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A METHOD FOR CALCULATING THE NATURAL FREQUENCIES OF CONTINUOUS BEAMS, FRAMES and CERTAIN TYPES OF PLATES

By
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I. INTRODUCTION

1. Object and Scope of Investigation.

The purpose of this report is to present a method for calculating the undamped natural frequencies of flexural vibration of elastic structures. The method is applicable to continuous beams on rigid or flexible supports, to rigid jointed plane frameworks, and to certain types of continuous plates. The beams may have any number of spans of arbitrary length and any condition of restraint at the far ends. In general, the frames are assumed to be fixed against lateral displacement, but consideration is also given to symmetrical, single-bay, multi-story frames free to undergo sidesway. The plates are assumed to be simply supported along two opposite edges and, in one direction, continuous over a series of rigid supports transverse to the simply supported edges. The mass of the members composing the structure is assumed to be uniformly distributed along each member. A system which has distributed mass and elasticity has an infinite number of natural frequencies. With the method presented herein one is capable of determining all natural frequencies as well as the corresponding natural modes of vibration of a system. The assumptions made in the analysis are those of the ordinary theory of flexure of beams and of medium-thick plates.

Knowledge of the natural frequencies of structures is important for the analysis and design of structures subjected to time-dependent forces.

This knowledge is particularly significant in the case of stationary periodic forces such as those resulting from rotating machinery. If the operating frequency of the machinery is sufficiently close to one of the natural frequencies of the structure supporting it, violent vibrations will ensue which, in the absence of dissipative forces, may attain extremely large amplitudes. In order to avoid, by proper design, the destructive condition of resonance, it is necessary to have a workable method for predicting the natural frequencies of structures. It has been the object of this investigation to attempt to meet this need.

The problem of calculating natural frequencies of dynamic systems has been the subject of discussion for a long period of time. The natural frequencies of single span members having different boundary conditions have been investigated rather exhaustively; yet, comparatively little has been done for multiple member systems. Except for structures consisting of only a few members, the classical method of determining natural frequencies becomes so cumbersome that it tends to be entirely useless for practical purposes. Several considerably more efficient methods have been developed, but these seem to be applicable to limited types of structures.

Among the available methods, Mudrak's method (1),* (2), (3) which utilizes the effective stiffness criterion for determining natural frequencies, is by far the most efficient. This method has been applied only to continuous beams on rigid supports and apparently is not capable of extension to continuous frames involving closed panels. The natural frequencies of continuous beams on intermediate flexible supports may be best determined by the method developed by Lee and Saibel (4), (5). This method utilizes principles

*Numbers in parentheses, unless otherwise identified, refer to the Bibliography at the end of this report.

that are not well known to engineers; furthermore, it presupposes that the natural modes of vibration of the beam without the intermediate supports are known. This assumption restricts seriously the range of applicability of this procedure. The determination of the natural frequencies of continuous frames, involving closed panels, has apparently been attempted only by use of the classical method (6), which, as already stated, is too laborious for practical applications.

The method described herein is a generalization of Holzer's method (7) for calculating the natural frequencies of torsional vibration of shafts. It utilizes well known engineering principles and, like Holzer's method, it is reduced to a routine scheme of computation which, when repeated a sufficient number of times, will give the natural frequencies of the system to any desired degree of accuracy. Holzer's method has been applied to the determination of the natural frequencies of flexural vibration of beams, first, by Myklestad (8) and later by Prohl (9), Rankin (10), and Bellin (11). In these studies, distributed masses were assumed to be lumped at a number of stations along the length of the beam while the portion of the beam between these stations was assumed to be massless. In the method to be presented, the mass is assumed to be uniformly distributed along each member of the structure. Abrupt changes in the magnitude of the distributed mass or of the flexural rigidity within a member may be treated by assuming that the member is supported by a flexible support of zero stiffness at the point of the discontinuity.

The principles underlying the method are presented in Chapter II of this report. This chapter also includes definitions of the several physical quantities which are necessary in the analysis. These quantities are the dynamic stiffnesses and the dynamic carry-over factors which are analogous to those introduced by Hardy Cross in connection with the method of moment distribution (12). Extensive tables of numerical values of these quantities are presented in Appendix A, while the derivation of the governing equations is given in Appendix B. With these tabulated values, the calculations required in the application of the method to particular problems are simplified immensely.

The application of the method to continuous beams on rigid supports is discussed in Chapter III. In addition, several alternate methods of analysis are considered and the range of applicability and the relative merits of each are discussed.

Chapter IV presents the extension of the method to frames without sidesway. For continuous beams, a single procedure is capable of determining all possible natural frequencies. For frames, however, this procedure may fail to detect those natural frequencies for which only a portion of the structure vibrates while the rest remains stationary. A technique for overcoming this difficulty has been developed and is presented also in Chapter IV. In the study of frames, the effect of permanent axial forces is neglected. The concluding section of Chapter IV is devoted to a discussion of the manner in which this effect may be taken into account. Also, it is pointed out that problems of framework instability are special cases of the more general problem of the vibration of axially compressed members.

In Chapter V, the method is extended to beams continuous over supports that are flexible instead of rigid. The resistance to deformation of the supports is represented by an equivalent set of mutually independent deflectional and rotational springs. It is shown that the method may be modified readily to include the influence of concentrated rigid masses, of concentrated sprung masses, and of an elastic subgrade of the Winkler type.

Chapter VI shows the application of the method to summetrical, one-bay, multi-story building frames which are free to undergo sidesway.

Chapter VII is concerned with the extension of the method to continuous plates having two opposite edges simply supported. The pertinent expressions for dymanic stiffness and dynamic carry-over factor for plates are presented in Appendix B. It is also shown that numerical values of these quantities may be obtained from available tables of stiffness and carry-over factor for compressed plates.

Table II in Appendix A gives influence coefficients for calculating the natural frequencies of vibration of systems composed of bars. It is pointed out that Müller-Breslau's principle of influence lines is applicable in the case of steady-state forced vibrations, so that these coefficients may be interpreted also as coefficients for dynamic fixed-end moments produced by a concentrated harmonic force.

Appendix C includes a brief account of the manner in which the information presented in this report may be used in the analysis of the steady-state forced vibration of frames.

For convenience of reference, a detailed outline of the steps involved in the application of the method to each class of problems is included in each chapter. In addition, several numerical examples are given to illustrate the application of the method and to indicate convenient schemes for arranging the computations. Throughout this report, special effort has been made to present each chapter as independently of the others as possible and to discuss the numerical examples adequately.

2. Sign Convention and Notation.

The following sign convention is used throughout this report with the exception of Appendix B. Clockwise rotations are positive. Internal bending moments acting at the ends of a member (not a joint) and external moments, except for the restraining moments provided by rotational springs, are positive clockwise. The restraining moment of a rotational spring is positive when it acts in a counter-clockwise direction. Deflections are positive downward. Shears acting at the ends of a member (not a joint) and external forces, except for the forces produced by deflectional springs, are positive downward. The restraining force of a deflectional spring is positive upward.

The letter symbols used are defined where they first appear in the text or by illustration, and they are assembled in this section for convenience of reference.

General:

ω = circular frequency of vibration

ω_N = natural circular frequency of vibration

f = frequency of vibration, in cycles per second

t = time

E = modulus of elasticity

For structures composed of bars:

x = horizontal coordinate

I = moment of inertia of the cross section of a bar about its centroidal axis

L = span length of a bar

m = mass per unit of length of a bar

$\bar{m}$ = magnitude of a concentrated rigid mass

$(\bar{m})_{eq.}$ = magnitude of an equivalent concentrated rigid mass

$\lambda = \sqrt[4]{\frac{m\omega^2}{EI}}\,L$ = a dimensionless parameter for a bar

λ_N = λ value corresponding to a natural frequency

w_x = deflection amplitude at a point of a bar defined by the coordinate x

θ_j = rotation amplitude at support or joint j

δ_j = deflection amplitude at support j

M_j = amplitude of internal bending moment at support j of a continuous beam

M_{ji} = amplitude of internal bending moment at end j of a bar ji in a continuous frame

$\overline{M}_j$ = external moment at support or joint j of a continuous beam or frame

V_j = amplitude of shear at support j of a continuous beam

$\overline{F}_j$ = external force at support j

u, v = variable parameters

$\theta', \delta', \overline{M}', \overline{F}'$ = values of θ, δ, $\overline{M}$, and $\overline{F}$ due to $u = 1.00$ and $v = 0$

$\theta'', \delta'', \overline{M}'', \overline{F}''$ = values of θ, δ, $\overline{M}$, and $\overline{F}$ due to $u = 0$ and $v = 1.00$

K, Q, T = dynamic stiffnesses for a bar, defined in Section 5

k, q, t = dynamic carry-over factors for a bar, defined in Section 5

C_K, C_Q, C_T = dimensionless coefficients in expressions for K, Q, and T

K'', K^S, K^A = modified stiffnesses for a bar, defined by Eqs. (9), (10) and (11)

K' = effective flexural stiffness for a bar, defined in Section 13

$\overline{K}_j, \overline{Q}_j, \overline{T}_j$ = stiffnesses K, Q, T of all bars meeting at support or joint j

$\overline{K}'_j$ = effective flexural stiffness of all bars meeting at support or joint j

D = stiffness of a deflectional elastic spring

$(D)_{eq.}$ = stiffness of an equivalent deflectional spring

R = stiffness of a rotational restraint

C = dimensionless coefficient in Eq. (14) for the deflection of a bar

P = axial force in a bar

P_o = buckling load for a bar hinged at both ends

d = modulus of a continuous elastic subgrade

ω_s = circular vibration frequency for a bar on a continuous elastic subgrade

δ^* = deflection amplitude of a sprung mass

$\alpha, \beta, \gamma, \eta$ = dimensionless coefficients defined, respectively, by Eqs. (24), (73), (74), and (50)

For continuous plates:

x, y = horizontal rectangular coordinates. The y-axis is taken parallel to the pair of simply supported edges

a, b = span lengths in the x and y directions, respectively, for a panel of a plate

ρ = density of plate material in a particular panel

h = thickness of plate in a particular panel

I = $h^3/12$ = moment of inertia, per unit width, for a particular panel of the plate

ν = Poisson's ratio

N = $EI/1-\nu^2$ = flexural rigidity of a particular panel of a plate

λ^* = $\frac{b^2}{\pi^2}\sqrt{\frac{\rho h \omega^2}{N}}$ = a dimensionless parameter for a panel of a plate

n = integer representing number of half sine waves in the distribution of deflections, slopes, moments, etc., across a plate width $\underline{a}$

θ_j = maximum rotation amplitude along support $\underline{j}$

M_j = maximum amplitude of bending moment at support $\underline{j}$

K, k = flexural stiffness and flexural carry-over factor for a panel of a plate, defined in Section 35

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The writers wish to thank Dr. L. E. Goodman, Research Associate Professor in Civil Engineering, for calling their attention to Lord Rayleigh's paper (14), and Dr. J. G. Sutherland, formerly Research Assistant in Civil Engineering, for first calling their attention to Mr. Gaskell's paper (13).

The numerical values reported in Appendix A of this report were calculated on the Electronic Digital Computer of the University of Illinois. The governing expressions for these constants were coded for machine solution by Mr. A. J. Carlson, Jr., Research Associate in Civil Engineering. Acknowledgment of Mr. Carlson's part in this work is made gratefully. Appreciation is finally due to Mr. D. Trimakas for the tracing of the diagrams.

II. METHOD OF ANALYSIS

4. Basis of Method of Analysis.

The method used in this report is based on the fact that the exciting couple or the exciting force which is necessary to maintain a dynamical system in a steady-state forced vibration with finite amplitudes becomes equal to zero at any one of the natural frequencies of the system.

Figure 1 shows $\underline{z}$ members of a plane framework rigidly connected at their common joint $\underline{o}$. The far ends of the members may be considered elastically restrained against both rotation and translation. These restraints are furnished by the portion of the structure not shown on the figure.

Consider that the frame undergoes a steady-state forced vibration under the action of a harmonically varying exciting couple applied at joint $\underline{z}$. Joint $\underline{z}$ is assumed to be different from joint $\underline{o}$. The magnitude of the exciting moment is assumed to be such that the amplitude of either the slope or of the internal bending moment at a joint of the structure different from joint $\underline{z}$, say at joint 1, has a prescribed finite value. The vibration of the structure is harmonic and its frequency is equal to the frequency of the exciting couple; since the effect of damping is neglected, the amplitudes of vibration are constant and the response is either in phase with, or 180 degrees out of phase with, the exciting couple.

For a given system, the magnitude of the exciting moment necessary to maintain the prescribed amplitude of vibration at joint 1 depends on the frequency of vibration. For the limiting case of no vibration, the magnitude of the moment is obviously finite; its actual value may, if desired, be calculated by any of the available methods of indeterminate stress analysis. As the frequency of vibration increases above zero, the structure is acted upon by inertia forces of increasingly greater magnitudes. These forces, which are distributed along the length of the individual members, bring about distortions in addition to those produced by the external moment acting statically. Therefore, the amplitude of the dynamic moment necessary to produce the prescribed distortion at joint 1 may be quite different from the magnitude of the corresponding static moment. As the vibration frequency approaches any one of the natural frequencies of the structure considered, the inertia effects predominate, and at a natural frequency the vibration is maintained without any exciting moment acting permanently on the structure.

Briefly, the method presented herein consists of (a) choosing a frequency of vibration, (b) determining the magnitude of the exciting moment which, when applied at joint z, will produce a vibration configuration with a fixed amplitude of slope or bending moment at joint 1, (c) repeating these steps for a number of assumed frequencies, and (d) plotting the magnitude of the exciting moment as a function of the frequency of vibration. The frequencies for which the exciting moment vanishes represent natural frequencies of the system.

In the application of this procedure, the following two conditions must be satisfied: (1) The joint to which a finite amplitude of slope is assigned should be one that is known to rotate for all the natural frequencies that need to be determined. If instead of fixing the amplitude of slope the amplitude of moment is fixed, it must be known that this moment amplitude remains finite. (2) The exciting couple must be applied at a joint which is known to rotate for all the natural frequencies that need to be determined. A couple applied at a joint which does not rotate acts through zero displacement and imparts no energy to the structure; therefore, it does not influence the natural frequencies or the vibration modes of the system. In such a case, the exciting moment may not vanish at a natural frequency.

If either of these conditions is not satisfied, the procedure will fail to reveal some of the natural frequencies of the system. It will be shown later that, for certain structures, it is impossible to satisfy these requirements. In such cases, in order to obtain the natural frequencies which the basic procedure fails to reveal, it becomes necessary to use a supplementary technique as described in Chapter IV.

For an assumed frequency of vibration, the magnitude of the exciting moment may be determined by a number of different procedures. The conditions to be satisfied are simply those of equilibrium and continuity for each joint of the structure. To satisfy the condition of equilibrium, the sum of the moments and of the forces at the ends of the members meeting at a joint must be respectively equal to zero. To satisfy the condition of continuity, the slopes of the members meeting at a joint must be equal and also the deflection of the members meeting at the joint must have the same magnitude. These conditions may be expressed in equation form in a number of different ways and the equations may be solved by a number of procedures. In the method adopted, these conditions are expressed in the form of a generalized slope deflection equation, and the distortions of the structure at the supports are computed by the repeated application of this equation, working progressively from one end of the structure to the other.

5. Elastic Constants for a Vibrating Bar.

The various quantities necessary to express the resistance to deformation of a bar undergoing steady-state forced vibration are defined in this section.

Consider a bar fg with the far end g fixed. Let the near end be subjected to a harmonically varying bending moment of a circular frequency ω producing at that end a steady-state forced rotation

$$\theta(t) = \theta \cos \omega t \ .$$

The amplitude of the impressed moment may be related to the amplitude of the resulting rotation at end f by the equation

$$M_f = K\theta \ . \tag{1}$$

The quantity K represents the moment required to produce a rotation of unit amplitude and is defined herein as the "dynamic flexural stiffness" of the end of the bar being rotated.

The moment induced at the fixed end g may be written as

$$M_g = kK\theta \ . \tag{2}$$

The quantity k is the ratio of the dynamic moment at the far fixed end to the moment at the near end and is defined as the "dynamic flexural carry-over factor".

In the analysis of continuous beams on rigid supports and of continuous frames for which the joints do not translate, the flexural stiffness and the product of the flexural stiffness and the flexural carry-over factor are the only two quantities needed.

The foregoing definitions are generalizations of those originally introduced by Hardy Cross (12) for the analysis of frames subjected to static loads, and they were first used by Gaskell (13), who extended and applied the method of moment distribution to the problem of determining the steady-state forced vibration of continuous beams and frames subjected to pulsating loads.

For static conditions, the end reactions of the bar resulting from the rotation of one end, may be determined from the end moments by statics. For dynamical conditions, this is not possible, since the bar is acted upon by inertia forces the distribution of which along the length of the bar is generally unknown. The reactions must, therefore, be defined by additional constants.

The amplitude of the vertical reaction at the end being rotated may be written as

$$V_f = Q\theta \, , \tag{3}$$

and the reaction at the fixed end may be written as

$$V_g = -qQ\theta = -qV_f \, . \tag{4}$$

The quantity Q represents the reaction at end f produced by a rotation of unit amplitude at that end and is defined as the "dynamic flexural shear stiffness". The quantity q represents the ratio of the reaction induced at the far fixed end to that produced at the near end and is called the "dynamic flexure-shear carry-over factor".

Consider now that end f is subjected to a harmonically varying force producing at that end a steady-state forced deflection without rotation, such that the magnitude of the deflection is

$$\delta(t) = \delta \cos \omega t \, .$$

The amplitudes of the force and of the deflection at the left end may be related by the expression

$$V_f = T\delta \tag{5}$$

The quantity T denotes the force necessary to cause a deflection of unit amplitude and is defined as the "dynamic shear stiffness" for the end being deflected. The reaction at the far fixed end may be written as

$$V_g = -tT\delta = -t\,V_f \, . \tag{6}$$

The quantity t shows the ratio of the reaction at the far end to that at the near end and is called the "dynamic shear carry-over factor". The amplitude of the moment induced at the end being deflected is

$$M_f = Q\delta \,, \tag{7}$$

and the moment at the far fixed end is

$$M_g = qQ\delta = qM_f \,. \tag{8}$$

The quantities Q and q are the same as those used in Eqs. (3) and (4). That these should be the same follows from a reciprocal theorem given by Lord Rayleigh (14), which is the dynamic equivalent of Maxwell's Law of reciprocal relations.

Throughout this presentation, the members composing the structure are considered to be uniform. The stiffness and the carry-over factors for both ends of such members are equal.

The notation used for the various stiffnesses and carry-over factors is the same as that used by Newmark (15) in his static analysis of slabs continuous over flexible supports. The notation is summarized in Fig. 2. The derivation of the algebraic expressions for the various stiffnesses and carry-over factors is given in Appendix B.

For certain conditions of symmetry, antisymmetry, and for those cases for which the degree of restraint at the far end of a member is known, it is convenient to use effective stiffnesses. The pertinent expressions for these stiffnesses are the same as those for the static case. Expressions are given here only for effective flexural stiffnesses. The particular cases considered and the symbols used to identify them are: K', when the far end of the bar is prevented from deflecting and is elastically restrained against rotation; K'', when the far end is simply supported; K^S, when the bar is on rigid supports and its deformation is symmetrical, and K^A, when the bar is on rigid supports and its deformation is antisymmetrical. It can be proved readily that

$$K'' = K(1-k^2) \ , \tag{9}$$

$$K^S = K(1-k) \ , \tag{10}$$

$$K^A = K(1+k) \ . \tag{11}$$

The stiffness K^A of a given bar is also equal to the stiffness K'' of a similar bar one half as long. The expression for K' is given in article 12, where it is used first.

It is possible to derive also expressions for effective shear stiffnesses. However, these are not, in general, as simple and convenient to use as the effective flexural stiffnesses.

6. Numerical Values of Stiffness and Carry-Over Factors.

All carry-over factors are dimensionless and depend on a single dimensionless parameter

$$\lambda = \sqrt{\frac{m\omega^2}{EI}}\, L \ , \tag{12}$$

in which m = the mass per unit of length of the bar,

ω = the circular frequency of vibration, as previously noted,

E = the modulus of elasticity of the material in the bar,

I = the moment of inertia of the cross section of the bar about its centroidal axis, and

L = the span length of the bar.

The various stiffnesses are determined from the expressions

$$K = C_K \frac{EI}{L} \ , \qquad Q = C_Q \frac{EI}{L^2} \ , \qquad T = C_T \frac{EI}{L^3} \ , \tag{13}$$

where the C's are dimensionless coefficients depending on the parameter λ.

A graphical representation of the variation with λ of the various carry-over factors, stiffness coefficients, and of their products is given in

Figs. 3 through 12. It is noted that the curves in these figures range between minus infinity and plus infinity. The λ values corresponding to the zero ordinates and to the discontinuities of the curves, represent natural frequencies of bars having standard boundary conditions. For example, consider the curves in Figs. 3 and 5 for the flexural stiffness and the flexural carry-over factor. Values of λ equal to 3.927, 7.069 and 10.210 correspond, respectively, to the first, the second, and the third natural frequencies of a hinged-fixed bar. At these frequencies, no exciting moment is required to maintain the vibration; consequently, the value of dynamic stiffness is equal to zero. Furthermore, since the moment at the fixed end of the bar has a finite magnitude, the carry-over factor for the member becomes infinite at these frequencies. Values of λ equal to 4.730 and 7.853 correspond, respectively, to the first and the second natural frequencies of a bar fixed at both ends. At these frequencies, the end moments have a finite value while the rotations of the ends are zero; accordingly, the stiffness of the member has an infinite value. For the case of no vibration, $\lambda = 0$, the various quantities in Fig. 3 through 12 assume the well known static values of

$$k = 0.5 \qquad K = 4\,\frac{EI}{L} \qquad kK = 2\,\frac{EI}{L} \qquad K'' = 3\,\frac{EI}{L}$$

$$q = 1.00 \qquad Q = 6\,\frac{EI}{L^2} \qquad qQ = 6\,\frac{EI}{L^2}$$

$$t = 1.00 \qquad T = 12\,\frac{EI}{L^3} \qquad tT = 12\,\frac{EI}{L^3}$$

Numerical values of the carry-over factors, of the coefficients of the various stiffnesses, and of their products are given in Table I of Appendix A. All values are reported to seven significant figures for the range of λ from zero to 10.20 at increments of 0.01. In some cases, the accuracy of the seventh significant figure reported is uncertain. These values were computed by use of the Electronic Digital Computer of the University of Illinois.

7. Numerical Values of Deflections due to End Rotations.

Consider the beam shown in Fig. 2a with the left end subjected to a steady-state forced rotation θ. The deflection amplitude of the beam at a distance $\bar{x}$ from the left end may be written as

$$Y_{\bar{x}} = C\theta L \, , \qquad (14)$$

where C is a dimensionless coefficient dependent on the value of $\bar{x}$ and the parameter λ. If θ represents the rotation at the right end instead of at the left end of the beam, the deflection amplitude at a distance $\bar{x}$ from the right end will be equal to the right hand side of Eq. (14) multiplied by minus one. The minus sign is a consequence of the sign convention adopted.

Numerical values of C are given in Table II of Appendix A for successive twelveth points of the beam for values of λ ranging between zero and 10.20. These values were computed from Eq. (B-32) in Appendix B, by use of the Electronic Digital Computer of the University of Illinois. The values are reported to five significant figures, but to no more than six decimal places.

The values in Table II may also be interpreted as coefficients of dynamic fixed-end moment for a beam subjected to a pulsating concentrated force. This follows from Müller-Breslau's principle, which it can be shown to hold true for dynamical systems undergoing steady-state forced vibration. This principle, as applied to the dynamical case, is presented in Appendix B.

III. APPLICATION OF METHOD TO CONTINUOUS BEAMS ON RIGID SUPPORTS

8. General.

The beams considered are assumed to be straight and may have any number of spans of arbitrary length. At their extreme ends they may be hinged, fixed, or only partially fixed by means of rotational restraints which are assumed to be proportional to the end rotations. The cross section and the mass per unit of length of the beam may vary from one span to the other, but in any one span these quantities are considered constant. It is assumed that vibration is restricted to one of the principal planes of flexure of the beam, and that the cross sectional dimensions of each span are small in comparison to its length so that the effects of shearing deformation and rotatory inertia are negligible.

The supports of the beam are numbered successively from left to right starting with 1 at the extreme left end and terminating with z at the extreme right end.

The portion of the beam between two consecutive supports j and $j+1$ is referred to as the j-th span. The quantities L_j, E_j, I_j, λ_j, K_j, and k_j refer to the j-th span.

Θ_j denotes the amplitude of rotation of the deflected beam over the j-th support and M_j denotes the amplitude of bending moment across a section at the same support. The subscripts L and R designate, respectively, sections just to the left and just to the right of the support. $\bar{M}_j$ denotes the amplitude of the external couple at support j.

9. Development of the Basic Equations.

Figure 13 shows the extreme deflected position of spans j-1 and j of a continuous beam undergoing a steady-state forced vibration. The vibration is assumed to be maintained by an exciting couple applied at the extreme right end of the beam. There is no other exciting force or moment acting on the system.

In Fig. 13, the rotations and bending moments at the ends of each span are indicated in their positive directions. The slope and the bending moment at a time t for support j are

$$\theta_j(t) = \theta_j \cos \omega t , \quad \text{and} \quad M_j(t) = M_j \cos \omega t \quad . \tag{15}$$

In the equations to be used the $\cos \omega t$ appears as a common factor; for convenience, this will be omitted, and in the remainder of this discussion the terms "amplitude of slope" and "slope" and the terms "amplitude of moment" and "moment" will be used interchangeably.

To insure continuity and equilibrium of the beam over the interior support j, it is required that

$$(\theta_j)_L = (\theta_j)_R = \theta_j \ , \tag{16}$$

$$\bar{M}_j = (M_j)_L + (M_j)_R = 0 \ . \tag{17}$$

The moments $(M_j)_L$ and $(M_j)_R$ can now be expressed as functions of the end rotations of the two spans as follows: Consider span j. First, assume that the right end of the span is held fixed while the left end is rotated through an angle θ_j; then, the moment at the end being rotated is equal to the product of the rotation θ_j and the flexural stiffness of the member K_j. Next, imagine that the left end of the span is kept fixed while the right end is rotated through θ_{j+1}; the moment induced at the fixed left end is equal to the product of the rotation θ_{j+1} and the product of the flexural stiffness and flexural carry-over factor of the member $(kK)_j$. Since the principle of superposition holds true, the moment $(M_j)_R$ corresponding to the rotations θ_j and θ_{j+1} is the sum of these partial moments.

$$(M_j)_R = K_j\theta_j + (kK)_j\,\theta_{j+1} \quad . \tag{18a}$$

Considering span j-1, one obtains in a similar manner:

$$(M_j)_L = K_{j-1}\theta_j + (kK)_{j-1}\,\theta_{j-1} \quad . \tag{18b}$$

Substituting Eqs. (18a) and (18b) in Eq. (17) and solving for θ_{j+1}, one obtains the following equation relating the slopes over three consecutive supports of a continuous beam:

$$\theta_{j+1} = -\frac{(K_{j-1} + K_j)\theta_j + (kK)_{j-1}\,\theta_{j-1}}{(kK)_j} \quad . \tag{19a}$$

This equation is a generalized slope deflection equation with the deflection term missing. It will be referred to as the "three slope equation". Eq. (19a) is applicable only to interior supports; the appropriate relations for the end supports are given in the following paragraphs.

It is assumed that the extreme ends of the beam are elastically restrained against rotation. The relationship between the end moments and end rotations are

$$M_1 = -R_1\theta_1 \; , \tag{20}$$

$$M_z = -R_z\theta_z \; , \tag{21}$$

where, R_1 and R_z are the known stiffnesses of the rotational restraints at the left and the right ends, respectively. For a hinged end, R = 0, and for a clamped end, R = infinity. The negative signs in these expressions follow from the sign convention used and indicate that for a positive restraint, the moment exerted on the beam by the restraint acts in a direction opposite to the direction of rotation of the beam.

The moments M_1 and M_z can also be expressed by the following equations, obtained respectively from Eqs. (18a) and (18b).

$$M_1 = K_1\theta_1 + (kK)_1\theta_2 , \tag{18a'}$$

$$M_z = K_{z-1}\theta_z + (kK)_{z-1}\theta_z . \tag{18b'}$$

Eliminating M_1 between Eqs. (18a') and (20) and M_z between Eqs. (18b') and (21), one obtains

$$(R_1 + K_1)\theta_1 + (kK)_1\theta_2 = 0 , \tag{21a}$$

$$(K_{z-1} + R_z)\theta_z + (kK)_{z-1}\theta_{z-1} = 0 . \tag{22a}$$

At a natural frequency, both of these equations must be satisfied identically.

Equations (21a) and (22a) apply only to hinged and to partially fixed ends. For fixed ends, the equations are specialized as follows: For $\theta_1 = 0$, the relation between the moment at the fixed end and the rotation of the beam over the second support is obtained from Eq. (18a') as

$$M_1 = (kK)_1\theta_2 . \tag{21b}$$

If the right end is fixed,

$$\theta_z = 0 , \tag{22b}$$

and the criterion for a natural frequency is that Eq. (22b) be satisfied. The magnitude of the moment at the fixed end is of no interest but, should it be desired, it may be calculated from Eq. (18b'), keeping in mind that $\theta_z = 0$.

10. Outline of the Procedure.

The procedure for arriving at the natural frequencies of a continuous beam may be outlined as follows:

1. A fixed value is assigned to the amplitude of slope or bending moment at the first support of the beam. Since the natural frequencies of a system depend only on the relative values of the deflection, any arbitrary amplitude consistent with the actual boundary conditions may be chosen. For a hinged or for a partially fixed end, θ_1 is taken, for convenience, equal to unity; for a clamped end, θ_1 is equal to zero, and M_1 or M_1 times the L/EI of some reference span is taken equal to unity instead.
2. A trial frequency of vibration, ω, is chosen and the λ values for all spans are evaluated. These calculations are carried out conveniently in a tabular form, as illustrated in Example 2.
3. With the λ values available, the flexural stiffness and the product of the flexural stiffness and flexural carry-over factor for each span of the beam are found from Table I in Appendix A.
4. The rotation of the beam over the second support is determined from Eq. (21a) or (21b).
5. By successive applications of Eq. (19), the rotations θ_3 to θ_z are evaluated. A convenient tabular scheme for arranging the computations is described in Example 2.
6. If support $\underline{z}$ is fixed, the determination of the rotation θ_z completes one cycle of the procedure (see Eq. 22b). However, if this support is hinged or is only partially fixed, it is necessary to carry out the additional step of evaluating the left hand side of Eq. (22a).
7. Steps 1 through 6 are repeated for different assumed frequencies, and the values calculated for the left hand side of Eq. (22a) or (22b) are plotted as a function of the assumed frequencies, or what is usually more convenient, as a function of the corresponding λ

values for some reference span. The zero intercepts of the resulting curve which is, in general, similar in shape to that shown in Figs. 16 and 18, correspond to the natural frequencies of the system.

11. Determination of Modes of Vibration

Since the rotations of the beam over the supports are evaluated in each cycle of this procedure, the deflection configuration of the beam for any desired frequency can ordinarily be sketched from these rotations. The natural modes of free vibration may be determined from the rotations corresponding to the natural frequencies in the same manner.

If it is desired to compute these deflections accurately, it is necessary to use the numerical coefficients given in Table II. The deflection at any point within a span may be obtained by adding (a) the deflection produced by the rotation of the left end of the span, assuming that the right end is fixed and (b) the deflection produced by the rotation of the right end of the span, assuming that the left end is fixed.

12. Illustrative Examples.

Example 1. Consider a uniform beam continuous over five rigid supports spaced equidistantly. The beam is simply supported at the left end and fixed at the right end, as shown in Fig. 14. It is desired to calculate its first eight natural frequencies and the corresponding natural modes of vibration. It is assumed that the beam is cut at the extreme right end and then an exciting moment is applied there. At a natural frequency, the magnitude of this moment must be such that the condition $\theta_5 = 0$ is satisfied. The amplitude of slope at the extreme left end is taken equal to unity. Since all spans are identical, rather than repeating the procedure outlined in Section 10 for each assumed frequency of vibration, it is more convenient to derive a general expression for Θ_5 and determine directly from this expression the natural frequencies of the beam.

Let K be the flexural stiffness and k the flexural carry-over factor for each span. These quantities depend, of course, on the parameter λ. From Eq. (21a) one obtains

$$\theta_2 = -\frac{K}{kK} = -\frac{1}{k} .$$

Applying Eq. (19) successively to joints 2, 3, and 4, one obtains

$$\theta_3 = -K\left[2\left(-\frac{1}{k}\right) + k\right] \div kK = \frac{2-k^2}{k^2} ,$$

$$\theta_4 = -K\left[2\,\frac{2-k^2}{k^2} + k\left(-\frac{1}{k}\right)\right] \div kK = \frac{3k^2-4}{k^3} ,$$

$$\theta_5 = -K\left[2\,\frac{3k^2-4}{k^3} + k\,\frac{2-k^2}{k^2}\right] \div kK = \frac{k^4-8k^2+8}{k^4} .$$

The expression for Θ_5 was evaluated for several values of λ and the results were used to plot the curve shown in Fig. 16. The values of k corresponding to the assumed values of λ were obtained from Table I. The λ values corresponding to the natural frequencies are

Order of λ_N	Value of λ_N	Order of λ_N	Value of λ_N
1	3.21	5	6.36
2	3.65	6	6.79
3	4.21	7	7.34
4	4.655	8	7.78

These values agree with those reported elsewhere (16). The circular natural frequencies ω_N are obtained from the expression

$$\omega_N = \frac{\lambda_N^2}{L^2}\sqrt{\frac{EI}{m}} ,$$

and the natural frequencies, in cycles per second, are computed from

$$f_N = \frac{\omega_N}{2\pi} .$$

In order to determine the natural modes of vibration, first, the k values corresponding to the natural frequencies were determined, and then the expressions of θ_2, θ_3, and θ_4 were evaluated. The results are summarized in the following:

Order of Mode	θ_1	θ_2	θ_3	θ_4	θ_5
1	1.00	-.924	.707	-.383	0
2	1.00	-.383	-.707	.924	0
3	1.00	.383	-.707	-.924	0
4	1.00	.924	.707	.383	0
5	1.00	.924	.707	.383	0
6	1.00	.383	-.707	-.924	0
7	1.00	-.383	-.707	.924	0
8	1.00	-.924	.707	-.383	0

From these rotations, the shapes of the natural modes of vibration can be sketched. For this particular problem, the vibration modes were computed by use of the numerical values given in Table II, following the procedure described in the preceding Article. For the purpose of illustration, the computations involved in the determination of the first natural mode (λ = 3.21) are presented in detail.

Deflection of span 1 at successive 1/6 - points:

for θ_1 = 1.00, θ_2 = 0:	0	.128	.179	.163	.103	.033	0
for θ_1 = 0, θ_2 = -.924:	0	.031	.095	.151	.166	.118	0
total:	0	.159	.274	.314	.269	.151	0

Deflection of span 2 at successive 1/6 - points:

for $\theta_2 = -.924$, $\theta_3 = 0$:	0	-.118	-.166	-.151	-.095	-.031	0
for $\theta_2 = 0$, $\theta_3 = .707$:	0	-.024	-.072	-.115	-.127	-.090	0
total:	0	-.142	-.238	-.266	-.222	-.121	0

Deflection of span 3 at successive 1/6 - points:

for $\theta_3 = .707$, $\theta_4 = 0$:	0	.090	.127	.115	.072	.024	0
for $\theta_3 = 0$, $\theta_4 = -.383$:	0	.013	.039	.062	.069	.049	0
total:	0	.103	.166	.177	.141	.073	0

Deflection of span 4 at successive 1/6 - points:

for $\theta_4 = -.383$, $\theta_5 = 0$:	0	-.049	-.069	-.062	-.039	-.013	0

The first six natural modes of vibration are shown in Fig. 17

Example 2. In order to illustrate several additional details of the procedure and present a convenient tabular scheme for recording the computations for the general case in which the dimensions of the beam may vary from span to span, we consider the four-span continuous beam shown in Fig. 15. Only the first five natural frequencies will be evaluated. The beam is assumed to be elastically restrained at the left end and hinged at the right end. The stiffness of the end restraint and the characteristics of the various spans are shown in Fig. 15.

For convenience in carrying out the calculations, the natural frequencied of the system are expressed in terms of the pertinent properties of some reference span, say span r. In this particular example we take r = 1. In terms of the λ value of the r-th span, the λ value for any span j is

$$\lambda_j = \sqrt[4]{\frac{m_j}{m_r} \frac{E_r I_r}{E_j I_j}} \frac{L_j}{L_r} \cdot \lambda_r \quad . \tag{23}$$

In terms of the $\frac{EI}{L}$ of the r-th span, the stiffness and the product of the stiffness and of the carry-over factor for any span j are equal to the values obtained from Table I multiplied by the dimensionless factor

$$\alpha_j = \frac{E_j I_j}{E_r I_r} \frac{L_r}{L_j} . \tag{24}$$

Equations (23) and (24) can be verified readily.

The quantities λ_j/λ_r and α_j are evaluated in Table 1A. It should be noted that the calculations in this table are independent of the frequency of vibration.

The trial-and-error produce for determining the natural frequencies of the system is carried out in Table 1B. As an example of the use of this table, a complete cycle of calculations is carried out for a trial value of $\lambda_r = \lambda_1 = 2.40$. This value, shown encircled in the r-th line of Column (2), corresponds to a circular frequency of vibration $\omega = \frac{(2.40)^2}{L_1^2}\sqrt{\frac{E_1 I_1}{m_1}}$. The arrangement of the various quantities in this table is believed to facilitate the computational work and to reduce substantially the probability for errors. The order in which the columns in this table are filled in is indicated by the following sequence of column numbers: (1), (3), (2), (4 and 8), (5), (7), and (6). Columns (1) and (3) are reproduced, respectively, from Columns (5) and (6) of Table A. The λ values for the various spans in Column (2) are obtained as the product of the assumed λ_r and each of the values in Column (1). Columns (4) and (8) give, respectively, values of the stiffness and of the product of the stiffness and the carry-over factor for each span, in terms of $\frac{E_j I_j}{L_j}$; these quantities are obtained directly from Table I in Appendix A, using the λ values computed in Column (2). Column (5) gives the total stiffness of the spans adjoining each support in terms of $\frac{E_r I_r}{L_r}$. The value for the j-th line in this column is determined by taking the sum of the products of the values in Columns (3) and (4) for lines j-1 and j. Column (7) gives the product of the stiffness and of the carry-over factor for each span in terms of $\frac{E_r I_r}{L_r}$; the entries in this column are obtained

TABLE 1. CALCULATIONS FOR EXAMPLE 2

TABLE A

Span	(1) $\frac{m_j}{m_r}$	(2) $\frac{E_j I_j}{E_r I_r}$	(3) $\sqrt[4]{\frac{(1)}{(2)}}$	(4) $\frac{L_j}{L_r}$	(5) $\frac{\lambda_j}{\lambda_r} = (3)(4)$	(6) $\alpha_j = \frac{(2)}{(4)}$
1=r	1.00	1.00	1.00	1.00	1.00	1.00
2	0.80	1.00	0.9457	1.25	1.182	0.80
3	1.20	1.35	0.9710	1.00	0.9710	1.35
4	1.00	1.35	0.9277	1.50	1.392	0.90

TABLE B

Span or Support	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	$\frac{\lambda_j}{\lambda_r}$	λ_j	α_j	$K_j \frac{L_j}{E_j I_j}$	$(K_{j-1}+K_j)\frac{L_r}{E_r I_r}$	θ_j	$(kK)_j \frac{L_r}{E_r I_r}$	$(kK)_j \frac{L_j}{E_j I_j}$
				from Table I	$(3)_{j-1}(4)_{j-1}+(3)_j(4)_j$	Eq. (19)	$(3)_j(8)_j$	from Table I
1=r	1.00	2.40	1.00	3.6649		1.0000	2.2555	2.2555
2	1.182	2.84	0.80	3.3015	6.3061	-1.8466	2.0330	2.5412
3	0.9710	2.33	1.35	3.7043	7.6420	4.6185	3.0038	2.2250
4	1.392	3.34	0.90	2.4810	7.2337	-10.500	2.8924	3.2138
5						21.463		

$$\text{Eq'n (22a)} = (0 + 2.4810 \times 0.90)\ 21.463 + 2.8924\ (-10.500)\ \frac{E_1 I_1}{L_1} = 17.55\ \frac{E_1 I_1}{L_1}$$

by multiplying the entries in Column (8) by those in Column (3). Column (6) gives the rotation of the beam over the supports. The first value in this column is unity. (Had the beam been fixed at the left end, this value would have been zero). The second value in the column, θ_2, is evaluated from Eq. (21a)

$$\theta_2 = -\frac{(0.5000 + 3.6649)\ 1.0000}{2.2555} = -1.8466$$

This operation is not indicated in the Table. (Had support 1 been fixed, Eq. (21b) would have been used instead). The values of θ_3 to θ_z are determined from the values in Columns (5) and (7) by use of Eq. (19a), which, in terms of column members, takes the form:

$$\theta_{j+1} = -\frac{(5)_j(6)_j + (6)_{j-1}(7)_{j-1}}{(7)_j} \qquad (\text{for } j \geqq 2). \tag{19b}$$

Thus, $$\theta_3 = -\frac{6.3061(-1.8466) + 1.0000(2.2555)}{2.0330} = 4.6185$$

The left hand side of Eq. (22a), evaluated at the bottom of the table, is found to be equal to $17.55 \dfrac{E_1 I_1}{L_1}$.

Since, for the assumed value of $\lambda_1 = 2.40$, Eq. (22a) was not satisfied, this value does not correspond to a natural frequency of the system. The physical significance of the computed value of $17.55 \dfrac{E_1 I_1}{L_1}$ is as follows: the negative of this value divided by the rotation θ_z,

$$-\frac{17.55}{21.463}\ \frac{E_1 I_1}{L_1} = -0.8179\ \frac{E_1 I_1}{L_1};$$

represents the stiffness of a rotational constraint which, if it were imposed at the right end of the beam, would have made the assumed frequency correspond to a natural frequency of the system.

By repeating several such cycles of computation for different values of λ_1, the curve in Fig. 18 was obtained. The first five critical values are recorded on the figure. The corresponding circular natural frequencies are

$$(\omega_N)_1 = \frac{6.27}{L_1^2}\sqrt{\frac{E_1I_1}{m_1}} ,$$

$$(\omega_N)_2 = \frac{9.42}{L_1^2}\sqrt{\frac{E_1I_1}{m_1}} ,$$

$$(\omega_N)_3 = \frac{13.7}{L_1^2}\sqrt{\frac{E_1I_1}{m_1}} ,$$

$$(\omega_N)_4 = \frac{16.9}{L_1^2}\sqrt{\frac{E_1I_1}{m_1}} ,$$

$$(\omega_N)_5 = \frac{24.0}{L_1^2}\sqrt{\frac{E_1I_1}{m_1}} .$$

If it is desired to evaluate these quantities more precisely, the computations should be repeated for several additional values of λ_1 in the neighborhood of the critical values, and the results should be plotted on a larger scale.

The natural modes of vibration, determined in the manner described in Section 11, are shown in Fig. 19. It should be stated that, in general, for the fundamental or lowest natural frequency, the rotations of the beam over the supports are not very sensitive to the magnitude of the frequency of vibration. For some of the higher vibration frequencies, however, a slight variation in the value of the frequency may affect the rotations materially. Accordingly, the accurate evaluation of the rotations in these latter cases may become somewhat cumbersome.

13. Alternate Methods of Analysis.

As applied to continuous beams, the criterion for a natural frequency is that

$$\bar{M}_z = 0 . \tag{22a}$$

It is presumed that support z is not fixed.

As has already been remarked, the method used to evaluate $\bar{M}_z$ is merely one of a number of possible methods. It is the purpose of the following discussion to present several alternate procedures for arriving at the same result.

The Effective Stiffness Criterion. The moment at a joint of a structure necessary to produce at that joint a rotation of unit amplitude, while all other joints are in their actual condition of restraint, is defined as the "effective flexural stiffness" of the joint. This quantity depends on the properties of all the members of the structure, and it will be denoted by $\bar{K}'$.

Let $\bar{K}'_z$ represent the total effective stiffness at the right hand support z of a continuous beam; then, Eq. (22a) may be written as

$$\bar{K}'_z \, \theta_z = 0. \tag{25}$$

Since θ_z is assumed to be different from zero, this equation is satisfied only if

$$\bar{K}'_z = 0. \tag{26a}$$

It should be emphasized that Eqs. (22a) and (26a) express identically the same condition, only in slightly different forms.

Equation (26a) represents the effective stiffness criterion for determining natural frequencies. This criterion will now be applied by use of the moment distribution procedure and a procedure which, for want of any better term, will be referred to as the "direct" procedure.

The Moment Distribution Procedure. Gaskell's adaptation of the method of moment distribution (13) may be applied as follows:

1. A frequency of vibration is assumed, and the flexural stiffness, K, and the flexural carry-over factor, k, for each span of the beam are computed from Table I in Appendix A.

2. With all joints of the structure, except joint z, fixed against rotation, an exciting moment is applied at joint z producing a rotation of unit amplitude at that joint. Obviously, the frequency of this moment is equal to the assumed frequency of vibration and the magnitude of the moment is equal to $K'_{z-1} + R_z$.

3. This moment is distributed to the adjacent members in proportion to their relative stiffness, and the proper proportion of the balancing moment is carried-over to joint z-1.

4. Joint z is then locked, and the unbalanced moment at joint z-1 is distributed, carried over, and balanced through the rest of the structure. During this process of moment balancing, joint z is maintained locked.

5. The total moment carried back to joint z is determined. Finally, the effective stiffness of the joint is computed as the algebraic sum of the moment applied initially to the joint and the moment carried back after all the other joints have been balanced.

6. Steps 1 through 5 are repeated for several frequencies of vibration, and the natural frequencies are determined as those frequencies for which the effective stiffness vanishes.*

The "Direct" Procedure. The second procedure for applying the effective stiffness criterion is presented in this section. Consider a bar on unyielding supports at each end with one end elastically restrained against rotation. The restraint may be due to an actual coil spring or it may symbolize the continuity of the bar with adjoining members. The stiffness of the restraint is

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*At this point, attention should be called to the fact that the method of moment distribution does not converge always to an answer. Therefore, this method, which probably would appeal to many engineers, is restricted in its practical application. This fact is considered in somewhat greater detail in Section 14.

denoted by $\underline{R}$. It can be shown (17) that the effective flexural stiffness of the opposite end of the bar is given by the expression

$$K' = K - \frac{(kK)^2}{K+R} \; . \tag{27}$$

This equation may be used to calculate the effective stiffness of a continuous beam as follows:

1. A frequency of vibration is assumed, and the corresponding values of $\underline{K}$ and $\underline{kK}$ for each span of the beam are determined from Table I in Appendix A.
2. With the stiffness of the restraint R_1 at the extreme left end of the beam known, the value of effective stiffness of the first span, K_1', is computed from Eq. (27). This value represents also the stiffness of the rotational restraint R_2 exerted by the first span on the left end of the second span. By application of Eq. (27) to consecutive spans, the effective stiffness of spans 2 to $\underline{z-1}$ are evaluated.
3. Having determined K_{z-1}', the effective stiffness at joint $\underline{z}$ is computed as $K_{z-1}' + R_z$.
4. As usual, the foregoing steps are repeated for a number of frequencies and the natural frequencies are determined as those frequencies for which the effective stiffness vanishes.

In the foregoing discussion it was assumed that $\theta_z = 0$. Consider now that support $\underline{z}$ is fixed; then, R_z = infinite, $\theta_z = 0$, and M_z is finite. But, since

$$M_z = K_{z-1}' \theta_z \; ,$$

$$K_{z-1}' = \textit{infinity} \tag{26b}$$

becomes the modified criterion for a natural frequency.

The curve in Fig. 20 shows the variation of the effective stiffness $\bar{K}_z'$ of a continuous beam as a function of the frequency of vibration. The curve was determined by the "direct" method and is applicable to the particular beam considered in Example 2. The zeros of the curve correspond to the natural frequencies of the beam; the discontinuities correspond to the natural frequencies of the beam assuming that its right end is fixed. The abscissa of any other point of the curve corresponds to the natural frequency of the beam, provided its right end is subjected to a restraint, the stiffness of which is equal to the negative of the value represented by the ordinate of the curve.

It should be pointed out again that the main method, which was presented at the beginning of this Chapter, and the effective stiffness method presented in the preceding paragraphs, are fundamentally alike. In the former method, the moment at joint z necessary to produce a rotation of unit amplitude at joint 1 is determined, while in the latter method, the moment at joint z necessary to produce a rotation of unit amplitude at the same joint z is determined. The correspondence of the two methods can be demonstrated further by noting that, if each ordinate of the curve in Fig. 18 is divided by the rotation of the beam θ_z corresponding to that ordinate, the curve will be transformed into that shown in Fig. 20. For example, for $\lambda_1 = 2.40$ the ordinate of the curve in Fig. 18 is $17.55 \frac{E_1 I_1}{L_1}$, and the corresponding rotation $\theta_z = \theta_5 = 21.463$. The ratio $\frac{17.55}{21.463} \frac{E_1 I_1}{L_1} = .8179 \frac{E_1 I_1}{L_1}$ is identical to the corresponding ordinate of the curve in Fig. 20.

The effective stiffness method is similar to Porter's (18) and Manley's (19), (20) methods of determining natural frequencies of torsional vibration of shafts, and is similar also to Lundquist's stiffness and series criteria for determining the critical buckling loads of structures (21). The stiffness criterion has been applied to the determination of the natural frequencies of continuous beams previously by Mudrak (1), (2), (3). However, Mudrak's method differs from the procedures described in this section both in its development and in the form of its application.

The Method of Three Moments. An alternate procedure for calculating the magnitude of the exciting moment $\bar{M}_z$ is provided by the use of the equation of three moments, first applied to the study of steady-state forced vibrations by W. Prager (22). Numerical values of the various coefficients appearing in these equations have been published (23), (24) but, unfortunately, these references are not readily accessible. In general, the three-moment equation can be applied in the same manner as the three-slope equation.

14. Range of Applicability and Relative Merits of Various Procedures.

As previously remarked, the moment distribution procedure is of restricted practical value. Convergence of this procedure can be insured only for vibration frequencies which are smaller that the (unknown) fundamental or lowest natural frequency of the system considered (13). Consequently, the method can, in general, be used to determine only the lowest natural frequency of a structure. Also, it might be important to note that, even for vibration frequencies which are below the fundamental natural frequency of a structure, the moment distribution procedure may be so slow to converge that it may be necessary to carry out a large number of distributions to affect a solution. This process may become rather time consuming, especially when applied to structures involving a large number of members.

The "direct" method does not offer any difficulty of convergence.

It can, therefore, be used to calculate the higher natural frequencies of continuous beams. In general, this procedure requires a much larger number of trials than the main procedure of this report. In addition, it cannot be extended to continuous frames involving closed panels. It is, therefore, of restricted applicability, too.

For continuous beams only, the choice between the main method of this report and the procedure based on the use of the three moment equation depends, to a large extent, on personal preference and on one's familiarity with the particular procedure. One major advantage of the use of the three slope equation is that it gives a clear picture of the distortions which the structure undergoes during vibration. This feature is particularly important because, in practice, it is frequently desirable to have a rapid means of sketching the vibration configuration corresponding to a given frequency. For the analysis of continuous frames, equations involving the rotation of the joints as unknowns are remarkably better suited than equations involving moments as the redundant quantities. The extension of the main method of this report to the determination of the natural frequencies of continuous frames without sidesway is presented in the following Chapter.

IV. APPLICATION OF METHOD TO CONTINUOUS FRAMES WITHOUT SIDESWAY

15. General

This Chapter is concerned with the determination of the natural frequencies of flexural vibration of rigid jointed plane frameworks for which the joints do not move. The extension of the method to some relatively simple frames with sidesway will be presented in Chapter VI.

The frames considered may have any number of members of arbitrary length; the mass per unit of length and the flexural rigidity of cross section of the members may differ from one member to the other, but in any one member, these quantities are assumed to remain constant. The simplifying assumptions made in the analysis are as follows: The vibrations are assumed to take place in the plane of the framework. The change in length of the members due to axial deformation, and the effect of the axial forces on the bending moment in the members are neglected. In addition, no account is taken of the influence of axial vibrations. As before, the cross sectional dimensions of the members are considered to be small in comparison to their length, so that the effects of shearing deformation and rotatory inertia may be neglected.

16. Basic Relations

Figure 21 shows $\underline{s}$ members of a structure rigidly connected at their common intersection $\underline{o}$. The far ends of the members are assumed to be fixed against translation, but free to rotate subject to the restraint imposed by the adjoining members. Assume that the structure is in a steady-state forced vibration under the action of some exciting moment applied at a joint different from joint $\underline{o}$.

Let θ_{oj} denote the amplitude of rotation at end o of member oj, and θ_{jo} denote the amplitude of rotation at end j of the same member. Similarly, let M_{oj} and M_{jo} be the corresponding moment amplitudes at the same ends.

Since all members are rigidly connected at their joints,

$$\theta_{o1} = \theta_{o2} = \cdots\cdots = \theta_{oj} = \cdots\cdots = \theta_{os} = \theta_o \quad , \tag{28a}$$

and

$$\theta_{jo} = \theta_j \quad . \tag{28b}$$

Furthermore, since no external moment acts at joint o,

$$\bar{M}_o = M_{o1} + M_{o2} + \cdots\cdots + M_{oj} + \cdots\cdots + M_{os} = \sum_{j=1}^{s} M_{oj} = 0 \quad . \tag{29}$$

The moment M_{oj} may be expressed in terms of the end rotations of member oj by the relation

$$M_{oj} = K_{oj}\theta_o + (kK)_{oj}\theta_j \quad . \tag{30}$$

Substituting this expression into Eq. (29), one obtains Eq. (31a)

$$\sum_{j=1}^{s} K_{oj}\theta_o + \sum_{j=1}^{s} (kK)_{oj}\theta_j = 0 \quad . \tag{31a}$$

which expresses the conditions of both equilibrium and continuity for joint o of the structure. If only two members meet at joint o, Eq. (31a) reduces to Eq. (19a) for continuous beams.

If the degree of restraint at the far ends of the members meeting at a joint are known, it is convenient to use effective stiffnesses. Assume that the restraints at ends 1 and 2 of the portion of the structure shown in Fig. 21 are known. Let K'_{o1} and K'_{o2} represent the effective stiffness of members o1 and o2. Then, Eq. (31a) may be written as

$$(K'_{o1} + K'_{o2})\theta_o + \sum_{j=3}^{s} K_{oj}\theta_o + \sum_{j=3}^{s} (kK)_{oj}\theta_j = 0 \quad . \tag{31b}$$

Equations (30) and (31) are the only two relations needed in the analysis of frames without sideway.

17. Open Frames.

The frames considered in this section do not involve any closed panels and have known conditions of restraint at all external terminals. It is assumed that the joints of the frame do not translate.

Simple L-frames and portal frames, such as those shown in Fig. 22, act as continuous beams on rigid supports. Their natural frequencies can therefore be calculated by the procedure outlined in Section 10 of the preceding Chapter.

When applied to the analysis of continuous frames, such as those shown in Fig. 23, this procedure will, in general, reveal only a portion of the natural frequencies of the frame considered. The failure of the procedure to identify the complete set of natural frequencies results from the fact that, for certain natural frequencies, only a portion of the frame may vibrate with finite amplitudes while the rest may remain stationary. The natural frequencies corresponding to these modes, which will be referred to as "modes of partial vibration", must be determined by a supplementary procedure.

Consider any of the frames shown in Fig. 23. Let 1 denote the joint of the frame at the extreme left terminal and $\underline{z}$ denote the joint at the extreme right terminal. Without loss of generality, it may be assumed that joint $\underline{z}$ is either hinged or elastically restrained. A fixed end may be handled in the manner described in Illustrative Example 1. Assume that the structure is in a steady-state forced vibration under the action of an exciting couple $\overline{M}_z$ applied at joint $\underline{z}$. The amplitude of the slope or of the bending moment at joint 1 is assumed to have some fixed value. If the joint is hinged or elastically restrained, the amplitude of slope is taken equal to unity. If the joint is fixed, the amplitude of bending moment is taken equal to unity instead. For an assumed frequency of vibration, it is generally possible to calculate the magnitude of the exciting moment $\bar{M}_z$, in a manner entirely analogous to that used for continuous beams, by working progressively from one end of the frame to the other. By repeating this procedure for several frequencies of vibration, the magnitude of $\bar{M}_z$ may be plotted as a function of the frequency. All frequencies for which the magnitude of the exciting moment vanishes are natural frequencies of the frame.

The natural frequencies determined by the previous procedure may not represent the complete set of natural frequencies of the frame. The procedure is based on the assumption that the amplitude of slope or bending moment at joint 1 is finite. For continuous beams this condition is satisfied for all non-trivial natural frequencies. For continuous frames, however, bar 1-2 may be still even though the rest of the structure, or some portion of it, vibrates with finite amplitudes. Obviously then, the procedure fails to reveal those natural frequencies for which bar 1-2 remains still. A second assumption implicit in the procedure described is that the rotation of joint z is finite for the natural frequencies to be determined. For continuous frames, this condition is not satisfied always. Therefore, the procedure fails also to reveal the natural frequencies for which the bar meeting at joint z is stationary.

Figure 24 presents several natural modes of vibration for which either bar 1-2 or the bar meeting at joint z is stationary. The modes are applicable to the particular structures shown in Fig. 23 and can, of course, exist only if the dimensions of the various members composing these structures satisfy certain definite relations. It should be emphasized that natural modes of partial vibration are peculiar to frames and cannot exist in the case of continuous beams.

The technique for determining the natural frequencies for which either bar 1-2 or the bar meeting at joint z is stationary consists of (a) calculating the frequencies for which these conditions can occur, and (b) ascertaining whether or not these frequencies are natural frequencies of the system. The details of this supplementary technique will be explained in the examples to be presented.

18. Illustrative Examples.

Example 3. The simple frame shown in Fig. 25a has been selected for analysis. To illustrate several features of the method, the bars identified by (1) are taken identical while the bar designated by (2) is considered to have such dimensions that

$$\frac{E_2 I_2}{L_2} = \frac{E_1 I_1}{L_1}, \; L_2 = 0.80 L_1$$

and

$$\lambda_2 = \frac{3.927}{4.730}\lambda_1 = 0.8302\,\lambda_1$$

The subscripts 1 and 2 refer to bars (1) and (2), respectively.

For the sake of brevity, only one cycle of the procedure is presented. The computations are given for a value of $\lambda_1 = 3.30$; this corresponds to a value of $\lambda_2 = 2.74$. The appropriate values of K and kK are obtained from Table I in Appendix A.

for bars (1) $\quad K_1 = 2.5720 \dfrac{E_1 I_1}{L_1}$ and $(kK)_1 = 3.1375 \dfrac{E_1 I_1}{L_1}$

for bar (2) $\quad K_2 = 3.4051 \dfrac{E_1 I_1}{L_1}$ and $(kK)_2 = 2.4589 \dfrac{E_1 I_1}{L_1}$

The data necessary for the analysis are compiled on the diagram in Fig. 25b. The number in parentheses opposite each joint gives the sum of the stiffnesses of the members meeting at that joint. The parenthesized number at the middle of each member gives the product of the stiffness and the carry-over factor for the member. Both quantities are expressed in terms of $\frac{E_1 I_1}{L_1}$. The numbers without parentheses denote the rotations of the various joints. These rotations are evaluated in the manner described below, and they are recorded on the diagram as they are computed.

The procedure is started by taking $M_1 = 1.00 \frac{E_1 I_1}{L_1}$. Then, θ_2 is computed by application to joint 1 of Eq. (30), as

$$\theta_2 = \frac{1}{3.1375} = 0.31873$$

The rotations θ_4 and θ_5 are determined successively by application of Eq. (31a) to joints 2 and 4.

$$\theta_4 = -\frac{(5.1440)\ 0.31873 + 3.1375\ (0)}{3.1375} = -0.52256$$

$$\theta_5 = -\frac{8.5491\ (-.52256) + 3.1375\ (.31873)}{2.4589} = 1.4101$$

The magnitude of the exciting moment is determined from Eq. (30), as

$$\bar{M}_5 = (1.4101)3.4051 \frac{E_1 I_1}{L_1} + (-.52256)2.4589 \frac{E_1 I_1}{L_1} = 3.517 \frac{E_1 I_1}{L_1}$$

It should be noted that each of the foregoing operations can be carried out with a single set-up on a desk calculator.

Repeating this procedure for several values of λ_1, the curve shown in Fig. 26 was obtained. The λ_1 values corresponding to the natural frequencies are recorded on the figure. The corresponding natural modes of free vibration are given in Figs. 27a to 27f.

The foregoing procedure is based on the assumption that the amplitudes of both the bending moment at joint 1 and of the rotation at joint z = 5 are finite. To obtain the natural frequencies for which $M_1 = 0$, the following reasoning is used. In order for M_1 to be equal to zero, bar 1-2 must be stationary; then, the rotation of joint 2 and the internal bending moment at the joint are also equal to zero. This condition requires that bar 2-4 be stationary and that $\theta_4 = M_{42} = 0$. But with joint 4 remaining fixed against rotation, bar 3-4 can oscillate freely only for frequencies represented by values of

$$\lambda_1 = 4.730,\ 7.853,\ \ldots\ldots \tag{32}$$

During a natural frequency, every member of the structure vibrates with the same frequency. Therefore, in order for these frequencies to be natural frequencies of the entire frame, they must be also natural frequencies of the remaining portion of the frame (bar 4-5, in this particular case). The natural frequencies of bar 4-5, considering that its left end is fixed, are

$$\lambda_2 = 3.927,\ 7.069,\ \ldots\ldots\ ;$$

these correspond to values of

$$\lambda_1 = 4.730,\ 8.514,\ \ldots\ldots$$

Comparing the latter values with those given in Eq. (32), one concludes that, within the range of frequencies considered in Fig. 26, $\lambda_1 = 7.853$ and $\lambda_1 = 8.514$ do not represent a natural frequency, while $\lambda_1 = 4.730$ does. The natural mode of vibration for $\lambda_1 = 4.730$ is shown by the solid curve in Fig. 27g.

The natural frequencies, if any, for which $\theta_5 = 0$, are determined in a similar manner. θ_5 can be equal to zero only if bar 4-5 is stationary. Under this condition, $\theta_4 = M_{45} = 0$; then, bar 3-4 may vibrate freely only at frequencies represented by the λ_1 values given in Eq. (32). Since members 1-2 and 2-4 of the remaining portion of the frame are identical to member 3-4, each of these members can vibrate with its ends fixed for the same frequency; therefore, the λ_1 values given in Eq. (32) correspond to natural frequencies of the frame. The natural mode corresponding to $\lambda_1 = 4.730$ is shown by the dotted curve in Fig. 27g, while the mode corresponding to $\lambda_1 = 7.853$ is shown in Fig. 27h.

It should be observed that the natural modes shown in Figs. 27c and 27g can exist for the same frequency. Of these three modes, however, only the two are independent; the third is a linear combination of the other two. In fact, from any two of these three modes, one can obtain an infinite number of combination modes.

Within the range of frequencies considered in Fig. 26, the complete set of λ_1 values corresponding to natural frequencies is

$$(\lambda_1)_N = 3.59,\ 4.22,\ 4.73\ \text{(double)},\ 6.80,\ 7.44,\ 7.85,\ 8.35$$

More involved frames may be handled by the same procedure. The technique for obtaining the natural frequencies corresponding to modes of partial vibration is illustrated further by the examples given in Section 21.

Example 4. A sketch of the frame considered is shown in Fig. 28. This frame is similar to that analyzed in the preceding example. In this case, the dimensions of the structure are assumed to be such that

$$\frac{E_1 I_1}{L_1} = \frac{E_2 I_2}{L_2} = \frac{E_3 I_3}{L_3} = \frac{E_4 I_4}{L_4}$$

and

$$\lambda_2 = 0.75\lambda_1, \qquad \lambda_3 = 1.30\lambda_1, \qquad \lambda_4 = 0.90\lambda_1 .$$

To determine the natural frequencies of this frame, we proceed in the usual manner and plot the magnitude of the exciting moment at joint 5 as a function of the assumed frequency of vibration. The curve in Fig. 28 summarizes the results obtained. It should be noted that this curve, unlike that shown in Fig. 26, is not continuous. The values of λ_1 corresponding to the first few natural frequencies are recorded on Fig. 28. It can be shown that, within the range of the frequencies considered, there are no natural frequencies corresponding to modes of partial vibration.

19. Closed Frames.

The application of the method to frames involving closed panels, such as those shown in Fig. 29, is described in this section by reference to a simple example. The hypothertical two celled rectangular frame shown in Fig. 29a is selected for this purpose.

The first step in the analysis is to assume that the frame is cut at some convenient joint. In the example considered, the cut is introduced at joint 6. Next, it is assumed that the structure undergoes a steady-state forced vibration with finite amplitudes and known frequency. The vibration is assumed to be maintained by an exciting couple applied at the cut joint (joint 6). If the frequency of vibration is equal to the natural frequency of the frame, the magnitude of the exciting moment must vanish and the amplitudes of slope on either side of the cut must be equal.

For an assumed frequency, the discontinuity of slope and the magnitude of the exciting moment may be determined in the same way as for open frames, by working progressively from joint to joint across the structure.

For the frame considered, Eq. (31a) is first applied to joint 1. It is noted that the resulting expression involves three unknowns: θ_1, θ_2, and θ_3. Therefore, θ_2 and θ_3 cannot be solved directly in terms of θ_1 alone, as it was possible in the case of continuous beams and continuous open frames. Instead, it is necessary to express θ_3 in terms of both θ_1 and θ_2. Next, Eq. (31a) is applied to joint 2, and θ_4 is determined in terms of the same two permanent unknowns θ_1 and θ_2. By successive applications of the same equation to joints 3, 4, and 5, the rotations θ_5, θ_{64}, and θ_{65} may also be determined in terms of θ_1 and θ_2. Having computed the rotations of all joints, the exciting moment at joint 6 may also be expressed in terms of the two permanent unknowns θ_1 and θ_2. In the application of this technique, consecutive joints must be selected in such an order that, when Eq. (31a) is applied to a joint, the resulting expression involves only one new unknown. This technique is an adaption of Wilbur's scheme (25) of solving the set of simultaneous equations resulting from the use of the slope deflection equation.

The joint for which the rotation is evaluated last in this procedure (joint 6 in this case), is the one at which the structure is generally assumed to be cut. The number of members meeting at this joint must be equal to the number of permanent unknowns used.

At a natural frequency

$$\theta_{64} - \theta_{65} = 0,$$
and
$$\bar{M}_6 = 0. \tag{33a}$$

In terms of the two permanent unknowns θ_1 and θ_2, these conditions may be written as

$$\begin{aligned} c_{11}\theta_1 + c_{12}\theta_2 &= 0, \\ c_{21}\theta_1 + c_{22}\theta_2 &= 0, \end{aligned} \tag{33b}$$

where the c's are constants, the magnitudes of which depend on the assumed frequency of vibration and on the characteristics of all the members in the structure. Equation (33b) represents a set of linear and homogeneous equations for the unknowns θ_1 and θ_2; these equations have a solution different from zero when the determinant of their coefficient is zero,

$$\Delta_6(\theta_1, \theta_2) = \begin{vmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{vmatrix} = 0. \tag{34}$$

This criterion of vanishing determinant will fail to reveal

(a) the natural frequencies for which θ_1 and θ_2 are simultaneously equal to zero, and

(b) the natural frequencies for which the bars meeting at the cut joint (joint 6) are stationary.

The natural frequencies corresponding to the foregoing two conditions may be calculated by the supplementary technique described in connection with continuous open frames.

The single-bay multi-story frame shown in Fig. 29b may be handled by the same procedure. The rotations of its joints may be expressed in terms of θ_1 and θ_2; the cut may be introduced at joint 9 or 10. For the analysis of the two-bay multi-story frame shown in Fig. 29c, one needs to take three quantities, say θ_1, M_2, and θ_3, as permanent unknowns. The cut must be introduced at joint 14. The conditions of continuity and equilibrium for this joint may then be expressed as

$$\begin{aligned} \theta_{14,13} - \theta_{14,11} &= c_{11}\theta_1 + c_{12}\theta_2 + c_{13}\theta_3 = 0 \,, \\ \theta_{14,11} - \theta_{14,15} &= c_{21}\theta_1 + c_{22}\theta_2 + c_{23}\theta_3 = 0 \,, \\ \bar{M}_{14} &= c_{31}\theta_1 + c_{32}\theta_3 + c_{33}\theta_3 = 0 \,, \end{aligned}$$

where the c's are numerical constants. The criterion for a natural frequency is

$$\Delta_{14}(\theta_1, M_1, \theta_3) = \begin{vmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{vmatrix} = 0 \,.$$

It is seen that, for this problem, it becomes necessary to evaluate a third order determinant. For the general case, the order of the determinant is equal to the number of the permanent unknowns that must be used in evaluating the rotations of the joints.

The computational work required to calculate the exciting moment and the discontinuity of slope at the cut, may get fairly involved, particularly if more than two quantities must be used as permanent unknowns. One may simplify this work considerably by carrying out the computations in parts: Consider again the frame shown in Fig. 29a. First, assume that $\theta_1 = 1.00$ and that $\theta_2 = 0$. Calculate the rotations of the joints in the usual manner, and designate them by θ'. Calculate also the magnitude of the exciting moment at the cut joint. Designate this by $\bar{M}'$. Next, assume that $\theta_1 = 0$ and that $\theta_2 = 1.00$, and calculate the corresponding rotations and moment. Denote these by θ'' and $\bar{M}''$, respectively. Then, the actual rotation at a joint j is

$$\theta_j = \theta_j'\theta_1 + \theta_j''\theta_2 , \tag{35a}$$

and the total exciting moment at the cut joint, say joint j, is

$$\bar{M}_j = \bar{M}_j'\theta_1 + \bar{M}_j''\theta_2 . \tag{35b}$$

The direct combination of these partial effects is justified by the fact that the differential equation for steady-state forced vibration is linear.

20. Outline of Procedure.

The procedure for determining the natural frequencies of continuous frames involving closed panels may be outlined as follows:

1. For some member of the frame, say member r, assume a value of $\lambda_r = \sqrt[4]{\frac{m_r \omega^2}{E_r I_r}}\, L_r$. This is equivalent to assuming a frequency of vibration ω.
2. From Eq. (23) compute the λ values for the remaining members of the frame.
3. From Table I in Appendix A, calculate the appropriate values of K and kK for each member. These values, as obtained from Table I, are expressed in terms of $\frac{E_j I_j}{L_j}$, where the subscript j refers to the particular member considered.

4. Express the quantities determined in step (3) in terms of the $\frac{EI}{L}$ of the reference member r, by multiplying them by the dimensionless coefficient α_j (Eq. 24).

5. Compute the sum of the stiffnesses of the members meeting at the various joints and record these values on a diagram of the frame. On the same diagram record also the product of the stiffness and the carry-over factor for each member. These quantities must be expressed in terms of the $\frac{EI}{L}$ of the same reference member r. A convenient scheme for arranging the computations is shown in the illustrative examples presented in the next section.

6. Choose the unknowns in terms of which the distortions of the frame will be expressed. In general, the number of unknowns that must be selected is equal to the number of the main longitudinal members in the frame. For the frame shown in Fig. 29a, one may take θ_1 and θ_2 as the two permanent unknowns.

7. Consider the first of these quantities equal to unity and the other equal to zero ($\theta_1 = 1.00$ and $\theta_2 = 0$). Working across the structure, as described in the preceding section, compute the rotations of the joints. Denote these by θ'. Compute also the exciting moment and denote it by $\overline{M}'$.

8. Repeat step (7), taking the second quantity equal to unity and the first equal to zero ($\theta_1 = 0$ and $\theta_2 = 1.00$). Denote the resulting rotations by θ'' and the exciting moment by $\overline{M}''$. In general, steps (7) and (8) must be repeated as many times as there are permanent unknowns.

9. Determine the total discontinuity of slope and the total exciting moment and set each expression equal to zero. For the

frame considered, these expressions will be

$$\begin{aligned}(\theta'_{64}-\theta'_{65})\theta_1+(\theta''_{64}-\theta''_{65})\theta_2&=0,\\ \bar{M}'_6\theta_1 \quad + \quad \bar{M}''_6\theta_2&=0.\end{aligned} \tag{33c}$$

10. Evaluate the determinant of the coefficients of θ_1 and θ_2 in the expressions of Eq. (33c).
11. Repeat steps (1) to (10) for different values of λ_r.
12. Plot the variation of the determinant evaluated in step (10) as a function of the λ_r values. The frequencies for which the determinant becomes equal to zero are natural frequencies of the frame.

The foregoing procedure fails to reveal the natural frequencies for which the permanent unknowns (θ_1 and θ_2) are simultaneously equal to zero. In addition, it fails to reveal the natural frequencies for which the members meeting at the cut joint are stationary. A supplementary procedure for determining these natural frequencies has been described, and its details are illustrated further in the two numerical examples that follow.

21. Illustrative Examples.

Example 5. The structure considered is shown in Fig. 30a. All members are assumed to be uniform and identical to each other. It is desired to calculate the natural frequencies and the corresponding natural modes of vibration of this frame for a range of values of λ less than 6.50. Since all members are identical, rather than repeating the procedure outlined in Section 20 for each assumed frequency, it is more convenient to derive a general expression for the criterion for a natural frequency and determine directly from this expression the desired quantities. This is similar to what was done in Section 12 for Example 1.

