & G^{(p')} & \to & Q & \to & 1 \end{array} \quad \left(\textit{resp.} \quad \begin{array}{ccccccccc} 1 & \to & N & \to & G & \to & Q & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & N^{l} & \to & G^{(l)} & \to & Q & \to & 1 \end{array}\right). \tag{3.3}$$

Proof. Since every continuous homomorphism from N to a pro-p' profinite group factors $N \to N^{p'}$ (resp. $N \to N^{l}$), $\mathrm{Ker}\,(N \to N^{p'})$ (resp. $\mathrm{Ker}\,(N \to N^{l})$) is a characteristic subgroup of N. □

We start the proof of Theorem 1.1, 1.2, and 1.3.

Proofs of Theorem 1.1.1 and Theorem 1.2.1 (cf. Remark 2). Suppose that (A) holds. Take a smooth model $\mathfrak{X}$ of X (resp. the scheme $\mathfrak{X}$ in the condition (B)) if we are in the situation of Theorem 1.1.1 (resp. Theorem 1.2.2). We have the following diagram:

$$\begin{array}{ccccccc} X \otimes_K K^{\mathrm{sep}} & \to & X \otimes_K \mathrm{Frac}\, O_K^{\mathrm{h}} & \to & \mathrm{Spec}\,(\mathrm{Frac}\, O_K^{\mathrm{h}}) & \leftarrow & \mathrm{Spec}\, K^{\mathrm{sep}} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathfrak{X} \otimes_{O_K} O_K^{\mathrm{sh}} & \to & \mathfrak{X} \otimes_{O_K} O_K^{\mathrm{h}} & \to & \mathrm{Spec}\, O_K^{\mathrm{h}} & \leftarrow & \mathrm{Spec}\, O_K^{\mathrm{sh}}. \end{array}$$

Take a geometric point $\overline{\eta}$ of $X \otimes_K K^{\mathrm{sep}}$. By Notation-Proposition 3.1, [15, Exposé IX, Theorem 6.1], and the same argument as the proof of [15, Exposé IX, Theorem 6.1], we have the following commutative diagram of profinite groups with exact horizontal lines:

(3.4)

$$\begin{array}{ccccccccc} 1 & \to & \pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{\eta})^{p'} & \to & \pi_1(X \otimes_K \operatorname{Frac} O_K^{\mathrm{h}}, \overline{\eta})^{(p')} & \to & \mathrm{Gal}(K^{\mathrm{sep}}/\operatorname{Frac} O_K^{\mathrm{h}}) & \to & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & \pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{sh}}, \overline{\eta})^{p'} & \longrightarrow & \pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{h}}, \overline{\eta})^{(p')} & \longrightarrow & \pi_1(\operatorname{Spec} O_K^{\mathrm{h}}, \overline{\eta}) & \longrightarrow & 1. \end{array}$$

Since the first vertical arrow in the diagram (3.4) is an isomorphism by Theorem [11, Theorem 8.3] and the action I_K on $\pi_1(\mathfrak{X} \otimes_{O_K} O_K^{\mathrm{sh}}, \overline{\eta})^{p'}$ is trivial by the above diagram, the action of I_K on $\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{\eta})^{p'}$ is also trivial. Hence, we finished the proof of Theorem 1.1.1 and 1.2.1. □

To prove Theorem 1.3 and the rest of the assertions in Theorem 1.1 and 1.2, we need the following proposition:

Proposition 3.2. *Let T be a regular integral separated scheme. Let Y be a scheme over T satisfying the following condition: There exists a factorization*

$$Y = Y_n \to \ldots \to Y_0 = \operatorname{Spec} O_K$$

such that there exist a proper smooth morphism $\overline{Y}_{i+1} \to Y_i$ with geometrically connected fibers and a normal crossing divisor $E_{i+1} \subset \overline{Y}_{i+1}$ of the scheme $\overline{Y}_{i+1}$ relative to Y_i satisfying that the complement $\overline{Y}_{i+1} \setminus E_{i+1}$ is isomorphic to Y_{i+1} for each $0 \leq i \leq n-1$. Suppose that T is a scheme over $\mathbb{Z}_{(p)}$ (resp. $\mathbb{Q}$) if $p > 0$ (resp. $p = 0$). Let $\eta = \operatorname{Spec} K(T)$ be the generic point of T, $K(T)^{\mathrm{sep}}$ a separable closure of $K(T)$, $\overline{\eta}$ the scheme Spec $K(T)^{\mathrm{sep}}$, *and s a geometric point of $Y \times_T \overline{\eta}$. By* [15, Exposé IX, Theorem 6.1] *and* [10, Theorem 0.2], *we have the following commutative diagram with exact horizontal lines:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(Y \times_T \overline{\eta}, s) & \longrightarrow & \pi_1(Y \times_T \eta, s) & \longrightarrow & \pi_1(\eta, s) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \pi_1(Y \times_T \overline{\eta}, s) & \longrightarrow & \pi_1(Y, s) & \longrightarrow & \pi_1(T, s) & \longrightarrow & 1. \end{array}$$

Then the conjugate action of $\pi_1(Y \times_T \eta, s)$ on $\pi_1(Y \times_T \overline{\eta}, s)$ induces a natural action $\pi_1(Y, s) \to \mathrm{Aut}(\pi_1(Y \times_T \overline{\eta}, s)^{p'})$. Thus, we also obtain a natural outer action $\pi_1(T, s) \to \mathrm{Out}(\pi_1(Y \times_T \overline{\eta}, s)^{p'})$.

Proof. We need to show that the action $\pi_1(Y \times_T \eta, s) \to \mathrm{Aut}(\pi_1(Y \times_T \overline{\eta}, s)^{p'})$ factors through $\pi_1(Y \times_T \eta, s) \to \pi_1(Y, s)$.

First, we show that we can reduce to the case where the scheme T is the spectrum of a strictly henselian discrete valuation ring. Since the scheme Y is regular, the kernel of the morphism $\pi_1(Y \times_T \eta, s) \to \pi_1(Y, s)$ is generated by inertia subgroups at points of codimension 1 in $Y \setminus (Y \times_T \eta)$. Consider such a point y and write t for the image of y in T. Since Y is flat over T, t is codimension 1 in T. Choose a strict henselization $O_{T,t}^{\mathrm{sh}} \subset K(T)^{\mathrm{sep}}$ of the discrete valuation ring $O_{T,t}$ and write η^{sh} for the scheme $\operatorname{Spec}\operatorname{Frac}(O_{T,t}^{\mathrm{sh}})$. Then the Galois group $\mathrm{Gal}(K(T)^{\mathrm{sep}}/\mathrm{Frac}(O_{T,t}^{\mathrm{sh}}))$ is an inertia subgroup of t in $\pi_1(\eta, s)$. Fix a separable closure $K(Y \times_T \overline{\eta})^{\mathrm{sep}}$ of the function field of $Y \times_T \overline{\eta}$. Take a strict henselization $O_{Y,y}^{\mathrm{sh}}$ of the local ring $O_{Y,y}$ such that the diagram

$$\begin{array}{ccc}
\operatorname{Spec} K(Y \times_T \overline{\eta})^{\mathrm{sep}} & \longrightarrow & \operatorname{Spec} O_{Y,y} \times_T \overline{\eta} \\
\downarrow & & \downarrow \\
\operatorname{Spec} O_{Y,y}^{\mathrm{sh}} & \longrightarrow & \operatorname{Spec} O_{Y,y} \times_T \operatorname{Spec} O_{T,t}^{\mathrm{sh}}
\end{array}$$

commutes. Then the inertia subgroup of y in $\pi_1(Y \times_T \eta, s)$ associated with $\operatorname{Spec} O_{Y,y}^{\mathrm{sh}}$ is sent into $\mathrm{Gal}(K(T)^{\mathrm{sep}}/\mathrm{Frac}(O_{T,t}^{\mathrm{sh}})) \subset \pi_1(\eta, s)$. Therefore, we have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(Y \times_T \overline{\eta}, s) & \longrightarrow & \pi_1(Y \times_T \eta^{\mathrm{sh}}, s) & \longrightarrow & \pi_1(\eta^{\mathrm{sh}}, s) & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi_1(Y \times_T \overline{\eta}, s) & \longrightarrow & \pi_1(Y \times_T \eta, s) & \longrightarrow & \pi_1(\eta, s) & \longrightarrow & 1.
\end{array}$$

The inertia subgroup of y in $\pi_1(Y \times_T \eta, s)$ associated with $O_{Y,y}^{\mathrm{sh}}$ is contained in $\pi_1(Y \times_T \eta^{\mathrm{sh}}, s)$ and coincides with the inertia subgroup in $\pi_1(Y \times_T \eta^{\mathrm{sh}}, s)$ associated with the strict localization $\operatorname{Spec} O_{Y,y}^{\mathrm{sh}} \to Y \times_T \operatorname{Spec} O_{T,t}^{\mathrm{sh}}$. Thus, to prove this proposition, it suffices to show that the homomorphism $\pi_1(Y \times_T \eta^{\mathrm{sh}}, s) \to \mathrm{Aut}(\pi_1(Y \times_T \overline{\eta})^{p'})$ factors through $\pi_1(Y \times_T \eta^{\mathrm{sh}}, s) \to \pi_1(Y \times_T \operatorname{Spec} O_{T,t}^{\mathrm{sh}}, s)$. Therefore, we can reduce 3.2 to the case where T is the spectrum of a strictly henselian discrete valuation ring.

Suppose that T is the spectrum of a strictly henselian discrete valuation ring. We have a commutative diagram with an exact horizontal line

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(Y \times_T \overline{\eta}, s)^{p'} & \longrightarrow & \pi_1(Y \times_T \eta, s)^{(p')} & \longrightarrow & \pi_1(\eta, s) & \longrightarrow & 1 \\
 & & \downarrow & \swarrow & & & & & \\
 & & \pi_1(Y, s)^{p'}. & & & & & &
\end{array}$$

by Notation-Proposition 3.1. Since the specialization homomorphism $\pi_1(Y \times_T \overline{\eta}, s)^{p'} \to \pi_1(Y, s)^{p'}$ is an isomorphism by Theorem [11, Theorem 8.3], the assertions follow. □

We start the proof of Theorem 1.3 and the rest of the assertions in Theorem 1.1 and 1.2. When we are in the situation of Theorem 1.1, fix a sequence of parameterizing morphisms

$$X = X_n \overset{f_n}{\to} X_{n-1} \overset{f_{n-1}}{\to} \ldots \overset{f_{i+1}}{\to} X_i \overset{f_i}{\to} X_{i-1} \overset{f_{i-1}}{\to} \ldots \overset{f_2}{\to} X_1 \overset{f_1}{\to} X_0 = \operatorname{Spec} K$$

of $X \to \operatorname{Spec} K$ whose maximal genus is g_X and write b for $2g_X$. When we are in the situation of Theorem 1.2 or 1.3, we write b for $b_{\mathcal{S}}$. Take a geometric point $*$ of the scheme $X \otimes_K K^{\mathrm{sep}}$. Write Δ_i (resp. Π_i) for the étale fundamental group $\pi_1(X_i \otimes_K K^{\mathrm{sep}}, *)$ (resp. $\pi_1(X_i, *)$). By [15, Exposé IX, Theorem 6.1], we have the following homotopy exact sequences of profinite groups

$$1 \to \Delta_i \to \Pi_i \to G_K \to 1. \tag{3.5}$$

Assume that (B) or the conditions in Theorem 1.1.4 holds. Under the hypotheses given in Theorem 1.1 and Theorem 1.2, we will show that (A) holds by induction on n. For $n = 1$, this is proved in Proposition 2.1 and 2.2. Hence, we assume that $n \geq 2$.

