Math. J. Okayama Univ. **64** (2022), 47–61

ON WEAKLY SEPARABLE POLYNOMIALS IN SKEW POLYNOMIAL RINGS

Satoshi Yamanaka

Abstract. The notion of weakly separable extensions was introduced by N. Hamaguchi and A. Nakajima as a generalization of separable extensions. The purpose of this article is to give a characterization of weakly separable polynomials in skew polynomial rings. Moreover, we shall show the relation between separability and weak separability in skew polynomial rings of derivation type.

1. Introduction

This paper is the continuation of the author's previous paper [9].

Let A/B be a ring extension with common identity 1, and M an A-A-bimodule. An additive map $\delta : A \to M$ is called a *B-derivation of A to M* if $\delta(zw) = \delta(z)w + z\delta(w)$ for any $z, w \in A$ and $\delta(B) = \{0\}$. Moreover, a B-derivation δ of A to M is called *inner* if there exists $m \in M$ such that $\delta(z) = mz - zm$ for any $z \in A$. We say that A/B is *separable* if the A-A-homomorphism of $A \otimes_B A$ onto A defined by $\sum_j z_j \otimes w_j \mapsto \sum_j z_j w_j$ $(z_j, w_j \in A)$ splits. It is well known that A/B is separable if and only if every B-derivation of A to N is inner for any A-A-bimodule N (cf. [1, Satz 4.2]). A/B is called *weakly separable* if every B-derivation of A to A is inner. The notion of weakly separable extensions was introduced by N. Hamaguchi and A. Nakajima as a generalization of separable extensions (cf. [2]). Obviously, a separable extension is weakly separable.

Throughout this article, let B be a ring, ρ an automorphism of B, and D a ρ-derivation of B (i.e. D is an additive endomorphism of B such that $D(\alpha\beta) = D(\alpha)\rho(\beta) + \alpha D(\beta)$ for any α, $\beta \in B$). By $B[X;\rho,D]$ we denote the skew polynomial ring in which the multiplication is given by $\alpha X = X\rho(\alpha) + D(\alpha)$ for any $\alpha \in B$. We write $B[X;\rho] = B[X;\rho,0]$ and $B[X;D] = B[X;1,D]$. Moreover, by $B[X;\rho,D]_{(0)}$ we denote the set of all monic polynomials f in $B[X;\rho,D]$ such that $fB[X;\rho,D] = B[X;\rho,D]f$. For each polynomial $f \in B[X;\rho,D]_{(0)}$, the quotient ring $B[X;\rho,D]/fB[X;\rho,D]$

Mathematics Subject Classification. Primary 16S36; Secondary 16S32.

Key words and phrases. separable extension, weakly separable extension, skew polynomial ring, derivation.

is a free ring extension of B. A polynomial f in $B[X;\rho,D]_{(0)}$ is called *separable* (resp. *weakly separable*) in $B[X;\rho,D]$ if $B[X;\rho,D]/fB[X;\rho,D]$ is separable (resp. weakly separable) over B.

Let $B^\rho=\{\alpha\in B\mid\rho(\alpha)=\alpha\}$. In the previous paper [9], we studied weakly separable polynomials over rings. In particular, we showed a necessary and sufficient condition for a polynomial $f\in B[X;\rho]_{(0)}\cap B^\rho[X]$ (resp. a p-polynomial $f\in B[X;D]_{(0)}$ with a prime number p) to be weakly separable in $B[X;\rho]$ (resp. $B[X;D]$) (cf. [9, Theorem 3.2 and Theorem 3.8]). The purpose of this paper is to give some improvements and generalizations of our results for the general skew polynomial ring $B[X;\rho,D]$. In section 2, we shall mention briefly on some properties for polynomials in $B[X;\rho,D]_{(0)}$. In section 3, we shall give a necessary and sufficient condition for a polynomial f in $B[X;\rho,D]_{(0)}\cap B^\rho[X]$ to be weakly separable in $B[X;\rho,D]$. Moreover, we shall show the relation between separability and weak separability in $B[X;D]$.

2. Polynomials in $B[X;\rho,D]_{(0)}$

In this section, we shall mention briefly on polynomials in $B[X;\rho,D]_{(0)}$. We inductively define additive endomorphisms $\Phi_{[i,j]}$ $(0\le j\le i)$ of B as follows:

$$\Phi_{[i,j]}=\begin{cases}1\ (=\text{the identity map}) & (i=j=0)\\ D^i & (j=0,\ i\ge 1)\\ \rho^i & (i=j\ge 1)\\ \rho\Phi_{[i-1,j-1]}+D\Phi_{[i-1,j]} & (i\ge 2,\ 1\le j\le i-1)\end{cases}$$

First we shall state the following.