Let K be the flexural stiffness and k the flexural carry-over factor for each member of the frame. These quantities are, of course, functions of the parameter λ. The sum of the stiffnesses for the various joints and the product of the stiffness and the carry-over factor for each member of the frame are shown in Fig. 30b. The rotations of the joints will be expressed, not in terms of θ_1 and θ_2, as it was suggested in the preceding discussion, but rather, in terms of θ_1 and the internal bending moment M_{13}. The reason for this choice will become apparent shortly. The frame is assumed to be cut at joint 6. First, it is assumed that $M_{13} = M_1 = 1.00$ and $\theta_1 = 0$. Applying Eq. (30) to ends 1 of members (1) and (2), one obtains

$$\theta_2' = -\frac{1}{kK} ,$$

$$\theta_3' = \frac{1}{kK} .$$

Similarly, applying Eq. (31a) successively to joints 2, 3, 4, and 5, one obtains

$$\theta_4' = -K\left[2\left(-\frac{1}{kK}\right) + k(0)\right] \div kK = \frac{2}{k^2K} ,$$

$$\theta_5' = -K\left[3\left(\frac{1}{kK}\right) + k(0) + k\left(\frac{2}{k^2K}\right)\right] \div kK = -\frac{5}{k^2K} ,$$

$$\theta_{64}' = -K\left[3\left(\frac{2}{k^2K}\right) + k\left(-\frac{1}{kK}\right) + k\left(\frac{1}{kK}\right)\right] \div kK = -\frac{6}{k^3K} ,$$

$$\theta_{65}' = -K\left[2\left(-\frac{5}{k^2K}\right) + k\left(\frac{1}{kK}\right)\right] \div kK = \frac{10-k^2}{k^3K} .$$

The discontinuity of slope at joint 6 is

$$\theta_{64}' - \theta_{65}' = \frac{k^2-16}{k^3K} ,$$

and the exciting moment at joint 6 is

$$\bar{M}_6' = K\left[2\left(-\frac{6}{k^3K}\right) + k\left(\frac{2}{k^2K}\right) + k\left(-\frac{5}{k^2K}\right)\right] = -\frac{3}{k^3}(4+k^2) .$$

Next, it is assumed that $M_1 = 0$ and $\theta_1 = 1.00$. The rotations θ'' are obtained in a similar manner. The results are

$$\theta_2'' = -\frac{1}{k} ,$$

$$\theta_3'' = -\frac{1}{k} ,$$

$$\theta_4'' = -K\left[2\left(-\frac{1}{k}\right) + k\right] \div kK = \frac{2-k^2}{k^2} ,$$

$$\theta_5'' = -K\left[3\left(-\frac{1}{k}\right) + k + k\frac{2-k^2}{k^2}\right] \div kK = \frac{1}{k^2} ,$$

$$\theta_{64}'' = -K\left[3\frac{2-k^2}{k^2} + k\left(-\frac{1}{k}\right) + k\left(-\frac{1}{k}\right)\right] \div kK = \frac{5k^2-6}{k^3} ,$$

$$\theta_{65}'' = -K\left[2\frac{1}{k^2} + k\left(-\frac{1}{k}\right)\right] \div kK = \frac{k^2-2}{k^3} .$$

The discontinuity of slope and the exciting moment at joint 6 are

$$\theta_{64}'' - \theta_{65}'' = \frac{5k^2-6}{k^3} - \frac{k^2-2}{k^3} = \frac{4}{k^3}(k^2-1) ,$$

$$\bar{M}_6'' = K\left[2\frac{5k^2-6}{k^3} + k\frac{2-k^2}{k^2} + k\frac{1}{k^2}\right] = \frac{K}{k^3}(13k^2-12-k^4)\theta_1 .$$

The total discontinuity of slope and the total exciting moment at joint 6 are

$$\begin{aligned} \theta_{64} - \theta_{65} &= \frac{1}{k^3K}(k^2-16)M_1 + \frac{4}{k^3}(k^2-1)\theta_1 , \\ \bar{M}_6 &= -\frac{3}{k^3}(4+k^2)M_1 + \frac{K}{k^3}(13k^2-12-k^4)\theta_1 . \end{aligned} \tag{36}$$

The determinant of the coefficients of θ_1 and M_1 is

$$\Delta_6(M_1,\theta_1) = \frac{144}{k^6} - \frac{k^4-41k^2+184}{k^4} . \tag{37}$$

The curve in Fig. 30d shows the variation of this determinant with λ. The values of $\underline{k}$ and $\underline{K}$ corresponding to the various values of λ were obtained from Table I in Appendix A. The zero intercepts of this curve represent natural frequencies of vibration. These values are indicated on the figure.

The vibration modes corresponding to a natural frequency are determined as follows: First, the relationship between θ_1 and M_1 is determined by setting either of the expressions in Eq. (36) equal to zero. Then, the rotations of the joints are evaluated, in terms of either θ_1 or M_1, by use of the relation

$$\theta_j = \theta_j' M_1 \frac{L}{EI} + \theta_j'' \theta_1$$

In this particular case, the rotations were expressed in terms of θ_1. The results are summarized in the following table.

Value of λ_N	θ_2/θ_1	θ_3/θ_1	θ_4/θ_1	θ_5/θ_1	θ_6/θ_1
π	-1.00	-1.00	1.00	1.00	-1.00
3.556	-1.00	0	0	-1.00	1.00
3.805	1.00	-1.333	-1.333	1.00	1.00
4.048	-1.00	1.333	-1.333	1.00	-1.00
4.298	1.00	0	0	-1.00	-1.00
2π	1.00	1.00	1.00	1.00	1.00

It should be noted that the value of $\lambda_N = 4.730$ is not included in this table. For this value, as it will follow from the discussion of the succeeding paragraphs, there is an infinite number of possible natural modes. Such modes cannot be determined by the procedure described. From the rotations in the above table, the deflections at the interior points of the members may be obtained by use of the influence coefficients given in Table II of Appendix A. The natural modes corresponding to the natural frequencies determined in Fig. 30d, are shown in Fig. 31a through 31g.

The next step in the solution is the determination of the natural frequencies for which θ_1 and M_1 vanish simultaneously. θ_1 and M_1 may be equal to zero only if both bars (1) and (2) are stationary. Under this condition, bar (3) also is stationary and joints 3 and 4 remain fixed against rotation as shown in Fig. 30c. For the range of λ values considered, bar (4) can execute free vibrations with fixed ends only at a frequency represented by

$$\lambda = 4.730.$$

Since all members of panel 3-4-6-5 are identical, each member can vibrate at this frequency with fixed ends; therefore, $\lambda = 4.730$ corresponds to a natural frequency of the frame. The vibration mode corresponding to this natural frequency is shown in Fig. 31h.

The concluding step in the solution of the problem under consideration is the determination of the natural frequencies for which the members meeting at a joint 6 remain stationary. Proceeding in the manner described in the preceding case, it can be shown that, within the range of λ values considered, $\lambda = 4.730$ represents a natural frequency for which bars (6) and (7) are stationary. The vibration mode corresponding to this natural frequency is shown in Fig. 31i.

It should be observed that the natural modes shown in Figs. 31f, 31h, and 31i can exist for the same frequency of vibration. However, of these three modes only two are independent. The third is a linear combination of the other two.

Had the distortions of the frame and the exciting moment $\bar{M}_6$ been expressed in terms of θ_1 and θ_2 instead of in terms of θ_1 and M_1, the basic procedure would have failed to reveal the natural frequencies for which $\theta_1 = \theta_2 = 0$. In other words, the curve in Fig. 30d would not have intersected the λ-axis at $\lambda = 4.730$. Of course, this natural frequency would have been determined by the supplementary technique.

Example 6. As a further illustration of the application of the method to continuous closed frames, the symmetrical frame shown in Fig. 32a is selected for analysis. All columns are considered hinged at the base. It is desired to calculate natural frequencies corresponding only to symmetrical modes of vibration. Since for symmetrical vibrations the joints of the frame, even though free from external restraining forces, do not translate, the method of this Chapter is directly applicable to the problem considered. The effect of symmetry is taken into account by using for the girders of the central bay modified stiffnesses $\underline{K}^S$ instead of the usual stiffnesses $\underline{K}$.

The dimensions of the frame and some additional data pertinent in the analysis are assembled in the table below. Member (2) of the frame is taken as the reference member. Columns (5) and (6) of this table give,

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Member	$\frac{m_j}{m_r}$	$\frac{E_jI_j}{E_rI_r}$	$\sqrt[4]{\frac{(1)}{(2)}}$	$\frac{L_j}{L_r}$	$\frac{\lambda_j}{\lambda_r} = (3)(4)$	$\alpha_j = \frac{(2)}{(4)}$	λ_j
1	0.50	0.67	0.9294	1.00	0.9294	0.67	2.04
2=r	1.00	1.00	1.000	1.00	1.000	1.00	2.20
3	3.00	1.75	1.144	1.50	1.716	1.1667	3.78
4	3.00	1.50	1.189	1.125	1.338	1.3333	2.94
5	0.40	0.45	0.9710	0.75	0.7282	0.600	1.60
6	0.50	0.625	0.9457	0.75	0.7093	0.83333	1.56
7	2.00	1.25	1.125	1.50	1.687	0.83333	3.71
8	2.00	1.25	1.125	1.125	1.265	1.1111	2.78

respectively, the ratio of λ_j/λ_2 (Eq. 23) and the α_j values (Eq. 24) for each member of the frame. For the sake of brevity, only one cycle of the procedure is presented. The computations are given for a value of $\lambda_2 = 2.20$; this corresponds to a circular vibration frequency $\omega = \frac{(2.20)^2}{L^2}\sqrt{\frac{EI}{m}}$.

The α values of the various members are recorded also on the diagram in Fig. 32b. The numbers without parentheses above these values give the stiffness of the various members, while the parenthesized numbers below the α values give the product of the stiffness and the carry-over factor of the members. These quantities are expressed in terms of the $\frac{EI}{L}$ of the member to which they refer, and they are obtained directly from Table I in Appendix A.

On the diagram in Fig. 32c, the number in parenthesis opposite each joint represents the total stiffness of the members framing into that joint. These stiffnesses are expressed in terms of the $\frac{EI}{L}$ of the reference member (2). Thus, the total stiffness of the members connecting at joint 4 is $\bar{K}_4 = 3.7675x1.00 + 0.91884x1.1667 + 3.9430x0.83333 + 0.54510x1.3333 = 8.8521$. The parenthesized number at the middle of each member represents the product of the stiffness and the carry-over factor for that member; these values are also expressed in terms of the $\frac{EI}{L}$ of the reference member (2).

The rotations of the joints are expressed in terms of θ_1 and θ_2. The frame is assumed to be cut at joint 5 (and joint $5'$). First, consider that

$$\theta_1 = 1.00 \text{ and } \theta_2 = 0 \text{ .}$$

Applying Eq. (30) to joints 1 and 2, one obtains

$$\theta_3' = -\frac{2.5661 \text{ x } 1.00}{1.4262} = -1.7993 \text{ ,}$$

$$\theta_4' = -\frac{3.7675 \text{ x } 0}{2.1764} = 0 \text{ .}$$

As usual, these values are recorded on the diagram as they are computed. Applying Eq. (31) successively to joints 3, 4, and 6, one obtains

$$\theta_{53}' = -\frac{6.0003(-1.7993) + 1.4262(1.00) + 5.3324(0)}{1.2285} = 7.6273 \text{ ,}$$

$$\theta_6' = -\frac{0 + 5.3324(-1.7993) + 0}{1.7024} = 5.6359 \text{ ,}$$

$$\theta'_{56} = -\frac{5.3103(5.6359) + 0}{3.5544} = -8.4201 \ .$$

The unbalanced exciting moment at joint 5 is computed as

$$\bar{M}'_5 = 1.2285(-1.7993) + 2.3621(7.6273) + 1.0522(-8.4201) + 3.5544(5.3103) = 26.979 \ .$$

Next, consider that

$$\theta_1 = 0 \text{ and } \theta_2 = 1.00 \ .$$

The rotations θ'' are computed in the same manner and the results are recorded on the diagram above the values of θ'.

The total slope at a joint, say at joint 6, is

$$\theta_6 = 5.6359\theta_1 + 7.7229\theta_2 \ .$$

The total discontinuity of slope and the total exciting moment at joint 5 are

$$\theta_{56} - \theta_{53} = -16.047\theta_1 - 18.223\theta_2 \ ,$$
$$\bar{M}_5 = 26.979\theta_1 + 33.931\theta_2 \ .$$

The value of the determinant of the coefficients of θ_1 and θ_2 is

$$\Delta_5(\theta_1, \theta_2) = \begin{vmatrix} -16.047 & -18.223 \\ 26.979 & 33.931 \end{vmatrix} = -52.85 \ .$$

Since this is different from zero, the assumed value of $\lambda_2 = 2.20$ does not correspond to a natural frequency. Repeating this procedure for several values of λ_2 and evaluating, in each case, the resulting determinant, the curve given in Fig. 33 was obtained. The values of λ_2 corresponding to the natural frequencies of the frame are recorded on the figure.

The next step in the solution is the determination of the natural frequencies for which θ_1 and θ_2 are simultaneously equal to zero. θ_1 and θ_2 can be equal to zero only when bars (1) and (2) are stationary as shown in Fig. 34a.

Then, $$M_{31} = M_{42} = 0 \text{ and } \theta_3 = \theta_4 = 0$$

Under this condition, bars (3) and (4) can vibrate freely only at frequencies equal to the natural frequencies of these bars for the condition of fixed ends. For bar (3), these natural frequencies are represented by values of

$$\lambda_3 = 4.730,\ 7.853,\ 10.996,\ \ldots \tag{38a}$$

The equivalent values of λ_2 are

$$\lambda_2 = 2.756,\ 4.576,\ 6.408,\ \ldots \tag{38b}$$

For bar (4), natural frequencies corresponding only to symmetrical modes must be considered; these are given by values of

$$\lambda_4 = 4.730,\ 10.996,\ \ldots \tag{39a}$$

which are equivalent to

$$\lambda_2 = 3.535,\ 8.218,\ \ldots \tag{39b}$$

It is now necessary to ascertain whether or not these frequencies are natural frequencies of the portion of the frame composed of bars (5), (6), (7), and (8). To do this, it is necessary to carry out one cycle of the basic procedure for each frequency to be investigated. For the purpose of illustration, three different frequencies will be considered in detail.

(a) $\lambda_2 = 2.756$ ($\lambda_3 = 4.730$). Since none of the values given in Eq. (39b) is equal to 2.756, bar (4) must be stationary; this means that

$$M_{44'} = 0 .$$

The condition of equilibrium at joints 3 and 4 requires that

$$M_{34} = -M_{35} \text{ and } M_{46} = -M_{43} .$$

Since, for the λ value considered, the deflection of bar (3) is symmetrical about its midspan,

$$-M_{43} = +M_{34} = -M_{35}$$

The joint rotations of the frame may now be expressed in terms of M_{35}, which, for convenience, may be taken equal to $1.00 \frac{E_2 I_2}{L_2}$. Starting from joint 3 and considering that $\theta_3 = 0$, one may determine the rotations of joints 5, 6, and 4 in the usual manner. If $\lambda_2 = 2.756$ is a natural frequency, the computed value of θ_4 must be equal to zero and M_{46} must be equal to $-1.00 \frac{E_2 I_2}{L_2}$.

The data necessary in carrying out these computations are assembled in the following table and in Figs. 34b and 34c.

Member	2	5	7	8	6
λ_j / λ_2	1.00	0.7282	1.687	1.265	0.7093
λ_j	2.756	2.01	4.65	3.49	1.95

In Fig. 34b the stiffnesses are expressed in terms of the $\frac{EI}{L}$ of the member to which they refer, while in Fig. 34c they are expressed in terms of $\frac{E_2 I_2}{L_2}$. The rotation θ_5 is computed by application of Eq. (30) to joint 3, while θ_6 and θ_4 are computed by use of Eq. (31). Since the computed value of θ_4 is different from zero, $\lambda_2 = 2.756$ does not represent a natural frequency of the frame. If θ_4 were found to be equal to zero, it would have been necessary to investigate also if M_{46} were equal to $-1.00 \frac{E_2 I_2}{L_2}$.

(b) $\lambda_2 = 4.576$ ($\lambda_3 = 7.853$). In this case also, bar (4) is stationary, but M_{46} is equal to $+M_{35}$ instead of $-M_{35}$. In all other respects, the details of the analysis are similar to those of the previous case. The pertinent computations are given in the following table and in Figs. 34d and 34e.

Member	2	5	7	8	6
λ_j / λ_2	1.00	0.7282	1.687	1.265	0.7093
λ_j	4.576	3.33	7.72	5.79	3.25

Since the computed value of Θ_4 is different from zero, $\lambda_2 = 4.576$ does not correspond to a natural frequency.

(c) $\lambda_2 = 3.535$ ($\lambda_4 = 4.730$). Since none of the values given in Eq. (38b) are equal to 3.535, bar (3) in Fig. 34a must be stationary in this case. This means that bars (5) and (7) are stationary also and that joints 4 and 6 remain fixed against rotation. It follows that, if $\lambda_2 = 3.535$ is a natural frequency, each of the members composing panel 4-6-6'-4' must be capable of vibrating freely with the ends fixed. For bar (6) this is possible for values of

$$\lambda_6 = 4.730,\ 7.853,\ \ldots . \tag{40a}$$

while for bar (8) it is possible for values of

$$\lambda_8 = 4.730,\ 10.996,\ \ldots . \tag{41a}$$

The corresponding λ_2 values are, respectively

$$\lambda_2 = 6.669,\ 11.071,\ \ldots . \tag{40b}$$

$$\lambda_2 = 3.739,\ 8.692,\ \ldots . \tag{41b}$$

Since these are different from 3.535, $\lambda_2 = 3.535$ does not correspond to a natural frequency. It is worth noting also that the λ_2 values given in Eqs. (40b), (41b), and (39b) are different. It is, therefore, concluded that, within the range of the frequencies considered, the panel formed by the members (4), (6), and (8) cannot vibrate freely while the rest of the frame remains stationary.

The concluding step in the solution of the problem under consideration consists of investigating if there are any natural frequencies for which the bars meeting at joint 6 are stationary.

When bars (5) and (7) are stationary $M_{35} = M_{65} \neq 0$ and joints 3 and 6 remain fixed against rotation as shown in Fig. 35a. Excluding the trivial case of no vibration, this condition can occur only at frequencies equal to the natural frequencies of bar (8), assuming that it is fixed at the ends, and of bar (1), assuming that joint 3 is fixed. For bar (8) the first two of these natural frequencies are represented by the λ values given in Eq. (41). For bar (1) the first two natural frequencies are given by values of

$$\lambda_1 = 3.927,\ 7.069 \tag{42a}$$

which are equivalent to

$$\lambda_2 = 4.225,\ 7.606 \tag{42b}$$

It remains now to investigate whether these values represent natural frequencies of the frame shown in Fig. 35a. Only two frequencies will be considered in detail; the others may be handled in a similar manner.

(a) $\lambda_2 = 3.739$ ($\lambda_8 = 4.730$). Since none of the values given in Eq. (42b) are equal to 3.739, bar (1) cannot vibrate freely in this case; consequently, both bars (1) and (3) must remain still and joint 4 must remain fixed against rotation as shown in Fig. 35b. If $\lambda_2 = 3.739$ represents a natural frequency, each member of the frame in this figure must be capable of vibrating at the same frequency. It has already been shown that this condition is not possible for panel $4\text{-}6\text{-}6'\text{-}4'$. It remains, therefore, to ascertain whether panel $2\text{-}4\text{-}4'\text{-}2'$ can vibrate freely. The first two natural frequencies of bars (4) and (2) are represented respectively by values of

$$\lambda_2 = 3.535,\ 8.218$$

$$\lambda_2 = 3.927,\ 7.069$$

Since the λ_2 values for the two bars are unequal, it is concluded also that panel $2\text{-}4\text{-}4'\text{-}2'$ cannot vibrate freely while the rest of the frame remains still.

(b) $\lambda_2 = 4.225$ ($\lambda_1 = 3.927$). In this case, bar (8) cannot vibrate freely; therefore, bars (8) and (6) in Fig. 35a are stationary, and the rotation of joint 4 is zero as shown in Fig. 35c. This means that each member of the frame shown in Fig. 35c must behave as if it were fixed at joints 3 and 4. The λ_2 values for which this condition can be realized are different for the different members; consequently, $\lambda_2 = 4.225$ does not correspond to a natural frequency.

In summary, it should be stated that for this particular problem and for the range of frequencies considered, the zero intercepts of the curve in Fig. 33 represent the complete set of natural frequencies of the frame.

22. Comments on Natural Modes of Partial Vibration.

The determination of the natural frequencies corresponding to modes of partial vibration is not as tedious as it might appear from the space devoted to its discussion in the preceding sections. Furthermore, it should be noted that, for most practical applications, it may be entirely unnecessary to calculate these natural frequencies. As already explained, the procedure for determining these natural frequencies consists of (a) computing a number of frequencies, and (b) establishing whether or not these frequencies are natural frequencies of the frame. In most actual problems, one is interested in determining natural frequencies comprised within specified ranges of frequencies. Therefore, if the values calculated in the first step of this procedure are found to lie outside the ranges of interest, it will be superfluous to carry out the second step. The first step of the procedure can usually be carried out almost by inspection.

Natural modes of partial vibration correspond always to higher natural frequencies. Therefore, no consideration need be given to these modes, if only the fundamental natural frequency of a frame is to be determined. The fundamental natural frequency of a frame may be determined also by the moment distribution procedure described in Section 13.

23. Need for Approximate Methods of Analysis.

The method that has been described, even though simple bothoth principle and in the details of its application, may become time consuming when applied to structures comprising a very large number of members. For very complex structures, such as multi-story multi-bay building frames, it may be desirable to have a simpler, though less accurate, method of analysis.

Strictly speaking, the dynamic response of a member of a framework and, as a consequence, the natural frequencies of the structure depend on the properties of all the members in the structure. Intuition leads one to expect, however, that the importance of this influence diminishes rapidly as the distance from the member concerned increases. For example, the natural frequencies of a two-span beam may vary greatly, depending on whether the extreme ends are fixed or hinged; on the other hand, for a multiple-span beam the natural frequencies may be almost independent of the condition of restraint at the extreme ends. For example, for a uniform beam of two equal spans, the fundamental natural frequency for fixed ends is 56 percent higher than the corresponding natural frequency for hinged ends. For a uniform beam, of seven equal spans, the fundamental natural frequency for fixed ends is only 6 percent higher than for hinged ends. (The latter value was obtained from reference (16)). This condition indicates that it may be possible to determine the natural frequencies of complicated systems by considering only a portion of the structure instead of the entire structure.

It is believed that an investigation aimed at determining the possibilities of such an approximate procedure will be most rewarding. The procedure described herein will be of great value in such a study.

24. Effect of Axial Forces; Problems of Instability.

In the discussion thus far, the effect of permanent axial forces on the natural frequencies of flexural vibration has been omitted. This effect, which may be quite important when the magnitude of the external force on the structure is a sizeable fraction of the critical buckling load, may be taken into account by use of modified stiffness and carry-over factors which include the influence of the axial thrust. It is assumed that the axial loads are independent of time and that they are applied at the ends of the members.

The modified stiffnesses and carry-over factors may be expressed conveniently in terms of the dimensionless parameter λ, which was used previously, and the ratio P/P_o, where P is the axial compressive or tensile force and P_o is the fundamental buckling load of the member assuming its ends to be simply supported.

Algebraic expressions for the dynamic flexural stiffness and the dynamic flexural carry-over factor are given in Appendix II. Numerical values of these two quantities and of their product have been computed for values of λ between zero and 4.75 at increments of 0.05 and for values of P/P_o between -4.00 and 4.00 at increments of 0.1. It is expected that these results will be made available soon.

For the limiting case of $\lambda = 0$, the values of stiffness and carry-over factors are those for an axially loaded bar which does not vibrate. Detailed tabulations of these quantities have been presented previously by James (26), by Lundquist and Kroll (27), and by Hu and Libove (28). It becomes apparent that the problem of elastic instability is a limiting case of the more general problem of the vibration of structures for which the members are subjected to axial forces.

The three slope equation has been applied to the investigation of the instability of frameworks by Chwalla and Jokish (29) and by Winter and his associates (30). The procedure described in this report has the advantage that it can be used to obtain solutions with considerably less effort.

V. APPLICATION OF METHOD TO CONTINUOUS BEAMS ON FLEXIBLE SUPPORTS

25. General.

Thus far application of the method has been restricted to continuous beams on rigid supports and to continuous frames for which the joint do not translate. In this chapter, the method is extended to continuous beams on flexible supports. The flexibility of the supports is represented by a set of mutually independent deflectional and rotational springs. To start with, it is assumed that the stiffnesses of the restraints are independent of the vibration frequency. The assumptions made in the analysis are similar to those made in the case of continuous beams on rigid supports.

The continuous beam considered is shown in Fig. 36. As in the case of continuous beams on rigid supports, consecutive supports are numbered from left to right, starting with 1 at the extreme left end and terminating with z at the extreme right end.

26. Basic Relations.

Figure 37 shows the extreme deflected position of spans $j-1$ and j of a continuous beam undergoing steady-state forced vibrations. The vibrations are assumed to be maintained by an exciting moment applied at the extreme right end of the beam.

The forces acting at the ends of each span and the deflections and rotations at the supports are indicated in their positive directions. In addition to the symbols used previously, the symbol δ is used to designate the deflection of the beam at a support; the stiffness of a deflectional spring is denoted by D, while that of a rotational spring is denoted by R.

Since the beam is continuous at the supports,

$$(\theta_j)_L = (\theta_j)_R = \theta_j\,, \quad \text{and} \quad (\delta_j)_L = (\delta_j)_R = \delta_j\,. \tag{43}$$

Also, since no external moment or force acts at the joint, and since the internal moments and forces must be in equilibrium,

$$\bar{M}_j = (M_j)_L + (M_j)_R + R_j\theta_j = 0\,, \tag{44}$$

$$\bar{F}_j = (V_j)_L + (V_j)_R + D_j\delta_j = 0\,. \tag{45}$$

The moments and shears acting at the ends of a span may be expressed in terms of the rotations and deflections of the ends of the span. For example, the moment or shear at the left end of span j may be obtained by the addition of the following four component effects:

(1) Moment or shear produced at the left end, when that end is rotated by θ_j without deflection and the right end is held fixed.

(2) Moment or shear produced at the left end, when that end is held fixed while the right end is rotated by θ_{j+1} without deflection.

(3) Moment or shear produced at the left end, when that end is displaced by δ_j without rotation and right end is held fixed.

(4) Moment or shear produced at the left end, when that end is held fixed and the other end is deflected by δ_{j+1} without rotation.

The direct superposition of these effects is justified by the fact that, for a given frequency of vibration, the moments and shears are linear functions of the distortions. Thus,

$$(M_j)_R = K_j\theta_j + (kK)_j\theta_{j+1} + Q_j\delta_j - (qQ)_j\delta_{j+1}\,, \tag{46a}$$

$$(V_j)_R = T_j\delta_j - (tT)_j\delta_{j+1} + Q_j\theta_j + (qQ)_j\theta_{j+1}\,. \tag{47a}$$

Similarly,

$$(M_j)_L = K_{j-1}\theta_j + (kK)_{j-1}\theta_{j-1} - Q_{j-1}\delta_j + (qQ)_{j-1}\delta_{j-1}\,, \tag{46b}$$

$$(V_j)_L = T_{j-1}\delta_j - (tT)_{j-1}\delta_{j-1} - Q_{j-1}\theta_j - (qQ)_{j-1}\theta_{j-1}\,. \tag{47b}$$

Substituting Eqs. (46a) and (46b) in Eq. (44) and Eqs. (47a) and (47b) in Eq. (45), one obtains

$$\bar{M}_j = (qQ)_{j-1}\delta_{j-1} + (kK)_{j-1}\theta_{j-1} + (Q_j - Q_{j-1})\delta_j + (K_{j-1} + R_j + K_j)\theta_j - (qQ)_j\delta_{j+1} + (kK)_j\theta_{j+1} = 0\,, \tag{48a}$$

$$\bar{F}_j = -(tT)_{j-1}\delta_{j-1} - (qQ)_{j-1}\theta_{j-1} + (T_{j-1} + D_j + T_j)\delta_j + (Q_j - Q_{j-1})\theta_j - (tT)_j\delta_{j+1} + (qQ)_j\theta_{j+1} = 0\,. \tag{49}$$

Eliminating θ_{j+1} from Eq. (49) by use of Eq. (48a), one obtains

$$[\eta_j(qQ)_{j-1} + (tT)_{j-1}]\delta_{j-1} + [\eta_j(kK)_{j-1} + (qQ)_{j-1}]\theta_{j-1} + [\eta_j\bar{Q}_j - \bar{T}_j]\delta_j + [\eta_j\bar{K}_j - \bar{Q}_j]\theta_j - [\eta_j(qQ)_j - (tT)_j]\delta_{j+1} = 0,$$

$$\text{in which} \qquad \eta_j = \frac{(qQ)_j}{(kK)_j}\,, \tag{50}$$

$$\bar{K}_j = K_{j-1} + R_j + K_j\,,$$
$$\bar{T}_j = T_{j-1} + D_j + T_j\,,$$
$$\bar{Q}_j = Q_j - Q_{j-1}\,.$$

Equation (50) expresses the deflection at support j+1 in terms of the deflections and rotations at the two preceding supports. The rotation at support j+1 may be obtained in terms of the deflection δ_{j+1} and the distortions at supports j and j-1 from Eq. (48a), which may be rewritten as

$$-(qQ)_{j-1}\delta_{j-1} - (kK)_{j-1}\theta_{j-1} - \bar{Q}_j\delta_j - \bar{K}_j\theta_j + (qQ)_j\delta_{j+1} - (kK)_j\theta_{j+1} = 0\,. \tag{48b}$$

Equation (50) assures equilibrium of shears, while Eq. (48) assures equilibrium of moments for support j. Of course, both equations satisfy the conditions of continuity. If support j is rigid, Eq. (50) is satisfied automatically, and need not be used. If $\delta_{j-1} = \delta_j = \delta_{j+1} = 0$, that is, when three consecutive supports are rigid, then Eq. (48) reduces to Eq. (19a).

The foregoing relations are applicable to intermediate supports only. For the end supports the following specialized relations must be used. First, the boundary conditions for the extreme left support are considered. Four different cases must be distinguished:

Case I. Both D_1 and R_1 are considered finite or zero in this case. Then, the equilibrium condition for moments at support 1 is

$$\bar{M}_1 = Q_1\delta_1 + (R_1+K_1)\theta_1 - (qQ)_1\delta_2 + (kK)_1\theta_2 = 0. \tag{51a}$$

The corresponding condition for shears is

$$\bar{F}_1 = (D_1+T_1)\delta_1 + Q_1\theta_1 - (tT)_1\delta_2 + (qQ)_1\theta_2 = 0. \tag{52a}$$

Eliminating θ_2 from Eq. (52), one obtains an equation similar to Eq. (50),

$$[\eta_1 Q_1 - (D_1+T_1)]\delta_1 + [\eta_1(R_1+K_1) - Q_1]\,\theta_1 - [\eta_1(qQ)_1 - (tT)_1]\delta_2 = 0. \tag{53}$$

From this equation, δ_2 may be determined in terms of θ_1 and δ_1. With δ_2 determined, θ_2 also may be computed from Eq. (51a) in terms of θ_1 and δ_1.

Case II. D_1 is considered infinite (rigid deflectional support) and R_1 finite or zero. Then, the equilibrium condition for moments is expressed by the equation

$$\bar{M}_1 = (R_1+K_1)\theta_1 - (qQ)_1\delta_2 + (kK)_1\theta_2 = 0. \tag{51b}$$

No equilibrium equation for shears need be written, since at a rigid support this is satisfied automatically.

Case III. D_1 is considered finite or zero, but R_1 infinite (clamped end). In this case, it is necessary to write only one equilibrium equation for shears; this is

$$\bar{F}_1 = (D_1+T_1)\delta_1 - (tT)_1\delta_2 + (qQ)_1\theta_2 = 0. \tag{52b}$$

Case IV. Both D_1 and R_1 are considered infinite in this case (fixed end). The relation between the moment or shear at the fixed end and the distortions at the second support are

$$M_1 = (kK)_1\theta_2 - (qQ)_1\delta_2 \,, \tag{54}$$

$$V_1 = (qQ)_1\theta_2 - (tT)_1\delta_2 \,. \tag{55}$$

It should be noted that Eqs. (53), (51b), (52b), and (54) or (55) involve only three unknown quantities.

The conditions that must be satisfied at the right end of the beam are as follows:

For Case I.

$$\bar{M}_z = (K_{z-1} + R_z)\theta_z + (kK)_{z-1}\theta_{z-1} - Q_{z-1}\delta_z + (qQ)_{z-1}\delta_{z-1} = 0 \,, \tag{56}$$

$$\bar{F}_z = (T_{z-1} + D_z)\delta_z - (tT)_{z-1}\delta_{z-1} - Q_{z-1}\theta_z - (qQ)_{z-1}\theta_{z-1} = 0 \,. \tag{57}$$

For Case II.

$$\delta_z = 0 \,, \tag{58}$$

$$\bar{M}_z = (K_{z-1} + R_z)\theta_z + (kK)_{z-1}\theta_{z-1} + (qQ)_{z-1}\delta_{z-1} = 0 \,. \tag{59}$$

For Case III.

$$\theta_z = 0 \,, \tag{60}$$

$$\bar{F}_z = (T_{z-1} + D_z)\delta_z - (tT)_{z-1}\delta_{z-1} - (qQ)_{z-1}\theta_{z-1} = 0 \,. \tag{61}$$

For Case IV.

$$\theta_z = 0 \,, \tag{62}$$

$$\delta_z = 0 \,. \tag{63}$$

27. Outline of Procedure.

The rotation and deflection of the beam at support j may be written as

$$\theta_j = \theta_j' u + \theta_j'' v \quad ,$$
$$\delta_j = \delta_j' u + \delta_j'' v \quad , \qquad (64a)$$

where u and v are dimensionless parameters which represent two of the three unknowns in the equations expressing the boundary conditions for the left end of the beam. θ_j' and δ_j' are, respectively, the rotation and deflection at support j when u = 1.00 and v = 0, and θ_j'' and δ_j'' are the corresponding rotation and deflection when u = 0 and v = 1.00. Since the natural frequencies of a system depend on the relative values of the deflection, any arbitrary amplitude consistent with the actual boundary condition may be chosen for either u or v. For convenience, the following values are selected:

For Case I:	$u = \theta_1 = 1.00$	and	$v = \dfrac{\delta_1}{L_r}$
For Case II:	$u = \theta_1 = 1.00$	and	$v = \theta_2$
For Case III:	$u = \dfrac{\delta_1}{L_r} = 1.00$	and	$v = \theta_2$
For Case IV:	$u = M_1 \dfrac{L_r}{E_r I_r} = 1.00$	and	$v = \theta_2$.

The details of the procedure are:

1. For some reference span of the beam, say span r, assume a value of λ_r; this is equivalent to assuming a frequency of vibration
$$\omega = \frac{\lambda_r^2}{L_r^2}\sqrt{\frac{E_r I_r}{m_r}} \quad .$$
2. From Eq. (23) compute the λ values for the remaining spans of the beam.
3. From Table I in Appendix A and the λ values computed in the previous step, determine the stiffnesses and the product of the stiffnesses and the carry-over factors for each span.
4. Identify the parameters u and v.
5. Consider that u = 1.00 and v = 0. Progressing from support

to support across the beam, determine the deflections and rotations at all supports. Denote these by θ' and δ'. If necessary, evaluate also the unbalanced moment or shear at the extreme right hand support. Denote this by $\bar{M}_s'$ or $\bar{F}_s'$. The distortions of the second support are determined from the appropriate expressions given in Eqs. (51) through (55). The distortions of the remaining supports are evaluated by the repeated application to each support of Eqs. (50) and (48b).

6. Repeat the preceding step by considering that u = 0 and v = 1.00. Denote the resulting distortions by θ'' and δ''. If necessary, determine also the magnitude of the unbalanced moment or shear at the extreme right hand end. Denote this by $\bar{M}_s''$ or $\bar{F}_s''$.

7. The actual distortions at a support, say at support j, are

$$\begin{aligned} \theta_j &= \theta_j' + \theta_j'' v \ , \\ \delta_j &= \delta_j' + \delta_j'' v \ . \end{aligned} \tag{64b}$$

Similarly the total unbalanced moment or shear at support z is

$$\begin{aligned} \bar{M}_z &= \bar{M}_z' + \bar{M}_z'' v \ , \\ \bar{F}_z &= \bar{F}_z' + \bar{F}_z'' v \ . \end{aligned} \tag{65}$$

8. From one of the two equations expressing the boundary conditions for the right end of the beam, determine the unknown parameter v.

9. Evaluate the second boundary equation.

10. Repeat steps 1 through 9 for different assumed values of λ_r and plot the value calculated in step (9) against the assumed values of λ_r. The values of λ_r for which the ordinates of the resulting curve are equal to zero represent the natural frequencies of the beam.

28. Effect of Various Intermediate Constraints.

The foregoing procedure can be modified readily to include concentrated rigid masses, concentrated sprung masses, intermediate rigid supports, flexible supports for which the stiffness depends on the frequency of vibration, and a continuous elastic subgrade of the Winkler type. For convenience, these effects are discussed separately.

Rigid Concentrated Masses. Assume that at station j the beam has no deflectional support, instead it carries a concentrated mass $\bar{m}_j$ as shown in Fig. 38a. Assume that the beam undergoes a steady-state forced vibration with a frequency ω. Then for a downward deflection δ_j, the force exerted on the beam by the mass is $\bar{m}_j \omega^2 \delta_j$, downward. Had the beam been elastically supported against deflection at that point, the corresponding force would have been $D_j \delta_j$, upward. Thus, the effect of the concentrated mass is equivalent to that of a deflectional spring of stiffness

$$(D_j)_{eq.} = -\bar{m}_j \omega^2 = -\lambda_r^4 \frac{\bar{m}_j}{m_r L_r} \frac{E_r I_r}{L_r^3} . \tag{66a}$$

If the mass is applied at a point on the beam that is supported by a deflectional spring of stiffness D_j, as shown in Fig. 38b, the stiffness of the equivalent spring is

$$(D_j)_{eq.} = D_j - \bar{m}_j \omega^2 . \tag{66b}$$

After the concentrated mass has been replaced by the equivalent deflectional spring, the natural frequencies of the system may be determined in the usual manner. It must be noted, however, that the stiffness of the equivalent spring is a function of the assumed frequency of vibration and must be evaluated for each cycle of the procedure.

Concentrated Sprung Masses. Assume that the mass $\bar{m}_j$ is spring borne as shown in Fig. 38c. Let S_j be the stiffness of the spring. The influence of the flexible support may be taken into account by replacing the actual mass by an equivalent rigid mass of magnitude

$$(\bar{m}_j)_{eq.} = \frac{\bar{m}_j}{1 - \frac{\bar{m}_j \omega^2}{S_j}} . \tag{67}$$

The amplification factor $\frac{1}{1-\frac{\bar{m}_j\omega^2}{S_j}}$ is determined as follows: Let δ_j^* be the deflection of the mass while δ_j is, as before, the deflection of the elastic axis of the beam. The forces acting on the mass are the inertia force and the spring reaction; these must be in equilibrium; therefore,

$$\bar{m}_j \omega^2 \delta_j^* = S_j(\delta_j^* - \delta_j) ,$$

whence, the amplification factor is

$$\frac{\delta_j^*}{\delta_j} = \frac{1}{1 - \frac{\bar{m}_j \omega^2}{S_j}}$$

It should be noted that, for vibration frequencies greater than the natural frequency of the spring borne mass, the amplification factor becomes negative, and Eq. (67) results in a negative equivalent mass.

Rigid Intermediate Supports. Consider the beam shown in Fig. 38d for which support 4 is rigid instead of flexible. Let the outer supports be flexible. The natural frequencies of this beam may be determined by a combination of the principles that have been described. As usual, the procedure may be initiated at support 1 by taking $\theta_1 = 1.00$. For the portion of the beam between supports 1 and 4 the distortions at the supports may be expressed as the sum of a constant term and a term involving $\frac{\delta_1}{L_r}$ as unknown. The magnitude of this unknown may be determined from the condition that $\delta_4 = 0$. For the portion of the beam between supports 4 and 7, θ_5 may be selected as the new unknown, and the distortions at the supports may be computed by the repeated application of Eqs. (50) and (48b). As usual, the value of θ_5 may be determined from one of the two equations describing the boundary condition for the right end of the beam. The natural frequencies are those frequencies for which the second boundary condition is satisfied identically.

In the application of this procedure, one must, in general, change temporary unknowns as many times as there are rigid intermediate supports.

Abrupt changes in the magnitude of the distributed mass or in the flexural rigidity within a span may be handled by assuming that the beam is supported by a flexible support of zero stiffness at the point of the discontinuity.

Supports Having Mass. Consider a construction consisting of a cross beam supported on a series of longitudinal girders as shown in Fig. 39a. For simplicity's sake, the connections between the beam and the girders are considered hinged. It is assumed that the mass of the supports cannot be neglected; consequently, the stiffnesses of the supports are functions of the frequency of vibration.

The natural frequencies of this system are the same as those of the cross beam assuming that it is supported on deflectional springs as shown in Fig. 39b. The stiffness of each spring is equal to the stiffness of the corresponding girder. The stiffness of each girder is, in turn, equal to the magnitude of a harmonically varying concentrated force which, when applied at the point of intersection of the girder and the beam, will deflect the girder by a unit amount. The magnitude of this force may be calculated in terms of the numerical values tabulated in Table I, Appendix A. As an example, consider a girder fixed at both ends. For the point of application of the force, the conditions of equilibrium and continuity are expressed by the equations

$$(Q_2 - Q_1)\,w_x + (K_1 + K_2)\,\theta_x = 0 \ ,$$
$$(T_1 - T_2)\,w_x + (Q_2 - Q_1)\,\theta_x = \bar{F}_x \ .$$

where the subscripts 1 and 2 refer, respectively, to the portions of the girder to the left and the right of the concentrated force, and the subscript x refers to the point of application of the force. From the simultaneous solution of these equations one obtains

$$\frac{\bar{F}_x}{w_x} = D = T_1 + T_2 - \frac{(Q_2 - Q_1)^2}{K_1 + K_2} \quad . \tag{68a}$$

For a force at midspan, this equation reduces to

$$D = 2T_1 \quad . \tag{68b}$$

The stiffness of a girder with different boundary conditions may be determined in a similar manner.

With the expression for the effective stiffness of the deflectional spring available, the natural frequencies of the system may be determined in the usual manner. It is only necessary to evaluate the stiffness of the springs for each assumed frequency of vibration and to use this value in the calculations.

Continuous Elastic Subgrade. Consider that the beam is supported along a portion of its length on a continuous elastic subgrade of the Winkler type, as shown in Fig. 39c. Let d be the foundation modulus, defined as the force per unit length necessary to compress the foundation by unit amount. The mass of the foundation is assumed to be negligible.

The natural frequencies of such a system may be determined by the same procedure, except that the influence of the elastic foundation must be taken into account in evaluating the stiffnesses and the carry-over factors of the elastically supported spans.

A vibrating beam which is not supported by any foundation, is acted upon only by distributed inertia forces, the intensity of which is

$$m\omega^2 w$$

The intensity of the reactive forces produced by the elastic foundation is dw. These forces act opposite to the inertia forces and, in effect, reduce the intensity of the latter to

$$m\omega^2 w - dw \;.$$

Now, the coefficients of dynamic stiffnesses and the carry-over factors depend solely on the dimensionless parameter $\lambda = \sqrt[4]{\frac{m\omega^2}{EI}}\,L$; therefore, the effect of the foundation can be taken into account by replacing in this expression the quantity $m\omega^2$ by the reduced value of $m\omega^2 - d$.

If $m\omega^2 - d$ is positive, λ is real, and the various stiffness and carry-over factors may be obtained directly from Table I in Appendix A. If, however, $m\omega^2 - d$ is negative, λ is imaginary and the values of Table I can no longer be used. In this case, the expressions for dynamic stiffness and dymanic carry-over factors may be obtained from the expressions for static stiffness and static carry-over factors of a bar on elastic foundation. It is only necessary to replace in the latter expressions the quantity $\underline{d}$ by the reduced value $d - m\omega^2$. These expressions may be obtained readily from solutions given by Hetényi (31).

Consider now the case for which the foundation extends along the entire length of the beam as shown in Fig. 39d. The modulus of the foundation may differ from one span to the other. Let d_j be the foundation modulus for span $\underline{j}$. In what follows, it is shown that, if the natural frequencies of the system without the subgrade are known, under certain conditions, the natural frequencies of the system with the subgrade may be computed directly.

Assume that the beam without subgrade and the beam with the subgrade are in a steady-state of vibration. Let ω be the circular vibration frequency of the beam without the subgrade and ω_s be the frequency of the beam with the subgrade. Then, for span $\underline{r}$ the λ value of the beam without the subgrade is

$$\lambda_r = \sqrt[4]{\frac{m_r\, \omega^2}{E_r\, I_r}}\; L_r \;, \tag{12}$$

and the value for the beam with the subgrade is

$$(\lambda_r)_s = \sqrt{\frac{m_r}{E_r\, I_r}\left(\omega^2 - \frac{d_r}{m_r}\right)}\; L_r \;. \tag{69}$$

For any other span $\underline{j}$, the values for the two cases are

$$\lambda_j = \sqrt[4]{\frac{m_j\, \omega^2}{E_j\, I_j}}\; L_j \;.$$

$$(\lambda_j)_s = \sqrt[4]{\frac{m_j}{E_j\, I_j}\left(\omega^2 - \frac{d_j}{m_j}\right)}\; L_j$$

and if

$$\frac{d_r}{m_r} = \frac{d_j}{m_j} = constant \;, \tag{70}$$

$$\frac{(\lambda_j)_s}{(\lambda_r)_s} = \sqrt[4]{\frac{m_j}{m_r}\;\frac{E_r I_r}{E_j\, I_j}} \times \frac{L_j}{L_r} = \frac{\lambda_j}{\lambda_r} \;.$$

Now, if λ_r and $(\lambda_r)_s$ are equal, the numerical calculations involved in carrying out a cycle of the procedure will be identical for the two systems. Therefore, if λ_r corresponds to a natural frequency of the beam without the subgrade, then $(\lambda_r)_s$ will correspond to a natural frequency of the beam with the subgrade. Equating Eqs. (12) and (69), one obtains the following relationship between the natural frequencies of the two systems.

$$(\omega_s)_N = \sqrt{\omega_N^2 + (d_j/m_j)} \;. \tag{71}$$

The subscript $\underline{N}$ designates natural frequencies. Equation (71) can be applied to beams having any number of spans and any boundary conditions, provided that the relationship given in Eq. (70) is satisfied.

29. Determination of Modes of Vibration.

After the distortions of the beam at the ends of a span have been evaluated, the deflection configuration of the span may be determined by superimposing the following four component configurations:

(1) Deflection configuration produced by the rotation of the left end of the span when that end is restrained against deflection and the right end is fixed.

(2) Deflection configuration produced by the rotation of the right end of the span when that end is restrained against deflection and the left end is fixed.

(3) Deflection configuration produced by the deflection of the left end of the span when that end is restrained against rotation and the right end is fixed, and

(4) Deflection configuration produced by the deflection of the right end of the span when that end is restrained against rotation and the left end is fixed.

Effects (1) and (2) can be obtained readily by multiplying the end rotations by the appropriate values given in Table II, paying proper attention to signs. Effects (3) and (4) may be calculated from Eq. (B-34) in Appendix B. These effects may also be expressed in terms of the quantities given in Table I as follows: Consider a clamped ended uniform beam, the left end of which is displaced by unit amount. The deflection amplitude at a point, a distance x from the end being displaced, may be obtained by writing for the point the two equilibrium equations (see Eqs. 48a and 49),

$$\bar{M}_x = (Q_2 - Q_1) w_x + (K_1 + K_2)\theta_x + (qQ)_1 = 0 ,$$
$$\bar{F}_x = (T_1 + T_2) w_x + (Q_2 - Q_1)\theta_x - (tT)_1 = 0 .$$

The subscripts 1 and 2 denote, respectively, the left and the right portions of the beam. Eliminating from the first of the foregoing two equations the unknown θ_x, and solving for the deflection w_x, one obtains

$$w_x = \frac{(tT)_1 + (qQ)_1 \dfrac{Q_2 - Q_1}{K_1 + K_2}}{(T_1 + T_2) - \dfrac{(Q_2 - Q_1)^2}{K_1 + K_2}} . \tag{72a}$$

The deflection at the same point of the beam due to a unit deflection at the right end may be obtained from Eq. (72a) by interchanging the subscripts 1 and 2. When $x = \frac{L}{2}$, this equation reduces to

$$w_x = \frac{(tT)_1}{2T_1} . \tag{72b}$$

Note that for $\lambda = 0$ this equation yields $\frac{1}{2}$, which is, of course, the correct value for the static deflection at midspan.

30. Illustrative Example.

Example 7. In order to illustrate the details of the procedure and present a tabular scheme for arranging the computations, we shall calculate the first three natural frequencies of the hypothetical continuous beam shown in Fig. 40. The dimensions of this beam and the stiffnesses of the rotational and deflectional restraints are shown in Fig. 40. The various quantities are expressed in terms of the physical properties of a reference span r. In this particular case, we choose r = 4.

It is convenient to express the stiffnesses and the carry-over effects for each span in terms of the EI and L of the reference span r. To do this, the values obtained from Table I in Appendix A must be multiplied by certain dimensionless factors as follows: The coefficients of K and kK for a span j must be multiplied by the factor α_j (Eq. 24). The coefficients of Q and qQ must be multiplied by

$$\beta_j = \frac{E_j I_j}{E_r I_r} \frac{L_r^2}{L_j^2} , \tag{73}$$

and the coefficients of T and tT by

$$\gamma_j = \frac{E_j I_j}{E_r I_r} \frac{L_r^3}{L_j^3} \quad . \tag{74}$$

These relations may be verified readily.

For the beam considered, the ratio λ_j / λ_r and the factors α_j, β_j and γ_j are evaluated in Table 2A. This table is independent of the frequency of vibration and is computed but once.

Table 2B presents one complete cycle of the procedure for a trial value of $\lambda_4 = \lambda_r = 1.30$. This value, shown encircled in the r-th line of Column (1), corresponds to a circular frequency of vibration $\omega = \frac{1.69}{L^2}\sqrt{\frac{EI}{m}}$. Columns (1) through (18) in this table are conveniently filled in the following order: (2), (9), (1), (3 and 8), (4 and 7), (10 and 12), (5), (6), (11), (13), (14), (15), (16), (17), and (18). It should be noted that all quantities in Columns (2) through (12) are expressed in terms of the EI and L of the reference span r. The value of -5.7122 in Column (9), which represents the stiffness of an equivalent deflectional spring having the same effect as the concentrated mass $\bar{m}_3$, is computed from Eq. (66a). The partial distortions δ', θ', δ'', and θ'' are evaluated in Columns (19) through (22). The parameter v is determined from the condition that

$$\frac{\delta_5}{L} = -87.5781 + 3436.36\ v = 0 \ ;$$

whence,

$$v = 0.0254857 \ .$$

The total distortions at supports 4 and 5 are given in Columns (23) and (24). The magnitude of the exciting moment

$$\bar{M}_5 = \left[4.473(2.327) + 2.021(-1.177) + 6.089(0.3438)\right] \frac{EI}{L} = 10.12\ \frac{EI}{L} \ .$$

Under the action of this moment the distortions at the extreme left support are

$$\theta_1 = 1.00 \quad \text{and} \quad \delta_1 = 0.0254857L.$$

Had δ_1/L, instead of θ_1, been taken equal to unity, the magnitude of the exciting moment would have been

$$\bar{M}_5 = \frac{10.12}{0.0254857}\,\frac{EI}{L} = 397.1\,\frac{EI}{L} .$$

Since $\bar{M}_5$ is different from zero, the assumed value of $\lambda_4 = 1.30$ does not correspond to a natural frequency. By repeating several such cycles of computation for different values of λ_4, the curves shown in Fig. 41 were obtained. The solid curve in this figure shows the moment $\bar{M}_5$ necessary to produce a rotation $\theta_1 = 1.00$, while the dashed curve shows the moment $\bar{M}_5$ necessary to produce a deflection $\delta_1 = 1.00L$. The λ intercepts of these curves correspond to natural frequencies of vibration. The first three critical values are

$$(\lambda_4)_1 = 0.85 , \quad (\lambda_4)_2 = 2.01 , \quad (\lambda_4)_3 = 2.55 .$$

It should be noted that the distortions δ and θ are obtained as small differences between large quantities. It becomes necessary, therefore, to use in the computations a relatively large number of significal figures, particularly if the higher natural frequencies of the beam are to be evaluated. It is recommended that at least 6 or 7 significant figures be retained throughout the computations.

TABLE 2. CALCULATIONS FOR EXAMPLE 7

TABLE A

Span	(1) $\frac{m_j}{m_r}$	(2) $\frac{E_j I_j}{E_r I_r}$	(3) $\sqrt[4]{\frac{(1)}{(2)}}$	(4) $\frac{L_j}{L_r}$	(5) $(4)^2$	(6) $(4)^3$	(7) $\alpha_j = \frac{(2)}{(4)}$	(8) $\beta_j = \frac{(2)}{(5)}$	(9) $\gamma_j = \frac{(2)}{(6)}$	(10) $\frac{\lambda_j}{\lambda_r} = (3)(4)$
1	0.60	1.00	0.880112	1.30	1.69	2.8561	0.7692308	0.5917160	0.3501278	1.14415
2	1.30	1.50	0.964857	0.60	0.36	0.1296	2.500000	4.166667	11.57407	0.57891
3	1.30	1.50	0.964857	0.70	0.49	0.2401	2.142857	3.061224	6.247397	0.67540
4=r	1.00	1.00	1.00	1.00	1.00	1.00	1.000000	1.000000	1.000000	1.0000

Table 2. – Calculations for Example 7 – Continued

Table B

Span or Support	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)
	λ_j	$R_j \frac{L_r}{E_r I_r}$	$K_j \frac{L_r}{E_r I_r}$	$Q_j \frac{L_r^2}{E_r I_r}$	$\bar{Q}_j \frac{L_r^2}{E_r I_r}$ $= (4)_j - (4)_{j-1}$	$\bar{K}_j \frac{L_r}{E_r I_r}$ $= (3)_{j-1} + (2)_j + (3)_j$	$(qQ)_j \frac{L_r^2}{E_r I_r}$	$(kK)_j \frac{L_r}{E_r I_r}$	$D_j \frac{L_r^3}{E_r I_r}$	$T_j \frac{L_r^3}{E_r I_r}$	$\bar{T}_j \frac{L_r^3}{E_r I_r}$ $= (10)_{j-1} + (9)_j + (10)_j$	$(tT)_j \frac{L_r^3}{E_r I_r}$	η_j $= (7)/(8)$
1	1.49		3.040508	3.396416	3.396416	3.040508	3.641616	1.565839	3.00	3.557417	6.557417	4.426246	2.325664
2	0.75	0.70	9.992462	24.93091	21.53449	13.73297	25.04084	5.005655		137.52819	141.08561	139.36002	5.002510
3	0.88		8.559177	18.27110	−6.65981	18.551639	18.42425	4.294905	−5.7122	73.57634	205.39233	75.45119	4.289792
4=r	1.30		3.972666	5.849766	−12.42133	12.531843	6.088995	2.020530	1.50	10.93617	86.01251	12.36992	3.013563
5		0.50				4.472666			∞				

(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)
Coef. of δ_{j-1} in Eq. (50) $= (13)_j (7)_{j-1} + (12)_{j-1}$	Coef. of θ_{j-1} in Eq. (50) $= (13)_j (8)_{j-1} + (7)_{j-1}$	Coef. of δ_j in Eq. (50) $= (13)_j (5)_j - (11)_j$	Coef. of θ_j in Eq. (50) $= (13)_j (6)_j - (5)_j$	Coef. of θ_{j+1} in Eq. (50) $= (13)_j (7)_j - (12)_j$	$\delta'_{j+1}/L_r = [(14)_j (19)_{j-1} + (15)_j (20)_{j-1} + (16)_j (19)_j + (17)_j (20)_j] \div (18)_j$	$\theta'_{j+1} = [-(7)_{j-1} (19)_{j-1} - (8)_{j-1} (20)_{j-1} - (5)_j (19)_j - (6)_j (20)_j + (7)_j (19)_{j+1}] \div (8)_j$	$\delta''_{j+1}/L = [(14)_j (21)_{j-1} + (15)_j (22)_{j-1} + (16)_j (21)_j + (17)_j (22)_j] \div (18)_j$	$\theta''_{j+1} = [-(7)_{j-1} (21)_{j-1} - (8)_{j-1} (22)_{j-1} - (5)_j (21)_j - (6)_j (22)_j + (7)_j (21)_{j+1}] \div (8)_j$	$\delta_j/L = (19)_j + v(21)_j$	$\theta_j = (20)_j + v(22)_j$
		1.341505	3.674784	4.042929	0	1.00	1.00	0		
22.64347	11.47474	−33.3591	47.16483	−14.09297	0.9089410	0.1721161	0.3318151	−1.397380		
246.7800	46.51406	−233.9615	86.24248	3.58501	0.7612935	−0.886928	3.85531	20.9649		
130.9738	31.36721	−123.44497	50.18683	5.97965	−6.21754	−27.1605	257.449	1019.52	0.3438	−1.177
					−87.5781	−138.745	3436.36	5535.33	0	2.327

$$v = \frac{\delta_1}{L} = \frac{87.5781}{3436.36} = 0.0254857$$

$$\bar{M}_5 = \left[4.473(2.327) + 2.021(-1.177) + 6.089(0.3438)\right] \frac{EI}{L}$$

$$\underline{\bar{M}_5 = 10.12 \frac{EI}{L}}$$

VI. APPLICATION OF METHOD TO FRAMES WITH SIDESWAY

31. Symmetrical, Single-Bay, Multi-Story Frames.

The method described in the preceding Chapter can also be used to determine the natural frequencies of lateral vibration of frames for which the joints are free to translate. Though this method can be applied to frames of any complexity, it can efficiently be used only for symmetrical, single-bay, multi-story frames. For such frames, the details of application of the procedure are similar to those for a continuous beam on flexible supports.

Consider the symmetrical frame shown in Fig. 42a. Because of symmetry, the end rotations of each girder are algebraically equal, and the midpoints of the girders are points of inflection. Consequently, it is possible to consider in the analysis only one half of the structure as shown in Fig. 42b. In this figure, the right ends of the girders can rotate and slide freely in the horizontal direction but can not deflect in the vertical direction. The influence of these horizontal members may be represented by a set of concentrated masses and concentrated rotational springs, attached at the points of intersection of the horizontal members and the main column as shown in Fig. 42c. The magnitude of each concentrated mass in Fig. 42c is equal to the total distributed mass of the corresponding girder in Fig. 42b. Similarly, the stiffness of each rotational spring in Fig. 42c is equal to the flexural stiffness $\underline{K}''$ of the corresponding girder in Fig. 42b.

The natural frequencies of this system may be determined by the procedure described in Sections 27 and 28. It must be borne in mind, however, that the stiffnesses of the rotational restraints are not constant in this case, but rather depend on the frequency of vibration, and must be evaluated for each cycle of the procedure.

VII. EXTENSION OF METHOD TO CONTINUOUS PLATES

32. General.

This chapter is concerned with the determination of the natural frequencies of flexural vibration of rectangular plates which are simply supported along two opposite edges and which, in one direction, are continuous over rigid supports transverse to the simply supported edges. The plate may have any number of panels of arbitrary length. The mass per unit of area and the flexural rigidity of the plate may vary from one panel to the other, but, in any one panel, these quantities are assumed to remain constant.

The method of analysis is similar to that described in Chapter III for the case of continuous beams on rigid supports. The assumptions made in the analysis are those embodied in the ordinary flexure theory of medium thick plates composed of an elastic, homogeneous, and isotropic material. In addition, it is assumed that the supports can offer no torsional restraint and that no frictional forces or horizontal shearing forces may develop between the plate and the supports. Throughout this discussion, the effects of damping, rotatory inertia, and shearing deformation are disregarded.

33. Basis of the Method.

Figure 43 shows the type of the continuous plate considered. The coordinate axes are taken parallel to the edges as shown in Fig. 43. The sides parallel to the y-axis are assumed to be simply supported. All transverse supports are considered to be rigid. Along the extreme edges the plate may be hinged, fixed or subjected to a rotational elastic restraint of constant stiffness. For simplicity, it will first be assumed that the extreme right edge is either hinged or elastically restrained against rotation. A fixed edge will be treated separately at the end.

For the plate shown in Fig. 43, the solution of the differential equation for transverse vibration of plates reveals that, during free vibration, the deflections, rotations, shears, and bending moments along sections perpendicular to the simply supported edges are proportional to

$$\sin \frac{n\pi x}{a} \cos \omega_N t \quad . \tag{75}$$

In this expression, ω_N represents a circular natural frequency, t time, and a the width of the plate; n is an integer which designates the number of half-sine waves, alternately upward and downward, in the distribution of the deflection, slope, shear, or bending moment along the plate width. Corresponding to each value of n there is an infinite number of natural frequencies.

Consider now that the plate undergoes a steady-state forced vibration, such that the rotation of the plate at the extreme left support is

$$\theta_1(x,t) = \theta_1 \sin \frac{n\pi x}{a} \cos \omega t \ ,$$

where ω is the circular frequency of vibration, n is the same integer used in Eq. (75), and θ_1 is the maximum amplitude of slope at the left support. The magnitude of θ_1 is assumed to have a prescribed finite value. If support 1 is fixed instead of hinged or elastically restrained, θ_1 is equal to zero, and it becomes necessary to assume that the bending moment at that support is

$$M_1(x,t) = M_1 \sin \frac{n\pi x}{a} \cos \omega t \ .$$

M_1 is assumed to have a fixed value. In either case, the vibration of the plate is presumed to be maintained by an exciting couple applied at the extreme right hand support z. Then, the solution for steady-state forced vibration of the governing differential equation reveals (a) that the deflections, slopes, shears, and bending moments along sections perpendicular to the simply supported edges are proportional to, and in phase with, the rotation (or bending moment) at the extreme left edge, and (b) that the exciting moment may be expressed as

$$\bar{M}_z(x,t) = \bar{M}_z \sin\frac{n\pi x}{a}\cos\omega t \;.$$

Obviously, the magnitude of $\bar{M}_z$ depends upon the frequency of vibration. For a frequency equal to a natural frequency of the plate ($\omega = \omega_N$), $\bar{M}_z$ becomes equal to zero, but the deflection of the plate remains finite. The deflected surface of the plate, which represents a natural mode of vibration, is then expressed as

$$w(x,y,t) = Y_n \sin\frac{n\pi x}{a}\cos\omega t \;.$$

where, Y_n is a function of the y-coordinate only; its absolute value depends on the value assigned to the amplitude of slope (or bending moment) at support 1.

In the discussion thus far, it was assumed that the extreme right support of the plate was either hinged or elastically restrained. If support z is fixed, the criterion for a natural frequency is that $\theta_z = 0$.

34. Details of the Method.

The foregoing considerations suggest the following procedure for calculating the natural vibration frequencies of continuous plates having two opposite edges simply supported.

1. Assume that the amplitude of slope or bending moment at the extreme left edge of the plate is distributed sinusoidally, and that it has a fixed value. Since the natural frequencies of a system depend only on the relative values of the deflection, any arbitrary amplitude, consistent with the actual boundary condition, may be chosen. For a

hinged edge or for a partially fixed edge, take

$$\theta_1(x) = 1.00 \sin \frac{n\pi x}{a}$$

For a fixed end, $\theta_1 = 0$; therefore, take

$$M_1(x) = 1.00 \sin \frac{n\pi x}{a}$$

In the application of this procedure, it is necessary to consider a specific value of n in each case. Let $n = n_o$.

2. Choose a frequency of vibration and determine the rotations of the plate over the supports, and from these determine the magnitude of the exciting moment at the extreme right edge. (For a fixed edge the moment at that edge need not be evaluated). These quantities are determined by identically the same procedure that was used for continuous beams. The pertinent relations are reviewed in subsequent paragraphs.
3. Repeat the previous step for different frequencies and, from the results obtained, plot a curve of the variation of the exciting moment or of the slope at the right edge as a function of the frequency of vibration. If the right edge is hinged or elastically restrained, plot the variation of the exciting moment; then, the natural frequencies are those frequencies for which the exciting moment vanishes. If the right end is fixed, establish the variation of the rotation θ_z; the frequencies for which θ_z vanishes are natural frequencies.
4. For the natural frequencies determined in step (3), the deflection of the plate along sections perpendicular to the hinged edges has n_o half-sine waves. In general, steps (1) to (3) must be repeated for as many values of n as may be necessary for a given problem.

For a given frequency of vibration, the rotations of the plate over the supports and the magnitude of the exciting moment may be determined from the same equations that were used to analyze continuous beams on rigid supports, namely from Eqs. (19), (21), and (22). Equation (19) expresses the condition of equilibrium and continuity at an interior support of a continuous beam, by relating the rotations of the beam over three consecutive supports. Equations (21) and (22) express, respectively, the boundary conditions for the extreme left and the extreme right ends of the beam.

In the application of these equations to continuous plates, it is only necessary to interpret θ_j as the maximum amplitude of slope along the j-th support, and M_j as the maximum amplitude of moment at that support. In addition, it is necessary to replace the stiffness and carry-over factors for beams by equivalent quantities for plates. Such quantities are defined in the section that follows.

35. Elastic Constants for a Panel of a Plate.

Consider a rectangular plate simply supported on three edges and fixed on the fourth as shown in Fig. 44. Let edge f be subjected to a steady-state rotation without deflection, such that the magnitude of the rotation is given by the relation

$$\theta_f(x,t) = \theta_f \sin\frac{n\pi x}{a}\cos\omega t$$

The amplitude of the moment at edge f necessary to produce this rotation is proportional to $\sin\frac{n\pi x}{a}$ and may be written as

$$M_f = K\theta_f \ . \qquad (76)$$

The quantity K is a measure of the resistance to steady-state forced rotation of edge f of the plate, when edge g is fixed, and is referred to as the "dynamic flexural stiffness" of edge $\underline{f}$.

The amplitude of the moment induced at the fixed edge is also proportional to $\sin\frac{n\pi x}{a}$, and may be expressed as

$$M_g = kK\theta_f = kM_f \quad . \tag{77}$$

The quantity $\underline{k}$, which represents the ratio of the moment at the far fixed edge to that at the edge being rotated, is defined as the "dynamic flexural carry-over factor".

These quantities are strictly analogous to those defined for beams, and they are used in the analysis of plates in identically the same way as the quantities for beams are used in the analysis of continuous beams.

The quantity $\underline{K}$ may be expressed as

$$K = C_K \frac{N}{b} , \tag{78}$$

where $\underline{N}$ is the flexural rigidity of the plate per unit width and $\underline{b}$ is its span length in a direction parallel to the simply supported edges. The coefficient C_K is dimensionless and depends on the ratio $\frac{a}{nb}$ and the dimensionless parameter

$$\lambda^* = \frac{b^2}{\pi^2}\sqrt{\frac{\rho h \omega^2}{N}} \tag{79}$$

where ρ denotes the density and $\underline{h}$ the thickness of the plate. The carry-over factor $\underline{k}$ is dimensionless and depends on the same two quantities as the coefficient C_K.

The pertinent analytical expressions for $\underline{K}$ and $\underline{k}$ are given in Appendix B. For $\lambda^* = 0$, that is, when the plate does not vibrate, the values of $\underline{K}$ and $\underline{k}$ reduce to those given by Newmark (15).

Assume now that the plate is not vibrating but is instead subjected along its simply supported edges to uniformly distributed compressive forces p_x. Let

$$k' = \frac{b^2 P_x}{\pi^2 N} \quad . \tag{80}$$

Then, it can be shown* that the values of $\underline{K}$ and $\underline{k}$ of a vibrating plate for given values of $\frac{a}{nb}$ and λ^* are numerically equal to the stiffness and carry-over factor of an equivalent compressed plate having the same $\frac{a}{nb}$ value but a value of

$$k'_{eq.} = (\lambda^*)^2 \left(\frac{a}{nb}\right)^2 . \tag{81}$$

Numerical values of $\underline{K}$ and $\underline{k}$ for compressed plates have been tabulated by W. D. Kroll (32)+ as a function of $\frac{a}{nb}$ and $\underline{k}'$. It should be noted that Kroll defines stiffness as the moment necessary to produce a maximum rotation amplitude of 1/4 radian instead of one radian. Consequently, the stiffness values obtained from Kroll's tables must be multiplied by 4 to make them conform to the definition given in this report.

As a simple illustration, we shall determine the dynamic flexural stiffness and dynamic flexural carry-over factor for a rectangular plate having a ratio of sides a/b = 0.5 when n = 1 and $\lambda^* = 4.00$. The equivalent value of $\underline{k}'$,

$$k'_{eq.} = (16)(0.25) = 4.00 .$$

From page 14 of reference (32) we find

$$\frac{K}{4} = 2.54385 \frac{N}{b} \quad \text{and} \quad k = 0.126499 ,$$

whence

$$K = 10.1754 \frac{N}{b} \quad \text{and} \quad k = 0.126499 .$$

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*See Section 2c of Appendix B.

+Kroll uses the symbols λ for $\frac{a}{nb}$, $\underline{k}$ for $\underline{k}'$, $\underline{S}$ for $\underline{K}$, and $\underline{C}$ for $\underline{k}$.

36. Illustrative Example.

Example 8. This example involves a rectangular plate simply supported along two opposite edges and continuous over five equally spaced rigid supports as shown by the sketch in Fig. 45. The plate is simply supported at the extreme left edge and fixed at the extreme right edge. All panels are square. It is desired to calculate the first few natural frequencies of this structure.

The plate is analyzed in identically the same manner as the continuous beam considered in Example 1. In fact the expression for θ_5 is the same as that for the beam:

$$\theta_5 = \frac{k^4 - 8k^2 + 8}{k^4} \quad .$$

The carry-over factor $\underline{k}$, which depends on the parameters $\frac{a}{nb}$ and λ^*, was determined from the numerical values reported in reference (32) as described in the preceding section. In Fig. 45 θ_5 has been plotted as a function of $(\lambda^*)^2\left(\frac{a}{nb}\right)^2$ for values of n = 1 and n = 2. The values of $(\lambda^*)^2\left(\frac{a}{nb}\right)^2$ corresponding to natural frequencies are recorded on the figure.

The lowest circular natural frequency for n = 1 is

$$\omega_N = \sqrt{4.11}\,\frac{\pi^2}{b^2}\sqrt{\frac{N}{\rho h}} = \frac{20.01}{b^2}\sqrt{\frac{N}{\rho h}}$$

The lowest frequency ω_N for n = 2 is

$$\omega_N = 2\sqrt{6.29}\,\frac{\pi^2}{b^2}\sqrt{\frac{N}{\rho h}} = \frac{49.51}{b^2}\sqrt{\frac{N}{\rho h}}$$

For purposes of comparison, the corresponding circular natural frequencies of a square plate having (a) all four edges simply supported and (b) three edges simply supported and one fixed are given in the following. For edge condition (a):

when n = 1 $$\omega_N = \frac{19.74}{b^2}\sqrt{\frac{N}{\rho h}}\,,$$

when n = 2 $$\omega_N = \frac{49.35}{b^2}\sqrt{\frac{N}{\rho h}}\,.$$

For edge condition (b):

when n = 1 $$\omega_N = \frac{23.65}{b^2}\sqrt{\frac{N}{\rho h}}\,,$$

when n = 2 $$\omega_N = \frac{51.68}{b^2}\sqrt{\frac{N}{\rho h}}\,.$$

These values have been obtained from reference (33).

VIII SUMMARY

37. General.

A method has been presented in this report for the determination of the undamped natural frequencies and of the corresponding natural modes of flexural vibration of elastic structures. The method has been applied to continuous beams on rigid or flexible supports, to continuous frames without sidesway, to symmetrical single-bay multi-story frames for which the joints are free to translate, and to continuous plates having two opposite edges simply supported.

The method is a generalization of Holzer's method for calculating the natural frequencies of torsional vibration of shafts and, like Holzer's method, it has been reduced to a routine scheme of computation which, when repeated a sufficient number of times, will give the natural frequencies of the system to any desired degree of accuracy. The method is based on the fact that the exciting couple necessary to maintain a dynamical system in a steady-state of forced vibration with finite amplitudes becomes equal to zero at any one of the natural frequencies of the system.

Extensive tables of numerical values for the various quantities entering in the analysis are presented in Appendix A of this report. With these tables the calculations required in the application of the method to particular problems are simplified greatly. The tabulated values may be used also with other analytical techniques as well as for the analysis of the steady-state forced vibration of structures.

APPENDIX A

TABLE I

NUMERICAL VALUES OF DYNAMIC CARRY-OVER FACTORS, OF COEFFICIENTS FOR DYNAMIC STIFFNESSES, AND OF THEIR PRODUCTS FOR A UNIFORM BAR UNDERGOING STEADY-STATE FORCED VIBRATIONS

The various quantities are defined in Fig. 2, and are tabulated here as a function of the dimensionless parameter

$$\lambda = \sqrt[4]{\frac{m\omega^2}{EI}}\, L$$

in which m is the mass per unit of length of the bar; ω is the circular frequency of vibration; E is the modulus of elasticity of the material in the bar; I is the moment of inertia of the cross section of the bar about its centroidal axis; and L is the span length of the bar.