First, we prove that (A) holds for X_{n-1}. Since the morphism $X_{n-1} \to \operatorname{Spec} K$ has a structure of a hyperbolic polycurve associated with the above sequence of parameterizing morphisms, we can apply the main theorems by the induction hypothesis if we show that the outer Galois representation

$$I_K \to \operatorname{Out}(\Delta_{n-1}^{p'}) \tag{3.6}$$

is trivial. Since $X_n \to X_{n-1}$ has geometrically connected fibers, the homomorphism $\Delta_n \to \Delta_{n-1}$ is surjective. Then we can show that the outer Galois representation (3.6) is trivial by using the commutative diagram of profinite groups with exact horizontal lines

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_n & \longrightarrow & \Pi_n & \longrightarrow & G_K & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & \Delta_{n-1} & \longrightarrow & \Pi_{n-1} & \longrightarrow & G_K & \longrightarrow & 1. \end{array}$$

When we are in the situation of Theorem 1.1, we may assume that there exists a smooth model $\mathfrak{X}_{n-1}$ of X_{n-1}. When we are in the situation of Theorem 1.2, we may assume that there exists a hyperbolic polycurve

$$\mathfrak{X}_{n-1} \to \ldots \to \mathfrak{X}_1 \to \mathfrak{X}_0 = \operatorname{Spec} O_K \tag{3.7}$$

whose generic fiber is isomorphic to the sequence of parameterizing morphisms

$$X_{n-1} \overset{f_{n-1}}{\to} \dots \overset{f_2}{\to} X_1 \overset{f_1}{\to} \operatorname{Spec} K. \tag{3.8}$$

Consider the following diagram:

$$\begin{array}{ccccccc}
X_n \times_{X_{n-1}} \operatorname{Spec} K(X_{n-1}) & \longrightarrow & X_n & \dashrightarrow & \cdot & \dashleftarrow & \\
\big\downarrow f'_n & & \big\downarrow f_n & & \vdots & & \vdots \\
\operatorname{Spec} K(X_{n-1}) & \longrightarrow & X_{n-1} & \longrightarrow & \mathfrak{X}_{n-1} & \longleftarrow & \operatorname{Spec} O_{\mathfrak{X}_{n-1},\xi},
\end{array}$$

where $K(X_{n-1})$ is the function field of X_{n-1}, ξ is the generic point of the special fiber $\mathfrak{X}_{n-1} \setminus X_{n-1}$, f'_n is the base change of f_n, and $O_{\mathfrak{X}_{n-1},\xi}$ is the local ring of $\mathfrak{X}_{n-1}$ at ξ. It suffices to show that there exists a hyperbolic curve $(\overline{\mathfrak{X}}_n, \mathfrak{D}_\mathfrak{n})$ over $\mathfrak{X}_{n-1}$ such that the scheme $(\overline{\mathfrak{X}}_n \setminus \mathfrak{D}_n) \times_{\mathfrak{X}_{n-1}} X_{n-1}$ is isomorphic to X_n over X_{n-1}. By Proposition 7.1 (cf. [8]), it suffices to show that there exists a hyperbolic curve $(\overline{\mathfrak{X}}, \mathfrak{D})$ over $\operatorname{Spec} O_{\mathfrak{X}_{n-1},\xi}$ such that the scheme $(\overline{\mathfrak{X}} \setminus \mathfrak{D}) \otimes_{O_{\mathfrak{X}_{n-1},\xi}} K(X_{n-1})$ is isomorphic to $X_n \times_{X_{n-1}} \operatorname{Spec} K(X_{n-1})$ over $\operatorname{Spec} K(X_{n-1})$. Let $I_\xi \subset G_{K(X_{n-1})}$ be an inertia subgroup ξ and $\overline{t}$ a geometric point of $X_n \times_{X_{n-1}} \operatorname{Spec} K(X_{n-1})$. Write $\Delta_{n,n-1}$ for the étale fundamental group $\pi_1(X_n \times_{X_{n-1}} \overline{t}, \overline{t})$. To complete the proof, it suffices to show that the outer representation of I_ξ on $\Delta^l_{n,n-1}$ is trivial for some prime number $l \neq p$ by Proposition 2.1. Note that the homomorphism

$$I_\xi \to \operatorname{Out} \Delta^l_{n,n-1}$$

is the composite of the natural morphisms

$$I_\xi \to G_{K(X_{n-1})} \to \pi_1(X_{n-1}, \overline{t}) \tag{3.9}$$

and the outer representation $\pi_1(X_{n-1}, \overline{t}) \to \operatorname{Out} \Delta^l_{n,n-1}$ constructed in Proposition 3.2. Write I for the image of the inertia subgroup I_ξ in (3.9) to $\pi_1(X_{n-1}, \overline{t})$, which coincides with an inertia subgroup of $\pi_1(X_{n-1}, \overline{t})$ at ξ. Therefore, we have

$$\operatorname{Ker}(\Pi_{n-1} \to \pi_1(\mathfrak{X}_{n-1}, \overline{t})) = (\text{the subgroup topologically normally generated by } I).$$

Then we have the following proposition in summary:

Proposition 3.3. *To prove Theorem 1.1.2, 3, 4 and Theorem 1.2.2, 3, it suffices to show that the outer action $I \to \operatorname{Out} \Delta^l_{n,n-1}$ is trivial for some prime number $l \neq p$.*

By the argument as in the proof of Proposition 3.2, the image of I by the homomorphism $\Pi_{n-1} \to G_K$ is contained in some inertia subgroup of G_K. Therefore, we may assume that K is strictly henselian.

4. The case of residual characteristic 0

We prove Theorem 1.1.2 and Theorem 1.2.2 in this section. Hence, we may assume that $p = 0$ and the field K is strictly henselian. By Proposition 3.3, it suffices to show that the outer action $I \to \operatorname{Out} \Delta_{n,n-1}$ is trivial.

Since the characteristic of the field K is also 0, we have homotopy exact sequences

$$1 \to \Delta_{n,n-1} \to \Pi_n \to \Pi_{n-1} \to 1$$

and

$$1 \to \Delta_{n,n-1} \to \Delta_n \to \Delta_{n-1} \to 1 \tag{4.1}$$

by [5, PROPOSITION 2.5]. Note that Δ_n is center-free, which follows from [19, Proposition 1.11], the homotopy exact sequence (4.1), and the fact that an extension group of center-free groups is also center-free. For a group G and a subgroup H of G, we write $Z_G(H)$ for the centralizer subgroup of H in G.

Lemma 4.1. *There exist decompositions of profinite groups*

$$\Pi_n = \Delta_n \times Z_{\Pi_n}(\Delta_n)$$

and

$$\Pi_{n-1} = \Delta_{n-1} \times Z_{\Pi_{n-1}}(\Delta_{n-1}),$$

which are compatible with the natural surjection $\Pi_n \to \Pi_{n-1}$.

Proof. Since Δ_n is center-free, we have

$$\Delta_n \cap Z_{\Pi_n}(\Delta_n) = \{1\}.$$

Moreover, since K is strictly henselian and the outer action

$$I_K(\cong \Pi_n/\Delta_n) \to \operatorname{Out}(\Delta_n)$$

is trivial, we have a canonical decomposition $\Pi_n = \Delta_n \times Z_{\Pi_n}(\Delta_n)$. By the same argument, we obtain a canonical decomposition $\Pi_{n-1} = \Delta_{n-1} \times Z_{\Pi_{n-1}}(\Delta_{n-1})$. The homomorphism $\Pi_n \to \Pi_{n-1}$ is compatible with the homomorphism $\Delta_n \to \Delta_{n-1}$, and hence $Z_{\Pi_n}(\Delta_n) \to Z_{\Pi_n}(\Delta_{n-1})$ is well-defined. Therefore, these decompositions of the groups are compatible. □

Lemma 4.2.

$$I \subset Z_{\Pi_{n-1}}(\Delta_{n-1}).$$

Proof. We have the following commutative diagram with exact horizontal line:

$$\begin{array}{ccccccccc} & & & & I & & & & \\ & & & & \big\downarrow & & & & \\ 1 & \longrightarrow & \Delta_{n-1} & \longrightarrow & \Pi_{n-1} & \longrightarrow & G_K & \longrightarrow & 1 \\ & & & \searrow & \big\downarrow & & & & \\ & & & & \pi_1(\mathfrak{X}_{n-1}, \overline{t}). & & & & \end{array}$$

Here, the homomorphism $\Delta_{n-1} \to \pi_1(\mathfrak{X}_{n-1}, \overline{t})$ is an isomorphism by Theorem [11, Theorem 8.3]. Then we obtain a decomposition

$$\Pi_{n-1} \cong \Delta_{n-1} \times \mathrm{Ker}(\Pi_{n-1} \to \pi_1(\mathfrak{X}_{n-1}, \overline{t})). \tag{4.2}$$

Therefore, $I \subset \mathrm{Ker}(\Pi_{n-1} \to \pi_1(\mathfrak{X}_{n-1}, \overline{t})) \subset Z_{\Pi_{n-1}}(\Delta_{n-1})$ holds. □

Now, we prove Theorem 1.1.2 and Theorem 1.2.2.

Proofs of Theorem 1.1.2 and Theorem 1.2.2. By Lemma 4.2, it suffices to show that the outer action

$$Z_{\Pi_{n-1}}(\Delta_{n-1}) \to \mathrm{Out}\,\Delta_{n,n-1}$$

is trivial. By Lemma 4.1, the homomorphism $Z_{\Pi_n}(\Delta_n) \to Z_{\Pi_{n-1}}(\Delta_{n-1})$ is surjective. Therefore, the outer action (4.2) is trivial. We finished the proof of Theorem 1.1.2 and Theorem 1.2.2. □

5. The case of residual characteristic $p > 0$

We maintain the notation of Section 3. In this section, we will prove Theorem 1.1.3, 4 and Theorem 1.2.3. Hence, we assume that $p > b + 1$. It suffices to prove that the outer action $I \to \mathrm{Out}\,\Delta^l_{n,n-1}$ is trivial for some prime number $l \neq p$ by Proposition 3.3. Let us take a prime number l which is a generator of the cyclic group $(\mathbb{Z}/p\mathbb{Z})^*$, whose existence follows from the theorem on arithmetic progression. Since $l, l^2, \ldots, l^b$ are not $1 \in (\mathbb{Z}/p\mathbb{Z})^*$ by the hypothesis $p > b+1$, the order of the group $\mathrm{GL}(b, \mathbb{F}_l)$ is not divisible by p. Also, the profinite group $\mathrm{Ker}(\mathrm{Aut}(\Delta^l_{n,n-1}) \to \mathrm{Aut}((\Delta^l_{n,n-1})^{\mathrm{ab}}/l(\Delta^l_{n,n-1})^{\mathrm{ab}}))$ is pro-l by a well-known theorem of P.Hall. Here, the superscript "ab" denotes the abelianization of the profinite group. Therefore, the profinite groups $\mathrm{Aut}(\Delta^l_{n,n-1})$ and $\mathrm{Out}(\Delta^l_{n,n-1})$ are pro-prime-to-p.

