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THE d-SMITH SETS OF CARTESIAN PRODUCTS OF THE ALTERNATING GROUPS AND FINITE ELEMENTARY ABELIAN 2-GROUPS

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Abstract. Let G be a finite group. In 1970s, T. Petrie defined the Smith equivalence of real G-modules. The Smith set of G is the subset of the real representation ring consisting of elements obtained as differences of Smith equivalent real G-modules. Various results of the topic have been obtained. The d-Smith set of G is the set of all elements $[V] - [W]$ in the Smith set of G such that the H-fixed point sets of V and W have the same dimension for all subgroups H of G. The results of the Smith sets of the alternating groups and the symmetric groups are obtained by E. Laitinen, K. Pawałowski and R. Solomon. In this paper, we give the calculation results of the d-Smith sets of the alternating groups and the symmetric groups. In addition, we give the calculation results of the d-Smith sets of Cartesian products of the alternating groups and finite elementary abelian 2-groups.

1. Introduction

Throughout this paper, let G be a finite group. Let $\mathcal{S}(G)$ denote the set of all subgroups of G and $\mathcal{P}(G)$ the set of all subgroups with prime power order of G. Let $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ denote the rational, real, and complex number fields, respectively, and let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and the ring of integers, respectively. For a subfield F of $\mathbb{C}$, let $\mathrm{R}(G,F)$ denote the F-representation ring of G. In particular, We denote $\mathrm{R}(G,\mathbb{R})$ and $\mathrm{R}(G,\mathbb{C})$ by $\mathrm{RO}(G)$ and $\mathrm{R}(G)$, respectively. By canonical homomorphisms, we regard

$$\mathrm{R}(G,\mathbb{Q}) \subset \mathrm{RO}(G) \subset \mathrm{R}(G).$$

For an algebra A, we mean by an *A-module* a finitely generated module over A. For a commutative ring R, we denote by $R[G]$ the group algebra of G over R. Since $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$, we regard $\mathbb{Z}[G] \subset \mathbb{Q}[G] \subset \mathbb{R}[G] \subset \mathbb{C}[G]$. We refer to a $\mathbb{Q}[G]$-module, an $\mathbb{R}[G]$-module and a $\mathbb{C}[G]$-module as a *rational G-module*, a *real G-module* and a *complex G-module*, respectively. We say that real G-modules V and W are *dim-equivalent* if $\dim V^H = \dim W^H$ holds for any subgroup H of G. T. tom Dieck [6, p. 229] defined $\mathrm{RO}_0(G)$ to be the set of all elements $[V] - [W] \in \mathrm{RO}(G)$ such that V and W are dim-equivalent. Clearly, $\mathrm{RO}_0(G)$ is a submodule of $\mathrm{RO}(G)$.

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In 1960, P. A. Smith [22] asked whether the tangent spaces $T_a(S)$ and $T_b(S)$ are isomorphic as real G-modules for any sphere S with smooth G-action such that the G-fixed point set S^G consists of exactly two points a and b, in other words he asked whether the element $[T_a(S)] - [T_b(S)] \in \mathrm{RO}(G)$ is trivial. This isomorphism problem motivates various researchers to study transformation groups on spheres with finite fixed points. T. Petrie [17, 18] called real G-modules V and W *Smith equivalent* if there is a homotopy sphere Σ with a smooth G-action such that $\Sigma^G = \{a,\, b\}$, $a \neq b$, and the tangent spaces $T_a(\Sigma)$ and $T_b(\Sigma)$ are isomorphic to V and W as real G-modules, respectively. If V and W are Smith equivalent, we write $V \sim_{\mathfrak{S}} W$. In this paper, we call V and W *d-Smith equivalent* if V and W are Smith equivalent and dim-equivalent. If V and W are d-Smith equivalent, we write $V \sim_{\mathfrak{d}\mathfrak{S}} W$. Define the *Smith set* $\mathfrak{S}(G)$, and the *d-Smith set* $\mathfrak{d}\mathfrak{S}(G)$, respectively, by

$$\mathfrak{S}(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{S}} W\},$$
$$\mathfrak{d}\mathfrak{S}(G) = \{[V] - [W] \in \mathrm{RO}(G) \mid V \sim_{\mathfrak{d}\mathfrak{S}} W\}.$$

By definition, it holds that

$$\mathfrak{d}\mathfrak{S}(G) \subset \mathfrak{S}(G).$$

In this paper, let C_n, A_n, and S_n denote a cyclic group of order n, the alternating group of degree n, and the symmetric group of degree n, respectively. For a natural number n, let $C_2^n = C_2 \times \cdots \times C_2$ (n-fold). M. F. Atiyah–R. Bott [1, Theorem 7.15], and J. W. Milnor [12], proved $\mathfrak{S}(C_p) = 0$ for any prime p, and C. U. Sanchez [20, Corollary 1.11] proved $\mathfrak{S}(C_{q^k}) = 0$ for any odd prime q and any natural number k. Let $\mathcal{P}^*(G)$ denote the set of all subgroups H of G of which the order $|H|$ is either an odd prime power, 2, or 4. T. Petrie remarked that if real G-modules V and W are Smith equivalent then $\dim V^G = 0 = \dim W^G$ and $\mathrm{res}^G_H V \cong \mathrm{res}^G_H W$ for all $H \in \mathcal{P}^*(G)$, cf. [18, p. 61], [19, Section 4, Theorem 0.4].

On the other hand, T. Petrie [16, Theorem B], [17, 18] proved $\mathfrak{S}(G) \neq 0$ for abelian groups G having at least 4 noncyclic Sylow subgroups, and so did S. E. Cappell–J. L. Shaneson [4, Theorem A], [5] for $G = C_{4k}$, where $k \in \mathbb{N}$ with $k \geq 2$. In general, the sets $\mathfrak{S}(G)$, and $\mathfrak{d}\mathfrak{S}(G)$ are not additively closed subsets of $\mathrm{RO}(G)$, see [14, p. 62].

The next results related to the Smith sets of finite groups S_m, A_m, $S_m \times C_2^n$ and $A_m \times C_2^n$ are well known. It follows that $\mathfrak{S}(G)$ is trivial for each $G = A_m$, S_m with $m \leq 5$, cf. [11, Lemma 1.4], [15, Theorem C3]. Furthermore, K. Pawałowski and R. Solomon [15, Theorem C3] proved that $\mathfrak{S}(S_m)$ (resp. $\mathfrak{S}(A_m)$) is trivial if and only if $m \leq 5$ (resp. $m \leq 7$). X. -M. Ju [8, Theorems A and B] proved that $\mathfrak{S}(S_5 \times C_2^n)$ (resp. $\mathfrak{S}(A_5 \times C_2^n)$) is a free module over $\mathbb{Z}$ with rank $2^n - 1$ (resp. $2(2^n - 1)$). In this paper, we determine

$\mathfrak{d}\mathfrak{S}(G)$ and $\mathrm{RO}_0(G)$ when $G = S_m$, A_m, $S_m \times C_2^n$ or $A_m \times C_2^n$ for natural numbers m and n.

For a subset $\mathcal{G}$ of $\mathcal{S}(G)$, let $\mathrm{RO}_0(G)_{\mathcal{G}}$ denote the set of all elements $x \in \mathrm{RO}_0(G)$ such that $\mathrm{res}^G_H\, x = 0$ for all $H \in \mathcal{G}$. Clearly $\mathrm{RO}_0(G)_{\mathcal{G}}$ is a direct summand of $\mathrm{RO}(G)$ as a $\mathbb{Z}$-module.