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
0	4.000000	2.000000	0.5000000	3.000000	6.000000	6.000000	1.000000	12.00000	12.00000	1.000000
0.10	3.999999	2.000001	0.5000003	2.999998	5.999995	6.000003	1.000001	11.99996	12.00001	1.000004
0.20	3.999985	2.000011	0.5000048	2.999970	5.999916	6.000050	1.000022	11.99941	12.00021	1.000067
0.30	3.999923	2.000058	0.5000241	2.999846	5.999576	6.000251	1.000113	11.99699	12.00104	1.000338
0.40	3.999756	2.000183	0.5000762	2.999512	5.998659	6.000792	1.000356	11.99049	12.00329	1.001068
0.50	3.999405	2.000447	0.5001861	2.998809	5.996726	6.001935	1.000869	11.97678	12.00804	1.002609
0.55	3.999128	2.000654	0.5002724	2.998256	5.995206	6.002833	1.001272	11.96601	12.01177	1.003824
0.60	3.998766	2.000926	0.5003859	2.997530	5.993210	6.004013	1.001802	11.95186	12.01667	1.005423
0.65	3.998299	2.001276	0.5005317	2.996598	5.990647	6.005527	1.002484	11.93369	12.02296	1.007481
0.70	3.997712	2.001716	0.5007153	2.995422	5.987419	6.007436	1.003343	11.91080	12.03089	1.010082
0.75	3.996985	2.002262	0.5009430	2.993966	5.983419	6.009801	1.004409	11.88244	12.04071	1.013320
0.76	3.996821	2.002385	0.5009944	2.993637	5.982516	6.010334	1.004650	11.87604	12.04293	1.014053
0.77	3.996650	2.002513	0.5010478	2.993295	5.981577	6.010890	1.004900	11.86939	12.04524	1.014816
0.78	3.996473	2.002646	0.5011034	2.992940	5.980601	6.011467	1.005161	11.86247	12.04764	1.015610
0.79	3.996288	2.002785	0.5011612	2.992570	5.979586	6.012067	1.005432	11.85527	12.05013	1.016436
0.80	3.996096	2.002928	0.5012212	2.992186	5.978532	6.012690	1.005713	11.84780	12.05272	1.017296
0.81	3.995897	2.003078	0.5012836	2.991787	5.977437	6.013337	1.006006	11.84004	12.05541	1.018189
0.82	3.995691	2.003233	0.5013483	2.991374	5.976302	6.014009	1.006309	11.83199	12.05820	1.019118
0.83	3.995476	2.003393	0.5014154	2.990944	5.975124	6.014706	1.006624	11.82364	12.06109	1.020082
0.84	3.995254	2.003560	0.5014850	2.990499	5.973902	6.015428	1.006951	11.81499	12.06409	1.021084

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
0.85	3.995024	2.003733	0.5015572	2.990038	5.972636	6.016177	1.007290	11.80601	12.06720	1.022124
0.86	3.994785	2.003912	0.5016319	2.989559	5.971324	6.016953	1.007641	11.79672	12.07043	1.023202
0.87	3.994538	2.004097	0.5017093	2.989064	5.969966	6.017756	1.008005	11.78709	12.07377	1.024321
0.88	3.994283	2.004289	0.5017895	2.988551	5.968560	6.018588	1.008382	11.77712	12.07722	1.025481
0.89	3.994018	2.004488	0.5018725	2.988021	5.967105	6.019449	1.008772	11.76681	12.08080	1.026684
0.90	3.993744	2.004693	0.5019583	2.987472	5.965600	6.020339	1.009176	11.75615	12.08450	1.027930
0.91	3.993461	2.004906	0.5020471	2.986904	5.964044	6.021260	1.009593	11.74512	12.08832	1.029221
0.92	3.993169	2.005125	0.5021388	2.986318	5.962435	6.022211	1.010025	11.73372	12.09228	1.030558
0.93	3.992867	2.005352	0.5022337	2.985711	5.960773	6.023194	1.010472	11.72195	12.09636	1.031941
0.94	3.992554	2.005586	0.5023316	2.985085	5.959057	6.024210	1.010933	11.70978	12.10058	1.033374
0.95	3.992232	2.005828	0.5024328	2.984438	5.957284	6.025259	1.011410	11.69723	12.10494	1.034856
0.96	3.991899	2.006078	0.5025373	2.983770	5.955455	6.026341	1.011903	11.68426	12.10944	1.036389
0.97	3.991556	2.006336	0.5026451	2.983081	5.953567	6.027459	1.012411	11.67089	12.11408	1.037974
0.98	3.991202	2.006602	0.5027563	2.982370	5.951620	6.028611	1.012936	11.65710	12.11887	1.039613
0.99	3.990836	2.006876	0.5028710	2.981637	5.949612	6.029799	1.013478	11.64287	12.12381	1.041308
1.00	3.990460	2.007159	0.5029893	2.980881	5.947542	6.031025	1.014036	11.62821	12.12890	1.043059
1.01	3.990072	2.007450	0.5031112	2.980101	5.945409	6.032288	1.014613	11.61309	12.13415	1.044868
1.02	3.989672	2.007750	0.5032369	2.979298	5.943211	6.033589	1.015207	11.59753	12.13956	1.046737
1.03	3.989260	2.008059	0.5033663	2.978471	5.940948	6.034929	1.015819	11.58149	12.14513	1.048667
1.04	3.988836	2.008378	0.5034997	2.977619	5.938617	6.036309	1.016450	11.56498	12.15086	1.050660
1.05	3.988400	2.008705	0.5036370	2.976741	5.936217	6.037730	1.017101	11.54799	12.15677	1.052718
1.06	3.987950	2.009043	0.5037783	2.975838	5.933748	6.039192	1.017770	11.53050	12.16285	1.054841
1.07	3.987488	2.009390	0.5039237	2.974909	5.931207	6.040697	1.018460	11.51250	12.16910	1.057033
1.08	3.987013	2.009747	0.5040734	2.973953	5.928594	6.042244	1.019170	11.49400	12.17553	1.059295
1.09	3.986524	2.010114	0.5042273	2.972970	5.925907	6.043836	1.019901	11.47497	12.18215	1.061628
1.10	3.986021	2.010492	0.5043856	2.971958	5.923144	6.045473	1.020653	11.45541	12.18895	1.064035
1.11	3.985505	2.010880	0.5045483	2.970919	5.920305	6.047155	1.021426	11.43530	12.19594	1.066517
1.12	3.984974	2.011278	0.5047156	2.969850	5.917387	6.048883	1.022222	11.41464	12.20313	1.069077
1.13	3.984428	2.011688	0.5048875	2.968752	5.914390	6.050660	1.023040	11.39342	12.21051	1.071716
1.14	3.983868	2.012109	0.5050641	2.967624	5.911311	6.052484	1.023882	11.37163	12.21810	1.074437
1.15	3.983293	2.012541	0.5052456	2.966465	5.908150	6.054358	1.024747	11.34925	12.22588	1.077242
1.16	3.982702	2.012985	0.5054319	2.965276	5.904905	6.056281	1.025636	11.32628	12.23388	1.080133
1.17	3.982096	2.013440	0.5056232	2.964054	5.901574	6.058256	1.026549	11.30270	12.24209	1.083112
1.18	3.981474	2.013908	0.5058196	2.962800	5.898156	6.060282	1.027488	11.27851	12.25052	1.086182
1.19	3.980836	2.014387	0.5060212	2.961513	5.894649	6.062362	1.028452	11.25369	12.25916	1.089346

TABLE I (continued)

λ	$K\,\frac{L}{EI}$	$kK\,\frac{L}{EI}$	k	$K''\,\frac{L}{EI}$	$Q\,\frac{L^2}{EI}$	$qQ\,\frac{L^2}{EI}$	q	$T\,\frac{L^3}{EI}$	$tT\,\frac{L^3}{EI}$	t
1.20	3.980181	2.014879	0.5062280	2.960193	5.891052	6.064495	1.029442	11.22823	12.26803	1.092606
1.21	3.979510	2.015384	0.5064402	2.958838	5.887363	6.066682	1.030458	11.20213	12.27713	1.095964
1.22	3.978821	2.015901	0.5066579	2.957449	5.883581	6.068926	1.031502	11.17536	12.28645	1.099423
1.23	3.978116	2.016432	0.5068811	2.956025	5.879704	6.071226	1.032573	11.14793	12.29602	1.102987
1.24	3.977392	2.016975	0.5071100	2.954564	5.875730	6.073584	1.033673	11.11981	12.30582	1.106657
1.25	3.976651	2.017533	0.5073447	2.953067	5.871658	6.076000	1.034801	11.09101	12.31587	1.110437
1.26	3.975892	2.018104	0.5075852	2.951532	5.867486	6.078476	1.035959	11.06149	12.32617	1.114331
1.27	3.975114	2.018689	0.5078317	2.949960	5.863213	6.081012	1.037147	11.03127	12.33671	1.118341
1.28	3.974317	2.019288	0.5080842	2.948349	5.858836	6.083610	1.038365	11.00031	12.34752	1.122470
1.29	3.973501	2.019901	0.5083430	2.946698	5.854354	6.086271	1.039614	10.96862	12.35859	1.126722
1.30	3.972666	2.020530	0.5086080	2.945008	5.849766	6.088995	1.040896	10.93617	12.36992	1.131101
1.31	3.971811	2.021173	0.5088794	2.943277	5.845070	6.091785	1.042209	10.90296	12.38152	1.135611
1.32	3.970935	2.021831	0.5091574	2.941505	5.840263	6.094640	1.043556	10.86898	12.39339	1.140254
1.33	3.970040	2.022505	0.5094420	2.939691	5.835345	6.097562	1.044936	10.83420	12.40555	1.145036
1.34	3.969123	2.023194	0.5097333	2.937834	5.830313	6.100551	1.046351	10.79863	12.41799	1.149959
1.35	3.968186	2.023899	0.5100314	2.935934	5.825166	6.103610	1.047800	10.76225	12.43071	1.155029
1.36	3.967227	2.024621	0.5103365	2.933989	5.819901	6.106740	1.049286	10.72504	12.44373	1.160251
1.37	3.966247	2.025359	0.5106487	2.932000	5.814517	6.109940	1.050808	10.68699	12.45705	1.165627
1.38	3.965244	2.026113	0.5109681	2.929965	5.809013	6.113213	1.052367	10.64809	12.47066	1.171164
1.39	3.964219	2.026885	0.5112948	2.927883	5.803386	6.116559	1.053964	10.60833	12.48459	1.176866
1.40	3.963172	2.027674	0.5116290	2.925755	5.797634	6.119981	1.055600	10.56770	12.49882	1.182739
1.41	3.962101	2.028480	0.5119708	2.923578	5.791756	6.123478	1.057275	10.52617	12.51338	1.188787
1.42	3.961006	2.029304	0.5123202	2.921353	5.785749	6.127052	1.058990	10.48374	12.52825	1.195017
1.43	3.959888	2.030146	0.5126775	2.919078	5.779612	6.130704	1.060747	10.44040	12.54345	1.201434
1.44	3.958746	2.031006	0.5130427	2.916753	5.773343	6.134436	1.062545	10.39612	12.55898	1.208045
1.45	3.957579	2.031885	0.5134160	2.914377	5.766940	6.138249	1.064386	10.35091	12.57485	1.214855
1.46	3.956388	2.032782	0.5137976	2.911949	5.760400	6.142143	1.066270	10.30473	12.59106	1.221871
1.47	3.955171	2.033699	0.5141874	2.909468	5.753722	6.146121	1.068199	10.25759	12.60762	1.229101
1.48	3.953928	2.034635	0.5145858	2.906934	5.746904	6.150183	1.070173	10.20946	12.62453	1.236552
1.49	3.952660	2.035591	0.5149928	2.904345	5.739943	6.154331	1.072194	10.16034	12.64180	1.244230
1.50	3.951365	2.036567	0.5154086	2.901700	5.732838	6.158566	1.074261	10.11020	12.65943	1.252144
1.51	3.950043	2.037564	0.5158333	2.899000	5.725587	6.162889	1.076377	10.05904	12.67743	1.260303
1.52	3.948694	2.038581	0.5162670	2.896242	5.718187	6.167302	1.078541	10.00683	12.69580	1.268714
1.53	3.947318	2.039618	0.5167100	2.893427	5.710637	6.171806	1.080756	9.953569	12.71456	1.277387
1.54	3.945913	2.040677	0.5171623	2.890552	5.702934	6.176402	1.083022	9.899238	12.73370	1.286331

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
1.55	3.944480	2.041758	0.5176241	2.887617	5.695075	6.181091	1.085340	9.843821	12.75323	1.295557
1.56	3.943019	2.042860	0.5180955	2.884622	5.687060	6.185876	1.087711	9.787304	12.77316	1.305074
1.57	3.941528	2.043985	0.5185768	2.881565	5.678885	6.190757	1.090136	9.729673	12.79349	1.314894
1.58	3.940008	2.045132	0.5190680	2.878445	5.670549	6.195736	1.092617	9.670911	12.81423	1.325028
1.59	3.938458	2.046302	0.5195693	2.875262	5.662049	6.200814	1.095154	9.611005	12.83538	1.335488
1.60	3.936877	2.047495	0.5200810	2.872014	5.653383	6.205992	1.097748	9.549937	12.85696	1.346287
1.61	3.935266	2.048711	0.5206031	2.868700	5.644549	6.211273	1.100402	9.487693	12.87896	1.357438
1.62	3.933623	2.049952	0.5211358	2.865320	5.635544	6.216657	1.103116	9.424257	12.90139	1.368956
1.63	3.931949	2.051216	0.5216793	2.861871	5.626366	6.222147	1.105891	9.359613	12.92427	1.380855
1.64	3.930242	2.052505	0.5222338	2.858355	5.617012	6.227742	1.108729	9.293745	12.94759	1.393151
1.65	3.928503	2.053819	0.5227994	2.854768	5.607481	6.233446	1.111630	9.226637	12.97136	1.405860
1.66	3.926731	2.055158	0.5233763	2.851110	5.597770	6.239260	1.114597	9.158273	12.99559	1.419000
1.67	3.924925	2.056522	0.5239647	2.847380	5.587877	6.245185	1.117631	9.088637	13.02029	1.432590
1.68	3.923086	2.057912	0.5245648	2.843577	5.577798	6.251222	1.120733	9.017712	13.04546	1.446648
1.69	3.921212	2.059329	0.5251767	2.839700	5.567532	6.257373	1.123904	8.945481	13.07110	1.461196
1.70	3.919303	2.060772	0.5258007	2.835747	5.557076	6.263641	1.127147	8.871928	13.09724	1.476256
1.71	3.917359	2.062242	0.5264369	2.831718	5.546428	6.270026	1.130462	8.797035	13.12386	1.491851
1.72	3.915379	2.063740	0.5270856	2.827611	5.535585	6.276530	1.133851	8.720786	13.15099	1.508005
1.73	3.913363	2.065265	0.5277469	2.823425	5.524544	6.283155	1.137316	8.643164	13.17862	1.524745
1.74	3.911310	2.066818	0.5284211	2.819159	5.513303	6.289902	1.140859	8.564150	13.20676	1.542098
1.75	3.909220	2.068400	0.5291082	2.814812	5.501859	6.296774	1.144481	8.483728	13.23543	1.560095
1.76	3.907092	2.070011	0.5298087	2.810382	5.490210	6.303772	1.148184	8.401879	13.26462	1.578768
1.77	3.904926	2.071651	0.5305225	2.805868	5.478353	6.310897	1.151970	8.318586	13.29435	1.598150
1.78	3.902721	2.073321	0.5312501	2.801269	5.466286	6.318151	1.155840	8.233832	13.32462	1.618277
1.79	3.900477	2.075021	0.5319915	2.796583	5.454004	6.325537	1.159797	8.147597	13.35544	1.639188
1.80	3.898193	2.076751	0.5327471	2.791810	5.441507	6.333056	1.163842	8.059863	13.38682	1.660924
1.81	3.895869	2.078512	0.5335170	2.786947	5.428790	6.340709	1.167978	7.970613	13.41876	1.683530
1.82	3.893504	2.080305	0.5343015	2.781994	5.415852	6.348500	1.172207	7.879827	13.45128	1.707053
1.83	3.891098	2.082129	0.5351007	2.776949	5.402689	6.356428	1.176530	7.787486	13.48439	1.731546
1.84	3.888649	2.083986	0.5359151	2.771810	5.389299	6.364498	1.180951	7.693572	13.51808	1.757061
1.85	3.886159	2.085875	0.5367446	2.766577	5.375678	6.372709	1.185471	7.598065	13.55237	1.783660
1.86	3.883625	2.087797	0.5375898	2.761247	5.361824	6.381065	1.190092	7.500947	13.58726	1.811407
1.87	3.881048	2.089753	0.5384507	2.755819	5.347734	6.389567	1.194818	7.402197	13.62278	1.840369
1.88	3.878427	2.091743	0.5393276	2.750293	5.333405	6.398218	1.199650	7.301796	13.65891	1.870623
1.89	3.875761	2.093767	0.5402208	2.744665	5.318833	6.407018	1.204591	7.199724	13.69568	1.902250

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
1.90	3.873050	2.095826	0.5411306	2.738935	5.304016	6.415971	1.209644	7.095962	13.73308	1.935338
1.91	3.870293	2.097920	0.5420572	2.733101	5.288950	6.425078	1.214812	6.990488	13.77114	1.969983
1.92	3.867490	2.100050	0.5430009	2.727161	5.273634	6.434342	1.220097	6.883283	13.80986	2.006289
1.93	3.864640	2.102217	0.5439619	2.721114	5.258062	6.443764	1.225502	6.774326	13.84924	2.044372
1.94	3.861742	2.104420	0.5449407	2.714958	5.242233	6.453347	1.231030	6.663596	13.88930	2.084355
1.95	3.858796	2.106661	0.5459374	2.708691	5.226143	6.463093	1.236685	6.551072	13.93005	2.126377
1.96	3.855801	2.108939	0.5469523	2.702312	5.209788	6.473004	1.242470	6.436734	13.97149	2.170587
1.97	3.852757	2.111256	0.5479858	2.695818	5.193166	6.483083	1.248387	6.320559	14.01364	2.217151
1.98	3.849662	2.113611	0.5490381	2.689209	5.176273	6.493330	1.254441	6.202527	14.05650	2.266254
1.99	3.846517	2.116006	0.5501096	2.682482	5.159106	6.503750	1.260635	6.082615	14.10009	2.318096
2.00	3.843321	2.118441	0.5512006	2.675635	5.141661	6.514344	1.266973	5.960802	14.14441	2.372904
2.01	3.840072	2.120916	0.5523114	2.668666	5.123936	6.525115	1.273458	5.837065	14.18948	2.430927
2.02	3.836771	2.123431	0.5534423	2.661574	5.105925	6.536064	1.280094	5.711382	14.23530	2.492444
2.03	3.833417	2.125989	0.5545937	2.654357	5.087627	6.547195	1.286886	5.583731	14.28189	2.557768
2.04	3.830008	2.128588	0.5557660	2.647011	5.069037	6.558510	1.293837	5.454088	14.32925	2.627250
2.05	3.826545	2.131230	0.5569594	2.639536	5.050153	6.570011	1.300953	5.322431	14.37741	2.701286
2.06	3.823026	2.133915	0.5581743	2.631930	5.030969	6.581701	1.308237	5.188736	14.42636	2.780322
2.07	3.819452	2.136644	0.5594111	2.624190	5.011483	6.593582	1.315695	5.052981	14.47612	2.864867
2.08	3.815820	2.139417	0.5606702	2.616313	4.991691	6.605658	1.323331	4.915140	14.52670	2.955500
2.09	3.812132	2.142235	0.5619520	2.608299	4.971589	6.617929	1.331150	4.775191	14.57811	3.052885
2.10	3.808384	2.145098	0.5632568	2.600143	4.951173	6.630401	1.339158	4.633109	14.63036	3.157785
2.11	3.804578	2.148008	0.5645850	2.591845	4.930440	6.643074	1.347359	4.488869	14.68347	3.271085
2.12	3.800713	2.150964	0.5659370	2.583402	4.909385	6.655952	1.355761	4.342448	14.73745	3.393811
2.13	3.796786	2.153968	0.5673134	2.574812	4.888005	6.669037	1.364368	4.193820	14.79230	3.527167
2.14	3.792799	2.157019	0.5687144	2.566071	4.866295	6.682332	1.373187	4.042960	14.84805	3.672569
2.15	3.788749	2.160119	0.5701405	2.557178	4.844253	6.695841	1.382224	3.889843	14.90470	3.831697
2.16	3.784637	2.163269	0.5715922	2.548130	4.821873	6.709566	1.391486	3.734442	14.96227	4.006560
2.17	3.780461	2.166468	0.5730698	2.538924	4.799151	6.723510	1.400979	3.576732	15.02076	4.199576
2.18	3.776221	2.169718	0.5745740	2.529557	4.776083	6.737676	1.410712	3.416688	15.08020	4.413689
2.19	3.771915	2.173019	0.5761050	2.520028	4.752666	6.752067	1.420690	3.254281	15.14059	4.652514
2.20	3.767544	2.176373	0.5776635	2.510333	4.728894	6.766686	1.430924	3.089486	15.20195	4.920543
2.21	3.763106	2.179779	0.5792499	2.500469	4.704764	6.781537	1.441419	2.922276	15.26429	5.223427
2.22	3.758599	2.183238	0.5808647	2.490434	4.680271	6.796621	1.452185	2.752623	15.32763	5.568374
2.23	3.754025	2.186751	0.5825085	2.480224	4.655410	6.811944	1.463232	2.580499	15.39198	5.964730
2.24	3.749381	2.190320	0.5841817	2.469836	4.630178	6.827507	1.474567	2.405877	15.45736	6.424831

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
2.25	3.744666	2.193943	0.5858849	2.459268	4.604569	6.843315	1.486201	2.228729	15.52377	6.965302
2.26	3.739881	2.197624	0.5876186	2.448516	4.578579	6.859371	1.498144	2.049025	15.59123	7.609098
2.27	3.735023	2.201361	0.5893835	2.437577	4.552203	6.875677	1.510406	1.866738	15.65977	8.388842
2.28	3.730092	2.205156	0.5911800	2.426448	4.525437	6.892238	1.522999	1.681837	15.72939	9.352502
2.29	3.725087	2.209010	0.5930089	2.415125	4.498276	6.909057	1.535935	1.494294	15.80011	10.57362
2.30	3.720008	2.212923	0.5948706	2.403605	4.470714	6.926138	1.549224	1.304079	15.87194	12.17099
2.31	3.714852	2.216897	0.5967658	2.391884	4.442747	6.943484	1.562881	1.111161	15.94490	14.34977
2.32	3.709620	2.220932	0.5986952	2.379958	4.414370	6.961099	1.576918	0.9155105	16.01901	17.49736
2.33	3.704309	2.225028	0.6006594	2.367825	4.385578	6.978986	1.591349	0.7170960	16.09429	22.44370
2.34	3.698920	2.229188	0.6026591	2.355480	4.356366	6.997151	1.606190	0.5158866	16.17074	31.34554
2.35	3.693451	2.233411	0.6046949	2.342919	4.326728	7.015596	1.621455	0.3118507	16.24840	52.10313
2.36	3.687901	2.237699	0.6067676	2.330138	4.296660	7.034325	1.637161	0.1049565	16.32727	155.5623
2.37	3.682270	2.242052	0.6088778	2.317134	4.266156	7.053343	1.653325	-0.1048284	16.40737	-156.5164
2.38	3.676555	2.246472	0.6110263	2.303901	4.235210	7.072654	1.669965	-0.3175365	16.48872	-51.92700
2.39	3.670756	2.250959	0.6132140	2.290437	4.203817	7.092261	1.687100	-0.5332011	16.57134	-31.07897
2.40	3.664872	2.255514	0.6154414	2.276735	4.171971	7.112169	1.704750	-0.7518555	16.65525	-22.15219
2.41	3.658902	2.260138	0.6177095	2.262793	4.139668	7.132382	1.722936	-0.9735334	16.74046	-17.19557
2.42	3.652844	2.264833	0.6200191	2.248604	4.106900	7.152905	1.741680	-1.198269	16.82700	-14.04276
2.43	3.646698	2.269599	0.6223709	2.234165	4.073663	7.173742	1.761005	-1.426098	16.91489	-11.86096
2.44	3.640462	2.274437	0.6247659	2.219471	4.039950	7.194898	1.780937	-1.657054	17.00414	-10.26167
2.45	3.634135	2.279348	0.6272050	2.204517	4.005756	7.216376	1.801502	-1.891173	17.09478	-9.039246
2.46	3.627716	2.284333	0.6296890	2.189297	3.971074	7.238183	1.822727	-2.128491	17.18682	-8.074649
2.47	3.621204	2.289394	0.6322189	2.173806	3.935898	7.260322	1.844642	-2.369045	17.28029	-7.294200
2.48	3.614597	2.294531	0.6347956	2.158039	3.900222	7.282798	1.867278	-2.612871	17.37521	-6.649852
2.49	3.607895	2.299745	0.6374201	2.141991	3.864040	7.305617	1.890668	-2.860007	17.47159	-6.108934
2.50	3.601095	2.305038	0.6400935	2.125656	3.827344	7.328783	1.914848	-3.110491	17.56947	-5.648458
2.51	3.594198	2.310411	0.6428168	2.109027	3.790130	7.352302	1.939855	-3.364360	17.66887	-5.251777
2.52	3.587200	2.315864	0.6455910	2.092099	3.752389	7.376178	1.965728	-3.621654	17.76980	-4.906543
2.53	3.580102	2.321400	0.6484172	2.074866	3.714116	7.400418	1.992511	-3.882413	17.87230	-4.603400
2.54	3.572901	2.327018	0.6512966	2.057322	3.675303	7.425025	2.020248	-4.146675	17.97638	-4.335131
2.55	3.565596	2.332722	0.6542304	2.039459	3.635944	7.450006	2.048988	-4.414482	18.08207	-4.096080
2.56	3.558187	2.338510	0.6572197	2.021272	3.596031	7.475367	2.078783	-4.685873	18.18940	-3.881752
2.57	3.550671	2.344386	0.6602658	2.002753	3.555558	7.501113	2.109687	-4.960892	18.29838	-3.688526
2.58	3.543046	2.350350	0.6633699	1.983895	3.514516	7.527249	2.141760	-5.239579	18.40905	-3.513460
2.59	3.535313	2.356404	0.6665333	1.964691	3.472899	7.553783	2.175065	-5.521978	18.52143	-3.354130

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
2.60	3.527468	2.362548	0.6697575	1.945134	3.430699	7.580719	2.209672	-5.808131	18.63555	-3.208528
2.61	3.519511	2.368785	0.6730437	1.925216	3.387909	7.608064	2.245652	-6.098082	18.75144	-3.074973
2.62	3.511440	2.375115	0.6763934	1.904928	3.344520	7.635823	2.283085	-6.391877	18.86912	-2.952046
2.63	3.503253	2.381540	0.6798081	1.884263	3.300525	7.664005	2.322056	-6.689559	18.98862	-2.838545
2.64	3.494949	2.388061	0.6832893	1.863212	3.255916	7.692614	2.362658	-6.991174	19.10997	-2.733442
2.65	3.486526	2.394681	0.6868386	1.841767	3.210684	7.721658	2.404988	-7.296770	19.23320	-2.635851
2.66	3.477983	2.401399	0.6904575	1.819918	3.164822	7.751143	2.449156	-7.606393	19.35834	-2.545009
2.67	3.469317	2.408219	0.6941478	1.797657	3.118321	7.781076	2.495277	-7.920090	19.48542	-2.460252
2.68	3.460527	2.415141	0.6979112	1.774974	3.071172	7.811464	2.543480	-8.237910	19.61447	-2.381001
2.69	3.451612	2.422167	0.7017494	1.751858	3.023367	7.842314	2.593901	-8.559902	19.74553	-2.306747
2.70	3.442569	2.429298	0.7056643	1.728300	2.974897	7.873634	2.646692	-8.886117	19.87862	-2.237042
2.71	3.433397	2.436537	0.7096578	1.704290	2.925752	7.905430	2.702017	-9.216604	20.01378	-2.171492
2.72	3.424094	2.443885	0.7137317	1.679816	2.875924	7.937711	2.760056	-9.551416	20.15104	-2.109744
2.73	3.414658	2.451343	0.7178883	1.654868	2.825403	7.970484	2.821008	-9.890604	20.29044	-2.051487
2.74	3.405087	2.458913	0.7221294	1.629433	2.774180	8.003757	2.885089	-10.23422	20.43202	-1.996441
2.75	3.395378	2.466598	0.7264574	1.603500	2.722245	8.037539	2.952540	-10.58232	20.57580	-1.944356
2.76	3.385532	2.474398	0.7308744	1.577057	2.669589	8.071836	3.023625	-10.93496	20.72183	-1.895007
2.77	3.375544	2.482316	0.7353827	1.550091	2.616200	8.106658	3.098638	-11.29219	20.87014	-1.848192
2.78	3.365413	2.490354	0.7399847	1.522589	2.562070	8.142013	3.177904	-11.65407	21.02077	-1.803728
2.79	3.355137	2.498513	0.7446830	1.494536	2.507187	8.177909	3.261787	-12.02066	21.17376	-1.761448
2.80	3.344714	2.506796	0.7494800	1.465920	2.451541	8.214357	3.350691	-12.39201	21.32916	-1.721202
2.81	3.334141	2.515204	0.7543785	1.436725	2.395122	8.251364	3.445071	-12.76819	21.48699	-1.682853
2.82	3.323417	2.523740	0.7593812	1.406936	2.337917	8.288940	3.545437	-13.14925	21.64731	-1.646276
2.83	3.312538	2.532405	0.7644908	1.376538	2.279917	8.327094	3.652367	-13.53526	21.81015	-1.611357
2.84	3.301504	2.541202	0.7697105	1.345514	2.221110	8.365837	3.766512	-13.92628	21.97556	-1.577892
2.85	3.290311	2.550133	0.7750432	1.313848	2.161483	8.405178	3.888616	-14.32237	22.14358	-1.546083
2.86	3.278957	2.559200	0.7804922	1.281521	2.101026	8.445127	4.019525	-14.72360	22.31426	-1.515544
2.87	3.267439	2.568405	0.7860607	1.248517	2.039726	8.485694	4.160218	-15.13002	22.48765	-1.486293
2.88	3.255755	2.577751	0.7917521	1.214815	1.977571	8.526890	4.311800	-15.54172	22.66378	-1.458255
2.89	3.243903	2.587240	0.7975701	1.180398	1.914548	8.568726	4.475588	-15.95875	22.84272	-1.431361
2.90	3.231880	2.596875	0.8035184	1.145243	1.850644	8.611213	4.653089	-16.38118	23.02451	-1.405547
2.91	3.219682	2.606657	0.8096006	1.109331	1.785847	8.654361	4.846082	-16.80909	23.20920	-1.380753
2.92	3.207309	2.616590	0.8158210	1.072640	1.720143	8.698183	5.056662	-17.24255	23.39685	-1.356925
2.93	3.194755	2.626675	0.8221836	1.035146	1.653519	8.742690	5.287325	-17.68162	23.58750	-1.334012
2.94	3.182020	2.636917	0.8286928	0.9968259	1.585960	8.787894	5.541057	-18.12639	23.78121	-1.311966

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
2.95	3.169099	2.647316	0.8353530	0.9576554	1.517452	8.833808	5.821473	-18.57693	23.97804	-1.290743
2.96	3.155990	2.657877	0.8421690	0.9176088	1.447982	8.880444	6.132979	-19.03331	24.17804	-1.270301
2.97	3.142690	2.668602	0.8491456	0.8766592	1.377534	8.927815	6.481011	-19.49562	24.38127	-1.250602
2.98	3.129196	2.679493	0.8562879	0.8347789	1.306094	8.975934	6.872351	-19.96393	24.58779	-1.231611
2.99	3.115505	2.690554	0.8636013	0.7919389	1.233645	9.024814	7.315569	-20.43832	24.79766	-1.213293
3.00	3.101613	2.701788	0.8710913	0.7481089	1.160172	9.074470	7.821658	-20.91889	25.01095	-1.195616
3.01	3.087517	2.713198	0.8787636	0.7032575	1.085660	9.124915	8.404949	-21.40570	25.22772	-1.178551
3.02	3.073214	2.724786	0.8866244	0.6573517	1.010091	9.176164	9.084492	-21.89886	25.44803	-1.162071
3.03	3.058700	2.736557	0.8946799	0.6103571	0.9334494	9.228231	9.886161	-22.39844	25.67195	-1.146149
3.04	3.043971	2.748513	0.9029367	0.5622378	0.8557178	9.281133	10.84602	-22.90454	25.89955	-1.130761
3.05	3.029025	2.760658	0.9114017	0.5129562	0.7768785	9.334883	12.01589	-23.41724	26.13090	-1.115883
3.06	3.013857	2.772996	0.9200821	0.4624731	0.6969139	9.389499	13.47297	-23.93664	26.36607	-1.101494
3.07	2.998464	2.785529	0.9289855	0.4107474	0.6158056	9.444996	15.33763	-24.46284	26.60514	-1.087573
3.08	2.982841	2.798262	0.9381198	0.3577361	0.5335350	9.501392	17.80838	-24.99594	26.84817	-1.074101
3.09	2.966986	2.811199	0.9474932	0.3033940	0.4500828	9.558702	21.23765	-25.53602	27.09526	-1.061060
3.10	2.950893	2.824342	0.9571145	0.2476740	0.3654297	9.616945	26.31682	-26.08320	27.34648	-1.048433
3.11	2.934558	2.837697	0.9669927	0.1905265	0.2795555	9.676139	34.61258	-26.63757	27.60190	-1.036202
3.12	2.917978	2.851266	0.9771375	0.1318995	0.1924399	9.736302	50.59399	-27.19925	27.86162	-1.024353
3.13	2.901148	2.865055	0.9875588	0.07173840	0.1040619	9.797453	94.15024	-27.76834	28.12572	-1.012870
3.14	2.884064	2.879067	0.9982673	0.009985909	0.01440000	9.859610	684.6952	-28.34494	28.39429	-1.001741
3.15	2.866720	2.893306	1.009274	-0.05341830	-0.07656766	9.922795	-129.5951	-28.92918	28.66741	-0.9909513
3.16	2.849113	2.907778	1.020591	-0.1185377	-0.1688636	9.987028	-59.14257	-29.52117	28.94519	-0.9804892
3.17	2.831237	2.922486	1.032229	-0.1854389	-0.2625108	10.05233	-38.29301	-30.12102	29.22771	-0.9703426
3.18	2.813088	2.937436	1.044203	-0.2541924	-0.3575328	10.11872	-28.30151	-30.72886	29.51508	-0.9605002
3.19	2.794660	2.952632	1.056526	-0.3248723	-0.4539539	10.18622	-22.43889	-31.34481	29.80739	-0.9509514
3.20	2.775949	2.968079	1.069212	-0.3975567	-0.5517986	10.25486	-18.58442	-31.96899	30.10474	-0.9416859
3.21	2.756949	2.983782	1.082277	-0.4723283	-0.6510925	10.32465	-15.85742	-32.60153	30.40725	-0.9326940
3.22	2.737655	2.999746	1.095736	-0.5492742	-0.7518615	10.39562	-13.82652	-33.24257	30.71502	-0.9239666
3.23	2.718060	3.015977	1.109606	-0.6284867	-0.8541323	10.46781	-12.25548	-33.89224	31.02817	-0.9154947
3.24	2.698160	3.032479	1.123906	-0.7100633	-0.9579322	10.54122	-11.00418	-34.55068	31.34680	-0.9072702
3.25	2.677948	3.049260	1.138655	-0.7941073	-1.063289	10.61588	-9.984004	-35.21802	31.67103	-0.8992848
3.26	2.657419	3.066324	1.153873	-0.8807283	-1.170232	10.69183	-9.136507	-35.89442	32.00100	-0.8915312
3.27	2.636566	3.083677	1.169581	-0.9700424	-1.278791	10.76910	-8.421313	-36.58003	32.33681	-0.8840019
3.28	2.615383	3.101326	1.185801	-1.062173	-1.388995	10.84769	-7.809746	-37.27499	32.67861	-0.8766900
3.29	2.593864	3.119276	1.202560	-1.157251	-1.500876	10.92766	-7.280857	-37.97945	33.02651	-0.8695889

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
3.30	2.572001	3.137534	1.219881	-1.255416	-1.614465	11.00902	-6.818991	-38.69359	33.38066	-0.8626922
3.31	2.549788	3.156107	1.237792	-1.356816	-1.729796	11.09181	-6.412207	-39.41757	33.74120	-0.8559939
3.32	2.527218	3.175002	1.256323	-1.461609	-1.846903	11.17606	-6.051245	-40.15154	34.10826	-0.8494883
3.33	2.504283	3.194225	1.275505	-1.569965	-1.965819	11.26180	-5.728808	-40.89569	34.48201	-0.8431697
3.34	2.480976	3.213784	1.295370	-1.682064	-2.086581	11.34906	-5.439072	-41.65019	34.86258	-0.8370328
3.35	2.457290	3.233686	1.315956	-1.798098	-2.209224	11.43788	-5.177330	-42.41522	35.25013	-0.8310727
3.36	2.433216	3.253938	1.337299	-1.918274	-2.333787	11.52830	-4.939738	-43.19098	35.64484	-0.8252844
3.37	2.408746	3.274550	1.359442	-2.042812	-2.460308	11.62034	-4.723123	-43.97764	36.04685	-0.8196632
3.38	2.383872	3.295528	1.382426	-2.171951	-2.588827	11.71405	-4.524847	-44.77541	36.45635	-0.8142048
3.39	2.358586	3.316881	1.406301	-2.305945	-2.719385	11.80946	-4.342696	-45.58448	36.87351	-0.8089048
3.40	2.332878	3.338617	1.431116	-2.445069	-2.852024	11.90662	-4.174798	-46.40507	37.29851	-0.8037592
3.41	2.306739	3.360746	1.456925	-2.589618	-2.986786	12.00557	-4.019560	-47.23739	37.73153	-0.7987640
3.42	2.280160	3.383277	1.483790	-2.739911	-3.123717	12.10634	-3.875618	-48.08166	38.17278	-0.7939156
3.43	2.253131	3.406218	1.511771	-2.896293	-3.262863	12.20898	-3.741799	-48.93810	38.62245	-0.7892101
3.44	2.225642	3.429581	1.540940	-3.059136	-3.404271	12.31354	-3.617084	-49.80695	39.08074	-0.7846443
3.45	2.197683	3.453374	1.571370	-3.228845	-3.547989	12.42005	-3.500590	-50.68844	39.54787	-0.7802148
3.46	2.169243	3.477607	1.603143	-3.405858	-3.694068	12.52858	-3.391541	-51.58282	40.02406	-0.7759184
3.47	2.140312	3.502293	1.636347	-3.590652	-3.842559	12.63917	-3.289257	-52.49034	40.50953	-0.7717521
3.48	2.110878	3.527441	1.671078	-3.783749	-3.993516	12.75186	-3.193141	-53.41126	41.00452	-0.7677129
3.49	2.080929	3.553063	1.707440	-3.985714	-4.146994	12.86672	-3.102661	-54.34585	41.50926	-0.7637981
3.50	2.050454	3.579170	1.745550	-4.197168	-4.303050	12.98379	-3.017345	-55.29439	42.02400	-0.7600048
3.51	2.019439	3.605776	1.785533	-4.418793	-4.461742	13.10313	-2.936775	-56.25716	42.54901	-0.7563306
3.52	1.987873	3.632891	1.827527	-4.651333	-4.623130	13.22480	-2.860573	-57.23445	43.08455	-0.7527730
3.53	1.955741	3.660530	1.871684	-4.895613	-4.787277	13.34886	-2.788403	-58.22657	43.63089	-0.7493295
3.54	1.923031	3.688705	1.918172	-5.152540	-4.954248	13.47537	-2.719963	-59.23383	44.18831	-0.7459979
3.55	1.889728	3.717430	1.967177	-5.423117	-5.124108	13.60439	-2.654977	-60.25655	44.75712	-0.7427760
3.56	1.855817	3.746720	2.018906	-5.708458	-5.296928	13.73600	-2.593200	-61.29507	45.33761	-0.7396616
3.57	1.821284	3.776590	2.073586	-6.009800	-5.472777	13.87025	-2.534407	-62.34972	45.93010	-0.7366529
3.58	1.786113	3.807054	2.131475	-6.328526	-5.651730	14.00722	-2.478395	-63.42086	46.53491	-0.7337477
3.59	1.750287	3.838128	2.192857	-6.666179	-5.833863	14.14698	-2.424977	-64.50886	47.15239	-0.7309444
3.60	1.713790	3.869830	2.258054	-7.024495	-6.019254	14.28961	-2.373984	-65.61410	47.78288	-0.7282410
3.61	1.676604	3.902175	2.327428	-7.405426	-6.207984	14.43519	-2.325262	-66.73697	48.42674	-0.7256359
3.62	1.638712	3.935182	2.401386	-7.811180	-6.400139	14.58380	-2.278669	-67.87787	49.08436	-0.7231275
3.63	1.600096	3.968868	2.480394	-8.244263	-6.595805	14.73551	-2.234073	-69.03723	49.75611	-0.7207142
3.64	1.560734	4.003253	2.564980	-8.707532	-6.795073	14.89043	-2.191357	-70.21547	50.44241	-0.7183945

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
3.65	1.520609	4.038357	2.655749	-9.204254	-6.998036	15.04864	-2.150409	-71.41305	51.14366	-0.7161669
3.66	1.479699	4.074199	2.753397	-9.738188	-7.204793	15.21023	-2.111127	-72.63043	51.86032	-0.7140302
3.67	1.437982	4.110800	2.858728	-10.31368	-7.415443	15.37531	-2.073417	-73.86809	52.59282	-0.7119830
3.68	1.395437	4.148183	2.972677	-10.93577	-7.630091	15.54397	-2.037193	-75.12653	53.34164	-0.7100240
3.69	1.352040	4.186371	3.096338	-11.61038	-7.848846	15.71632	-2.002373	-76.40628	54.10726	-0.7081520
3.70	1.307767	4.225387	3.230995	-12.34444	-8.071820	15.89246	-1.968882	-77.70786	54.89019	-0.7063660
3.71	1.262593	4.265255	3.378172	-13.14617	-8.299131	16.07252	-1.936651	-79.03184	55.69096	-0.7046649
3.72	1.216492	4.306002	3.539688	-14.02541	-8.530900	16.25662	-1.905616	-80.37879	56.51011	-0.7030476
3.73	1.169437	4.347654	3.717731	-14.99397	-8.767254	16.44486	-1.875714	-81.74932	57.34822	-0.7015131
3.74	1.121401	4.390238	3.914957	-16.06619	-9.008323	16.63739	-1.846891	-83.14405	58.20587	-0.7000605
3.75	1.072354	4.433783	4.134627	-17.25969	-9.254245	16.83434	-1.819094	-84.56363	59.08368	-0.6986890
3.76	1.022265	4.478320	4.380782	-18.59628	-9.505163	17.03584	-1.792272	-86.00874	59.98230	-0.6973978
3.77	0.9711032	4.523880	4.658495	-20.10337	-9.761223	17.24204	-1.766381	-87.48008	60.90240	-0.6961859
3.78	0.9188353	4.570494	4.974226	-21.81584	-10.02258	17.45309	-1.741377	-88.97837	61.84467	-0.6950528
3.79	0.8654270	4.618198	5.336323	-23.77877	-10.28940	17.66916	-1.717220	-90.50438	62.80984	-0.6939977
3.80	0.8108425	4.667027	5.755775	-26.05152	-10.56184	17.89039	-1.693871	-92.05892	63.79867	-0.6930200
3.81	0.7550443	4.717017	6.247338	-28.71376	-10.84008	18.11697	-1.671295	-93.64279	64.81196	-0.6921191
3.82	0.6979935	4.768208	6.831307	-31.87510	-11.12431	18.34907	-1.649457	-95.25686	65.85053	-0.6912943
3.83	0.6396492	4.820639	7.536379	-35.69051	-11.41470	18.58688	-1.628328	-96.90204	66.91525	-0.6905453
3.84	0.5799690	4.874353	8.404506	-40.38656	-11.71147	18.83060	-1.607876	-98.57926	68.00702	-0.6898715
3.85	0.5189083	4.929393	9.499546	-46.30809	-12.01483	19.08042	-1.588073	-100.2895	69.12679	-0.6892725
3.86	0.4564206	4.985807	10.92371	-54.00709	-12.32498	19.33657	-1.568894	-102.0338	70.27556	-0.6887478
3.87	0.3924572	5.043642	12.85144	-64.42561	-12.64215	19.59927	-1.550312	-103.8132	71.45435	-0.6882973
3.88	0.3269673	5.102948	15.60691	-79.31428	-12.96659	19.86875	-1.532304	-105.6289	72.66425	-0.6879204
3.89	0.2598972	5.163779	19.86854	-102.3369	-13.29854	20.14526	-1.514848	-107.4819	73.90640	-0.6876170
3.90	0.1911912	5.226191	27.33489	-142.6661	-13.63826	20.42905	-1.497922	-109.3736	75.18199	-0.6873868
3.91	0.1207906	5.290240	43.79677	-231.5747	-13.98603	20.72040	-1.481507	-111.3053	76.49228	-0.6872296
3.92	0.04863390	5.355990	110.1287	-589.7997	-14.34213	21.01958	-1.465583	-113.2782	77.83857	-0.6871452
3.93	-0.02534862	5.423503	-213.9988	1160.597	-14.70687	21.32691	-1.450132	-115.2938	79.22224	-0.6871335
3.94	-0.1012098	5.492847	-54.27190	298.0060	-15.08056	21.64268	-1.435137	-117.3536	80.64475	-0.6871945
3.95	-0.1790359	5.564093	-31.07809	172.7423	-15.46354	21.96722	-1.420582	-119.4591	82.10763	-0.6873281
3.96	-0.2588970	5.637315	-21.77435	122.4899	-15.85615	22.30090	-1.406451	-121.6121	83.61246	-0.6875342
3.97	-0.3408721	5.712593	-16.75876	95.39508	-16.25876	22.64406	-1.392730	-123.8141	85.16095	-0.6878130
3.98	-0.4250443	5.790008	-13.62213	78.44719	-16.67176	22.99709	-1.379403	-126.0671	86.75487	-0.6881644
3.99	-0.5115012	5.869648	-11.47534	66.84468	-17.09557	23.36039	-1.366459	-128.3729	88.39610	-0.6885885

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
4.00	-0.6003354	5.951605	-9.913800	58.40269	-17.53060	23.73440	-1.353884	-130.7336	90.08661	-0.6890856
4.01	-0.6916444	6.035976	-8.726994	51.98429	-17.97732	24.11957	-1.341667	-133.1512	91.82850	-0.6896557
4.02	-0.7855311	6.122864	-7.794553	46.93946	-18.43620	24.51636	-1.329795	-135.6281	93.62398	-0.6902991
4.03	-0.8821045	6.212376	-7.042676	42.86965	-18.90776	24.92529	-1.318257	-138.1667	95.47538	-0.6910161
4.04	-0.9814799	6.304628	-6.423594	39.51689	-19.39252	25.34688	-1.307044	-140.7693	97.38516	-0.6918068
4.05	-1.083779	6.399742	-5.905024	36.70685	-19.89106	25.78170	-1.296145	-143.4387	99.35595	-0.6926718
4.06	-1.189131	6.497845	-5.464362	34.31744	-20.40398	26.23035	-1.285551	-146.1777	101.3905	-0.6936113
4.07	-1.297674	6.599074	-5.085809	32.26066	-20.93191	26.69346	-1.275252	-148.9893	103.4918	-0.6946257
4.08	-1.409552	6.703575	-4.755819	30.47144	-21.47554	27.17170	-1.265240	-151.8766	105.6630	-0.6957156
4.09	-1.524921	6.811501	-4.466791	28.90063	-22.03558	27.66579	-1.255506	-154.8431	107.9073	-0.6968813
4.10	-1.643944	6.923017	-4.211225	27.51044	-22.61280	28.17650	-1.246042	-157.8922	110.2282	-0.6981235
4.11	-1.766796	7.038296	-3.983649	26.27131	-23.20802	28.70462	-1.236841	-161.0278	112.6297	-0.6994427
4.12	-1.893664	7.157524	-3.779722	25.15979	-23.82209	29.25102	-1.227895	-164.2539	115.1156	-0.7008395
4.13	-2.024747	7.280900	-3.595956	24.15705	-24.45595	29.81663	-1.219197	-167.5748	117.6902	-0.7023146
4.14	-2.160257	7.408636	-3.429517	23.24778	-25.11059	30.40243	-1.210741	-170.9952	120.3582	-0.7038687
4.15	-2.300420	7.540957	-3.278078	22.41943	-25.78705	31.00946	-1.202521	-174.5199	123.1243	-0.7055025
4.16	-2.445480	7.678106	-3.139714	21.66157	-26.48647	31.63885	-1.194529	-178.1543	125.9938	-0.7072169
4.17	-2.595696	7.820343	-3.012811	20.96552	-27.21006	32.29181	-1.186760	-181.9040	128.9723	-0.7090128
4.18	-2.751348	7.967945	-2.896015	20.32394	-27.95912	32.96964	-1.179209	-185.7750	132.0658	-0.7108909
4.19	-2.912735	8.121212	-2.788174	19.73062	-28.73503	33.67371	-1.171870	-189.7738	135.2807	-0.7128523
4.20	-3.080179	8.280466	-2.688306	19.18025	-29.53931	34.40553	-1.164737	-193.9074	138.6240	-0.7148980
4.21	-3.254028	8.446053	-2.595569	18.66828	-30.37355	35.16670	-1.157806	-198.1834	142.1032	-0.7170290
4.22	-3.434655	8.618346	-2.509232	18.19077	-31.23951	35.95895	-1.151073	-202.6097	145.7263	-0.7192464
4.23	-3.622464	8.797749	-2.428664	17.74431	-32.13907	36.78417	-1.144532	-207.1953	149.5021	-0.7215515
4.24	-3.817893	8.984698	-2.353313	17.32591	-33.07424	37.64438	-1.138178	-211.9495	153.4398	-0.7239453
4.25	-4.021415	9.179666	-2.282696	16.93297	-34.04725	38.54178	-1.132009	-216.8826	157.5499	-0.7264293
4.26	-4.233543	9.383166	-2.216386	16.56318	-35.06046	39.47875	-1.126019	-222.0057	161.8432	-0.7290047
4.27	-4.454836	9.595756	-2.154009	16.21451	-36.11649	40.45787	-1.120205	-227.3308	166.3318	-0.7316730
4.28	-4.685901	9.818043	-2.095230	15.88516	-37.21815	41.48198	-1.114563	-232.8713	171.0290	-0.7344356
4.29	-4.927402	10.05069	-2.039754	15.57353	-38.36853	42.55415	-1.109090	-238.6413	175.9488	-0.7372941
4.30	-5.180064	10.29442	-1.987315	15.27819	-39.57101	43.67774	-1.103781	-244.6566	181.1071	-0.7402501
4.31	-5.444680	10.55003	-1.937676	14.99785	-40.82926	44.85644	-1.098635	-250.9343	186.5208	-0.7433053
4.32	-5.722121	10.81838	-1.890624	14.73137	-42.14734	46.09429	-1.093646	-257.4933	192.2088	-0.7464614
4.33	-6.013346	11.10044	-1.845967	14.47769	-43.52971	47.39574	-1.088814	-264.3544	198.1918	-0.7497203
4.34	-6.319410	11.39726	-1.803532	14.23591	-44.98127	48.76571	-1.084134	-271.5402	204.4926	-0.7530838

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
4.35	-6.641483	11.71000	-1.763161	14.00514	-46.50745	50.20960	-1.079603	-279.0762	211.1362	-0.7565541
4.36	-6.980856	12.03997	-1.724713	13.78463	-48.11427	51.73343	-1.075220	-286.9902	218.1508	-0.7601332
4.37	-7.338968	12.38859	-1.688057	13.57368	-49.80841	53.34388	-1.070981	-295.3133	225.5672	-0.7638232
4.38	-7.717421	12.75748	-1.653075	13.37165	-51.59731	55.04838	-1.066885	-304.0802	233.4200	-0.7676265
4.39	-8.118004	13.14841	-1.619660	13.17794	-53.48930	56.85525	-1.062927	-313.3295	241.7479	-0.7715453
4.40	-8.542724	13.56339	-1.587712	12.99204	-55.49370	58.77381	-1.059108	-323.1047	250.5943	-0.7755823
4.41	-8.993839	14.00468	-1.557142	12.81344	-57.62101	60.81555	-1.055423	-333.4544	260.0077	-0.7797399
4.42	-9.473899	14.47484	-1.527865	12.64169	-59.88307	62.98931	-1.051872	-344.4338	270.0433	-0.7840209
4.43	-9.985791	14.97674	-1.499805	12.47639	-62.29331	65.31151	-1.048451	-356.1053	280.7634	-0.7884281
4.44	-10.53280	15.51367	-1.472891	12.31714	-64.86701	67.79642	-1.045160	-368.5399	292.2390	-0.7929644
4.45	-11.11869	16.08939	-1.447058	12.16359	-67.62163	70.46151	-1.041997	-381.8188	304.5512	-0.7976328
4.46	-11.74777	16.70822	-1.422246	12.01542	-70.57721	73.32679	-1.038958	-396.0351	317.7931	-0.8024366
4.47	-12.42501	17.37512	-1.398398	11.87232	-73.75688	76.41540	-1.036044	-411.2964	332.0721	-0.8073790
4.48	-13.15619	18.09587	-1.375464	11.73401	-77.18747	79.75416	-1.033253	-427.7273	347.5129	-0.8124636
4.49	-13.94805	18.87720	-1.353394	11.60023	-80.90027	83.37435	-1.030582	-445.4734	364.2609	-0.8176939
4.50	-14.80848	19.72701	-1.332143	11.47073	-84.93202	87.31272	-1.028031	-464.7053	382.4867	-0.8230738
4.51	-15.74682	20.65466	-1.311671	11.34529	-89.32615	91.61266	-1.025597	-485.6248	402.3922	-0.8286072
4.52	-16.77421	21.67124	-1.291938	11.22370	-94.13432	96.32586	-1.023281	-508.4722	424.2174	-0.8342982
4.53	-17.90398	22.79012	-1.272908	11.10575	-99.41856	101.5143	-1.021080	-533.5357	448.2506	-0.8401510
4.54	-19.15229	24.02745	-1.254547	10.99126	-105.2539	107.2531	-1.018994	-561.1642	474.8405	-0.8461703
4.55	-20.53888	25.40295	-1.236823	10.88007	-111.7322	113.6340	-1.017021	-591.7841	504.4136	-0.8523608
4.56	-22.08813	26.94103	-1.219706	10.77200	-118.9667	120.7702	-1.015160	-625.9221	537.4963	-0.8587272
4.57	-23.83053	28.67215	-1.203169	10.66691	-127.0991	128.8035	-1.013411	-664.2364	574.7470	-0.8652748
4.58	-25.80468	30.63493	-1.187185	10.56465	-136.3087	137.9133	-1.011772	-707.5610	616.9995	-0.8720089
4.59	-28.06021	32.87899	-1.171730	10.46509	-146.8263	148.3301	-1.010242	-756.9678	665.3257	-0.8789352
4.60	-30.66203	35.46924	-1.156781	10.36810	-158.9534	160.3557	-1.008822	-813.8585	721.1271	-0.8860595
4.61	-33.69666	38.49220	-1.142315	10.27356	-173.0922	174.3920	-1.007508	-880.1018	786.2725	-0.8933881
4.62	-37.28208	42.06585	-1.128313	10.18135	-189.7910	190.9875	-1.006304	-958.2458	863.3098	-0.9009273
4.63	-41.58344	46.35533	-1.114755	10.09138	-209.8172	210.9095	-1.005206	-1051.857	955.8058	-0.9086839
4.64	-46.83925	51.59916	-1.101622	10.00355	-234.2793	235.2665	-1.004214	-1166.088	1068.912	-0.9166650
4.65	-53.40723	58.15507	-1.088899	9.917749	-264.8398	265.7210	-1.003327	-1308.662	1210.353	-0.9248781
4.66	-61.84939	66.58504	-1.076567	9.833897	-304.1103	304.8846	-1.002546	-1491.719	1392.268	-0.9333309
4.67	-73.10214	77.82550	-1.064613	9.751908	-356.4427	357.1092	-1.001870	-1735.482	1634.879	-0.9420317
4.68	-88.85041	93.56137	-1.053021	9.671701	-429.6673	430.2252	-1.001298	-2076.341	1974.578	-0.9509890
4.69	-112.4624	117.1608	-1.041778	9.593198	-539.4370	539.8852	-1.000831	-2587.039	2484.106	-0.9602119

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
4.70	-151.7910	156.4769	-1.030870	9.516329	-722.2467	722.5844	-1.000468	-3437.178	3333.066	-0.9697099
4.71	-230.3633	235.0364	-1.020286	9.441023	-1087.433	1087.659	-1.000208	-5134.874	5029.573	-0.9794930
4.72	-465.4320	470.0923	-1.010013	9.367215	-2179.901	2180.015	-1.000052	-10212.46	10105.96	-0.9895717
4.73	-116083.6	116088.3	-1.000040	9.294843	-539489.4	539489.4	-1.000000	-2507238.	2507131.	-0.9999570
4.74	480.5811	-475.9468	-0.9903569	9.223847	2216.403	-2216.517	-1.000051	10217.21	-10326.13	-1.010661
4.75	242.6212	-238.0001	-0.9809535	9.154169	1110.498	-1110.727	-1.000206	5077.167	-5187.317	-1.021695
4.76	163.5132	-158.9054	-0.9718200	9.085757	742.8208	-743.1663	-1.000465	3367.874	-3479.261	-1.033073
4.77	123.9967	-119.4023	-0.9629474	9.018559	559.1359	-559.5986	-1.000828	2513.639	-2626.271	-1.044809
4.78	100.2974	-95.71650	-0.9543271	8.952524	448.9574	-449.5384	-1.001294	2001.003	-2114.892	-1.056916
4.79	84.50119	-79.93396	-0.9459507	8.887607	375.5067	-376.2070	-1.001865	1659.047	-1774.203	-1.069411
4.80	73.21909	-68.66562	-0.9378103	8.823762	323.0339	-323.8545	-1.002540	1414.578	-1531.009	-1.082308
4.81	64.75741	-60.21782	-0.9298985	8.760945	283.6680	-284.6100	-1.003321	1231.015	-1348.733	-1.095627
4.82	58.17555	-53.64996	-0.9222081	8.699117	253.0379	-254.1022	-1.004206	1088.046	-1207.060	-1.109384
4.83	52.90931	-48.39785	-0.9147322	8.638236	228.5216	-229.7093	-1.005197	973.4838	-1093.806	-1.123599
4.84	48.59974	-44.10253	-0.9074643	8.578265	208.4507	-209.7629	-1.006295	879.5773	-1001.217	-1.138293
4.85	45.00756	-40.52473	-0.9003982	8.519168	191.7135	-193.1512	-1.007499	801.1584	-924.1255	-1.153487
4.86	41.96715	-37.49882	-0.8935279	8.460909	177.5401	-179.1044	-1.008811	734.6501	-858.9557	-1.169204
4.87	39.36023	-34.90653	-0.8868477	8.403456	165.3810	-167.0729	-1.010231	677.4978	-803.1525	-1.185469
4.88	37.10005	-32.66111	-0.8803521	8.346775	154.8329	-156.6536	-1.011759	627.8280	-754.8426	-1.202308
4.89	35.12157	-30.69752	-0.8740360	8.290836	145.5936	-147.5442	-1.013397	584.2359	-712.6213	-1.219749
4.90	33.37505	-28.96602	-0.8678943	8.235609	137.4319	-139.5135	-1.015146	545.6473	-675.4143	-1.237822
4.91	31.82181	-27.42792	-0.8619223	8.181064	130.1681	-132.3818	-1.017006	511.2266	-642.3863	-1.256559
4.92	30.43130	-26.05271	-0.8561155	8.127175	123.6603	-126.0072	-1.018979	480.3144	-612.8779	-1.275993
4.93	29.17910	-24.81593	-0.8504693	8.073915	117.7948	-120.2761	-1.021065	452.3828	-586.3614	-1.296161
4.94	28.04544	-23.69783	-0.8449797	8.021257	112.4798	-115.0967	-1.023265	427.0049	-562.4097	-1.317104
4.95	27.01414	-22.68222	-0.8396427	7.969176	107.6402	-110.3938	-1.025581	403.8311	-540.6737	-1.338861
4.96	26.07182	-21.75574	-0.8344543	7.917649	103.2138	-106.1052	-1.028014	382.5728	-520.8646	-1.361478
4.97	25.20736	-20.90726	-0.8294109	7.866653	99.14877	-102.1793	-1.030566	362.9893	-502.7419	-1.385005
4.98	24.41139	-20.12741	-0.8245089	7.816164	95.40173	-98.57256	-1.033237	344.8787	-486.1037	-1.409492
4.99	23.67601	-19.40829	-0.8197450	7.766161	91.93585	-95.24818	-1.036029	328.0698	-470.7791	-1.434997
5.00	22.99447	-18.74315	-0.8151159	7.716623	88.71981	-92.17486	-1.038943	312.4170	-456.6225	-1.461580
5.01	22.36098	-18.12622	-0.8106185	7.667529	85.72673	-89.32574	-1.041982	297.7954	-443.5092	-1.489308
5.02	21.77057	-17.55252	-0.8062497	7.618859	82.93346	-86.67767	-1.045147	284.0974	-431.3315	-1.518252
5.03	21.21892	-17.01772	-0.8020068	7.570594	80.31992	-84.21060	-1.048440	271.2297	-419.9962	-1.548489
5.04	20.70227	-16.51807	-0.7978870	7.522716	77.86865	-81.90706	-1.051862	259.1110	-409.4223	-1.580104

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
5.05	20.21732	-16.05028	-0.7938875	7.475206	75.56432	-79.75177	-1.055416	247.6701	-399.5387	-1.613189
5.06	19.76119	-15.61146	-0.7900059	7.428046	73.39351	-77.73128	-1.059103	236.8446	-390.2829	-1.647844
5.07	19.33132	-15.19905	-0.7862398	7.381219	71.34436	-75.83376	-1.062926	226.5792	-381.6000	-1.684179
5.08	18.92545	-14.81081	-0.7825868	7.334709	69.40634	-74.04870	-1.066887	216.8252	-373.4411	-1.722314
5.09	18.54158	-14.44472	-0.7790446	7.288499	67.57013	-72.36679	-1.070988	207.5891	-365.7630	-1.762382
5.10	18.17790	-14.09898	-0.7756112	7.242573	65.82738	-70.77969	-1.075232	198.6820	-358.5270	-1.804527
5.11	17.83282	-13.77201	-0.7722843	7.196915	64.17065	-69.27997	-1.079621	190.2194	-351.6986	-1.848910
5.12	17.50490	-13.46236	-0.7690622	7.151511	62.59324	-67.86095	-1.084158	182.1201	-345.2466	-1.895708
5.13	17.19284	-13.16873	-0.7659428	7.106345	61.08913	-66.51662	-1.088845	174.3561	-339.1433	-1.945119
5.14	16.89548	-12.88997	-0.7629244	7.061403	59.65288	-65.24156	-1.093687	166.9020	-333.3634	-1.997360
5.15	16.61174	-12.62501	-0.7600051	7.016671	58.27957	-64.03086	-1.098685	159.7350	-327.8841	-2.052676
5.16	16.34069	-12.37290	-0.7571834	6.972135	56.96474	-62.88008	-1.103842	152.8342	-322.6848	-2.111338
5.17	16.08144	-12.13276	-0.7544576	6.927781	55.70433	-61.78516	-1.109163	146.1809	-317.7468	-2.173654
5.18	15.83319	-11.90381	-0.7518263	6.883595	54.49464	-60.74244	-1.114650	139.7577	-313.0528	-2.239968
5.19	15.59523	-11.68532	-0.7492878	6.839566	53.33230	-59.74854	-1.120307	133.5490	-308.5874	-2.310669
5.20	15.36689	-11.47662	-0.7468408	6.795679	52.21421	-58.80039	-1.126138	127.5403	-304.3363	-2.386198
5.21	15.14755	-11.27711	-0.7444841	6.751922	51.13754	-57.89517	-1.132146	121.7184	-300.2863	-2.467059
5.22	14.93667	-11.08624	-0.7422162	6.708283	50.09969	-57.03030	-1.138336	116.0711	-296.4255	-2.553826
5.23	14.73372	-10.90348	-0.7400359	6.664750	49.09826	-56.20339	-1.144712	110.5874	-292.7428	-2.647162
5.24	14.53823	-10.72837	-0.7379422	6.621309	48.13105	-55.41225	-1.151279	105.2569	-289.2281	-2.747831
5.25	14.34976	-10.56047	-0.7359338	6.577950	47.19600	-54.65486	-1.158040	100.0701	-285.8720	-2.856719
5.26	14.16791	-10.39939	-0.7340098	6.534661	46.29125	-53.92936	-1.165001	95.01810	-282.6658	-2.974863
5.27	13.99230	-10.24473	-0.7321690	6.491429	45.41503	-53.23400	-1.172167	90.09284	-279.6015	-3.103482
5.28	13.82259	-10.09617	-0.7304106	6.448244	44.56572	-52.56718	-1.179543	85.28671	-276.6717	-3.244019
5.29	13.65845	-9.953372	-0.7287336	6.405094	43.74183	-51.92743	-1.187134	80.59267	-273.8695	-3.398194
5.30	13.49958	-9.816046	-0.7271371	6.361968	42.94195	-51.31334	-1.194947	76.00415	-271.1884	-3.568074
5.31	13.34570	-9.683912	-0.7256204	6.318855	42.16477	-50.72364	-1.202986	71.51505	-268.6226	-3.756168
5.32	13.19655	-9.556712	-0.7241827	6.275743	41.40908	-50.15713	-1.211259	67.11967	-266.1664	-3.965550
5.33	13.05188	-9.434204	-0.7228232	6.232622	40.67373	-49.61268	-1.219772	62.81266	-263.8146	-4.200023
5.34	12.91148	-9.316162	-0.7215412	6.189481	39.95767	-49.08925	-1.228531	58.58905	-261.5626	-4.464359
5.35	12.77511	-9.202375	-0.7203361	6.146309	39.25989	-48.58587	-1.237545	54.44416	-259.4057	-4.764619
5.36	12.64259	-9.092647	-0.7192074	6.103095	38.57948	-48.10162	-1.246819	50.37360	-257.3397	-5.108622
5.37	12.51373	-8.986792	-0.7181544	6.059829	37.91554	-47.63565	-1.256362	46.37326	-255.3607	-5.506637
5.38	12.38835	-8.884636	-0.7171766	6.016499	37.26725	-47.18714	-1.266182	42.43926	-253.4650	-5.972417
5.39	12.26629	-8.786017	-0.7162736	5.973095	36.63385	-46.75536	-1.276288	38.56796	-251.6490	-6.524821

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
5.40	12.14738	-8.690782	-0.7154448	5.929607	36.01460	-46.33959	-1.286689	34.75592	-249.9096	-7.190418
5.41	12.03149	-8.598786	-0.7146900	5.886024	35.40882	-45.93916	-1.297393	30.99991	-248.2437	-8.007884
5.42	11.91847	-8.509894	-0.7140087	5.842335	34.81586	-45.55345	-1.308411	27.29686	-246.6483	-9.035776
5.43	11.80820	-8.423979	-0.7134006	5.798530	34.23511	-45.18187	-1.319753	23.64389	-245.1209	-10.36720
5.44	11.70055	-8.340919	-0.7128655	5.754598	33.66598	-44.82387	-1.331429	20.03826	-243.6588	-12.15968
5.45	11.59541	-8.260601	-0.7124030	5.710528	33.10794	-44.47892	-1.343452	16.47739	-242.2596	-14.70256
5.46	11.49265	-8.182919	-0.7120130	5.666310	32.56047	-44.14652	-1.355832	12.95878	-240.9211	-18.59135
5.47	11.39219	-8.107770	-0.7116952	5.621932	32.02307	-43.82622	-1.368583	9.480135	-239.6411	-25.27823
5.48	11.29392	-8.035059	-0.7114496	5.577385	31.49528	-43.51756	-1.381717	6.039237	-238.4176	-39.47809
5.49	11.19775	-7.964694	-0.7112761	5.532657	30.97667	-43.22013	-1.395248	2.633976	-237.2487	-90.07246
5.50	11.10359	-7.896590	-0.7111745	5.487738	30.46680	-42.93353	-1.409198	-0.7376535	-236.1326	320.1131
5.51	11.01135	-7.830666	-0.7111448	5.442616	29.96528	-42.65740	-1.423561	-4.077559	-235.0676	57.64909
5.52	10.92096	-7.766844	-0.7111870	5.397280	29.47174	-42.39137	-1.438374	-7.387561	-234.0521	31.68191
5.53	10.83233	-7.705051	-0.7113012	5.351720	28.98580	-42.13512	-1.453647	-10.66940	-233.0845	21.84608
5.54	10.74540	-7.645218	-0.7114875	5.305923	28.50714	-41.88832	-1.469398	-13.92473	-232.1635	16.67275
5.55	10.66009	-7.587278	-0.7117459	5.259880	28.03540	-41.65066	-1.485645	-17.15513	-231.2876	13.48212
5.56	10.57635	-7.531168	-0.7120766	5.213577	27.57030	-41.42188	-1.502410	-20.36213	-230.4556	11.31785
5.57	10.49409	-7.476829	-0.7124798	5.167004	27.11151	-41.20167	-1.519712	-23.54718	-229.6662	9.753445
5.58	10.41328	-7.424205	-0.7129556	5.120149	26.65877	-40.98982	-1.537574	-26.71167	-228.9182	8.569970
5.59	10.33384	-7.373241	-0.7135045	5.073000	26.21178	-40.78604	-1.556019	-29.85694	-228.2107	7.643471
5.60	10.25573	-7.323886	-0.7141265	5.025545	25.77030	-40.59012	-1.575074	-32.98426	-227.5424	6.898514
5.61	10.17888	-7.276090	-0.7148222	4.977771	25.33407	-40.40183	-1.594763	-36.09488	-226.9125	6.286556
5.62	10.10326	-7.229808	-0.7155918	4.929667	24.90285	-40.22097	-1.615115	-39.18997	-226.3200	5.774946
5.63	10.02881	-7.184994	-0.7164357	4.881219	24.47640	-40.04732	-1.636160	-42.27068	-225.7640	5.340911
5.64	9.955479	-7.141607	-0.7173544	4.832416	24.05451	-39.88070	-1.657930	-45.33811	-225.2436	4.968085
5.65	9.883234	-7.099606	-0.7183484	4.783243	23.63696	-39.72092	-1.680458	-48.39332	-224.7581	4.644404
5.66	9.812028	-7.058952	-0.7194183	4.733689	23.22354	-39.56782	-1.703781	-51.43732	-224.3067	4.360777
5.67	9.741819	-7.019609	-0.7205645	4.683738	22.81407	-39.42123	-1.727935	-54.47111	-223.8887	4.110228
5.68	9.672568	-6.981541	-0.7217877	4.633379	22.40834	-39.28099	-1.752963	-57.49565	-223.5033	3.887309
5.69	9.604238	-6.944714	-0.7230885	4.582596	22.00617	-39.14695	-1.778908	-60.51185	-223.1500	3.687708
5.70	9.536793	-6.909098	-0.7244676	4.531375	21.60739	-39.01896	-1.805816	-63.52061	-222.8282	3.507967
5.71	9.470196	-6.874660	-0.7259259	4.479702	21.21183	-38.89690	-1.833736	-66.52280	-222.5372	3.345276
5.72	9.404415	-6.841373	-0.7274639	4.427563	20.81932	-38.78064	-1.862724	-69.51927	-222.2765	3.197336
5.73	9.339416	-6.809207	-0.7290827	4.374942	20.42970	-38.67004	-1.892834	-72.51082	-222.0455	3.062240
5.74	9.275169	-6.778136	-0.7307830	4.321823	20.04282	-38.56500	-1.924130	-75.49825	-221.8439	2.938398

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
5.75	9.211643	-6.748135	-0.7325658	4.268191	19.65853	-38.46539	-1.956678	-78.48234	-221.6711	2.824471
5.76	9.148809	-6.719179	-0.7344320	4.214029	19.27668	-38.37113	-1.990547	-81.46383	-221.5267	2.719326
5.77	9.086639	-6.691245	-0.7363828	4.159322	18.89713	-38.28209	-2.025815	-84.44347	-221.4103	2.621994
5.78	9.025105	-6.664310	-0.7384191	4.104051	18.51975	-38.19820	-2.062566	-87.42195	-221.3214	2.531646
5.79	8.964181	-6.638353	-0.7405421	4.048201	18.14440	-38.11935	-2.100887	-90.39999	-221.2598	2.447564
5.80	8.903841	-6.613354	-0.7427530	3.991752	17.77096	-38.04546	-2.140878	-93.37825	-221.2250	2.369128
5.81	8.844060	-6.589294	-0.7450530	3.934687	17.39931	-37.97644	-2.182641	-96.35742	-221.2168	2.295794
5.82	8.784815	-6.566153	-0.7474435	3.876987	17.02931	-37.91223	-2.226293	-99.33813	-221.2349	2.227089
5.83	8.726082	-6.543913	-0.7499257	3.818633	16.66085	-37.85273	-2.271957	-102.3210	-221.2789	2.162594
5.84	8.667838	-6.522558	-0.7525012	3.759605	16.29382	-37.79789	-2.319768	-105.3068	-221.3486	2.101941
5.85	8.610061	-6.502072	-0.7551714	3.699882	15.92810	-37.74764	-2.369876	-108.2959	-221.4438	2.044803
5.86	8.552730	-6.482439	-0.7579379	3.639444	15.56359	-37.70190	-2.422442	-111.2891	-221.5642	1.990889
5.87	8.495825	-6.463643	-0.7608024	3.578269	15.20017	-37.66063	-2.477645	-114.2869	-221.7096	1.939939
5.88	8.439324	-6.445672	-0.7637664	3.516336	14.83775	-37.62377	-2.535679	-117.2899	-221.8799	1.891722
5.89	8.383208	-6.428511	-0.7668318	3.453621	14.47621	-37.59126	-2.596760	-120.2987	-222.0748	1.846029
5.90	8.327458	-6.412147	-0.7700005	3.390101	14.11546	-37.56305	-2.661127	-123.3139	-222.2943	1.802671
5.91	8.272055	-6.396569	-0.7732744	3.325752	13.75540	-37.53909	-2.729043	-126.3359	-222.5380	1.761479
5.92	8.216982	-6.381764	-0.7766555	3.260549	13.39594	-37.51935	-2.800800	-129.3654	-222.8061	1.722300
5.93	8.162219	-6.367722	-0.7801459	3.194467	13.03697	-37.50377	-2.876725	-132.4030	-223.0982	1.684994
5.94	8.107749	-6.354431	-0.7837478	3.127478	12.67840	-37.49232	-2.957180	-135.4491	-223.4143	1.649434
5.95	8.053556	-6.341882	-0.7874635	3.059555	12.32015	-37.48497	-3.042574	-138.5042	-223.7544	1.615506
5.96	7.999623	-6.330065	-0.7912954	2.990671	11.96212	-37.48168	-3.133364	-141.5690	-224.1183	1.583103
5.97	7.945933	-6.318972	-0.7952460	2.920796	11.60423	-37.48242	-3.230067	-144.6440	-224.5060	1.552129
5.98	7.892471	-6.308593	-0.7993179	2.849900	11.24637	-37.48716	-3.333266	-147.7295	-224.9175	1.522495
5.99	7.839219	-6.298920	-0.8035137	2.777951	10.88848	-37.49587	-3.443627	-150.8263	-225.3527	1.494121
6.00	7.786164	-6.289946	-0.8078363	2.704917	10.53046	-37.50854	-3.561908	-153.9348	-225.8116	1.466930
6.01	7.733290	-6.281664	-0.8122887	2.630765	10.17224	-37.52514	-3.688977	-157.0555	-226.2942	1.440855
6.02	7.680582	-6.274067	-0.8168739	2.555460	9.813712	-37.54565	-3.825835	-160.1889	-226.8005	1.415832
6.03	7.628025	-6.267148	-0.8215951	2.478967	9.454811	-37.57005	-3.973644	-163.3355	-227.3306	1.391801
6.04	7.575605	-6.260902	-0.8264557	2.401247	9.095451	-37.59833	-4.133751	-166.4959	-227.8844	1.368709
6.05	7.523307	-6.255322	-0.8314591	2.322263	8.735550	-37.63048	-4.307740	-169.6705	-228.4620	1.346504
6.06	7.471119	-6.250404	-0.8366089	2.241974	8.375028	-37.66648	-4.497475	-172.8599	-229.0634	1.325139
6.07	7.419024	-6.246144	-0.8419090	2.160339	8.013805	-37.70634	-4.705173	-176.0646	-229.6888	1.304572
6.08	7.367011	-6.242535	-0.8473634	2.077315	7.651801	-37.75003	-4.933483	-179.2850	-230.3382	1.284760
6.09	7.315065	-6.239576	-0.8529761	1.992856	7.288936	-37.79756	-5.185608	-182.5217	-231.0118	1.265667