Note that we have two exact sequences of profinite groups

$$\Delta_{n,n-1} \to \Pi_n \to \Pi_{n-1} \to 1$$

and

$$\Delta_{n,n-1} \to \Delta_n \to \Delta_{n-1} \to 1$$

by [5, PROPOSITION 1.10]. Unfortunately, if the characteristic of K is equal to p, the homomorphism $\Delta_{n,n-1} \to \Delta_n$ is not always injective (cf. [20, Theorem (0.3)]).

We have the following commutative diagram with exact horizontal lines:

$$
\begin{array}{ccccccccc}
 & & \Delta_{n,n-1} & \longrightarrow & \Delta_n & \longrightarrow & \Delta_{n-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Delta^l_{n,n-1}) & \longrightarrow & \mathrm{Aut}(\Delta^l_{n,n-1}) & \longrightarrow & \mathrm{Out}(\Delta^l_{n,n-1}) & \longrightarrow & 1.
\end{array}
$$

Here, the homomorphism $\Delta_n \to \mathrm{Aut}(\Delta^l_{n,n-1})$ is constructed in Proposition 3.2. Since the profinite group $\mathrm{Out}(\Delta^l_{n,n-1})$ is a pro-prime-to-p group, the outer action $\Delta_{n-1} \to \mathrm{Out}(\Delta^l_{n,n-1})$ factors through $\Delta^{p'}_{n-1}$. We write $\Delta^{(l,p')}_n$ for the pull-back $\mathrm{Aut}(\Delta^l_{n,n-1}) \times_{\mathrm{Out}(\Delta^l_{n,n-1})} \Delta^{p'}_{n-1}$. Then we have the following commutative diagram with exact horizontal lines:

$$
\begin{array}{ccccccccc}
 & & \Delta_{n,n-1} & \longrightarrow & \Delta_n & \longrightarrow & \Delta_{n-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Delta^l_{n,n-1}) & \longrightarrow & \Delta^{(l,p')}_n & \longrightarrow & \Delta^{p'}_{n-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Delta^l_{n,n-1}) & \longrightarrow & \mathrm{Aut}(\Delta^l_{n,n-1}) & \longrightarrow & \mathrm{Out}(\Delta^l_{n,n-1}) & \longrightarrow & 1.
\end{array}
$$

Since the profinite group $\Delta^l_{n,n-1}$ is center-free, the homomorphism $\Delta^l_{n,n-1} \to \mathrm{Inn}(\Delta^l_{n,n-1})$ is an isomorphism. Therefore, we obtain an exact sequence

$$1 \to \Delta^l_{n,n-1} \to \Delta^{(l,p')}_n \to \Delta^{p'}_{n-1} \to 1. \tag{5.1}$$

Next, consider the following diagram of exact sequences with exact horizontal lines:

$$
\begin{array}{ccccccccc}
 & & \Delta_{n,n-1} & \longrightarrow & \Pi_n & \longrightarrow & \Pi_{n-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Delta^l_{n,n-1}) & \longrightarrow & \mathrm{Aut}(\Delta^l_{n,n-1}) & \longrightarrow & \mathrm{Out}(\Delta^l_{n,n-1}) & \longrightarrow & 1.
\end{array}
$$

Here, the homomorphism $\Pi_n \to \mathrm{Aut}(\Delta^l_{n,n-1})$ is constructed in Proposition 3.2. By applying Notation-Proposition 3.1 to the exact sequences (3.5), we have an exact sequence of profinite groups

$$1 \to \Delta_{n-1}^{p'} \to \Pi_{n-1}^{(p')} \to G_K \to 1. \tag{5.2}$$

We write $\Pi_n^{((l,p'))}$ for the pull-back $\mathrm{Aut}(\Delta_{n,n-1}^{l}) \times_{\mathrm{Out}(\Delta_{n,n-1}^{l})} \Pi_{n-1}^{(p')}$. Then we have the following commutative diagram with exact horizontal lines:

$$\begin{array}{ccccccccc}
 & & \Delta_{n,n-1} & \longrightarrow & \Pi_n & \longrightarrow & \Pi_{n-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Delta_{n,n-1}^{l}) & \longrightarrow & \Pi_n^{((l,p'))} & \longrightarrow & \Pi_{n-1}^{(p')} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\Delta_{n,n-1}^{l}) & \longrightarrow & \mathrm{Aut}(\Delta_{n,n-1}^{l}) & \longrightarrow & \mathrm{Out}(\Delta_{n,n-1}^{l}) & \longrightarrow & 1.
\end{array}$$

Therefore, we obtain an exact sequence

$$1 \to \Delta_{n,n-1}^{l} \to \Pi_n^{((l,p'))} \to \Pi_{n-1}^{(p')} \to 1. \tag{5.3}$$

From the three exact sequences (5.1), (5.2), and (5.3), we have an exact sequence

$$1 \to \Delta_n^{(l,p')} \to \Pi_n^{((l,p'))} \to G_K \to 1.$$

Proofs of Theorem 1.1.3 and Theorem 1.2.3. If $n = 2$, the profinite groups $\Delta_{2,1}^{l}$ and $\Delta_1^{p'}$ are center-free. Therefore, the profinite group $\Delta_2^{(l,p')}$ is also center-free. Since Lemma 4.1 and Lemma 4.2 work if we replace $\Pi_n, \Delta_n, \Pi_{n-1}$ and Δ_{n-1} by $\Pi_2^{((l,p'))}, \Delta_2^{(l,p')}, \Pi_1^{(p')}$ and $\Delta_1^{p'}$, we can prove that the outer action $I \to \mathrm{Out}\,\Delta_{2,1}^{l}$ is trivial. Thus, we finished the proofs of Theorem 1.1.3 and Theorem 1.2.3. $\square$

Proof of Theorem 1.1.4. Suppose that X has a K-rational point x and that the Galois representation $I_{K(x)} \to \mathrm{Aut}(\pi_1(X \otimes_K K^{\mathrm{sep}}, \overline{x})^{p'})$ defined as in (2.4) in Section 2 is trivial. Take a path from the fundamental functor defined by $\overline{x} \to X \otimes_K K^{\mathrm{sep}}$ to that defined by $\overline{t} \to X \otimes_K K^{\mathrm{sep}}$. Then we have an induced homomorphism $I_{K(x)} \to \Pi_n$ and the induced action of $I_{K(x)}$

on $\Delta_n^{p'}$ is trivial. We have a commutative diagram with exact horizontal lines

$$\begin{array}{ccccccccc}
 & & & & I_{K(x)} & & I & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & \Delta_{n,n-1} & \longrightarrow & \Pi_n & \longrightarrow & \Pi_{n-1} & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta^l_{n,n-1} & \longrightarrow & \Pi_n^{((l,p'))} & \longrightarrow & \Pi_{n-1}^{(p')} & \longrightarrow & 1 \\
 & & \cap\!\downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta_n^{(l,p')} & \longrightarrow & \Pi_n^{((l,p'))} & \longrightarrow & G_K & \longrightarrow & 1.
\end{array}$$

Since the action of $I_{K(x)}$ on $\Delta_n^{p'}$ is trivial, the action of $I_{K(x)}$ on $\Delta_n^{(l,p')}$ is also trivial. Hence, the action of $I_{K(x)}$ on $\Delta^l_{n,n-1}$ is trivial. Since the image of I in $\Pi_{n-1}^{(p')}$ is contained in $\mathrm{Ker}(\Pi_{n-1}^{(p')} \to \pi_1(\mathfrak{X}_{n-1}, \bar{t})^{p'})$, it suffices to show that the image of $I_{K(x)}$ in $\Pi_{n-1}^{(p')}$ coincides with $\mathrm{Ker}(\Pi_{n-1}^{(p')} \to \pi_1(\mathfrak{X}_{n-1}, \bar{t})^{p'})$ to show that the outer action $I \to \mathrm{Out}\Delta^l_{n,n-1}$ is trivial.

By valuative criterion, the composite morphism $\mathrm{Spec}\, K(x) \to X_{n-1} \to \mathfrak{X}_{n-1}$ factors the morphism $\mathrm{Spec}\, K(x) \to \mathrm{Spec}\, O_{K(x)}$. Therefore, we have a natural homomorphism $(I_{K(x)} =)\, G_{K(x)} \to \mathrm{Ker}(\Pi_{n-1}^{(p')} \to \pi_1(\mathfrak{X}_{n-1}, \bar{t})^{p'})$. Since the composite morphism $\Delta^{p'}_{n-1} \to \Pi_{n-1}^{(p')} \to \pi_1(\mathfrak{X}_{n-1}, \bar{t})^{p'}$ is an isomorphism by Theorem [11, Theorem 8.3], the composite homomorphism $\mathrm{Ker}(\Pi_{n-1}^{(p')} \to \pi_1(\mathfrak{X}_{n-1}, \bar{t})^{p'}) \hookrightarrow \Pi_{n-1}^{(p')} \to G_K$ is an isomorphism by the exactness of the sequence (5.2). Therefore, the homomorphism $(I_{K(x)} =)\, G_{K(x)} \to \mathrm{Ker}(\Pi_{n-1}^{(p')} \to \pi_1(\mathfrak{X}_{n-1}, \bar{t})^{p'})$ is an isomorphism. Thus, we finished the proof of Theorem 1.1.4. □

6. The case of residual characteristic $p \gg 0$

In this section, we prove Theorem 1.3. To prove Theorem 1.3, we need a very strong condition on p. In this section, we maintain the notation of Section 5 and suppose that the outer Galois representation $I_K \to \mathrm{Out}\, \Delta^l_X$ is trivial.

Lemma 6.1 (cf. [14, Lemma 2.18] and [1, Proposition 3])**.** *Let*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & N & \longrightarrow & \tilde{G} & \longrightarrow & \tilde{H} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ & & N & \longrightarrow & G & \longrightarrow & H & \longrightarrow & 1 \end{array} \tag{6.1}$$

be a commutative diagram of profinite groups with exact horizontal lines such that the middle and the right vertical lines are surjective. Suppose that the conjugate action $\tilde{G} \to \mathrm{Aut}(N^l)$ *admits a factorization*

$$\tilde{G} \to G \to \mathrm{Aut}(N^l). \tag{6.2}$$

Moreover, suppose that N^l *is topologically finitely generated and center-free. Then the following are equivalent:*

(1) *The pro-*l *completion of the sequence*

$$1 \to N^l \to G^l \to H^l \to 1$$

induced by the lower line of (6.1) is exact.