E. Laitinen and M. Morimoto [10] called a finite group G an *Oliver group* if there is no normal series $P \trianglelefteq H \trianglelefteq G$ such that $P \in \mathcal{P}(G)$, H/P is cyclic, and G/H is of prime power order. The next theorem is useful to compute the d-Smith sets of various finite Oliver groups G.

Theorem 1.1. *For an arbitrary Oliver group G such that $G^{\mathrm{nil}} = G^{\cap 2}$, it holds that*

$$\mathrm{RO}_0(G)_{\mathcal{P}(G)} \subset \mathfrak{d}\mathfrak{S}(G) \subset \mathrm{RO}_0(G)_{\mathcal{P}^*(G)}.$$

Next we give basic facts.

Theorem 1.2. *Let m be a natural number and $G = S_m$. Then* $\mathrm{RO}_0(G)$ *is trivial, and hence so is $\mathfrak{d}\mathfrak{S}(G)$, for any m.*

Theorem 1.3. *Let m and n be natural numbers and $G = S_m \times C_2^n$. Then* $\mathrm{RO}_0(G)$ *is trivial, and hence so is $\mathfrak{d}\mathfrak{S}(G)$, for any m and n.*

Let m be a natural number. A partition of m is a tuple $t = (t_1, t_2, \dots, t_r)$ consisting of natural numbers $t_1 \geq t_2 \geq \cdots \geq t_r$ that add up to m. One usually denotes the partition t by $m = t_1 + t_2 + \cdots + t_r$. We call the natural number r the *length* of t. For $m \geq 2$, let $\pi(m)$ denote the number of all partitions $t = (t_1, t_2, \dots, t_r)$ satisfying the conditions (P1)–(P3):

(P1) $t_1, t_2, \dots, t_r$ are odd natural numbers.
(P2) $t_1 > t_2 > \cdots > t_r$.
(P3) $m - r \equiv 0 \mod 4$.

For convenience, we define $\pi(1) = 0$. Let $\rho(m)$ denote the number of all partitions $t = (t_1, t_2, \dots, t_r)$ satisfying the above three conditions (P1)–(P3) and the next condition (P4):

(P4) $t_1 t_2 \cdots t_r$ is not any prime power.

For $m \leq 27$, $\pi(m)$ and $\rho(m)$ are as in the next table.

m	≤ 4	5	6	7	8	9	10	11	12	13	14	15
$\pi(m)$	0	1	1	0	0	1	2	1	0	1	3	3
$\rho(m)$	0	0	0	0	0	0	1	1	0	0	2	3

m	16	17	18	19	20	21	22	23	24	25	26	27
$\pi(m)$	1	1	4	5	2	1	5	8	5	2	6	12
$\rho(m)$	1	0	3	5	2	1	5	8	5	1	5	12

Table 1.1. The values of $\pi(m)$ and $\rho(m)$

Theorem 1.4. *Let m be a natural number and $G = A_m$. Then $\mathfrak{d}\mathfrak{S}(G)$ coincides with* $\mathrm{RO}_0(G)_{\mathcal{P}(G)}$ *and the $\mathbb{Z}$-rank of* $\mathrm{RO}_0(G)_{\mathcal{P}(G)}$ *is equal to $\rho(m)$.*

Corollary 1.5. *The set $\mathfrak{d}\mathfrak{S}(A_m)$ is trivial if and only if $m \le 9$ or $m \in \{12,\ 13,\ 17\}$.*

Remark. For a natural number m, $\pi(m) = 0$ if and only if $m \in \{1,2,3,4,7,8,12\}$.

For natural numbers m and n, let $\kappa(m,n)$ denote the number $(2^n - 1)\pi(m) + \rho(m)$. We exhibit the values $\kappa(m,n)$ for $(m,n) \in \{1,\ 2,\dots,\ 18\} \times \{1,\ 2,\dots,\ 6\}$ in the next table.

n \ m	≤ 4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
1	0	1	1	0	0	1	3	2	0	1	5	6	2	1	7
2	0	3	3	0	0	3	7	4	0	3	11	12	4	3	15
3	0	7	7	0	0	7	15	8	0	7	23	24	8	7	31
4	0	15	15	0	0	15	31	16	0	15	47	48	16	15	63
5	0	31	31	0	0	31	63	32	0	31	95	96	32	31	127
6	0	63	63	0	0	63	127	64	0	63	191	192	64	63	255

Table 1.2. The values of $\kappa(m,n)$

Theorem 1.6. *Let m and n be natural numbers and $G = A_m \times C_2^n$. Then $\mathfrak{d}\mathfrak{S}(G)$ coincides with* $\mathrm{RO}_0(G)_{\mathcal{P}(G)}$*, and the $\mathbb{Z}$-rank of* $\mathrm{RO}_0(G)_{\mathcal{P}(G)}$ *is equal to $\kappa(m,n)$.*

Corollary 1.7. *Let m and n be natural numbers and $G = A_m \times C_2^n$. Then, the set $\mathfrak{d}\mathfrak{S}(G)$ is trivial if and only if $m \in \{1,\ 2,\ 3,\ 4,\ 7,\ 8,\ 12\}$.*

We prove Theorem 1.1 in Section 4, Theorems 1.2, 1.3 and 1.4 in Section 5, Corollary 1.5 in Section 6, Theorem 1.6 in Section 7, and Corollary 1.7 in Section 8. The proofs will be understood without difficulties by readers with basic knowledge of the representation theory of finite groups.

2. Preparation of notation and terminology

For subfields $K \subset F$ of $\mathbb{C}$, let $\varphi_{K,F} : \mathrm{R}(G,K) \to \mathrm{R}(G,F)$ denote the ring homomorphism of changing rings, i.e. $\varphi_{K,F}([V]) = [V \otimes_K F]$ for $K[G]$-modules V. Here we recall that $\varphi_{K,F}$ is a monomorphism. Set

$$\mathrm{RO}_{\mathbb{Q}}(G) = \varphi_{\mathbb{Q},\mathbb{R}}(\mathrm{R}(G,\mathbb{Q})),$$
$$\mathrm{R}_{\mathbb{Q}}(G) = \varphi_{\mathbb{Q},\mathbb{C}}(\mathrm{R}(G,\mathbb{Q})),$$
$$\mathrm{R}_{\mathbb{R}}(G) = \varphi_{\mathbb{R},\mathbb{C}}(\mathrm{R}(G,\mathbb{R})),$$
$$\overline{\mathrm{RO}}_{\mathbb{Q}}(G) = \{x \in \mathrm{R}(G) \mid kx \in \mathrm{RO}_{\mathbb{Q}}(G) \text{ for some } k \in \mathbb{N}\},$$
$$\overline{\mathrm{R}}_{\mathbb{Q}}(G) = \{x \in \mathrm{R}(G) \mid kx \in \mathrm{R}_{\mathbb{Q}}(G) \text{ for some } k \in \mathbb{N}\},$$
$$\overline{\mathrm{R}}_{\mathbb{R}}(G) = \{x \in \mathrm{R}(G) \mid kx \in \mathrm{R}_{\mathbb{R}}(G) \text{ for some } k \in \mathbb{N}\}.$$

For a subset A of $\mathrm{RO}(G)$ and subsets $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{S}(G)$, set

$$A^{\mathcal{F}} = \{[V] - [W] \in A \mid V^H = 0,\ W^H = 0 \text{ for all } H \in \mathcal{F}\},$$
$$A_{\mathcal{G}} = \{[V] - [W] \in A \mid \mathrm{res}^G_K V \cong \mathrm{res}^G_K W \text{ for all } K \in \mathcal{G}\},$$
$$A^{\mathcal{F}}_{\mathcal{G}} = (A^{\mathcal{F}})_{\mathcal{G}}.$$

We call a real G-module V *$\mathcal{F}$-free* if $V^H = 0$ for all $H \in \mathcal{F}$, and we call real G-modules V and W *$\mathcal{G}$-matched* if $\mathrm{res}^G_K V \cong \mathrm{res}^G_K W$ for all $K \in \mathcal{G}$. In the current paper, let E denote the trivial group, i.e. $E = \{e\}$, and use the notation:

$G^{\{p\}}$: the smallest normal subgroup $H \le G$ such that $|G/H|$ is a power of p.