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
6.10	7.263173	-6.237261	-0.8587515	1.906916	6.925131	-37.84893	-5.465446	-185.7752	-231.7095	1.247257
6.11	7.211322	-6.235587	-0.8646941	1.819447	6.560308	-37.90413	-5.777798	-189.0460	-232.4315	1.229497
6.12	7.159499	-6.234552	-0.8708085	1.730397	6.194388	-37.96317	-6.128640	-192.3347	-233.1780	1.212356
6.13	7.107690	-6.234153	-0.8770998	1.639715	5.827293	-38.02605	-6.525509	-195.6417	-233.9492	1.195804
6.14	7.055883	-6.234388	-0.8835731	1.547345	5.458944	-38.09278	-6.978049	-198.9675	-234.7450	1.179816
6.15	7.004064	-6.235254	-0.8902337	1.453231	5.089263	-38.16336	-7.498800	-202.3127	-235.5658	1.164365
6.16	6.952222	-6.236750	-0.8970873	1.357313	4.718171	-38.23780	-8.104371	-205.6779	-236.4117	1.149427
6.17	6.900343	-6.238875	-0.9041398	1.259528	4.345589	-38.31612	-8.817244	-209.0634	-237.2828	1.134980
6.18	6.848415	-6.241627	-0.9113973	1.159813	3.971440	-38.39833	-9.668617	-212.4700	-238.1794	1.121003
6.19	6.796425	-6.245006	-0.9188664	1.058098	3.595643	-38.48444	-10.70308	-215.8981	-239.1018	1.107475
6.20	6.744360	-6.249012	-0.9265538	0.9543144	3.218120	-38.57447	-11.98665	-219.3482	-240.0501	1.094379
6.21	6.692208	-6.253645	-0.9344665	0.8483868	2.838791	-38.66844	-13.62145	-222.8209	-241.0245	1.081696
6.22	6.639957	-6.258904	-0.9426121	0.7402381	2.457574	-38.76638	-15.77424	-226.3168	-242.0254	1.069409
6.23	6.587593	-6.264791	-0.9509984	0.6297872	2.074391	-38.86829	-18.73721	-229.8365	-243.0529	1.057503
6.24	6.535105	-6.271306	-0.9596336	0.5169493	1.689159	-38.97422	-23.07315	-233.3804	-244.1074	1.045968
6.25	6.482480	-6.278451	-0.9685262	0.4016353	1.301796	-39.08419	-30.02327	-236.9493	-245.1892	1.034775
6.26	6.429704	-6.286228	-0.9776853	0.2837520	0.9122206	-39.19822	-42.97011	-240.5436	-246.2985	1.023925
6.27	6.376766	-6.294637	-0.9871205	0.1632018	0.5203484	-39.31634	-75.55773	-244.1640	-247.4357	1.013400
6.28	6.323653	-6.303681	-0.9968417	0.03988056	0.1260954	-39.43860	-312.7679	-247.8110	-248.6012	1.003189
6.29	6.270352	-6.313364	-1.006860	-0.08631841	-0.2706234	-39.56503	146.1996	-251.4853	-249.7952	0.9932795
6.30	6.216850	-6.323687	-1.017185	-0.2155092	-0.6698942	-39.69565	59.25660	-255.1875	-251.0181	0.9836615
6.31	6.163135	-6.334654	-1.027830	-0.3478113	-1.071804	-39.83052	37.16214	-258.9182	-252.2704	0.9743245
6.32	6.109192	-6.346268	-1.038806	-0.4833506	-1.476441	-39.96968	27.07164	-262.6781	-253.5523	0.9652586
6.33	6.055010	-6.358533	-1.050127	-0.6222597	-1.883895	-40.11316	21.29268	-266.4679	-254.8644	0.9564546
6.34	6.000575	-6.371453	-1.061807	-0.7646785	-2.294255	-40.26102	17.54862	-270.2881	-256.2070	0.9479036
6.35	5.945874	-6.385033	-1.073859	-0.9107544	-2.707615	-40.41330	14.92579	-274.1394	-257.5806	0.9395972
6.36	5.890893	-6.399277	-1.086300	-1.060643	-3.124068	-40.57006	12.98629	-278.0226	-258.9857	0.9315272
6.37	5.835618	-6.414192	-1.099145	-1.214510	-3.543708	-40.73135	11.49399	-281.9384	-260.4226	0.9236862
6.38	5.780036	-6.429781	-1.112412	-1.372528	-3.966631	-40.89722	10.31032	-285.8874	-261.8920	0.9160668
6.39	5.724134	-6.446051	-1.126118	-1.534882	-4.392936	-41.06773	9.348583	-289.8704	-263.3943	0.9086620
6.40	5.667896	-6.463009	-1.140284	-1.701768	-4.822723	-41.24294	8.551796	-293.8882	-264.9300	0.9014652
6.41	5.611308	-6.480661	-1.154929	-1.873393	-5.256092	-41.42294	7.880935	-297.9415	-266.4997	0.8944701
6.42	5.554357	-6.499013	-1.170075	-2.049976	-5.693148	-41.60773	7.308388	-302.0310	-268.1040	0.8876706
6.43	5.497027	-6.518074	-1.185745	-2.231750	-6.133995	-41.79745	6.814066	-306.1576	-269.7435	0.8810610
6.44	5.439303	-6.537851	-1.201965	-2.418965	-6.578742	-41.99213	6.383004	-310.3221	-271.4188	0.8746357

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
6.45	5.381170	-6.558352	-1.218760	-2.611885	-7.027497	-42.19187	6.003825	-314.5253	-273.1304	0.8683894
6.46	5.322613	-6.579586	-1.236157	-2.810790	-7.480374	-42.39673	5.667728	-318.7681	-274.8791	0.8623171
6.47	5.263616	-6.601562	-1.254188	-3.015982	-7.937485	-42.60679	5.367795	-323.0512	-276.6656	0.8564139
6.48	5.204162	-6.624289	-1.272883	-3.227781	-8.398949	-42.82214	5.098512	-327.3757	-278.4905	0.8506753
6.49	5.144237	-6.647777	-1.292277	-3.446530	-8.864884	-43.04287	4.855436	-331.7425	-280.3545	0.8450968
6.50	5.083822	-6.672036	-1.312406	-3.672597	-9.335413	-43.26907	4.634939	-336.1524	-282.2585	0.8396741
6.51	5.022901	-6.697078	-1.333309	-3.906372	-9.810661	-43.50083	4.434036	-340.6064	-284.2031	0.8344033
6.52	4.961457	-6.722913	-1.355028	-4.148280	-10.29075	-43.73824	4.250246	-345.1055	-286.1892	0.8292804
6.53	4.899471	-6.749554	-1.377609	-4.398771	-10.77583	-43.98141	4.081488	-349.6508	-288.2177	0.8243016
6.54	4.836927	-6.777011	-1.401099	-4.658335	-11.26601	-44.23045	3.926008	-354.2431	-290.2893	0.8194635
6.55	4.773805	-6.805300	-1.425551	-4.927494	-11.76145	-44.48546	3.782312	-358.8837	-292.4051	0.8147626
6.56	4.710086	-6.834431	-1.451021	-5.206815	-12.26227	-44.74655	3.649124	-363.5736	-294.5658	0.8101957
6.57	4.645751	-6.864421	-1.477570	-5.496909	-12.76864	-45.01385	3.525345	-368.3139	-296.7724	0.8057595
6.58	4.580781	-6.895282	-1.505264	-5.798436	-13.28068	-45.28747	3.410026	-373.1058	-299.0260	0.8014511
6.59	4.515154	-6.927031	-1.534174	-6.112114	-13.79857	-45.56754	3.302338	-377.9504	-301.3276	0.7972675
6.60	4.448851	-6.959682	-1.564377	-6.438719	-14.32245	-45.85418	3.201560	-382.8490	-303.6781	0.7932060
6.61	4.381849	-6.993252	-1.595959	-6.779096	-14.85249	-46.14755	3.107058	-387.8028	-306.0788	0.7892640
6.62	4.314127	-7.027759	-1.629011	-7.134166	-15.38885	-46.44776	3.018274	-392.8131	-308.5307	0.7854388
6.63	4.245662	-7.063219	-1.663632	-7.504932	-15.93170	-46.75498	2.934713	-397.8813	-311.0350	0.7817281
6.64	4.176432	-7.099651	-1.699932	-7.892493	-16.48123	-47.06935	2.855937	-403.0087	-313.5929	0.7781294
6.65	4.106412	-7.137074	-1.738032	-8.298050	-17.03760	-47.39103	2.781555	-408.1967	-316.2057	0.7746406
6.66	4.035577	-7.175508	-1.778062	-8.722924	-17.60102	-47.72017	2.711217	-413.4468	-318.8747	0.7712594
6.67	3.963904	-7.214974	-1.820169	-9.168568	-18.17166	-48.05696	2.644611	-418.7604	-321.6012	0.7679838
6.68	3.891365	-7.255493	-1.864511	-9.636583	-18.74978	-48.40156	2.581454	-424.1392	-324.3866	0.7648117
6.69	3.817934	-7.297087	-1.911266	-10.12874	-19.33543	-48.75415	2.521493	-429.5847	-327.2325	0.7617414
6.70	3.743584	-7.339780	-1.960629	-10.64700	-19.92898	-49.11493	2.464498	-435.0986	-330.1402	0.7587710
6.71	3.668286	-7.383596	-2.012819	-11.19355	-20.53058	-49.48408	2.410262	-440.6825	-333.1113	0.7558986
6.72	3.592012	-7.428559	-2.068078	-11.77083	-21.14047	-49.86182	2.358595	-446.3383	-336.1475	0.7531227
6.73	3.514730	-7.474697	-2.126677	-12.38154	-21.75888	-50.24834	2.309326	-452.0677	-339.2504	0.7504416
6.74	3.436410	-7.522035	-2.188922	-13.02874	-22.38605	-50.64387	2.262296	-457.8726	-342.4218	0.7478537
6.75	3.357020	-7.570603	-2.255156	-13.71587	-23.02222	-51.04864	2.217364	-463.7551	-345.6634	0.7453577
6.76	3.276526	-7.620429	-2.325765	-14.44680	-23.66766	-51.46289	2.174397	-469.7170	-348.9772	0.7429521
6.77	3.194895	-7.671544	-2.401188	-15.22593	-24.32262	-51.88684	2.133275	-475.7604	-352.3651	0.7406355
6.78	3.112090	-7.723981	-2.481927	-16.05827	-24.98739	-52.32077	2.093887	-481.8876	-355.8290	0.7384067
6.79	3.028075	-7.777771	-2.568553	-16.94954	-25.66224	-52.76494	2.056131	-488.1008	-359.3712	0.7362644

TABLE I (continued)

λ	$K\,\frac{L}{EI}$	$kK\,\frac{L}{EI}$	k	$K''\,\frac{L}{EI}$	$Q\,\frac{L^2}{EI}$	$qQ\,\frac{L^2}{EI}$	Q	$T\,\frac{L^3}{EI}$	$\bar{t}T\,\frac{L^3}{EI}$	$\bar{t}$
6.80	2.942811	-7.832950	-2.661723	-17.90634	-26.34748	-53.21962	2.019913	-494.4022	-362.9938	0.7342074
6.81	2.856260	-7.889553	-2.762197	-18.93624	-27.04341	-53.68509	1.985145	-500.7944	-366.6990	0.7322347
6.82	2.768381	-7.947617	-2.870854	-20.04807	-27.75035	-54.16166	1.951747	-507.2797	-370.4892	0.7303451
6.83	2.679131	-8.007183	-2.988724	-21.25213	-28.46862	-54.64963	1.919645	-513.8607	-374.3669	0.7285377
6.84	2.588466	-8.068289	-3.117015	-22.56051	-29.19856	-55.14933	1.888769	-520.5402	-378.3346	0.7268115
6.85	2.496341	-8.130978	-3.257158	-23.98754	-29.94054	-55.66109	1.859054	-527.3209	-382.3950	0.7251656
6.86	2.402709	-8.195295	-3.410857	-25.55027	-30.69492	-56.18527	1.830442	-534.2058	-386.5508	0.7235990
6.87	2.307519	-8.261284	-3.580158	-27.26918	-31.46209	-56.72222	1.802875	-541.1977	-390.8049	0.7221110
6.88	2.210722	-8.328995	-3.767544	-29.16914	-32.24243	-57.27233	1.776308	-548.3000	-395.1603	0.7207008
6.89	2.112265	-8.398476	-3.976053	-31.28052	-33.03637	-57.83599	1.750676	-555.5158	-399.6201	0.7193677
6.90	2.012091	-8.469781	-4.209443	-33.64097	-33.84434	-58.41362	1.725950	-562.8485	-404.1876	0.7181109
6.91	1.910143	-8.542963	-4.472420	-36.29757	-34.66678	-59.00565	1.702080	-570.3016	-408.8663	0.7169299
6.92	1.806362	-8.618079	-4.770958	-39.31013	-35.50417	-59.61253	1.679029	-577.8789	-413.6596	0.7158240
6.93	1.700686	-8.695189	-5.112754	-42.75568	-36.35699	-60.23472	1.656758	-585.5841	-418.5712	0.7147927
6.94	1.593049	-8.774354	-5.507902	-46.73523	-37.22575	-60.87272	1.635232	-593.4213	-423.6051	0.7138354
6.95	1.483383	-8.855641	-5.969897	-51.38388	-38.11097	-61.52705	1.614418	-601.3946	-428.7652	0.7129516
6.96	1.371617	-8.939115	-6.517208	-56.88646	-39.01323	-62.19823	1.594286	-609.5084	-434.0559	0.7121409
6.97	1.257679	-9.024850	-7.175799	-63.50283	-39.93308	-62.88684	1.574806	-617.2673	-439.4815	0.7114030
6.98	1.141490	-9.112918	-7.983354	-71.61016	-40.87114	-63.59345	1.555950	-626.1759	-445.0466	0.7107373
6.99	1.022970	-9.203398	-8.996743	-81.77764	-41.82803	-64.31869	1.537693	-634.7393	-450.7561	0.7101437
7.00	0.9020346	-9.296371	-10.30600	-94.90640	-42.80443	-65.06319	1.520011	-643.4627	-456.6151	0.7096217
7.01	0.7785956	-9.391924	-12.06265	-112.5129	-43.80101	-65.82764	1.502879	-652.3515	-462.6289	0.7091712
7.02	0.6525604	-9.490145	-14.54294	-137.3620	-44.81851	-66.61274	1.486277	-661.4114	-468.8031	0.7087919
7.03	0.5238323	-9.591130	-18.30954	-175.0854	-45.85769	-67.41923	1.470184	-670.6484	-475.1434	0.7084837
7.04	0.3923097	-9.694976	-24.71256	-239.1953	-46.91933	-68.24790	1.454580	-680.0687	-481.6561	0.7082463
7.05	0.2578861	-9.801789	-38.00821	-372.2905	-48.00428	-69.09956	1.439446	-689.6790	-488.3476	0.7080797
7.06	0.1204498	-9.911677	-82.28889	+815.5004	-49.11341	-69.97508	1.424765	-699.4861	-495.2247	0.7079837
7.07	-0.02011682	-10.02475	498.3270	+4995.585	-50.24765	-70.87536	1.410521	-709.4972	-502.2945	0.7079584
7.08	-0.1639370	-10.14114	61.86000	627.1671	-51.40795	-71.80135	1.396697	-719.7201	-509.5645	0.7080037
7.09	-0.3111406	-10.26097	32.97856	338.0809	-52.59535	-72.75406	1.383279	-730.1626	-517.0426	0.7081197
7.10	-0.4618643	-10.38437	22.48360	233.0161	-53.81091	-73.73454	1.370253	-740.8334	-524.7370	0.7083064
7.11	-0.6162518	-10.51148	17.05712	178.6794	-55.05576	-74.74391	1.357604	-751.7411	-532.6567	0.7085639
7.12	-0.7744550	-10.64246	13.74187	145.4729	-56.33109	-75.78334	1.345320	-762.8953	-540.8107	0.7088924
7.13	-0.9366336	-10.77746	11.50659	123.0752	-57.63817	-76.85407	1.333389	-774.3058	-549.2088	0.7092920
7.14	-1.102957	-10.91664	9.897618	106.9458	-58.97830	-77.95742	1.321798	-785.9829	-557.8615	0.7097628

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
7.15	-1.273603	-11.06019	8.684180	94.77510	-60.35290	-79.09477	1.310538	-797.9378	-566.7795	0.7103053
7.16	-1.448760	-11.20829	7.736474	85.26390	-61.76345	-80.26759	1.299597	-810.1822	-575.9748	0.7109196
7.17	-1.628628	-11.36114	6.975896	77.62549	-63.21151	-81.47742	1.288965	-822.7282	-585.4583	0.7116060
7.18	-1.813418	-11.51894	6.352059	71.35557	-64.69875	-82.72592	1.278632	-835.5890	-595.2448	0.7123649
7.19	-2.003355	-11.68192	5.831179	66.11601	-66.22691	-84.01483	1.268590	-848.7785	-605.3460	0.7131967
7.20	-2.198675	-11.85081	5.389751	61.67157	-67.79788	-85.34599	1.258830	-862.3112	-615.7780	0.7141018
7.21	-2.399682	-12.02437	5.010923	57.85356	-69.41363	-86.72139	1.249342	-876.2029	-626.5558	0.7150807
7.22	-2.606493	-12.20436	4.682290	54.53785	-71.07627	-88.14310	1.240120	-890.4701	-637.6958	0.7161339
7.23	-2.819544	-12.39056	4.394525	51.63107	-72.78804	-89.61336	1.231155	-905.1304	-649.2155	0.7172619
7.24	-3.039089	-12.58327	4.140475	49.06163	-74.55132	-91.13454	1.222440	-920.2025	-661.1335	0.7184652
7.25	-3.265451	-12.78282	3.914566	46.77374	-76.36867	-92.70917	1.213969	-935.7066	-673.4698	0.7197447
7.26	-3.498976	-12.98955	3.712387	44.72326	-78.24280	-94.33996	1.205733	-951.6640	-686.2456	0.7211008
7.27	-3.740034	-13.20383	3.530404	42.87481	-80.17661	-96.02978	1.197728	-968.0975	-699.4836	0.7225343
7.28	-3.989019	-13.42605	3.365751	41.19971	-82.17321	-97.78173	1.189947	-985.0314	-713.2080	0.7240460
7.29	-4.246354	-13.65662	3.216082	39.67446	-84.23591	-99.59912	1.182388	-1002.492	-727.4450	0.7256366
7.30	-4.512494	-13.89601	3.079453	38.27962	-86.36829	-101.4855	1.175032	-1020.507	-742.2222	0.7273071
7.31	-4.787924	-14.14470	2.954245	36.99898	-88.57417	-103.4447	1.167888	-1039.107	-757.5698	0.7290582
7.32	-5.073169	-14.40321	2.839095	35.81890	-90.85765	-105.4807	1.160945	-1058.325	-773.5199	0.7308910
7.33	-5.368790	-14.67209	2.732849	34.72782	-93.22317	-107.5981	1.154199	-1078.194	-790.1072	0.7328068
7.34	-5.675395	-14.95196	2.634523	33.71589	-95.67550	-109.8016	1.147645	-1098.752	-807.3690	0.7348054
7.35	-5.993640	-15.24347	2.543274	32.77467	-98.21978	-112.0962	1.141279	-1120.041	-825.3458	0.7368892
7.36	-6.324231	-15.54732	2.458373	31.89687	-100.8616	-114.4876	1.135096	-1142.103	-844.0810	0.7390589
7.37	-6.667936	-15.86427	2.379188	31.07616	-103.6069	-116.9817	1.129092	-1164.986	-863.6220	0.7413157
7.38	-7.025586	-16.19517	2.305170	30.30703	-106.4623	-119.5851	1.123262	-1188.741	-884.0200	0.7436608
7.39	-7.398085	-16.54090	2.235836	29.58466	-109.4349	-122.3049	1.117604	-1213.424	-905.3307	0.7460956
7.40	-7.786414	-16.90246	2.170763	28.90481	-112.5324	-125.1487	1.112113	-1239.097	-927.6146	0.7486215
7.41	-8.191645	-17.28090	2.109576	28.26373	-115.7632	-128.1250	1.106785	-1265.825	-950.9379	0.7512399
7.42	-8.614949	-17.67740	2.051945	27.65810	-119.1365	-131.2430	1.101618	-1293.680	-975.3725	0.7539522
7.43	-9.057605	-18.09323	1.997574	27.08496	-122.6624	-134.5127	1.096608	-1322.741	-1000.998	0.7567601
7.44	-9.521018	-18.52981	1.946200	26.54169	-126.3520	-137.9451	1.091752	-1353.096	-1027.900	0.7596652
7.45	-10.00673	-18.98866	1.897588	26.02592	-130.2172	-141.5523	1.087047	-1384.839	-1056.174	0.7626693
7.46	-10.51645	-19.47149	1.851527	25.53554	-134.2716	-145.3477	1.082491	-1418.076	-1085.926	0.7657740
7.47	-11.05204	-19.98017	1.807826	25.06864	-138.5298	-149.3461	1.078079	-1452.923	-1117.271	0.7689813
7.48	-11.61560	-20.51679	1.766314	24.62350	-143.0083	-153.5637	1.073810	-1489.508	-1150.337	0.7722931
7.49	-12.20942	-21.08864	1.726835	24.19856	-147.7251	-158.0188	1.069681	-1527.974	-1185.267	0.7757114

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
7.50	-12.83607	-21.68330	1.689248	23.79240	-152.7005	-162.7314	1.065690	-1568.480	-1222.220	0.7792384
7.51	-13.49843	-22.31864	1.653425	23.40375	-157.9569	-167.7241	1.061834	-1611.202	-1261.372	0.7828762
7.52	-14.19971	-22.99285	1.619248	23.03142	-163.5197	-173.0221	1.058112	-1656.339	-1302.921	0.7866272
7.53	-14.94351	-23.70956	1.586612	22.67486	-169.4173	-178.6539	1.054520	-1704.115	-1347.092	0.7904937
7.54	-15.73391	-24.47282	1.555419	22.33157	-175.6815	-184.6513	1.051057	-1754.780	-1394.135	0.7944783
7.55	-16.57552	-25.28726	1.525578	22.00276	-182.3487	-191.0506	1.047721	-1808.620	-1444.334	0.7985835
7.56	-17.47358	-26.15809	1.497008	21.68530	-189.4599	-197.8929	1.044511	-1865.958	-1498.018	0.8028120
7.57	-18.43405	-27.09131	1.469634	21.38024	-197.0623	-205.2252	1.041423	-1927.163	-1555.542	0.8071667
7.58	-19.46380	-28.09375	1.443384	21.08628	-205.2095	-213.1014	1.038457	-1992.658	-1617.342	0.8116506
7.59	-20.57071	-29.17330	1.418196	20.80276	-213.9637	-221.5833	1.035612	-2062.931	-1683.902	0.8162666
7.60	-21.76392	-30.33911	1.394009	20.52908	-223.3967	-230.7429	1.032884	-2138.547	-1755.786	0.8210181
7.61	-23.05408	-31.60181	1.370769	20.26470	-233.5920	-240.6638	1.030274	-2220.162	-1833.650	0.8259082
7.62	-24.45364	-32.97387	1.348424	20.00911	-244.6478	-251.4438	1.027779	-2308.548	-1918.266	0.8309406
7.63	-25.97732	-34.46999	1.326926	19.76181	-256.6794	-263.1987	1.025398	-2404.611	-2010.541	0.8361189
7.64	-27.64257	-36.10762	1.306232	19.52237	-269.8242	-276.0654	1.023131	-2509.430	-2111.552	0.8414468
7.65	-29.47029	-37.90766	1.286301	19.29037	-284.2464	-290.2084	1.020975	-2624.297	-2222.592	0.8469284
7.66	-31.48569	-39.89538	1.267094	19.06543	-300.1442	-305.8257	1.018929	-2750.769	-2345.217	0.8525677
7.67	-33.71951	-42.10135	1.248575	18.84719	-317.7590	-323.1588	1.016993	-2890.744	-2481.325	0.8583690
7.68	-36.20957	-44.56354	1.230712	18.63531	-337.3882	-342.5051	1.015166	-3046.558	-2633.258	0.8643370
7.69	-39.00296	-47.32900	1.213472	18.42947	-359.4019	-364.2345	1.013446	-3221.121	-2803.909	0.8704762
7.70	-42.15907	-50.45712	1.196827	18.22938	-384.2668	-388.8139	1.011833	-3418.100	-2996.962	0.8767917
7.71	-45.75387	-54.02386	1.180749	18.03476	-412.5801	-416.8404	1.010326	-3642.190	-3217.104	0.8832885
7.72	-49.88617	-58.12802	1.165218	17.84535	-445.1181	-449.0903	1.008924	-3899.491	-3470.438	0.8899720
7.73	-54.68682	-62.90046	1.150194	17.66091	-482.9095	-486.5923	1.007626	-4198.088	-3765.046	0.8968479
7.74	-60.33294	-68.51828	1.135670	17.48120	-527.3462	-530.7381	1.006432	-4548.920	-4111.869	0.9039220
7.75	-67.07026	-75.22725	1.121619	17.30602	-580.3595	-583.4592	1.005341	-4967.168	-4526.086	0.9112006
7.76	-75.25001	-83.37856	1.108021	17.13514	-644.7097	-647.5159	1.004353	-5474.524	-5029.391	0.9186901
7.77	-85.39224	-93.49226	1.094857	16.96839	-724.4843	-726.9954	1.003466	-6108.112	-5658.906	0.9263972
7.78	-98.30117	-106.3726	1.082109	16.80557	-826.0036	-828.2184	1.002681	-6902.609	-6449.308	0.9343290
7.79	-115.2900	-123.3327	1.069761	16.64652	-959.5886	-961.5055	1.001998	-7954.127	-7496.709	0.9424930
7.80	-138.6592	-146.6731	1.057796	16.49108	-1143.319	-1144.937	1.001415	-9399.764	-8938.208	0.9508970
7.81	-172.8389	-180.8240	1.046200	16.33908	-1412.015	-1413.331	1.000933	-11513.17	-11047.46	0.9595492
7.82	-227.5966	-235.5527	1.034957	16.19038	-1842.438	-1843.452	1.000551	-14897.66	-14427.76	0.9684581
7.83	-329.5360	-337.4631	1.024055	16.04485	-2643.678	-2644.389	1.000269	-21196.52	-20722.41	0.9776328
7.84	-585.8507	-593.7487	1.013481	15.90235	-4658.202	-4658.607	1.000087	-37030.96	-36552.68	0.9870828

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
7.85	-2441.721	-2449.590	1.003223	15.76276	-19244.13	-19244.28	1.000005	-151667.9	-151185.3	0.9968181
7.86	1164.517	1156.678	0.9932681	15.62595	9098.344	9098.554	1.000023	71082.61	71569.47	1.006849
7.87	476.4183	468.6034	0.9836068	15.49182	3690.255	3690.775	1.000141	28577.07	29068.22	1.017187
7.88	301.8962	294.1158	0.9742283	15.36026	2318.602	2319.433	1.000359	17795.11	18290.59	1.027844
7.89	222.2283	214.4776	0.9651226	15.23116	1692.397	1693.542	1.000676	12871.86	13871.68	1.038831
7.90	176.6028	168.8818	0.9562805	15.10442	1333.742	1335.201	1.001094	10051.35	10555.54	1.050162
7.91	147.0380	139.3469	0.9476929	14.97996	1101.312	1103.088	1.001613	8222.874	8731.462	1.061850
7.92	126.3206	118.6594	0.9393514	14.85769	938.4160	940.5100	1.002281	6940.877	7453.882	1.073911
7.93	110.9942	103.3631	0.9312479	14.73751	817.8893	820.3032	1.002951	5991.867	6509.315	1.086358
7.94	99.19553	91.59465	0.9233747	14.61935	725.0872	727.8228	1.003773	5260.745	5782.659	1.099209
7.95	89.83116	82.26059	0.9157244	14.50313	651.4167	654.4755	1.004696	4679.974	5206.380	1.112481
7.96	82.21726	74.67712	0.9082900	14.38878	591.5031	594.8870	1.005721	4207.814	4738.237	1.126191
7.97	75.90419	68.39459	0.9010649	14.27622	541.8125	545.5233	1.006849	3814.991	4350.457	1.140359
7.98	70.58414	63.10522	0.8940425	14.16539	499.9258	503.9653	1.008080	3483.992	4024.027	1.155005
7.99	66.03932	58.59120	0.8872169	14.05621	464.1314	468.5014	1.009415	3200.865	3745.494	1.170151
8.00	62.11128	54.69409	0.8805822	13.94864	433.1839	437.8862	1.010855	2955.823	3505.072	1.185820
8.01	58.68201	51.29587	0.8741329	13.84260	406.1558	411.1922	1.012400	2741.574	3295.470	1.202086
8.02	55.66176	48.30681	0.8678635	13.73804	382.3418	387.7143	1.014052	2552.577	3111.147	1.218826
8.03	52.98113	45.65750	0.8617691	13.63491	361.1963	366.9068	1.015810	2384.544	2947.813	1.236217
8.04	50.58554	43.29337	0.8558447	13.53914	342.2906	348.3409	1.017676	2234.104	2802.101	1.254239
8.05	48.43154	41.17097	0.8500858	13.43269	325.2830	331.6752	1.019651	2098.573	2671.324	1.272924
8.06	46.48407	39.25523	0.8444877	13.33351	309.8980	316.6339	1.021736	1975.784	2553.317	1.292306
8.07	44.71453	37.51756	0.8390463	13.23556	295.9108	302.9925	1.023932	1863.972	2446.315	1.312420
8.08	43.09937	35.93442	0.8337575	13.13877	283.1364	290.5659	1.026240	1761.682	2348.863	1.333307
8.09	41.61902	34.48624	0.8286174	13.04312	271.4209	279.2002	1.028661	1667.705	2259.752	1.355007
8.10	40.25708	33.15662	0.8236221	12.94855	260.6354	268.7666	1.031198	1581.028	2177.969	1.377565
8.11	38.99970	31.93171	0.8187682	12.85503	250.6712	259.1564	1.033850	1500.795	2102.660	1.401030
8.12	37.83512	30.79976	0.8140521	12.76252	241.4357	250.2769	1.036619	1426.280	2033.097	1.425454
8.13	36.75327	29.75069	0.8094705	12.67097	232.8498	242.0492	1.039508	1356.860	1968.659	1.450893
8.14	35.74548	28.77584	0.8050203	12.58035	224.8453	234.4051	1.042517	1292.001	1908.811	1.477407
8.15	34.80426	27.86771	0.8006984	12.49062	217.3634	227.2857	1.045649	1231.238	1853.089	1.505062
8.16	33.92307	27.01980	0.7965020	12.40175	210.3528	220.6398	1.048904	1174.170	1801.092	1.533928
8.17	33.09624	26.22639	0.7924282	12.31371	203.7687	214.4227	1.052285	1120.444	1752.468	1.564084
8.18	32.31875	25.48251	0.7884743	12.22645	197.5719	208.5951	1.055794	1069.751	1706.907	1.595612
8.19	31.58621	24.78373	0.7846377	12.13996	191.7276	203.1223	1.059432	1021.819	1664.138	1.628604

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
8.20	30.89470	24.12617	0.7809161	12.05419	186.2053	197.9738	1.063202	976.4073	1623.921	1.663160
8.21	30.24078	23.50637	0.7773070	11.96912	180.9777	193.1224	1.067106	933.3019	1586.042	1.699388
8.22	29.62136	22.92125	0.7738081	11.88471	176.0207	188.5438	1.071146	892.3123	1550.310	1.737408
8.23	29.03368	22.36805	0.7704173	11.80095	171.3125	184.2165	1.075324	853.2681	1516.556	1.777350
8.24	28.47527	21.84431	0.7671325	11.71779	166.8337	180.1209	1.079644	816.0165	1484.627	1.819359
8.25	27.94390	21.34779	0.7639517	11.63522	162.5668	176.2398	1.084107	780.4201	1454.386	1.863593
8.26	27.43758	20.87651	0.7608729	11.55321	158.4961	172.5573	1.088716	746.3550	1425.709	1.910229
8.27	26.95449	20.42865	0.7578943	11.47173	154.6073	169.0592	1.093475	713.7092	1398.485	1.959460
8.28	26.49298	20.00258	0.7550142	11.39075	150.8875	165.7326	1.098385	682.3811	1372.612	2.011508
8.29	26.05158	19.59680	0.7522308	11.31026	147.3250	162.5659	1.103451	652.2785	1347.999	2.066600
8.30	25.62890	19.20995	0.7495426	11.23022	143.9092	159.5484	1.108674	623.3174	1324.561	2.125019
8.31	25.22373	18.84081	0.7469479	11.15062	140.6301	156.6703	1.114059	595.4210	1302.223	2.187063
8.32	24.83491	18.48824	0.7444453	11.07143	137.4790	153.9228	1.119609	568.5193	1280.915	2.253071
8.33	24.46142	18.15119	0.7420334	10.99263	134.4477	151.2977	1.125328	542.5479	1260.572	2.323429
8.34	24.10231	17.82874	0.7397108	10.91420	131.5287	148.7877	1.131219	517.4479	1241.136	2.398572
8.35	23.75668	17.51999	0.7374762	10.83611	128.7150	146.3857	1.137286	493.1650	1222.554	2.478995
8.36	23.42374	17.22414	0.7353284	10.75834	126.0003	144.0855	1.143533	469.6492	1204.774	2.565264
8.37	23.10274	16.94046	0.7332661	10.68088	123.3788	141.8812	1.149965	446.8544	1187.753	2.658032
8.38	22.79299	16.66825	0.7312883	10.60370	120.8450	139.7676	1.156586	424.7378	1171.447	2.758048
8.39	22.49385	16.40688	0.7293939	10.52678	118.3939	137.7395	1.163400	403.2599	1155.817	2.866184
8.40	22.20474	16.15576	0.7275818	10.45010	116.0209	135.7923	1.170413	382.3842	1140.827	2.983458
8.41	21.92510	15.91436	0.7258511	10.37364	113.7216	133.9218	1.177629	362.0766	1126.444	3.111065
8.42	21.65442	15.68215	0.7242008	10.29739	111.4920	132.1241	1.185054	342.3055	1112.635	3.250416
8.43	21.39222	15.45867	0.7226302	10.22133	109.3284	130.3953	1.192693	323.0414	1099.373	3.403196
8.44	21.13807	15.24347	0.7211382	10.14542	107.2274	128.7320	1.200552	304.2568	1086.630	3.571423
8.45	20.89156	15.03616	0.7197242	10.06967	105.1855	127.1311	1.208637	285.9261	1074.380	3.757544
8.46	20.65228	14.83634	0.7183874	9.994042	103.1999	125.5895	1.216953	268.0253	1062.600	3.964552
8.47	20.41989	14.64366	0.7171272	9.918525	101.2676	124.1043	1.225509	250.5320	1051.269	4.196145
8.48	20.19405	14.45778	0.7159428	9.843102	99.38602	122.6731	1.234309	233.4252	1040.365	4.456951
8.49	19.97443	14.27839	0.7148336	9.767752	97.55263	121.2933	1.243362	216.6851	1029.869	4.752837
8.50	19.76074	14.10520	0.7137992	9.692460	95.76508	119.9626	1.252675	200.2932	1019.763	5.091353
8.51	19.55270	13.93792	0.7128388	9.617206	94.02117	118.6788	1.262256	184.2321	1010.031	5.482382
8.52	19.35005	13.77631	0.7119522	9.541975	92.31883	117.4399	1.272112	168.4854	1000.655	5.939122
8.53	19.15253	13.62011	0.7111387	9.466748	90.65610	116.2440	1.282253	153.0377	991.6221	6.479594
8.54	18.95993	13.46909	0.7103980	9.391508	89.03115	115.0894	1.292687	137.8744	982.9171	7.129075

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
8.55	18.77200	13.32305	0.7097297	9.316238	87.44222	113.9742	1.303423	122.9818	974.5269	7.924153
8.56	18.58856	13.18177	0.7091335	9.240921	85.88769	112.8969	1.314472	108.3470	966.4391	8.919852
8.57	18.40939	13.04506	0.7086091	9.165540	84.36601	111.8561	1.325843	93.95760	958.6419	10.20292
8.58	18.23432	12.91275	0.7081562	9.090078	82.87569	110.8501	1.337547	79.80203	951.1242	11.91855
8.59	18.06317	12.78465	0.7077746	9.014517	81.41537	109.8779	1.349596	65.86930	943.8753	14.32952
8.60	17.89577	12.66061	0.7074640	8.938841	79.98372	108.9380	1.362002	52.14897	936.8855	17.96556
8.61	17.73197	12.54048	0.7072244	8.863033	78.57949	108.0292	1.374776	38.63113	930.1453	24.07761
8.62	17.57161	12.42410	0.7070556	8.787074	77.20150	107.1504	1.387932	25.30640	923.6458	36.49851
8.63	17.41455	12.31135	0.7069575	8.710950	75.84863	106.3006	1.401484	12.16582	917.3787	75.40622
8.64	17.26066	12.20208	0.7069301	8.634641	74.51980	105.4787	1.415446	-0.7990917	911.3359	-1140.465
8.65	17.10981	12.19618	0.7069734	8.558132	73.21399	104.6838	1.429833	-13.59644	905.5099	-66.59904
8.66	16.96187	11.99352	0.7070873	8.481403	71.93024	103.9149	1.444662	-26.23394	899.8936	-34.30265
8.67	16.81674	11.89401	0.7072720	8.404440	70.66763	103.1711	1.459949	-38.71893	894.4802	-23.10188
8.68	16.67429	11.79752	0.7075275	8.327222	69.42527	102.4517	1.475712	-51.05844	889.2631	-17.41657
8.69	16.53443	11.70396	0.7078539	8.249734	68.20232	101.7558	1.491969	-63.25918	884.2364	-13.97799
8.70	16.39705	11.61323	0.7082515	8.171957	66.99798	101.0827	1.508742	-75.32754	879.3941	-11.67427
8.71	16.26205	11.52525	0.7087204	8.093873	65.81149	100.4316	1.526050	-87.26966	874.7308	-10.02331
8.72	16.12935	11.43992	0.7092609	8.015465	64.64212	99.80201	1.543916	-99.09139	870.2411	-8.782207
8.73	15.99885	11.35716	0.7098731	7.936713	63.48915	99.19314	1.562364	-110.7984	865.9202	-7.815280
8.74	15.87048	11.27689	0.7105576	7.857599	62.35192	98.60440	1.581417	-122.3959	861.7632	-7.040782
8.75	15.74414	11.19903	0.7113144	7.778105	61.22979	98.03522	1.601103	-133.8893	857.7657	-6.406528
8.76	15.61977	11.12353	0.7121442	7.698212	60.12214	97.48503	1.621450	-145.2835	853.9239	-5.877636
8.77	15.49728	11.05029	0.7130472	7.617901	59.02838	96.95331	1.642486	-156.5831	850.2318	-5.429909
8.78	15.37662	10.97927	0.7140240	7.537152	57.94794	96.43954	1.664244	-167.7928	846.6875	-5.046030
8.79	15.25770	10.91040	0.7150750	7.455945	56.88028	95.94325	1.686758	-178.9170	843.2866	-4.713284
8.80	15.14046	10.84361	0.7162008	7.374260	55.82486	95.46397	1.710062	-189.9600	840.0256	-4.422119
8.81	15.02485	10.77886	0.7174020	7.292078	54.78119	95.00125	1.734195	-200.9257	836.9010	-4.165226
8.82	14.91080	10.71608	0.7186791	7.209377	53.74878	94.55469	1.759197	-211.8183	833.9097	-3.936911
8.83	14.79825	10.65523	0.7200329	7.126136	52.72717	94.12387	1.785112	-222.6414	831.0486	-3.732679
8.84	14.68714	10.59625	0.7214640	7.042335	51.71589	93.70842	1.811985	-233.3987	828.3148	-3.548927
8.85	14.57743	10.53909	0.7229732	6.957950	50.71452	93.30797	1.839867	-244.0938	825.7055	-3.382738
8.86	14.46906	10.48372	0.7245613	6.872961	49.72264	92.92216	1.868810	-254.7301	823.2180	-3.231726
8.87	14.36198	10.43009	0.7262291	6.787345	48.73984	92.55067	1.898871	-265.3110	820.8499	-3.093916
8.88	14.25614	10.37815	0.7279776	6.701077	47.76573	92.19318	1.930111	-275.8395	818.5986	-2.967662
8.89	14.15149	10.32786	0.7298075	6.614136	46.79994	91.84937	1.962596	-286.3190	816.4620	-2.851582

TABLE I (continued)

λ	$K\frac{L}{EI}$	$kK\frac{L}{EI}$	k	$K''\frac{L}{EI}$	$Q\frac{L^2}{EI}$	$qQ\frac{L^2}{EI}$	q	$T\frac{L^3}{EI}$	$tT\frac{L^3}{EI}$	t
8.90	14.04799	10.27920	0.7317201	6.526497	45.84209	91.51897	1.996396	-296.7523	814.4378	-2.744504
8.91	13.94560	10.23211	0.7337161	6.438135	44.89183	91.20168	2.031588	-307.1423	812.5240	-2.645431
8.92	13.84427	10.18657	0.7357969	6.349025	43.94881	90.89725	2.068253	-317.4921	810.7186	-2.553508
8.92	13.74397	10.14255	0.7379634	6.259141	43.01272	90.60542	2.106480	-327.8042	809.0196	-2.467996
8.94	13.64465	10.10000	0.7402169	6.168458	42.08321	90.32596	2.146366	-338.0814	807.4254	-2.388257
8.95	13.54627	10.05890	0.7425586	6.076947	41.15999	90.05862	2.188014	-348.3264	805.9342	-2.313733
8.96	13.44880	10.01922	0.7449899	5.984582	40.24273	89.80319	2.231538	-358.5416	804.5444	-2.243936
8.97	13.35221	9.980935	0.7475121	5.891335	39.33115	89.55946	2.277061	-368.7295	803.2545	-2.178438
8.98	13.25645	9.944014	0.7501267	5.797175	38.42497	89.32722	2.324718	-378.8926	802.0630	-2.116861
8.99	13.16149	9.908430	0.7528351	5.702074	37.52389	89.10629	2.374655	-389.0833	800.9685	-2.058869
9.00	13.06730	9.874161	0.7556389	5.606000	36.62764	88.89648	2.427032	-399.1537	799.9698	-2.004165
9.01	12.97385	9.841183	0.7585398	5.508923	35.73597	88.69762	2.482026	-409.2563	799.0656	-1.952482
9.02	12.88111	9.809472	0.7615394	5.410809	34.84861	88.50955	2.539830	-419.3432	798.2547	-1.903583
9.03	12.78904	9.779007	0.7646396	5.311626	33.96530	88.33210	2.600657	-429.4166	797.5360	-1.857255
9.04	12.69762	9.749769	0.7678421	5.211339	33.08580	88.16513	2.664742	-439.4786	796.9085	-1.813305
9.05	12.60682	9.721736	0.7711490	5.109913	32.20987	88.00850	2.732345	-449.5313	796.3713	-1.771559
9.06	12.51661	9.694890	0.7745622	5.007311	31.33727	87.86206	2.803756	-459.5767	795.9234	-1.731862
9.07	12.42696	9.669213	0.7780838	4.903497	30.46777	87.72570	2.879295	-469.6169	795.5640	-1.694070
9.08	12.33784	9.644688	0.7817161	4.798431	29.60114	87.59929	2.959322	-479.6538	795.2923	-1.658055
9.09	12.24923	9.621299	0.7854614	4.692074	28.73715	87.48272	3.044238	-489.6893	795.1075	-1.623698
9.10	12.16111	9.599029	0.7893219	4.584385	27.87560	87.37588	3.134493	-499.7255	795.0089	-1.590891
9.11	12.07344	9.577865	0.7933002	4.475320	27.01626	87.27868	3.230598	-509.7641	794.9960	-1.559537
9.12	11.98621	9.557793	0.7973990	4.364837	26.15893	87.19100	3.333125	-519.8070	795.0681	-1.529545
9.13	11.89939	9.538798	0.8016208	4.252890	25.30340	87.11277	3.442730	-529.8561	795.2247	-1.500832
9.14	11.81295	9.520868	0.8059685	4.139432	24.44946	87.04390	3.560157	-539.9132	795.4654	-1.473321
9.15	11.72688	9.503991	0.8104451	4.024415	23.59691	86.98432	3.686259	-549.9801	795.7896	-1.446943
9.16	11.64114	9.488156	0.8150536	3.907787	22.74556	86.93394	3.822018	-560.0585	796.1971	-1.421632
9.17	11.55573	9.473353	0.8197972	3.789499	21.89520	86.89271	3.968573	-570.1503	796.6874	-1.397329
9.18	11.47060	9.459570	0.8246792	3.669494	21.04566	86.86056	4.127244	-580.2571	797.2603	-1.373978
9.19	11.38576	9.446799	0.8297032	3.547718	20.19673	86.83744	4.299580	-590.3808	797.9155	-1.351527
9.20	11.30116	9.435030	0.8348726	3.424113	19.34822	86.82329	4.487404	-600.5229	798.6529	-1.329929
9.21	11.21680	9.424256	0.8401914	3.298618	18.49996	86.81807	4.692878	-610.6854	799.4722	-1.309139
9.22	11.13264	9.414468	0.8456634	3.171171	17.65175	86.82173	4.918589	-620.8697	800.3733	-1.289116
9.23	11.04868	9.405660	0.8512928	3.041707	16.80342	86.83424	5.167653	-631.0777	801.3563	-1.269822
9.24	10.96488	9.397824	0.8570839	2.910159	15.95478	86.85557	5.443859	-641.3111	802.4209	-1.251220

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
9.25	10.88124	9.390955	0.8630411	2.776458	15.10565	86.88568	5.751868	-651.5714	803.5673	-1.233276
9.26	10.79772	9.385047	0.8691692	2.640529	14.25585	86.92457	6.097468	-661.8605	804.7955	-1.215959
9.27	10.71432	9.380095	0.8754730	2.502297	13.40520	86.97219	6.487945	-672.1799	806.1056	-1.199241
9.28	10.63100	9.376094	0.8819577	2.361683	12.55353	87.02856	6.932597	-682.5314	807.4977	-1.183092
9.29	10.54776	9.373041	0.8886287	2.218605	11.70066	87.09365	7.443483	-692.9166	808.9720	-1.167488
9.30	10.46457	9.370931	0.8954915	2.072978	10.84641	87.16745	8.036528	-703.3372	810.5287	-1.152404
9.31	10.38141	9.369761	0.9025521	1.924711	9.990604	87.24998	8.733204	-713.7949	812.1681	-1.137817
9.32	10.29826	9.369530	0.9098165	1.773711	9.133072	87.34122	9.563182	-724.2915	813.8904	-1.123706
9.33	10.21511	9.370235	0.9172912	1.619881	8.273635	87.44120	10.56866	-734.8285	815.6960	-1.110050
9.34	10.13194	9.371874	0.9249829	1.463119	7.412119	87.54993	11.81173	-745.4077	817.5853	-1.096830
9.35	10.04873	9.374445	0.9328988	1.303319	6.548346	87.66741	13.38772	-756.0308	819.5588	-1.084028
9.36	9.965451	9.377950	0.9410461	1.140368	5.682142	87.79367	15.45081	-766.6995	821.6167	-1.071628
9.37	9.882096	9.382386	0.9494328	0.9741515	4.813329	87.92874	18.26776	-777.4156	823.7598	-1.059613
9.38	9.798643	9.387754	0.9580668	0.8045464	3.941731	88.07265	22.34365	-788.1808	825.9884	-1.047968
9.39	9.715072	9.394056	0.9669569	0.6314250	3.067170	88.22543	28.76444	-798.9969	828.3033	-1.036679
9.40	9.631366	9.401292	0.9761119	0.4546533	2.189466	88.38712	40.36926	-809.8656	830.7049	-1.025732
9.41	9.547507	9.409463	0.9855414	0.2740907	1.308441	88.55777	67.68189	-820.7887	833.1941	-1.015114
9.42	9.463474	9.418573	0.9952553	+0.08958941	+0.4239135	88.73741	+209.3290	-831.7681	835.7715	-1.004813
9.43	9.379250	9.428623	1.005264	-0.09900557	-0.4642990	88.92611	-191.5277	-842.8055	838.4379	-0.9948178
9.44	9.294815	9.439616	1.015579	-0.2918573	-1.356380	89.12392	-65.70720	-853.9029	841.1941	-0.9851168
9.45	9.210151	9.451556	1.026211	-0.4891373	-2.252514	89.33090	-39.65830	-865.0620	844.0409	-0.9756998
9.46	9.125239	9.464447	1.037173	-0.6910262	-3.152890	89.54712	-28.40160	-876.2849	846.9792	-0.9665563
9.47	9.040059	9.478294	1.048477	-0.8977140	-4.057694	89.77265	-22.12406	-887.5733	850.0101	-0.9576787
9.48	8.954592	9.493101	1.060138	-1.109401	-4.967118	90.00756	-18.12068	-898.9294	853.1344	-0.9490561
9.49	8.868820	9.508873	1.072169	-1.326300	-5.881355	90.25194	-15.34543	-910.3549	856.3532	-0.9406806
9.50	8.782721	9.525618	1.084586	-1.548632	-6.800602	90.50586	-13.30851	-921.8520	859.6677	-0.9325441
9.51	8.696277	9.543340	1.097405	-1.776635	-7.725055	90.76942	-11.75000	-933.4227	863.0789	-0.9246388
9.52	8.609467	9.562048	1.110643	-2.010558	-8.654914	91.04272	-10.51919	-945.0690	866.5880	-0.9169574
9.53	8.522272	9.581747	1.124318	-2.250664	-9.590385	91.32585	-9.522648	-956.7931	870.1964	-0.9094927
9.54	8.434670	9.602448	1.138450	-2.497234	-10.53167	91.61893	-8.699371	-968.5971	873.9052	-0.9022381
9.55	8.346641	9.624157	1.153057	-2.750564	-11.47899	91.92205	-8.007854	-980.4832	877.7159	-0.8951871
9.56	8.258165	9.646884	1.168163	-3.010969	-12.43254	92.23534	-7.418866	-992.4536	881.6299	-0.8883337
9.57	8.169220	9.670638	1.183790	-3.278783	-13.39255	92.55892	-6.911225	-1004.511	885.6487	-0.8816718
9.58	8.079784	9.695431	1.199962	-3.554362	-14.35924	92.89292	-6.469210	-1016.656	889.7737	-0.8751961
9.59	7.989835	9.721271	1.216705	-3.838083	-15.33282	93.23748	-6.080907	-1028.894	894.0066	-0.8689010

TABLE I (continued)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L^2}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
9.60	7.899352	9.748172	1.234047	-4.130349	-16.31354	93.59272	-5.737118	-1041.224	898.3491	-0.8627815
9.61	7.808312	9.776143	1.252018	-4.431591	-17.30162	93.95880	-5.430636	-1053.651	902.8028	-0.8568327
9.62	7.716692	9.805199	1.270648	-4.742266	-18.29730	94.33587	-5.155726	-1066.177	907.3695	-0.8510499
9.63	7.624468	9.835352	1.289972	-5.062864	-19.30082	94.72410	-4.907775	-1078.803	912.0512	-0.8454286
9.64	7.531617	9.866616	1.310026	-5.393911	-20.31243	95.12363	-4.683025	-1091.534	916.8498	-0.8399645
9.65	7.438114	9.899006	1.330849	-5.735967	-21.33239	95.53466	-4.478386	-1104.371	921.7671	-0.8346536
9.66	7.343936	9.932536	1.352481	-6.089635	-22.36094	95.95735	-4.291293	-1117.317	926.8055	-0.8294918
9.67	7.249055	9.967223	1.374968	-6.455561	-23.39836	96.39190	-4.119600	-1130.375	931.9669	-0.8244755
9.68	7.153448	10.00308	1.398358	-6.834442	-24.44491	96.83849	-3.961499	-1143.549	937.2536	-0.8196009
9.69	7.057087	10.04013	1.422702	-7.227026	-25.50088	97.29734	-3.815451	-1156.840	942.6680	-0.8148646
9.70	6.959947	10.07839	1.448056	-7.634121	-26.56654	97.76865	-3.680143	-1170.252	948.2125	-0.8102634
9.71	6.861998	10.11787	1.474479	-8.056598	-27.64218	98.25264	-3.554446	-1183.788	953.8895	-0.8057939
9.72	6.763213	10.15861	1.502038	-8.495401	-28.72810	98.74954	-3.437385	-1197.452	959.7017	-0.8014532
9.73	6.663564	10.20060	1.530803	-8.951550	-29.82461	99.25959	-3.328110	-1211.246	965.6517	-0.7972383
9.74	6.563021	10.24389	1.560850	-9.426155	-30.93203	99.78302	-3.225881	-1225.174	971.7424	-0.7931465
9.75	6.461553	10.28849	1.592262	-9.920419	-32.05066	100.3201	-3.130048	-1239.239	977.9765	-0.7891749
9.76	6.359131	10.33442	1.625131	-10.43565	-33.18084	100.8711	-3.040040	-1253.446	984.3572	-0.7853211
9.77	6.255721	10.38171	1.659554	-10.97329	-34.32292	101.4363	-2.955351	-1267.796	990.8876	-0.7815825
9.78	6.151291	10.43038	1.695641	-11.53489	-35.47723	102.0159	-2.875533	-1282.296	997.5707	-0.7779568
9.79	6.045809	10.48046	1.733509	-12.12216	-36.64414	102.6104	-2.800184	-1296.947	1004.410	-0.7744417
9.80	5.939238	10.53198	1.773288	-12.73699	-37.82402	103.2199	-2.728950	-1311.755	1011.409	-0.7710349
9.81	5.831545	10.58496	1.815121	-13.38144	-39.01724	103.8448	-2.661509	-1326.723	1018.571	-0.7677345
9.82	5.722692	10.63944	1.859166	-14.05779	-40.22421	104.4854	-2.597575	-1341.856	1025.901	-0.7645383
9.83	5.612642	10.69544	1.905598	-14.76856	-41.44531	105.1421	-2.536887	-1357.158	1033.401	-0.7614444
9.84	5.501357	10.75299	1.954608	-15.51653	-42.68097	105.8152	-2.479213	-1372.634	1041.076	-0.7584511
9.85	5.388796	10.81214	2.006411	-16.30480	-43.93162	106.5052	-2.424340	-1388.287	1048.930	-0.7555565
9.86	5.274920	10.87291	2.061247	-17.13683	-45.19769	107.2123	-2.372075	-1404.124	1056.967	-0.7527588
9.87	5.159684	10.93534	2.119382	-18.01647	-46.47966	107.9371	-2.322243	-1420.149	1065.192	-0.7500566
9.88	5.043047	10.99947	2.181115	-18.94806	-47.77797	108.6798	-2.274685	-1436.367	1073.610	-0.7474482
9.89	4.924962	11.06533	2.246785	-19.93645	-49.09313	109.4410	-2.229254	-1452.783	1082.225	-0.7449321
9.90	4.805384	11.13297	2.316770	-20.98714	-50.42564	110.2212	-2.185816	-1469.403	1091.042	-0.7425069
9.91	4.684264	11.20243	2.391502	-22.10636	-51.77602	111.0207	-2.144249	-1486.233	1100.067	-0.7401712
9.92	4.561552	11.27375	2.471471	-23.30119	-53.14480	111.8400	-2.104440	-1503.279	1109.305	-0.7379237
9.93	4.437198	11.34697	2.557238	-24.57972	-54.53255	112.6797	-2.066284	-1520.546	1118.762	-0.7357633
9.94	4.311149	11.42216	2.649446	-25.95125	-55.93984	113.5404	-2.029687	-1538.041	1128.443	-0.7336885

TABLE I (concluded)

λ	$K \frac{L}{EI}$	$kK \frac{L}{EI}$	k	$K'' \frac{L}{EI}$	$Q \frac{L}{EI}$	$qQ \frac{L^2}{EI}$	q	$T \frac{L^3}{EI}$	$tT \frac{L^3}{EI}$	t
9.95	4.183348	11.49935	2.748838	-27.42649	-57.36727	114.4225	-1.994560	-1555.771	1138.355	-0.7316985
9.96	4.053739	11.57859	2.856274	-29.01788	-58.81546	115.3266	-1.960821	-1573.741	1148.504	-0.7297920
9.97	3.922263	11.65994	2.972759	-30.73993	-60.28506	116.2533	-1.928393	-1591.961	1158.896	-0.7279680
9.98	3.788860	11.74346	3.099471	-32.60967	-61.77672	117.2033	-1.897207	-1610.435	1169.539	-0.7262256
9.99	3.653464	11.82921	3.237805	-34.64720	-63.29115	118.1771	-1.867198	-1629.173	1180.440	-0.7245637
10.00	3.516011	11.91723	3.389419	-36.87649	-64.82907	119.1755	-1.838304	-1648.182	1191.606	-0.7229817
10.01	3.376432	12.00761	3.556303	-39.32625	-66.39121	120.1992	-1.810469	-1667.471	1203.044	-0.7214785
10.02	3.234656	12.10040	3.740862	-42.08128	-67.97836	121.2489	-1.783640	-1687.047	1214.764	-0.7200534
10.03	3.090608	12.19568	3.946044	-45.03406	-69.59132	122.3254	-1.757767	-1706.920	1226.773	-0.7187056
10.04	2.944213	12.29350	4.175481	-48.38708	-71.23094	123.4294	-1.732806	-1727.099	1239.080	-0.7174346
10.05	2.795390	12.39396	4.433716	-52.15593	-72.89807	124.5618	-1.708711	-1747.593	1251.695	-0.7162395
10.06	2.644055	12.49713	4.726502	-56.42367	-74.59365	125.7234	-1.685444	-1768.412	1264.627	-0.7151198
10.07	2.490123	12.60309	5.061233	-61.29707	-76.31862	126.9153	-1.662966	-1789.568	1277.885	-0.7140748
10.08	2.333502	12.71193	5.447575	-66.91569	-78.07396	128.1382	-1.641242	-1811.070	1291.481	-0.7131042
10.09	2.174099	12.82373	5.898411	-73.46555	-79.86070	129.3933	-1.620237	-1832.930	1305.426	-0.7122073
10.10	2.011816	12.93860	6.431300	-81.20017	-81.67992	130.6815	-1.599922	-1855.160	1319.731	-0.7113838
10.11	1.846549	13.05662	7.070820	-90.47443	-83.53275	132.0039	-1.580265	-1877.772	1334.407	-0.7106331
10.12	1.678192	13.17790	7.852440	-101.8005	-85.42036	133.3617	-1.561240	-1900.779	1349.467	-0.7099549
10.13	1.506632	13.30255	8.829332	-115.9460	-87.34398	134.7561	-1.542820	-1924.195	1364.926	-0.7093489
10.14	1.331752	13.43069	10.08497	-134.1164	-89.30488	136.1882	-1.524981	-1948.033	1380.795	-0.7088148
10.15	1.153431	13.56242	11.75833	-158.3179	-91.30441	137.6595	-1.507698	-1972.309	1397.090	-0.7083523
10.16	0.9715397	13.69788	14.09914	-192.1568	-93.34399	139.1712	-1.490950	-1997.038	1413.825	-0.7079611
10.17	0.7859439	13.83719	17.60582	-242.8292	-95.42507	140.7248	-1.474716	-2022.235	1431.017	-0.7076411
10.18	0.5965028	13.98050	23.43743	-327.0704	-97.54921	142.3219	-1.458973	-2047.919	1448.682	-0.7073921
10.19	0.4030689	14.12793	35.05091	-494.7939	-99.71802	143.9640	-1.443710	-2074.106	1466.837	-0.7072139
10.20	0.2054873	14.27966	69.49167	-992.1118	-101.9332	145.6527	-1.428903	-2100.816	1485.501	-0.7071066

TABLE II

VALUES OF THE COEFFICIENT C FOR THE STEADY - STATE FORCED DEFLECTION OF A UNIFORM BAR FIXED AT ONE END AND SUBJECTED TO A HARMONICALLY VARYING ROTATION AT THE OTHER END.

For an end rotation $\theta(t) = \Theta\cos\omega t$, the deflection of the bar at a distance $\bar{x}$ from the end being rotated is expressed as $w(\bar{x},t) = Y_{\bar{x}}\cos\omega t$, where $Y_{\bar{x}} = C\Theta L$, if Θ represents the rotation amplitude at the left end, and $Y_{\bar{x}} = -C\Theta L$, if Θ represents the rotation amplitude at the right end. Values of C are tabulated for successive twelveth points of the bar as a function of the dimensionless parameter

$$\lambda = \sqrt[4]{\frac{m\omega^2}{EI}}\,L$$

in which m is the mass per unit of length of the bar; ω is the circular frequency of vibration; E is the modulus of elasticity of the material in the bar; I is the moment of inertia of the cross section of the bar about its centroidal axis; and L is the span length of the bar.

λ	RATIO $\bar{x}$/L										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
0	0.070023	0.11574	0.14062	0.14815	0.14178	0.12500	0.10127	0.074074	0.046875	0.023148	0.006366
0.50	0.070025	0.11575	0.14064	0.14816	0.14180	0.12502	0.10129	0.074087	0.046884	0.023153	0.006367
0.60	0.070027	0.11575	0.14065	0.14818	0.14182	0.12504	0.10131	0.074101	0.046894	0.023158	0.006369
0.70	0.070030	0.11576	0.14067	0.14820	0.14185	0.12507	0.10134	0.074124	0.046909	0.023166	0.006371
0.80	0.070035	0.11578	0.14069	0.14824	0.14189	0.12512	0.10138	0.074160	0.046934	0.023179	0.006375
0.90	0.070042	0.11580	0.14074	0.14830	0.14196	0.12519	0.10144	0.074211	0.046969	0.023198	0.006380
1.00	0.070051	0.11583	0.14079	0.14838	0.14206	0.12528	0.10153	0.074284	0.047019	0.023224	0.006388
1.10	0.070064	0.11588	0.14087	0.14849	0.14218	0.12541	0.10165	0.074381	0.047085	0.023259	0.006398
1.20	0.070082	0.11593	0.14097	0.14863	0.14235	0.12559	0.10181	0.074509	0.047173	0.023305	0.006411
1.30	0.070104	0.11601	0.14111	0.14882	0.14257	0.12581	0.10202	0.074675	0.047287	0.023365	0.006428
1.40	0.070132	0.11610	0.14128	0.14905	0.14284	0.12609	0.10228	0.074884	0.047430	0.023440	0.006450
1.50	0.070166	0.11621	0.14148	0.14934	0.14318	0.12644	0.10260	0.075144	0.047608	0.023534	0.006477
1.55	0.070187	0.11628	0.14161	0.14951	0.14338	0.12665	0.10278	0.075295	0.047712	0.023589	0.006493
1.60	0.070209	0.11636	0.14174	0.14969	0.14360	0.12687	0.10299	0.075463	0.047827	0.023649	0.006511
1.65	0.070234	0.11644	0.14189	0.14990	0.14384	0.12712	0.10322	0.075648	0.047954	0.023716	0.006530
1.70	0.070261	0.11653	0.14205	0.15013	0.14410	0.12740	0.10347	0.075851	0.048093	0.023789	0.006551

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO : $\bar{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
1.75	0.070291	0.11663	0.14223	0.15037	0.14439	0.12770	0.10375	0.076074	0.048246	0.023870	0.006575
1.76	0.070297	0.11665	0.14227	0.15043	0.14446	0.12776	0.10381	0.076121	0.048278	0.023887	0.006580
1.77	0.070303	0.11667	0.14231	0.15048	0.14452	0.12782	0.10387	0.076169	0.048311	0.023904	0.006585
1.78	0.070310	0.11669	0.14234	0.15053	0.14458	0.12789	0.10393	0.076217	0.048344	0.023922	0.006590
1.79	0.070316	0.11671	0.14238	0.15059	0.14465	0.12796	0.10399	0.076267	0.048378	0.023940	0.006595
1.80	0.070323	0.11673	0.14242	0.15064	0.14471	0.12803	0.10405	0.076318	0.048413	0.023958	0.006600
1.81	0.070330	0.11676	0.14247	0.15070	0.14478	0.12809	0.10411	0.076369	0.048448	0.023976	0.006606
1.82	0.070337	0.11678	0.14251	0.15076	0.14485	0.12817	0.10418	0.076421	0.048484	0.023995	0.006611
1.83	0.070344	0.11680	0.14255	0.15082	0.14492	0.12824	0.10424	0.076475	0.048520	0.024014	0.006617
1.84	0.070351	0.11683	0.14259	0.15088	0.14499	0.12831	0.10431	0.076529	0.048558	0.024034	0.006622
1.85	0.070359	0.11685	0.14264	0.15094	0.14506	0.12838	0.10438	0.076584	0.048595	0.024054	0.006628
1.86	0.070366	0.11688	0.14268	0.15100	0.14513	0.12846	0.10445	0.076640	0.048634	0.024074	0.006634
1.87	0.070374	0.11690	0.14273	0.15107	0.14521	0.12854	0.10452	0.076697	0.048673	0.024095	0.006640
1.88	0.070382	0.11693	0.14277	0.15113	0.14528	0.12861	0.10459	0.076755	0.048713	0.024116	0.006646
1.89	0.070389	0.11695	0.14282	0.15120	0.14536	0.12869	0.10466	0.076814	0.048753	0.024137	0.006652
1.90	0.070397	0.11698	0.14287	0.15126	0.14544	0.12878	0.10474	0.076874	0.048795	0.024159	0.006658
1.91	0.070405	0.11701	0.14292	0.15133	0.14552	0.12886	0.10481	0.076936	0.048836	0.024181	0.006665
1.92	0.070414	0.11703	0.14297	0.15140	0.14560	0.12894	0.10489	0.076998	0.048879	0.024203	0.006671
1.93	0.070422	0.11706	0.14302	0.15147	0.14568	0.12903	0.10497	0.077061	0.048922	0.024226	0.006678
1.94	0.070431	0.11709	0.14307	0.15154	0.14577	0.12911	0.10505	0.077125	0.048967	0.024249	0.006685
1.95	0.070439	0.11712	0.14312	0.15161	0.14585	0.12920	0.10513	0.077191	0.049011	0.024273	0.006691
1.96	0.070448	0.11715	0.14318	0.15169	0.14594	0.12929	0.10521	0.077257	0.049057	0.024297	0.006698
1.97	0.070457	0.11718	0.14323	0.15176	0.14603	0.12938	0.10530	0.077325	0.049103	0.024321	0.006705
1.98	0.070466	0.11721	0.14328	0.15184	0.14611	0.12947	0.10538	0.077393	0.049150	0.024346	0.006713
1.99	0.070475	0.11724	0.14334	0.15191	0.14621	0.12957	0.10547	0.077463	0.049198	0.024372	0.006720
2.00	0.070485	0.11727	0.14340	0.15199	0.14630	0.12966	0.10555	0.077534	0.049247	0.024397	0.006727
2.01	0.070494	0.11730	0.14345	0.15207	0.14639	0.12976	0.10564	0.077606	0.049297	0.024423	0.006735
2.02	0.070504	0.11733	0.14351	0.15215	0.14649	0.12986	0.10573	0.077680	0.049347	0.024450	0.006743
2.03	0.070514	0.11737	0.14357	0.15224	0.14658	0.12996	0.10583	0.077754	0.049398	0.024477	0.006750
2.04	0.070524	0.11740	0.14363	0.15232	0.14668	0.13006	0.10592	0.077830	0.049450	0.024504	0.006758
2.05	0.070534	0.11743	0.14369	0.15241	0.14678	0.13017	0.10602	0.077907	0.049503	0.024532	0.006766
2.06	0.070545	0.11747	0.14376	0.15249	0.14689	0.13027	0.10611	0.077986	0.049557	0.024560	0.006775
2.07	0.070555	0.11750	0.14382	0.15258	0.14699	0.13038	0.10621	0.078065	0.049611	0.024589	0.006783
2.08	0.070566	0.11754	0.14388	0.15267	0.14709	0.13049	0.10631	0.078146	0.049667	0.024618	0.006791
2.09	0.070577	0.11757	0.14395	0.15276	0.14720	0.13060	0.10641	0.078228	0.049723	0.024648	0.006800

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{X}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
2.10	0.070588	0.11761	0.14402	0.15285	0.14731	0.13071	0.10652	0.078312	0.049780	0.024678	0.006809
2.11	0.070599	0.11765	0.14408	0.15295	0.14742	0.13082	0.10662	0.078396	0.049838	0.024709	0.006818
2.12	0.070611	0.11769	0.14415	0.15304	0.14753	0.13094	0.10673	0.078482	0.049898	0.024740	0.006827
2.13	0.070622	0.11772	0.14422	0.15314	0.14765	0.13106	0.10684	0.078570	0.049958	0.024772	0.006836
2.14	0.070634	0.11776	0.14429	0.15324	0.14776	0.13118	0.10695	0.078659	0.050019	0.024804	0.006845
2.15	0.070646	0.11780	0.14436	0.15334	0.14788	0.13130	0.10706	0.078749	0.050081	0.024836	0.006855
2.16	0.070658	0.11784	0.14444	0.15344	0.14800	0.13142	0.10717	0.078841	0.050144	0.024870	0.006864
2.17	0.070670	0.11788	0.14451	0.15354	0.14812	0.13155	0.10729	0.078934	0.050207	0.024903	0.006874
2.18	0.070683	0.11793	0.14459	0.15365	0.14824	0.13167	0.10740	0.079029	0.050272	0.024938	0.006884
2.19	0.070696	0.11797	0.14466	0.15375	0.14837	0.13180	0.10752	0.079125	0.050338	0.024972	0.006894
2.20	0.070708	0.11801	0.14474	0.15386	0.14849	0.13193	0.10764	0.079223	0.050405	0.025008	0.006904
2.21	0.070722	0.11805	0.14482	0.15397	0.14862	0.13207	0.10776	0.079322	0.050474	0.025043	0.006915
2.22	0.070735	0.11810	0.14490	0.15408	0.14875	0.13220	0.10789	0.079423	0.050543	0.025080	0.006925
2.23	0.070748	0.11814	0.14498	0.15420	0.14889	0.13234	0.10802	0.079525	0.050613	0.025117	0.006936
2.24	0.070762	0.11819	0.14506	0.15431	0.14902	0.13248	0.10814	0.079629	0.050684	0.025154	0.006947
2.25	0.070776	0.11824	0.14515	0.15443	0.14916	0.13262	0.10827	0.079734	0.050757	0.025193	0.006958
2.26	0.070790	0.11828	0.14523	0.15454	0.14930	0.13277	0.10841	0.079841	0.050830	0.025231	0.006969
2.27	0.070804	0.11833	0.14532	0.15466	0.14944	0.13291	0.10854	0.079950	0.050905	0.025271	0.006980
2.28	0.070819	0.11838	0.14541	0.15479	0.14958	0.13306	0.10868	0.080061	0.050981	0.025311	0.006992
2.29	0.070834	0.11843	0.14550	0.15491	0.14973	0.13321	0.10882	0.080173	0.051058	0.025351	0.007004
2.30	0.070849	0.11848	0.14559	0.15504	0.14988	0.13336	0.10896	0.080287	0.051136	0.025393	0.007016
2.31	0.070864	0.11853	0.14568	0.15516	0.15003	0.13352	0.10910	0.080402	0.051215	0.025434	0.007028
2.32	0.070879	0.11858	0.14577	0.15529	0.15018	0.13368	0.10924	0.080520	0.051296	0.025477	0.007040
2.33	0.070895	0.11863	0.14587	0.15542	0.15033	0.13384	0.10939	0.080639	0.051377	0.025520	0.007053
2.34	0.070911	0.11868	0.14596	0.15556	0.15049	0.13400	0.10954	0.080760	0.051360	0.025564	0.007065
2.35	0.070927	0.11874	0.14606	0.15569	0.15065	0.13417	0.10969	0.080883	0.051545	0.025608	0.007078
2.36	0.070944	0.11879	0.14616	0.15583	0.15081	0.13433	0.10985	0.081007	0.051630	0.025653	0.007091
2.37	0.070960	0.11885	0.14626	0.15597	0.15098	0.13450	0.11000	0.081134	0.051717	0.025699	0.007105
2.38	0.070977	0.11890	0.14636	0.15611	0.15114	0.13468	0.11016	0.081262	0.051806	0.025746	0.007118
2.39	0.070994	0.11896	0.14646	0.15626	0.15131	0.13485	0.11032	0.081393	0.051895	0.025793	0.007132
2.40	0.071012	0.11902	0.14657	0.15640	0.15148	0.13503	0.11049	0.081525	0.051986	0.025841	0.007146
2.41	0.071029	0.11908	0.14667	0.15655	0.15166	0.13521	0.11065	0.081660	0.052078	0.025890	0.007160
2.42	0.071047	0.11913	0.14678	0.15670	0.15183	0.13539	0.11082	0.081796	0.052172	0.025939	0.007174
2.43	0.071065	0.11920	0.14689	0.15685	0.15201	0.13558	0.11099	0.081935	0.052267	0.025989	0.007189
2.44	0.071084	0.11926	0.14700	0.15701	0.15220	0.13577	0.11117	0.082075	0.052364	0.026040	0.007203