(2) *The induced homomorphism* $N^l \to G^l$ *is injective.*

(3) *The action* $G \to \mathrm{Aut}(N^l)$ *given in (6.2) factors through* $G \to G^l$*.*

(4) *The outer action* $H \to \mathrm{Out}(N^l)$ *induced by (6.2) factors through* $H \to H^l$*.*

(5) *The action* $H \to \mathrm{Aut}(N^{\mathrm{ab},l}/lN^{\mathrm{ab},l})$ *induced by (6.2) factors through* $H \to H^l$*.*

Proof. The implications $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 5$ are trivial. Since N^l is topologically finitely generated, the profinite group $\mathrm{Ker}(\mathrm{Aut}(N^l) \to \mathrm{Aut}((N^{\mathrm{ab}}/l(N^l)^{\mathrm{ab}}))$ is pro-l by a well-known theorem of P.Hall. Therefore, the implication $5 \Rightarrow 4$ holds. The image of the homomorphism $G \to \mathrm{Aut}(N^l)$ is pro-l if and only if the image of the homomorphism $H \to \mathrm{Out}(N^l)$ is pro-l. Therefore, the implication $4 \Rightarrow 3$ holds. Since N^l is center-free, the homomorphism $N^l \to \mathrm{Inn}(N^l)$ is isomorphic, and hence the composite homomorphism $N^l \to \mathrm{Inn} \to \mathrm{Aut}(N^l)$ is injective. Therefore, the implication $3 \Rightarrow 2$ holds. Since taking pro-l completion is a right exact functor, the implication $2 \Rightarrow 1$ holds. □

Proposition 6.2. *If there exist a quotient group* $\Delta_n^{(l,p')} \to \overline{\Delta}_n$ *and a center-free quotient group* $\Delta_{n-1}^{p'} \to \overline{\Delta}_{n-1}$ *such that the diagram*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{n,n-1}^l & \longrightarrow & \Delta_n^{(l,p')} & \longrightarrow & \Delta_{n-1}^{p'} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_{n,n-1}^l & \longrightarrow & \overline{\Delta}_n & \longrightarrow & \overline{\Delta}_{n-1} & \longrightarrow & 1 \end{array}$$

commutes and the second horizontal line is exact, then the outer action $I \to \mathrm{Out}(\Delta^l_{n,n-1})$ *is trivial.*

Proof. Since the outer actions $I_K \to \mathrm{Out}(\Delta^{p'}_n)$ and $I_K \to \mathrm{Out}(\Delta^{p'}_{n-1})$ are trivial, we have $\Pi^{((l,p'))}_n = \Delta^{(l,p')}_n Z_{\Pi^{((l,p'))}_n}(\Delta^{(l,p')}_n)$ and $\Pi^{(p')}_{n-1} = \Delta^{p'}_{n-1} Z_{\Pi^{(p')}_{n-1}}(\Delta^{p'}_{n-1})$. Therefore, the normal subgroup $K_n \overset{\mathrm{def}}{=} \mathrm{Ker}(\Delta^{(l,p')}_n \to \overline{\Delta}_n)$ (resp. $K_{n-1} \overset{\mathrm{def}}{=} \mathrm{Ker}(\Delta^{p'}_{n-1} \to \overline{\Delta}_{n-1})$) of $\Delta^{(l,p')}_n$ (resp. $\Delta^{p'}_{n-1}$) is also a normal subgroup of $\Pi^{((l,p'))}_n$ (resp. $\Pi^{(p')}_{n-1}$). Thus, we have quotient groups $\overline{\Pi}_n \overset{\mathrm{def}}{=} \Pi^{((l,p'))}_n / K_n$ and $\overline{\Pi}_{n-1} \overset{\mathrm{def}}{=} \Pi^{(p')}_{n-1}/K_{n-1}$ such that the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^l_{n,n-1} & \longrightarrow & \overline{\Delta}_n & \longrightarrow & \overline{\Delta}_{n-1} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^l_{n,n-1} & \longrightarrow & \overline{\Pi}_n & \longrightarrow & \overline{\Pi}_{n-1} & \longrightarrow & 1 \end{array}$$

commutes. Since the group $\overline{\Delta}_{n-1}$ is center-free, we have a decomposition $\overline{\Pi}_{n-1} = \overline{\Delta}_{n-1} \times Z_{\overline{\Pi}_{n-1}}(\overline{\Delta}_{n-1})$. If we replace Δ_{n-1}, Π_{n-1}, and $\pi_1(\mathfrak{X}_{n-1}, \bar{t})$ in Lemma 4.2 by $\overline{\Delta}_{n-1}$, $\overline{\Pi}_{n-1}$, and $\pi_1(\mathfrak{X}_{n-1}, \bar{t})/\mathrm{Im}(K_{n-1} \to \pi_1(\mathfrak{X}_{n-1}, \bar{t}))$, it follows that the image of the group I in $\overline{\Pi}_{n-1}$ is contained in $Z_{\overline{\Pi}_{n-1}}(\overline{\Delta}_{n-1})$. Since the homomorphism $\overline{\Pi}_n \to \overline{\Pi}_{n-1}$ is surjective and compatible with the homomorphism $\overline{\Delta}_n \to \overline{\Delta}_{n-1}$, the image of $Z_{\overline{\Pi}_n}(\overline{\Delta}_n)$ in $\overline{\Pi}_{n-1}$ coincides with $Z_{\overline{\Pi}_{n-1}}(\overline{\Delta}_{n-1})$. Thus, the outer action $I \to \mathrm{Out}\,\Delta^l_{n,n-1}$, which factors through $Z_{\overline{\Pi}_{n-1}}(\overline{\Delta}_{n-1}) \to \mathrm{Out}\,\Delta^l_{n,n-1}$, is trivial (cf. the proof at the end of Section 4). □

Lemma 6.3. *Assume that* $n \geq 3$. *Let* Δ'_{n-1} *be an open normal subgroup of* Δ_{n-1}. *Write* Δ'_i *(resp.* Δ'_n*) for the images of* Δ'_{n-1} *in* Δ_i *for* $1 \leq i \leq n-2$ *(resp. the inverse image of* Δ'_{n-1} *in* Δ_n*),* $\Delta'_{i,i-1}$ *for the inverse images of* Δ'_i *in* $\Delta_{i,i-1}$ *for* $2 \leq i \leq n-1$*,* Γ_i *(resp.* $\Gamma_{i,i-1}$*) for the group* Δ_i/Δ'_i *(resp.* $\Delta_{i,i-1}/\Delta'_{i,i-1}$*) for* $1 \leq i \leq n$ *(resp.* $2 \leq i \leq n-1$*),* $\Delta^{(l)}_i$ *(resp.* $\Delta^{(l)}_{i,i-1}$*) for the quotient group of* Δ_i *(resp.* $\Delta_{i,i-1}$*) defined by Notation-Proposition 3.1 and the exact sequence* $1 \to \Delta'_i \to \Delta_i \to \Gamma_i \to 1$ *(resp.* $1 \to \Delta'_{i,i-1} \to \Delta_{i,i-1} \to \Gamma_{i,i-1} \to 1$ *) for* $1 \leq i \leq n$ *(resp.* $2 \leq i \leq n-1$*). Suppose that the sequence* $1 \to (\Delta'_{i,i-1})^l \to (\Delta'_i)^l \to (\Delta'_{i-1})^l \to 1$ *is exact for each* $2 \leq i \leq n-1$.

1. *The profinite group* $(\Delta'_i)^l$ *is center-free for each* $2 \leq i \leq n-1$.
2. *Write* $\tilde{X}'_i$ *for the Galois covering of* $\tilde{X}_i \overset{\mathrm{def}}{=} X_i \otimes_K K^{\mathrm{sep}}$ *corresponding to the normal subgroup* Δ'_i *of* Δ_i. *Then the composite homomorphism*

$\Gamma_i(= \mathrm{Aut}(\tilde{X}'_i/\tilde{X}_i)) \hookrightarrow \mathrm{Aut}(\tilde{X}'_i/\mathrm{Spec}\, K^{\mathrm{sep}}) \to \mathrm{Out}((\Delta'_i)^l)$ *is injective for each* $1 \leq i \leq n-1$.

(3) Suppose moreover that the composite homomorphism

$$\Delta'_{n-1} \hookrightarrow \Delta_{n-1} \to \mathrm{Out}(\Delta^l_{n,n-1}) \to \mathrm{Aut}(\Delta^{l,\mathrm{ab}}_{n,n-1}/l\Delta^{l,\mathrm{ab}}_{n,n-1}),$$

where $\Delta_{n-1} \to \mathrm{Out}(\Delta^l_{n,n-1})$ *is constructed in Proposition 3.2, is trivial and* Γ_{n-1} *is of order prime-to-p. Then the group* $\Delta^{(l)}_{n-1}$ *is center-free and we have the following commutative diagram with exact horizontal lines:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta^l_{n,n-1} & \longrightarrow & \Delta^{(l,p')}_n & \longrightarrow & \Delta^{p'}_{n-1} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta^l_{n,n-1} & \longrightarrow & \Delta^{(l)}_n & \longrightarrow & \Delta^{(l)}_{n-1} & \longrightarrow & 1. \end{array}$$

Proof. Assertion 1 follows from [19, Proposition 1.11] and the fact that an extension group of center-free groups is also center-free.

Next, We prove assertion 2. Note that, if the characteristic of K is 0, this is a special case of [14, Proposition 3.2]. We will show the injectivity of the outer action $\Gamma_i \to \mathrm{Out}((\Delta'_i)^l)$ by induction on i. To show that the outer action $\Gamma_1 \to \mathrm{Out}((\Delta'_1)^l)$ is injective, it suffices to show that the action $\Gamma_1 \to \mathrm{Aut}((\Delta'_1)^{l,\mathrm{ab}})$ is injective. Let K^{alg} be an algebraic closure of K^{sep} and write Y (resp. Y') for the scheme $\tilde{X}_1 \otimes_{K^{\mathrm{sep}}} K^{\mathrm{alg}}$ (resp. $\tilde{X}'_1 \otimes_{K^{\mathrm{sep}}} K^{\mathrm{alg}}$). There exists a hyperbolic curve $(\overline{Y'}, E')$ over K^{alg} such that $\overline{Y'} \setminus E'$ is isomorphic to Y' over K^{alg}. Such a pair always exists since we work over K^{alg}. Let ϕ be an element of the group $\Gamma_1(= \mathrm{Aut}(\tilde{X}'_1/\tilde{X}_1) = \mathrm{Aut}(Y'/Y))$ whose image in the group

$$\mathrm{Aut}((\Delta'_1)^{l,\mathrm{ab}})(= \mathrm{Aut}((\pi_1(\tilde{X}'_1, *)^{l,\mathrm{ab}}) = \mathrm{Aut}((\pi_1(Y', *)^{l,\mathrm{ab}}))$$

is trivial. Here, $*$ is a geometric point of Y'. Then ϕ induces the identity on E'. Therefore, if the genus of $\overline{Y'}$ is equal to or less than 1 (resp. equal to or more than 2), ϕ is the identity by the theory of automorphisms of rational curves and elliptic curves (resp. by [3, Theorem 1.13]).