$\mathcal{L}(G) = \{H \in \mathcal{S}(G) \mid H \supset G^{\{p\}} \text{ for some prime } p\}$.

G^{nil} : the smallest normal subgroup H of G such that G/H is nilpotent.

$G^{\cap 2}$: the intersection of all normal subgroups H of G such that $|G/H| \le 2$.

The group G^{nil} coincides with $\bigcap_p G^{\{p\}}$, where p runs over the set of all primes dividing $|G|$.

Let $\mathrm{Gal}(G)$ denote the group of field automorphisms $\mathbb{Q}(\zeta) \to \mathbb{Q}(\zeta)$, where $\zeta = \exp(2\pi\sqrt{-1}/|G|)$. For $\psi \in \mathrm{Gal}(G)$ and a complex G-module V, there is a complex G-module ψV such that $\chi_{\psi V}(g) = \psi(\chi_V(g))$ for all $g \in G$, where χ_V is the character function associated with V. This induces $\mathrm{Gal}(G)$-actions on $\mathrm{R}(G,\mathbb{Q}(\zeta))$ and $\mathrm{R}(G)$, see [21, Section 12.4], and $\varphi_{\mathbb{Q}(\zeta),\mathbb{C}} : \mathrm{R}(G,\mathbb{Q}(\zeta)) \to \mathrm{R}(G)$ is a $\mathrm{Gal}(G)$-isomorphism. The action of $\mathrm{Gal}(G)$ on $\mathrm{R}(G)$ factors through a homomorphism $\mathrm{Gal}(G) \to \mathbb{Z}^{\times}_{|G|}$, where $\mathbb{Z}^{\times}_{|G|}$ stands for the group of units in $\mathbb{Z}_{|G|} = \mathbb{Z}/(|G|)$, hence for $\psi \in \mathrm{Gal}(G)$, there is an element $t \in \mathbb{Z}^{\times}_{|G|}$ such that $\chi_{\psi V}(g) = \chi_V(g^t)$ for all $g \in G$ and complex G-modules V. Let $\sigma : \mathbb{C} \to \mathbb{C}$ be the field automorphism of complex conjugation, i.e.

$\sigma(\sqrt{-1}) = -\sqrt{-1}$ and $\sigma(x) = x$ for $x \in \mathbb{R}$. Then $\overline{\mathrm{R}}_{\mathbb{R}}(G)$ and $\overline{\mathrm{R}}_{\mathbb{Q}}(G)$ coincide with the fixed point sets $\mathrm{R}(G)^{\sigma}$ and $\mathrm{R}(G)^{\mathrm{Gal}(G)}$, respectively. In particular, $\overline{\mathrm{R}}_{\mathbb{R}}(G)$ is $\mathrm{Gal}(G)$-invariant, furthermore $\mathrm{R}_{\mathbb{R}}(G)$ is also $\mathrm{Gal}(G)$-invariant. Therefore $\mathrm{Gal}(G)$ acts on $\mathrm{RO}(G)$ and $\mathrm{RO}(G)^{\mathrm{Gal}(G)} = \overline{\mathrm{RO}}_{\mathbb{Q}}(G)$. We call real G-modules V and W *Galois conjugate* if there is $\psi \in \mathrm{Gal}(G)$ such that W is isomorphic to ψV as real G-modules.

For an element g of G, we denote by $(g)_G$ the G-conjugacy class of g in G, i.e.

$$(g)_G = \{xgx^{-1} \mid x \in G\},$$

which is a subset of G. We mean by the *real G-conjugacy class* the set $(g)_G^{\pm} = (g)_G \cup (g^{-1})_G$.

3. Elements from the representation theory

Let Γ be a finite group, e.g. a quotient group of $\mathrm{Gal}(G)$. Let $\mathbb{Z}[\Gamma]$ denote the integral group ring of Γ, let $\varepsilon_\Gamma : \mathbb{Z}[\Gamma] \to \mathbb{Z}$ be the augmentation homomorphism, i.e. $\varepsilon_\Gamma(\sum_{h\in\Gamma} a_h h) = \sum_{h\in\Gamma} a_h$, where $a_h \in \mathbb{Z}$, and let $\mathfrak{I}_\Gamma$ be the augmentation ideal, i.e. $\mathfrak{I}_\Gamma = \ker(\varepsilon_\Gamma)$. We remark that

$$\mathfrak{I}_\Gamma = \langle (1-h)x \mid h \in \Gamma,\ x \in \mathbb{Z}[\Gamma]\rangle_{\mathbb{Z}}.$$

The next lemma immediately follows.

Lemma 3.1. *Let Γ and $\mathfrak{I}_\Gamma$ be as above. Then $\mathfrak{I}_\Gamma$ is a direct summand of $\mathbb{Z}[\Gamma]$ as a $\mathbb{Z}$-module.*

Let $\mathcal{B} = \{[V_i]\}_i$ be the set of all isomorphism classes of irreducible real G-modules. The group $\mathrm{Gal}(G)$ acts on $\mathcal{B}$ as permutations. The $\mathrm{Gal}(G)$-orbit $\mathrm{Gal}(G)[V_i]$ of $[V_i]$ is isomorphic to a quotient group Γ_i of $\mathrm{Gal}(G)$.

By Lemma 3.1, $\mathfrak{I}_{\mathrm{Gal}(G)}\mathrm{RO}(G)$ is a direct summand of $\mathrm{RO}(G)$ as a $\mathbb{Z}$-module. Therefore $\mathfrak{I}_{\mathrm{Gal}(G)}\mathrm{RO}(G)$, $\mathrm{RO}_0(G)$, $\mathrm{RO}(G)^{\mathrm{Gal}(G)}$ and $\overline{\mathrm{RO}}_{\mathbb{Q}}(G)$ all are direct summands of $\mathrm{RO}(G)$. The next lemma is a known fact, but we give a proof for the reader's convenience.

Lemma 3.2 ([6, Proposition 9.2.6])**.** *Let $\Gamma = \mathrm{Gal}(G)$. Then $\mathfrak{I}_\Gamma\mathrm{RO}(G)$ and $\mathrm{RO}(G)^\Gamma$ coincide with $\mathrm{RO}_0(G)$ and $\overline{\mathrm{RO}}_{\mathbb{Q}}(G)$, respectively, and $\mathrm{RO}(G) = \mathrm{RO}_0(G) \oplus \overline{\mathrm{RO}}_{\mathbb{Q}}(G)$.*

Proof. The equality $\mathrm{RO}(G)^\Gamma = \overline{\mathrm{RO}}_{\mathbb{Q}}(G)$ is shown in [21, Section 12.4].

For a real G-module V and a subgroup H, the formula

$$\dim V^H = \frac{1}{|H|}\sum_{g\in H} \chi_V(g),$$

implies the equality $\dim V^H = \dim(\psi V)^H$ for any $\psi \in \mathrm{Gal}(G)$, hence we see the inclusion $\mathfrak{I}_\Gamma\mathrm{RO}(G) \subset \mathrm{RO}_0(G)$.