TABLE 11 - VALUES OF THE COEFFICIENT C - CHANNELS

λ	RATIO X/L										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
2.45	0.071102	0.11932	0.14712	0.15716	0.15238	0.13596	0.11134	0.082218	0.052462	0.026092	0.007218
2.46	0.071121	0.11938	0.14723	0.15732	0.15257	0.13615	0.11152	0.082363	0.052561	0.026144	0.007234
2.47	0.071141	0.11945	0.14735	0.15748	0.15276	0.13635	0.11170	0.082510	0.052662	0.026197	0.007249
2.48	0.071160	0.11951	0.14746	0.15765	0.15295	0.13655	0.11189	0.082659	0.052765	0.026251	0.007265
2.49	0.071180	0.11958	0.14758	0.15781	0.15315	0.13675	0.11207	0.082810	0.052869	0.026306	0.007281
2.50	0.071200	0.11964	0.14770	0.15798	0.15335	0.13696	0.11226	0.082964	0.052974	0.026362	0.007297
2.51	0.071220	0.11971	0.14783	0.15815	0.15355	0.13717	0.11246	0.083120	0.053081	0.026418	0.007313
2.52	0.071241	0.11978	0.14795	0.15833	0.15375	0.13738	0.11265	0.083278	0.053190	0.026476	0.007330
2.53	0.071262	0.11985	0.14808	0.15851	0.15396	0.13759	0.11285	0.083439	0.053300	0.026534	0.007347
2.54	0.071283	0.11992	0.14821	0.15868	0.15417	0.13781	0.11305	0.083602	0.053412	0.026593	0.007364
2.55	0.071305	0.11999	0.14834	0.15887	0.15439	0.13803	0.11325	0.083767	0.053526	0.026653	0.007381
2.56	0.071327	0.12006	0.14847	0.15905	0.15460	0.13826	0.11346	0.083935	0.053642	0.026714	0.007399
2.57	0.071349	0.12014	0.14861	0.15924	0.15482	0.13849	0.11367	0.084106	0.053759	0.026776	0.007417
2.58	0.071371	0.12021	0.14874	0.15943	0.15505	0.13872	0.11389	0.084279	0.053878	0.026839	0.007435
2.59	0.071394	0.12029	0.14888	0.15962	0.15527	0.13896	0.11410	0.084454	0.053998	0.026902	0.007453
2.60	0.071418	0.12037	0.14902	0.15981	0.15551	0.13919	0.11432	0.084632	0.054121	0.026967	0.007472
2.61	0.071441	0.12044	0.14916	0.16001	0.15574	0.13944	0.11454	0.084813	0.054245	0.027032	0.007491
2.62	0.071465	0.12052	0.14931	0.16021	0.15598	0.13968	0.11477	0.084997	0.054371	0.027099	0.007510
2.63	0.071489	0.12060	0.14945	0.16042	0.15622	0.13993	0.11500	0.085183	0.054499	0.027166	0.007530
2.64	0.071514	0.12069	0.14960	0.16062	0.15646	0.14018	0.11523	0.085372	0.054629	0.027235	0.007550
2.65	0.071539	0.12077	0.14975	0.16083	0.15671	0.14044	0.11547	0.085564	0.054761	0.027305	0.007570
2.66	0.071564	0.12085	0.14991	0.16105	0.15696	0.14070	0.11571	0.085758	0.054895	0.027375	0.007591
2.67	0.071589	0.12094	0.15006	0.16126	0.15721	0.14097	0.11595	0.085956	0.055030	0.027447	0.007611
2.68	0.071615	0.12102	0.15022	0.16148	0.15747	0.14123	0.11620	0.086156	0.055168	0.027520	0.007633
2.69	0.071642	0.12111	0.15038	0.16171	0.15774	0.14151	0.11645	0.086360	0.055308	0.027594	0.007654
2.70	0.071669	0.12120	0.15054	0.16193	0.15800	0.14178	0.11671	0.086566	0.055450	0.027668	0.007676
2.71	0.071696	0.12129	0.15070	0.16216	0.15827	0.14206	0.11697	0.086776	0.055594	0.027745	0.007698
2.72	0.071723	0.12138	0.15087	0.16239	0.15855	0.14235	0.11723	0.086989	0.055740	0.027822	0.007720
2.73	0.071751	0.12147	0.15104	0.16263	0.15882	0.14264	0.11749	0.087204	0.055889	0.027900	0.007743
2.74	0.071779	0.12157	0.15121	0.16287	0.15911	0.14293	0.11776	0.087424	0.056039	0.027980	0.007766
2.75	0.071808	0.12166	0.15139	0.16311	0.15939	0.14323	0.11804	0.087646	0.056192	0.028060	0.007789
2.76	0.071837	0.12176	0.15156	0.16336	0.15968	0.14353	0.11832	0.087872	0.056348	0.028142	0.007813
2.77	0.071867	0.12186	0.15174	0.16361	0.15998	0.14383	0.11860	0.088101	0.056505	0.028225	0.007837
2.78	0.071897	0.12196	0.15192	0.16386	0.16028	0.14415	0.11888	0.088333	0.056665	0.028310	0.007862
2.79	0.071927	0.12206	0.15211	0.16412	0.16058	0.14446	0.11918	0.088569	0.056827	0.028396	0.007887

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{X}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
2.80	0.071958	0.12216	0.15229	0.16438	0.16089	0.14478	0.11947	0.088808	0.056992	0.028483	0.007912
2.81	0.071989	0.12227	0.15248	0.16464	0.16120	0.14511	0.11977	0.089051	0.057159	0.028571	0.007938
2.82	0.072021	0.12237	0.15268	0.16491	0.16152	0.14543	0.12007	0.089298	0.057329	0.028661	0.007964
2.83	0.072053	0.12248	0.15287	0.16519	0.16184	0.14577	0.12038	0.089548	0.057501	0.028752	0.007990
2.84	0.072086	0.12259	0.15307	0.16546	0.16217	0.14611	0.12070	0.089803	0.057676	0.028844	0.008017
2.85	0.072119	0.12270	0.15327	0.16574	0.16250	0.14645	0.12101	0.090061	0.057854	0.028938	0.008044
2.86	0.072153	0.12281	0.15348	0.16603	0.16284	0.14680	0.12134	0.090322	0.058034	0.029033	0.008072
2.87	0.072187	0.12293	0.15368	0.16632	0.16318	0.14716	0.12166	0.090588	0.058217	0.029129	0.008100
2.88	0.072222	0.12304	0.15389	0.16661	0.16353	0.14752	0.12200	0.090858	0.058402	0.029228	0.008128
2.89	0.072257	0.12316	0.15411	0.16691	0.16388	0.14788	0.12233	0.091132	0.058591	0.029327	0.008157
2.90	0.072292	0.12328	0.15432	0.16721	0.16424	0.14825	0.12268	0.091410	0.058782	0.029428	0.008186
2.91	0.072328	0.12340	0.15454	0.16752	0.16460	0.14863	0.12302	0.091692	0.058977	0.029531	0.008216
2.92	0.072365	0.12352	0.15477	0.16783	0.16497	0.14901	0.12338	0.091979	0.059174	0.029635	0.008247
2.93	0.072402	0.12364	0.15499	0.16814	0.16534	0.14940	0.12373	0.092270	0.059374	0.029741	0.008277
2.94	0.072440	0.12377	0.15522	0.16846	0.16572	0.14979	0.12410	0.092565	0.059577	0.029848	0.008308
2.95	0.072478	0.12390	0.15545	0.16879	0.16610	0.15019	0.12447	0.092865	0.059784	0.029958	0.008340
2.96	0.072517	0.12403	0.15569	0.16912	0.16649	0.15060	0.12484	0.093169	0.059993	0.030068	0.008372
2.97	0.072557	0.12416	0.15593	0.16946	0.16689	0.15101	0.12522	0.093478	0.060206	0.030181	0.008405
2.98	0.072597	0.12429	0.15617	0.16980	0.16729	0.15143	0.12561	0.093792	0.060422	0.030295	0.008438
2.99	0.072637	0.12443	0.15642	0.17014	0.16770	0.15185	0.12600	0.094111	0.060642	0.030411	0.008472
3.00	0.072678	0.12456	0.15667	0.17049	0.16811	0.15229	0.12640	0.094434	0.060865	0.030529	0.008506
3.01	0.072720	0.12470	0.15693	0.17085	0.16854	0.15272	0.12680	0.094763	0.061091	0.030649	0.008541
3.02	0.072763	0.12484	0.15718	0.17121	0.16896	0.15317	0.12721	0.095097	0.061321	0.030770	0.008576
3.03	0.072806	0.12499	0.15745	0.17158	0.16940	0.15362	0.12763	0.095436	0.061554	0.030894	0.008612
3.04	0.072849	0.12513	0.15771	0.17195	0.16984	0.15408	0.12805	0.095780	0.061791	0.031019	0.008648
3.05	0.072894	0.12528	0.15798	0.17233	0.17028	0.15454	0.12848	0.096129	0.062032	0.031146	0.008685
3.06	0.072939	0.12543	0.15826	0.17271	0.17074	0.15501	0.12892	0.096484	0.062277	0.031276	0.008723
3.07	0.072984	0.12558	0.15853	0.17310	0.17120	0.15549	0.12936	0.096845	0.062525	0.031407	0.008761
3.08	0.073031	0.12574	0.15882	0.17349	0.17167	0.15598	0.12981	0.097211	0.062778	0.031540	0.008800
3.09	0.073078	0.12590	0.15910	0.17389	0.17214	0.15648	0.13027	0.097583	0.063034	0.031676	0.008839
3.10	0.073125	0.12605	0.15940	0.17430	0.17263	0.15698	0.13074	0.097961	0.063294	0.031814	0.008879
3.11	0.073174	0.12622	0.15969	0.17472	0.17312	0.15749	0.13121	0.098344	0.063559	0.031954	0.008920
3.12	0.073223	0.12638	0.15999	0.17514	0.17361	0.15801	0.13169	0.098734	0.063828	0.032096	0.008961
3.13	0.073273	0.12655	0.16030	0.17556	0.17412	0.15853	0.13217	0.099130	0.064101	0.032240	0.009003
3.14	0.073324	0.12672	0.16061	0.17600	0.17463	0.15907	0.13267	0.099533	0.064378	0.032387	0.009046

TABLE 11 - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO X̄/L										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
3.15	0.073375	0.12689	0.16092	0.17644	0.17515	0.15961	0.13317	0.099942	0.064660	0.032536	0.009089
3.16	0.073428	0.12706	0.16124	0.17688	0.17568	0.16016	0.13368	0.10036	0.064946	0.032688	0.009133
3.17	0.073481	0.12724	0.16156	0.17734	0.17622	0.16072	0.13420	0.10078	0.065238	0.032842	0.009178
3.18	0.073534	0.12742	0.16189	0.17780	0.17677	0.16129	0.13473	0.10121	0.065533	0.032998	0.009224
3.19	0.073589	0.12760	0.16223	0.17827	0.17733	0.16187	0.13526	0.10164	0.065834	0.033157	0.009270
3.20	0.073645	0.12779	0.16257	0.17875	0.17789	0.16246	0.13581	0.10209	0.066140	0.033319	0.009317
3.21	0.073701	0.12798	0.16291	0.17923	0.17846	0.16306	0.13636	0.10254	0.066450	0.033484	0.009365
3.22	0.073759	0.12817	0.16326	0.17972	0.17905	0.16367	0.13692	0.10300	0.066766	0.033651	0.009413
3.23	0.073817	0.12836	0.16362	0.18022	0.17964	0.16428	0.13749	0.10346	0.067087	0.033821	0.009463
3.24	0.073876	0.12856	0.16398	0.18073	0.18024	0.16491	0.13808	0.10393	0.067413	0.033992	0.009513
3.25	0.073936	0.12876	0.16435	0.18124	0.18085	0.16555	0.13867	0.10441	0.067745	0.034169	0.009564
3.26	0.073997	0.12897	0.16472	0.18177	0.18147	0.16620	0.13927	0.10490	0.068082	0.034347	0.009616
3.27	0.074059	0.12917	0.16510	0.18230	0.18210	0.16686	0.13988	0.10540	0.068425	0.034529	0.009669
3.28	0.074122	0.12939	0.16549	0.18284	0.18275	0.16753	0.14050	0.10590	0.068774	0.034714	0.009722
3.29	0.074186	0.12960	0.16588	0.18339	0.18340	0.16821	0.14113	0.10642	0.069128	0.034901	0.009777
3.30	0.074251	0.12982	0.16628	0.18395	0.18406	0.16890	0.14177	0.10694	0.069489	0.035092	0.009833
3.31	0.074317	0.13004	0.16669	0.18452	0.18474	0.16960	0.14242	0.10747	0.069856	0.035287	0.009889
3.32	0.074384	0.13026	0.16710	0.18509	0.18542	0.17032	0.14308	0.10801	0.070229	0.035484	0.009947
3.33	0.074452	0.13049	0.16752	0.18568	0.18612	0.17105	0.14376	0.10856	0.070608	0.035685	0.010005
3.34	0.074522	0.13072	0.16794	0.18628	0.18683	0.17179	0.14444	0.10912	0.070994	0.035890	0.010065
3.35	0.074592	0.13096	0.16837	0.18689	0.18755	0.17254	0.14514	0.10969	0.071387	0.036098	0.010125
3.36	0.074664	0.13120	0.16881	0.18750	0.18828	0.17330	0.14585	0.11027	0.071787	0.036310	0.010187
3.37	0.074737	0.13144	0.16926	0.18813	0.18903	0.17408	0.14657	0.11086	0.072193	0.036525	0.010249
3.38	0.074811	0.13169	0.16971	0.18877	0.18979	0.17488	0.14731	0.11146	0.072607	0.036744	0.010313
3.39	0.074886	0.13194	0.17018	0.18942	0.19056	0.17568	0.14806	0.11207	0.073028	0.036967	0.010378
3.40	0.074962	0.13220	0.17065	0.19008	0.19135	0.17650	0.14882	0.11269	0.073457	0.037195	0.010444
3.41	0.075040	0.13246	0.17113	0.19075	0.19215	0.17734	0.14959	0.11332	0.073893	0.037426	0.010512
3.42	0.075119	0.13272	0.17161	0.19144	0.19296	0.17819	0.15038	0.11396	0.074337	0.037661	0.010580
3.43	0.075200	0.13299	0.17211	0.19213	0.19379	0.17905	0.15118	0.11462	0.074790	0.037901	0.010650
3.44	0.075282	0.13327	0.17261	0.19284	0.19463	0.17993	0.15200	0.11528	0.075250	0.038145	0.010721
3.45	0.075365	0.13355	0.17312	0.19356	0.19549	0.18083	0.15283	0.11596	0.075719	0.038393	0.010793
3.46	0.075450	0.13383	0.17365	0.19429	0.19636	0.18174	0.15368	0.11665	0.076196	0.038647	0.010867
3.47	0.075536	0.13412	0.17418	0.19504	0.19725	0.18267	0.15454	0.11735	0.076682	0.038904	0.010942
3.48	0.075623	0.13442	0.17472	0.19580	0.19815	0.18361	0.15541	0.11807	0.077177	0.039167	0.011018
3.49	0.075713	0.13471	0.17527	0.19657	0.19907	0.18457	0.15631	0.11880	0.077682	0.039434	0.011096

TABLE 11 - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
3.50	0.075804	0.13502	0.17583	0.19736	0.20001	0.18555	0.15722	0.11954	0.078196	0.039707	0.011176
3.51	0.075896	0.13533	0.17640	0.19816	0.20096	0.18655	0.15815	0.12030	0.078719	0.039985	0.011256
3.52	0.075990	0.13565	0.17698	0.19898	0.20193	0.18757	0.15909	0.12107	0.079253	0.040268	0.011339
3.53	0.076086	0.13597	0.17757	0.19981	0.20293	0.18861	0.16005	0.12186	0.079797	0.040556	0.011423
3.54	0.076183	0.13629	0.17817	0.20066	0.20393	0.18966	0.16103	0.12266	0.080351	0.040850	0.011509
3.55	0.076283	0.13663	0.17878	0.20152	0.20496	0.19074	0.16203	0.12348	0.080916	0.041150	0.011596
3.56	0.076384	0.13697	0.17941	0.20240	0.20601	0.19184	0.16305	0.12431	0.081492	0.041455	0.011685
3.57	0.076487	0.13731	0.18004	0.20330	0.20708	0.19295	0.16409	0.12516	0.082079	0.041767	0.011776
3.58	0.076592	0.13767	0.18069	0.20421	0.20817	0.19409	0.16515	0.12602	0.082678	0.042085	0.011868
3.59	0.076699	0.13802	0.18135	0.20514	0.20927	0.19525	0.16623	0.12690	0.083288	0.042409	0.011963
3.60	0.076808	0.13839	0.18203	0.20609	0.21041	0.19644	0.16733	0.12780	0.083911	0.042739	0.012059
3.61	0.076919	0.13876	0.18271	0.20706	0.21156	0.19765	0.16846	0.12872	0.084546	0.043077	0.012157
3.62	0.077032	0.13914	0.18341	0.20804	0.21273	0.19888	0.16960	0.12966	0.085195	0.043421	0.012258
3.63	0.077147	0.13953	0.18413	0.20905	0.21393	0.20013	0.17077	0.13061	0.085856	0.043772	0.012360
3.64	0.077264	0.13993	0.18485	0.21008	0.21516	0.20142	0.17196	0.13159	0.086531	0.044130	0.012464
3.65	0.077384	0.14033	0.18560	0.21112	0.21641	0.20272	0.17318	0.13258	0.087220	0.044496	0.012571
3.66	0.077506	0.14074	0.18635	0.21219	0.21768	0.20406	0.17442	0.13360	0.087924	0.044870	0.012680
3.67	0.077631	0.14116	0.18712	0.21328	0.21898	0.20542	0.17569	0.13464	0.088642	0.045251	0.012791
3.68	0.077758	0.14159	0.18791	0.21439	0.22030	0.20681	0.17698	0.13570	0.089375	0.045640	0.012905
3.69	0.077888	0.14202	0.18872	0.21552	0.22166	0.20823	0.17830	0.13678	0.090124	0.046038	0.013021
3.70	0.078020	0.14247	0.18954	0.21668	0.22304	0.20968	0.17965	0.13788	0.090889	0.046445	0.013139
3.71	0.078155	0.14292	0.19037	0.21786	0.22445	0.21116	0.18103	0.13901	0.091671	0.046860	0.013260
3.72	0.078293	0.14339	0.19123	0.21907	0.22589	0.21268	0.18244	0.14016	0.092470	0.047284	0.013384
3.73	0.078434	0.14386	0.19210	0.22030	0.22737	0.21422	0.18388	0.14134	0.093286	0.047718	0.013510
3.74	0.078577	0.14434	0.19299	0.22156	0.22887	0.21580	0.18535	0.14254	0.094120	0.048161	0.013640
3.75	0.078724	0.14484	0.19391	0.22285	0.23041	0.21741	0.18686	0.14377	0.094973	0.048615	0.013772
3.76	0.078874	0.14534	0.19484	0.22417	0.23198	0.21906	0.18839	0.14503	0.095846	0.049078	0.013907
3.77	0.079027	0.14586	0.19579	0.22551	0.23359	0.22075	0.18996	0.14632	0.096738	0.049552	0.014045
3.78	0.079183	0.14638	0.19676	0.22688	0.23523	0.22247	0.19157	0.14764	0.097650	0.050037	0.014187
3.79	0.079343	0.14692	0.19775	0.22829	0.23691	0.22424	0.19322	0.14898	0.098584	0.050534	0.014332
3.80	0.079506	0.14747	0.19877	0.22972	0.23863	0.22604	0.19490	0.15036	0.099540	0.051042	0.014480
3.81	0.079673	0.14804	0.19981	0.23119	0.24039	0.22789	0.19662	0.15177	0.10052	0.051562	0.014632
3.82	0.079844	0.14861	0.20087	0.23270	0.24218	0.22978	0.19838	0.15322	0.10152	0.052094	0.014787
3.83	0.080019	0.14920	0.20196	0.23424	0.24402	0.23171	0.20019	0.15469	0.10254	0.052640	0.014946
3.84	0.080197	0.14980	0.20307	0.23581	0.24591	0.23369	0.20203	0.15621	0.10360	0.053198	0.015109

TABLE 11 - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{X}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
3.85	0.080380	0.15042	0.20421	0.23742	0.24784	0.23572	0.20393	0.15776	0.10467	0.053771	0.015276
3.86	0.080567	0.15105	0.20538	0.23908	0.24982	0.23780	0.20586	0.15935	0.10577	0.054357	0.015447
3.87	0.080759	0.15170	0.20658	0.24077	0.25184	0.23993	0.20785	0.16098	0.10690	0.054958	0.015623
3.88	0.080955	0.15236	0.20780	0.24250	0.25392	0.24211	0.20989	0.16265	0.10806	0.055575	0.015803
3.89	0.081156	0.15304	0.20905	0.24427	0.25604	0.24435	0.21197	0.16436	0.10925	0.056207	0.015987
3.90	0.081362	0.15373	0.21034	0.24610	0.25822	0.24664	0.21412	0.16611	0.11047	0.056855	0.016177
3.91	0.081573	0.15444	0.21166	0.24796	0.26046	0.24900	0.21631	0.16792	0.11172	0.057521	0.016371
3.92	0.081789	0.15518	0.21301	0.24988	0.26275	0.25141	0.21857	0.16976	0.11300	0.058204	0.016570
3.93	0.082011	0.15592	0.21439	0.25184	0.26511	0.25389	0.22088	0.17166	0.11432	0.058905	0.016775
3.94	0.082238	0.15669	0.21582	0.25386	0.26752	0.25643	0.22325	0.17361	0.11568	0.059626	0.016986
3.95	0.082472	0.15748	0.21728	0.25592	0.27000	0.25904	0.22569	0.17561	0.11707	0.060366	0.017202
3.96	0.082711	0.15829	0.21877	0.25805	0.27255	0.26172	0.22820	0.17767	0.11849	0.061126	0.017424
3.97	0.082957	0.15912	0.22031	0.26023	0.27516	0.26448	0.23077	0.17978	0.11996	0.061908	0.017652
3.98	0.083210	0.15998	0.22189	0.26247	0.27785	0.26731	0.23342	0.18196	0.12147	0.062711	0.017887
3.99	0.083469	0.16085	0.22352	0.26478	0.28062	0.27022	0.23614	0.18419	0.12302	0.063538	0.018128
4.00	0.083736	0.16175	0.22519	0.26715	0.28346	0.27322	0.23894	0.18649	0.12462	0.064389	0.018377
4.01	0.084010	0.16268	0.22691	0.26958	0.28638	0.27630	0.24182	0.18886	0.12627	0.065264	0.018633
4.02	0.084292	0.16363	0.22867	0.27209	0.28939	0.27947	0.24479	0.19129	0.12796	0.066166	0.018896
4.03	0.084581	0.16461	0.23049	0.27467	0.29249	0.28273	0.24784	0.19380	0.12970	0.067095	0.019167
4.04	0.084880	0.16562	0.23236	0.27733	0.29568	0.28610	0.25099	0.19638	0.13150	0.068052	0.019447
4.05	0.085187	0.16666	0.23429	0.28007	0.29897	0.28956	0.25423	0.19905	0.13335	0.069038	0.019735
4.06	0.085504	0.16774	0.23628	0.28289	0.30236	0.29314	0.25757	0.20179	0.13526	0.070056	0.020033
4.07	0.085830	0.16884	0.23833	0.28580	0.30585	0.29682	0.26102	0.20463	0.13723	0.071106	0.020340
4.08	0.086166	0.16998	0.24044	0.28879	0.30945	0.30062	0.26458	0.20755	0.13927	0.072189	0.020656
4.09	0.086512	0.17115	0.24262	0.29189	0.31317	0.30455	0.26825	0.21057	0.14137	0.073308	0.020984
4.10	0.086870	0.17236	0.24487	0.29509	0.31701	0.30860	0.27204	0.21369	0.14354	0.074464	0.021321
4.11	0.087239	0.17361	0.24719	0.29839	0.32098	0.31279	0.27596	0.21691	0.14578	0.075659	0.021671
4.12	0.087621	0.17491	0.24959	0.30180	0.32508	0.31711	0.28002	0.22025	0.14810	0.076895	0.022032
4.13	0.088015	0.17626	0.25207	0.30532	0.32932	0.32159	0.28421	0.22369	0.15049	0.078174	0.022406
4.14	0.088422	0.17762	0.25463	0.30897	0.33371	0.32622	0.28855	0.22726	0.15298	0.079498	0.022793
4.15	0.088844	0.17905	0.25729	0.31275	0.33825	0.33102	0.29304	0.23096	0.15555	0.080869	0.023194
4.16	0.089280	0.18053	0.26004	0.31666	0.34295	0.33599	0.29769	0.23478	0.15821	0.082289	0.023610
4.17	0.089732	0.18206	0.26288	0.32071	0.34783	0.34113	0.30252	0.23875	0.16098	0.083763	0.024040
4.18	0.090201	0.18365	0.26584	0.32491	0.35288	0.34647	0.30752	0.24287	0.16384	0.085292	0.024488
4.19	0.090686	0.18530	0.26890	0.32927	0.35813	0.35202	0.31272	0.24715	0.16682	0.086879	0.024952

TABLE 11 - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\overline{X}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
4.20	0.091190	0.18701	0.27208	0.33379	0.36357	0.35777	0.31811	0.25159	0.16991	0.088528	0.025434
4.21	0.091713	0.18879	0.27538	0.33849	0.36923	0.36375	0.32372	0.25620	0.17313	0.090243	0.025936
4.22	0.092257	0.19063	0.27881	0.34338	0.37512	0.36997	0.32955	0.26100	0.17647	0.092026	0.026458
4.23	0.092823	0.19255	0.28238	0.34846	0.38124	0.37645	0.33562	0.26600	0.17995	0.093884	0.027001
4.24	0.093411	0.19455	0.28610	0.35376	0.38762	0.38319	0.34194	0.27121	0.18358	0.095819	0.027567
4.25	0.094024	0.19663	0.28997	0.35927	0.39426	0.39022	0.34854	0.27664	0.18736	0.097837	0.028158
4.26	0.094663	0.19880	0.29401	0.36503	0.40119	0.39755	0.35541	0.28230	0.19131	0.099943	0.028774
4.27	0.095330	0.20106	0.29822	0.37103	0.40843	0.40521	0.36260	0.28822	0.19543	0.10214	0.029418
4.28	0.096026	0.20343	0.30263	0.37731	0.41599	0.41321	0.37010	0.29440	0.19974	0.10444	0.030091
4.29	0.096754	0.20590	0.30723	0.38387	0.42390	0.42158	0.37796	0.30087	0.20425	0.10685	0.030795
4.30	0.097515	0.20849	0.31205	0.39074	0.43218	0.43034	0.38618	0.30765	0.20897	0.10937	0.031533
4.31	0.098313	0.21120	0.31710	0.39794	0.44086	0.43953	0.39481	0.31476	0.21392	0.11201	0.032307
4.32	0.099149	0.21404	0.32239	0.40549	0.44997	0.44917	0.40385	0.32222	0.21912	0.11479	0.033120
4.33	0.10003	0.21703	0.32795	0.41342	0.45953	0.45930	0.41336	0.33005	0.22458	0.11771	0.033974
4.34	0.10095	0.22016	0.33380	0.42176	0.46959	0.46995	0.42336	0.33830	0.23033	0.12078	0.034872
4.35	0.10192	0.22347	0.33995	0.43054	0.48019	0.48117	0.43390	0.34698	0.23639	0.12401	0.035819
4.36	0.10294	0.22695	0.34644	0.43980	0.49136	0.49300	0.44501	0.35614	0.24278	0.12742	0.036818
4.37	0.10403	0.23062	0.35329	0.44957	0.50315	0.50549	0.45675	0.36582	0.24952	0.13102	0.037873
4.38	0.10517	0.23451	0.36053	0.45991	0.51563	0.51871	0.46916	0.37606	0.25666	0.13484	0.038989
4.39	0.10638	0.23862	0.36820	0.47086	0.52884	0.53270	0.48231	0.38690	0.26423	0.13888	0.040173
4.40	0.10766	0.24298	0.37633	0.48247	0.54286	0.54756	0.49627	0.39841	0.27226	0.14317	0.041429
4.41	0.10902	0.24761	0.38497	0.49481	0.55775	0.56335	0.51110	0.41065	0.28079	0.14773	0.042764
4.42	0.11047	0.25254	0.39417	0.50794	0.57362	0.58016	0.52690	0.42369	0.28989	0.15258	0.044187
4.43	0.11201	0.25780	0.40398	0.52196	0.59054	0.59810	0.54377	0.43760	0.29959	0.15777	0.045706
4.44	0.11366	0.26342	0.41447	0.53694	0.60864	0.61729	0.56180	0.45248	0.30997	0.16332	0.047331
4.45	0.11543	0.26945	0.42571	0.55300	0.62804	0.63785	0.58113	0.46843	0.32111	0.16926	0.049073
4.46	0.11733	0.27592	0.43778	0.57025	0.64887	0.65994	0.60191	0.48557	0.33307	0.17566	0.050945
4.47	0.11938	0.28288	0.45078	0.58882	0.67132	0.68375	0.62428	0.50404	0.34596	0.18254	0.052963
4.48	0.12159	0.29040	0.46482	0.60889	0.69556	0.70946	0.64847	0.52400	0.35989	0.18999	0.055144
4.49	0.12398	0.29855	0.48003	0.63063	0.72184	0.73733	0.67467	0.54563	0.37498	0.19806	0.057508
4.50	0.12658	0.30740	0.49656	0.65426	0.75040	0.76763	0.70317	0.56915	0.39140	0.20683	0.060079
4.51	0.12941	0.31706	0.51460	0.68004	0.78156	0.80069	0.73426	0.59483	0.40932	0.21641	0.062885
4.52	0.13251	0.32764	0.53435	0.70828	0.81570	0.83691	0.76833	0.62296	0.42896	0.22690	0.065961
4.53	0.13593	0.33927	0.55608	0.73935	0.85326	0.87676	0.80582	0.65391	0.45057	0.23846	0.069345
4.54	0.13970	0.35212	0.58009	0.77369	0.89478	0.92081	0.84727	0.68813	0.47446	0.25123	0.073089

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\overline{X}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
4.55	0.14389	0.36640	0.60677	0.81184	0.94091	0.96978	0.89334	0.72617	0.50102	0.26543	0.077249
4.56	0.14857	0.38236	0.63658	0.85449	0.99248	1.0245	0.94484	0.76870	0.53072	0.28130	0.081902
4.57	0.15384	0.40031	0.67012	0.90247	1.0505	1.0861	1.0028	0.81657	0.56414	0.29917	0.087138
4.58	0.15981	0.42066	0.70813	0.95684	1.1163	1.1559	1.0685	0.87082	0.60203	0.31943	0.093075
4.59	0.16662	0.44390	0.75157	1.0190	1.1914	1.2357	1.1436	0.93285	0.64534	0.34259	0.099863
4.60	0.17449	0.47072	0.80169	1.0907	1.2782	1.3278	1.2302	1.0044	0.69534	0.36932	0.10770
4.61	0.18366	0.50200	0.86015	1.1743	1.3794	1.4352	1.3314	1.0880	0.75368	0.40051	0.11684
4.62	0.19450	0.53896	0.92924	1.2732	1.4990	1.5622	1.4509	1.1867	0.82264	0.43739	0.12765
4.63	0.20751	0.58331	1.0121	1.3918	1.6425	1.7146	1.5943	1.3052	0.90541	0.48165	0.14062
4.64	0.22340	0.63751	1.1134	1.5368	1.8179	1.9009	1.7697	1.4501	1.0066	0.53576	0.15648
4.65	0.24326	0.70524	1.2401	1.7180	2.0372	2.1337	1.9889	1.6312	1.1331	0.60340	0.17631
4.66	0.26879	0.79231	1.4028	1.9510	2.3191	2.4331	2.2707	1.8640	1.2957	0.69037	0.20181
4.67	0.30281	0.90837	1.6198	2.2616	2.6949	2.8322	2.6464	2.1744	1.5126	0.80634	0.23580
4.68	0.35043	1.0708	1.9235	2.6963	3.2209	3.3908	3.1724	2.6090	1.8161	0.96869	0.28339
4.69	0.42184	1.3144	2.3789	3.3481	4.0098	4.2286	3.9611	3.2607	2.2714	1.2122	0.35476
4.70	0.54077	1.7201	3.1374	4.4339	5.3238	5.6242	5.2751	4.3464	3.0298	1.6178	0.47366
4.71	0.77839	2.5306	4.6530	6.6033	7.9492	8.4126	7.9004	6.5157	4.5452	2.4282	0.71124
4.72	1.4893	4.9556	9.1872	13.094	15.804	16.755	15.755	13.006	9.0793	4.8531	1.4221
4.730+	±∞	±∞	±∞	±∞	±∞	±∞	±∞	±∞	±∞	±∞	±∞
4.74	-1.3717	-4.8038	-9.0609	-13.028	-15.809	-16.821	-15.858	-13.116	-9.1691	-4.9065	-1.4389
4.75	-0.65203	-2.3489	-4.4709	-6.4575	-7.8575	-8.3758	-7.9066	-6.5457	-4.5792	-2.4517	-0.71931
4.76	-0.41280	-1.5329	-2.9450	-4.2733	-5.2142	-5.5684	-5.2634	-4.3616	-3.0535	-1.6357	-0.48011
4.77	-0.29329	-1.1252	-2.1829	-3.1823	-3.8939	-4.1662	-3.9431	-3.2708	-2.2915	-1.2282	-0.36064
4.78	-0.22162	-0.88077	-1.7258	-2.5281	-3.1021	-3.3253	-3.1515	-2.6167	-1.8346	-0.98385	-0.28901
4.79	-0.17386	-0.71784	-1.4212	-2.0921	-2.5745	-2.7650	-2.6240	-2.1808	-1.5301	-0.82102	-0.24128
4.80	-0.13974	-0.60148	-1.2036	-1.7807	-2.1978	-2.3649	-2.2473	-1.8696	-1.3127	-0.70476	-0.20720
4.81	-0.11416	-0.51421	-1.0405	-1.5472	-1.9152	-2.0649	-1.9649	-1.6362	-1.1497	-0.61760	-0.18165
4.82	-0.094255	-0.44634	-0.91363	-1.3656	-1.6955	-1.8316	-1.7453	-1.4548	-1.0230	-0.54984	-0.16179
4.83	-0.078333	-0.39204	-0.81214	-1.2204	-1.5198	-1.6450	-1.5696	-1.3097	-0.92165	-0.49564	-0.14590
4.84	-0.065304	-0.34761	-0.72909	-1.1016	-1.3760	-1.4923	-1.4259	-1.1910	-0.83876	-0.45132	-0.13291
4.85	-0.054445	-0.31058	-0.65989	-1.0025	-1.2563	-1.3652	-1.3062	-1.0922	-0.76971	-0.41440	-0.12209
4.86	-0.045254	-0.27925	-0.60133	-0.91875	-1.1549	-1.2576	-1.2050	-1.0085	-0.71131	-0.38317	-0.11294
4.87	-0.037374	-0.25238	-0.55113	-0.84693	-1.0681	-1.1654	-1.1182	-0.93686	-0.66127	-0.35642	-0.10510
4.88	-0.030543	-0.22910	-0.50762	-0.78470	-0.99279	-1.0855	-1.0431	-0.87478	-0.61791	-0.33324	-0.098305
4.89	-0.024564	-0.20872	-0.46955	-0.73024	-0.92693	-1.0156	-0.97730	-0.82048	-0.58000	-0.31297	-0.092366

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
4.90	-0.019286	-0.19073	-0.43595	-0.68219	-0.86884	-0.95396	-0.91930	-0.77258	-0.54656	-0.29510	-0.087128
4.91	-0.014593	-0.17474	-0.40608	-0.63948	-0.81720	-0.89917	-0.86776	-0.73003	-0.51685	-0.27922	-0.082475
4.92	-0.010392	-0.16042	-0.37935	-0.60126	-0.77100	-0.85015	-0.82166	-0.69197	-0.49029	-0.26502	-0.078314
4.93	-0.006609	-0.14754	-0.35528	-0.56686	-0.72943	-0.80605	-0.78018	-0.65774	-0.46639	-0.25225	-0.074572
4.94	-0.003185	-0.13587	-0.33351	-0.53574	-0.69182	-0.76617	-0.74267	-0.62678	-0.44478	-0.24070	-0.071189
4.95	-0.000071	-0.12527	-0.31371	-0.50744	-0.65763	-0.72991	-0.70859	-0.59864	-0.42515	-0.23021	-0.068116
4.96	+0.002774	-0.11558	-0.29563	-0.48161	-0.62642	-0.69682	-0.67748	-0.57297	-0.40724	-0.22064	-0.065313
4.97	0.005384	-0.10669	-0.27905	-0.45792	-0.59781	-0.66649	-0.64897	-0.54946	-0.39083	-0.21187	-0.062745
4.98	0.007787	-0.098514	-0.26379	-0.43613	-0.57150	-0.63860	-0.62276	-0.52783	-0.37575	-0.20381	-0.060385
4.99	0.010006	-0.090961	-0.24970	-0.41601	-0.54721	-0.61286	-0.59858	-0.50788	-0.36183	-0.19638	-0.058208
5.00	0.012063	-0.083963	-0.23665	-0.39738	-0.52472	-0.58904	-0.57619	-0.48943	-0.34896	-0.18951	-0.056195
5.01	0.013974	-0.077461	-0.22453	-0.38008	-0.50384	-0.56692	-0.55542	-0.47230	-0.33702	-0.18313	-0.054328
5.02	0.015755	-0.071404	-0.21324	-0.36397	-0.48441	-0.54634	-0.53609	-0.45636	-0.32591	-0.17720	-0.052591
5.03	0.017419	-0.065747	-0.20270	-0.34893	-0.46627	-0.52713	-0.51806	-0.44150	-0.31555	-0.17167	-0.050972
5.04	0.018977	-0.060451	-0.19283	-0.33486	-0.44930	-0.50917	-0.50120	-0.42761	-0.30587	-0.16650	-0.049459
5.05	0.020438	-0.055483	-0.18358	-0.32167	-0.43340	-0.49234	-0.48541	-0.41460	-0.29680	-0.16166	-0.048043
5.06	0.021813	-0.050812	-0.17488	-0.30927	-0.41846	-0.47653	-0.47058	-0.40238	-0.28829	-0.15712	-0.046714
5.07	0.023108	-0.046413	-0.16669	-0.29760	-0.40440	-0.46166	-0.45663	-0.39090	-0.28029	-0.15285	-0.045465
5.08	0.024331	-0.042261	-0.15897	-0.28660	-0.39114	-0.44765	-0.44349	-0.38008	-0.27275	-0.14883	-0.044289
5.09	0.025487	-0.038337	-0.15167	-0.27620	-0.37862	-0.43442	-0.43109	-0.36987	-0.26564	-0.14504	-0.043180
5.10	0.026582	-0.034621	-0.14476	-0.26637	-0.36678	-0.42191	-0.41936	-0.36022	-0.25893	-0.14146	-0.042132
5.11	0.027621	-0.031097	-0.13821	-0.25705	-0.35557	-0.41006	-0.40826	-0.35109	-0.25257	-0.13807	-0.041141
5.12	0.028607	-0.027751	-0.13199	-0.24820	-0.34493	-0.39882	-0.39774	-0.34243	-0.24655	-0.13486	-0.040203
5.13	0.029546	-0.024568	-0.12608	-0.23979	-0.33482	-0.38816	-0.38775	-0.33422	-0.24084	-0.13181	-0.039313
5.14	0.030440	-0.021538	-0.12045	-0.23180	-0.32521	-0.37801	-0.37826	-0.32642	-0.23541	-0.12892	-0.038468
5.15	0.031293	-0.018648	-0.11509	-0.22418	-0.31605	-0.36836	-0.36922	-0.31900	-0.23025	-0.12617	-0.037665
5.16	0.032108	-0.015889	-0.10997	-0.21691	-0.30732	-0.35915	-0.36062	-0.31193	-0.22534	-0.12356	-0.036900
5.17	0.032887	-0.013253	-0.10508	-0.20997	-0.29899	-0.35037	-0.35241	-0.30519	-0.22066	-0.12106	-0.036172
5.18	0.033632	-0.010730	-0.10040	-0.20333	-0.29103	-0.34199	-0.34457	-0.29875	-0.21619	-0.11869	-0.035478
5.19	0.034346	-0.008314	-0.095925	-0.19698	-0.28342	-0.33397	-0.33708	-0.29261	-0.21192	-0.11642	-0.034815
5.20	0.035032	-0.005997	-0.091634	-0.19090	-0.27613	-0.32630	-0.32992	-0.28674	-0.20784	-0.11425	-0.034182
5.21	0.035690	-0.003774	-0.087519	-0.18507	-0.26914	-0.31895	-0.32306	-0.28111	-0.20394	-0.11217	-0.033576
5.22	0.036322	-0.001638	-0.083567	-0.17947	-0.26244	-0.31190	-0.31649	-0.27573	-0.20021	-0.11019	-0.032997
5.23	0.036930	+0.000416	-0.079769	-0.17410	-0.25601	-0.30514	-0.31019	-0.27057	-0.19663	-0.10828	-0.032442
5.24	0.037516	0.002393	-0.076116	-0.16893	-0.24983	-0.29865	-0.30414	-0.26561	-0.19320	-0.10646	-0.031910

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
5.25	0.038080	0.004296	-0.072600	-0.16396	-0.24389	-0.29242	-0.29833	-0.26086	-0.18990	-0.10471	-0.031400
5.26	0.038625	0.006131	-0.069212	-0.15917	-0.23817	-0.28642	-0.29274	-0.25629	-0.18674	-0.10303	-0.030911
5.27	0.039150	0.007902	-0.065945	-0.15456	-0.23266	-0.28064	-0.28737	-0.25190	-0.18370	-0.10142	-0.030441
5.28	0.039658	0.009611	-0.062794	-0.15011	-0.22735	-0.27508	-0.28220	-0.24768	-0.18078	-0.099868	-0.029989
5.29	0.040149	0.011268	-0.059750	-0.14582	-0.22224	-0.26972	-0.27722	-0.24361	-0.17797	-0.098377	-0.029555
5.30	0.040623	0.012859	-0.056810	-0.14168	-0.21730	-0.26456	-0.27243	-0.23969	-0.17526	-0.096942	-0.029138
5.31	0.041083	0.014404	-0.053966	-0.13767	-0.21253	-0.25957	-0.26780	-0.23592	-0.17265	-0.095561	-0.028736
5.32	0.041528	0.015900	-0.051215	-0.13380	-0.20792	-0.25476	-0.26333	-0.23228	-0.17014	-0.094230	-0.028349
5.33	0.041960	0.017349	-0.048552	-0.13006	-0.20347	-0.25011	-0.25902	-0.22877	-0.16772	-0.092948	-0.027976
5.34	0.042378	0.018754	-0.045971	-0.12643	-0.19916	-0.24561	-0.25486	-0.22538	-0.16538	-0.091711	-0.027616
5.35	0.042785	0.020117	-0.043470	-0.12292	-0.19499	-0.24126	-0.25084	-0.22211	-0.16312	-0.090518	-0.027270
5.36	0.043179	0.021440	-0.041044	-0.11952	-0.19096	-0.23706	-0.24695	-0.21896	-0.16095	-0.089367	-0.026936
5.37	0.043563	0.022725	-0.038690	-0.11621	-0.18704	-0.23299	-0.24318	-0.21589	-0.15884	-0.088255	-0.026613
5.38	0.043936	0.023973	-0.036404	-0.11301	-0.18325	-0.22904	-0.23954	-0.21293	-0.15681	-0.087181	-0.026301
5.39	0.044299	0.025187	-0.034183	-0.10990	-0.17957	-0.22522	-0.23602	-0.21007	-0.15484	-0.086144	-0.026001
5.40	0.044652	0.026368	-0.032024	-0.10689	-0.17601	-0.22152	-0.23260	-0.20730	-0.15294	-0.085142	-0.025710
5.41	0.044997	0.027517	-0.029924	-0.10395	-0.17254	-0.21792	-0.22929	-0.20462	-0.15110	-0.084172	-0.025429
5.42	0.045332	0.028637	-0.027881	-0.10110	-0.16918	-0.21443	-0.22609	-0.20203	-0.14932	-0.083234	-0.025158
5.43	0.045659	0.029727	-0.025892	-0.098327	-0.16591	-0.21105	-0.22297	-0.19951	-0.14760	-0.082327	-0.024895
5.44	0.045978	0.030790	-0.023955	-0.095628	-0.16273	-0.20776	-0.21996	-0.19707	-0.14598	-0.081449	-0.024641
5.45	0.046290	0.031827	-0.022067	-0.093000	-0.15964	-0.20457	-0.21703	-0.19471	-0.14431	-0.080600	-0.024396
5.46	0.046594	0.032839	-0.020227	-0.090441	-0.15663	-0.20147	-0.21418	-0.19242	-0.14274	-0.079777	-0.024158
5.47	0.046891	0.033827	-0.018432	-0.087948	-0.15370	-0.19845	-0.21142	-0.19019	-0.14122	-0.078980	-0.023928
5.48	0.047182	0.034792	-0.016680	-0.085518	-0.15085	-0.19551	-0.20874	-0.18804	-0.13975	-0.078208	-0.023705
5.49	0.047466	0.035735	-0.014971	-0.083148	-0.14808	-0.19266	-0.20613	-0.18594	-0.13832	-0.077459	-0.023490
5.50	0.047744	0.036657	-0.013301	-0.080836	-0.14537	-0.18988	-0.20360	-0.18391	-0.13694	-0.076734	-0.023281
5.51	0.048016	0.037558	-0.011670	-0.078580	-0.14274	-0.18718	-0.20113	-0.18193	-0.13559	-0.076032	-0.023079
5.52	0.048282	0.038440	-0.010075	-0.076378	-0.14017	-0.18454	-0.19873	-0.18001	-0.13429	-0.075350	-0.022883
5.53	0.048544	0.039303	-0.008515	-0.074226	-0.13766	-0.18198	-0.19640	-0.17814	-0.13302	-0.074690	-0.022693
5.54	0.048799	0.040149	-0.006991	-0.072124	-0.13521	-0.17947	-0.19413	-0.17633	-0.13179	-0.074049	-0.022509
5.55	0.049050	0.040977	-0.005498	-0.070070	-0.13282	-0.17704	-0.19192	-0.17457	-0.13060	-0.073427	-0.022331
5.56	0.049297	0.041789	-0.004037	-0.068061	-0.13049	-0.17466	-0.18977	-0.17286	-0.12944	-0.072824	-0.022158
5.57	0.049538	0.042584	-0.002606	-0.066096	-0.12821	-0.17234	-0.18768	-0.17119	-0.12832	-0.072239	-0.021990
5.58	0.049775	0.043365	-0.001204	-0.064174	-0.12599	-0.17008	-0.18563	-0.16957	-0.12722	-0.071672	-0.021828
5.59	0.050008	0.044131	0.000170	-0.062293	-0.12381	-0.16787	-0.18365	-0.16799	-0.12616	-0.071122	-0.021671

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO x̄/L										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
5.60	0.050237	0.044882	0.001517	-0.060451	-0.12169	-0.16572	-0.18171	-0.16645	-0.12513	-0.070587	-0.021518
5.61	0.050462	0.045621	0.002838	-0.058647	-0.11960	-0.16362	-0.17982	-0.16496	-0.12418	-0.070069	-0.021371
5.62	0.050683	0.046346	0.004134	-0.056879	-0.11757	-0.16156	-0.17798	-0.16351	-0.12315	-0.069566	-0.021228
5.63	0.050901	0.047058	0.005406	-0.055147	-0.11558	-0.15955	-0.17618	-0.16209	-0.12220	-0.069077	-0.021088
5.64	0.051115	0.047759	0.006655	-0.053449	-0.11363	-0.15759	-0.17443	-0.16071	-0.12128	-0.068603	-0.020954
5.65	0.051326	0.048447	0.007882	-0.051783	-0.11173	-0.15568	-0.17272	-0.15937	-0.12039	-0.068143	-0.020823
5.66	0.051534	0.049125	0.009086	-0.050149	-0.10986	-0.15380	-0.17105	-0.15807	-0.11951	-0.067697	-0.020697
5.67	0.051736	0.049791	0.010270	-0.048546	-0.10803	-0.15197	-0.16943	-0.15679	-0.11867	-0.067264	-0.020575
5.68	0.051940	0.050448	0.011435	-0.046972	-0.10624	-0.15018	-0.16784	-0.15555	-0.11785	-0.066844	-0.020456
5.69	0.052138	0.051095	0.012580	-0.045427	-0.10448	-0.14843	-0.16629	-0.15435	-0.11705	-0.066436	-0.020341
5.70	0.052334	0.051732	0.013707	-0.043910	-0.10276	-0.14671	-0.16477	-0.15317	-0.11627	-0.066040	-0.020230
5.71	0.052528	0.052359	0.014815	-0.042419	-0.10107	-0.14504	-0.16330	-0.15202	-0.11552	-0.065657	-0.020122
5.72	0.052718	0.052978	0.015906	-0.040954	-0.099409	-0.14339	-0.16185	-0.15091	-0.11478	-0.065284	-0.020018
5.73	0.052907	0.053588	0.016981	-0.039513	-0.097783	-0.14179	-0.16044	-0.14982	-0.11407	-0.064923	-0.019917
5.74	0.053093	0.054189	0.018039	-0.038097	-0.096187	-0.14021	-0.15906	-0.14876	-0.11338	-0.064573	-0.019819
5.75	0.053277	0.054783	0.019082	-0.036704	-0.094621	-0.13867	-0.15772	-0.14773	-0.11270	-0.064234	-0.019724
5.76	0.053458	0.055369	0.020109	-0.035333	-0.093084	-0.13716	-0.15641	-0.14672	-0.11205	-0.063905	-0.019633
5.77	0.053638	0.055948	0.021123	-0.033985	-0.091574	-0.13568	-0.15512	-0.14574	-0.11141	-0.063586	-0.019544
5.78	0.053815	0.056519	0.022122	-0.032657	-0.090090	-0.13423	-0.15387	-0.14478	-0.11079	-0.063277	-0.019459
5.79	0.053991	0.057084	0.023108	-0.031349	-0.088633	-0.13281	-0.15264	-0.14385	-0.11019	-0.062978	-0.019376
5.80	0.054165	0.0576[illegible]2	0.024080	-0.030062	-0.087201	-0.13142	-0.15144	-0.14295	-0.10961	-0.062688	-0.019297
5.81	0.054337	0.058193	0.025041	-0.028793	-0.085793	-0.13005	-0.15027	-0.14206	-0.10904	-0.062408	-0.019220
5.82	0.054507	0.058739	0.025989	-0.027543	-0.084409	-0.12872	-0.14913	-0.14120	-0.10849	-0.062137	-0.019146
5.83	0.054675	0.059279	0.026925	-0.026311	-0.083047	-0.12740	-0.14801	-0.14036	-0.10796	-0.061874	-0.019074
5.84	0.054843	0.059813	0.027850	-0.025096	-0.081708	-0.12612	-0.14691	-0.13954	-0.10744	-0.061621	-0.019005
5.85	0.055008	0.060341	0.028764	-0.023897	-0.080391	-0.12485	-0.14584	-0.13875	-0.10694	-0.061376	-0.018939
5.86	0.055172	0.060865	0.029668	-0.022715	-0.079095	-0.12361	-0.14479	-0.13797	-0.10645	-0.061139	-0.018875
5.87	0.055335	0.061383	0.030561	-0.021548	-0.077819	-0.12240	-0.14377	-0.13722	-0.10598	-0.060910	-0.018814
5.88	0.055497	0.061896	0.031445	-0.020397	-0.076563	-0.12120	-0.14277	-0.13648	-0.10552	-0.060690	-0.018755
5.89	0.055657	0.062405	0.032320	-0.019260	-0.075326	-0.12003	-0.14179	-0.13577	-0.10507	-0.060478	-0.018699
5.90	0.055816	0.062910	0.033185	-0.018137	-0.074108	-0.11888	-0.14084	-0.13507	-0.10464	-0.060273	-0.018645
5.91	0.055974	0.063410	0.034042	-0.017028	-0.072908	-0.11775	-0.13990	-0.13439	-0.10423	-0.060076	-0.018593
5.92	0.056131	0.063906	0.034890	-0.015933	-0.071725	-0.11664	-0.13898	-0.13373	-0.10383	-0.059886	-0.018544
5.93	0.056287	0.064398	0.035730	-0.014850	-0.070560	-0.11555	-0.13809	-0.13309	-0.10344	-0.059704	-0.018496
5.94	0.056441	0.064887	0.036562	-0.013779	-0.069411	-0.11448	-0.13722	-0.13246	-0.10306	-0.059529	-0.018451

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
5.95	0.056595	0.065372	0.037387	-0.012721	-0.068278	-0.11343	-0.13636	-0.13186	-0.10270	-0.059361	-0.018408
5.96	0.056748	0.065854	0.038205	-0.011674	-0.067161	-0.11240	-0.13552	-0.13127	-0.10234	-0.059201	-0.018368
5.97	0.056900	0.066332	0.039016	-0.010638	-0.066060	-0.11139	-0.13471	-0.13069	-0.10200	-0.059047	-0.018329
5.98	0.057052	0.066807	0.039820	-0.009613	-0.064973	-0.11039	-0.13391	-0.13014	-0.10168	-0.058900	-0.018293
5.99	0.057203	0.067280	0.040618	-0.008599	-0.063900	-0.10941	-0.13312	-0.12959	-0.10136	-0.058760	-0.018258
6.00	0.057352	0.067749	0.041410	-0.007594	-0.062841	-0.10845	-0.13236	-0.12907	-0.10106	-0.058627	-0.018226
6.01	0.057502	0.068216	0.042196	-0.006599	-0.061796	-0.10750	-0.13161	-0.12856	-0.10077	-0.058500	-0.018196
6.02	0.057651	0.068681	0.042977	-0.005614	-0.060764	-0.10657	-0.13088	-0.12806	-0.10049	-0.058380	-0.018167
6.03	0.057799	0.069143	0.043752	-0.004638	-0.059745	-0.10566	-0.13017	-0.12758	-0.10022	-0.058266	-0.018141
6.04	0.057946	0.069603	0.044522	-0.003670	-0.058739	-0.10476	-0.12947	-0.12712	-0.099963	-0.058159	-0.018116
6.05	0.058094	0.070061	0.045288	-0.002711	-0.057744	-0.10387	-0.12879	-0.12667	-0.099716	-0.058057	-0.018094
6.06	0.058240	0.070517	0.046049	-0.001760	-0.056761	-0.10300	-0.12812	-0.12623	-0.099481	-0.057963	-0.018073
6.07	0.058387	0.070971	0.046805	-0.000817	-0.055790	-0.10214	-0.12747	-0.12581	-0.099256	-0.057874	-0.018055
6.08	0.058533	0.071424	0.047557	0.000119	-0.054830	-0.10130	-0.12683	-0.12540	-0.099042	-0.057791	-0.018038
6.09	0.058678	0.071875	0.048306	0.001048	-0.053880	-0.10047	-0.12621	-0.12500	-0.098839	-0.057715	-0.018023
6.10	0.058824	0.072324	0.049051	0.001970	-0.052941	-0.099658	-0.12560	-0.12462	-0.098646	-0.057644	-0.018010
6.11	0.058969	0.072773	0.049792	0.002885	-0.052012	-0.098856	-0.12501	-0.12426	-0.098463	-0.057580	-0.017999
6.12	0.059114	0.073220	0.050530	0.003794	-0.051092	-0.098066	-0.12443	-0.12390	-0.098291	-0.057521	-0.017990
6.13	0.059259	0.073666	0.051265	0.004696	-0.050182	-0.097289	-0.12387	-0.12356	-0.098129	-0.057468	-0.017983
6.14	0.059403	0.074111	0.051997	0.005593	-0.049282	-0.096524	-0.12332	-0.12323	-0.097976	-0.057422	-0.017977
6.15	0.059548	0.074555	0.052727	0.006485	-0.048390	-0.095771	-0.12278	-0.12292	-0.097834	-0.057381	-0.017973
6.16	0.059692	0.074999	0.053454	0.007371	-0.047507	-0.095029	-0.12226	-0.12261	-0.097702	-0.057345	-0.017971
6.17	0.059837	0.075442	0.054179	0.008252	-0.046633	-0.094299	-0.12175	-0.12232	-0.097580	-0.057316	-0.017971
6.18	0.059981	0.075885	0.054901	0.009128	-0.045766	-0.093581	-0.12125	-0.12205	-0.097467	-0.057292	-0.017973
6.19	0.060126	0.076327	0.055622	0.010000	-0.044908	-0.092873	-0.12076	-0.12178	-0.097364	-0.057274	-0.017976
6.20	0.060271	0.076769	0.056342	0.010868	-0.044057	-0.092176	-0.12029	-0.12153	-0.097271	-0.057262	-0.017981
6.21	0.060416	0.077211	0.057060	0.011732	-0.043214	-0.091489	-0.11983	-0.12128	-0.097188	-0.057256	-0.017988
6.22	0.060560	0.077653	0.057776	0.012591	-0.042377	-0.090813	-0.11938	-0.12106	-0.097114	-0.057255	-0.017997
6.23	0.060706	0.078095	0.058492	0.013448	-0.041548	-0.090147	-0.11894	-0.12084	-0.097049	-0.057260	-0.018008
6.24	0.060851	0.078537	0.059207	0.014301	-0.040725	-0.089491	-0.11852	-0.12063	-0.096995	-0.057270	-0.018020
6.25	0.060997	0.078979	0.059921	0.015150	-0.039908	-0.088844	-0.11811	-0.12044	-0.096949	-0.057286	-0.018035
6.26	0.061143	0.079422	0.060634	0.015997	-0.039098	-0.088207	-0.11770	-0.12026	-0.096914	-0.057308	-0.018051
6.27	0.061289	0.079866	0.061348	0.016842	-0.038294	-0.087579	-0.11732	-0.12009	-0.096887	-0.057335	-0.018068
6.28	0.061436	0.080310	0.062061	0.017684	-0.037495	-0.086961	-0.11694	-0.11993	-0.096871	-0.057369	-0.018088
6.29	0.061583	0.080755	0.062774	0.018524	-0.036702	-0.086351	-0.11657	-0.11978	-0.096863	-0.057407	-0.018110

TABLE II - VALUES OF THE COEFFICIENT $\underline{C}$ - CONTINUED

λ	RATIO $\bar{X}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
6.30	0.061731	0.081201	0.063487	0.019362	-0.035915	-0.085751	-0.11622	-0.11965	-0.096865	-0.057452	-0.018133
6.31	0.061879	0.081648	0.064201	0.020198	-0.035132	-0.085159	-0.11587	-0.11952	-0.096877	-0.057502	-0.018158
6.32	0.062027	0.082096	0.064915	0.021032	-0.034354	-0.084575	-0.11554	-0.11941	-0.096898	-0.057558	-0.018185
6.33	0.062177	0.082545	0.065631	0.021866	-0.033581	-0.084000	-0.11522	-0.11931	-0.096928	-0.057620	-0.018214
6.34	0.062326	0.082996	0.066347	0.022698	-0.032812	-0.083433	-0.11490	-0.11922	-0.096968	-0.057687	-0.018244
6.35	0.062477	0.083448	0.067064	0.023530	-0.032048	-0.082874	-0.11460	-0.11914	-0.097018	-0.057760	-0.018277
6.36	0.062628	0.083902	0.067783	0.024361	-0.031288	-0.082323	-0.11431	-0.11907	-0.097077	-0.057840	-0.018311
6.37	0.062780	0.084357	0.068503	0.025191	-0.030531	-0.081779	-0.11403	-0.11902	-0.097145	-0.057924	-0.018347
6.38	0.062932	0.084814	0.069225	0.026022	-0.029778	-0.081243	-0.11377	-0.11897	-0.097223	-0.058015	-0.018386
6.39	0.063086	0.085273	0.069949	0.026852	-0.029029	-0.080715	-0.11351	-0.11894	-0.097311	-0.058112	-0.018426
6.40	0.063240	0.085735	0.070675	0.027683	-0.028282	-0.080194	-0.11326	-0.11892	-0.097408	-0.058215	-0.018468
6.41	0.063395	0.086198	0.071404	0.028514	-0.027539	-0.079680	-0.11302	-0.11891	-0.097516	-0.058323	-0.018512
6.42	0.063551	0.086663	0.072135	0.029346	-0.026799	-0.079173	-0.11279	-0.11891	-0.097633	-0.058438	-0.018558
6.43	0.063708	0.087132	0.072868	0.030179	-0.026061	-0.078673	-0.11258	-0.11892	-0.097759	-0.058559	-0.018606
6.44	0.063866	0.087602	0.073605	0.031013	-0.025326	-0.078180	-0.11237	-0.11895	-0.097896	-0.058686	-0.018656
6.45	0.064024	0.088075	0.074344	0.031848	-0.024593	-0.077694	-0.11218	-0.11898	-0.098043	-0.058819	-0.018708
6.46	0.064184	0.088552	0.075087	0.032685	-0.023862	-0.077214	-0.11199	-0.11903	-0.098200	-0.058958	-0.018762
6.47	0.064346	0.089031	0.075834	0.033524	-0.023133	-0.076741	-0.11181	-0.11909	-0.098367	-0.059104	-0.018818
6.48	0.064508	0.089513	0.076584	0.034365	-0.022406	-0.076274	-0.11165	-0.11916	-0.098544	-0.059256	-0.018876
6.49	0.064671	0.089998	0.077338	0.035208	-0.021680	-0.075813	-0.11149	-0.11924	-0.098731	-0.059415	-0.018936
6.50	0.064836	0.090487	0.078097	0.036054	-0.020955	-0.075358	-0.11135	-0.11934	-0.098929	-0.059580	-0.018999
6.51	0.065002	0.090979	0.078859	0.036903	-0.020232	-0.074910	-0.11121	-0.11945	-0.099138	-0.059752	-0.019063
6.52	0.065170	0.091475	0.079627	0.037755	-0.019509	-0.074467	-0.11109	-0.11956	-0.099357	-0.059931	-0.019130
6.53	0.065339	0.091974	0.080399	0.038610	-0.018787	-0.074031	-0.11097	-0.11969	-0.099587	-0.060116	-0.019199
6.54	0.065509	0.092478	0.081176	0.039469	-0.018066	-0.073600	-0.11087	-0.11984	-0.099827	-0.060309	-0.019271
6.55	0.065681	0.092985	0.081958	0.040331	-0.017344	-0.073174	-0.11078	-0.11999	-0.10008	-0.060508	-0.019344
6.56	0.065854	0.093497	0.082746	0.041198	-0.016623	-0.072755	-0.11069	-0.12016	-0.10034	-0.060714	-0.019420
6.57	0.066029	0.094013	0.083540	0.042069	-0.015902	-0.072340	-0.11062	-0.12034	-0.10062	-0.060928	-0.019499
6.58	0.066206	0.094534	0.084340	0.042944	-0.015181	-0.071931	-0.11056	-0.12053	-0.10090	-0.061148	-0.019579
6.59	0.066384	0.095059	0.085146	0.043825	-0.014459	-0.071528	-0.11050	-0.12074	-0.10120	-0.061377	-0.019663
6.60	0.066564	0.095590	0.085959	0.044710	-0.013736	-0.071129	-0.11046	-0.12096	-0.10151	-0.061612	-0.019749
6.61	0.066746	0.096125	0.086778	0.045601	-0.013012	-0.070736	-0.11043	-0.12119	-0.10183	-0.061856	-0.019837
6.62	0.066930	0.096666	0.087605	0.046498	-0.012287	-0.070348	-0.11041	-0.12144	-0.10216	-0.062107	-0.019928
6.63	0.067116	0.097212	0.088439	0.047401	-0.011561	-0.069965	-0.11040	-0.12169	-0.10251	-0.062366	-0.020021
6.64	0.067303	0.097763	0.089280	0.048310	-0.010833	-0.069587	-0.11040	-0.12197	-0.10287	-0.062633	-0.020118

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
6.65	0.067493	0.098321	0.090130	0.049225	-0.010103	-0.069213	-0.11041	-0.12225	-0.10324	-0.062909	-0.020217
6.66	0.067685	0.098884	0.090988	0.050148	-0.009371	-0.068845	-0.11043	-0.12255	-0.10362	-0.063193	-0.020318
6.67	0.067880	0.099454	0.091854	0.051078	-0.008637	-0.068481	-0.11047	-0.12287	-0.10402	-0.063485	-0.020423
6.68	0.068076	0.10003	0.092729	0.052015	-0.007901	-0.068121	-0.11051	-0.12320	-0.10443	-0.063786	-0.020531
6.69	0.068275	0.10061	0.093614	0.052961	-0.007161	-0.067766	-0.11057	-0.12354	-0.10485	-0.064096	-0.020641
6.70	0.068476	0.10120	0.094508	0.053915	-0.006419	-0.067416	-0.11064	-0.12390	-0.10529	-0.064414	-0.020755
6.71	0.068680	0.10180	0.095411	0.054877	-0.005673	-0.067070	-0.11071	-0.12427	-0.10574	-0.064742	-0.020871
6.72	0.068887	0.10240	0.096325	0.055848	-0.004924	-0.066729	-0.11080	-0.12466	-0.10621	-0.065080	-0.020991
6.73	0.069096	0.10301	0.097250	0.056829	-0.004171	-0.066391	-0.11091	-0.12507	-0.10669	-0.065427	-0.021114
6.74	0.069308	0.10363	0.098186	0.057820	-0.003414	-0.066058	-0.11102	-0.12549	-0.10718	-0.065784	-0.021240
6.75	0.069523	0.10426	0.099133	0.058821	-0.002652	-0.065729	-0.11115	-0.12592	-0.10770	-0.066151	-0.021370
6.76	0.069740	0.10490	0.10009	0.059832	-0.001886	-0.065404	-0.11128	-0.12638	-0.10822	-0.066528	-0.021503
6.77	0.069961	0.10554	0.10106	0.060854	-0.001115	-0.065083	-0.11143	-0.12685	-0.10877	-0.066916	-0.021640
6.78	0.070185	0.10620	0.10205	0.061888	-0.000339	-0.064767	-0.11160	-0.12734	-0.10933	-0.067315	-0.021780
6.79	0.070412	0.10686	0.10304	0.062934	+0.000443	-0.064454	-0.11177	-0.12784	-0.10990	-0.067724	-0.021924
6.80	0.070642	0.10753	0.10405	0.063992	0.001230	-0.064145	-0.11196	-0.12837	-0.11050	-0.068145	-0.022072
6.81	0.070876	0.10821	0.10508	0.065063	0.002024	-0.063840	-0.11217	-0.12891	-0.11111	-0.068577	-0.022224
6.82	0.071114	0.10890	0.10611	0.066147	0.002824	-0.063538	-0.11238	-0.12947	-0.11174	-0.069021	-0.022379
6.83	0.071355	0.10961	0.10717	0.067245	0.003631	-0.063241	-0.11261	-0.13005	-0.11238	-0.069478	-0.022539
6.84	0.071600	0.11032	0.10824	0.068358	0.004445	-0.062947	-0.11285	-0.13065	-0.11305	-0.069946	-0.022703
6.85	0.071849	0.11104	0.10932	0.069485	0.005266	-0.062656	-0.11311	-0.13127	-0.11374	-0.070428	-0.022872
6.86	0.072101	0.11178	0.11042	0.070628	0.006096	-0.062369	-0.11338	-0.13191	-0.11444	-0.070923	-0.023045
6.87	0.072358	0.11253	0.11154	0.071787	0.006934	-0.062086	-0.11367	-0.13257	-0.11517	-0.071431	-0.023222
6.88	0.072620	0.11329	0.11268	0.072963	0.007780	-0.061806	-0.11397	-0.13326	-0.11592	-0.071954	-0.023404
6.89	0.072886	0.11406	0.11383	0.074156	0.008636	-0.061530	-0.11429	-0.13396	-0.11669	-0.072490	-0.023591
6.90	0.073156	0.11485	0.11501	0.075367	0.009501	-0.061257	-0.11462	-0.13469	-0.11748	-0.073041	-0.023783
6.91	0.073431	0.11565	0.11620	0.076597	0.010376	-0.060987	-0.11497	-0.13544	-0.11829	-0.073608	-0.023980
6.92	0.073711	0.11646	0.11741	0.077846	0.011261	-0.060721	-0.11534	-0.13622	-0.11913	-0.074190	-0.024183
6.93	0.073996	0.11729	0.11865	0.079115	0.012158	-0.060458	-0.11572	-0.13702	-0.11999	-0.074788	-0.024391
6.94	0.074287	0.11813	0.11991	0.080405	0.013066	-0.060198	-0.11612	-0.13785	-0.12088	-0.075402	-0.024604
6.95	0.074583	0.11899	0.12119	0.081717	0.013986	-0.059941	-0.11654	-0.13870	-0.12179	-0.076034	-0.024823
6.96	0.074885	0.11987	0.12249	0.083052	0.014918	-0.059687	-0.11697	-0.13958	-0.12273	-0.076683	-0.025048
6.97	0.075192	0.12076	0.12382	0.084410	0.015863	-0.059437	-0.11743	-0.14049	-0.12369	-0.077350	-0.025280
6.98	0.075505	0.12167	0.12517	0.085792	0.016822	-0.059189	-0.11790	-0.14143	-0.12469	-0.078036	-0.025518
6.99	0.075825	0.12260	0.12655	0.087199	0.017795	-0.058945	-0.11840	-0.14239	-0.12571	-0.078742	-0.025762

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{X}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
7.00	0.076152	0.12355	0.12796	0.088633	0.018783	-0.058703	-0.11891	-0.14339	-0.12676	-0.079467	-0.026013
7.01	0.076485	0.12451	0.12940	0.090094	0.019787	-0.058464	-0.11945	-0.14442	-0.12785	-0.080213	-0.026271
7.02	0.076825	0.12550	0.13086	0.091584	0.020807	-0.058229	-0.12000	-0.14548	-0.12896	-0.080981	-0.026537
7.03	0.077172	0.12651	0.13236	0.093103	0.021843	-0.057996	-0.12058	-0.14657	-0.13011	-0.081770	-0.026810
7.04	0.077527	0.12754	0.13388	0.094653	0.022898	-0.057766	-0.12118	-0.14770	-0.13130	-0.082582	-0.027091
7.05	0.077890	0.12859	0.13544	0.096235	0.023970	-0.057539	-0.12181	-0.14887	-0.13252	-0.083418	-0.027380
7.06	0.078260	0.12967	0.13704	0.097850	0.025063	-0.057314	-0.12246	-0.15007	-0.13377	-0.084279	-0.027677
7.07	0.078640	0.13077	0.13867	0.099500	0.026175	-0.057093	-0.12313	-0.15131	-0.13506	-0.085165	-0.027983
7.08	0.079028	0.13189	0.14034	0.10119	0.027308	-0.056874	-0.12383	-0.15259	-0.13640	-0.086078	-0.028298
7.09	0.079425	0.13304	0.14204	0.10291	0.028464	-0.056657	-0.12456	-0.15391	-0.13777	-0.087018	-0.028622
7.10	0.079832	0.13422	0.14379	0.10467	0.029642	-0.056444	-0.12532	-0.15528	-0.13919	-0.087986	-0.028956
7.11	0.080248	0.13543	0.14558	0.10648	0.030844	-0.056233	-0.12610	-0.15669	-0.14065	-0.088985	-0.029301
7.12	0.080675	0.13667	0.14741	0.10832	0.032072	-0.056024	-0.12692	-0.15815	-0.14216	-0.090014	-0.029655
7.13	0.081113	0.13794	0.14929	0.11021	0.033326	-0.055818	-0.12776	-0.15965	-0.14371	-0.091075	-0.030021
7.14	0.081562	0.13924	0.15121	0.11215	0.034608	-0.055615	-0.12864	-0.16120	-0.14531	-0.092169	-0.030398
7.15	0.082023	0.14058	0.15319	0.11414	0.035919	-0.055414	-0.12955	-0.16281	-0.14697	-0.093298	-0.030787
7.16	0.082496	0.14195	0.15522	0.11618	0.037260	-0.055215	-0.13050	-0.16447	-0.14868	-0.094464	-0.031188
7.17	0.082981	0.14336	0.15730	0.11827	0.038632	-0.055019	-0.13148	-0.16619	-0.15045	-0.095667	-0.031603
7.18	0.083480	0.14480	0.15944	0.12041	0.040038	-0.054826	-0.13250	-0.16796	-0.15227	-0.096909	-0.032030
7.19	0.083993	0.14629	0.16164	0.12262	0.041479	-0.054635	-0.13356	-0.16980	-0.15416	-0.098193	-0.032472
7.20	0.084520	0.14782	0.16390	0.12488	0.042957	-0.054446	-0.13466	-0.17170	-0.15611	-0.099520	-0.032929
7.21	0.085063	0.14939	0.16622	0.12721	0.044473	-0.054259	-0.13580	-0.17367	-0.15812	-0.10089	-0.033401
7.22	0.085622	0.15101	0.16861	0.12961	0.046029	-0.054075	-0.13699	-0.17571	-0.16021	-0.10231	-0.033889
7.23	0.086197	0.15268	0.17108	0.13207	0.047628	-0.053893	-0.13822	-0.17782	-0.16237	-0.10378	-0.034394
7.24	0.086790	0.15440	0.17362	0.13462	0.049272	-0.053713	-0.13951	-0.18001	-0.16461	-0.10530	-0.034917
7.25	0.087401	0.15618	0.17624	0.13723	0.050963	-0.053536	-0.14084	-0.18228	-0.16693	-0.10687	-0.035458
7.26	0.088032	0.15801	0.17894	0.13993	0.052703	-0.053361	-0.14223	-0.18464	-0.16933	-0.10851	-0.036019
7.27	0.088684	0.15990	0.18173	0.14272	0.054496	-0.053188	-0.14367	-0.18708	-0.17182	-0.11020	-0.036600
7.28	0.089356	0.16185	0.18461	0.14560	0.056343	-0.053017	-0.14518	-0.18962	-0.17441	-0.11195	-0.037203
7.29	0.090052	0.16386	0.18759	0.14857	0.058249	-0.052848	-0.14674	-0.19225	-0.17709	-0.11377	-0.037829
7.30	0.090771	0.16595	0.19067	0.15164	0.060216	-0.052682	-0.14837	-0.19499	-0.17988	-0.11566	-0.038478
7.31	0.091516	0.16811	0.19386	0.15482	0.062248	-0.052517	-0.15007	-0.19784	-0.18278	-0.11763	-0.039153
7.32	0.092287	0.17035	0.19717	0.15812	0.064349	-0.052355	-0.15184	-0.20080	-0.18579	-0.11967	-0.039855
7.33	0.093086	0.17267	0.20059	0.16153	0.066523	-0.052195	-0.15369	-0.20389	-0.18893	-0.12180	-0.040585
7.34	0.093915	0.17507	0.20414	0.16507	0.068774	-0.052036	-0.15562	-0.20710	-0.19219	-0.12401	-0.041345