Suppose that assertion 2 holds for each $1 \leq i \leq n-2$. Write $\mathrm{Aut}((\Delta'_{n-1})^l, (\Delta'_{n-2})^l)$ for the subgroup of $\mathrm{Aut}((\Delta'_{n-1})^l)$ consisting of automorphisms of $(\Delta'_{n-1})^l$ inducing automorphisms of the quotient group $(\Delta'_{n-2})^l$ and $\mathrm{Out}((\Delta'_{n-1})^l, (\Delta'_{n-2})^l)$ for the quotient group of $\mathrm{Aut}((\Delta'_{n-1})^l, (\Delta'_{n-2})^l)$ by the inner subgroup $\mathrm{Inn}((\Delta'_{n-1})^l)$.

Then we have the following diagram:

$$\begin{array}{ccccc}
\Gamma_{n-1} & \longrightarrow & \mathrm{Out}((\Delta'_{n-1})^l, (\Delta'_{n-2})^l) & \hookrightarrow & \mathrm{Out}((\Delta'_{n-1})^l) \\
\downarrow & & \downarrow & & \\
\Gamma_{n-2} & \longrightarrow & \mathrm{Out}((\Delta'_{n-2})^l), & &
\end{array}$$

where the homomorphism $\Gamma_{n-2} \to \mathrm{Out}((\Delta'_{n-2})^l)$ is injective by the induction hypothesis. Write ζ' for the spectrum of a separable closure of the function field of $\tilde{X}'_{n-2}$, $\tilde{X}_{n-1,n-2}$ for the scheme $\tilde{X}_{n-1} \times_{\tilde{X}_{n-2}} \zeta'$, and $\tilde{X}'_{n-1,n-2}$ for the scheme $\tilde{X}'_{n-1} \times_{\tilde{X}'_{n-2}} \zeta'$. Then the Galois covering $\tilde{X}'_{n-1,n-2} \to \tilde{X}_{n-1,n-2}$ corresponds to the normal subgroup $\Delta'_{n-1,n-2} \subset \Delta_{n-1,n-2}$. Therefore, we obtain the following diagram:

$$\begin{array}{ccccccc}
 & & \mathrm{Aut}(\tilde{X}'_{n-1,n-2}/\tilde{X}_{n-1,n-2}) & \hookrightarrow & \mathrm{Aut}(\tilde{X}'_{n-1}/\tilde{X}_{n-1}) & \rightarrow & \mathrm{Out}((\Delta'_{n-1})^l) \\
 & & \| & & \| & & \\
1 & \longrightarrow & \Gamma_{n-1,n-2} & \longrightarrow & \Gamma_{n-1} & \longrightarrow & \Gamma_{n-2},
\end{array}$$

where the lower horizontal line is exact. Thus, it suffices to show that the composite homomorphism $\Gamma_{n-1,n-2} \to \Gamma_{n-1} \to \mathrm{Out}((\Delta'_{n-1})^l)$ is injective.

Consider the following diagram with exact horizontal lines and vertical lines:

(6.3)

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & (\Delta'_{n-1,n-2})^l & \rightarrow & (\Delta'_{n-1})^l & \rightarrow & (\Delta'_{n-2})^l & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & (\Delta_{n-1,n-2})^{(l)} & \rightarrow & (\Delta_{n-1})^{(l)} & \rightarrow & (\Delta_{n-2})^{(l)} & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \rightarrow & \Gamma_{n-1,n-2} & \rightarrow & \Gamma_{n-1} & \rightarrow & \Gamma_{,n-2} & \rightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1. & &
\end{array}$$

Here, the second horizontal line in the diagram (6.3) is exact by nine lemma. Since the homomorphism $\Gamma_{n-1,n-2} \to \mathrm{Out}((\Delta'_{n-1,n-2})^l)$ is injective by the argument of the first step of the induction, it suffices to show that an element g of $(\Delta_{n-1,n-2})^{(l)}$, whose action on $(\Delta'_{n-1})^l$ is same as the inner action of some $h \in (\Delta'_{n-1})^l$, induces an inner action on $(\Delta'_{n-1,n-2})^l$. The image of g in $(\Delta_{n-2})^{(l)}$ is trivial, which induces a trivial action on $(\Delta'_{n-2})^l$. Since $(\Delta'_{n-2})^l$ is center-free, h is in $(\Delta'_{n-1,n-2})^l$. Hence, we finished the proof of assertion 2.

Finally, we prove assertion 3. We have a diagram with exact horizontal lines

$$\begin{array}{ccccccc}
\Delta_{n,n-1} & \longrightarrow & \Delta'_n & \longrightarrow & \Delta'_{n-1} & \longrightarrow & 1\\
\| & & \downarrow & & \downarrow & & \\
\Delta_{n,n-1} & \longrightarrow & \Delta_n & \longrightarrow & \Delta_{n-1} & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \\
 & & \Gamma_{n-1} & = & \Gamma_{n-1}, & &
\end{array}$$

which induces a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^l_{n,n-1} & \longrightarrow & (\Delta'_n)^l & \longrightarrow & (\Delta'_{n-1})^l & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta^l_{n,n-1} & \longrightarrow & \Delta^{(l)}_n & \longrightarrow & \Delta^{(l)}_{n-1} & \longrightarrow & 1\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Gamma_{n-1} & = & \Gamma_{n-1}. & &
\end{array} \tag{6.4}$$

Since the homomorphism $\Delta'_{n-1} \to \mathrm{Aut}(\Delta^{l,\mathrm{ab}}_{n,n-1}/l\Delta^{l,\mathrm{ab}}_{n,n-1})$ is trivial and the group $\Delta^l_{n,n-1}$ is center-free, the first horizontal line of the diagram (6.4) is exact by Lemma 6.1. (Note that we can construct the diagram of the form (6.1) as in the statement of Proposition 3.2 with $N = \Delta_{n-1,n}, G = \Delta'_n$, and $H = \Delta'_{n-1}$.) Hence, we see the exactness of the second line by diagram chasing. Since $(\Delta'_{n-1})^l$ is center-free by assertion 1 and the outer action $\Gamma_{n-1} \to \mathrm{Out}((\Delta'_{n-1})^l)$ is injective by assertion 2, the group $\Delta^{(l)}_{n-1}$ is center-free. The order of Γ_{n-1} is prime to p, and hence $\Delta^{(l)}_{n-1}$ is a pro-p' group. Therefore, we obtain the surjective homomorphisms $\Delta^{p'}_{n-1} \to \Delta^{(l)}_{n-1}$ and $\Delta^{(l,p')}_n \to \Delta^{(l)}_n$. Thus, we have the desired commutative diagram with exact horizontal lines

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \Delta^l_{n,n-1} & \longrightarrow & \Delta^{(l,p')}_n & \longrightarrow & \Delta^{p'}_{n-1} & \longrightarrow & 1\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Delta^l_{n,n-1} & \longrightarrow & \Delta^{(l)}_n & \longrightarrow & \Delta^{(l)}_{n-1} & \longrightarrow & 1.
\end{array}$$

□

Now we prove Theorem 1.3.

Proposition 6.4. *Assume that $n \geq 3$ and $p \neq 2$. Let $l \neq p$ be a prime number. Define a function $f_{b,l}(m)$ for $m \geq 3$ in the following way:*

- *For $m = 3$, $f_{b,l}(3) = l^{b^2}$.*
- *For $m \geq 3$,*

$$f_{b,l}(m+1) = (f_{b,l}(m)) \times (l^{b^2 \times f_{b,l}(m)^2})^{f_{b,l}(m)}.$$

If $p > l^{b \times f_{b,l}(n)}$, (B) of Theorem 1.3 implies (A) of Theorem 1.3.

Proof. If we can find an open normal subgroup Δ'_{n-1} of Δ_{n-1} satisfying the assumptions of Lemma 6.3.3, the groups $\Delta_n^{(l)}$ and $\Delta_{n-1}^{(l)}$ in Lemma 6.3.3 work as $\overline{\Delta}_n$ and $\overline{\Delta}_{n-1}$ in Proposition 6.2, and hence (A) holds by Proposition 3.3.

Let us construct Δ'_{n-1} in the following. Fix $2 \leq i \leq n-1$ and assume that there exists an open normal subgroup Δ_{n-1}^i of Δ_{n-1} such that the images Δ_j^i of Δ_{n-1}^i in Δ_j for $i \leq j \leq n-1$ and the inverse images $\Delta_{j,j-1}^i$ of Δ_j^i in $\Delta_{j,j-1}$ for $i \leq j \leq n-1$ induce exact sequences

$$1 \to (\Delta_{j,j-1}^i)^l \to (\Delta_j^i)^l \to (\Delta_{j-1}^i)^l \to 1$$

for all $i+1 \leq j \leq n-1$. Write $\tilde{\Delta}_{i-1}^i$ for the image of Δ_{n-1}^i in Δ_{i-1}. By Proposition 3.2, the exact sequence

$$\Delta_{i,i-1}^i \to \Delta_i^i \to \tilde{\Delta}_{i-1}^i \to 1$$

induces homomorphisms $\tilde{\Delta}_{i-1}^i \to \mathrm{Out}((\Delta_{i,i-1}^i)^l)$. Write α for the composite homomorphism $\tilde{\Delta}_{i-1}^i \to \mathrm{Out}((\Delta_{i,i-1}^i)^l) \to \mathrm{Aut}((\Delta_{i,i-1}^i)^{l,\mathrm{ab}}/l(\Delta_{i,i-1}^i)^{l,\mathrm{ab}})$, Δ_{i-1}^{i-1} for the maximum normal subgroup of Δ_{i-1} contained in $\mathrm{Ker}\,\alpha$, Δ_{n-1}^{i-1} for the inverse image of Δ_{i-1}^{i-1} in Δ_{n-1}^i, Δ_j^{i-1} for the image of Δ_{n-1}^{i-1} in Δ_j^i for each $i \leq j \leq n$, and $\Delta_{j,j-1}^{i-1}$ for the inverse image of Δ_j^{i-1} in $\Delta_{j,j-1}^i$ for each $i \leq j \leq n$. Note that Δ_j^{i-1} coincides with the inverse image of Δ_{i-1}^{i-1} in Δ_j^i for any $i \leq j \leq n$, $\Delta_{j,j-1}^i = \Delta_{j,j-1}^{i-1}$ for any $i \leq j \leq n$, and Δ_j^{i-1} is a normal subgroup of Δ_j for any $i \leq j \leq n-1$. Thus, we have exact sequences of profinite groups

$$\Delta_{j,j-1}^i \to \Delta_j^{i-1} \to \Delta_{j-1}^{i-1} \to 1$$

which induce the exact sequences

$$1 \to (\Delta_{j,j-1}^i)^l \to (\Delta_j^{i-1})^l \to (\Delta_{j-1}^{i-1})^l \to 1$$

for all $i \leq j \leq n-1$ by the same argument of the proof of the exactness of the first line of the diagram (6.4).

Starting from $\Delta_{n-1}^{n-1} = \mathrm{Ker}(\Delta_{n-1} \to \mathrm{Aut}((\Delta_{n,n-1}^{n-1})^{l,\mathrm{ab}}/l(\Delta_{n,n-1}^{n-1})^{l,\mathrm{ab}})$, we reach the construction of Δ_j^1 for $1 \leq j \leq n-1$. Then we define Δ'_{n-1} to be Δ_{n-1}^1. To finish the proof of Proposition 6.4, it suffices to show that the quotient group $\Gamma_{n-1} = \Delta_{n-1}/\Delta_{n-1}^1$ is of order prime to p.