Rational G-modules V and W are isomorphic if and only if $\dim V^C = \dim W^C$ for all cyclic subgroups C of G. This implies $\mathrm{RO}_0(G)\cap\overline{\mathrm{RO}}_{\mathbb{Q}}(G) = 0$, hence $\mathfrak{I}_\Gamma\mathrm{RO}(G)\cap\overline{\mathrm{RO}}_{\mathbb{Q}}(G) = 0$. Let A be the submodule $\mathfrak{I}_\Gamma\mathrm{RO}(G)\oplus\overline{\mathrm{RO}}_{\mathbb{Q}}(G)$ of $\mathrm{RO}(G)$. By the structure theorem of finitely generated free abelian groups, A is a direct summand of $\mathrm{RO}(G)$. For an element $x \in \mathrm{RO}(G)$, we have

$$|\Gamma|x = \sum_{\psi\in\Gamma}(1-\psi)x + \left(\sum_{\psi\in\Gamma}\psi\right) x \quad \in \mathfrak{I}_\Gamma\mathrm{RO}(G) + \mathrm{RO}(G)^\Gamma,$$

which implies

$$\mathrm{RO}(G)\otimes_{\mathbb{Z}}\mathbb{Q} = \langle\mathfrak{I}_\Gamma\mathrm{RO}(G)\rangle_{\mathbb{Q}} + \langle\overline{\mathrm{RO}}_{\mathbb{Q}}(G)\rangle_{\mathbb{Q}}.$$

Therefore the $\mathbb{Z}$-rank of A is equal to the $\mathbb{Z}$-rank of $\mathrm{RO}(G)$, which shows $\mathrm{RO}(G) = A$ $(= \mathfrak{I}_\Gamma\mathrm{RO}(G)\oplus\overline{\mathrm{RO}}_{\mathbb{Q}}(G))$.

Furthermore, since $\mathfrak{I}_\Gamma\mathrm{RO}(G) \subset \mathrm{RO}_0(G)$ and $\mathrm{RO}_0(G)\cap\overline{\mathrm{RO}}_{\mathbb{Q}}(G) = 0$, we get $\mathrm{RO}_0(G) = \mathfrak{I}_\Gamma\mathrm{RO}(G)$. □

4. Corollary and proof of Theorem 1.1

We begin the section with a corollary which immediately follows from Theorem 1.1.

Corollary 4.1. *Let G be an arbitrary Oliver group with $G^{\mathrm{nil}} = G^{\cap 2}$. Suppose*

(1) *$G^{\cap 2}$ is of odd order, or*
(2) *$\mathrm{res}^G_C\mathrm{RO}_0(G) = 0$ for all cyclic subgroups $C \in \mathcal{S}(G)$ of 2-power order.*

Then $\mathfrak{d}\mathfrak{S}(G)$ coincides with $\mathrm{RO}_0(G)_{\mathcal{P}(G)}$.

The next lemma is well-known but we give a proof for the sake of reader's convenience.

Lemma 4.2. *Let G be a finite group and H a subgroup of G with order 1, 2, or 4. If real G-modules V and W are Smith equivalent, then $\mathrm{res}^G_H V$ and $\mathrm{res}^G_H W$ are isomorphic.*

Proof. Let Σ be a homotopy sphere with G-action such that $\Sigma^G = \{x,\, y\}$, $V \cong T_x(\Sigma)$, and $W \cong T_y(\Sigma)$. Let K be a subgroup of H. Since $|K|$ is a power of 2, P. A. Smith's theorem says that Σ^K is a $\mathbb{Z}_2$-homology sphere, and hence Σ^K is either connected or equal to $\{x,\, y\}$, which implies $\dim V^K = \dim W^K$. Since $|H| = 1$, 2, or 4, we see that $\mathrm{res}^G_H V \cong \mathrm{res}^G_H W$. □

For a real G-module V, let $V^{\mathcal{L}(G)}$ denote the submodule $\sum_{L\in\mathcal{L}(G)} V^L$ and let $V_{\mathcal{L}(G)}$ denote the orthogonal complement of $V^{\mathcal{L}(G)}$ in V, with respect to a G-invariant inner-product on V.

Lemma 4.3. *Let G be an Oliver group and let V and W be $\mathcal{L}(G)$-free real G-modules. If V and W are dim-equivalent and $\mathcal{P}(G)$-matched, then the element $x = [V] - [W]$ belongs to $\mathfrak{d}\mathfrak{S}(G)$ (furthermore there exists an $\mathcal{L}(G)$-free real G-module U such that $V \oplus U \oplus \mathbb{R}[G]^{\oplus m}_{\mathcal{L}(G)}$ and $W \oplus U \oplus \mathbb{R}[G]^{\oplus m}_{\mathcal{L}(G)}$ are Smith equivalent for any $m \in \mathbb{N}$).*

Proof. By definition, $\mathfrak{d}\mathfrak{S}(G)$ coincides with $\mathfrak{S}(G) \cap \mathrm{RO}_0(G)$. Since V and W are dim-equivalent, $x = [V] - [W]$ belongs to $\mathrm{RO}_0(G)$. By [13, Theorem 6.7] (obtained by equivariant surgery theory [2, 3]), the element x belongs to $\mathfrak{S}(G)$. □

Proof of Theorem 1.1. Let G be an Oliver group. By Lemma 4.3, $\mathrm{RO}_0(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)}$ is contained in $\mathfrak{d}\mathfrak{S}(G)$. By Lemma 4.2 and C. U. Sanchez [20, Corollary 1.11], $\mathfrak{S}(G)$ is contained in $\mathrm{RO}(G)_{\mathcal{P}^*(G)}$. Therefore we obtain

$$\mathrm{RO}_0(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} \subset \mathfrak{d}\mathfrak{S}(G) \subset \mathrm{RO}_0(G)_{\mathcal{P}^*(G)}.$$

Since $\mathrm{RO}_0(G) = \mathrm{RO}_0(G/G^{\cap 2}) \oplus \mathrm{RO}_0(G)^{\{G^{\cap 2}\}}$ and $\mathrm{RO}_0(G/G^{\cap 2}) = 0$, $\mathrm{RO}_0(G)^{\{G^{\cap 2}\}}$ coincides with $\mathrm{RO}_0(G)$. Since $G^{\mathrm{nil}} = G^{\cap 2}$, we have $\mathrm{RO}_0(G)^{\mathcal{L}(G)}_{\mathcal{P}(G)} = \mathrm{RO}_0(G)^{\{G^{\cap 2}\}}_{\mathcal{P}(G)}$, and hence

$$\mathrm{RO}_0(G)_{\mathcal{P}(G)} \subset \mathfrak{d}\mathfrak{S}(G) \subset \mathrm{RO}_0(G)_{\mathcal{P}^*(G)}.$$

□

5. Proofs of Theorems 1.2, 1.3 and 1.4

Let m be a natural number. In this section, we recall basics of the representation theory of S_m and A_m. Details of the theory are given in [7, 9].