TABLE II - VALUES OF THE COEFFICIENT $\underline{C}$ - CONTINUED

λ	RATIO $\bar{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
7.35	0.094776	0.17757	0.20783	0.16874	0.071107	-0.051880	-0.15763	-0.21046	-0.19559	-0.12631	-0.042136
7.36	0.095670	0.18017	0.21166	0.17255	0.073527	-0.051726	-0.15974	-0.21395	-0.19914	-0.12872	-0.042961
7.37	0.096599	0.18287	0.21564	0.17652	0.076039	-0.051574	-0.16194	-0.21760	-0.20284	-0.13122	-0.043821
7.38	0.097567	0.18568	0.21979	0.18064	0.078651	-0.051424	-0.16424	-0.22141	-0.20670	-0.13384	-0.044720
7.39	0.098574	0.18860	0.22411	0.18494	0.081367	-0.051275	-0.16666	-0.22540	-0.21074	-0.13657	-0.045658
7.40	0.099625	0.19165	0.22861	0.18942	0.084195	-0.051129	-0.16918	-0.22957	-0.21497	-0.13943	-0.046640
7.41	0.10072	0.19484	0.23331	0.19409	0.087144	-0.050984	-0.17183	-0.23394	-0.21939	-0.14243	-0.047667
7.42	0.10187	0.19817	0.23823	0.19897	0.090220	-0.050842	-0.17462	-0.23852	-0.22403	-0.14557	-0.048744
7.43	0.10307	0.20165	0.24336	0.20408	0.093434	-0.050701	-0.17754	-0.24333	-0.22889	-0.14886	-0.049873
7.44	0.10432	0.20529	0.24874	0.20943	0.096796	-0.050563	-0.18062	-0.24837	-0.23400	-0.15231	-0.051059
7.45	0.10563	0.20911	0.25438	0.21503	0.10032	-0.050426	-0.18385	-0.25368	-0.23936	-0.15594	-0.052305
7.46	0.10701	0.21312	0.26031	0.22091	0.10401	-0.050291	-0.18726	-0.25927	-0.24501	-0.15976	-0.053616
7.47	0.10847	0.21734	0.26653	0.22710	0.10789	-0.050157	-0.19086	-0.26516	-0.25096	-0.16379	-0.054997
7.48	0.10999	0.22177	0.27308	0.23360	0.11196	-0.050026	-0.19467	-0.27138	-0.25724	-0.16804	-0.056455
7.49	0.11160	0.22645	0.27998	0.24046	0.11625	-0.049896	-0.19869	-0.27794	-0.26388	-0.17252	-0.057994
7.50	0.11330	0.23138	0.28727	0.24769	0.12078	-0.049768	-0.20295	-0.28489	-0.27090	-0.17727	-0.059623
7.51	0.11509	0.23660	0.29497	0.25534	0.12557	-0.049642	-0.20747	-0.29226	-0.27834	-0.18230	-0.061348
7.52	0.11699	0.24212	0.30313	0.26344	0.13063	-0.049518	-0.21227	-0.30008	-0.28623	-0.18764	-0.063179
7.53	0.11901	0.24799	0.31179	0.27204	0.13599	-0.049395	-0.21737	-0.30840	-0.29462	-0.19331	-0.065126
7.54	0.12115	0.25422	0.32099	0.28117	0.14169	-0.049274	-0.22282	-0.31725	-0.30356	-0.19936	-0.067199
7.55	0.12343	0.26085	0.33079	0.29090	0.14776	-0.049155	-0.22863	-0.32671	-0.31310	-0.20580	-0.069411
7.56	0.12586	0.26793	0.34126	0.30128	0.15423	-0.049037	-0.23486	-0.33682	-0.32330	-0.21270	-0.071776
7.57	0.12847	0.27551	0.35245	0.31239	0.16115	-0.048922	-0.24153	-0.34765	-0.33423	-0.22009	-0.074311
7.58	0.13126	0.28363	0.36445	0.32430	0.16857	-0.048807	-0.24871	-0.35930	-0.34597	-0.22802	-0.077033
7.59	0.13426	0.29236	0.37736	0.33711	0.17654	-0.048695	-0.25644	-0.37184	-0.35862	-0.23657	-0.079965
7.60	0.13750	0.30178	0.39127	0.35092	0.18514	-0.048584	-0.26479	-0.38538	-0.37228	-0.24580	-0.083132
7.61	0.14100	0.31197	0.40632	0.36586	0.19443	-0.048475	-0.27385	-0.40005	-0.38707	-0.25580	-0.086561
7.62	0.14479	0.32302	0.42265	0.38206	0.20450	-0.048367	-0.28369	-0.41600	-0.40314	-0.26667	-0.090288
7.63	0.14893	0.33505	0.44044	0.39971	0.21547	-0.048261	-0.29443	-0.43339	-0.42067	-0.27851	-0.094351
7.64	0.15344	0.34820	0.45988	0.41900	0.22746	-0.048157	-0.30619	-0.45262	-0.43986	-0.29148	-0.098799

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
7.65	0.15840	0.36264	0.48122	0.44018	0.24061	-0.048054	-0.31912	-0.47335	-0.46095	-0.30573	-0.10369
7.66	0.16387	0.37856	0.50476	0.46354	0.25512	-0.047953	-0.33341	-0.49645	-0.48423	-0.32147	-0.10909
7.67	0.16993	0.39622	0.53086	0.48944	0.27120	-0.047853	-0.34927	-0.52210	-0.51008	-0.33894	-0.11508
7.68	0.17669	0.41590	0.55996	0.51832	0.28913	-0.047755	-0.36698	-0.55073	-0.53893	-0.35843	-0.12177
7.69	0.18427	0.43798	0.59262	0.55072	0.30923	-0.047658	-0.38688	-0.58288	-0.57133	-0.38033	-0.12928
7.70	0.19284	0.46293	0.62952	0.58734	0.33195	-0.047563	-0.40939	-0.61925	-0.60798	-0.40510	-0.13777
7.71	0.20259	0.49136	0.67156	0.62906	0.35784	-0.047470	-0.43506	-0.66072	-0.64976	-0.43334	-0.14746
7.72	0.21381	0.52404	0.71989	0.67702	0.38759	-0.047378	-0.46461	-0.70844	-0.69785	-0.46583	-0.15861
7.73	0.22684	0.56201	0.77605	0.73276	0.42215	-0.047287	-0.49897	-0.76394	-0.75376	-0.50362	-0.17157
7.74	0.24217	0.60668	0.84212	0.79832	0.46281	-0.047198	-0.53943	-0.82926	-0.81957	-0.54810	-0.18682
7.75	0.26046	0.65998	0.92096	0.87658	0.51133	-0.047111	-0.58775	-0.90727	-0.89817	-0.60121	-0.20505
7.76	0.28267	0.72470	1.0167	0.97160	0.57024	-0.047025	-0.64647	-1.0021	-0.99366	-0.66575	-0.22718
7.77	0.31021	0.80496	1.1354	1.0894	0.64330	-0.046940	-0.71933	-1.1197	-1.1121	-0.74582	-0.25465
7.78	0.34526	0.90712	1.2866	1.2394	0.73629	-0.046857	-0.81213	-1.2694	-1.2630	-0.84780	-0.28963
7.79	0.39139	1.0416	1.4855	1.4369	0.85867	-0.046776	-0.93433	-1.4666	-1.4617	-0.98207	-0.33569
7.80	0.45485	1.2266	1.7592	1.7085	1.0270	-0.046696	-1.1025	-1.7380	-1.7351	-1.1669	-0.39908
7.81	0.54767	1.4971	2.1595	2.1058	1.2733	-0.046617	-1.3486	-2.1351	-2.1352	-1.4372	-0.49183
7.82	0.69638	1.9306	2.8008	2.7424	1.6679	-0.046540	-1.7430	-2.7715	-2.7763	-1.8705	-0.64046
7.83	0.97322	2.7376	3.9948	3.9276	2.4024	-0.046464	-2.4774	-3.9564	-3.9701	-2.6773	-0.91723
7.84	1.6693	4.7667	6.9972	6.9078	4.2495	-0.046389	-4.3242	-6.9364	-6.9722	-4.7062	-1.6132
7.85	6.7095	19.459	28.787	28.486	17.623	-0.046316	-17.698	-28.515	-28.711	-19.398	-6.6533
7.86	-3.0843	-9.0905	-13.507	-13.445	-8.3644	-0.046245	8.2900	13.416	13.532	9.1514	3.1405
7.87	-1.2156	-3.6431	-5.4463	-5.4438	-3.4057	-0.046175	3.3315	5.4159	5.4720	3.7041	1.2719
7.88	-0.74165	-2.2615	-3.4021	-3.4148	-2.1481	-0.046106	2.0741	3.3870	3.4281	2.3227	0.79800
7.89	-0.52529	-1.6308	-2.4690	-2.4885	-1.5741	-0.046038	1.5002	2.4610	2.4952	1.6922	0.58172
7.90	-0.40139	-1.2697	-1.9346	-1.9581	-1.2453	-0.045972	1.1716	1.9309	1.9611	1.3312	0.45789
7.91	-0.32111	-1.0356	-1.5884	-1.6145	-1.0323	-0.045908	0.95874	1.5874	1.6151	1.0974	0.37768
7.92	-0.26485	-0.87167	-1.3458	-1.3737	-0.88311	-0.045844	0.80966	1.3469	1.3728	0.93363	0.32150
7.93	-0.22323	-0.75037	-1.1663	-1.1956	-0.77273	-0.045782	0.69943	1.1690	1.1936	0.81252	0.27995
7.94	-0.19119	-0.65700	-1.0282	-1.0585	-0.68776	-0.045722	0.61462	1.0321	1.0557	0.71935	0.24799
7.95	-0.16577	-0.58291	-0.91863	-0.94978	-0.62035	-0.045663	0.54735	0.92357	0.94632	0.64545	0.22264
7.96	-0.14510	-0.52267	-0.82953	-0.86136	-0.56555	-0.045605	0.49270	0.83537	0.85747	0.58540	0.20204
7.97	-0.12796	-0.47274	-0.75568	-0.78808	-0.52012	-0.045548	0.44742	0.76230	0.78386	0.53566	0.18498
7.98	-0.11352	-0.43066	-0.69345	-0.72634	-0.48185	-0.045493	0.40929	0.70077	0.72188	0.49377	0.17062
7.99	-0.10118	-0.39472	-0.64031	-0.67362	-0.44917	-0.045439	0.37676	0.64825	0.66898	0.45803	0.15835

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\overline{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
8.00	-0.090521	-0.36367	-0.59439	-0.62806	-0.42094	-0.045387	0.34866	0.60291	0.62330	0.42717	0.14777
8.01	-0.081215	-0.33656	-0.55432	-0.58831	-0.39631	-0.045336	0.32416	0.56336	0.58347	0.40025	0.13854
8.02	-0.073019	-0.31270	-0.51904	-0.55332	-0.37462	-0.045286	0.30261	0.52857	0.54843	0.37658	0.13042
8.03	-0.065746	-0.29152	-0.48774	-0.52228	-0.35538	-0.045238	0.28350	0.49773	0.51738	0.35560	0.12322
8.04	-0.059247	-0.27260	-0.45978	-0.49455	-0.33820	-0.045190	0.26645	0.47021	0.48966	0.33687	0.11680
8.05	-0.053404	-0.25559	-0.43465	-0.46964	-0.32276	-0.045145	0.25113	0.44550	0.46477	0.32006	0.11104
8.06	-0.048122	-0.24022	-0.41194	-0.44712	-0.30881	-0.045100	0.23731	0.42318	0.44230	0.30488	0.10583
8.07	-0.043324	-0.22625	-0.39131	-0.42668	-0.29614	-0.045057	0.22477	0.40294	0.42192	0.29112	0.10111
8.08	-0.038945	-0.21351	-0.37249	-0.40804	-0.28459	-0.045015	0.21334	0.38449	0.40335	0.27857	0.096811
8.09	-0.034931	-0.20184	-0.35526	-0.39096	-0.27401	-0.044974	0.20288	0.36761	0.38635	0.26710	0.092877
8.10	-0.031240	-0.19110	-0.33941	-0.37526	-0.26429	-0.044935	0.19327	0.35211	0.37075	0.25656	0.089264
8.11	-0.027832	-0.18119	-0.32479	-0.36078	-0.25532	-0.044897	0.18442	0.33788	0.35637	0.24685	0.085936
8.12	-0.024677	-0.17202	-0.31125	-0.34738	-0.24703	-0.044860	0.17623	0.32462	0.34308	0.23788	0.082861
8.13	-0.021746	-0.16350	-0.29869	-0.33494	-0.23933	-0.044825	0.16864	0.31238	0.33076	0.22956	0.080010
8.14	-0.019017	-0.15557	-0.28700	-0.32337	-0.23216	-0.044791	0.16159	0.30100	0.31931	0.22183	0.077361
8.15	-0.016468	-0.14817	-0.27608	-0.31257	-0.22548	-0.044758	0.15501	0.29039	0.30864	0.21463	0.074893
8.16	-0.014083	-0.14124	-0.26587	-0.30247	-0.21923	-0.044727	0.14886	0.28048	0.29867	0.20790	0.072588
8.17	-0.011845	-0.13474	-0.25630	-0.29300	-0.21338	-0.044696	0.14311	0.27121	0.28935	0.20161	0.070432
8.18	-0.009741	-0.12864	-0.24730	-0.28411	-0.20788	-0.044667	0.13771	0.26251	0.28060	0.19571	0.068410
8.19	-0.007759	-0.12289	-0.23884	-0.27575	-0.20271	-0.044640	0.13264	0.25433	0.27238	0.19016	0.066510
8.20	-0.005889	-0.11746	-0.23085	-0.26786	-0.19784	-0.044614	0.12786	0.24663	0.26464	0.18494	0.064723
8.21	-0.004121	-0.11234	-0.22331	-0.26041	-0.19323	-0.044588	0.12336	0.23937	0.25734	0.18002	0.063037
8.22	-0.002447	-0.10748	-0.21617	-0.25337	-0.18888	-0.044565	0.11910	0.23251	0.25045	0.17538	0.061446
8.23	-0.000859	-0.10288	-0.20941	-0.24669	-0.18476	-0.044542	0.11507	0.22602	0.24393	0.17098	0.059942
8.24	0.000650	-0.098512	-0.20299	-0.24036	-0.18086	-0.044521	0.11125	0.21987	0.23776	0.16682	0.058518
8.25	0.002085	-0.094358	-0.19688	-0.23434	-0.17715	-0.044501	0.10762	0.21404	0.23190	0.16288	0.057167
8.26	0.003452	-0.090402	-0.19107	-0.22862	-0.17362	-0.044483	0.10418	0.20850	0.22634	0.15913	0.055885
8.27	0.004755	-0.086630	-0.18554	-0.22317	-0.17026	-0.044465	0.10090	0.20323	0.22106	0.15557	0.054667
8.28	0.006000	-0.083029	-0.18026	-0.21797	-0.16706	-0.044449	0.097776	0.19821	0.21603	0.15218	0.053508
8.29	0.007191	-0.079588	-0.17521	-0.21300	-0.16400	-0.044435	0.094798	0.19343	0.21123	0.14895	0.052404
8.30	0.008330	-0.076296	-0.17039	-0.20826	-0.16108	-0.044421	0.091956	0.18886	0.20666	0.14588	0.051351
8.31	0.009422	-0.073143	-0.16577	-0.20372	-0.15829	-0.044409	0.089239	0.18451	0.20229	0.14294	0.050346
8.32	0.010470	-0.070120	-0.16134	-0.19937	-0.15562	-0.044398	0.086641	0.18034	0.19812	0.14013	0.049386
8.33	0.011475	-0.067218	-0.15710	-0.19521	-0.15307	-0.044389	0.084152	0.17635	0.19412	0.13744	0.048469
8.34	0.012442	-0.064431	-0.15302	-0.19121	-0.15061	-0.044380	0.081767	0.17253	0.19030	0.13488	0.047591

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\overline{X}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
8.35	0.013372	-0.061751	-0.14910	-0.18737	-0.14826	-0.044373	0.079480	0.16887	0.18664	0.13241	0.046750
8.36	0.014267	-0.059172	-0.14534	-0.18368	-0.14600	-0.044368	0.077283	0.16535	0.18312	0.13005	0.045944
8.37	0.015130	-0.056687	-0.14171	-0.18013	-0.14383	-0.044363	0.075173	0.16198	0.17975	0.12779	0.045171
8.38	0.015963	-0.054293	-0.13822	-0.17671	-0.14174	-0.044360	0.073144	0.15874	0.17652	0.12562	0.044430
8.39	0.016766	-0.051982	-0.13485	-0.17342	-0.13973	-0.044359	0.071191	0.15562	0.17341	0.12353	0.043717
8.40	0.017542	-0.049752	-0.13160	-0.17025	-0.13779	-0.044358	0.069311	0.15262	0.17041	0.12152	0.043033
8.41	0.018293	-0.047597	-0.12846	-0.16719	-0.13593	-0.044359	0.067499	0.14974	0.16754	0.11959	0.042375
8.42	0.019019	-0.045513	-0.12543	-0.16423	-0.13413	-0.044361	0.065751	0.14696	0.16476	0.11774	0.041741
8.43	0.019722	-0.043497	-0.12250	-0.16138	-0.13239	-0.044365	0.064065	0.14427	0.16209	0.11595	0.041132
8.44	0.020402	-0.041545	-0.11967	-0.15862	-0.13072	-0.044370	0.062436	0.14169	0.15952	0.11423	0.040545
8.45	0.021063	-0.039654	-0.11693	-0.15595	-0.12910	-0.044376	0.060863	0.13919	0.15704	0.11256	0.039979
8.46	0.021703	-0.037821	-0.11427	-0.15337	-0.12754	-0.044383	0.059342	0.13678	0.15464	0.11096	0.039434
8.47	0.022325	-0.036043	-0.11169	-0.15087	-0.12602	-0.044392	0.057871	0.13445	0.15233	0.10942	0.038908
8.48	0.022928	-0.034317	-0.10919	-0.14845	-0.12456	-0.044403	0.056447	0.13220	0.15010	0.10792	0.038400
8.49	0.023515	-0.032641	-0.10677	-0.14610	-0.12314	-0.044414	0.055068	0.13002	0.14794	0.10648	0.037910
8.50	0.024086	-0.031012	-0.10442	-0.14382	-0.12177	-0.044427	0.053732	0.12792	0.14585	0.10509	0.037437
8.51	0.024641	-0.029429	-0.10213	-0.14161	-0.12044	-0.044441	0.052437	0.12588	0.14383	0.10375	0.036979
8.52	0.025182	-0.027888	-0.099910	-0.13947	-0.11916	-0.044457	0.051181	0.12390	0.14188	0.10244	0.036538
8.53	0.025708	-0.026389	-0.097750	-0.13739	-0.11791	-0.044474	0.049962	0.12199	0.13999	0.10119	0.036110
8.54	0.026221	-0.024930	-0.095649	-0.13536	-0.11670	-0.044492	0.048778	0.12013	0.13816	0.099968	0.035697
8.55	0.026722	-0.023507	-0.093604	-0.13340	-0.11552	-0.044512	0.047629	0.11833	0.13638	0.098789	0.035297
8.56	0.027210	-0.022121	-0.091613	-0.13148	-0.11438	-0.044533	0.046512	0.11659	0.13467	0.097647	0.034911
8.57	0.027686	-0.020769	-0.089674	-0.12962	-0.11328	-0.044556	0.045426	0.11489	0.13300	0.096541	0.034536
8.58	0.028151	-0.019450	-0.087783	-0.12781	-0.11220	-0.044580	0.044369	0.11325	0.13139	0.095470	0.034174
8.59	0.028606	-0.018163	-0.085940	-0.12605	-0.11115	-0.044605	0.043342	0.11165	0.12982	0.094432	0.033822
8.60	0.029050	-0.016906	-0.084143	-0.12433	-0.11014	-0.044632	0.042341	0.11010	0.12830	0.093426	0.033482
8.61	0.029484	-0.015678	-0.082389	-0.12266	-0.10915	-0.044660	0.041367	0.10860	0.12682	0.092450	0.033153
8.62	0.029909	-0.014477	-0.080676	-0.12103	-0.10819	-0.044689	0.040418	0.10713	0.12539	0.091503	0.032833
8.63	0.030325	-0.013304	-0.079004	-0.11944	-0.10726	-0.044720	0.039493	0.10571	0.12400	0.090585	0.032523
8.64	0.030732	-0.012156	-0.077371	-0.11789	-0.10635	-0.044753	0.038591	0.10432	0.12265	0.089694	0.032223
8.65	0.031131	-0.011032	-0.075774	-0.11637	-0.10547	-0.044787	0.037712	0.10297	0.12134	0.088830	0.031932
8.66	0.031522	-0.009933	-0.074213	-0.11489	-0.10460	-0.044822	0.036854	0.10166	0.12006	0.087991	0.031650
8.67	0.031905	-0.008855	-0.072686	-0.11345	-0.10377	-0.044859	0.036016	0.10038	0.11882	0.087176	0.031376
8.68	0.032281	-0.007800	-0.071192	-0.11204	-0.10295	-0.044897	0.035198	0.099138	0.11762	0.086385	0.031110
8.69	0.032650	-0.006766	-0.069730	-0.11067	-0.10215	-0.044937	0.034399	0.097926	0.11644	0.085617	0.030852

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO x̄/L										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
8.70	0.033011	-0.005752	-0.068299	-0.10932	-0.10138	-0.044978	0.033619	0.096745	0.11530	0.084871	0.030602
8.71	0.033367	-0.004758	-0.066897	-0.10801	-0.10063	-0.045021	0.032856	0.095594	0.11420	0.084146	0.030359
8.72	0.033716	-0.003782	-0.065524	-0.10672	-0.099890	-0.045066	0.032110	0.094472	0.11312	0.083442	0.030124
8.73	0.034059	-0.002824	-0.064177	-0.10546	-0.099172	-0.045111	0.031380	0.093378	0.11207	0.082757	0.029895
8.74	0.034396	-0.001884	-0.062858	-0.10423	-0.098473	-0.045159	0.030667	0.092312	0.11105	0.082092	0.029673
8.75	0.034727	-0.000961	-0.061564	-0.10303	-0.097791	-0.045208	0.029968	0.091271	0.11005	0.081446	0.029457
8.76	0.035053	-0.000053	-0.060294	-0.10185	-0.097126	-0.045258	0.029284	0.090256	0.10908	0.080818	0.029248
8.77	0.035374	0.000838	-0.059048	-0.10069	-0.096477	-0.045310	0.028614	0.089265	0.10814	0.080208	0.029045
8.78	0.035690	0.001715	-0.057825	-0.099563	-0.095844	-0.045364	0.027958	0.088298	0.10722	0.079614	0.028848
8.79	0.036001	0.002578	-0.056624	-0.098455	-0.095226	-0.045419	0.027315	0.087354	0.10633	0.079038	0.028657
8.80	0.036307	0.003426	-0.055445	-0.097370	-0.094623	-0.045476	0.026684	0.086432	0.10546	0.078477	0.028471
8.81	0.036609	0.004261	-0.054286	-0.096306	-0.094035	-0.045534	0.026066	0.085532	0.10461	0.077932	0.028291
8.82	0.036906	0.005083	-0.053148	-0.095263	-0.093461	-0.045594	0.025460	0.084652	0.10378	0.077402	0.028116
8.83	0.037200	0.005892	-0.052028	-0.094240	-0.092900	-0.045656	0.024865	0.083793	0.10298	0.076887	0.027946
8.84	0.037489	0.006689	-0.050928	-0.093236	-0.092353	-0.045719	0.024281	0.082953	0.10219	0.076386	0.027781
8.85	0.037774	0.007474	-0.049845	-0.092252	-0.091819	-0.045785	0.023708	0.082133	0.10143	0.075900	0.027621
8.86	0.038056	0.008249	-0.048780	-0.091286	-0.091297	-0.045851	0.023145	0.081331	0.10069	0.075427	0.027466
8.87	0.038334	0.009012	-0.047732	-0.090338	-0.090788	-0.045920	0.022592	0.080547	0.099961	0.074967	0.027316
8.88	0.038609	0.009765	-0.046700	-0.089408	-0.090291	-0.045990	0.022049	0.079780	0.099254	0.074521	0.027170
8.89	0.038880	0.010507	-0.045684	-0.088494	-0.089805	-0.046062	0.021515	0.079031	0.098566	0.074086	0.027029
8.90	0.039148	0.011240	-0.044684	-0.087597	-0.089331	-0.046135	0.020990	0.078298	0.097894	0.073665	0.026891
8.91	0.039413	0.011963	-0.043698	-0.086715	-0.088868	-0.046211	0.020474	0.077581	0.097240	0.073255	0.026759
8.92	0.039675	0.012677	-0.042727	-0.085849	-0.088416	-0.046288	0.019966	0.076879	0.096603	0.072857	0.026630
8.93	0.039934	0.013383	-0.041769	-0.084998	-0.087974	-0.046367	0.019466	0.076193	0.095982	0.072470	0.026505
8.94	0.040190	0.014080	-0.040825	-0.084161	-0.087543	-0.046448	0.018975	0.075521	0.095377	0.072095	0.026384
8.95	0.040444	0.014768	-0.039894	-0.083339	-0.087122	-0.046531	0.018490	0.074864	0.094787	0.071731	0.026267
8.96	0.040695	0.015449	-0.038976	-0.082530	-0.086711	-0.046615	0.018014	0.074221	0.094213	0.071377	0.026154
8.97	0.040943	0.016122	-0.038070	-0.081735	-0.086309	-0.046701	0.017544	0.073591	0.093653	0.071034	0.026045
8.98	0.041189	0.016787	-0.037176	-0.080953	-0.085917	-0.046790	0.017081	0.072975	0.093108	0.070701	0.025939
8.99	0.041433	0.017446	-0.036293	-0.080183	-0.085534	-0.046880	0.016625	0.072372	0.092577	0.070378	0.025837
9.00	0.041675	0.018097	-0.035421	-0.079426	-0.085160	-0.046972	0.016176	0.071781	0.092059	0.070065	0.025738
9.01	0.041914	0.018742	-0.034561	-0.078680	-0.084795	-0.047066	0.015733	0.071203	0.091556	0.069762	0.025643
9.02	0.042152	0.019380	-0.033710	-0.077946	-0.084439	-0.047162	0.015295	0.070636	0.091065	0.069468	0.025551
9.03	0.042387	0.020013	-0.032870	-0.077224	-0.084091	-0.047260	0.014864	0.070082	0.090588	0.069183	0.025462
9.04	0.042621	0.020639	-0.032040	-0.076513	-0.083752	-0.047360	0.014438	0.069538	0.090123	0.068908	0.025377

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
9.05	0.042853	0.021260	-0.031219	-0.075812	-0.083421	-0.047463	0.014018	0.069006	0.089670	0.068641	0.025295
9.06	0.043083	0.021875	-0.030407	-0.075122	-0.083097	-0.047567	0.013603	0.068485	0.089230	0.068383	0.025216
9.07	0.043312	0.022484	-0.029604	-0.074442	-0.082782	-0.047673	0.013193	0.067974	0.088802	0.068134	0.025140
9.08	0.043538	0.023089	-0.028810	-0.073772	-0.082474	-0.047782	0.012788	0.067474	0.088385	0.067893	0.025067
9.09	0.043764	0.023689	-0.028024	-0.073112	-0.082174	-0.047892	0.012388	0.066984	0.087980	0.067661	0.024997
9.10	0.043988	0.024284	-0.027246	-0.072461	-0.081882	-0.048005	0.011992	0.066504	0.087587	0.067437	0.024930
9.11	0.044210	0.024874	-0.026476	-0.071820	-0.081597	-0.048120	0.011601	0.066034	0.087204	0.067221	0.024866
9.12	0.044432	0.025460	-0.025713	-0.071187	-0.081319	-0.048237	0.011214	0.065573	0.086832	0.067013	0.024805
9.13	0.044652	0.026042	-0.024958	-0.070563	-0.081048	-0.048356	0.010832	0.065121	0.086471	0.066812	0.024746
9.14	0.044871	0.026620	-0.024210	-0.069948	-0.080784	-0.048478	0.010453	0.064679	0.086121	0.066620	0.024691
9.15	0.045088	0.027194	-0.023468	-0.069341	-0.080527	-0.048602	0.010078	0.064245	0.085781	0.066435	0.024638
9.16	0.045305	0.027764	-0.022733	-0.068742	-0.080277	-0.048728	0.009707	0.063820	0.085451	0.066258	0.024588
9.17	0.045521	0.028331	-0.022005	-0.068151	-0.080033	-0.048857	0.009340	0.063404	0.085131	0.066088	0.024540
9.18	0.045736	0.028895	-0.021283	-0.067568	-0.079796	-0.048988	0.008976	0.062996	0.084821	0.065925	0.024496
9.19	0.045949	0.029455	-0.020566	-0.066992	-0.079565	-0.049122	0.008615	0.062596	0.084521	0.065770	0.024454
9.20	0.046162	0.030012	-0.019855	-0.066423	-0.079341	-0.049258	0.008257	0.062205	0.084230	0.065622	0.024414
9.21	0.046375	0.030567	-0.019150	-0.065862	-0.079124	-0.049396	0.007903	0.061821	0.083949	0.065481	0.024378
9.22	0.046586	0.031118	-0.018450	-0.065308	-0.078912	-0.049537	0.007551	0.061445	0.083677	0.065347	0.024343
9.23	0.046797	0.031668	-0.017755	-0.064760	-0.078707	-0.049681	0.007203	0.061076	0.083414	0.065220	0.024312
9.24	0.047007	0.032214	-0.017065	-0.064219	-0.078507	-0.049827	0.006857	0.060715	0.083161	0.065100	0.024282
9.25	0.047217	0.032759	-0.016380	-0.063685	-0.078314	-0.049976	0.006513	0.060361	0.082916	0.064986	0.024256
9.26	0.047426	0.033301	-0.015699	-0.063156	-0.078127	-0.050128	0.006173	0.060015	0.082681	0.064880	0.024232
9.27	0.047635	0.033841	-0.015023	-0.062634	-0.077945	-0.050282	0.005834	0.059675	0.082454	0.064780	0.024210
9.28	0.047843	0.034380	-0.014350	-0.062118	-0.077769	-0.050439	0.005498	0.059343	0.082236	0.064686	0.024191
9.29	0.048051	0.034916	-0.013682	-0.061608	-0.077600	-0.050599	0.005164	0.059017	0.082026	0.064600	0.024174
9.30	0.048259	0.035451	-0.013018	-0.061104	-0.077435	-0.050762	0.004832	0.058698	0.081825	0.064520	0.024160
9.31	0.048466	0.035985	-0.012357	-0.060605	-0.077277	-0.050928	0.004502	0.058385	0.081632	0.064446	0.024148
9.32	0.048673	0.036517	-0.011699	-0.060112	-0.077124	-0.051096	0.004174	0.058079	0.081448	0.064379	0.024138
9.33	0.048880	0.037048	-0.011045	-0.059623	-0.076977	-0.051268	0.003847	0.057779	0.081272	0.064318	0.024131
9.34	0.049087	0.037578	-0.010394	-0.059141	-0.076835	-0.051442	0.003522	0.057486	0.081104	0.064264	0.024126
9.35	0.049294	0.038107	-0.009746	-0.058663	-0.076698	-0.051620	0.003199	0.057198	0.080944	0.064216	0.024124
9.36	0.049501	0.038635	-0.009100	-0.058190	-0.076567	-0.051801	0.002877	0.056917	0.080793	0.064175	0.024124
9.37	0.049708	0.039162	-0.008458	-0.057721	-0.076442	-0.051985	0.002557	0.056642	0.080649	0.064139	0.024127
9.38	0.049915	0.039689	-0.007817	-0.057258	-0.076321	-0.052171	0.002238	0.056373	0.080514	0.064111	0.024132
9.39	0.050122	0.040216	-0.007180	-0.056799	-0.076207	-0.052363	0.001920	0.056109	0.080386	0.064088	0.024139

TABLE II - VALUES OF THE COEFFICIENT $\underline{C}$ - CONTINUED

λ	RATIO $\bar{X}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
9.40	0.050329	0.040742	-0.006544	-0.056344	-0.076097	-0.052557	0.001603	0.055851	0.080266	0.064072	0.024149
9.41	0.050536	0.041268	-0.005910	-0.055894	-0.075993	-0.052754	0.001287	0.055599	0.080154	0.064062	0.024161
9.42	0.050744	0.041794	-0.005278	-0.055448	-0.075894	-0.052955	0.000972	0.055352	0.080050	0.064058	0.024175
9.43	0.050952	0.042319	-0.004648	-0.055006	-0.075800	-0.053159	0.000658	0.055111	0.079954	0.064061	0.024192
9.44	0.051160	0.042845	-0.004019	-0.054568	-0.075711	-0.053367	0.000344	0.054875	0.079865	0.064070	0.024211
9.45	0.051369	0.043372	-0.003392	-0.054134	-0.075628	-0.053578	0.000031	0.054645	0.079784	0.064085	0.024233
9.46	0.051578	0.043899	-0.002765	-0.053704	-0.075550	-0.053793	-0.000281	0.054420	0.079711	0.064106	0.024257
9.47	0.051788	0.044426	-0.002140	-0.053277	-0.075477	-0.054012	-0.000593	0.054200	0.079645	0.064134	0.024284
9.48	0.051999	0.044954	-0.001516	-0.052854	-0.075409	-0.054235	-0.000905	0.053985	0.079587	0.064168	0.024313
9.49	0.052209	0.045483	-0.000893	-0.052435	-0.075346	-0.054462	-0.001216	0.053776	0.079537	0.064208	0.024344
9.50	0.052421	0.046013	-0.000270	-0.052018	-0.075288	-0.054692	-0.001527	0.053571	0.079495	0.064255	0.024378
9.51	0.052633	0.046543	0.000352	-0.051606	-0.075236	-0.054927	-0.001838	0.053372	0.079460	0.064308	0.024414
9.52	0.052846	0.047075	0.000974	-0.051196	-0.075189	-0.055166	-0.002149	0.053177	0.079433	0.064368	0.024453
9.53	0.053060	0.047609	0.001596	-0.050789	-0.075147	-0.055409	-0.002461	0.052988	0.079413	0.064434	0.024494
9.54	0.053275	0.048143	0.002218	-0.050386	-0.075110	-0.055656	-0.002772	0.052803	0.079401	0.064506	0.024538
9.55	0.053491	0.048680	0.002840	-0.049985	-0.075078	-0.055908	-0.003084	0.052623	0.079397	0.064585	0.024584
9.56	0.053707	0.049218	0.003462	-0.049587	-0.075051	-0.056164	-0.003396	0.052448	0.079400	0.064671	0.024633
9.57	0.053925	0.049758	0.004085	-0.049192	-0.075030	-0.056425	-0.003708	0.052277	0.079412	0.064763	0.024685
9.58	0.054143	0.050299	0.004708	-0.048799	-0.075013	-0.056690	-0.004022	0.052112	0.079431	0.064861	0.024739
9.59	0.054363	0.050843	0.005332	-0.048409	-0.075002	-0.056960	-0.004335	0.051951	0.079457	0.064967	0.024795
9.60	0.054584	0.051390	0.005957	-0.048021	-0.074997	-0.057235	-0.004650	0.051794	0.079492	0.065079	0.024855
9.61	0.054806	0.051938	0.006583	-0.047636	-0.074996	-0.057515	-0.004965	0.051642	0.079535	0.065198	0.024917
9.62	0.055030	0.052489	0.007210	-0.047253	-0.075001	-0.057800	-0.005282	0.051495	0.079585	0.065323	0.024982
9.63	0.055255	0.053043	0.007839	-0.046872	-0.075011	-0.058090	-0.005599	0.051352	0.079643	0.065456	0.025049
9.64	0.055481	0.053600	0.008469	-0.046494	-0.075026	-0.058385	-0.005918	0.051214	0.079709	0.065596	0.025119
9.65	0.055708	0.054159	0.009101	-0.046117	-0.075047	-0.058686	-0.006237	0.051081	0.079784	0.065742	0.025193
9.66	0.055938	0.054722	0.009734	-0.045742	-0.075073	-0.058992	-0.006558	0.050951	0.079866	0.065896	0.025268
9.67	0.056168	0.055288	0.010370	-0.045369	-0.075105	-0.059303	-0.006881	0.050827	0.079957	0.066057	0.025347
9.68	0.056401	0.055857	0.011008	-0.044998	-0.075142	-0.059620	-0.007205	0.050706	0.080056	0.066226	0.025429
9.69	0.056635	0.056430	0.011649	-0.044629	-0.075185	-0.059943	-0.007531	0.050591	0.080163	0.066401	0.025514
9.70	0.056871	0.057007	0.012292	-0.044261	-0.075233	-0.060272	-0.007858	0.050479	0.080278	0.066585	0.025601
9.71	0.057109	0.057588	0.012937	-0.043894	-0.075287	-0.060608	-0.008188	0.050372	0.080402	0.066775	0.025692
9.72	0.057349	0.058173	0.013586	-0.043529	-0.075346	-0.060949	-0.008519	0.050270	0.080535	0.066974	0.025786
9.73	0.057590	0.058762	0.014238	-0.043166	-0.075412	-0.061297	-0.008852	0.050172	0.080676	0.067180	0.025883
9.74	0.057834	0.059355	0.014893	-0.042803	-0.075483	-0.061651	-0.009188	0.050078	0.080826	0.067394	0.025983

TABLE II - VALUES OF THE COEFFICIENT C - CONTINUED

λ	RATIO $\bar{x}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
9.75	0.058080	0.059953	0.015552	-0.042442	-0.075560	-0.062012	-0.009526	0.049989	0.080985	0.067617	0.026086
9.76	0.058328	0.060556	0.016214	-0.042082	-0.075643	-0.062379	-0.009866	0.049904	0.081153	0.067847	0.026193
9.77	0.058579	0.061164	0.016880	-0.041723	-0.075732	-0.062754	-0.010209	0.049823	0.081330	0.068086	0.026303
9.78	0.058831	0.061777	0.017551	-0.041364	-0.075827	-0.063136	-0.010555	0.049747	0.081516	0.068333	0.026416
9.79	0.059087	0.062395	0.018225	-0.041007	-0.075928	-0.063525	-0.010903	0.049676	0.081711	0.068589	0.026533
9.80	0.059344	0.063020	0.018905	-0.040650	-0.076035	-0.063921	-0.011254	0.049608	0.081916	0.068853	0.026654
9.81	0.059605	0.063649	0.019589	-0.040294	-0.076149	-0.064325	-0.011609	0.049546	0.082131	0.069126	0.026778
9.82	0.059868	0.064285	0.020278	-0.039938	-0.076269	-0.064737	-0.011967	0.049487	0.082355	0.069409	0.026905
9.83	0.060134	0.064927	0.020972	-0.039583	-0.076396	-0.065157	-0.012327	0.049434	0.082589	0.069700	0.027037
9.84	0.060403	0.065576	0.021672	-0.039228	-0.076530	-0.065586	-0.012692	0.049385	0.082834	0.070001	0.027172
9.85	0.060674	0.066231	0.022377	-0.038873	-0.076670	-0.066023	-0.013060	0.049340	0.083088	0.070312	0.027311
9.86	0.060949	0.066893	0.023088	-0.038519	-0.076817	-0.066468	-0.013432	0.049300	0.083353	0.070632	0.027455
9.87	0.061227	0.067563	0.023806	-0.038165	-0.076970	-0.066923	-0.013808	0.049264	0.083629	0.070962	0.027602
9.88	0.061509	0.068239	0.024530	-0.037810	-0.077131	-0.067386	-0.014188	0.049233	0.083915	0.071302	0.027753
9.89	0.061794	0.068924	0.025260	-0.037456	-0.077300	-0.067859	-0.014572	0.049207	0.084213	0.071653	0.027909
9.90	0.062082	0.069616	0.025998	-0.037101	-0.077475	-0.068342	-0.014961	0.049186	0.084521	0.072014	0.028069
9.91	0.062374	0.070317	0.026743	-0.036746	-0.077658	-0.068834	-0.015354	0.049169	0.084841	0.072387	0.028234
9.92	0.062670	0.071026	0.027495	-0.036390	-0.077849	-0.069337	-0.015752	0.049157	0.085173	0.072770	0.028403
9.93	0.062969	0.071745	0.028256	-0.036034	-0.078047	-0.069850	-0.016155	0.049150	0.085517	0.073165	0.028576
9.94	0.063273	0.072472	0.029025	-0.035677	-0.078253	-0.070373	-0.016563	0.049148	0.085873	0.073571	0.028755
9.95	0.063581	0.073209	0.029802	-0.035319	-0.078468	-0.070908	-0.016977	0.049151	0.086241	0.073989	0.028938
9.96	0.063893	0.073955	0.030588	-0.034961	-0.078690	-0.071455	-0.017396	0.049159	0.086622	0.074419	0.029127
9.97	0.064209	0.074712	0.031383	-0.034601	-0.078921	-0.072012	-0.017821	0.049172	0.087016	0.074862	0.029320
9.98	0.064531	0.075479	0.032188	-0.034240	-0.079161	-0.072582	-0.018252	0.049190	0.087424	0.075317	0.029519
9.99	0.064856	0.076257	0.033003	-0.033878	-0.079409	-0.073165	-0.018689	0.049213	0.087845	0.075786	0.029724
10.00	0.065187	0.077047	0.033828	-0.033515	-0.079667	-0.073759	-0.019133	0.049242	0.088280	0.076268	0.029934
10.01	0.065523	0.077848	0.034664	-0.033150	-0.079934	-0.074368	-0.019584	0.049276	0.088730	0.076764	0.030149
10.02	0.065864	0.078661	0.035511	-0.032783	-0.080210	-0.074989	-0.020041	0.049316	0.089194	0.077274	0.030371
10.03	0.066210	0.079486	0.036370	-0.032415	-0.080496	-0.075625	-0.020506	0.049361	0.089673	0.077799	0.030598
10.04	0.066563	0.080325	0.037241	-0.032044	-0.080792	-0.076275	-0.020978	0.049411	0.090168	0.078339	0.030832
10.05	0.066920	0.081177	0.038124	-0.031672	-0.081098	-0.076939	-0.021458	0.049468	0.090678	0.078894	0.031072
10.06	0.067284	0.082042	0.039021	-0.031297	-0.081414	-0.077619	-0.021946	0.049530	0.091205	0.079465	0.031319
10.07	0.067654	0.082922	0.039930	-0.030920	-0.081742	-0.078315	-0.022443	0.049599	0.091748	0.080053	0.031573
10.08	0.068031	0.083817	0.040854	-0.030540	-0.082080	-0.079027	-0.022948	0.049673	0.092309	0.080657	0.031833
10.09	0.068414	0.084728	0.041793	-0.030157	-0.082430	-0.079756	-0.023462	0.049754	0.092888	0.081278	0.032101

TABLE II - VALUES OF THE COEFFICIENT $\underline{C}$ - CONTINUED

λ	RATIO $\bar{X}/L$										
	1/12	2/12	3/12	4/12	5/12	1/2	7/12	8/12	9/12	10/12	11/12
10.10	0.068804	0.085654	0.042746	-0.029772	-0.082791	-0.080502	-0.023986	0.049841	0.093485	0.081918	0.032376
10.11	0.069201	0.086597	0.043715	-0.029383	-0.083165	-0.081266	-0.024520	0.049935	0.094100	0.082576	0.032659
10.12	0.069606	0.087558	0.044701	-0.028991	-0.083550	-0.082049	-0.025063	0.050035	0.094735	0.083253	0.032950
10.13	0.070018	0.088536	0.045703	-0.028596	-0.083949	-0.082851	-0.025618	0.050143	0.095391	0.083949	0.033249
10.14	0.070438	0.089533	0.046723	-0.028197	-0.084361	-0.083672	-0.026183	0.050257	0.096066	0.084666	0.033557
10.15	0.070866	0.090548	0.047762	-0.027794	-0.084786	-0.084515	-0.026759	0.050378	0.096763	0.085404	0.033873
10.16	0.071303	0.091584	0.048820	-0.027386	-0.085225	-0.085378	-0.027348	0.050507	0.097482	0.086163	0.034199
10.17	0.071749	0.092641	0.049897	-0.026975	-0.085679	-0.086264	-0.027949	0.050643	0.098224	0.086945	0.034533
10.18	0.072204	0.093719	0.050995	-0.026558	-0.086148	-0.087172	-0.028562	0.050787	0.098989	0.087750	0.034878
10.19	0.072669	0.094820	0.052115	-0.026137	-0.086632	-0.088105	-0.029189	0.050939	0.099778	0.088579	0.035232
10.20	0.073144	0.095944	0.053257	-0.025711	-0.087131	-0.089062	-0.029830	0.051100	0.10059	0.089433	0.035597

APPENDIX B. DERIVATION OF FORMULAS

1. Formulas for Bars Without Axial Forces

a. General Solution of Fundamental Equation for Vibration of Bars

Using Bernoulli-Euler's beam theory, the differential equation for transverse vibration of a uniform bar which is free from external loads is

$$EI\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} = 0 , \tag{B1}$$

where w(x,t) is the deflection of any point of the bar at a time t, m is the mass per unit of length of the bar, and EI is the flexural rigidity of its cross section. The effects of damping, shearing deformation, and rotatory inertia are disregarded.

We can let

$$w(x,t) = Y(x)\cos\omega t , \tag{B2}$$

where the amplitude of the motion Y(x) is a function of x only, and ω is the circular frequency of the motion.

Substituting (B2) in (B1), one obtains the following expression for determining the function Y(x):

$$Y'''' - \left(\frac{\lambda}{L}\right)^4 Y = 0 , \tag{B3}$$

where L is the span length of the bar and

$$\lambda = \sqrt[4]{\frac{m\omega^2}{EI}}\cdot L . \tag{B4}$$

Primes in Eq. (B3) indicate differentiations with respect to x.

The solution of Eq. (B3) is

$$Y(x) = c_1\cosh\lambda\frac{x}{L} + c_2\sinh\lambda\frac{x}{L} + c_3\cos\lambda\frac{x}{L} + c_4\sin\lambda\frac{x}{L} . \tag{B5}$$

The integration constants c_1, c_2, c_3, and c_4 must be determined from the boundary conditions of the particular problem considered.

b. Formulas for Elastic Constants

Consider the bar shown in Fig. B-1. At $x = 0$ the bar is fixed,

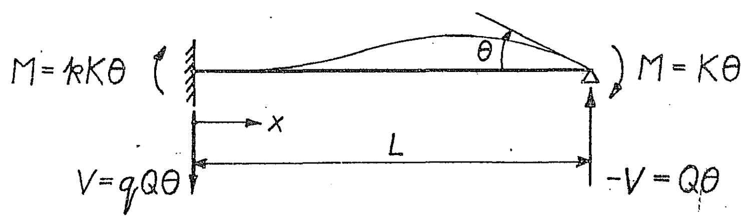

Fig. B-1

while at $x = L$ it is subjected to a moment

$$M(x,t) = M\cos\omega t \tag{B6}$$

producing at that end a steady-state forced rotation

$$\theta(x,t) = \theta\cos\omega t\ . \tag{B7}$$

With the notation and sign convention given in Section 2, the boundary conditions may be stated as follows:

At $x = 0$

$$\begin{aligned} \delta &= Y = 0\ , \\ \theta &= Y' = 0\ , \end{aligned} \tag{B8}$$

and at $x = L$

$$\begin{aligned} \delta &= Y = 0\ , \\ M &= EIY''\ . \end{aligned} \tag{B9}$$

From these boundary conditions, one finds the following values for the integration constants in Eq. (B5):

$$\begin{aligned} c_1 = -c_3 &= -\frac{ML^2}{2\lambda^2 EI}\cdot\frac{\sinh\lambda - \sin\lambda}{\cosh\lambda\sin\lambda - \sinh\lambda\cos\lambda}\ , \\ c_2 = -c_4 &= \frac{ML^2}{2\lambda^2 EI}\cdot\frac{\cosh\lambda - \cos\lambda}{\cosh\lambda\sin\lambda - \sinh\lambda\cos\lambda}\ . \end{aligned} \tag{B10}$$

The moments $\underline{M}$ and the reactions $\underline{V}$ at the ends of the bar are found to be as follows:

At $x = L$

$$M = K\theta , \tag{B11}$$

where

$$K = \frac{EIY''}{Y'} = \lambda \frac{\cosh\lambda \sin\lambda - \sinh\lambda \cos\lambda}{1 - \cosh\lambda \cos\lambda} \cdot \frac{EI}{L} , \tag{B12}$$

and

$$-V = Q\theta , \tag{B13}$$

where

$$Q = -\frac{EIY'''}{Y'} = \lambda^2 \frac{\sinh\lambda \sin\lambda}{1 - \cosh\lambda \cos\lambda} \cdot \frac{EI}{L^2} . \tag{B14}$$

At $x = 0$

$$M = kK\theta , \tag{B15}$$

where

$$k = -\frac{Y'']_{x=0}}{Y'']_{x=L}} = \frac{\sinh\lambda - \sin\lambda}{\cosh\lambda \sin\lambda - \sinh\lambda \cos\lambda} , \tag{B16}$$

and

$$V = qQ\theta , \tag{B17}$$

where

$$q = \frac{Y''']_{x=0}}{Y''']_{x=L}} = \frac{\cosh\lambda - \cos\lambda}{\sinh\lambda \sin\lambda} . \tag{B18}$$

For a simply supported beam, the moment at $x = L$ may be expressed as

$$M = K''\theta . \tag{B19}$$

It can be shown that

$$K'' = \lambda \frac{2 \sinh\lambda \sin\lambda}{\cosh\lambda \sin\lambda - \sinh\lambda \cos\lambda} . \tag{B20}$$

Equations (B12), (B16), and (B20) have been presented previously by Gaskell (13).

Now consider the case in which the right end of the beam is displaced without rotation by

$$\delta(x,t) = \delta \cos\omega t , \tag{B21}$$

as shown in Fig. B2.

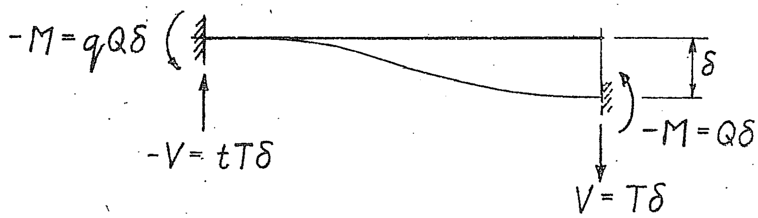

Fig. B2

In this case, the boundary conditions may be stated as follows:

At $x = 0$ $\qquad \delta = Y = 0,$ $\qquad$ (B22)

$\qquad \theta = Y' = 0.$

At $x = L$ $\qquad \theta = Y' = 0,$ $\qquad$ (B23)

$\qquad \delta = Y.$

For these conditions, one finds the following values for the integration constants in Eq. (B5).

$$c_1 = -c_3 = \frac{\delta}{2} \cdot \frac{\cosh\lambda - \cos\lambda}{1 - \cosh\lambda\cos\lambda},$$

$$c_2 = -c_4 = -\frac{\delta}{2} \cdot \frac{\sinh\lambda + \sin\lambda}{1 - \cosh\lambda\cos\lambda}. \tag{B24}$$

The moments M and the reactions V at the ends of the bar are found to be as follows:

At $x = L$ $\qquad M = -Q\delta,$ $\qquad$ (B25)

where Q is given by Eq. (B14), and

$$V = T\delta, \tag{B26}$$

where $$T = -\frac{EIY'''}{\delta} = \frac{\cosh\lambda\sin\lambda + \sinh\lambda\cos\lambda}{1 - \cosh\lambda\cos\lambda}. \tag{B27}$$

At $x = 0$ $\qquad M = -qQ\delta,$ $\qquad$ (B28)

where q is given by Eq. (B18), and

$$V = -tT\delta, \qquad \text{(B29)}$$

where
$$t = \frac{Y''']_{x=0}}{Y''']_{x=L}} = \frac{\sinh\lambda + \sin\lambda}{\cosh\lambda\,\sin\lambda + \sinh\lambda\,\cos\lambda}\,. \qquad \text{(B30)}$$

As already remarked, the quantities $\underline{Q}$ in Eqs. (B13) and (B25) are equal; likewise the quantities $\underline{q}$ in Eqs. (B17) and (B28) are equal. These equalities follow from Lord Rayleigh's reciprocal relations which, for convenience, are reviewed in Section 1d of this Appendix.

It should be stated that numerical values of the trigonometric and hyperbolic functions appearing in the numerator and denominator of most of the formulas given in this section have been tabulated previously in Reference (24). These functions were tabulated for values of λ between zero and 10.00 at increments of 0.02. The numerical values presented in this report have been tabulated at increments of λ of 0.01. Since all quantities were computed on the Electronic Digital Computer, there was no need to use the previously tabulated functions.

c. Formulas for Deflections of a Bar Due to Distortions at the Ends

It is desired to find the deflections of the bars shown in Figs. B1 and B2. The distortions θ and δ may be introduced at either end of the bar.

First consider the bar shown in Fig. B1. Let $\bar{x}$ denote the distance of any point of the bar from the end being rotated. Then, if θ represents the rotation at the left end, the deflection amplitude

$$Y_{\bar{x}} = C\theta L\,, \qquad \text{(B31a)}$$

and if θ represents the rotation at the right end

$$Y_{\bar{x}} = -C\theta L\,, \qquad \text{(B31b)}$$

where $\underline{C}$ is a dimensionless coefficient dependent on the coordinate $\bar{x}$ and the parameter λ.

Substituting the integration constants given in Eq. (B10) into Eq. (B5), the following expression is found for $\underline{C}$:

$$C = \frac{1}{2\lambda(1-\cosh\lambda\cos\lambda)}\Big\{[\sinh\lambda-\sin\lambda][\cosh(1-\xi)\lambda - \cos(1-\xi)\lambda] - [\cosh\lambda-\cos\lambda][\sinh(1-\xi)\lambda - \sin(1-\xi)\lambda]\Big\}, \qquad \text{(B32)}$$

where

$$\xi = \frac{\bar{x}}{L}. \qquad \text{(B33)}$$

Consider now the bar shown in Fig. B2. Let $\bar{x}$ denote the distance of a point of the bar from the deflected end. Then, for a deflection amplitude δ at either end

$$Y_{\bar{x}} = C'\delta, \qquad \text{(B34)}$$

where $\underline{C'}$ is a dimensionless coefficient.

Substituting the integration constants given in Eq. (B24) into Eq. (B5), the following expression is found for $\underline{C'}$:

$$C' = \frac{1}{2\lambda(1-\cosh\lambda\cos\lambda)}\Big\{[\cosh\lambda-\cos\lambda][\cosh(1-\xi)\lambda - \cos(1-\xi)\lambda] - [\sinh\lambda+\sin\lambda][\sinh(1-\xi)\lambda - \sin(1-\xi)\lambda]\Big\}. \qquad \text{(B35)}$$

d. Rayleigh's Reciprocal Relations and Principle of Influence Lines

Rayleigh's reciprocal relations for dynamics (14) are strictly analogous to Maxwell's law of reciprocal relations for statics. Rayleigh's relations may be stated as follows: If $\underline{A}$ and $\underline{B}$ denote two points in a given structure, the steady-state forced displacement at $\underline{A}$ produced by a harmonically varying load applied at $\underline{B}$ is equal to the steady-state displacement at $\underline{B}$ due to the same load applied at $\underline{A}$. The term displacement may be interpreted in a general sense as either linear or angular displacement; similarly, the term load may be interpreted in the same general sense as either force or couple. Couples correspond to rotations while forces correspond to linear displacements.

Using Rayleigh's reciprocal relations, one can prove readily that Müller-Breslau's principle is also applicable in the case of steady-state forced vibrations. The proof is identical to that for the static case and is, therefore, omitted. According to this principle: The influence line for any dynamic function, such as reaction, shear, bending moment, torque, at some point A of a structure due to harmonically varying load can be obtained as the deflected shape of the structure due to a very small unit displacement, linear or angular, introduced at point A.

It follows from this principle that the deflection of a fixed ended beam subjected to a unit rotation at one end represents an influence line for fixed end moment due to a unit concentrated force on the beam. Similarly, the deflection of a beam resulting from a unit deflection without rotation of one end represents an influence line for fixed end shear due to a concentrated unit force.

2. Formulas for Bars with Axial Forces

a. General Solution of Governing Differential Equation

The differential equation for transverse deflection $w(x,t)$ of a uniform bar which is free from lateral loads but is acted upon by constant axial forces P is

$$EI\frac{\partial^4 w}{\partial x^4} - P\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = 0. \tag{B36}$$

A positive P indicates a tensile force.

The deflection $w(x,t)$ may be expressed as

$$w(x,t) = Y(x) \cos \omega t \ , \tag{B37}$$

where $Y(x)$ is the amplitude of the harmonic motion and ω is its circular frequency.

Substituting Eq. (B37) into Eq. (B36) and using the symbols

$$P_o = \frac{\pi^2 EI}{L^2} \ , \tag{B38}$$

$$\lambda = \sqrt[4]{\frac{m\omega^2}{EI}} \ L \ . \tag{B4}$$

one obtains the following expression for determining the function $Y(x)$:

$$Y'''' - \frac{\pi^2}{L^2} \frac{P}{P_o} Y'' - \left(\frac{\lambda}{L}\right)^4 Y = 0 \ . \tag{B39}$$

The roots of the corresponding characteristic equation are

$$\begin{aligned} r_{1,2} &= \pm \frac{\phi}{L} \ , \\ r_{3,4} &= \pm \frac{\chi}{L} \ . \end{aligned} \tag{B40}$$

where

$$\phi = \frac{\pi}{\sqrt{2}} \sqrt{\sqrt{\left(\frac{P}{P_o}\right)^2 + \frac{4}{\pi^4}\lambda^4} + \frac{P}{P_o}} \ , \tag{B41}$$

$$\chi = \frac{\pi}{\sqrt{2}} \sqrt{\sqrt{\left(\frac{P}{P_o}\right)^2 + \frac{4}{\pi^4}\lambda^4} - \frac{P}{P_o}} = \frac{\lambda^2}{\phi} \ . \tag{B42}$$

Then, the solution of Eq. (B39) becomes

$$Y(x) = c_1 \cosh \phi \frac{x}{L} + c_2 \sinh \phi \frac{x}{L} + c_3 \cos \chi \frac{x}{L} + c_4 \sin \chi \frac{x}{L} \ . \tag{B43}$$

The integration constants c_1, c_2, c_3, and c_4 must be determined from the boundary conditions of the particular problem considered.

b. Formulas for Flexural Stiffness and Flexural Carry-Over Factor

Consider a bar, such as that shown in Fig. B1, fixed at $x = 0$ and subjected to a periodic moment

$$M(x,t) = M\cos\omega t$$

at x = L. Assume that the bar is acted upon by tensile end forces.

The boundary conditions in this case are the same as those given in Eqs. (B8) and (B9). For these boundary conditions, one finds the following values for the integration constants in Eq. (B43).

$$\begin{aligned} C_1 = -C_3 &= -\frac{ML^2}{EI}\cdot\frac{1}{\phi^2+\chi^2}\cdot\frac{\chi\sinh\phi-\phi\sin\chi}{\phi\cosh\phi\sin\chi-\chi\sinh\phi\cos\chi}\,, \\ C_2 = -\frac{\chi}{\phi}C_4 &= \frac{ML^2}{EI}\cdot\frac{\chi}{\phi^2+\chi^2}\cdot\frac{\cosh\phi-\cos\chi}{\phi\cosh\phi\sin\chi-\chi\sinh\phi\cos\chi}\,. \end{aligned} \tag{B44}$$

The moments at the ends of the bar are found to be as follows:

At x = L

$$M = K\Theta = KY', \tag{B45}$$

where

$$K = \frac{\phi\cosh\phi\sin\chi-\chi\sinh\phi\cos\chi}{\frac{2\phi\chi}{\phi^2+\chi^2}\left[1-\cosh\phi\cos\chi\right]+\frac{\phi^2-\chi^2}{\phi^2+\chi^2}\sinh\phi\sin\chi}\cdot\frac{EI}{L}\,. \tag{B46}$$

At x = 0

$$M = kK\Theta\,, \tag{B47}$$

where

$$k = -\frac{Y'']_{x=0}}{Y'']_{x=L}} = \frac{\sinh\phi-\frac{\phi}{\chi}\sin\chi}{\frac{\phi}{\chi}\cosh\phi\sin\chi-\sinh\phi\cos\chi}\,. \tag{B48}$$

Now, assume that the bar is subjected to compressive forces. In this case, it is necessary to replace +P by −P. Then, Eq. (B41) is changed to Eq. (B42) and vice versa. Therefore, the expressions of K and k for a compressive force can be obtained from the corresponding expressions for a tensile force simply by interchanging in the latter expressions the quantities ϕ and χ.

For the special case in which no axial force is present, $\phi = \chi = \lambda$, and Eqs. (B46) and (B48) reduce respectively to Eqs. (B12) and (B16).

3. Formulas for Plates Simply Supported Along Two Opposite Edges

a. General Solution of Fundamental Equation for Vibration of Plates

Using the ordinary flexure theory of medium thick plates, the differential equation for lateral deflection w(x,y,t) of a uniform plate which is free from lateral loads is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{\rho h}{N}\frac{\partial^2 w}{\partial t^2} = 0 , \tag{B49}$$

where ρ = the density of the plate material, h = the thickness of the plate, assumed to be constant, and N = the flexural rigidity of the plate.

The coordinate axes are taken in the middle plane of the plate parallel to the sides of the plate. The origin of the axes is taken at the upper left hand corner of the plate.

If the plate is simply supported along two opposite edges (say, along x = 0 and x = a) the solution of Eq. (B49) may be taken in the form

$$w(x,y,t) = Y_n \sin\frac{n\pi x}{a} \cos\omega t , \tag{B50}$$

where Y_n is a function of the y variable only. Substituting this equation into Eq. (B49), one obtains the following equation for determining the function Y_n:

$$Y_n'''' - \frac{2n^2\pi^2}{a^2} Y_n'' + \left(\frac{n^4\pi^4}{a^4} - \frac{\rho h \omega^2}{N}\right) Y_n = 0 . \tag{B51}$$

The roots of the corresponding characteristic equation are

$$r_{1,2,3,4} = \pm \frac{\pi}{b}\sqrt{\left(\frac{nb}{a}\right)^2 \pm \lambda^*} , \tag{B52}$$

where

$$\lambda^* = \frac{b^2}{\pi^2}\sqrt{\frac{\rho h \omega^2}{N}} . \tag{B53}$$

Two different cases must now be considered:

Case (I), when $\lambda^* > \left(\frac{nb}{a}\right)^2$, and

Case (II), when $\lambda^* < \left(\frac{nb}{a}\right)^2$.

The particular case in which $\lambda^* = 0$ is omitted.

In the first case two of the roots are real while the remaining two are imaginary. In the second case all four roots are real.

First consider Case (I). Using the notation

$$\bar{\phi} = \pi\sqrt{\lambda^* + \left(\frac{nb}{a}\right)^2}\,, \tag{B54}$$

and

$$\bar{\chi} = \pi\sqrt{\lambda^* - \left(\frac{nb}{a}\right)^2}\,. \tag{B55}$$

the solution of Eq. (B51) becomes

$$Y_n(y) = c_1 \cosh\bar{\phi}\frac{y}{b} + c_2 \sinh\bar{\phi}\frac{y}{b} + c_3 \cos\bar{\chi}\frac{y}{b} + c_4 \sin\bar{\chi}\frac{y}{b}\,. \tag{B56}$$

In Case (II) the $\bar{\chi}$ values in Eq. (B55) are imaginary and, using the notation

$$\bar{\chi}' = \pi\sqrt{\left(\frac{nb}{a}\right)^2 - \lambda^*}\,, \tag{B57}$$

the solution of Eq. (B51) becomes

$$Y_n(y) = c_1' \cosh\bar{\phi}\frac{y}{b} + c_2' \sinh\bar{\phi}\frac{y}{b} + c_3' \cosh\bar{\chi}'\frac{y}{b} + c_4' \sinh\bar{\chi}'\frac{y}{b}\,. \tag{B58}$$

The integration constants in Eqs. (B56) and (B58) must be determined from the boundary conditions of the particular problem considered.

b. Formulas for Flexural Stiffness and Flexural Carry-Over Factor

Consider a rectangular plate simply supported at x = 0 and x = a and fixed at y = 0. Let an exciting moment

$$M(x,t) = M \sin\frac{n\pi x}{a} \cos\omega t \tag{B59}$$

be applied along the edge y = b

The boundary conditions in this case are:

$$\text{at } y = 0 \qquad \begin{aligned} \delta &= Y_n = 0 \,, \\ \theta &= Y'_n = 0 \,. \end{aligned} \tag{B60}$$

$$\text{at } y = b \qquad \begin{aligned} \delta &= Y_n = 0 \,, \\ M &= NY''_n \,. \end{aligned} \tag{B61}$$

The moment amplitudes at the ends may be expressed in terms of the rotation amplitude θ as follows:

$$\text{At } y = 0 \qquad M = K\theta \,, \tag{B62}$$

$$\text{At } y = b \qquad M = kK\theta \,. \tag{B63}$$

where $\underline{K}$ and $\underline{k}$ are respectively the flexural stiffness and the flexural carry-over factor.

It should be noted that when $\lambda^* > \left(\frac{nb}{a}\right)^2$ Eq. (B51) for plates is similar to Eq. (B39) for bars subjected to fixed axial forces. And since, for the particular case considered, the boundary conditions given in Eqs. (B60) and (B61) are similar to those given in Eqs. (B8) and (B9), the expressions of $\underline{K}$ and $\underline{k}$ for a plate panel can be obtained directly from the corresponding expressions given in Eqs. (B46) and (B48). It is only necessary to replace in Eqs. (B46) and (B48) the quantities ϕ, χ, $\underline{EI}$, and $\underline{L}$ by $\bar{\phi}$, $\bar{\chi}$, $\underline{N}$, and $\underline{b}$, respectively. The results are as follows:

$$K = \frac{\bar{\phi}\cosh\bar{\phi}\sin\bar{\chi} - \bar{\chi}\sinh\bar{\phi}\cos\bar{\chi}}{\frac{2\bar{\phi}\bar{\chi}}{\bar{\phi}^2+\bar{\chi}^2}\left[1-\cosh\bar{\phi}\cos\bar{\chi}\right] + \frac{\bar{\phi}^2-\bar{\chi}^2}{\bar{\phi}^2+\bar{\chi}^2}\sinh\bar{\phi}\sin\bar{\chi}} \cdot \frac{N}{b} \,, \tag{B64}$$

$$k = \frac{\sinh\bar{\phi} - \frac{\bar{\phi}}{\bar{\chi}}\sin\bar{\chi}}{\frac{\bar{\phi}}{\bar{\chi}}\cosh\bar{\phi}\sin\bar{\chi} - \sinh\bar{\phi}\cos\bar{\chi}} \,. \tag{B65}$$

These relations apply to values of $\lambda^* > \left(\frac{nb}{a}\right)^2$.

For $\lambda^* < \left(\frac{nb}{a}\right)^2$ the expressions for K and k are obtained from those given in Eqs. (B64) and (B65) by replacing trigonometric functions by hyperbolic functions and the quantity $(\bar{\chi})^2$ by $-(\bar{\chi}')^2$. The results are:

$$K = \frac{\bar{\Phi}\cosh\bar{\Phi}\sinh\bar{\chi}' - \bar{\chi}'\sinh\bar{\Phi}\cosh\bar{\chi}'}{\frac{2\bar{\Phi}\bar{\chi}'}{\bar{\Phi}^2-(\bar{\chi}')^2}\left[1-\cosh\bar{\Phi}\cosh\bar{\chi}'\right] + \frac{\bar{\Phi}^2+(\bar{\chi}')^2}{\bar{\Phi}^2-(\bar{\chi}')^2}\sinh\bar{\Phi}\sinh\bar{\chi}'} \cdot \frac{N}{b}, \tag{B66}$$

$$k = \frac{\sinh\bar{\Phi} - \frac{\bar{\Phi}}{\bar{\chi}'}\sinh\bar{\chi}'}{\frac{\bar{\Phi}}{\bar{\chi}'}\cosh\bar{\Phi}\sinh\bar{\chi}' - \sinh\bar{\Phi}\cosh\bar{\chi}'}. \tag{B67}$$

c. Correlation Between Elastic Constants for Vibrating Plates and Compressed Plates

Consider a flat plate loaded on two edges parallel to the y-axis by uniformly distributed compressive forces p_x. Assume that no lateral load acts on the plate.

The differential equation for the deflection w(x,y) of the plate is

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{p_x}{N}\frac{\partial^2 w}{\partial x^2} = 0 \tag{B68}$$

For the case in which the loaded edges are simply supported, the solution of Eq. (B68) may be taken in the form

$$w(x,y) = Y_n \sin\frac{n\pi x}{a}, \tag{B69}$$

where Y_n is a function of y only. The unloaded edges may have any condition of restraint.

Substituting Eq. (B69) into Eq. (B68), one obtains the following equation for determining the function Y_n:

$$Y_n'''' - \frac{2n^2\pi^2}{a^2}Y_n'' + \left(\frac{n^4\pi^4}{a^4} - \frac{k' n^2\pi^4}{a^2 b^2}\right)Y = 0, \tag{B70}$$

where
$$k' = \frac{b^2 P_x}{\pi^2 N} \qquad \text{(B71)}$$

Comparing Eq. (B70) with Eq. (B51) it can readily be shown that if

$$k' = (\lambda^*)^2 \left(\frac{a}{nb}\right)^2 , \qquad \text{(B72)}$$

the two equations will be identical. Accordingly, the quantities $\underline{K}$ and $\underline{k}$ of a vibrating plate for a given value of $\frac{a}{nb}$ and λ^* are numerically equal to the corresponding quantities of a compressed plate for the same $\frac{a}{nb}$ value but for a value of $k' = (\lambda^*)^2 \left(\frac{a}{nb}\right)^2$. Since tabulated values of stiffness and carry-over factor for compressed plates do exist (see Reference 32), these values can be used to determine the corresponding dynamic quantities. It should be noted, however, that in Reference 32 stiffness has been defined as the moment necessary to produce a rotation of a maximum amplitude of a quarter radian rather than one radian. Therefore, the stiffness values obtained from Reference 32 must be multiplied by 4 to make them conform to the definition given in this report.

APPENDIX C. ANALYSIS OF STEADY-STATE FORCED VIBRATIONS

1. General Description of Method of Analysis

The information presented in this report may be used also to analyze the steady-state forced vibration of continuous beams and frames subjected to harmonically varying forces. The forces are presumed to have the same frequency and the same phase angle. The effect of damping is neglected.

The determination of the steady-state forced vibration of a system is a much simpler problem than the calculation of its natural frequencies, since in the former case it is only necessary to go through a single cycle of the trial-and-error procedure used in the determination of natural frequencies.

An analysis for steady-state forced vibrations can be carried out in substantially the same way as an analysis for static conditions. In either case, the analysis involves two basic steps:

a. Determination of the redundant quantities at the joints or supports of the continuous system. In general, the redundant quantities may be moments, rotations, deflections or any combination of these.

b. Determination of the moments, shears, deflections at points between joints or supports. The principal difference between a static and dynamic analysis is that while in a static analydis moments and shears within a member may be determined by statics from the moments and shears at the ends of the member, in a dynamic analysis these effects must be computed independently.

We shall now outline one of a number of possible procedures for executing these steps. For illustration, we shall consider the relatively simple frame shown in Fig. B3 and, for simplicity's sake, shall assume that the joints of the frame do not move. Joints 1 and 9 are assumed to be simply supported.

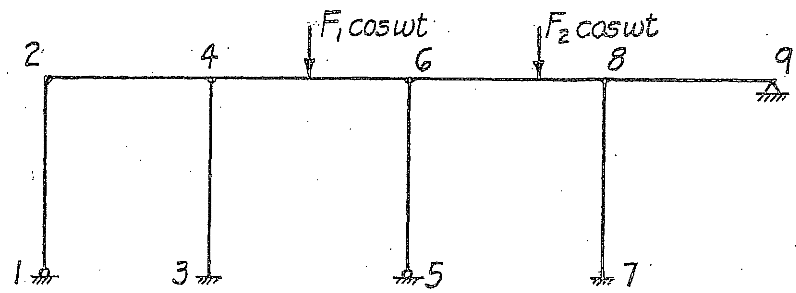

Fig. B3

Although this procedure, in the form presented herein, is applicable to continuous open frames only, it can readily be extended to frames involving closed panels. The details of the procedure are as follows:

Step a. (1) Replace the concentrated forces by equivalent fixed end moments. The magnitude of these moments may be obtained directly from Table II in Appendix A. (If the ends of the loaded members were free to deflect, it would have been necessary to calculate also fixed end shears. These could have been computed either from Eq. (72) or from Eq. (B31).

(2) For each span compute the quantities $\underline{K}$ and $\underline{kK}$. (If the joints of the frame were free to translate, it would have been necessary to compute also the quantities $\underline{Q}$, $\underline{qQ}$, $\underline{T}$, and $\underline{tT}$.) These quantities are obtained directly from Table I in Appendix A.

(3) Assume a slope at support 1. Progressing from joint to joint across the frame in the manner described in the body of this report, evaluate the rotations of the joints. It should be remembered that in this case the external moment $\overline{M}_j$ is, in general, different from zero.

(4) From the rotations determined in the preceding step, calculate the magnitude of the moment at the extreme right end of the frame. This

moment will, in general, be different from the actual moment at that end, a condition indicating that the assumed slope θ_1 was in error. It is therefore necessary to add a correction configuration.

(5) Disregarding the external moments, find the influence of a unit rotation at joint 1 on the distortions of the remaining joints. Calculate also the magnitude of the corresponding moment at the right end.

(6) Combine the configurations determined in (4) and (5) so as to eliminate the unbalanced moment at the right end.

(7) From the rotations determined in (6), compute the magnitude of the moments at the joints or supports of the frame using the relationship

$$M_{oj} = K\theta_o + kK\theta_j$$

where M_{oj} is the moment at end $\underline{o}$ of the member $\underline{oj}$. This concludes Step a.

Step b. Any desired effect (moment, shear, deflection, etc.) within a member $\underline{oj}$ may be determined by superimposing the following effects:

(1) The effect of the load on a simply supported member

(2) The effect of the moment at end $\underline{o}$

(3) The effect of the moment at end $\underline{j}$

If the ends of the member deflect, the following effects must also be added:

(4) The effect of the deflection at end $\underline{o}$

(5) The effect of the deflection at end $\underline{j}$

Each effect is computed on the assumption that the member is simply supported at the ends.

The quantities added may, if desired, be different from those outlined. For example, one may add the effect of the load on a member fixed at both ends, the effects of the end deflections, and the effects of the end rotations. Irrespective of the manner in which the computations are carried out, the facility with which results are computed depends on the information available for the various quantities added.

The appropriate expressions for effects (2) through (5) are given in Reference (24). Included in this reference are also tabulated numerical values of functions in terms of which these effects can easily be computed. The appropriate expression for effect (1) is given in the next section.