Lemma 2.1. *For any $\alpha\in B$, there holds*

$$\alpha X^i=\sum_{j=0}^{i}X^j\Phi_{[i,j]}(\alpha)\quad (i\ge 0).$$

Proof. We shall show it by induction. It is true when $i=0$. Let α be arbitrary element in B and assume that it is true when $i\ge 0$. We have then

$$\begin{aligned}\alpha X^{i+1}&=\alpha X^i\cdot X\\ &=\left(\sum_{j=0}^{i}X^j\Phi_{[i,j]}(\alpha)\right)X\\ &=\sum_{j=0}^{i}X^j\left(X\rho\Phi_{[i,j]}(\alpha)+D\Phi_{[i,j]}(\alpha)\right)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{i} X^{j+1}\rho\Phi_{[i,j]}(\alpha) + \sum_{j=0}^{i} X^{j} D\Phi_{[i,j]}(\alpha) \\
&= \sum_{j=1}^{i+1} X^{j}\rho\Phi_{[i,j-1]}(\alpha) + \sum_{j=0}^{i} X^{j} D\Phi_{[i,j]}(\alpha) \\
&= X^{i+1}\rho\Phi_{[i,i]}(\alpha) + \sum_{j=1}^{i} X^{j}\left(\rho\Phi_{[i,j-1]} + D\Phi_{[i,j]}\right)(\alpha) + D\Phi_{[i,0]}(\alpha) \\
&= X^{i+1}\Phi_{[i+1,i+1]}(\alpha) + \sum_{j=1}^{i} X^{j}\Phi_{[i+1,j]}(\alpha) + \Phi_{[i+1,0]}(\alpha) \\
&= \sum_{j=0}^{i+1} X^{j}\Phi_{[i+1,j]}(\alpha).
\end{aligned}
$$

This completes the proof. □

Lemma 2.2. *Let f be a monic polynomial in $B[X;\rho,D]$ of the form $f = \sum_{i=0}^{m} X^i a_i$ $(m \geq 1, a_m = 1)$. Then f is in $B[X;\rho,D]_{(0)}$ if and only if*

(1) $a_j\rho^m(\alpha) = \sum_{i=j}^{m} \Phi_{[i,j]}(\alpha)a_i$ *for any* $\alpha \in B$ $(0 \leq j \leq m-1)$.

(2) $D(a_i) = \begin{cases} a_{i-1} - \rho(a_{i-1}) + a_i\big(\rho(a_{m-1}) - a_{m-1}\big) & (1 \leq i \leq m-1) \\ a_0\big(\rho(a_{m-1}) - a_{m-1}\big) & (i = 0) \end{cases}$.

Proof. Let $f = \sum_{i=0}^{m} X^i a_i$ $(m \geq 1, a_m = 1)$ be in $B[X;\rho,D]$ and α arbitrary element in B. As was shown in [4, Lemma 1.1], f is in $B[X;\rho,D]_{(0)}$ if and only if $\alpha f = f\rho^m(\alpha)$ and $Xf = f(X - \big(\rho(a_{m-1}) - a_{m-1})\big)$. It follows from Lemma 2.1 that

$$\alpha f = \sum_{i=0}^{m} \alpha X^i a_i = \sum_{i=0}^{m}\left(\sum_{j=0}^{i} X^j\Phi_{[i,j]}(\alpha)\right)a_i = \sum_{j=0}^{m} X^j\left(\sum_{i=j}^{m}\Phi_{[i,j]}(\alpha)a_i\right).$$

Noting that $f\rho^m(\alpha) = \sum_{j=0}^{m-1} X^j a_j\rho^m(\alpha)$, the equation $\alpha f = f\rho^m(\alpha)$ implies the condition (1), and conversely. Next we see that

$$f\Big(X - \big(\rho(a_{m-1}) - a_{m-1}\big)\Big)$$

$$
\begin{aligned}
&= \sum_{i=0}^{m} X^i a_i X - \sum_{i=0}^{m} X^i a_i\big(\rho(a_{m-1}) - a_{m-1}\big)\\
&= \sum_{i=0}^{m} X^i\big(X\rho(a_i) + D(a_i)\big) - \sum_{i=0}^{m} X^i a_i\big(\rho(a_{m-1}) - a_{m-1}\big)\\
&= \sum_{i=0}^{m} X^{i+1}\rho(a_i) + \sum_{i=0}^{m} X^i D(a_i) - \sum_{i=0}^{m} X^i a_i\big(\rho(a_{m-1}) - a_{m-1}\big)\\
&= \sum_{i=1}^{m+1} X^i\rho(a_{i-1}) + \sum_{i=0}^{m} X^i\Big(D(a_i) - a_i\big(\rho(a_{m-1}) - a_{m-1}\big)\Big)\\
&= X^{m+1} + X^m a_{m-1} + \sum_{i=1}^{m-1} X^i\Big(\rho(a_{i-1}) + D(a_i) - a_i\big(\rho(a_{m-1}) - a_{m-1}\big)\Big)\\
&\quad + D(a_0) - a_0\big(\rho(a_{m-1}) - a_{m-1}\big).
\end{aligned}
$$