We recall the structure of the group $\Gamma_{n-1} = \Delta_{n-1}/\Delta_{n-1}^1$ and show the order of Γ_{n-1} is prime to p. Since we have

$$\begin{aligned}\Delta_{n-1}^1 &= \bigcap_{1\leq i\leq n-1} \Delta_{n-1}^i \\ &= \bigcap_{1\leq i\leq n-2} (\text{the inverse image of}\,\Delta_i^i\,\text{in}\,\Delta_{n-1}),\end{aligned}$$

it suffices to show that the order of the group Δ_i/Δ_i^i is prime-to-p for any $1 \leq i \leq n-2$.

Claim 6.5. *For each $1 \leq j \leq n-1$, the order of the group $\Delta_{n-j}/\Delta_{n-j}^{n-j}$ is prime-to-p and $\leq f_{b,l}(j+2)$.*

We show Claim 6.5 by induction on j. First, we consider the case $j = 1$. It holds that

$$\dim_{\mathbb{F}_l}((\Delta_{n,n-1})^{l,\mathrm{ab}}/l(\Delta_{n,n-1})^{l,\mathrm{ab}} \leq b.$$

Write $A_\sharp$ for the number $\#\mathrm{Aut}((\Delta_{n,n-1})^{l,\mathrm{ab}}/l(\Delta_{n,n-1})^{l,\mathrm{ab}})$. Then we have

$$A_\sharp = \prod_{0\leq j\leq \dim_{\mathbb{F}_l}(\Delta_{n,n-1})^{l,\mathrm{ab}}/l(\Delta_{n,n-1})^{l,\mathrm{ab}}-1} (l^{\dim_{\mathbb{F}_l}(\Delta_{n,n-1})^{l,\mathrm{ab}}/l(\Delta_{n,n-1})^{l,\mathrm{ab}}} - l^j)$$

and hence

$$A_\sharp \mid \prod_{0\leq j\leq b-1} (l^b - l^j).$$

Here, the notation $a \mid c$ means that a divides c. Since we have

$$\begin{aligned}\prod_{0\leq j\leq b-1} (l^b - l^j) &= l^{(b-1)b/2} \times \prod_{1\leq j\leq b} (l^j - 1) \\ &\leq l^{(b-1)b/2} \times l^{b(b+1)/2} = l^{b^2} = f_{b,l}(3),\end{aligned}$$

it holds that $A_\sharp \leq f_{b,l}(3)$. Since $l^b < p$, the group

$$\Delta_{n-1}/\Delta_{n-1}^{n-1}(\hookrightarrow \mathrm{Aut}((\Delta_{n,n-1})^{l,\mathrm{ab}}/l(\Delta_{n,n-1})^{l,\mathrm{ab}}))$$

is of order prime-to-p and $\leq f_{b,l}(3)$.

Fix $1 \leq j \leq n-2$ and assume that Claim 6.5 holds for each i satisfying that $1 \leq i \leq j$. We show that the order of the group $\Delta_{n-j-1}/\Delta_{n-j-1}^{n-j-1}$ is prime-to-p and $\leq f_{b,l}(j+3)$. We have a surjection

$$\Delta_{n-j}/\Delta_{n-j}^{n-j} \to \Delta_{n-j-1}/\tilde{\Delta}_{n-j-1}^{n-j},$$

which shows that the order of $\Delta_{n-j-1}/\tilde{\Delta}_{n-j-1}^{n-j}$ is prime-to-p. Recall that the group Δ_{n-j-1}^{n-j-1} is the maximum normal subgroup of Δ_{n-j-1} contained in $\mathrm{Ker}(\tilde{\Delta}_{n-j-1}^{n-j} \to \mathrm{Aut}((\Delta_{n-j,n-j-1}^{n-j})^{l,\mathrm{ab}}/l(\Delta_{n-j,n-j-1}^{n-j})^{l,\mathrm{ab}}))$. To see that the group $\Delta_{n-j-1}/\Delta_{n-j-1}^{n-j-1}$ is of order prime-to-p, it suffices to show that the image of the homomorphism

$$\tilde{\Delta}_{n-j-1}^{n-j} \to \mathrm{Aut}((\Delta_{n-j,n-j-1}^{n-j})^{l,\mathrm{ab}}/l(\Delta_{n-j,n-j-1}^{n-j})^{l,\mathrm{ab}})$$

is of order prime-to-p. We have inequalities

$$\begin{aligned}\dim_{\mathbb{F}_l}(\Delta_{n-j,n-j-1}^{n-j})^{l,\mathrm{ab}}/l(\Delta_{n-j,n-j-1}^{n-j})^{l,\mathrm{ab}} &\leq b \times (\Delta_{n-j,n-j-1} : \Delta_{n-j,n-j-1}^{n-j}) \\ &\leq b \times (\Delta_{n-j} : \Delta_{n-j}^{n-j}) \\ &= b \times f_{b,l}(j+2).\end{aligned}$$

Therefore, it holds that

$$\#\mathrm{Aut}((\Delta_{n-j,n-j-1}^{n-j})^{l,\mathrm{ab}}/l(\Delta_{n-j,n-j-1}^{n-j})^{l,\mathrm{ab}})$$

divides

$$(\Pi_1 :=) \prod_{0 \leq i \leq b \times (\Delta_{n-j,n-j-1}:\Delta_{n-j,n-j-1}^{n-j})-1} (l^{b \times (\Delta_{n-j,n-j-1}:\Delta_{n-j,n-j-1}^{n-j})} - l^i),$$

Π_1 divides

$$(\Pi_2 :=) \prod_{0 \leq i \leq b \times (\Delta_{n-j}:\Delta_{n-j}^{n-j})-1} (l^{b \times (\Delta_{n-j}:\Delta_{n-j}^{n-j})} - l^i),$$

and Π_2 divides

$$\prod_{0 \leq i \leq b \times f_{b,l}(j+2)-1} (l^{b \times f_{b,l}(j+2)} - l^i).$$

Then we have inequalities

$$\begin{aligned}&\prod_{0 \leq i \leq b \times f_{b,l}(j+2)-1} (l^{b \times f_{b,l}(j+2)} - l^i) \\ =&l^{(b \times f_{b,l}(j+2)-1)(b \times f_{b,l}(j+2))/2} \times \prod_{1 \leq i \leq b \times f_{b,l}(j+2)} (l^i - 1) \\ \leq&l^{b^2 \times f_{b,l}(j+2)^2}.\end{aligned}$$

and hence

$$\#\mathrm{Aut}((\Delta_{n-j,n-j-1}^{n-j})^{l,\mathrm{ab}}/l(\Delta_{n-j,n-j-1}^{n-j})^{l,\mathrm{ab}}) \leq l^{b^2 \times f_{b,l}(j+2)^2}.$$

Since $l^{b \times f_{b,l}(j+2)} \leq l^{b \times f_{b,l}(n)} < p$, the group $\tilde{\Delta}_{n-j-1}^{n-j}/\Delta_{n-j-1}^{n-j-1}$ is of order prime-to-p, and hence $\Delta_{n-j-1}/\Delta_{n-j-1}^{n-j-1}$ is also of order prime-to-p. The desired estimate of the order of $\Delta_{n-j-1}/\Delta_{n-j-1}^{n-j-1}$ is obtained in the following way:

$$(\Delta_{n-j-1} : \Delta_{n-j-1}^{n-j-1})$$

$$\leq (\Delta_{n-j-1} : \tilde{\Delta}^{n-j}_{n-j-1})$$
$$\times \#\mathrm{Im}(\tilde{\Delta}^{n-j}_{n-j-1} \to \mathrm{Aut}((\Delta^{i}_{i,i-1})^{l,\mathrm{ab}}/l(\Delta^{i}_{i,i-1})^{l,\mathrm{ab}}))^{(\Delta_{n-j-1}:\tilde{\Delta}^{n-j}_{n-j-1})}$$
$$\leq (\Delta_{n-j-1} : \tilde{\Delta}^{n-j}_{n-j-1}) \times (\#\mathrm{Aut}((\Delta^{n-j}_{n-j,n-j-1})^{l,\mathrm{ab}}/l(\Delta^{n-j}_{n-j,n-j-1})^{l,\mathrm{ab}}))^{(\Delta_{n-j-1}:\tilde{\Delta}^{n-j}_{n-j-1})}$$
$$\leq (\Delta_{n-j-1} : \tilde{\Delta}^{n-j}_{n-j-1}) \times (l^{b^2 \times f_{b,l}(j+2)^2})^{(\Delta_{n-j-1}:\tilde{\Delta}^{n-j}_{n-j-1})}$$
$$\leq (\Delta_{n-j} : \Delta^{n-j}_{n-j}) \times (l^{b^2 \times f_{b,l}(j+2)^2})^{(\Delta_{n-j}:\Delta^{n-j}_{n-j})}$$
$$\leq f_{b,l}(j+2) \times (l^{b^2 \times f_{b,l}(j+2)^2})^{f_{b,l}(j+2)} = f_{b,l}(j+3).$$

Therefore, Claim 6.5 holds. □

7. Appendix 1: Notes on extensions of smooth curves

In this section, we review the extension property of families of proper hyperbolic curves proved in [8] and prove a non-proper version of this property (cf. Remark 7). Hoshi informed the author of the proof of Proposition 7.1.

Proposition 7.1 (cf. [8]). *Let S be a connected regular Noetherian scheme, U an open subscheme of S such that the codimension of $S \setminus U$ in S is ≥ 2, $(\overline{X_U}, D_U) \to U$ a hyperbolic curve, and g the genus of $\overline{X_U}$. Then there exists a hyperbolic curve $(\overline{X}, D) \to S$ such that $(\overline{X} \times_S U, D \times_S U)$ is isomorphic to $(\overline{X_U}, D_U)$ over U, which is unique up to unique isomorphism.*

Proof. The uniqueness follows from the separatedness of the moduli stacks of hyperbolic curves (cf. [3] and [6]).

If D_U is zero, the assertion follows from [8, Théorème (ii)]. Suppose that the étale morphism $D_U \to U$ is of degree $r \geq 1$. By Galois descent, we may assume that the scheme D_U is the disjoint union $\coprod_{1 \leq i \leq r} D_{i,U}$ of r copies of S. Let us assume that the extension $\overline{X} \to S$ and $D_i \to \overline{X}$ of $\overline{X_U} \to U$ and $D_{i,U} \to \overline{X_U}$ exist. Then D_i and D_j are disjoint if $i \neq j$. Otherwise, if we denote the diagonal of $X \times_S X$ by $\Delta_{X/S}$, $\Delta_{X/S} \cap (D_i \times_S D_j)$ is codimension 1 in $D_i \times_S D_j \cong S$, which is a contradiction. Thus, it suffices to show that the extension $\overline{X} \to S$ (resp. $D_i \to \overline{X}$) of $\overline{X_U} \to U$ (resp. $D_{i,U} \to \overline{X_U}$) exists.

If $g \geq 1$, we have an extension $\overline{X} \to S$ of $\overline{X_U} \to U$ by [8, Théorème (ii)]. From [8, Lemme1], we have an extension $D_i \to \overline{X} \to S$ of the section $D_{i,U} \to \overline{X_U} \to U$ for each i.