2. Formulas for Steady-State Deflection of a Simply Supported Uniform Beam Carrying a Concentrated Force $F\cos\omega t$

The force is assumed to be applied at a distance ψL from the left end of the beam as shown in Fig. B4. The subscripts 1 and 2 will be used to

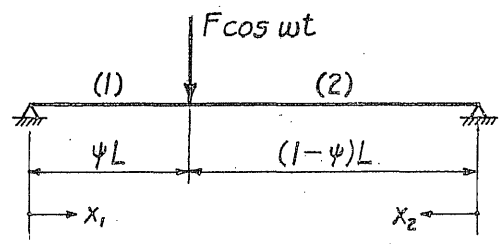

Fig. B4

designate quantities for the left and the right portions of the beam. The distances x_1 and x_2 are measured from the ends of the beam as shown on the figure.

The deflections w_1 and w_2 may be written as

$$w_1 = \left[c_1 \cosh \lambda \frac{x_1}{L} + c_2 \sinh \lambda \frac{x_1}{L} + c_3 \cos \lambda \frac{x_1}{L} + c_4 \sin \lambda \frac{x_1}{L}\right] \cos \omega t \ ,$$

$$w_2 = \left[\bar{c}_1 \cosh \lambda \frac{x_2}{L} + \bar{c}_2 \sinh \lambda \frac{x_2}{L} + \bar{c}_3 \cos \lambda \frac{x_2}{L} + \bar{c}_4 \sin \lambda \frac{x_2}{L}\right] \cos \omega t \ .$$

The eight integration constants have been evaluated from the four boundary conditions and from the four continuity and equilibrium conditions for the point of application of the force. The results are:

$$W_1(x_1) = \frac{FL^3}{EI} \cdot \frac{1}{2\lambda^3} \left[\frac{\sin(1-\psi)\lambda}{\sin\lambda} \sin\lambda \frac{x_1}{L} - \frac{\sinh(1-\psi)\lambda}{\sinh\lambda} \sinh\frac{x_1}{L} \right] \cos\omega t \ ,$$

$$W_2(x_2) = \frac{FL^3}{EI} \cdot \frac{1}{2\lambda^3} \left[\frac{\sin\psi\lambda}{\sin\lambda} \sin\lambda \frac{x_2}{L} - \frac{\sinh\psi\lambda}{\sinh\lambda} \sinh\frac{x_2}{L} \right] \cos\omega t \ .$$

From these equations, expressions for bending moment and shear can be obtained readily by differentiation.

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(32) "Tables of Stiffness and Carry-Over Factor for Flat Rectangular Plates Under Compression", by W. D. Kroll, Wartime Report L-398 (ARR No.3K27), National Advisory Committee for Aeronautics, Washington, D.C., Nov.1943

(33) "The Frequency of Vibration of Rectangular Isotropic Plates", by R. F. S. Hearmon, Journal of Applied Mechanics, vol. 19, 1952, pp. 402-403

(34) "On the Fundamental Frequencies of Vibration of Rigid Frames", by E. F. Masur; paper presented at the First Midwestern Conference on Solid Mechanics, University of Illinois, April 24, 1953

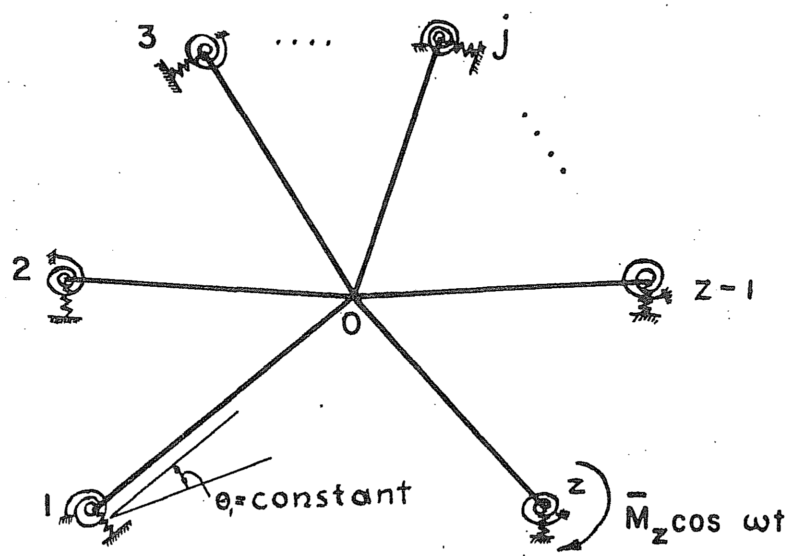
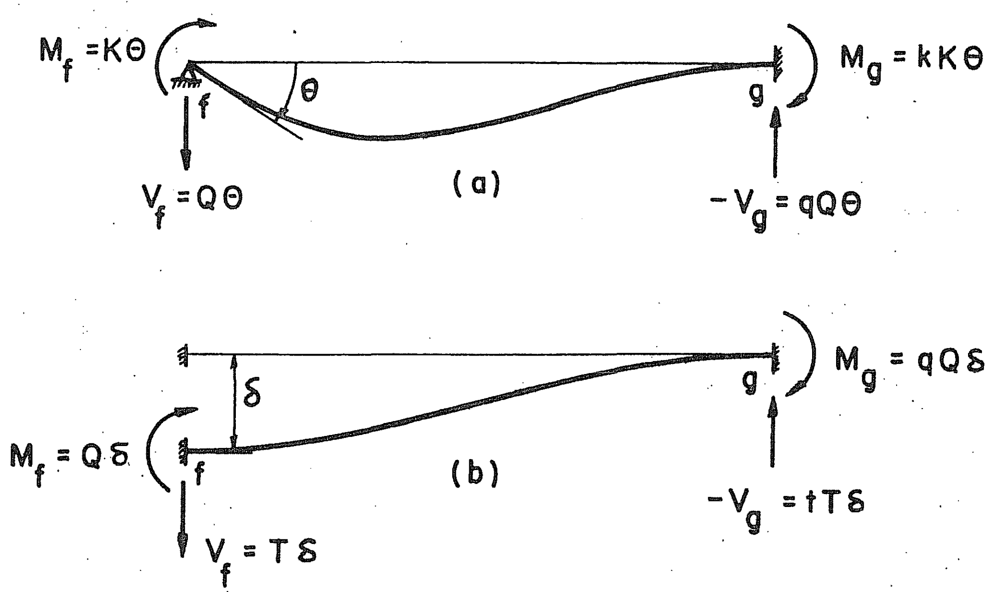


FIG. 1 TYPICAL JOINT OF A PLANE FRAMEWORK



Summary of Elastic Constants

Movement of End "f"	At End "f"		At End "g"	
	Moment M	Shear V	Moment M	Shear V
$\theta = 1$	K	Q	kK	$-qQ$
$\delta = 1$	Q	T	qQ	$-tT$

FIG. 2 DEFINITION OF ELASTIC CONSTANTS

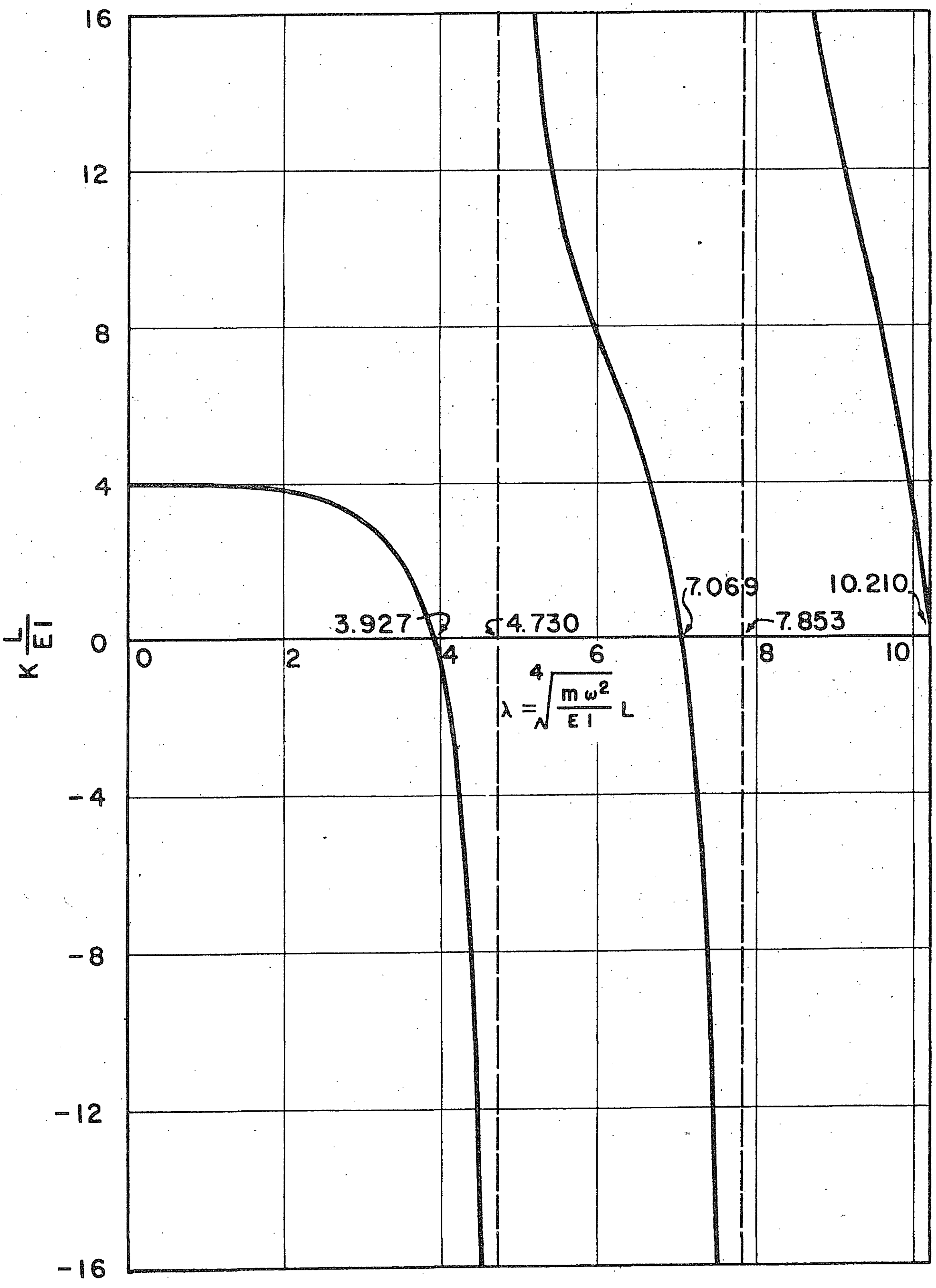


FIG. 3 COEFFICIENTS OF DYNAMIC FLEXURAL STIFFNESS $\underline{K}$ FOR A UNIFORM BAR FIXED AT THE FAR END

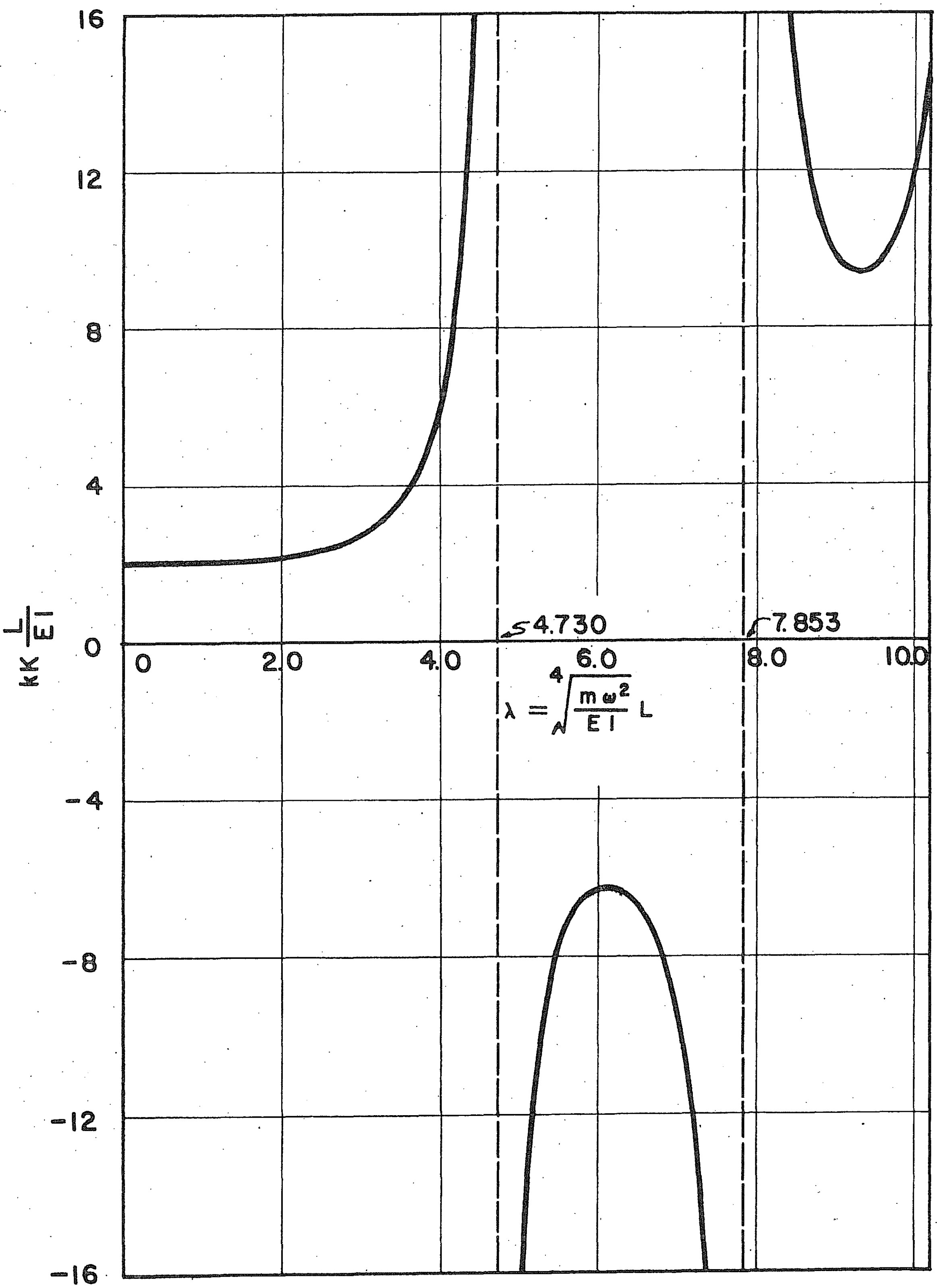


FIG. 4 COEFFICIENTS OF THE PRODUCT OF DYNAMIC FLEXURAL STIFFNESS AND DYNAMIC FLEXURAL CARRY-OVER FACTOR kK FOR A UNIFORM BAR FIXED AT THE FAR END

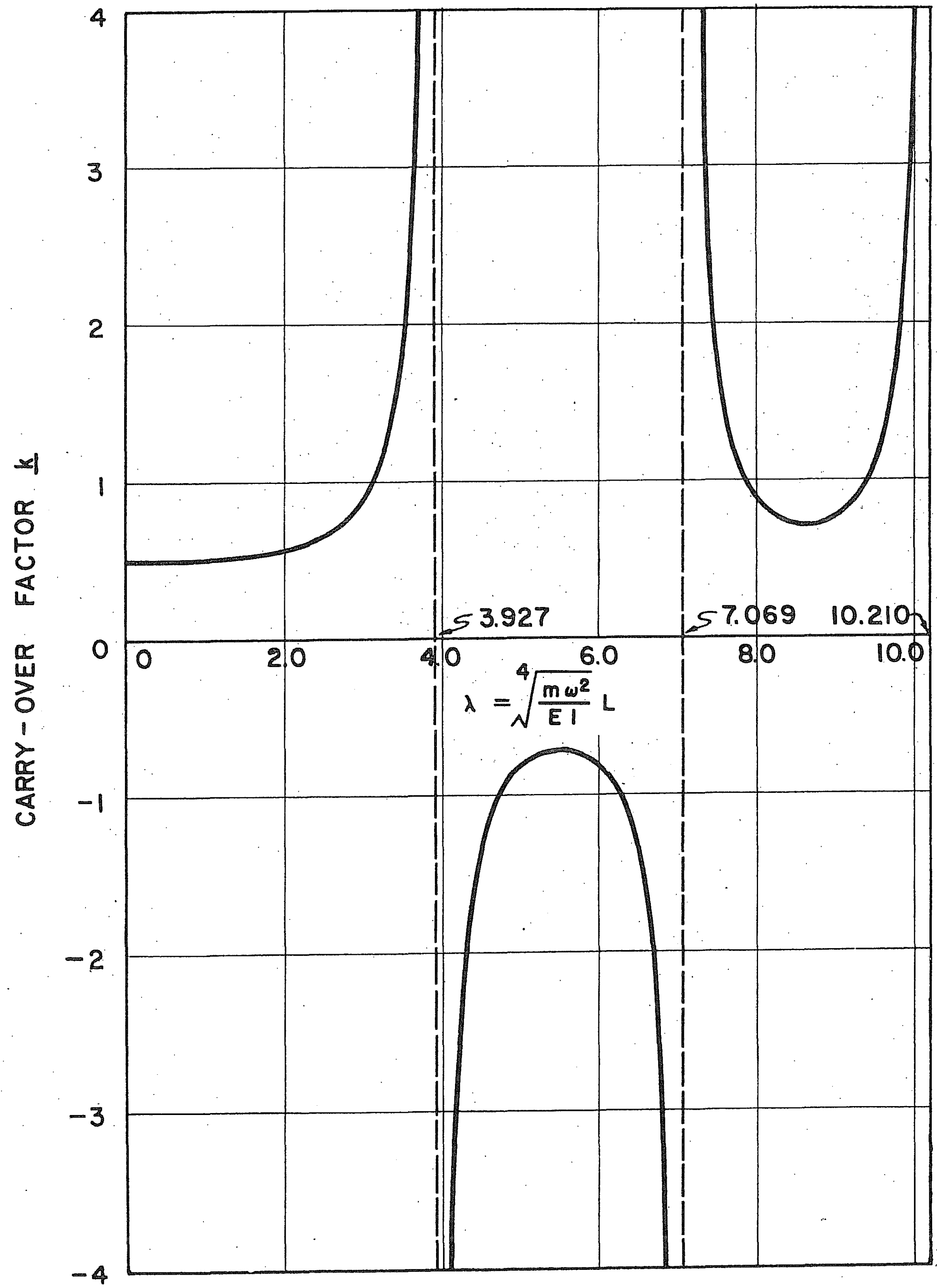


FIG. 5 DYNAMIC FLEXURAL CARRY-OVER FACTOR k FOR A UNIFORM BAR FIXED AT THE FAR END

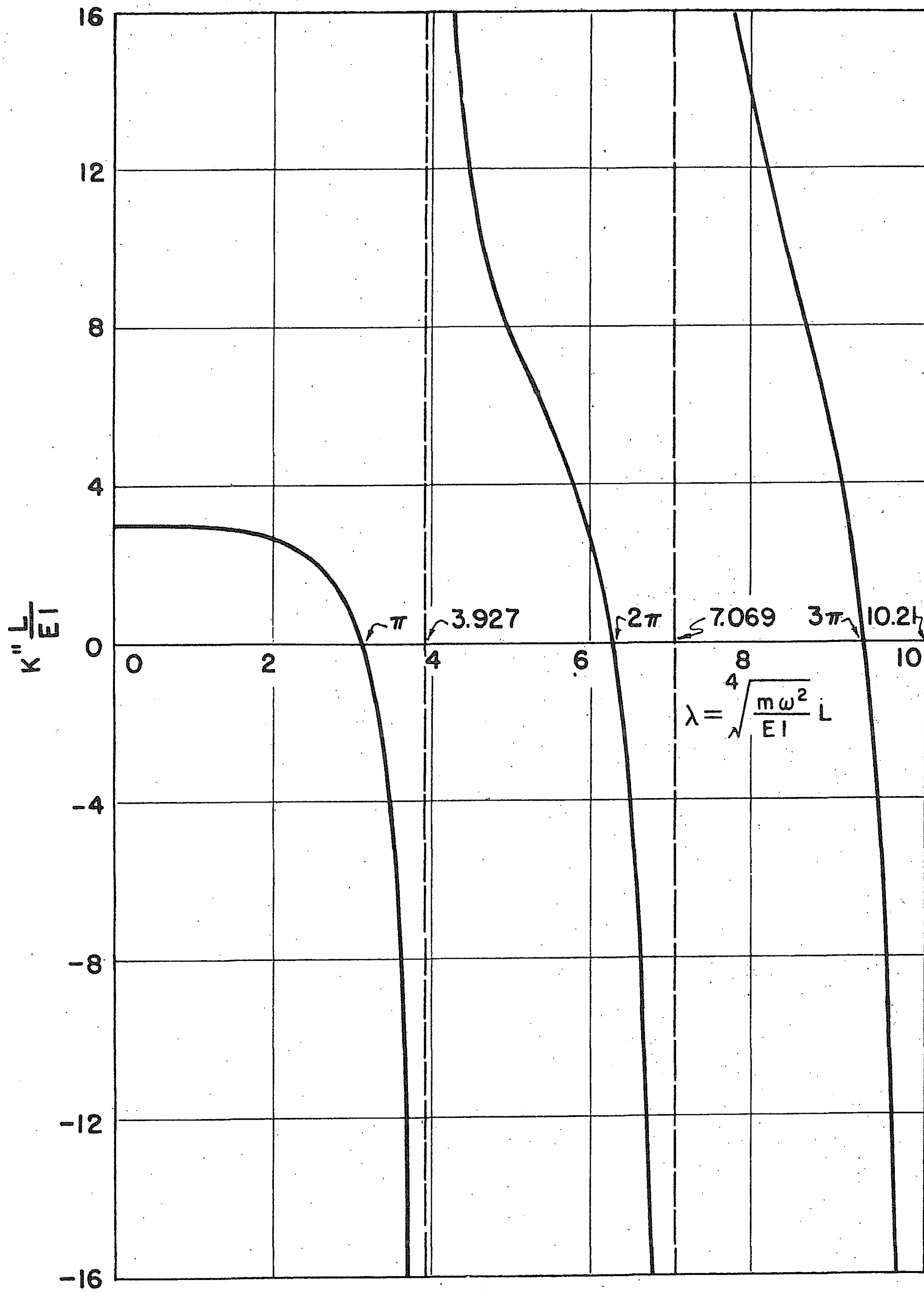


FIG. 6 COEFFICIENTS OF DYNAMIC FLEXURAL STIFFNESS $\underline{K''}$ FOR A UNIFORM BAR HINGED AT THE FAR END

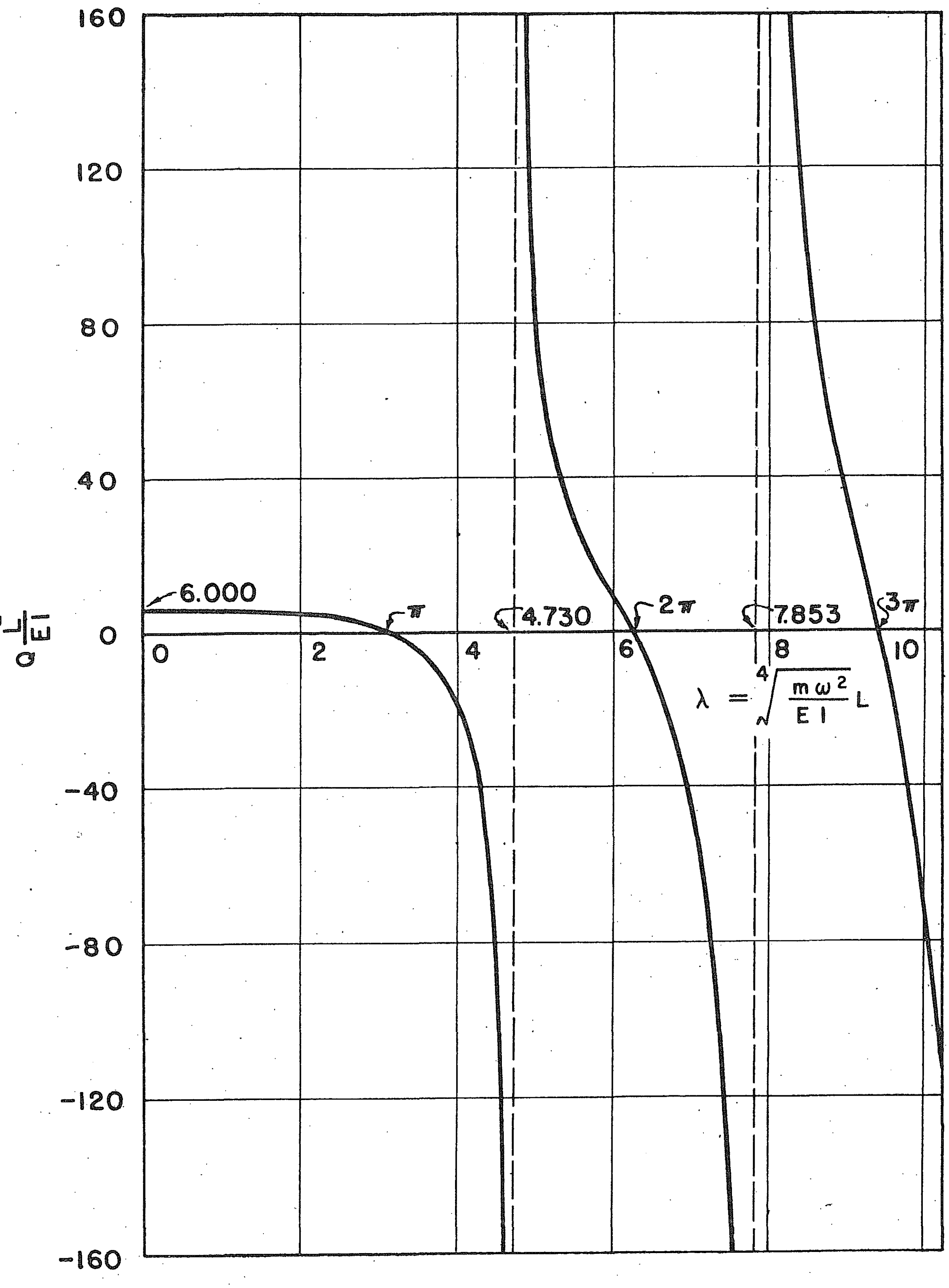


FIG. 7 COEFFICIENTS OF DYNAMIC FLEXURE-SHEAR STIFFNESS Q FOR A UNIFORM BAR FIXED AT THE FAR END

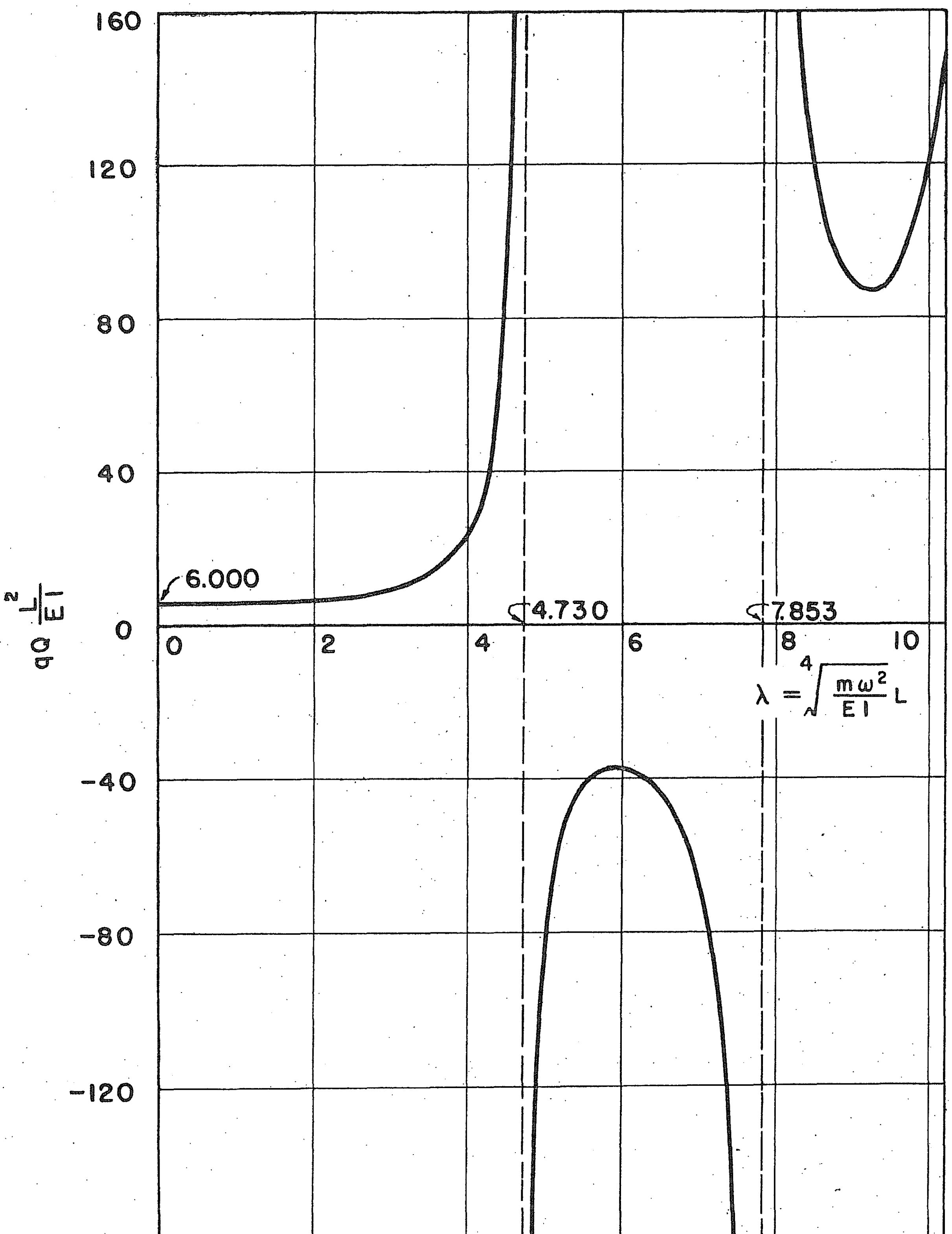


FIG. 8 COEFFICIENTS OF THE PRODUCT OF DYNAMIC FLEXURE-SHEAR STIFFNESS AND DYNAMIC FLEXURE-SHEAR CARRY-OVER FACTOR qQ FOR A UNIFORM BAR FIXED AT THE FAR END

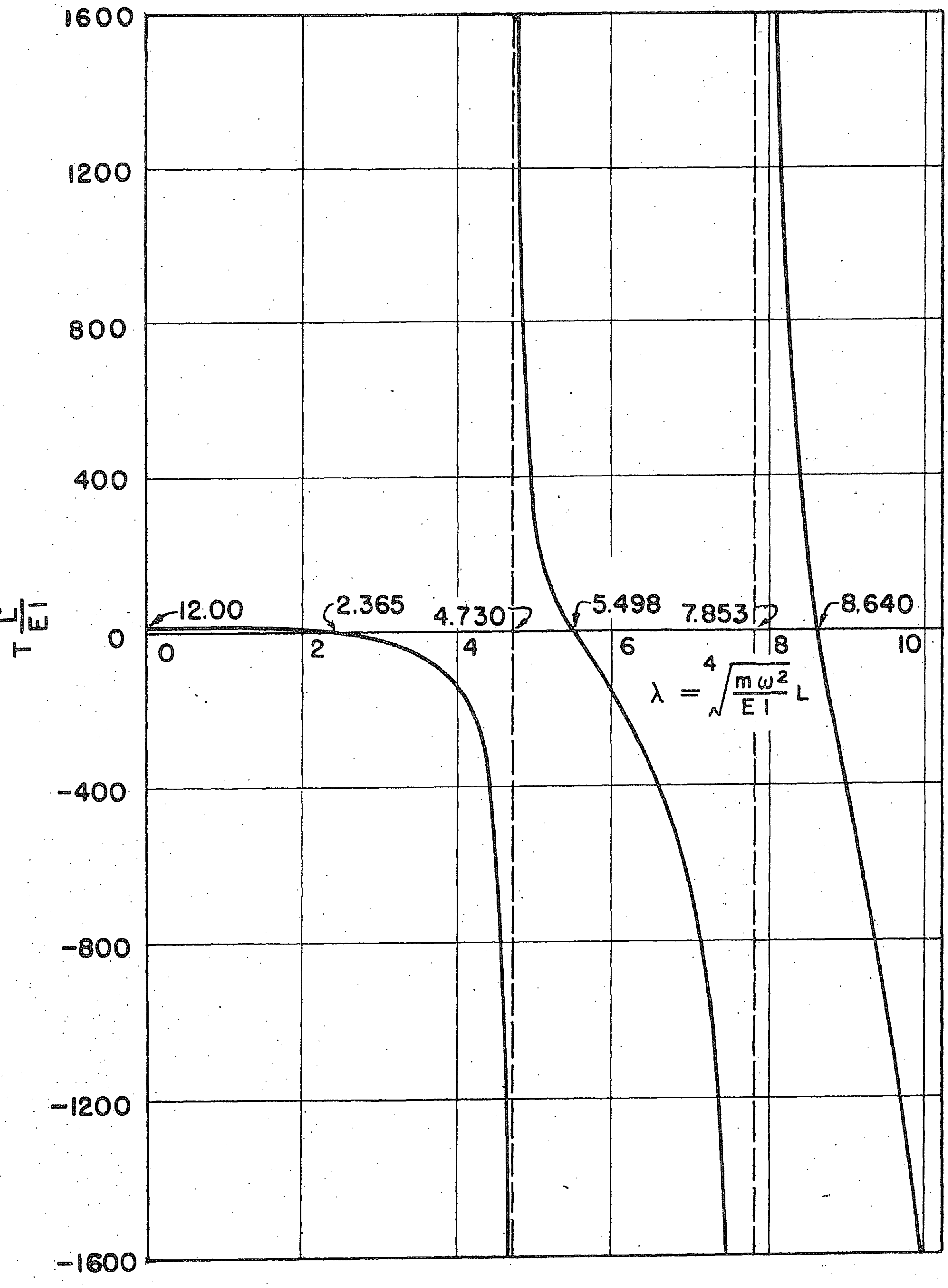


FIG. 9 COEFFICIENTS OF DYNAMIC SHEAR STIFFNESS $\underline{T}$ FOR A UNIFORM BAR FIXED AT THE FAR END

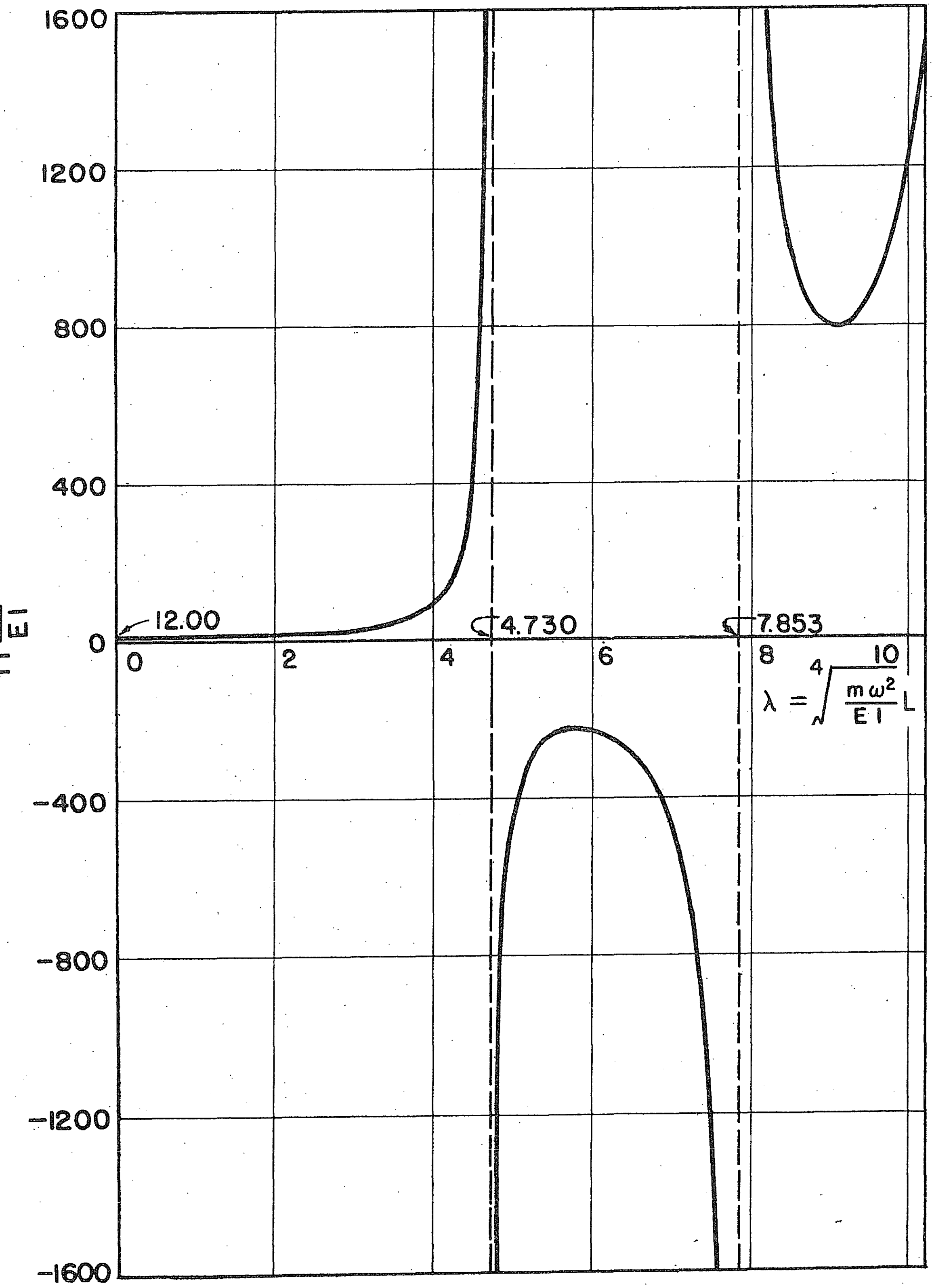


FIG. 10 COEFFICIENTS OF THE PRODUCT OF DYNAMIC SHEAR STIFFNESS AND DYNAMIC SHEAR CARRY-OVER FACTOR <u>tT</u> FOR A UNIFORM BAR FIXED AT THE FAR END

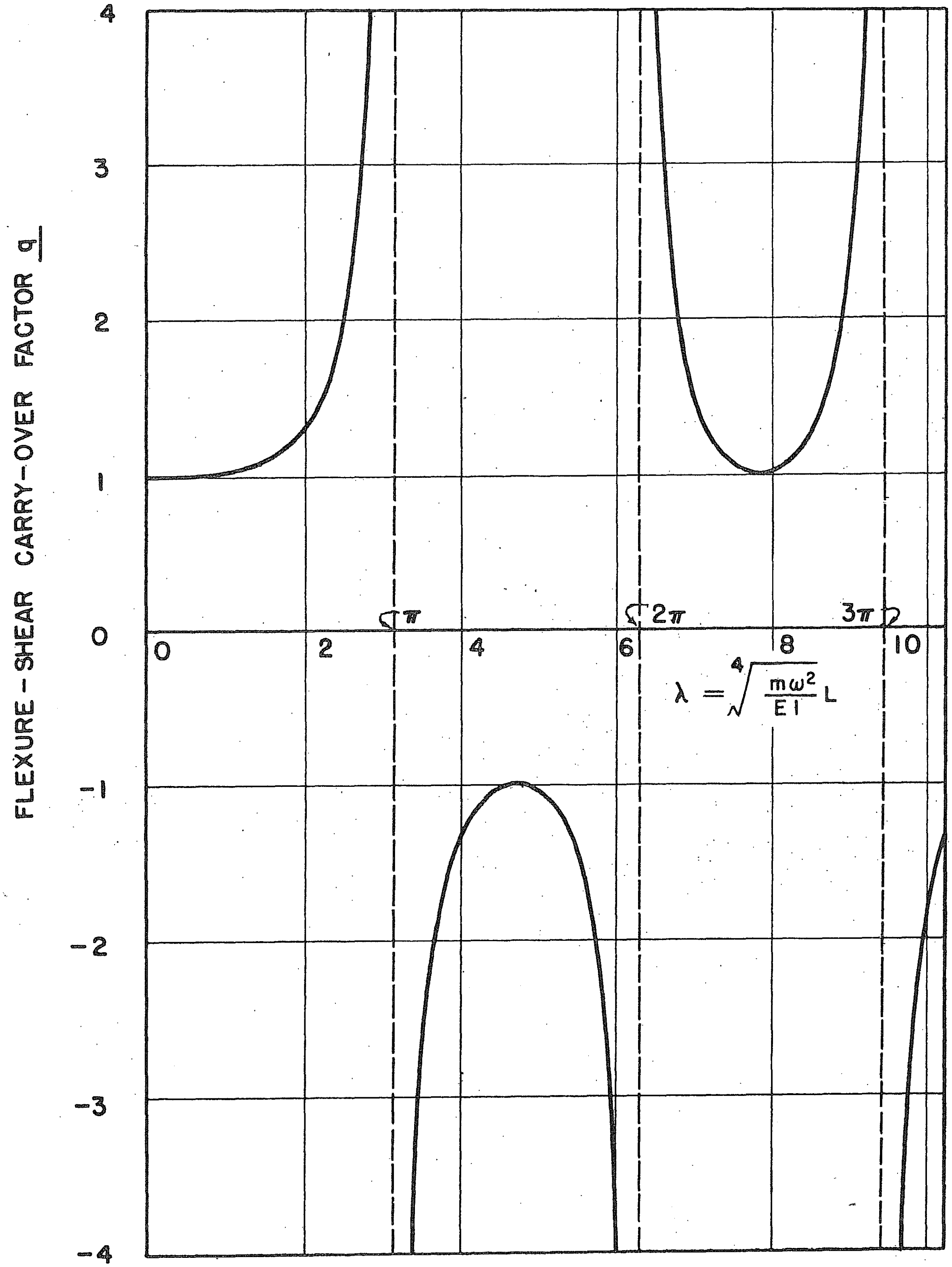


FIG. 11 DYNAMIC FLEXURE-SHEAR CARRY-OVER FACTOR q FOR A UNIFORM BAR FIXED AT THE FAR END

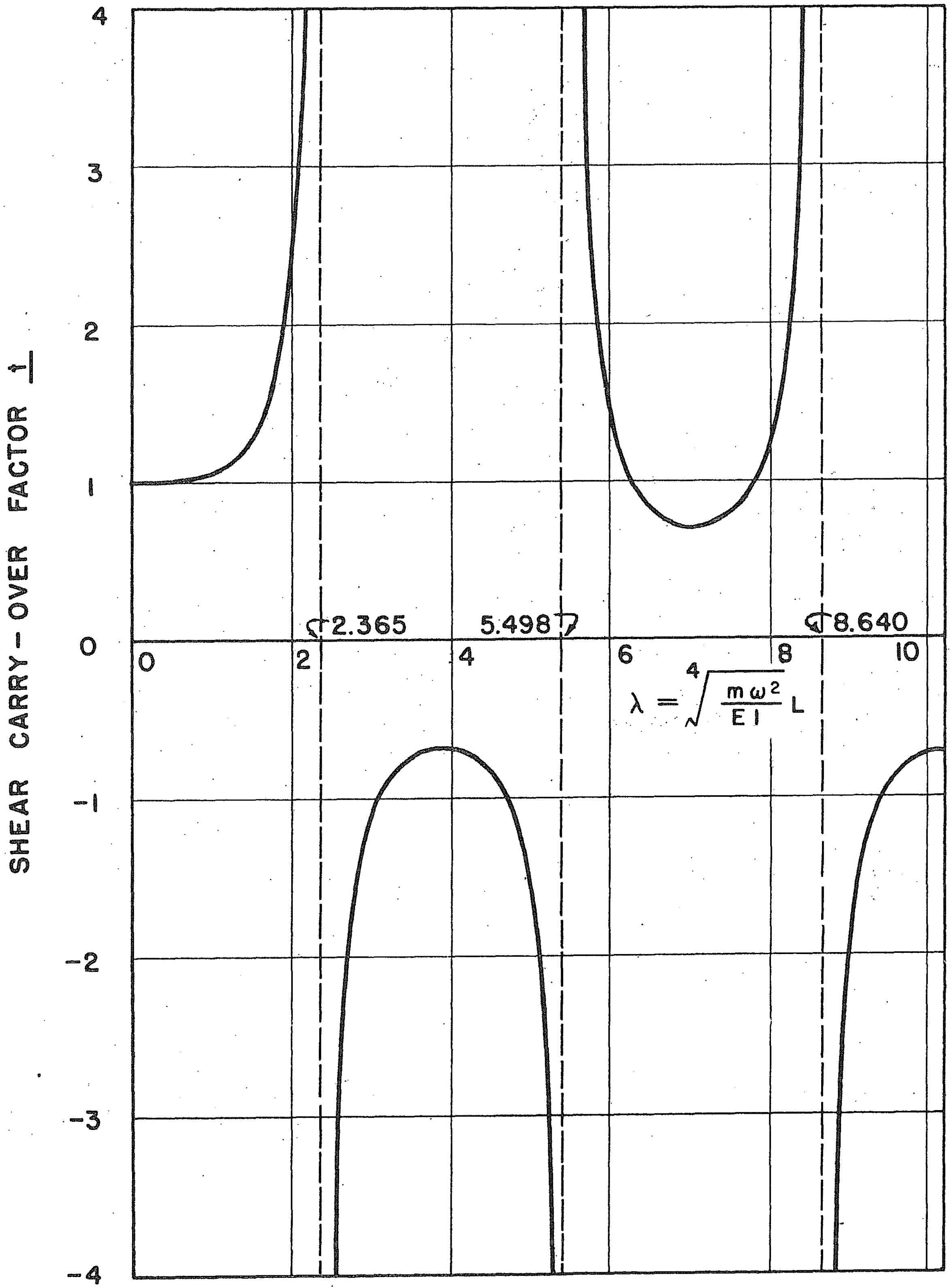


FIG. 12 DYNAMIC SHEAR CARRY-OVER FACTOR t FOR A UNIFORM BAR FIXED AT THE FAR END

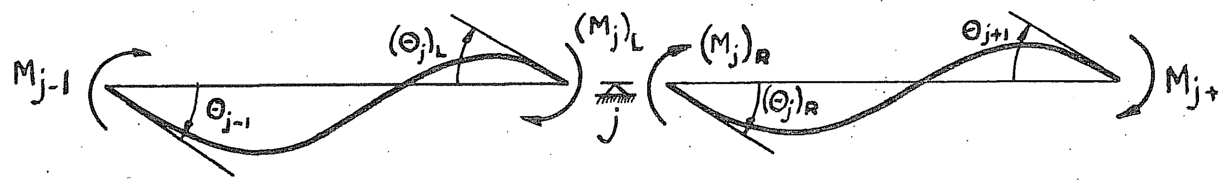


FIG. 13 DEFLECTED SHAPE OF TWO ADJACENT SPANS OF A CONTINUOUS BEAM ON RIGID SUPPORTS

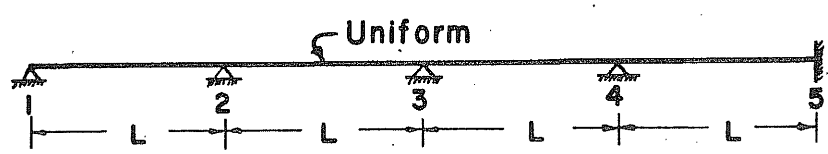


FIG. 14 BEAM CONSIDERED IN EXAMPLE 1

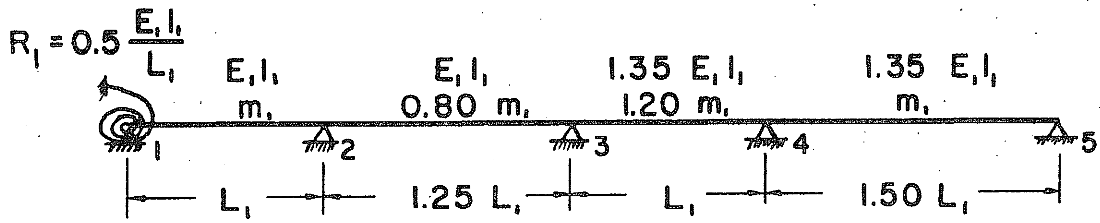


FIG. 15 BEAM CONSIDERED IN EXAMPLE 2

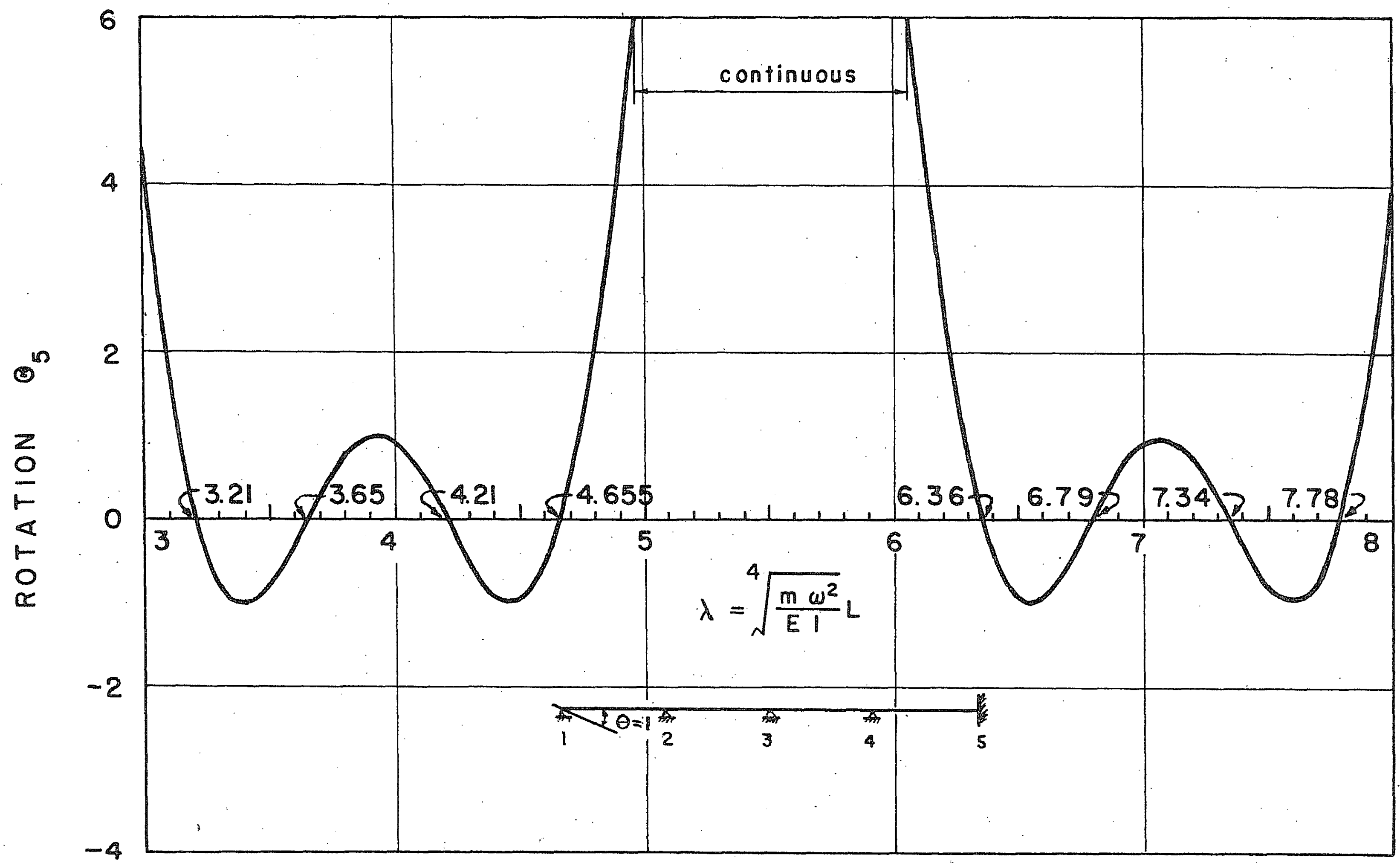


FIG. 16 VARIATION OF ROTATION Θ_5 AS A FUNCTION OF λ, EXAMPLE 1

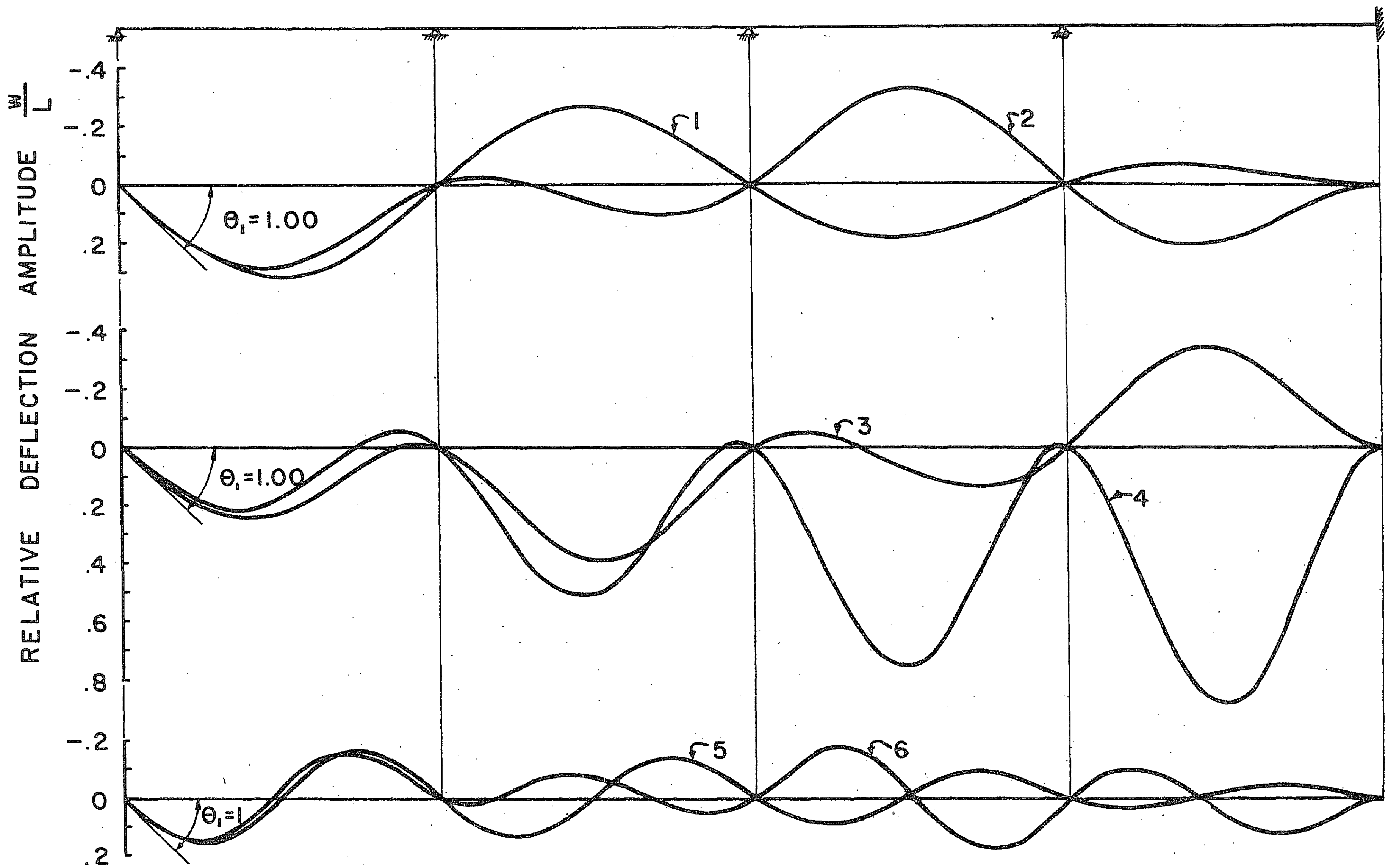


FIG. 17 FIRST SIX NATURAL VIBRATION MODES OF A UNIFORM, 4-SPAN CONTINUOUS BEAM HINGED AT LEFT END AND FIXED AT RIGHT END

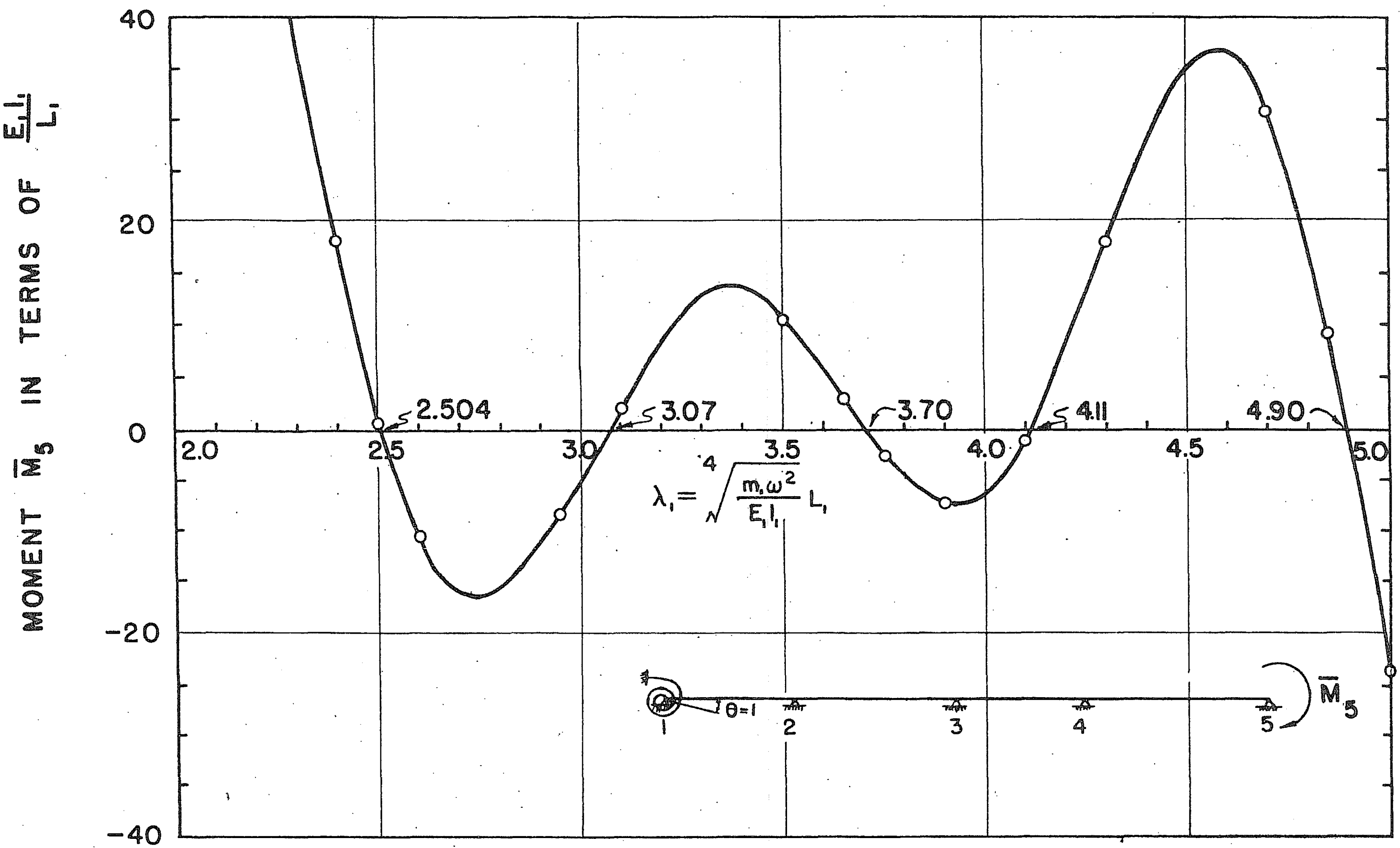


FIG. 18 VARIATION OF EXCITING MOMENT $\bar{M}_5$ AS A FUNCTION OF λ_1, EXAMPLE 2

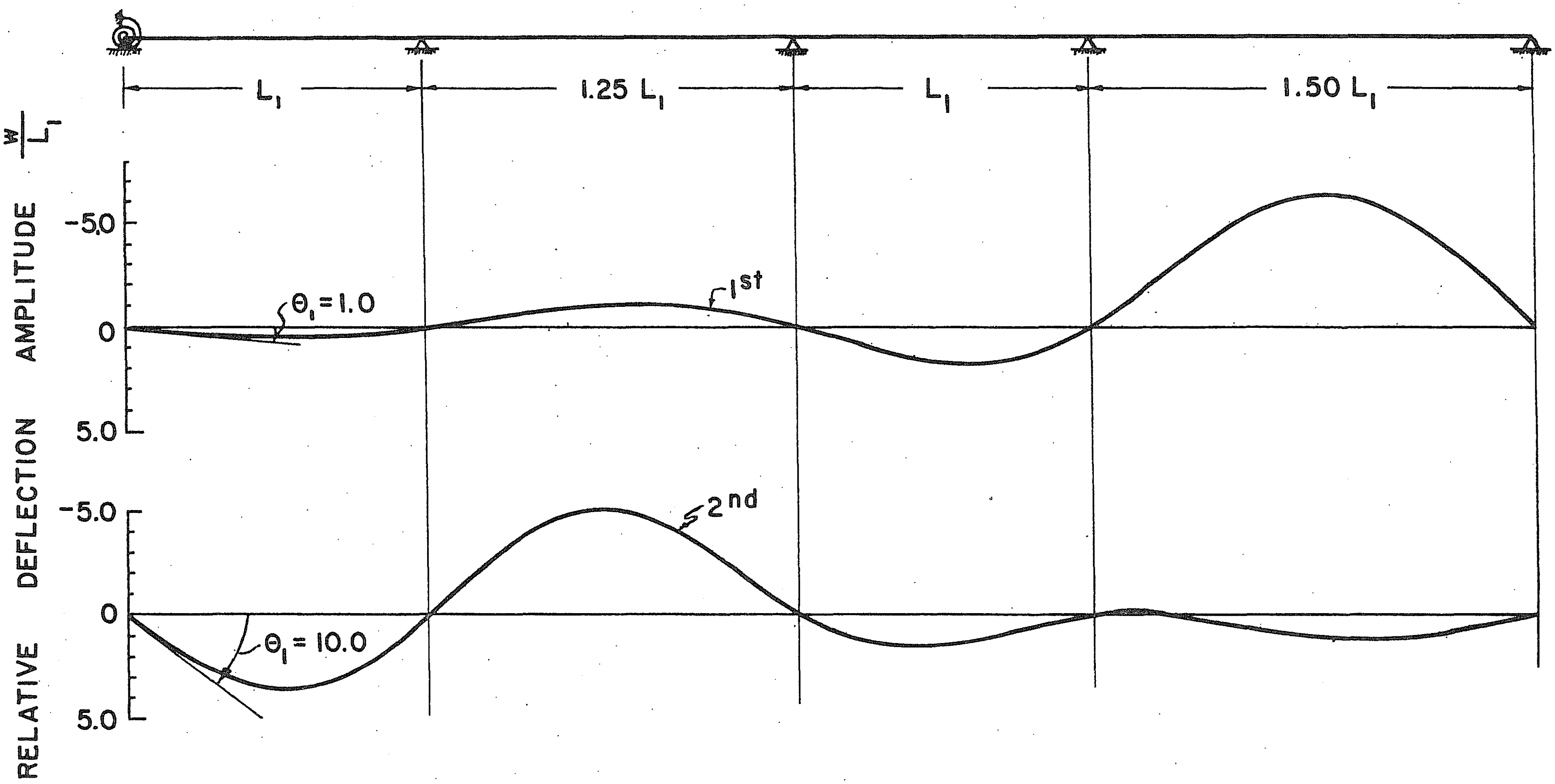


FIG. 19 FIRST TWO NATURAL VIBRATION MODES OF BEAM CONSIDERED IN EXAMPLE 2

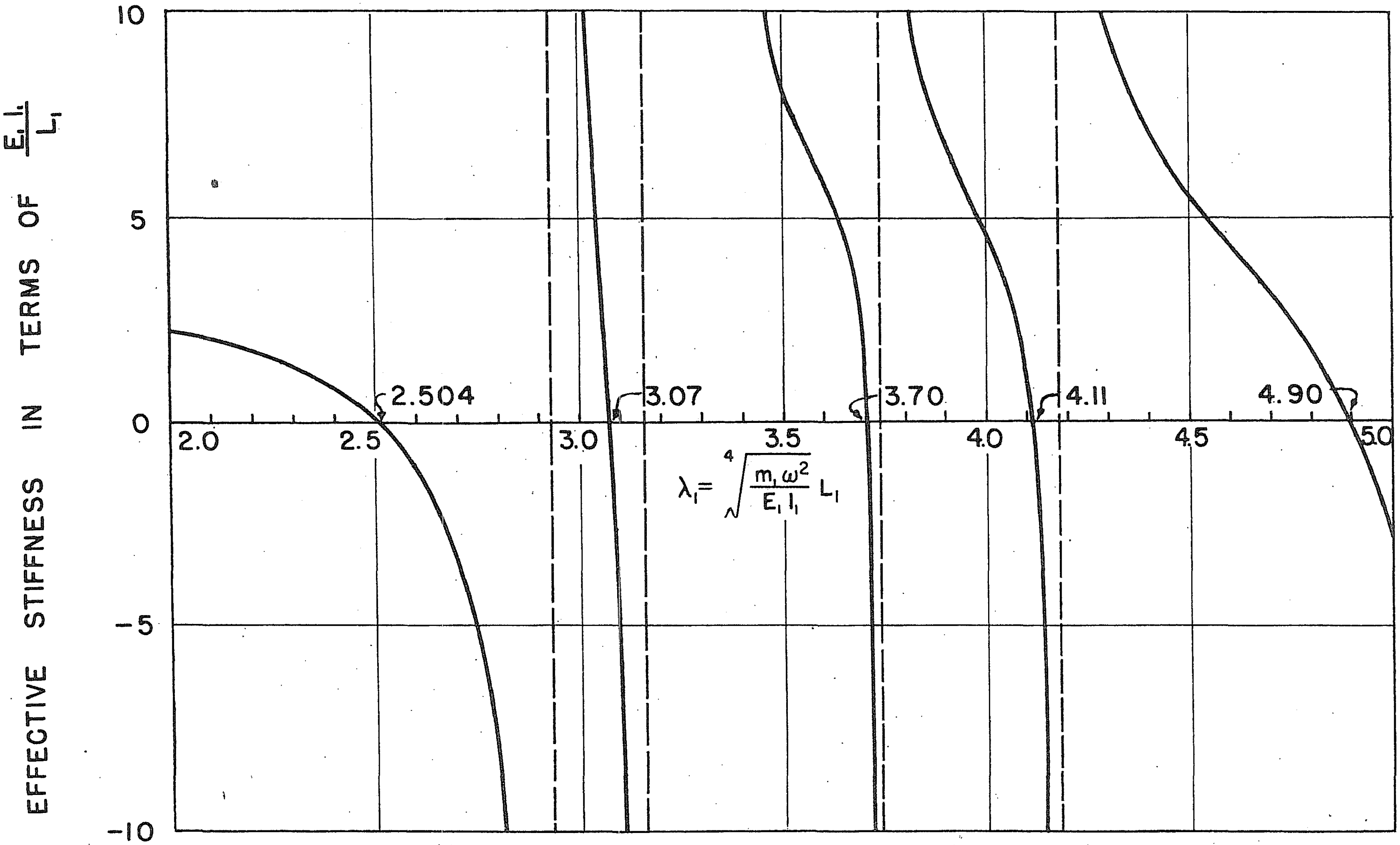


FIG. 20 EFFECTIVE STIFFNESS AT RIGHT END OF THE BEAM AS A FUNCTION OF λ_1, EXAMPLE 2

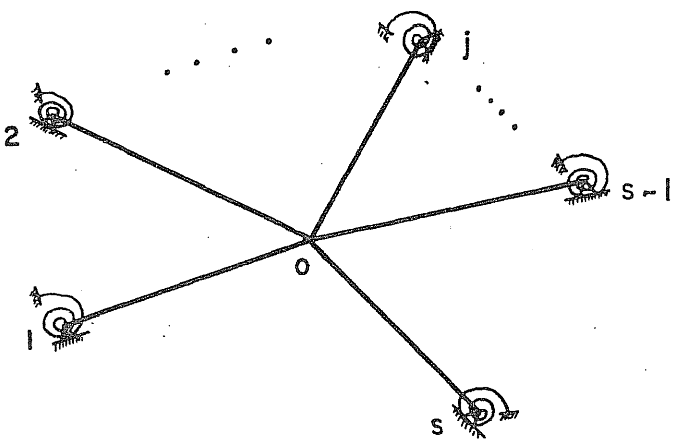


FIG. 21 TYPICAL JOINT OF A PLANE RIGID FRAME, NO SIDESWAY

FIG. 22 TYPICAL PORTAL AND L - FRAMES

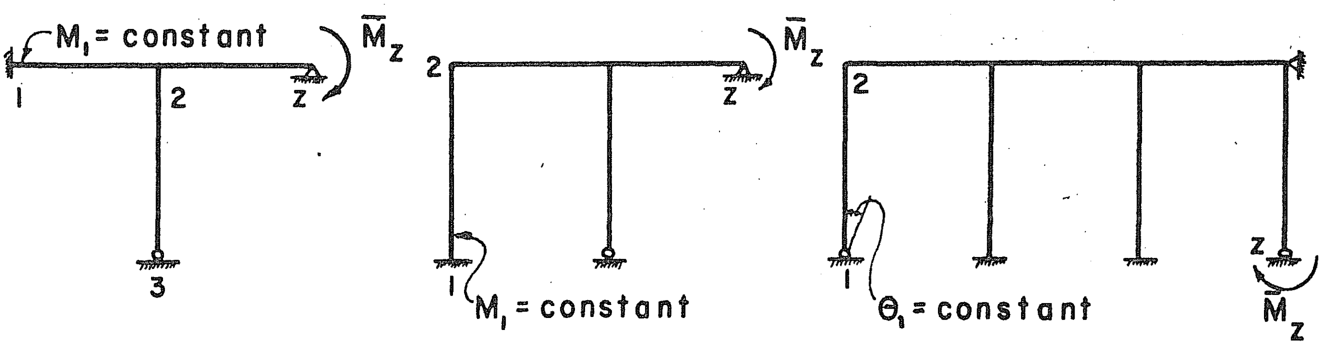


FIG. 23 TYPICAL OPEN FRAMES

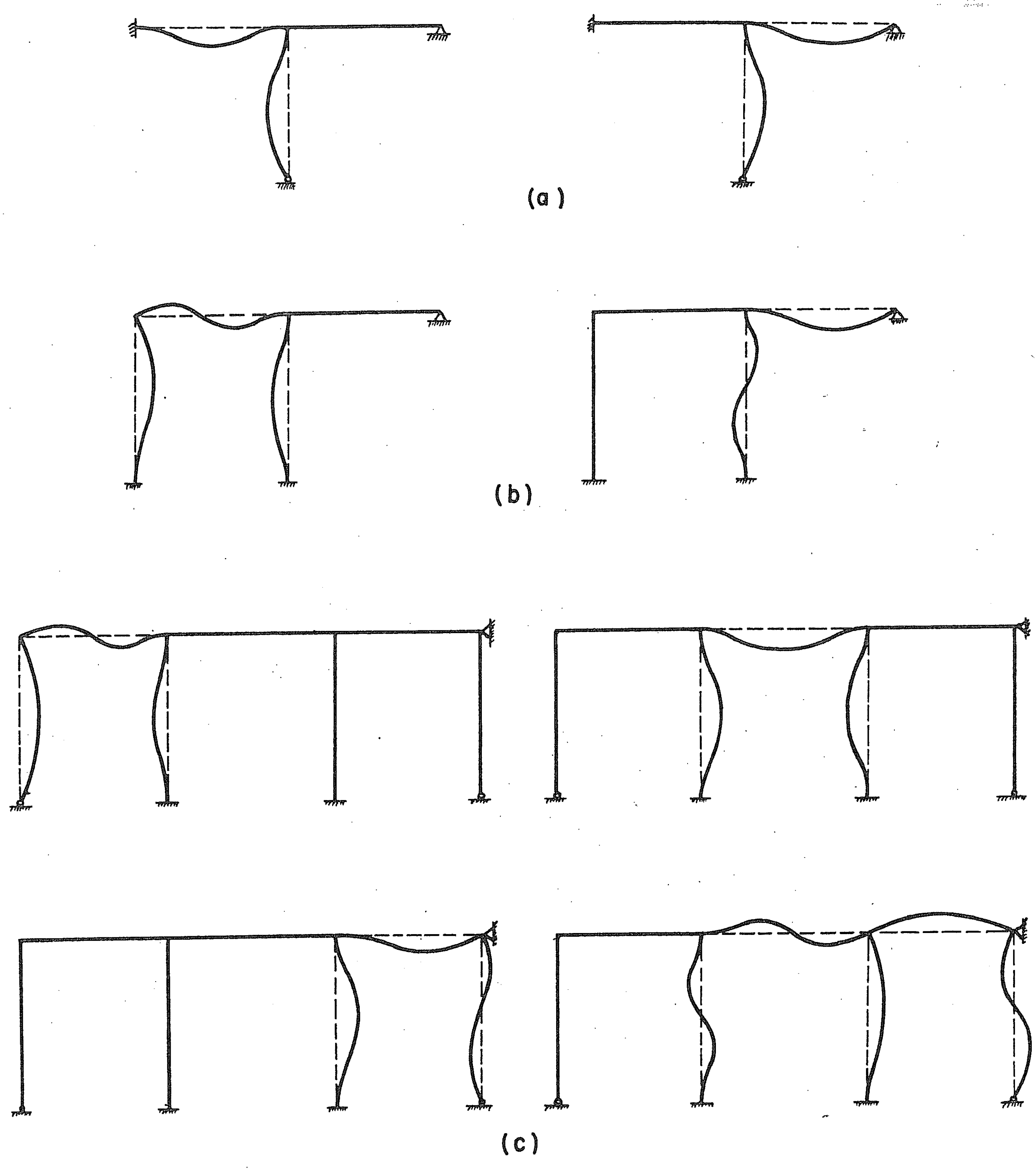


FIG. 24 POSSIBLE NATURAL "MODES, OF PARTIAL VIBRATION"