Noting that $Xf = \sum_{i=0}^{m} X^{i+1}a_i = X^{m+1} + X^m a_{m-1} + \sum_{i=1}^{m-1} X^i a_{i-1}$, the equation $Xf = f(X - \big(\rho(a_{m-1}) - a_{m-1})\big)$ implies that

$$
\begin{cases} a_{i-1} = \rho(a_{i-1}) + D(a_i) - a_i\big(\rho(a_{m-1}) - a_{m-1}\big) & (1 \le i \le m-1)\\ 0 = D(a_0) - a_0\big(\rho(a_{m-1}) - a_{m-1}\big) \end{cases}.
$$

Hence we have the condition (2). The converse is obvious. □

Recall that $B^\rho = \{\alpha \in B \mid \rho(\alpha) = \alpha\}$. In addition, let $B^D = \{\alpha \in B \mid D(\alpha) = 0\}$, $B^{\rho,D} = B^\rho \cap B^D$, and $C(B^{\rho,D})$ the center of $B^{\rho,D}$. We have then the following.

Corollary 2.3. *Let* $f = \sum_{i=0}^{m} X^i a_i$ $(m \ge 1, a_m = 1)$ *be in* $B[X;\rho,D]_{(0)}$. *If* $f \in B^\rho[X]$ *then* $f \in C(B^{\rho,D})[X]$.

Proof. Let $f = \sum_{i=0}^{m} X^i a_i$ $(m \ge 1, a_m = 1)$ be in $B[X;\rho,D]_{(0)}$ and assume that $f \in B^\rho[X]$. Then, by Lemma 2.2 (2), we have

$$
\begin{aligned}
D(a_i) &= a_{i-1} - \rho(a_{i-1}) + a_i\big(\rho(a_{m-1}) - a_{m-1}\big)\\
&= a_{i-1} - a_{i-1} + a_i\big(a_{m-1} - a_{m-1}\big)\\
&= 0 \quad (1 \le i \le m-1),\\
D(a_0) &= a_0\big(\rho(a_{m-1}) - a_{m-1}\big)
\end{aligned}
$$

$$= a_0\big(a_{m-1} - a_{m-1}\big)$$
$$= 0.$$

Hence $a_i \in B^D$, that is, $a_i \in B^{\rho,D}$ $(0 \leq i \leq m-1)$. Let β be arbitrary element in $B^{\rho,D}$. It is clear that $\Phi_{[i,j]}(\beta) = \begin{cases} \beta & (i=j) \\ 0 & (i>j) \end{cases}$. Therefore it follows from Lemma 2.2 (1) that

$$a_j\beta = a_j\rho^m(\beta) = \sum_{i=j}^{m} \Phi_{[i,j]}(\beta)a_i = \beta a_j \quad (0 \leq j \leq m-1).$$

Thus $a_j \in C(B^{\rho,D})$ $(0 \leq j \leq m-1)$. □

3. Weakly separable polynomials in $B[X;\rho,D]$

The conventions and notations employed in the preceding section will be used in this section. Throughout this section, let $R = B[X;\rho,D]$, $R_{(0)} = B[X;\rho,D]_{(0)}$, and f a monic polynomial in $R_{(0)} \cap B^\rho[X]$ of the form $f = \sum_{i=0}^{m} X^i a_i$ $(m \geq 1,\ a_m = 1)$. Note that f is in $C(B^{\rho,D})[X]$ by Corollary 2.3. We shall use the following conventions:

- $R_1 = \{g \in R \,|\, \alpha g = g\rho(\alpha)\ (\forall \alpha \in B)\}$
- $A = R/fR$ (the quotient ring of R modulo fR)
- $x = X + fR \in A$ (i.e. $\{1, x, x^2, \cdots, x^{m-1}\}$ is a free B-basis of A and $x^m = -\sum_{j=0}^{m-1} x^j a_j$)
- I_x is an inner derivation of A by x (i.e. $I_x(z) = zx - xz$ $(\forall z \in A)$)
- $C(A)$ is the center of A
- $A_k = \{u \in A \,|\, \alpha u = u\rho^k(\alpha)\ (\forall \alpha \in B)\}$ $(k \in \mathbb{Z})$
- $V = A_0 = \{z \in A \,|\, \alpha z = z\alpha\ (\forall \alpha \in B)\}$ (i.e. V is the centralizer of B in A)

Moreover, we define polynomials $Y_j \in R \cap C(B^{\rho,D})[X]$ $(0 \leq j \leq m-1)$ as follows:

$$Y_0 = X^{m-1} + X^{m-2}a_{m-1} + \cdots + Xa_2 + a_1,$$
$$Y_1 = X^{m-2} + X^{m-3}a_{m-1} + \cdots + Xa_3 + a_2,$$
$$\cdots\cdots$$

$$Y_j = X^{m-j-1} + X^{m-j-2}a_{m-1} + \cdots + Xa_{j+2} + a_{j+1} \left(= \sum_{k=j}^{m-1} X^{k-j}a_{k+1} \right),$$

$\cdots\cdots$

$Y_{m-2} = X + a_{m-1}$,

$Y_{m-1} = 1$.