First, let g be an element of S_m. We call a product $g_1 g_2 \cdots g_r$ of disjoint cycles $g_i = (g_{i,1}, g_{i,2}, \ldots, g_{i,\tau_i})$ the *cycle decomposition* of g if the conditions $g = g_1 g_2 \cdots g_r$, $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_r$, and $m = \tau_1 + \tau_2 + \cdots + \tau_r$ all are fulfilled. By virtue of cycle decomposition, each element $g \in S_m$ determines a partition $\tau(g) = (\tau_1, \tau_2, \ldots, \tau_r)$ of m for some $r \in \mathbb{N}$. Clearly an arbitrary partition t of m is obtainable as $\tau(g)$ for some $g \in S_m$. A partition t of m determines a Young diagram $YD(t)$, (as well as a typical standard Young tableau $YT(t)$,) and a Young symmetrizer $c_t \in \mathbb{Z}[S_m]$, see [7, p.46, (4.2)]. For a subfield F of $\mathbb{C}$, we mean by $F[S_m]c_t$ the image of c_t by right multiplication on $F[S_m]$. By [7, Theorem 4.3], we have $c_t^2 = n_t c_t$ for some $n_t \in \mathbb{N}$ and $V_t = \mathbb{C}[S_m]c_t$ is an irreducible complex S_m-module. Here we remark that $V_{\tau(g)}$ and $V_{\tau(h)}$, where $g, h \in S_m$, are isomorphic if and only if the S_m-conjugacy classes $(g)_{S_m}$ of g and $(h)_{S_m}$ of h coincide with each other. Let $\mathcal{F}_{S_m}$ be a complete set of representatives of S_m-conjugacy classes of elements of S_m. Then the set $\{[V_{\tau(g)}] \mid g \in \mathcal{F}_{S_m}\}$ is a basis of the free module $\mathrm{R}(S_m)$ over $\mathbb{Z}$. It is easy to see the next fact.

Proposition 5.1. *Let* $\mathcal{F}_{S_m}$ *be as above. Then the sets* $\{[\mathbb{Q}[S_m]c_{\tau(g)}] \mid g \in \mathcal{F}_{S_m}\}$ *and* $\{[\mathbb{R}[S_m]c_{\tau(g)}] \mid g \in \mathcal{F}_{S_m}\}$ *are bases of* $\mathrm{R}(S_m, \mathbb{Q})$ *and* $\mathrm{RO}(S_m)$*, respectively. Therefore* $\mathrm{RO}_{\mathbb{Q}}(S_m) = \mathrm{RO}(S_m)$ *and* $\mathrm{RO}_0(S_m) = 0$.

Theorem 1.2 immediately follows from Proposition 5.1.

For any $x \in \mathrm{RO}(S_m)$, $y \in \mathrm{RO}(C_2^n)$ and $\psi \in \mathrm{Gal}(S_m \times C_2^n)$, $\psi(x \otimes y)$ is isomorphic to $x \otimes y$, because the character of $x \otimes y$ has values in $\mathbb{Q}$. Therefore, we obtain Theorem 1.3.

For a partition t of m, the conjugate partition t' of m to t is defined by interchanging rows and columns in the Young diagram. We remark that $V_{t'}$ is isomorphic to $V_t \otimes_{\mathbb{C}} \mathbb{C}_{\pm}$ as complex S_m-modules, where $\mathbb{C}_{\pm}$ is the 1-dimensional nontrivial complex S_m-module. The set $\mathcal{T}$ of all partitions of m has the C_2-action given by conjugations. Let $\Lambda_{\text{s-conj}}$ be the set of self-conjugate partitions of m, i.e. $\Lambda_{\text{s-conj}} = \mathcal{T}^{C_2}$, and let $\Lambda^* \ (\subset \mathcal{T})$ be a complete set of representatives of the C_2-orbit set $(\mathcal{T} \smallsetminus \Lambda_{\text{s-conj}})/C_2$, e.g.

$$\Lambda^* = \{t \in \mathcal{T} \mid t > t'\},$$

with respect to the lexicographic order [7, Part I (4.22)].

Since A_1 is the trivial group, $\mathfrak{d}\mathfrak{S}(A_1) = \mathrm{RO}_0(A_1)_{\mathcal{P}(A_1)} = 0$. In the rest of this section, let m be a natural number satisfying $m \geq 2$, and let g be an element of A_m and a an odd permutation in S_m. We call g, or more precisely the S_m-conjugacy class $(g)_{S_m}$ of g, *split* if $(g)_{S_m} \neq (g)_{A_m}$, where $(g)_{A_m}$ stands for the A_m-conjugacy class of g. If g is split then $(g)_{S_m} = (g)_{A_m} \amalg (aga^{-1})_{A_m}$. We call g *real* if $(g)_{A_m} = (g^{-1})_{A_m}$. If g is not real then we call g *complex*. If g is complex then clearly, g is split and $(g^{-1})_{A_m} = (aga^{-1})_{A_m}$. We call g *rational* if $(g)_{A_m} = (aga^{-1})_{A_m}$. Therefore g is rational if and only if g is not split. We use the notation:

$$\mathcal{A}_1 = \{(x)_{S_m} \mid x \in A_m,\ x \text{ is split and real}\}.$$
$$\mathcal{A}_2 = \{(x)_{S_m} \mid x \in A_m,\ x \text{ is complex}\}.$$
$$\mathcal{A}_3 = \{(x)_{S_m} \mid x \in A_m,\ x \text{ is rational}\}.$$

For each $i = 1, 2, 3$, let $\mathcal{F}_i \ (\subset A_m)$ be a complete set of representatives of S_m-conjugacy classes belonging to $\mathcal{A}_i$.

Let $t = (t_1, t_2, \ldots, t_r)$ be a partition of m. We call t *split* if $t_1, t_2, \ldots, t_r$ are distinct odd natural numbers, therefore $t_1 > t_2 > \cdots > t_r$. The next two lemmas are classical results, see [7, Section 5.1], particularly see [7, Proposition 5.3].

Lemma 5.2. *Let* g *be an element of* A_m *and let* τ *be the partition of* m *obtained from (the cycle decomposition of)* g*. Then the following holds.*

(1) *The element* g *is split if and only if* τ *is split.*

(2) *Suppose g is split. Then the element g is real if and only if $m-r \equiv 0 \mod 4$, where r is the length of τ.*

Let Λ_{sp} be the set of all split partitions of m. For $t = (t_1, t_2, \ldots, t_\ell) \in \Lambda_{\text{s-conj}}$, taking the hook lengths of $YD(t)$, we obtain a split partition ω $(= \omega(t)) = (\omega_1, \omega_2, \ldots, \omega_r)$ of m, hence we have $\omega_1 = 2t_1 - 1$, $\omega_2 = 2t_2 - 3$, $\omega_3 = 2t_3 - 5$, $\ldots$. Conversely, for $t \in \Lambda_{\text{sp}}$, there is a unique partition λ $(= \lambda(t)) \in \Lambda_{\text{s-conj}}$ such that $\omega(\lambda) = t$. Therefore the correspondences $\Lambda_{\text{s-conj}} \xrightarrow{\omega} \Lambda_{\text{sp}}$ and $\Lambda_{\text{sp}} \xrightarrow{\lambda} \Lambda_{\text{s-conj}}$ are bijective.

Lemma 5.3. *Let g be an element of A_m and let a be an odd permutation in S_m. Then the following holds.*

(1) *Suppose g is split and real, and set $\lambda = \lambda(\tau(g))$. Then $\operatorname{res}^{S_m}_{A_m} \mathbb{Q}[S_m]c_\lambda$ is irreducible, and $\operatorname{res}^{S_m}_{A_m} \mathbb{R}[S_m]c_\lambda$ is the direct sum of non-isomorphic irreducible real A_m-modules $U_{g,+}$ and $U_{g,-}$ such that*

$$\operatorname{res}^{S_m}_{A_m} V_\lambda = (U_{g,+} \otimes_{\mathbb{R}} \mathbb{C}) \oplus (U_{g,-} \otimes_{\mathbb{R}} \mathbb{C}).$$

In addition, for the character $\chi_{U_{g,\pm}}$ of $U_{g,\pm}$, $\chi_{U_{g,+}}(aha^{-1}) = \chi_{U_{g,-}}(h)$ for $h \in A_m$, $\chi_{U_{g,+}}(h) = \chi_{U_{g,-}}(h)$ for $h \in A_m$ such that $(h)_{S_m} \neq (g)_{S_m}$, and

$$\chi_{U_{g,+}}(g) = \frac{1}{2}\left(1 + \sqrt{q(g)}\right), \quad \chi_{U_{g,-}}(g) = \frac{1}{2}\left(1 - \sqrt{q(g)}\right),$$

for certain $q(g) \in \mathbb{N}$ satisfying $\sqrt{q(g)} \notin \mathbb{Q}$.