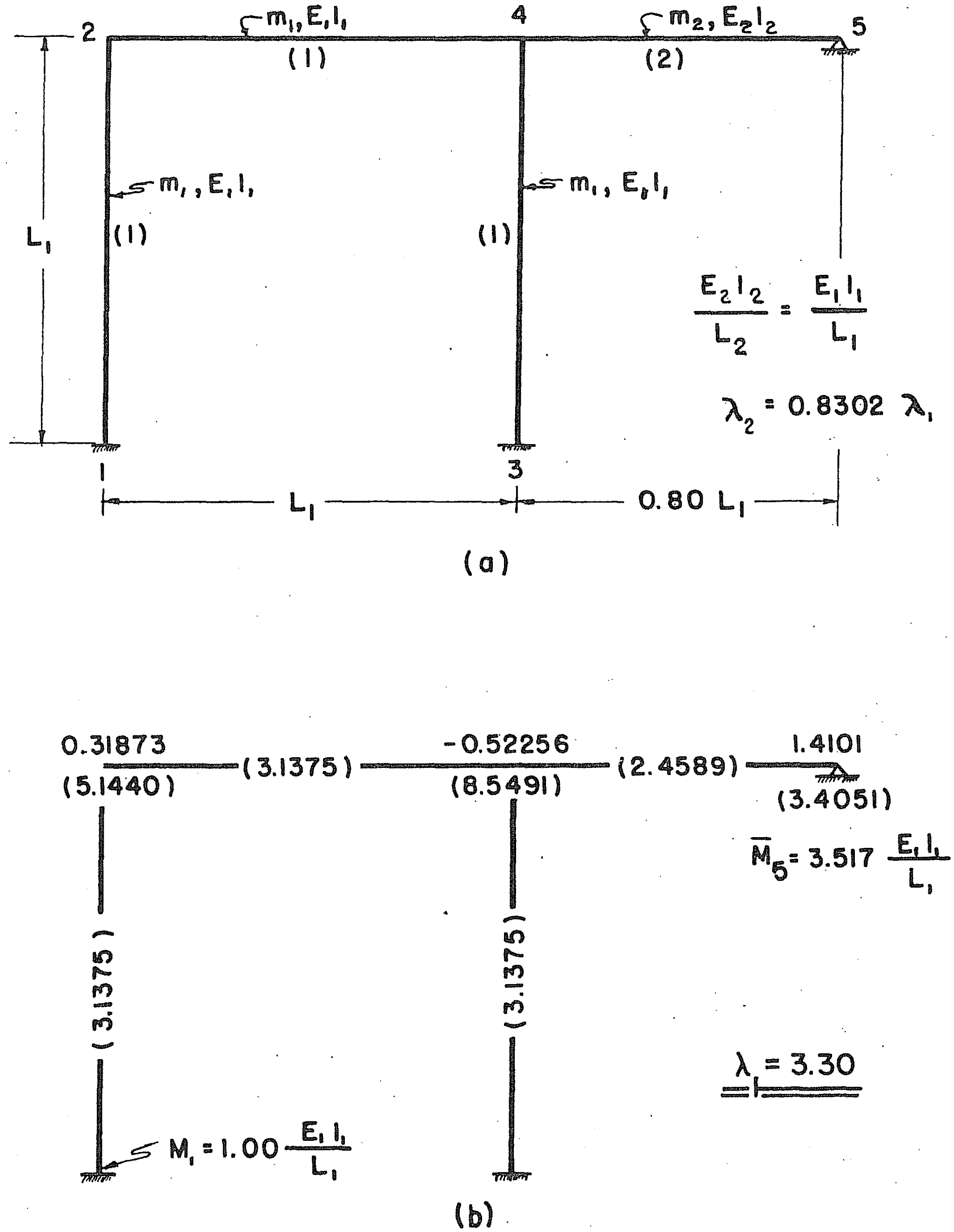


FIG. 25 ILLUSTRATIVE EXAMPLE 3

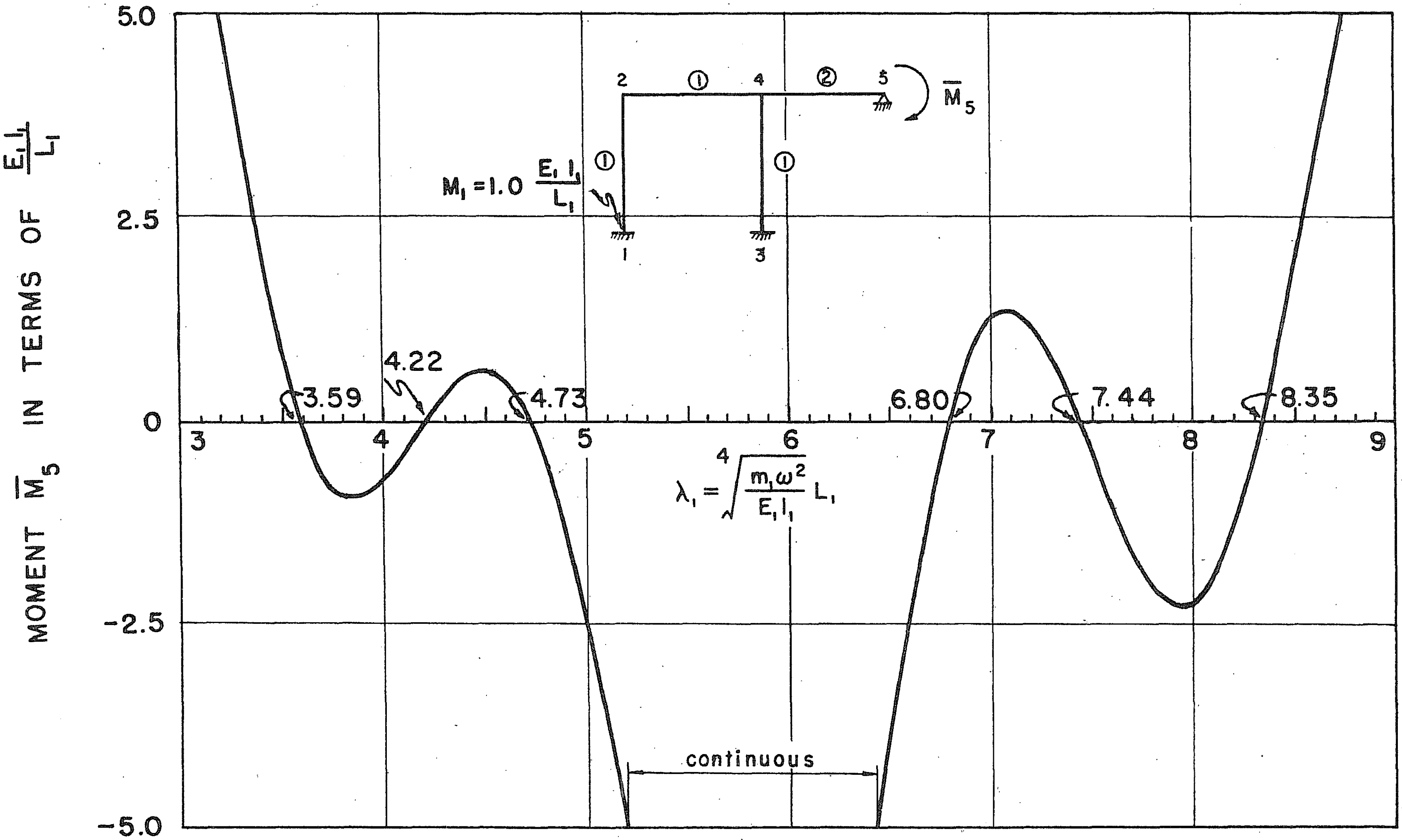


FIG. 26 VARIATION OF EXCITING MOMENT $\overline{M}_5$ AS A FUNCTION OF λ_1, EXAMPLE 3

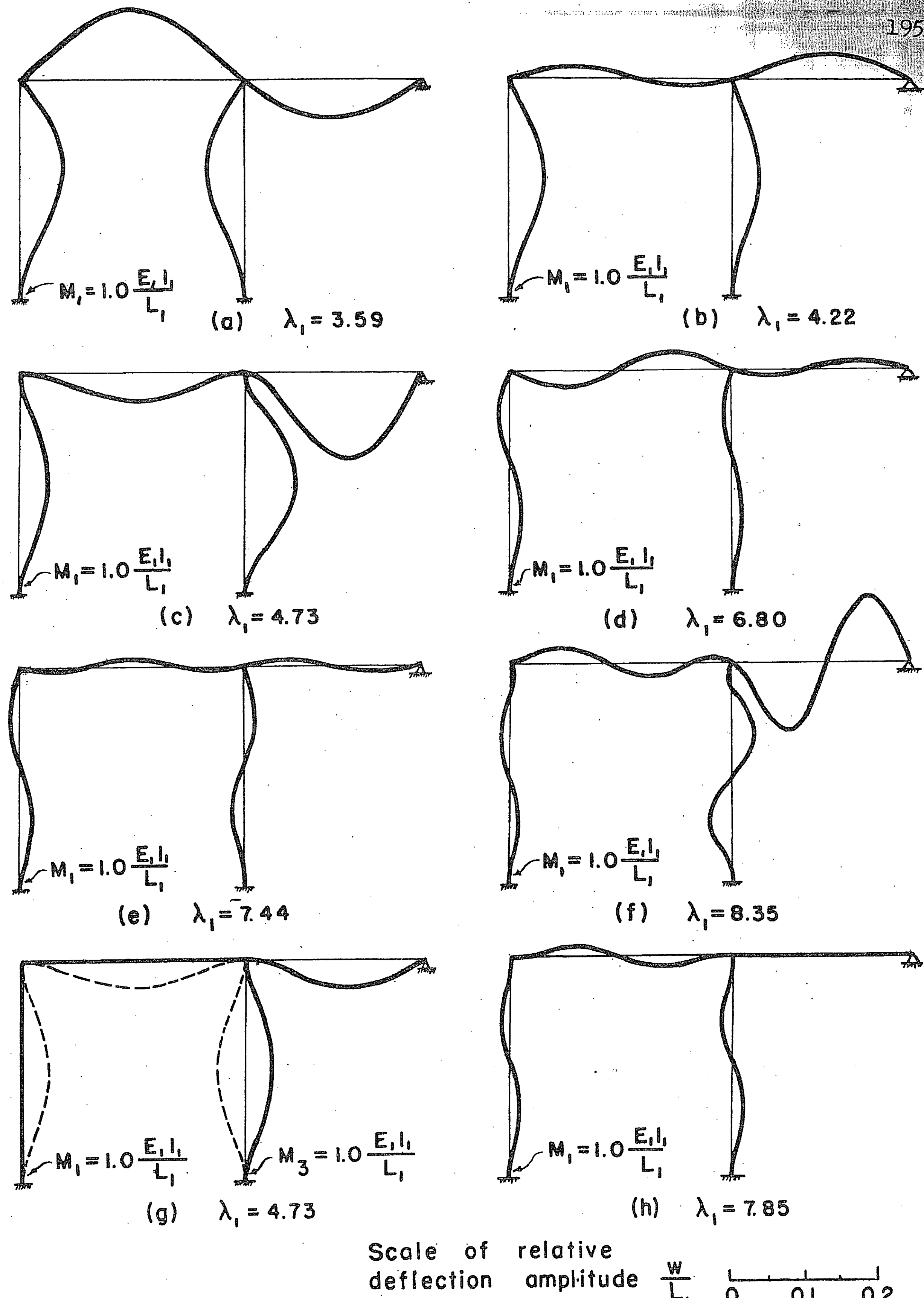


FIG. 27 NATURAL VIBRATION MODES FOR FRAME CONSIDERED IN EXAMPLE 3

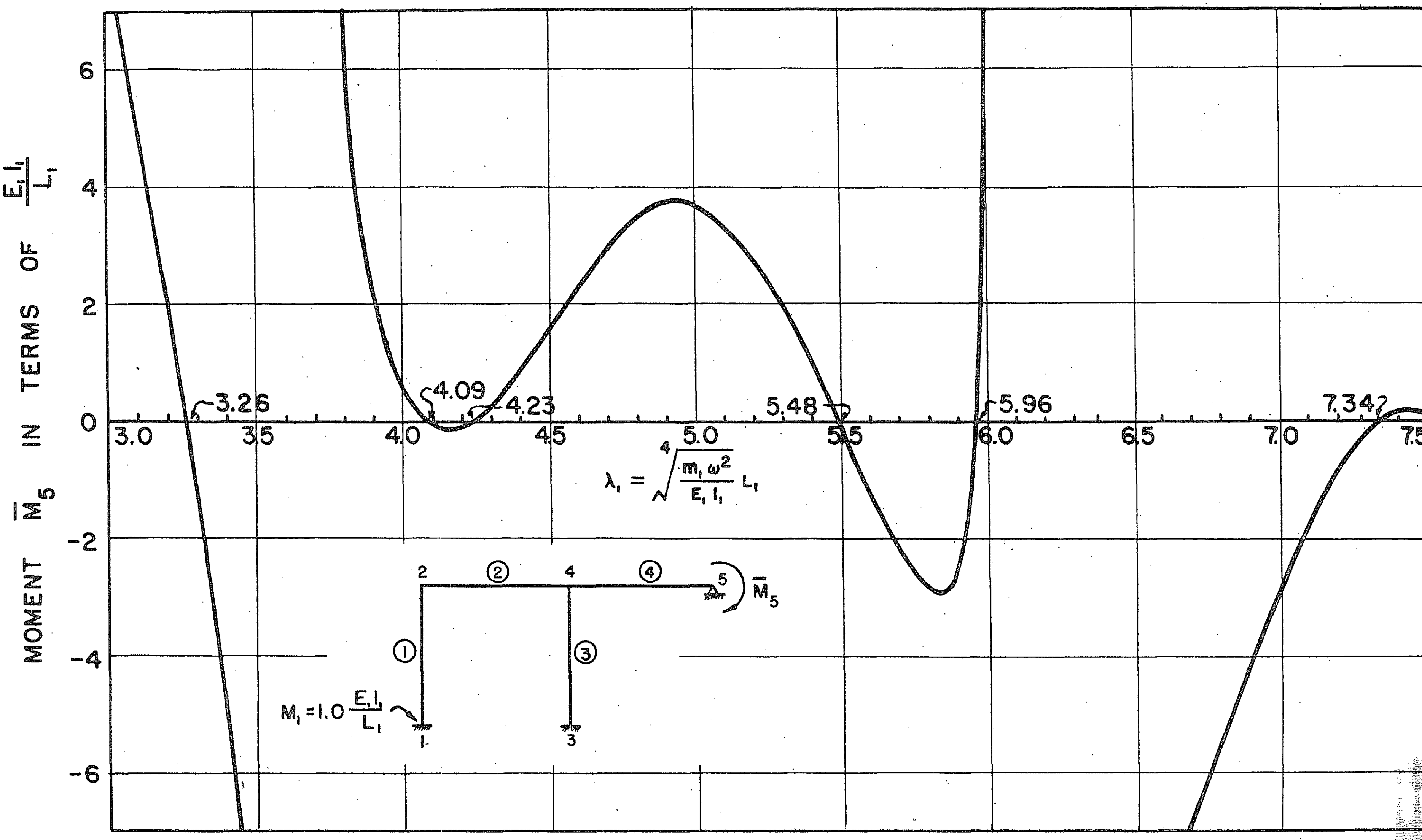


FIG. 28 VARIATION OF EXCITING MOMENT $\bar{M}_5$ AS A FUNCTION OF λ_1, EXAMPLE 4

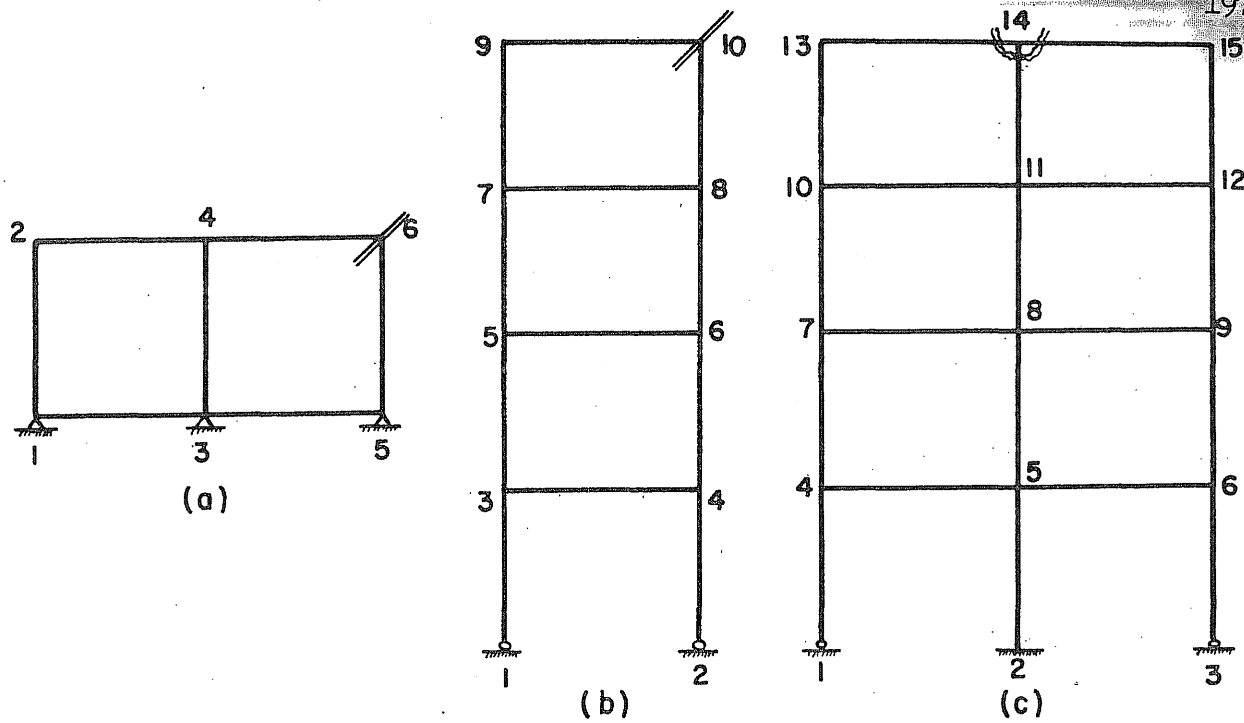


FIG. 29 TYPICAL CLOSED FRAMES, NO SIDESWAY

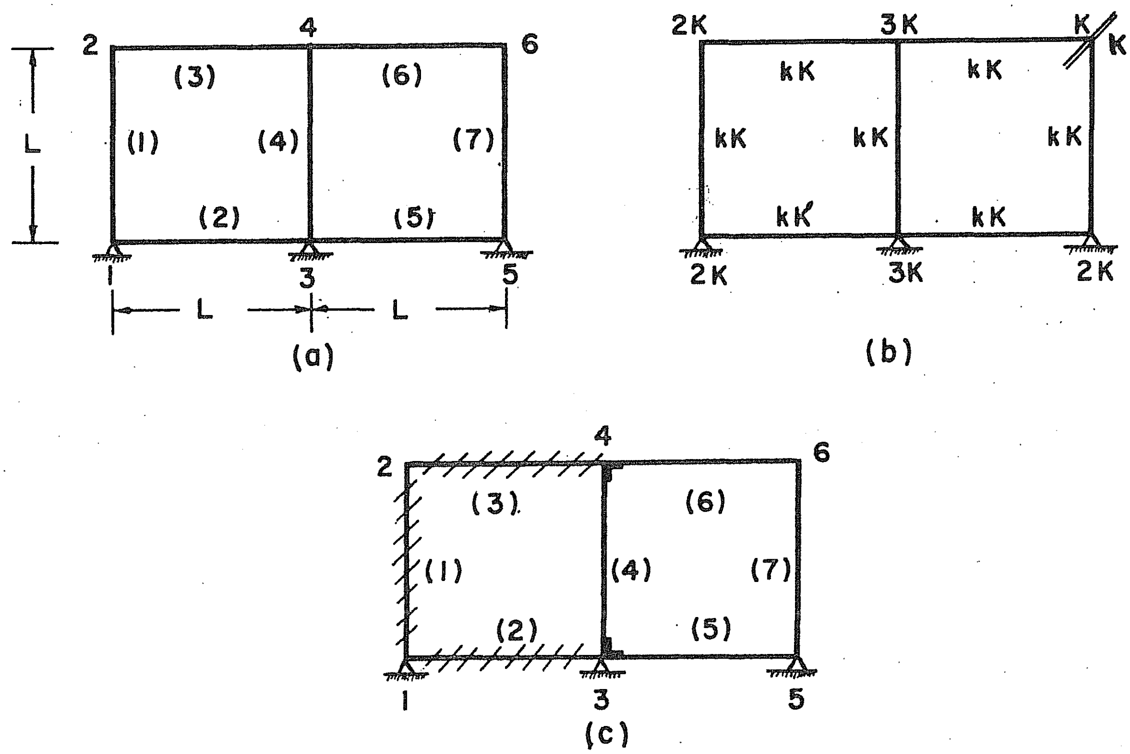


FIG. 30 ILLUSTRATIVE EXAMPLE 5

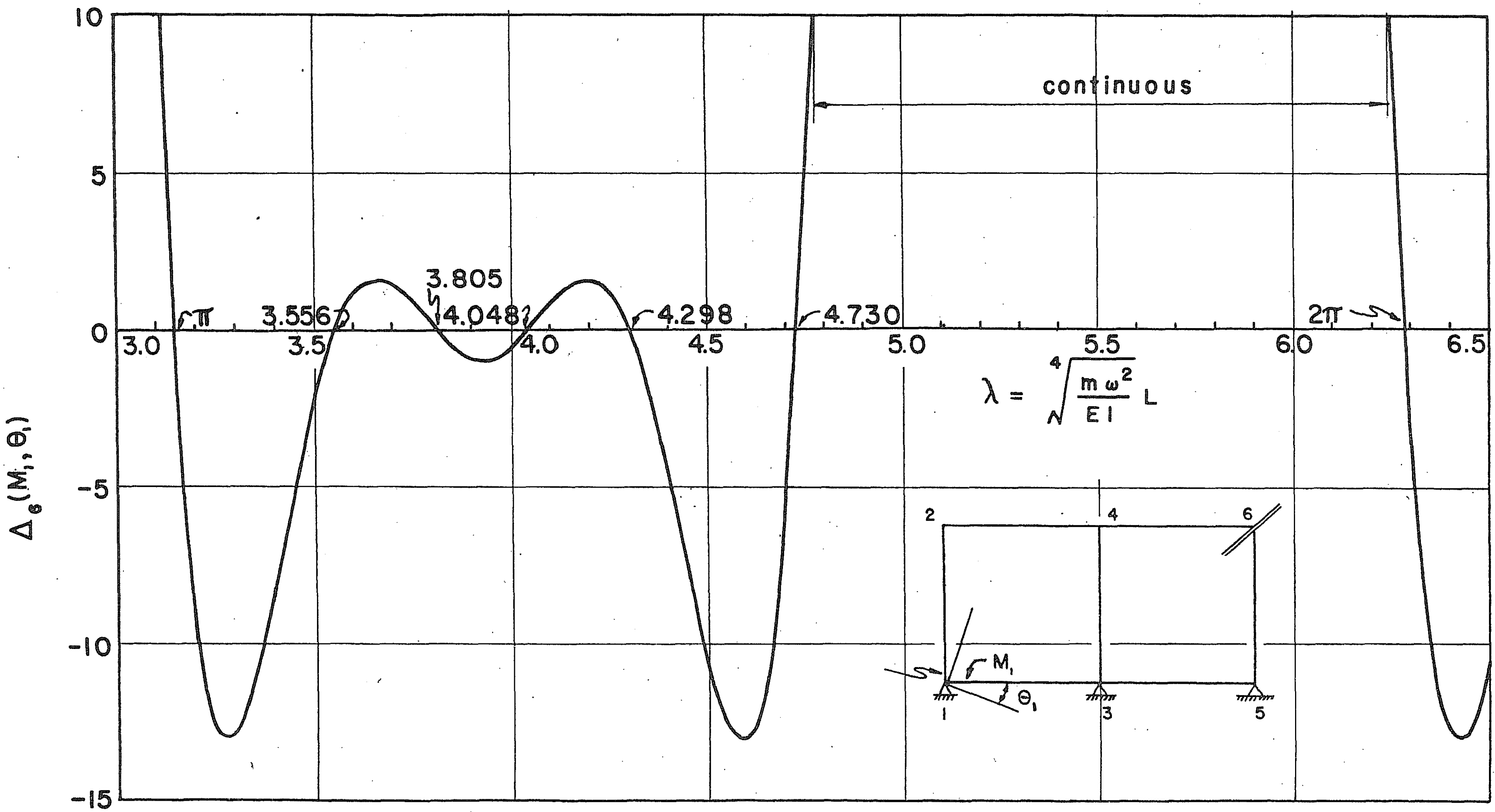


FIG. 30d VARIATION OF THE DETERMINANT $\Delta_6(M_1, \theta_1)$ AS A FUNCTION OF λ_1, EXAMPLE 5

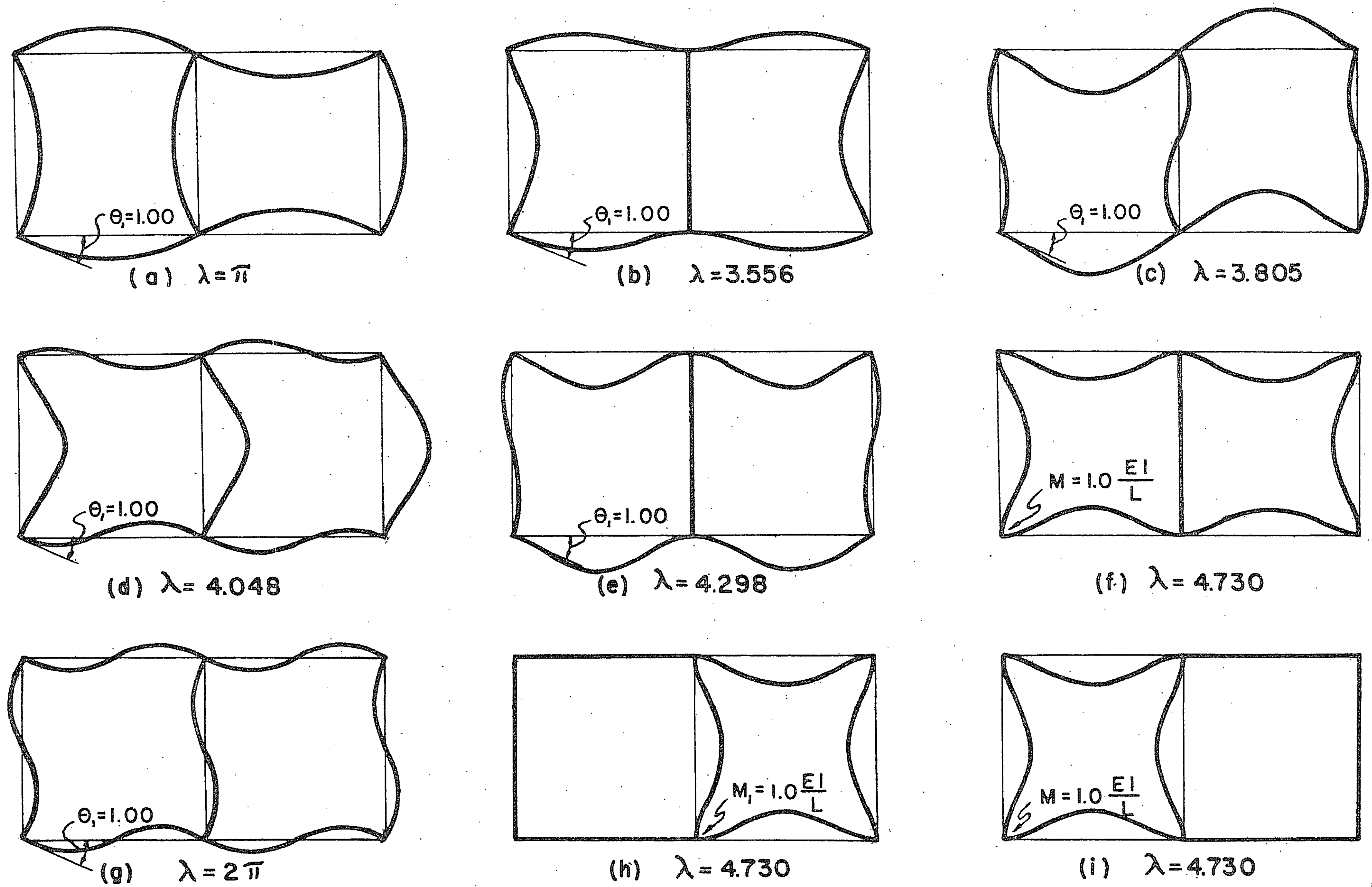


FIG. 31 NATURAL VIBRATION MODES FOR THE FRAME CONSIDERED IN EXAMPLE 5

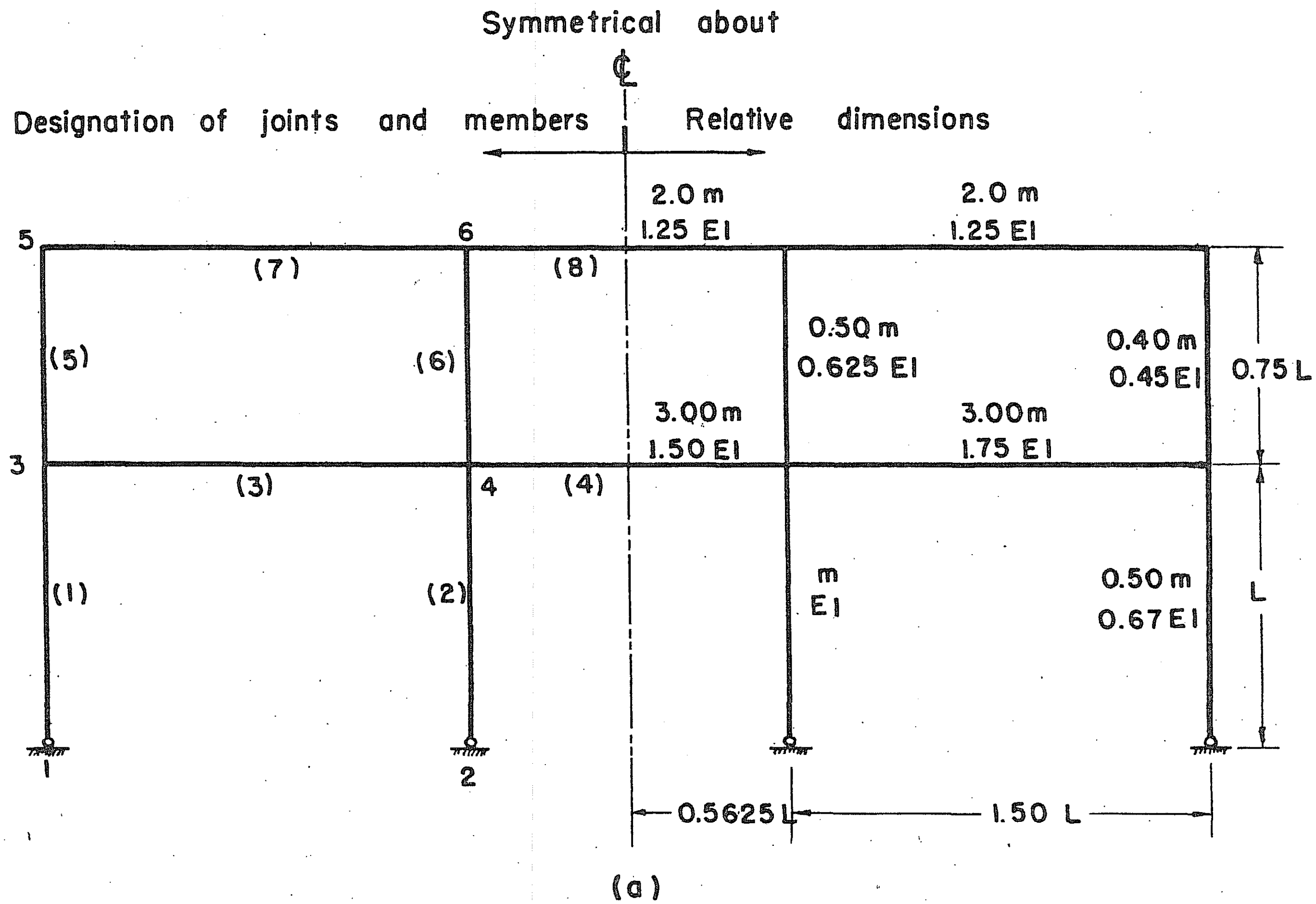
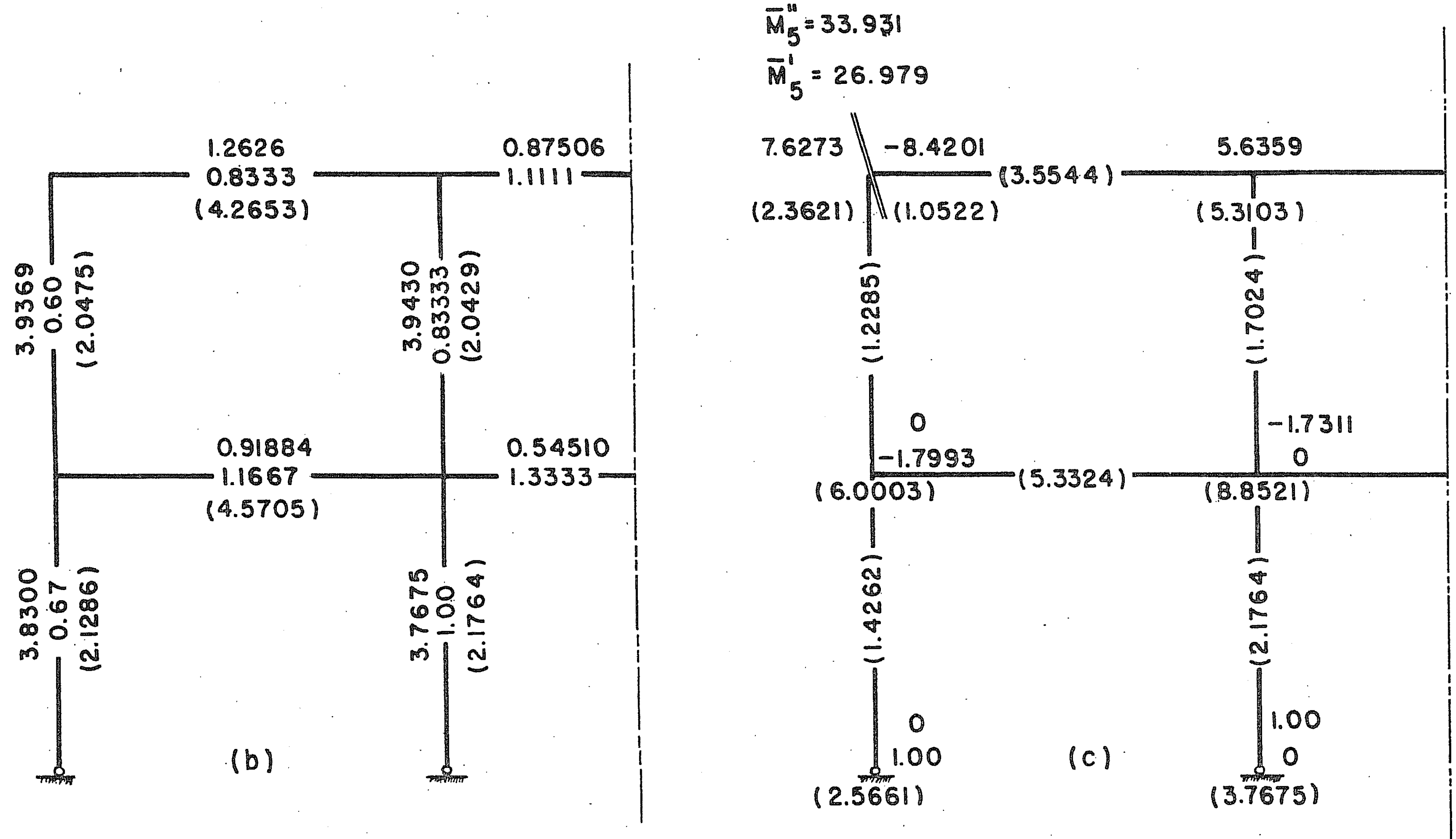


FIG. 32 ILLUSTRATIVE EXAMPLE 6



$$\lambda_2 = 2.20$$

$$\theta_{56} - \theta_{53} = -16.047\,\theta_1 - 18.223\,\theta_2$$

$$\bar{M}_5 = 26.979\,\theta_1 + 33.931\,\theta_2$$

$$\Delta_5\,(\theta_1, \theta_2) = -52.85$$

FIG. 32 (CONT.'D) ILLUSTRATIVE EXAMPLE 6

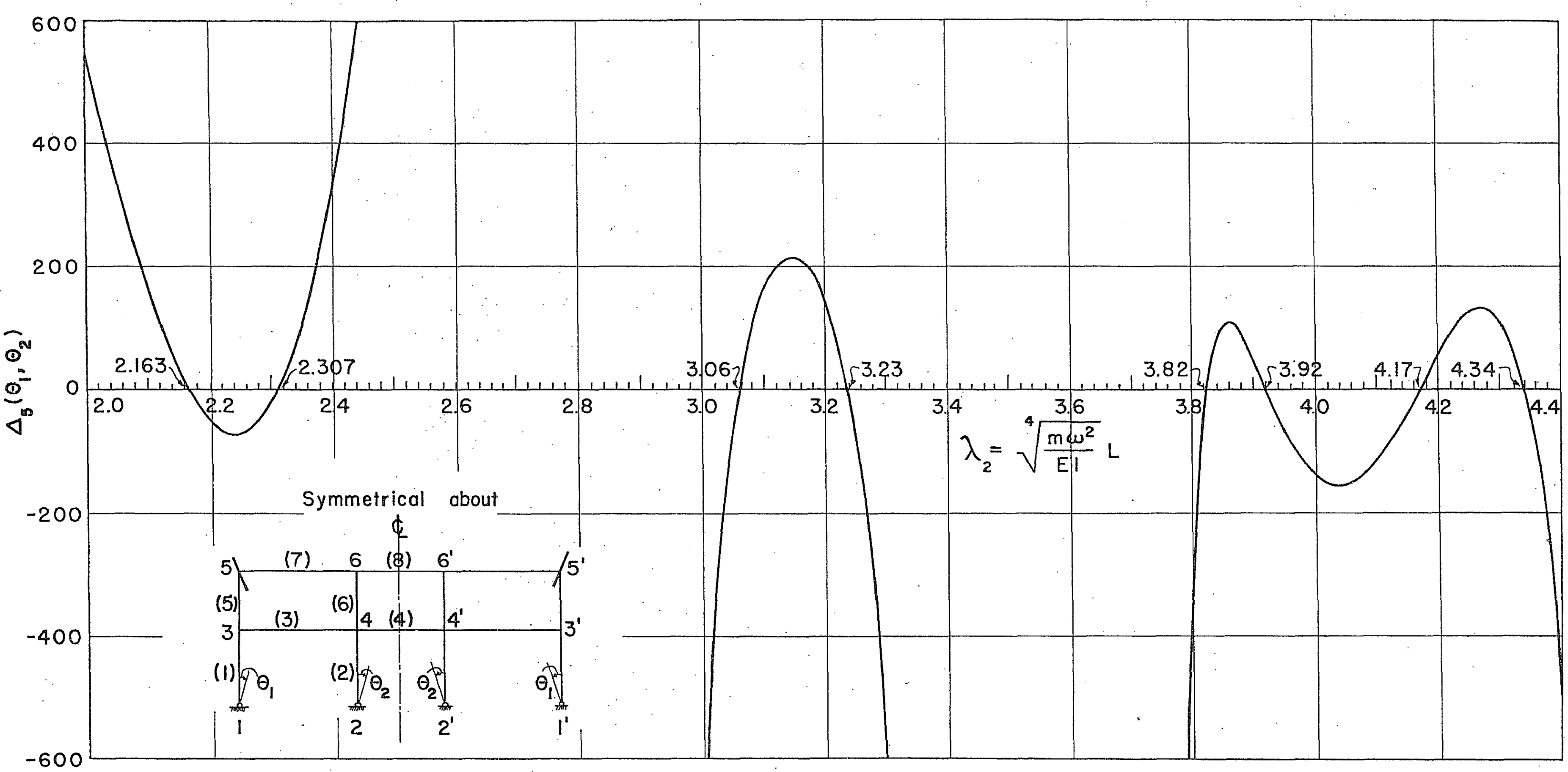


FIG. 33 VARIATION OF DETERMINANT $\Delta_5(\theta_1, \theta_2)$ AS A FUNCTION OF λ_2, EXAMPLE 6

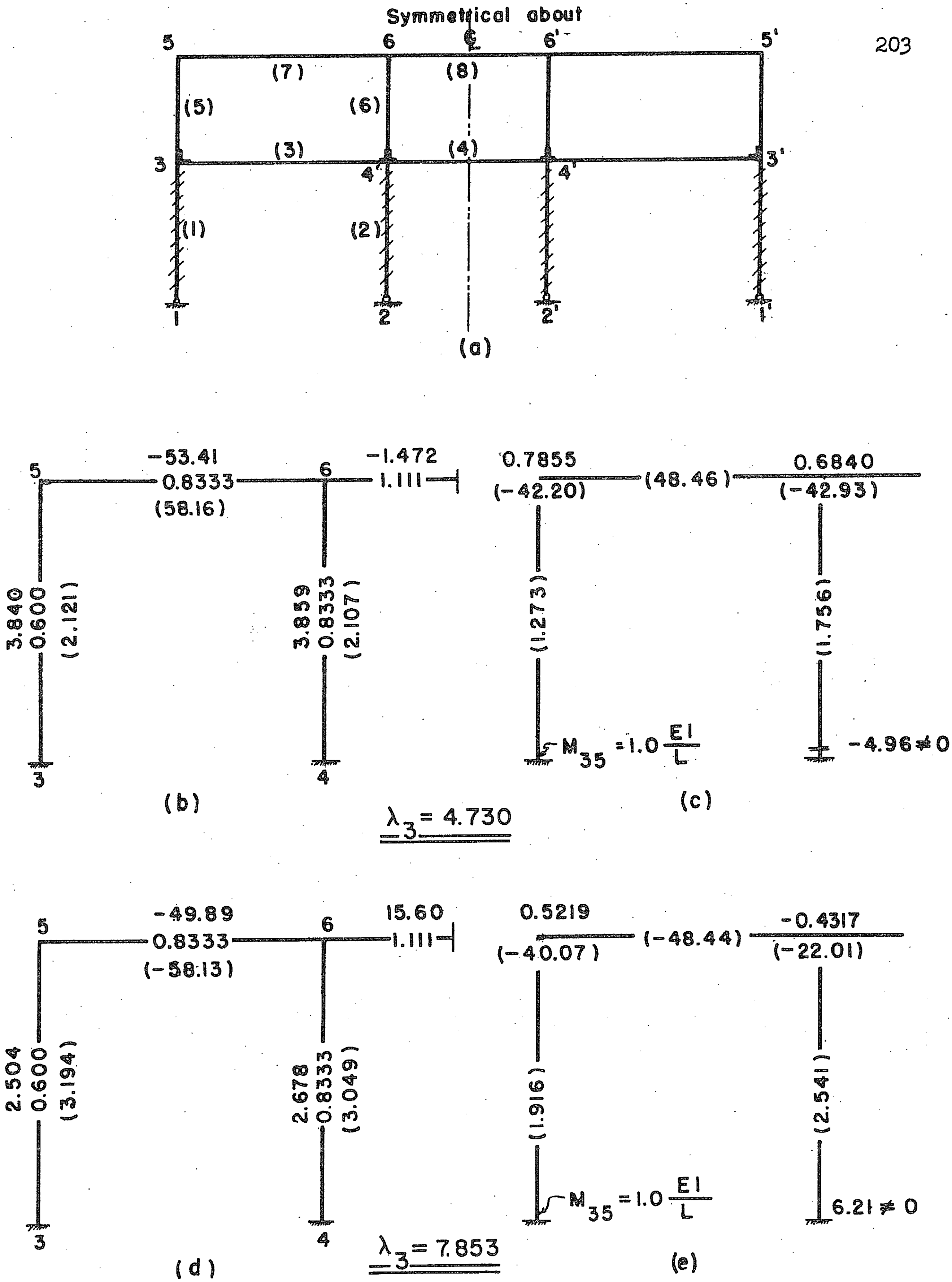


FIG. 34 ILLUSTRATIVE EXAMPLE 6 (CONTINUED)

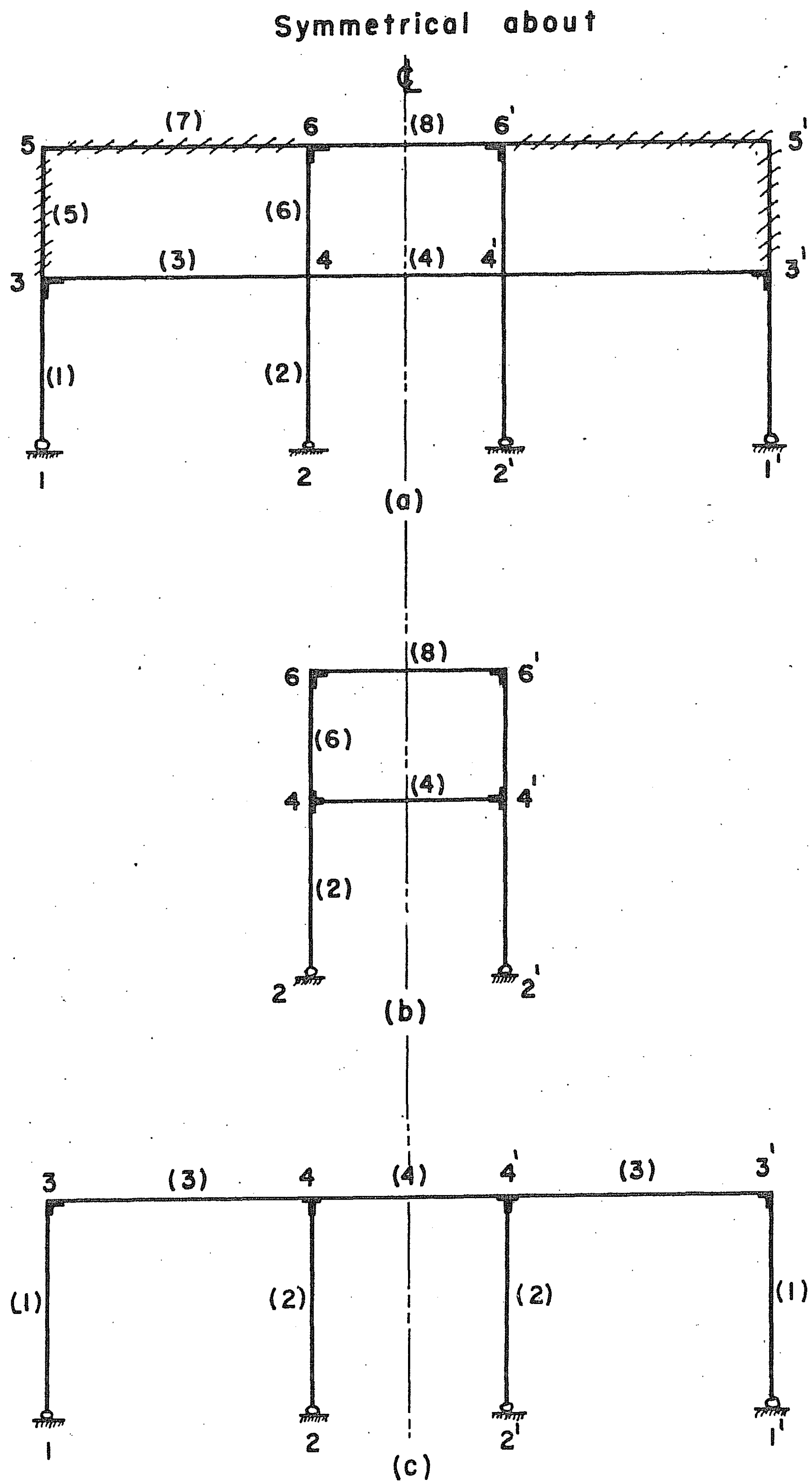


FIG. 35 ILLUSTRATIVE EXAMPLE 6 (CONTINUED)

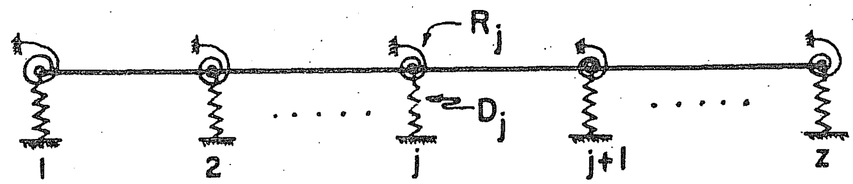


FIG. 36 TYPICAL CONTINUOUS BEAM ON FLEXIBLE SUPPORTS

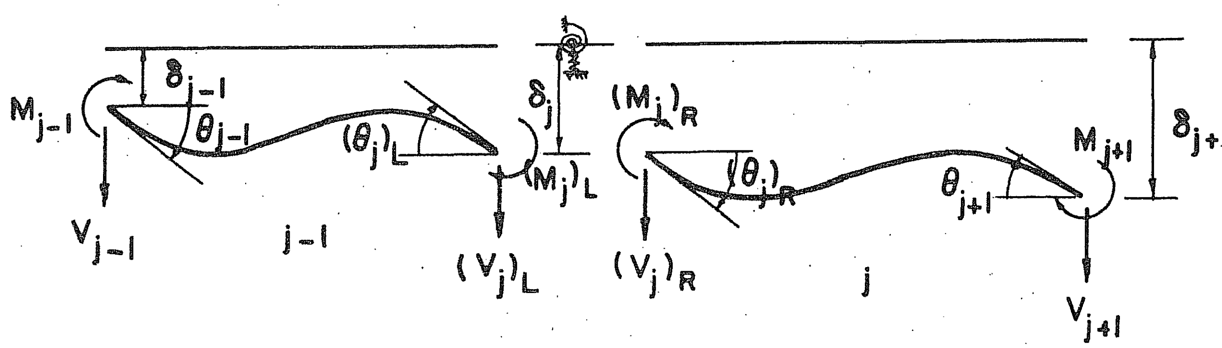


FIG. 37 SPANS j-1 AND j OF A CONTINUOUS BEAM ON FLEXIBLE SUPPORTS

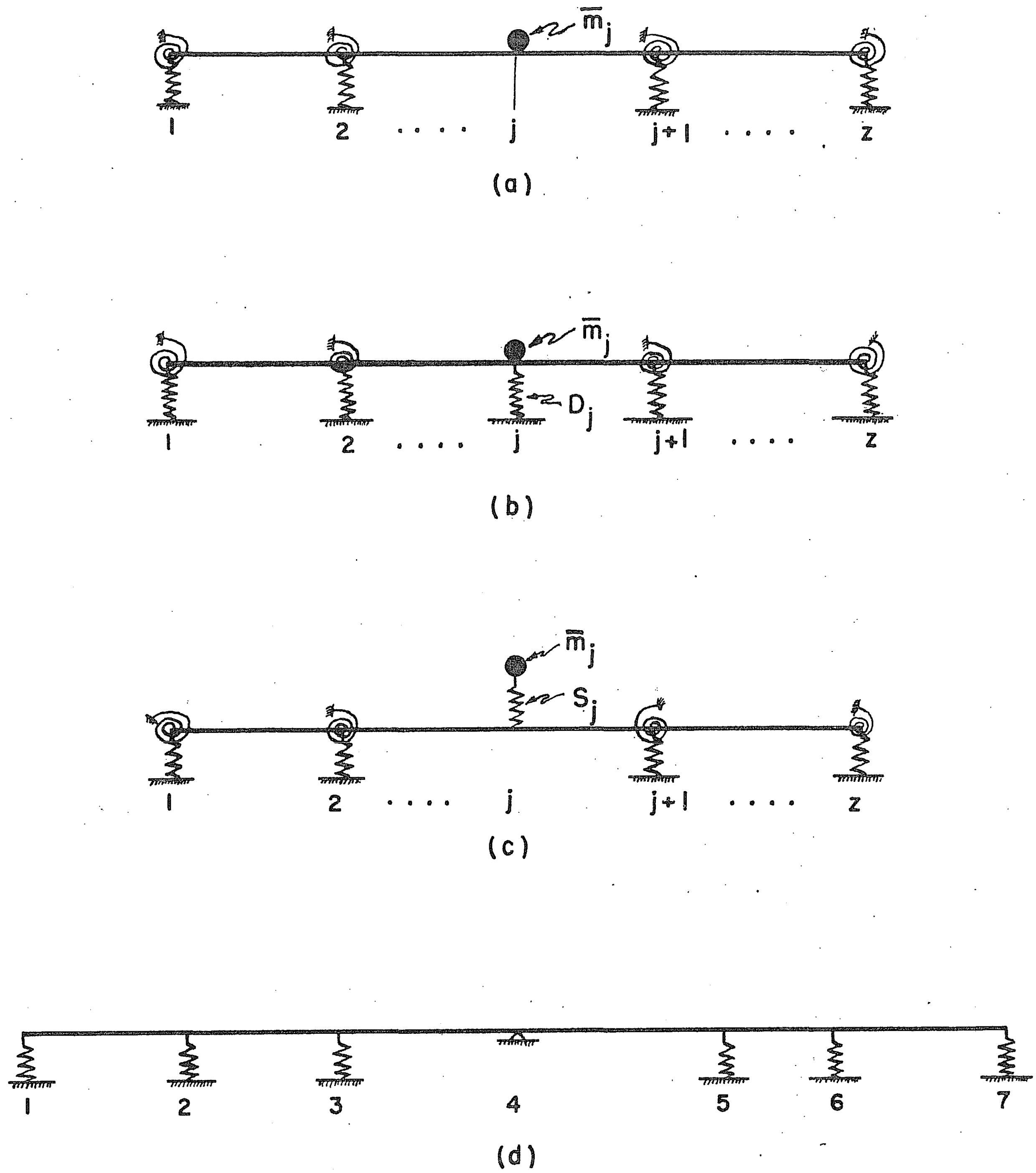
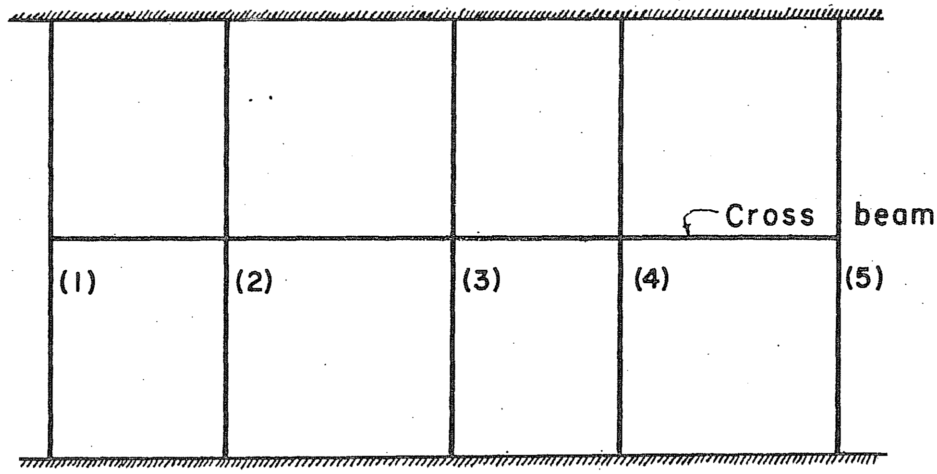
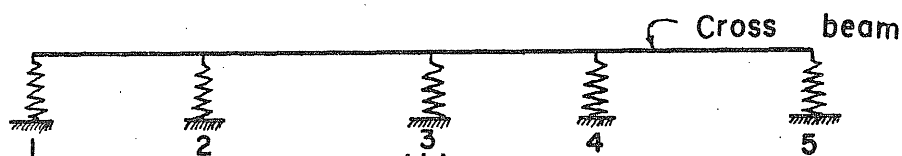


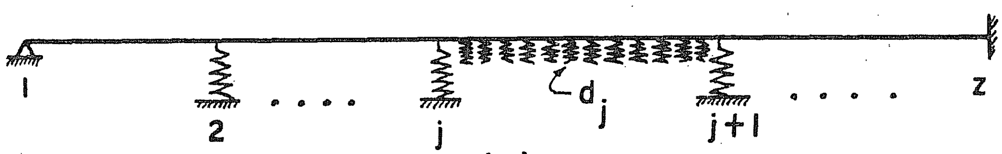
FIG. 38 BEAMS WITH VARIOUS INTERMEDIATE CONSTRAINTS



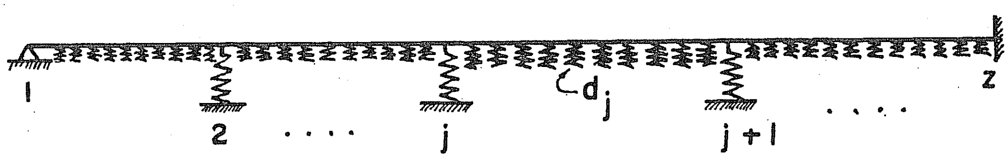
(a)



(b)



(c)



(d)

FIG. 39 BEAMS WITH VARIOUS INTERMEDIATE CONSTRAINTS

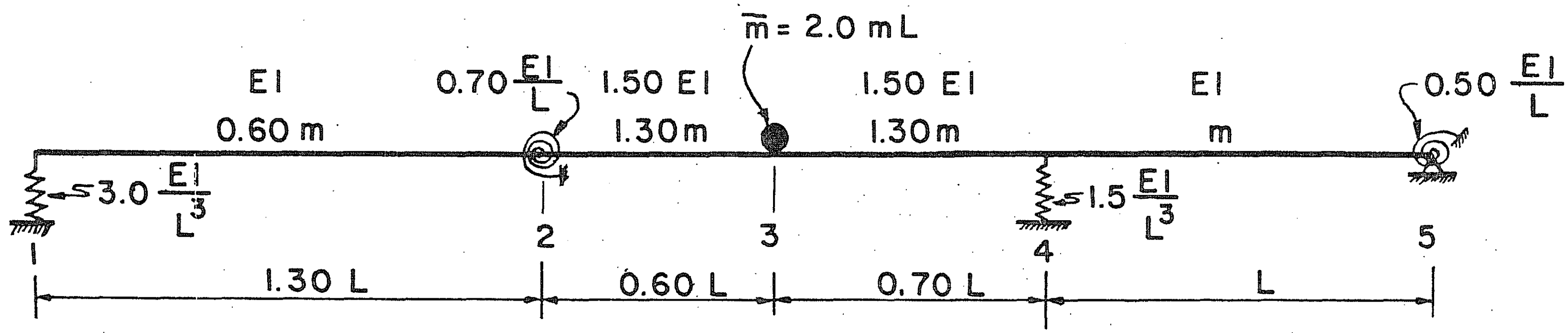


FIG. 40 PROPERTIES OF BEAM CONSIDERED IN EXAMPLE 7

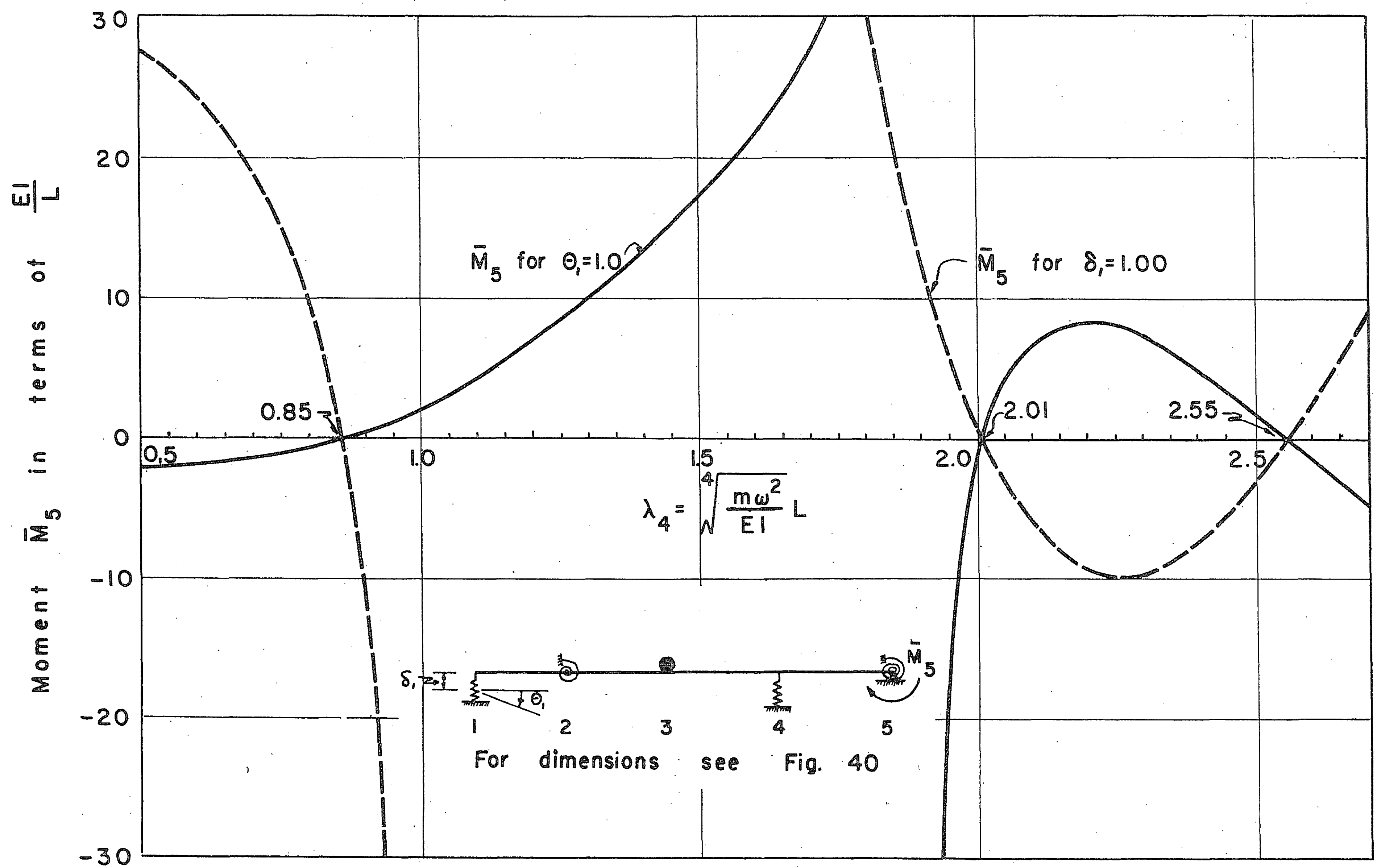


FIG. 41 VARIATION OF EXCITING MOMENT $\bar{M}_5$ AS A FUNCTION OF λ_4, EXAMPLE 7

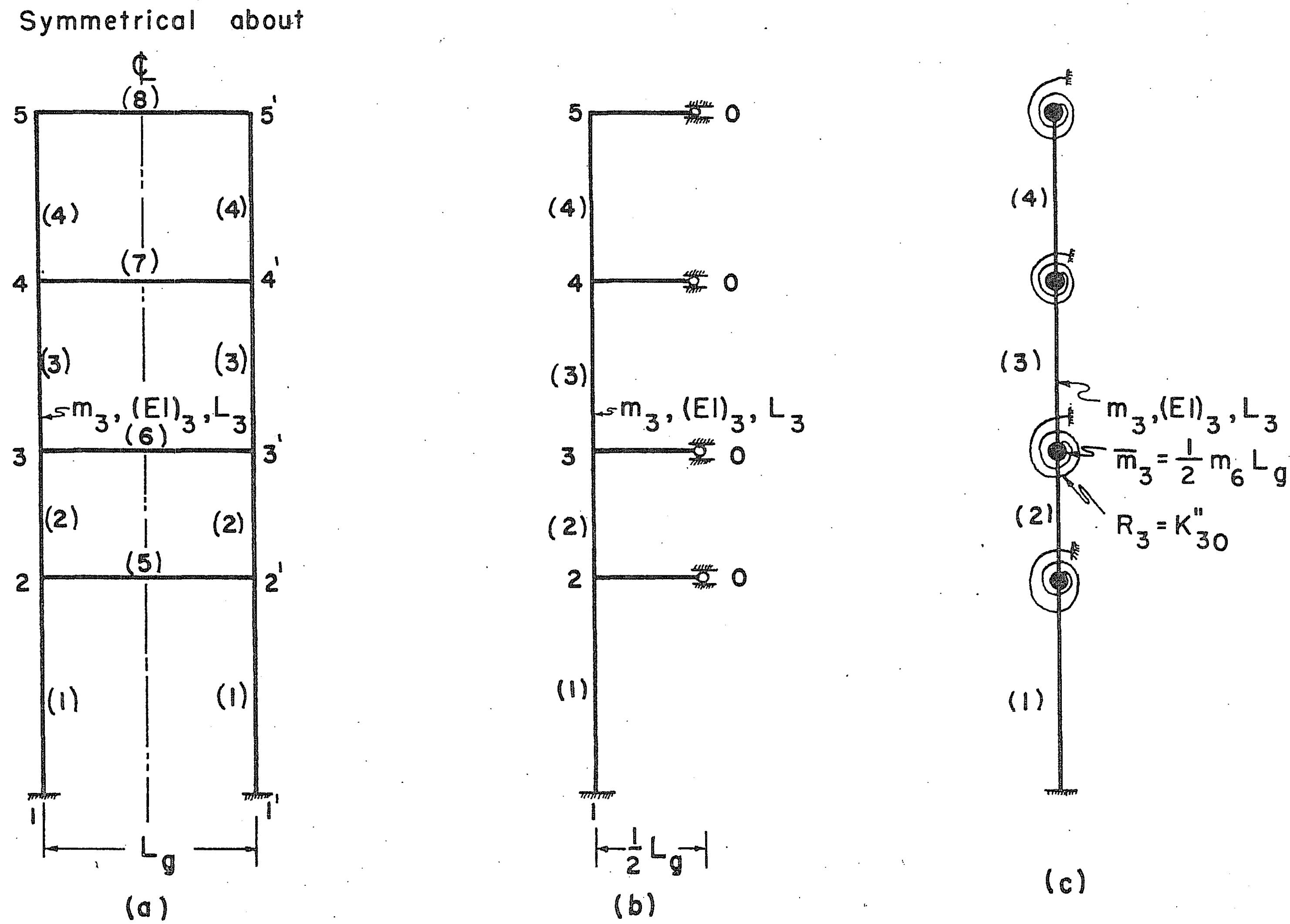


FIG. 42 SYMMETRICAL SINGLE-BAY FRAMES WITH SIDESWAY

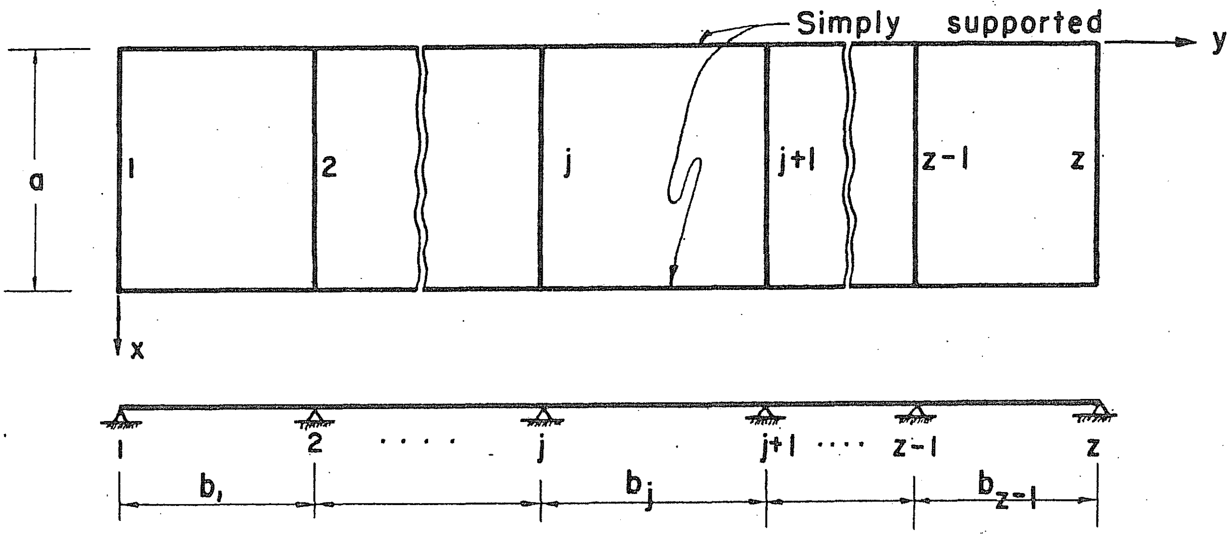


FIG. 43 TYPE OF CONTINUOUS PLATE CONSIDERED

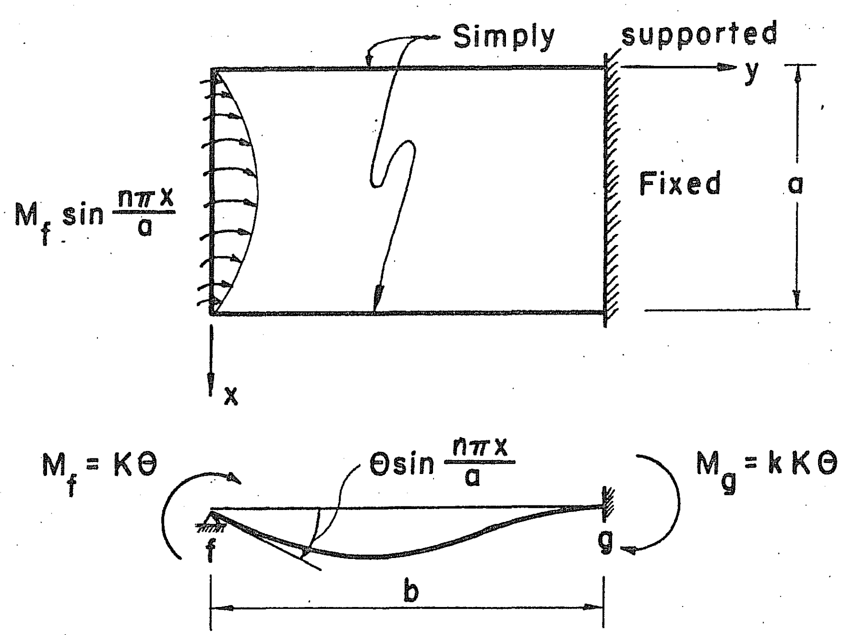


FIG. 44 ELASTIC CONSTANTS FOR A PANEL OF A SLAB ON RIGID SUPPORTS

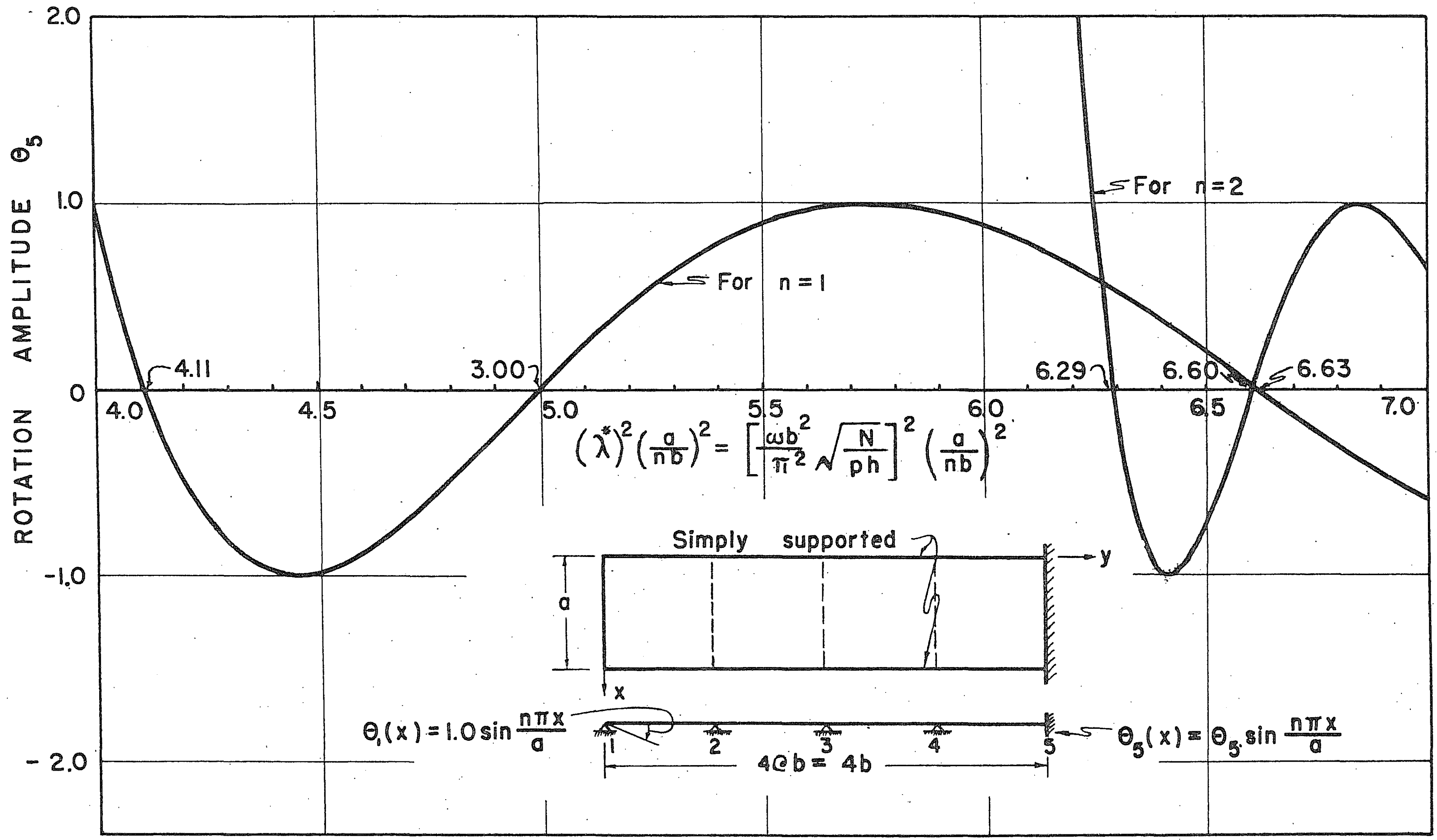


FIG. 45 VARIATION OF ROTATION AMPLITUDE Θ_5 WITH ω, FOR EXAMPLE 8