It is obvious that

$$XY_j = \begin{cases} Y_{j-1} - a_j & (1 \leq j \leq m-1) \\ f - a_0 & (j = 0) \end{cases}. \tag{3.1}$$

The polynomials Y_j $(0 \leq j \leq m-1)$ were introduced by Y. Miyashita to characterize separable polynomials in $B[X;\rho,D]$ (cf. [6]). Now let $y_j = Y_j + fR \in A$ $(0 \leq j \leq m-1)$. Since the equality (3.1), the following lemma is obvious.

Lemma 3.1.

$$xy_j = \begin{cases} y_{j-1} - a_j & (1 \leq j \leq m-1) \\ -a_0 & (j = 0) \end{cases}.$$

So we define a map $\tau : A \to A$ by

$$\tau(z) = \sum_{j=0}^{m-1} y_j z x^j \quad (z \in A).$$

Obviously, τ is a $C(A)$-$C(A)$-endomorphism of A. By making use of τ, separable polynomials in R are characterized as follows:

Lemma 3.2. ([6, Theorem 1.8] or [8, Theorem 1.3]) *f is separable in R if and only if there exists $u \in A_{1-m}$ (that is, $\rho^{m-1}(\alpha)u = u\alpha$ for any $\alpha \in B$) such that $\tau(u) = 1$.*

Remark. Needles to say, each a_i $(0 \leq i \leq m-1)$ satisfies that $a_i x = xa_i$ and $a_i u = ua_i$ for any $u \in A_k$ $(k \in \mathbb{Z})$. In particular, we see that $y_i x = xy_i$ $(0 \leq i \leq m-1)$.

First we shall prove the following lemma concerning the inner derivation I_x and the $C(A)$-$C(A)$-endomorphism τ.

Lemma 3.3. (1) *$I_x(A_k) \subset \mathrm{Ker}(\tau)$ for any integer k.*

(2) *$I_x(V) \subset \mathrm{Ker}(\tau) \cap A_1$.*

Proof. (1) Let k be arbitrary integer and $u \in A_k$. We obtain then

$$\begin{aligned}
\tau(I_x(u)) &= \tau(ux - xu) \\
&= \sum_{j=0}^{m-1} y_j(ux - xu)x^j \\
&= \left(\sum_{j=0}^{m-1} y_j u x^j\right) x - x\left(\sum_{j=0}^{m-1} y_j u x^j\right) \\
&= \tau(u)x - x\tau(u).
\end{aligned}$$

Therefore it suffice to prove that $\tau(u)x = x\tau(u)$. By Lemma 3.1, we have

$$\begin{aligned}
x\tau(u) &= x\left(\sum_{j=0}^{m-1} y_j u x^j\right) \\
&= xy_0u + \sum_{j=1}^{m-1} xy_j u x^j \\
&= -a_0u + \sum_{j=1}^{m-1} (-a_j + y_{j-1})ux^j \\
&= -ua_0 - u\sum_{j=1}^{m-1} x^j a_j + \sum_{j=1}^{m-1} y_{j-1}ux^j \\
&= u\left(-\sum_{j=0}^{m-1} x^j a_j\right) + \sum_{j=0}^{m-2} y_j u x^{j+1} \\
&= ux^m + \left(\sum_{j=0}^{m-2} y_j u x^j\right) x \\
&= (y_{m-1}ux^{m-1})x + \left(\sum_{j=0}^{m-2} y_j u x^j\right) x \\
&= \left(\sum_{j=0}^{m-1} y_j u x^j\right) x \\
&= \tau(u)x.
\end{aligned}$$

(2) Since the condition (1), it suffice to show that $I_x(V) \subset A_1$. For any $\alpha \in B$ and $u \in V$, we obtain

$$\begin{aligned}\alpha I_x(u) &= \alpha(ux - xu)\\ &= u\alpha x - \alpha xu\\ &= u(x\rho(\alpha) + D(\alpha)) - (x\rho(\alpha) + D(\alpha))u\\ &= ux\rho(\alpha) + uD(\alpha) - xu\rho(\alpha) - uD(\alpha)\\ &= (ux - xu)\rho(\alpha)\\ &= I_x(u)\rho(\alpha).\end{aligned}$$

Thus $I_x(V) \subset A_1$. □

To show the subsequent lemma (Lemma 3.6), we need the following two lemmas.