(2) *Suppose g is complex, and set $\lambda = \lambda(\tau(g))$. Then $\operatorname{res}^{S_m}_{A_m} \mathbb{Q}[S_m]c_\lambda$ and $\operatorname{res}^{S_m}_{A_m} \mathbb{R}[S_m]c_\lambda$ are irreducible, and $\operatorname{res}^{S_m}_{A_m} V_\lambda$ is the direct sum of non-isomorphic irreducible complex A_m-modules $W_{g,+}$ and $W_{g,-}$ such that $W_{g,+}$ and $W_{g,-}$ are of complex type and $\overline{W}_{g,+} \cong W_{g,-}$, where $\overline{W}_{g,+}$ is the complex conjugate of $W_{g,+}$. In addition, $\chi_{W_{g,+}}(aha^{-1}) = \chi_{W_{g,-}}(h)$ for $h \in A_m$, $\chi_{W_{g,+}}(h) = \chi_{W_{g,-}}(h)$ for $h \in A_m$ such that $(h)_{S_m} \neq (g)_{S_m}$, and*

$$\chi_{W_{g,+}}(g) = \frac{1}{2}\left(-1 + \sqrt{-q(g)}\right), \quad \chi_{W_{g,-}}(g) = \frac{1}{2}\left(-1 - \sqrt{-q(g)}\right),$$

for certain $q(g) \in \mathbb{N}$.

(3) *Let t be a partition of m which is not self-conjugate, i.e. $t \notin \Lambda_{\text{s-conj}}$. Then $\operatorname{res}^{S_m}_{A_m} \mathbb{Q}[S_m]c_t$, $\operatorname{res}^{S_m}_{A_m} \mathbb{R}[S_m]c_t$, and $\operatorname{res}^{S_m}_{A_m} V_t$ all are irreducible.*

The $q(g)$ in the lemma above is determined by $\tau(g) = (\tau_1, \tau_2, \ldots, \tau_r)$ with the formula $q(g) = \tau_1\tau_2 \cdots \tau_r$.

We can prove the next proposition without difficulties.

Proposition 5.4. *The following holds.*

(1) *The set*

$$\{[\mathrm{res}_{A_m}^{S_m}\mathbb{Q}[S_m]c_{\lambda(\tau(g))}] \mid g \in \mathcal{F}_1 \cup \mathcal{F}_2\} \cup \{[\mathrm{res}_{A_m}^{S_m}\mathbb{Q}[S_m]c_t] \mid t \in \Lambda^*\}$$

is a basis of $\mathrm{R}(A_m, \mathbb{Q})$.

(2) *The set*

$$\{[U_{g,+}],\ [U_{g,-}] \mid g \in \mathcal{F}_1\} \cup \{[W_{g,+\mathbb{R}}] \mid g \in \mathcal{F}_2\} \\ \cup \{[\mathrm{res}_{A_m}^{S_m}\mathbb{R}[S_m]c_t] \mid t \in \Lambda^*\}$$

is a basis of $\mathrm{RO}(A_m)$*, where* $W_{g,+\mathbb{R}}$ *is the realification of* $W_{g,+}$.

(3) *The set*

$$\{[U_{g,+} \otimes_{\mathbb{R}} \mathbb{C}],\ [U_{g,-} \otimes_{\mathbb{R}} \mathbb{C}] \mid g \in \mathcal{F}_1\} \cup \{[W_{g,+}],\ [W_{g,-}] \mid g \in \mathcal{F}_2\} \\ \cup \{[\mathrm{res}_{A_m}^{S_m} V_t] \mid t \in \Lambda^*\}$$

is a basis of $\mathrm{R}(A_m)$.

By virtue of Proposition 5.4 (3), we get $|\Lambda^*| = |\mathcal{A}_3| = |\mathcal{F}_3|$. We wonder which map $\mathcal{F}_3 \to \Lambda^*$ is a 'natural' one-to-one correspondence.

Proposition 5.5. *Let* $g \in \mathcal{F}_1$. *For any element* h *of* A_m *of even order, the equality* $\chi_{U_{g,+}}(h) = \chi_{U_{g,-}}(h)$ *holds.*

Proof. Let a be an odd permutation in S_m. Since h is of even order, h is rational, and hence $(h)_{A_m} = (aha^{-1})_{A_m}$. This implies

$$\chi_{U_{g,+}}(h) = \chi_{U_{g,+}}(aha^{-1}) = \chi_{U_{g,-}}(h).$$

□

Now we are ready to see the next proposition which we need in the study of the d-Smith set of A_m.

Proposition 5.6. *Let* $\mathcal{P}$ *be the set of all natural numbers being prime powers.*

(1) *The set* $\{[U_{g,+}] - [U_{g,-}] \mid g \in \mathcal{F}_1\}$ *is a basis of* $\mathrm{RO}_0(A_m)$.

(2) *The set* $\{[U_{g,+}]-[U_{g,-}] \mid g \in \mathcal{F}_1$ *and* $\mathrm{ord}(g) \notin \mathcal{P}\}$ *is a basis of* $\mathrm{RO}_0(A_m)_{\mathcal{P}(A_m)}$.

Proof. (1) Recall $\mathrm{RO}(A_m) = \mathrm{RO}_0(A_m) \oplus \overline{\mathrm{RO}}_{\mathbb{Q}}(A_m)$. By Proposition 5.4, the set

$$\{[U_{g,+}] - [U_{g,-}],\ [U_{g,+}] \mid g \in \mathcal{F}_1\} \cup \{[W_{g,+\mathbb{R}}] \mid g \in \mathcal{F}_2\} \\ \cup \{[\mathrm{res}_{A_m}^{S_m}\mathbb{R}[S_m]c_t] \mid t \in \Lambda^*\}$$

is a basis of $\mathrm{RO}(A_m)$. The set $\mathcal{U} = \{[U_{g,+}] - [U_{g,-}] \mid g \in \mathcal{F}_1\}$ is contained in $\mathrm{RO}_0(A_m)$, because $U_{g,+}$ and $U_{g,-}$ are Galois conjugate to each other, cf. [21, Section 12.4]. Since

$$\mathrm{rank}\,\mathrm{RO}_0(A_m) = \mathrm{rank}\,\mathrm{RO}(A_m) - \mathrm{rank}\,\mathrm{RO}_{\mathbb{Q}}(A_m) = |\mathcal{F}_1|,$$

$\mathcal{U}$ is a basis of $\mathrm{RO}_0(A_m)$.