If $g = 0$, there exists an isomorphism between $\overline{X_U}$ and $\mathbb{P}^1_U$ over U such that the sections $D_{1,U}$, $D_{2,U}$, and $D_{3,U}$ correspond to 0,1, and ∞. Therefore, we consider the extension $\overline{X} = \mathbb{P}^1_S \to S$ and we will show that extensions $D_i \to S$ of $D_{i,U} \to U$ exist for all $i \geq 4$. To give the divisors $D_{i,U} \to \mathbb{P}^1_U$ for

$i \geq 4$ is equivalent to give a section s_i of $\Gamma(U, O_U)$ other than 0, 1, and to give their extensions $D_i \to \mathbb{P}^1_S$ is equivalent to give extensions of s_i to elements of $\Gamma(S, O_S)$. Since $S \setminus U$ is of codimension ≥ 2 in the normal scheme S, we have $\Gamma(S, O_S) \cong \Gamma(U, O_U)$, and hence the morphisms $D_{i,U} \to \mathbb{P}^1_U$ extend to $D_i \to \mathbb{P}^1_S$. □

Remark. It is mentioned in [18, Remarks 2.6 (d)] that Proposition 7.1 is shown in [8]. Actually, in [8], only the case where D is empty is treated. As mentioned in [18, Remarks 2.6 (d)], Proposition 7.1 is equivalent to [18, Theorem 1.1] by the Zariski-Nagata purity theorem, under the hypothesis of Proposition 7.1. Here, we gave an elementary and direct proof.

8. Appendix 2: An example of the fundamental group of a hyperbolic polycurve

In this section, we give an example of a hyperbolic polycurve with bad reduction such that the pro-l outer Galois representation of the inertia subgroup is trivial for all but one prime l.

Lemma 8.1. *Let* $1 \to N \to G \to H \to 1$ *be an exact sequence of profinite groups.*

(1) *We have an exact sequence*

$$(N/[N, \mathrm{Ker}(G \to G^l)])^l \to G^l \to H^l \to 1.$$

Here, $[-,-]$ *denotes the closure of the commutator subgroup.*

(2) *Suppose that we have a section* s *of the homomorphism* $G \to H$. *Write* $N_{\mathrm{Ker}(H \to H^l)}$ *for the maximal quotient group of* N *on which* $\mathrm{Ker}(H \to H^l)$ *acts trivially. Then we have an exact sequence*

$$(N_{\mathrm{Ker}(H \to H^l)})^l \to G^l \to H^l \to 1.$$

Proof. Since the image of $[N, \mathrm{Ker}(G \to G^l)]$ in G^l is trivial, we have an exact sequence

$$N/[N, \mathrm{Ker}(G \to G^l)] \to G^l \to H^l \to 1,$$

and hence also the desired exact sequence

$$(N/[N, \mathrm{Ker}(G \to G^l)])^l \to G^l \to H^l \to 1.$$

Thus, assertion 1 holds. Since we have $s(\mathrm{Ker}(H \to H^l)) \subset \mathrm{Ker}(G \to G^l)$, assertion 2 follows from assertion 1. □

Example 8.2. *Let* $K, O_K, p, K^{\mathrm{sep}}, G_K$, *and* I_K *be as in Section 1. Suppose that* O_K *is strictly henselian and* $p = 0$. *Note that the Galois group* $G_K = I_K$ *is isomorphic to the profinite completion* $\widehat{\mathbb{Z}}$ *of* $\mathbb{Z}$. *We give an example of hyperbolic polycurve* Z *of bad reduction over a* K *whose pro-*l

outer Galois representation is trivial for all but one prime l. This example shows that, unlike the case of hyperbolic curves, to look at the pro-l outer Galois representation for a single prime number l is not enough to determine whether the hyperbolic polycurve has good reduction or not.

Fix a prime number l_1. Let X_1 and X_2 be proper hyperbolic curves over K which have good reduction. Suppose that there exist an automorphism ι_2 of X_2 over K of order l_1 and a rational point x_2 of X_2 fixed by ι_2. Take a geometric point $*_1$ of $X_1 \otimes_K K^{\mathrm{sep}}$ and write Π'_{X_1} (resp. Δ'_{X_1}) for $\pi_1(X_1, *_1)$ (resp. $\pi_1(X_1 \otimes_K K^{\mathrm{sep}}, *_1)$). By Lemma 4.1, we have a canonical decomposition $\Pi'_{X_1} \cong \Delta'_{X_1} \times G_K$. Take surjective group homomorphisms $\Pi'_{X_1} \to \mathbb{Z}/l_1\mathbb{Z}$ and $G_K \to \mathbb{Z}/l_1\mathbb{Z}$. Let $(\Pi'_{X_1} \cong)\Delta'_{X_1} \times G_K \to \mathbb{Z}/l_1\mathbb{Z}$ be the sum of these homomorphisms. Write X'_1 for the étale covering space of X_1 corresponding to $\mathrm{Ker}\,(\Pi'_{X_1} \to \mathbb{Z}/l_1\mathbb{Z})$ and ι_1 for a generator of $\mathrm{Aut}(X'_1/X_1)$. Consider the action of $\mathbb{Z}/l_1\mathbb{Z}$ on $X_2 \times_{\mathrm{Spec}\,K} X'_1$ induced by (ι_2, ι_1), and write Z for the quotient scheme of $X_2 \times_{\mathrm{Spec}\,K} X'_1$ by this $\mathbb{Z}/l_1\mathbb{Z}$-action.

Let l be a prime number. We will show that the pro-l outer Galois action $I_K \to \mathrm{Out}(\Delta^l_Z)$ is trivial if and only if $l \neq l_1$. Since the outer pro-l_1 Galois action $I_K \to \mathrm{Out}(\Delta^{l_1}_Z)$ is nontrivial, the outer pro-l_1 Galois action $I_K \to \mathrm{Out}(\Delta_Z)$ is also nontrivial and Z has bad reduction by Theorem 1.1.1. On the other hand, if $l \neq l_1$, we cannot judge whether or not Z has good reduction from the pro-l outer Galois action $I_K \to \mathrm{Out}(\Delta^l_Z)$.

By construction, we have a Cartesian diagram

$$\begin{array}{ccc} X_2 \times_{\mathrm{Spec}\,K} X'_1 & \longrightarrow & X'_1 \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X_1. \end{array}$$

Let L'' be an algebraic closure of the function field of X'_1 and $*$ a geometric point of $\{x_2\} \otimes_K L''$. Write Π_Z (resp. Π_{X_1}; Δ_{X_1}; Δ_{X_2}) for the étale fundamental group $\pi_1(Z, *)$ (resp. $\pi_1(X_1, *)$; $\pi_1(X_1 \otimes_K L'', *)$; $\pi_1(X_2 \otimes_K L'', *)$). By [5, PROPOSITION 2.5], *the sequence*

$$1 \to \Delta_{X_2} \to \Pi_Z \to \Pi_{X_1} \to 1 \tag{8.1}$$

is exact and Δ_{X_2} is isomorphic to $\pi_1(X_2 \otimes_K K^{\mathrm{sep}}, *)$. Moreover, the sequence

$$1 \to \Delta_{X_2} \to \Delta_Z \to \Delta_{X_1} \to 1. \tag{8.2}$$

is also exact. Since the section of $X_2 \times_{\mathrm{Spec}\,K} X'_1 \to X'_1$ determined by the point x_2 is compatible with the actions of $\mathbb{Z}/l_1\mathbb{Z}$, we obtain a section of $Z \to X_1$ by taking the quotient schemes by $\mathbb{Z}/l_1\mathbb{Z}$. This section defines a canonical section of $\Pi_Z \to \Pi_{X_1}$ (resp. $\Delta_Z \to \Delta_{X_1}$) in (8.1) (resp. (8.2)).

We calculate the action

$$\Pi_{X_1} \to \mathrm{Aut}(\Delta_{X_2}) \tag{8.3}$$

induced by the section. Write $\mathrm{Aut}_K(X_2, x_2)$ *for the subgroup of the group of automorphisms of* X_2 *over* K *consisting of automorphisms fixing* x_2*. Write* ψ *for the composite homomorphism of the natural surjection* $\Pi_{X_1} \to \Pi_{X_1}/\Pi_{X_1'}(= \langle \iota_1 \rangle)$*, two isomorphisms* $\langle \iota_1 \rangle \cong \mathbb{Z}/l_1\mathbb{Z} \cong \langle \iota_2 \rangle$*, the inclusion* $\langle \iota_2 \rangle \hookrightarrow \mathrm{Aut}_K(X_2, x_2)$*, and the action* $\mathrm{Aut}_K(X_2, x_2) \to \Delta_{X_2}$ *defined by the fixed point* x_2*. Write* ϕ *for the homomorphism* $\Pi_{X_1} \to \mathrm{Aut}(\Delta_{X_2})$ *defined by the splitting of the exact sequence*

$$1 \to \Delta_{X_2} \to \pi_1(X_2 \times_{\mathrm{Spec}\, K} X_1, *) \to \Pi_{X_1} \to 1$$

(, the exactness of which follows from [5, PROPOSITION 2.5]*,) determined by* x_2*. Then* ϕ *coincides with the composite homomorphism* $\Pi_{X_1} \to G_K \to \mathrm{Aut}(\Delta_{X_2})$*, where the second homomorphism is induced by* x_2*. Since* X_2 *has good reduction,* ϕ *is trivial. By the construction of* Z*, the action (8.3) coincides with* $\phi + \psi (= \psi)$*.*

First, we assume that $l \neq l_1$*. By taking pro-*l *completion of the exact sequence (8.2), we obtain a commutative diagram with exact horizontal lines*

$$\begin{array}{ccccccccc} & & \Delta^l_{X_2} & \longrightarrow & \Delta^l_Z & \longrightarrow & \Delta^l_{X_1} & \longrightarrow & 1 \\ & & \downarrow & & \| & & \| & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & \Delta^l_Z & \longrightarrow & \Delta^l_{X_1} & \longrightarrow & 1. \end{array}$$

Here, we write Δ *for the image of* $\Delta^l_{X_2} \to \Delta^l_Z$*. Since the composite homomorphism* $\mathrm{Ker}(\Delta_{X_1} \to \Delta^l_{X_1}) \hookrightarrow \Delta_{X_1} \hookrightarrow \Pi_{X_1} \to \langle \iota_2 \rangle$ *is surjective by the assumption* $l \neq l_1$*,* Δ *is a quotient group of* $(\Delta_{X_2})_{\langle \iota_2 \rangle}$ *by Lemma 8.1. Write* $\Pi^{(l)}_Z$ *(resp.* $\Pi^{(l)}_{X_1}$*) for the quotient group* $\Pi_Z/\mathrm{Ker}(\Delta_Z \to \Delta^l_Z)$ *(resp.* $\Pi_{X_1}/\mathrm{Ker}(\Delta_{X_1} \to \Delta^l_{X_1})$*). We have a commutative diagram with exact horizontal lines*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta & \longrightarrow & \Delta^l_Z & \longrightarrow & \Delta^l_{X_1} & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta & \longrightarrow & \Pi^{(l)}_Z & \longrightarrow & \Pi^{(l)}_{X_1} & \longrightarrow & 1. \end{array} \tag{8.4}$$