Lemma 3.4. *Let g_1 be arbitrary element in R. We define $g_0 = 0$ and $g_{j+1} = g_jX + X^jg_1$ $(j \geq 1)$, inductively.*

(1) $g_{i+k} = g_iX^k + X^ig_k$ $(i, k \geq 0)$.

(2) *If $g_1 \in R_1$ then* $\alpha g_j = \sum_{k=1}^{j} g_k \Phi_{[j,k]}(\alpha)$ $(j \geq 1)$ *for any* $\alpha \in B$.

Proof. (1) Fix $i \geq 0$ and we shall show it by induction for k. It is true when $k = 0$. Assume that it is true when $k \geq 1$. So we obtain

$$\begin{aligned}g_{i+k+1} &= g_{i+k}X + X^{i+k}g_1\\ &= \left(g_iX^k + X^ig_k\right)X + X^{i+k}g_1\\ &= g_iX^{k+1} + X^ig_kX + X^{i+k}g_1\\ &= g_iX^{k+1} + X^i\left(g_kX + X^kg_1\right)\\ &= g_iX^{k+1} + X^ig_{k+1}.\end{aligned}$$

This completes the proof.

(2) Let g_1 be arbitrary element in R_1. We shall show it by induction. It is true when $j = 1$. Assume that it is true when $j \geq 1$. Then, by Lemma 2.1, we have

$$\begin{aligned}\alpha g_{j+1} &= \alpha\left(g_jX + X^jg_1\right)\\ &= \alpha g_jX + \alpha X^jg_1\\ &= \sum_{k=1}^{j} g_k\Phi_{[j,k]}(\alpha)X + \sum_{k=0}^{j} X^k\Phi_{[j,k]}(\alpha)g_1\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{j} g_k \left(X\rho\Phi_{[j,k]}(\alpha) + D\Phi_{[j,k]}(\alpha) \right) + \sum_{k=0}^{j} X^k g_1 \rho\Phi_{[j,k]}(\alpha) \\
&= \sum_{k=1}^{j} g_k X\rho\Phi_{[j,k]}(\alpha) + \sum_{k=1}^{j} g_k D\Phi_{[j,k]}(\alpha) + \sum_{k=0}^{j} X^k g_1 \rho\Phi_{[j,k]}(\alpha) \\
&= \sum_{k=0}^{j} \left(g_k X + X^k g_1 \right) \rho\Phi_{[j,k]}(\alpha) + \sum_{k=1}^{j} g_k D\Phi_{[j,k]}(\alpha) \\
&= \sum_{k=0}^{j} g_{k+1} \rho\Phi_{[j,k]}(\alpha) + \sum_{k=1}^{j} g_k D\Phi_{[j,k]}(\alpha) \\
&= \sum_{k=1}^{j+1} g_k \rho\Phi_{[j,k-1]}(\alpha) + \sum_{k=1}^{j} g_k D\Phi_{[j,k]}(\alpha) \\
&= g_{j+1}\rho\Phi_{[j,j]}(\alpha) + \sum_{k=1}^{j} g_k \left(\rho\Phi_{[j,k-1]} + D\Phi_{[j,k]} \right)(\alpha) \\
&= g_{j+1}\Phi_{[j+1,j+1]}(\alpha) + \sum_{k=1}^{j} g_k \Phi_{[j+1,k]}(\alpha) \\
&= \sum_{k=1}^{j+1} g_k \Phi_{[j+1,k]}(\alpha).
\end{aligned}
$$

This completes the proof. □

Lemma 3.5. *For any $g \in R_1$, there exists a B-derivation Δ of R such that $\Delta(X) = g$.*

Proof. Let g be arbitrary element in R_1. We define $g_0 = 0$, $g_1 = g$, and $g_{j+1} = g_j X + X^j g_1$ $(j \geq 1)$, inductively. Moreover, let Δ be a right B-endomorphism of R defined by $\Delta(X^j) = g_j$ $(j \geq 0)$ (that is, $\Delta\left(\sum_j X^j c_j\right) = \sum_j g_j c_j$ $(c_j \in B, j \geq 0)$). For any $i, j \geq 1$ and $\alpha, \beta \in B$, it follows from Lemma 2.1 and Lemma 3.4 that