(2) Let $\mathcal{U}_{\mathcal{P}}$ be the set consisting of $[U_{g,+}] - [U_{g,-}]$ for $g \in \mathcal{F}_1$ such that g is not of prime power order. Let X_m be the set of real A_m-conjugacy classes of elements in A_m. We have the isomorphism

$$\Psi : \mathrm{RO}(A_m) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \mathrm{Map}(X_m, \mathbb{R})$$

of $\mathbb{R}$-modules with the formula

$$\Psi([U])((h)^{\pm}_{A_m}) = \chi_U(h)$$

for real A_m-modules U and $h \in A_m$. Let $g \in \mathcal{F}_1$ and $a \in S_m \smallsetminus A_m$. By Lemma 5.3, we get

$$\Psi([U_{g,+}] - [U_{g,-}])(h) = \begin{cases} 0 & \text{if } (h)_{S_m} \neq (g)_{S_m} \\ \sqrt{q(g)} & \text{if } (h)_{S_m} = (g)_{S_m}, \end{cases}$$

and

$$\Psi([U_{g,+}] - [U_{g,-}])(aha^{-1}) = -\Psi([U_{g,+}] - [U_{g,-}])(h)$$

for $h \in A_m$. Let

$$x = \sum_{g \in \mathcal{F}_1} a_g([U_{g,+}] - [U_{g,-}])$$

be an element in $\mathrm{RO}_0(A_m)_{\mathcal{P}(A_m)}$, where $a_g \in \mathbb{Z}$. For $g \in \mathcal{F}_1$ of prime power order, we have $a_g = 0$, because $\Psi(x)(g) = a_g\sqrt{q(g)} = 0$. Therefore, $\mathrm{RO}_0(A_m)_{\mathcal{P}(A_m)}$ is contained in the submodule $\langle \mathcal{U}_{\mathcal{P}} \rangle_{\mathbb{Z}}$ of $\mathrm{RO}(A_m)$ generated by $\mathcal{U}_{\mathcal{P}}$. Since $\mathcal{U}_{\mathcal{P}} \subset \mathrm{RO}_0(A_m)_{\mathcal{P}(A_m)}$, we obtain $\mathrm{RO}_0(A_m)_{\mathcal{P}(A_m)} = \langle \mathcal{U}_{\mathcal{P}} \rangle_{\mathbb{Z}}$. □

Recall that $g \in A_m$ is not of prime power order if and only if $\tau_1\tau_2\cdots\tau_r$ is not any prime power, where $\tau(g) = (\tau_1,\ \tau_2,\ \cdots,\ \tau_r)$.

Corollary 5.7. *Let $\mathcal{P}$ be the set of all natural numbers being prime powers.*

(1) *The $\mathbb{Z}$-rank of $\mathrm{RO}_0(A_m)$ is equal to the number of $(g)_{S_m}$, $g \in A_m$, such that the corresponding partition $\tau(g)$ of m consists of distinct odd integers, and $m - r \equiv 0 \mod 4$, where r is the length of $\tau(g)$.*

(2) *The $\mathbb{Z}$-rank of $\mathrm{RO}_0(A_m)_{\mathcal{P}(A_m)}$ is equal to the number of conjugacy classes $(g)_{S_m}$, $g \in A_m$, such that the corresponding partition $\tau(g)$ of m consists of distinct odd integers τ_i, and $m - r \equiv 0 \mod 4$, and $\tau_1\tau_2\ldots\tau_r \notin \mathcal{P}$, where $\tau(g) = (\tau_1,\ \tau_2,\ \ldots,\ \tau_r)$.*

By Lemma 5.2, Proposition 5.6 (1) and Corollary 5.7 (1), we obtain $|\mathcal{F}_1| = \pi(m)$ and the $\mathbb{Z}$-rank of $\mathrm{RO}_0(A_m)$ is equal to $\pi(m)$.

We can readily obtain Theorem 1.4 from Corollary 4.1, Proposition 5.5, and Corollary 5.7 (2). □

6. Proof of Corollary 1.5

In the case where $m \le 9$ or $m \in \{12, 13, 17\}$, it is straightforward to show that $\rho(m) = 0$, which implies $\mathfrak{d}\mathfrak{S}(A_m) = 0$. Let $I = \{m \in \mathbb{N} \mid m \ge 10\} \smallsetminus \{12, 13, 17\}$. In the following, we show $\rho(m) > 0$, which implies $\mathfrak{d}\mathfrak{S}(A_m) \ne 0$, for $m \in I$.

Case 1: $m = 4k+2 \ge 10$**.** First, let $m = 4k+2 \ge 10$, $k \in \mathbb{N}$ (hence $k \ge 2$), such that $m - 3$ is not any power of 3. Then the partition $t = (m-3, 3)$ of m satisfies the conditions (P1)–(P4) described before Theorem 1.4.

Second, let $m = 4k + 2 \ge 18$, $k \in \mathbb{N}$ (hence $k \ge 4$), such that $m - 7$ is not any power of 7. Then the partition $t = (m - 7, 7)$ of m satisfies the conditions (P1)–(P4).

Third, let $m = 4k + 2 \ge 38$, $k \in \mathbb{N}$ (hence $k \ge 9$). Then the partition $t = (m - 25, 9, 7, 5, 3, 1)$ of m satisfies the conditions (P1)–(P4).

Therefore we conclude $\rho(m) > 0$ for $m = 4k + 2 \ge 10$, $k \in \mathbb{N}$.

Case 2: $m = 4k+3 \ge 11$**.** Let $m = 4k+3 \ge 11$, $k \in \mathbb{N}$ (hence $k \ge 2$), such that $m-4$ is not any power of 3. Then the partition $t = (m-4, 3, 1)$ of m satisfies the conditions (P1)–(P4).

Let $m = 4k + 3 \ge 15$, $k \in \mathbb{N}$. Then the partition $t = (m - 8, 5, 3)$ of m satisfies the conditions (P1)–(P4).

Therefore we conclude $\rho(m) > 0$ for $m = 4k + 3 \ge 11$, $k \in \mathbb{N}$.

Case 3: $m = 4k \ge 16$**.** Let $m = 4k \ge 16$, $k \in \mathbb{N}$ (hence $k \ge 4$). Then the partition $t = (m-9, 5, 3, 1)$ of m satisfies the conditions (P1)–(P4), hence $\rho(m) > 0$.

Case 4: $m = 4k+1 \ge 21$**.** Let $m = 4k+1 \ge 21$, $k \in \mathbb{N}$ (hence $k \ge 5$), such that m is not any prime power. Then the partition $t = (m)$ of m satisfies the conditions (P1)–(P4).

Next let $m = 4k + 1 \ge 25$, $k \in \mathbb{N}$ (hence $k \ge 6$). Then the partition $t = (m - 16, 7, 5, 3, 1)$ satisfies the conditions (P1)–(P4).

Therefore we conclude $\rho(m) > 0$ for $m = 4k + 1 \ge 21$, $k \in \mathbb{N}$.

By the investigation in Cases 1–4, we have proved $\rho(m) > 0$ for all $m \in I$.

7. Proof of Theorem 1.6

Let m and n be natural numbers, and let $G = A_m \times C_2^n$. From here to Proposition 7.2, let m be a natural number satisfying $m \ge 2$. For natural number k, let I_k be the set $\{1, 2, \dots, k\}$. Let $V_1, V_2, \dots, V_{2^n}$ denote the all non-isomorphic irreducible real C_2^n-modules. Then, the set

$$\begin{aligned}\{[U_{g,+}\otimes V_i],\ [U_{g,-}\otimes V_i] \mid g\in\mathcal{F}_1, i\in I_{2^n}\}\\ \cup\{[W_{g,+_{\mathbb{R}}}\otimes V_i]\mid g\in\mathcal{F}_2,\ i\in I_{2^n}\}\\ \cup\{[(\mathrm{res}^{S_m}_{A_m}\mathbb{R}[S_m]c_t)\otimes V_i]\mid t\in\Lambda^*,\ i\in I_{2^n}\}\end{aligned}$$

is a basis of $\mathrm{RO}(G)$.