Both horizontal lines of (8.4) have a section determined by the point x_2*, which induces an action of* $\Pi^{(l)}_{X_1}$ *on* $\mathrm{Aut}(\Delta)$ *compatible with (8.3). Since* Δ *is a quotient group of* $(\Delta_{X_2})_{\langle \iota_2 \rangle}$*, this action is trivial. Hence, we have a canonical decomposition* $\Pi^{(l)}_Z \cong \Delta \times \Pi^{(l)}_{X_1}$*. Moreover, since* X_1 *has good* *reduction, we have a canonical decomposition* $\Pi_{X_1}^{(l)} \cong \Delta_{X_1}^{l} \times G_K$ *by* [19, Proposition 1.11] *and the proof of Lemma 4.1. Therefore, we can show that the outer Galois action* $I_K \to \mathrm{Out}(\Delta_Z^l)$ *is trivial by using these decompositions.*

Next, we assume that $l = l_1$ *and show that the pro-l outer Galois representation*

$$I_K \to \mathrm{Out}(\Delta_Z^l)$$

is nontrivial. Since the outer action $\Delta_{X_1} \to \mathrm{Out}(\Delta_{X_2}^l)$ *factors through* $\Delta_{X_1} \to \Delta_{X_1}^l$*, we have an exact sequence* $1 \to \Delta_{X_2}^l \to \Delta_Z^l \to \Delta_{X_1}^l \to 1$ *by Lemma 6.1.4. We have a commutative diagram with exact horizontal lines*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{X_2}^l & \longrightarrow & \Pi_Z^{(l)} & \longrightarrow & \Pi_{X_1}^{(l)} & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \Delta_Z^l & \longrightarrow & \Pi_Z^{(l)} & \longrightarrow & G_K & \longrightarrow & 1. \end{array} \tag{8.5}$$

Since X_1 *has good reduction, we have a decomposition* $\Pi_{X_1}^{(l)} = \Delta_{X_1}^l \times Z_{\Pi_{X_1}^{(l)}}(\Delta_{X_1}^l)$ *and* $Z_{\Pi_{X_1}^{(l)}}(\Delta_{X_1}^l)$ *is isomorphic to* G_K *by* [19, Proposition 1.11] *and the proof of Lemma 4.1. If the outer action* $I_K \to \mathrm{Out}\,\Delta_Z^l$ *is trivial, we would have a decomposition* $\Pi_Z^{(l)} = \Delta_Z^l \times Z_{\Pi_Z^{(l)}}(\Delta_Z^l)$ *by the proof of Lemma 4.1 and the center-freeness of* Δ_Z^l*, and the homomorphism* $\Pi_Z^{(l)} \to \Pi_{X_1}^{(l)}$ *would induce a surjection* $Z_{\Pi_Z^{(l)}}(\Delta_Z^l) \to Z_{\Pi_{X_1}^{(l)}}(\Delta_{X_1}^l)$*. Therefore, it suffices to show that the outer action* $Z_{\Pi_{X_1}^{(l)}}(\Delta_{X_1}^l) \to \mathrm{Out}\,(\Delta_{X_2}^l)$ *associated with the first horizontal line of the diagram (8.5) is nontrivial. We have a diagram with exact horizontal lines*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{X_2} & \longrightarrow & \Pi_Z \times_{\Pi_{X_1}} Z_{\Pi_{X_1}}(\Delta_{X_1}) & \longrightarrow & Z_{\Pi_{X_1}}(\Delta_{X_1}) & \longrightarrow & 1 \\ & & \Vert & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & \Delta_{X_2} & \longrightarrow & \Pi_Z & \longrightarrow & \Pi_{X_1} & \longrightarrow & 1. \end{array} \tag{8.6}$$

Write $\mathfrak{X}_1$ *for the smooth model of* X_1 *and* ξ_1 *for the generic point of the special fiber of* $\mathfrak{X}_1$*. Let* L *be the field of fractions of a strict henselization of the local ring of* $\mathfrak{X}_1$ *at* ξ_1 *in* L''*. By Lemma 4.2, the natural homomorphism* $(G_L :=)\mathrm{Gal}(L''/L) \to \Pi_{X_1}$ *induces* $G_L \to Z_{\Pi_{X_1}}(\Delta_{X_1})(\cong G_K)$*. Since a uniformizer of* K *is a uniformizer of* L*,* $G_L \to Z_{\Pi_{X_1}}(\Delta_{X_1})$ *is an isomorphism. Let* L' *be the finite extension field of* L *defined by* $\mathrm{Ker}(G_L \to \Pi_{X_1} \to \mathbb{Z}/l\mathbb{Z})$ *and* ι *a generator of* $\mathrm{Gal}(L'/L)$*. Consider the action of* $\mathbb{Z}/l\mathbb{Z}$ *on* $X_2 \otimes_K L'$

induced by (ι_2, ι) *and write* X'_2 *for the quotient scheme. Write* $\Pi_{X'_2}$ *for the étale fundamental group* $\pi_1(X_2, *)$. *Then we have an exact sequence*

$$1 \to \Delta_{X_2} \to \Pi_{X'_2} \to G_L(\cong Z_{\Pi_{X_1}}(\Delta_{X_1})) \to 1.$$

Since we have a diagram

$$\begin{array}{ccc} X_2 \otimes_K L' & \longrightarrow & \operatorname{Spec} L' \\ \downarrow & & \downarrow \\ X_2 \times_{\operatorname{Spec} K} X'_1 & \longrightarrow & X'_1 \end{array}$$

compatible with the actions of $\mathbb{Z}/l\mathbb{Z}$, *we obtain a commutative diagram with exact horizontal lines*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Delta_{X_2} & \longrightarrow & \Pi_{X'_2} & \longrightarrow & G_L & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \Delta_{X_2} & \longrightarrow & \Pi_Z \times_{\Pi_{X_1}} Z_{\Pi_{X_1}}(\Delta_{X_1}) & \longrightarrow & Z_{\Pi_{X_1}}(\Delta_{X_1}) & \longrightarrow & 1. \end{array} \tag{8.7}$$

Therefore, it suffices to show that the outer action $G_L \to \operatorname{Out}(\Delta^l_{X_2})$ *defined by the first line of (8.7) is nontrivial. As the calculation of the action (8.3), we can calculate the action of* G_L *on* $\Delta^l_{X_2}$ *determined by the rational point of* X'_2 *defined by* x_2. *By* [3, Theorem 1.13], *the action of* G_L *on* $\Delta^{\mathrm{ab},l}_{X_2}$ *is nontrivial, and hence the outer action* $G_L \to \operatorname{Out}(\Delta^l_{X_2})$ *is also nontrivial.*

Remark. Let p be a prime number and $K, O_K, K^{\mathrm{sep}}, G_K$, and I_K as in Example 8.2. We can give an example of hyperbolic polycurve Z of bad reduction over a K whose pro-l outer Galois representation is trivial for all but one prime l $(\neq p)$ in a similar way to that given in Example 8.2. In this case, we need to consider $\Delta^{p'}_{-}$ (resp. $\Pi^{(p')}_{-}$) in place of Δ_{-} (resp. Π_{-}) and use [15, Exposé XIII, Proposition 4.6] in place of [5, PROPOSITION 2.5].

References

[1] M. P. Anderson, *Exactness properties of profinite completion functors*, Topology, **13**, (1974), 229–239.

[2] M. Asada, M. Matsumoto, and T. Oda, *Local monodromy on the fundamental groups of algebraic curves along a degenerate stable curve*, J. Pure Appl. Algebra, **103**, (1995), 235–283.

[3] P. Deligne, and D. Mumford, *The irreducibility of the space of curves of given genus*, Publ. Math. IHES, **36**, (1969), 75–109.

[4] V. Drinfeld, *On a conjecture of Deligne*, Mosc. Math. J., **12**, no. 3 (2012), 515–542.

[5] Y. Hoshi, *The Grothendieck conjecture for hyperbolic polycurves of lower dimension*, J. Math. Sci. Univ. Tokyo, **21**, no. 2 (2014), 152–219.

[6] F. F. Knudsen, *The Projectivity of the Moduli Space of Stable Curves, II*, Math. Scand., **52**, (1983), 161–199.

[7] S. Mochizuki, *The local pro-p anabelian geometry of curves*, Invent. Math., **138**, no. 2 (1999), 319–423.

[8] L. Moret-Bailly, *Un theoreme de purete pour les familles de courbes lisses*, C. R. Acad. Sci. Paris, **300**, no. 14 (1985), 489–492.

[9] I. Nagamachi, *On a good reduction criterion for proper polycurves with sections*, Hiroshima Math. J, **48**, (2018), 223–251.

[10] I. Nagamachi, *On homotopy exact sequences for normal schemes*, arXiv:1811.11395[math.NT].

[11] I. Nagamachi, *Good reduction of hyperbolic polycurves and their fundamental groups: A survey*, to appear in Algebraic Number Theory and Related Topics 2018, RIMS Kôkyuroku Bessatsu.

[12] T. Oda, *A note on ramification of the Galois representation on the fundamental group of an algebraic curve*, J. Number Theory, **34**, (1990), 225–228.

[13] T. Oda, *A note on ramification of the Galois representation on the fundamental group of an algebraic curve II*, J. Number Theory, **53**, (1995), 342–355.

[14] K. Sawada, *Pro-p Grothendieck conjecture for hyperbolic polycurves*, Publ. Res. Inst. Math. Sci., **54**, no. 4 (2018), 781–853.

[15] A. Grothendieck, and Mme. Raynaud. M, Séminaire de Géometrie Algébrique du Bois Marie 1960/61, *Revétements Etales et Groupe Fondamental (SGA 1)*, Lecture Notes in Mathematics, 224, Springer-Verlag, Berlin/Heidelberg/New York, 1971.

[16] A. Schmidt, and J. Stix, *Anabelian geometry with étale homotopy types*, Annals of Math., **184**, Issue 3 (2016), 817–868.

[17] J.-P. Serre, and J. Tate, *Good reduction of abelian varieties*, Ann. of Math., **88**, no. 2 (1968), 492–517.

[18] J. Stix, *A monodromy criterion for extending curves*, Int. Math. Res. Notices, **29**, (2005), 1787–1802.

[19] A. Tamagawa, *The Grothendieck conjecture for affine curves*, Compositio Math., **109**, no. 2 (1997), 135–194.

[20] A. Tamagawa, *Finiteness of isomorphism classes of curves in positive characteristic with prescribed fundamental groups*, J. Algebraic Geom., **13**, (2004), 675–724.

[21] Z. Wojtkowiak, *On l-adic iterated integrals, IV, ramification and generators of Galois actions on fundamental groups and torsors of paths*, Mathematical Journal of Okayama University, **51**, (2009), 47–69.

Ippei Nagamachi
Research Institute for
Mathematical Sciences,
Kyoto University
Kyoto, 606-8502 Japan
e-mail address: nagachi@kurims.kyoto-u.ac.jp

(*Received March 2, 2020*)
(*Accepted July 19, 2021*)