$$
\begin{aligned}
\Delta(X^i \alpha X^j \beta) &= \Delta\left(X^i \left(\sum_{k=0}^{j} X^k \Phi_{[j,k]}(\alpha) \right) \beta \right) \\
&= \Delta\left(\sum_{k=0}^{j} X^{i+k} \Phi_{[j,k]}(\alpha)\beta \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=0}^{j} g_{i+k}\Phi_{[j,k]}(\alpha)\beta \\
&= \sum_{k=0}^{j} \left(g_i X^k + X^i g_k\right)\Phi_{[j,k]}(\alpha)\beta \\
&= g_i \sum_{k=0}^{j} X^k \Phi_{[j,k]}(\alpha)\beta + X^i \sum_{k=1}^{j} g_k \Phi_{[j,k]}(\alpha)\beta \\
&= g_i \alpha X^j \beta + X^i \alpha g_j \beta \\
&= \Delta(X^i\alpha)X^j\beta + X^i\alpha\Delta(X^j\beta).
\end{aligned}
$$

This implies that $\Delta(h_1h_2) = \Delta(h_1)h_2 + h_1\Delta(h_2)$ for any $h_1, h_2 \in R$, that is, Δ is a derivation of R. □

Now we shall characterize B-derivations of A as follows:

Lemma 3.6. *If δ is a B-derivation of A then $\delta(x) \in A_1 \cap \mathrm{Ker}(\tau)$. Conversely, if $u \in A_1 \cap \mathrm{Ker}(\tau)$ then there exists a B-derivation δ of A such that $\delta(x) = u$.*

Proof. Let δ be a B-derivation of A. We have $\alpha\delta(x) = \delta(\alpha x) = \delta(x\rho(\alpha) + D(\alpha)) = \delta(x)\rho(\alpha)$ for any $\alpha \in B$, and hence $\delta(x) \in A_1$. An easy induction shows that

$$\delta(x^{k+1}) = \sum_{j=0}^{k} x^{k-j}\delta(x)x^j \quad (k \geq 0).$$

Then, since $0 = \sum_{k=0}^{m} x^k a_k$ and $y_j = \sum_{k=j}^{m-1} x^{k-j} a_{k+1}$, we see that

$$
\begin{aligned}
0 &= \delta\left(\sum_{k=0}^{m} x^k a_k\right) \\
&= \sum_{k=1}^{m} \delta(x^k)a_k \\
&= \sum_{k=0}^{m-1} \delta(x^{k+1})a_{k+1} \\
&= \sum_{k=0}^{m-1} \left(\sum_{j=0}^{k} x^{k-j}\delta(x)x^j\right) a_{k+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{j=0}^{m-1} \left(\sum_{k=j}^{m-1} x^{k-j} a_{k+1} \right) \delta(x) x^j \\
&= \sum_{j=0}^{m-1} y_j \delta(x) x^j \\
&= \tau(\delta(x)).
\end{aligned}
$$

Therefore $\delta(x) \in \mathrm{Ker}(\tau)$.

Conversely, assume that $u \in A_1 \cap \mathrm{Ker}(\tau)$. Let u_0 be in R such that $u = u_0 + fR$ and $\deg(u_0) < m$. Obviously, $u_0 \in R_1$ because $u \in A_1$. Hence, by Lemma 3.5, there exists a B-derivation Δ of R such that $\Delta(X) = u_0$. Since $u \in \mathrm{Ker}(\tau)$, we have

$$
0 = \tau(u) = \sum_{j=0}^{m-1} y_j u x^j = \sum_{j=0}^{m-1} Y_j u_0 X^j + fR.
$$

This means that $\sum_{j=0}^{m-1} Y_j \Delta(X) X^j \in fR$. So we obtain

$$
\begin{aligned}
\Delta(f) &= \sum_{k=1}^{m} \Delta(X^k) a_k \\
&= \sum_{k=0}^{m-1} \Delta(X^{k+1}) a_{k+1} \\
&= \sum_{k=0}^{m-1} \left(\sum_{j=0}^{k} X^{k-j} \Delta(X) X^j \right) a_{k+1} \\
&= \sum_{j=0}^{m-1} \left(\sum_{k=j}^{m-1} X^{k-j} a_{k+1} \right) \Delta(X) X^j \\
&= \sum_{j=0}^{m-1} Y_j \Delta(X) X^j \\
&\in fR.
\end{aligned}
$$

This implies that $\Delta(fg) = \Delta(f)g + f\Delta(g) \in fR$ for any $g \in R$, namely, $\Delta(fR) \subset fR$. Thus there exists a B-derivation δ of A such that $\delta(x) = u$ which is naturally induced by Δ. □

Now we shall state the following theorem which is a generalization of [9, Theorem 3.2] and [9, Theorem 3.8].

Theorem 3.7. *f is weakly separable in R if and only if*

$$A_1 \cap \operatorname{Ker}(\tau) = I_x(V).$$

Proof. Note that $I_x(V) \subset \operatorname{Ker}(\tau) \cap A_1$ by Lemma 3.3 (2).