Proposition 7.1. *The set*

$$\{[U_{g,+}\otimes V_i]-[U_{g,-}\otimes V_i]\mid g\in\mathcal{F}_1,\ i\in I_{2^n}\}$$

is a basis of $\mathrm{RO}_0(G)$.

Proof. Recall $\mathrm{RO}(G)=\mathrm{RO}_0(G)\oplus\overline{\mathrm{RO}}_{\mathbb{Q}}(G)$. The set

$$\begin{aligned}\{[U_{g,+}\otimes V_i]-[U_{g,-}\otimes V_i],\ [U_{g,+}\otimes V_i]\mid g\in\mathcal{F}_1, i\in I_{2^n}\}\\ \cup\{[W_{g,+_{\mathbb{R}}}\otimes V_i]\mid g\in\mathcal{F}_2,\ i\in I_{2^n}\}\\ \cup\{[(\mathrm{res}^{S_m}_{A_m}\mathbb{R}[S_m]c_t)\otimes V_i]\mid t\in\Lambda^*,\ i\in I_{2^n}\}\end{aligned}$$

is a basis of $\mathrm{RO}(G)$. The set $\mathcal{V}=\{[U_{g,+}\otimes V_i]-[U_{g,-}\otimes V_i]\mid g\in\mathcal{F}_1, i\in I_{2^n}\}$ is contained in $\mathrm{RO}_0(G)$, because $U_{g,+}\otimes V_i$ and $U_{g,-}\otimes V_i$ are Galois conjugate to each other. If W is an irreducible A_m-module, then $\psi(W\otimes V_i)=(\psi W)\otimes V_i$ for any $\psi\in\mathrm{Gal}(G)$ and $i\in I_{2^n}$. Thus an arbitrary element of $\mathrm{RO}_0(G)$ is represented as a linear combination over $\mathbb{Z}$ of $[W\otimes V_i]-[(\psi W)\otimes V_i]$ such that W is an irreducible real A_m-module, $i\in I_{2^n}$, and $(\psi W)\otimes V_i$ is a real G-module. Furthermore, if $W\not\cong U_{g,\pm}$ for all $g\in\mathcal{F}_1$, then $(1-\psi)[W\otimes V_i]=0$ for any $\psi\in\mathrm{Gal}(G)$. Therefore, $\mathrm{RO}_0(G)\subset\langle\mathcal{V}\rangle_{\mathbb{Z}}$, i.e. $\mathrm{RO}_0(G)=\langle\mathcal{V}\rangle_{\mathbb{Z}}$. □

We remark that the $\mathbb{Z}$-rank of $\mathrm{RO}_0(G)$ is equal to $2^n\pi(m)$.

Let $\mathcal{F}_{11}$ (resp. $\mathcal{F}_{12}$) be the set of all elements $g\in\mathcal{F}_1$ such that the order of g is prime power (resp. not prime power). For $g\in\mathcal{F}_1$ and $i\in I_{2^n}$, let $u_{g,i}=[U_{g,+}\otimes V_i]-[U_{g,-}\otimes V_i]$.

Proposition 7.2. *The set*

$$\{u_{g,i}-u_{g,2^n}\mid g\in\mathcal{F}_{11},\ i\in I_{2^n-1}\}\cup\{u_{g,i}\mid g\in\mathcal{F}_{12},\ i\in I_{2^n}\}$$

is a basis of $\mathrm{RO}_0(G)_{\mathcal{P}(G)}$.

Proof. Let $\mathcal{W}=\{u_{g,i}-u_{g,2^n}\mid g\in\mathcal{F}_{11},\ i\in I_{2^n-1}\}\cup\{u_{g,i}\mid g\in\mathcal{F}_{12},\ i\in I_{2^n}\}$, and let $X_{m,n}$ be the set of real G-conjugacy classes of elements in G. We have the isomorphism

$$\Psi:\mathrm{RO}(G)\otimes_{\mathbb{Z}}\mathbb{R}\longrightarrow\mathrm{Map}(X_{m,n},\mathbb{R})$$

of $\mathbb{R}$-modules with the formula

$$\Psi([U])((h)^{\pm}_G)=\chi_U(h)$$

for real G-modules U and $h \in G$. Let

$$x = \sum_{g \in \mathcal{F}_1} \sum_{k=1}^{2^n} a_{g,k} u_{g,k}$$

be an element in $\mathrm{RO}_0(G)_{\mathcal{P}(G)}$, where $a_{g,k} \in \mathbb{Z}$. For $i \in I_{2^n}$, let χ_i be the character of V_i. By Lemma 5.3, we get

$$\Psi(x)(h,t) = \begin{cases} 0 & \text{if } h \notin \mathcal{F}_1 \\ \sqrt{q(h)} \sum_{k=1}^{2^n} a_{h,k} \chi_i(t) & \text{if } h \in \mathcal{F}_1, \end{cases}$$

for $(h,t) \in A_m \times C_2^n = G$. For any $g \in G$ of 2-power order, $\Psi(x)(g) = 0$. For any $h \in A_m$ of odd prime power order and the unit element $e \in C_2^n$,

$$\Psi(x)(h,e) = \sqrt{q(h)} \sum_{k=1}^{2^n} a_{h,k} = 0,$$

i.e. $a_{h,2^n} = -a_{h,1} - a_{h,2} - \cdots - a_{h,2^n-1}$. Therefore,

$$x = \sum_{g \in \mathcal{F}_{11}} \sum_{k=1}^{2^n-1} a_{g,k}(u_{g,k} - u_{g,2^n}) + \sum_{g \in \mathcal{F}_{12}} \sum_{k=1}^{2^n} a_{g,k} u_{g,k} \in \langle \mathcal{W} \rangle_{\mathbb{Z}},$$

i.e. $\mathrm{RO}_0(G)_{\mathcal{P}(G)} \subset \langle \mathcal{W} \rangle_{\mathbb{Z}}$. Since $\mathcal{W} \subset \mathrm{RO}_0(G)_{\mathcal{P}(G)}$, we obtain $\mathrm{RO}_0(G)_{\mathcal{P}(G)} = \langle \mathcal{W} \rangle_{\mathbb{Z}}$. □

Proof of Theorem 1.6. If $m = 1$, then G is isomorphic to C_2^n, and $\mathrm{RO}_0(G)$ is trivial. If $m \geq 2$, by Proposition 7.2, the $\mathbb{Z}$-rank of $\mathrm{RO}_0(G)_{\mathcal{P}(G)}$ is equal to $\kappa(m,n)$. If $2 \leq m \leq 4$, then $\mathrm{RO}_0(G)$ is trivial, because $\pi(m) = 0$. Therefore, if $1 \leq m \leq 4$, then $\mathfrak{d}\mathfrak{S}(G)$ and $\mathrm{RO}_0(G)_{\mathcal{P}(G)}$ are both trivial. If $m \geq 5$, $\mathfrak{d}\mathfrak{S}(G) = \mathrm{RO}_0(G)_{\mathcal{P}(G)}$ immediately follows from Corollary 4.1. □

8. Proof of Corollary 1.7

Let m and n be natural numbers, and let $G = A_m \times C_2^n$. For any m and n, $\kappa(m,n) = 0$ if and only if $\pi(m) = \rho(m) = 0$. By Remark and Theorem 1.6, $\mathfrak{d}\mathfrak{S}(G) = 0$ if and only if $m \in \{1, 2, 3, 4, 7, 8, 12\}$.

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