Assume that f is weakly separable in R, that is, every B-derivation of A is inner. Let $u \in A_1 \cap \operatorname{Ker}(\tau)$. By Lemma 3.6, there exists a B-derivation δ of A such that $\delta(x) = u$. Since δ is inner, we have $u = \delta(x) = vx - xv$ for some fixed element $v \in A$. In particular, we see that $v \in V$ because $0 = \delta(\alpha) = v\alpha - \alpha v$ for any $\alpha \in B$. We have then $u = \delta(x) \in I_x(V)$, namely, $A_1 \cap \operatorname{Ker}(\tau) \subset I_x(V)$. Therefore $A_1 \cap \operatorname{Ker}(\tau) = I_x(V)$ by Lemma 3.3 (2).

Conversely, assume that $A_1 \cap \operatorname{Ker}(\tau) = I_x(V)$, and let δ be a B-derivation of A. By Lemma 3.6, we see that $\delta(x) \in A_1 \cap \operatorname{Ker}(\tau) = I_x(V)$. Hence $\delta(x) = vx - xv$ for some $v \in V$. An easy induction shows that

$$\delta(x^j) = vx^j - x^j v \quad (j \geq 0).$$

So, for any $z = \sum_{j=0}^{m-1} x^j c_j \in A$ $(c_j \in B)$, we have

$$\begin{aligned}
\delta(z) &= \delta\left(\sum_{j=0}^{m-1} x^j c_j\right) \\
&= \sum_{j=0}^{m-1} \delta(x^j) c_j \\
&= \sum_{j=0}^{m-1} (vx^j - x^j v) c_j \\
&= v \sum_{j=0}^{m-1} x^j c_j - \sum_{j=0}^{m-1} x^j c_j v \\
&= vz - zv.
\end{aligned}$$

Therefore δ is inner, and hence f is weakly separable in R. □

In virtue of Theorem 3.7, we have the following.

Corollary 3.8. *f is weakly separable in R if and only if the following sequence of $C(A)$-$C(A)$-homomorphisms is exact:*

$$V \xrightarrow{I_x} A_1 \xrightarrow{\tau} A.$$

Proof. It is obvious by Theorem 3.7. □

Remark. There always holds $\mathrm{Ker}(I_x : V \to A_1) = C(A)$. Hence f is weakly separable in R if and only if the following sequence of $C(A)$-$C(A)$-homomorphisms is exact:

$$0 \longrightarrow C(A) \xrightarrow{\text{inclusion}} V \xrightarrow{I_x} A_1 \xrightarrow{\tau} A.$$

Concerning the relation between separability and weak separability in $B[X;D]$, we shall state the following theorem which is an improvement of [9, Theorem 3.10].

Theorem 3.9. *Let $R = B[X;D]$, $R_{(0)} = B[X;D]_{(0)}$, and f a monic polynomial in $R_{(0)}$ of the form $f = \sum_{i=0}^{m} X^i a_i$ $(a_m = 1, m \geq 1)$.*

(1) *f is weakly separable in R if and only if the following sequence of $C(A)$-$C(A)$-homomorphisms is exact:*

$$V \xrightarrow{I_x} V \xrightarrow{\tau} C(A).$$

(2) *f is separable in R if and only if the following sequence of $C(A)$-$C(A)$-homomorphisms is exact:*

$$V \xrightarrow{I_x} V \xrightarrow{\tau} C(A) \longrightarrow 0.$$

Proof. Note that $A_k = V$ $(k \in \mathbb{Z})$ in this case. First we shall show that $\tau(V) \subset C(A)$. Let φ be an A-A-homomorphism of $A \otimes_B A$ onto A defined by $\sum_j z_j \otimes w_j \mapsto \sum_j z_j w_j$ $(z_j, w_j \in A)$ and $(A \otimes_B A)^A = \{\mu \in A \otimes_B A \mid z\mu = \mu z \ (\forall z \in A)\}$. It is obvious that $\varphi\left((A \otimes_B A)^A\right) \subset C(A)$. Let v be arbitrary element in V. As was shown in [8, Lemma 3.1], we have already known that $\sum_{j=0}^{m-1} y_j v \otimes x^j \in (A \otimes_B A)^A$, and hence

$$\tau(v) = \sum_{j=0}^{m-1} y_j v x^j = \varphi\left(\sum_{j=0}^{m-1} y_j v \otimes x^j\right) \in C(A).$$

Therefore $\tau(V) \subset C(A)$.

(1) It is obvious by Corollary 3.8.

(2) If f is separable in R then f is always weakly separable in R, and therefore it suffices to show that $\tau(V) = C(A)$. By Lemma 3.2, f is separable in R if and only if there exists $v \in V$ such that $\tau(v) = 1$. This means that $\tau(V) = C(A)$ because τ is a $C(A)$-$C(A)$-homomorphism. □

Example. Let $B = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$ (the upper triangular matrix over $\mathbb{Z}$), D a derivation of B defined by $D\left(\begin{bmatrix} b_1 & b_2 \\ 0 & b_3 \end{bmatrix}\right) = \begin{bmatrix} 0 & b_2 \\ 0 & 0 \end{bmatrix}$ $(b_1, b_2, b_3 \in \mathbb{Z})$, $R = B[X; D]$, and $R_{(0)} = B[X; D]_{(0)}$. We put here $a = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \in B$ and $f = X^2 + Xa + a \in R$. It is easy to see that $\alpha f = f\alpha$ for any $\alpha \in B$ and $Xf = fX$, and hence $f \in R_{(0)}$. Now let $A = R/fR$ and $x = X + fR$. One easily see that

$$V = C(A) = \left\{ x \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} + \begin{bmatrix} s+t & 0 \\ 0 & t \end{bmatrix} \,\middle|\, s, t \in \mathbb{Z} \right\}.$$

Let $v = xb + c$ be arbitrary element in V such that $b = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}$, $c = \begin{bmatrix} s+t & 0 \\ 0 & t \end{bmatrix}$ $(s, t \in \mathbb{Z})$. Since $xb = bx$ and $xc = cx$, we obtain

$$I_x(v) = vx - xv = (xb + c)x - x(xb + c) = 0.$$

Thus $I_x(V) = \{0\}$. Recall that $y_0 = x + a$ and $y_1 = 1$ in this case. We have then

$$\begin{aligned}
\tau(v) &= y_0 v + y_1 v x \\
&= (x + a)(xb + c) + (xb + c)x \\
&= x^2 b + x(ab + c) + ac + x^2 b + xc \\
&= x^2 2b + x(ab + 2c) + ac \\
&= (-xa - a)2b + x(ab + 2c) + ac \\
&= x(2c - ab) + ac - 2ab \\
&= x\left(\begin{bmatrix} 2(s+t) & 0 \\ 0 & 2t \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}\right) \\
&\quad + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s+t & 0 \\ 0 & t \end{bmatrix} - 2\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} \\
&= x\begin{bmatrix} 2t - s & 0 \\ 0 & 2t - s \end{bmatrix} + \begin{bmatrix} 3t - 3s & 0 \\ 0 & t - 2s \end{bmatrix}.
\end{aligned}$$

So we see that $s = t = 0$ (i.e. $v = 0$) if $v \in \mathrm{Ker}(\tau)$. Therefore $\mathrm{Ker}(\tau) \cap V = \{0\} = I_x(V)$, and hence f is weakly separable in R by Theorem 3.9 (1). However, it is obvious that $\tau(V) \subsetneq C(A)$ (for example, there are no elements $u \in V$ such that $\tau(u) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in C(A)$). Thus f is not separable in R by Theorem 3.9 (2).

Acknowledgement

The author would like to thank the referee for his (her) valuable suggestions and comments.

References

[1] S. Elliger, *Über automorphismen und derivationen von ringen*, J. Reine Angew. Math., **277** (1975), 155–177.

[2] N. Hamaguchi and A. Nakajima, *On generalizations of separable polynomials over rings*, Hokkaido Math. J., **42** (2013), no.1, 53–68.

[3] K. Hirata and K. Sugano, *On semisimple extensions and separable extensions over noncommutative rings*, J. Math. Soc. Japan, **18** (1966), 360–373.

[4] S. Ikehata, *On separable polynomials and Frobenius polynomials in skew polynomial rings*, Math. J. Okayama Univ., **22** (1980), 115–129.

[5] S. Ikehata, *Azumaya algebras and skew polynomial rings*, Math, J. Okayama Univ., **23** (1981), 19–32.

[6] Y. Miyashita, *On a skew polynomial ring*, J. Math. Soc. Japan, **31** (1979), no.2, 317–330.

[7] K. Sugano, *Separable extensions and Frobenius extensions*, Osaka J. Math., **7** (1970), 29–40.

[8] S. Yamanaka and S. Ikehata, *An alternative proof of Miyasita's theorem in a skew polynomial rings*, Int. J. Algebra, **21** (2012), 1011–1023

[9] S. Yamanaka, *On weakly separable polynomials and weakly quasi-separable polynomials over rings*, Math. J. Okayama Univ., **58** (2016), 169–182.

Satoshi Yamanaka
Department of Integrated Science and Technology
National Institute of Technology, Tsuyama College
624-1 Numa, Tsuyama-shi, Okayama, 708-8509, Japan.
e-mail address: yamanaka@tsuyama.kosen-ac.jp

(*Received March 7, 2020*)
(*Accepted May 8, 2020*)