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A SIMPLE APPROXIMATION FOR THE NATURAL FREQUENCIES OF PARTLY RESTRAINED BARS

By
N. M. NEWMARK and A. S. VELETSOS

Technical Report
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OFFICE OF NAVAL RESEARCH
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Project NR-064-183

UNIVERSITY OF ILLINOIS
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University of Illinois

This paper presents a simple approximate formula for the natural frequencies of flexural vibration of a bar which rests at each end on non-deflecting supports but which is elastically restrained against rotation at the ends. The bar is of constant cross-section and density. The end restraints are assumed to be proportional to the end rotations, and the restraint stiffness may have any value in the range between perfectly hinged and completely fixed conditions. The restraints may be due to elements such as coil springs, or they may result from the continuity of the bar with adjoining members.

Consider the bar to be oscillating in one of its natural modes of free vibration. Let θ_L and θ_R be the angular rotations of the bar at the left and right ends respectively, and let $\underline{M_L}$ and $\underline{M_R}$ be the moments at the corresponding ends. The relation between moments and rotations at the ends is expressed by the equations:

$$M_L = K_L \theta_L \qquad \text{and} \qquad M_R = K_R \theta_R \tag{1}$$

where $\underline{K_L}$ and $\underline{K_R}$ represent the stiffnesses of the restraints at the left and the right ends. The stiffness $\underline{K}$ is defined as the moment necessary to rotate the spring restraint, or the restraining member which it may symbolize, by a unit amount. A restraint is referred to as positive when the moment which it exerts on the bar acts in a direction opposite to the direction of rotation of the bar; when the reactive moment acts so as to augment this rotation, the restraint is referred to as a negative restraint. A positive restraint, therefore, resists the tendency of the bar to rotate.

It is convenient to express the stiffnesses of the restraints by dimensionless coefficients which are related to the characteristics of the bar as follows:

$$\beta_L = K_L L/EI \qquad \text{and} \qquad \beta_R = K_R L/EI \tag{2}$$

where

L = the length of the span;

E = the modulus of elasticity of the material in the bar; and

I = the moment of inertia of the cross-section of the bar about its centroidal axis.

In terms of the coefficients β_L and β_R it is possible to derive "exact" values of the natural frequencies of flexural vibration. These appear as the roots of transcendental equations involving trigonometric and hyperbolic functions. In general the roots are tedious to evaluate, and the formulas do not give a simple picture of the mechanism of the action of the bar. Graphs or tables are required to obtain useful values of the frequencies for a range in the parameters. The frequencies can also be expressed by approximate formulas.

It is convenient to state the frequencies of the elastically restrained bar as the product of a dimensionless coefficient $\underline{C_n}$ multiplied by the fundamental frequency $\underline{f_o}$ of the same bar with hinged ends:

$$f_n = C_n f_o = C_n \frac{\pi}{2L^2} \sqrt{EI/m} \tag{3}$$

where E, I, and L are as previously defined, m is the mass per unit of length of the bar, and f_n is the n-th frequency of the bar in its elastically restrained condition, expressed in cycles per second.

From the results of numerical calculations based on the exact solutions the following simple approximation was developed for the coefficient C_n:

$$C_n \cong \left[n + \frac{1}{2} \left(\frac{\beta_L}{5n + \beta_L}\right)\right] \left[n + \frac{1}{2} \left(\frac{\beta_R}{5n + \beta_R}\right)\right] \tag{4}$$

In this expression, n is the order of the desired natural frequency and β_L and β_R are the restraint coefficients as defined by equation (2). For the fundamental frequency, n = 1. In general, for the n-th mode there are n waves, alternately upward and downward, in the deflection curve along the length of the bar.

The foregoing formula is applicable to bars having positive end restraints. It is valid for the fundamental as well as the higher natural frequencies and it is accurate within a maximum error of 4 per cent, with the maximum error occurring in the lowest or fundamental frequency.

For the first and second natural frequencies, the percentage error resulting from the use of this approximate formula is indicated in Fig. 1 in the form of a contour map. Since the error curves are symmetrical with respect to a diagonal line, only one-half of the contour map is plotted for each frequency. The curves above the diagonal correspond to the first natural frequency while those below the diagonal pertain to the second natural frequency. Because of symmetry, it is possible to specify that the subscript L always refers to the larger end restraint. The scale of the coordinates is chosen so that the entire range of positive restraints, from zero to infinity, is represented. A positive error indicates that the frequency given by equation (4) is too high.

It can be observed that the maximum error in the second frequency is negligible for all practical purposes. The errors become progressively smaller for the higher modes. The greatest error is obtained for a bar fixed at one end and hinged at the other, where the fundamental frequency given by equation (4) is 1.50 f_o, and the true value is 1.56 f_o. The large errors in the first mode are concentrated over an extremely small region, which corresponds to the cases of bars with one end practically unrestrained and the other end subject to a very stiff restraint. With the exception of this localized region, the maximum error over the remaining domain of end fixities is consistently less than 2 per cent.

When the end restraints are very small, i.e. for values of β approaching zero, the value of C_n is expressed ~~more conveniently~~ by the relation:

$$C_n = n^2 + 0.1\ (\beta_L + \beta_R) \tag{5}$$

Equation (5) is in error by less than 1.5 per cent when $\beta_L + \beta_R$ is less than 1.0, and the error is always positive.

The approximate formulas are based on exact solutions derived from the ordinary theory of flexure of beams, which neglects the effects of shearing deformation and of rotatory inertia. Because the influence of these neglected factors may become significant in the higher modes, the present formulas should not be used without regard for the limitations imposed by the ordinary theory of flexure.

This paper is based upon an observation of the senior author. The approximate equations are similar to those given in a previous paper* relating to buckling loads for partly restrained bars. The calculations and detailed study were made as a part of a program sponsored by the Office of Naval Research (Mechanics Branch) in the Structural Research Laboratory, Department of Civil Engineering, of the University of Illinois.

*N. M. Newmark, "A Simple Approximate Formula for Effective End-Fixity of Columns," Journal of the Aeronautical Sciences, Vol. 16, No. 2, February 1949, p. 116.

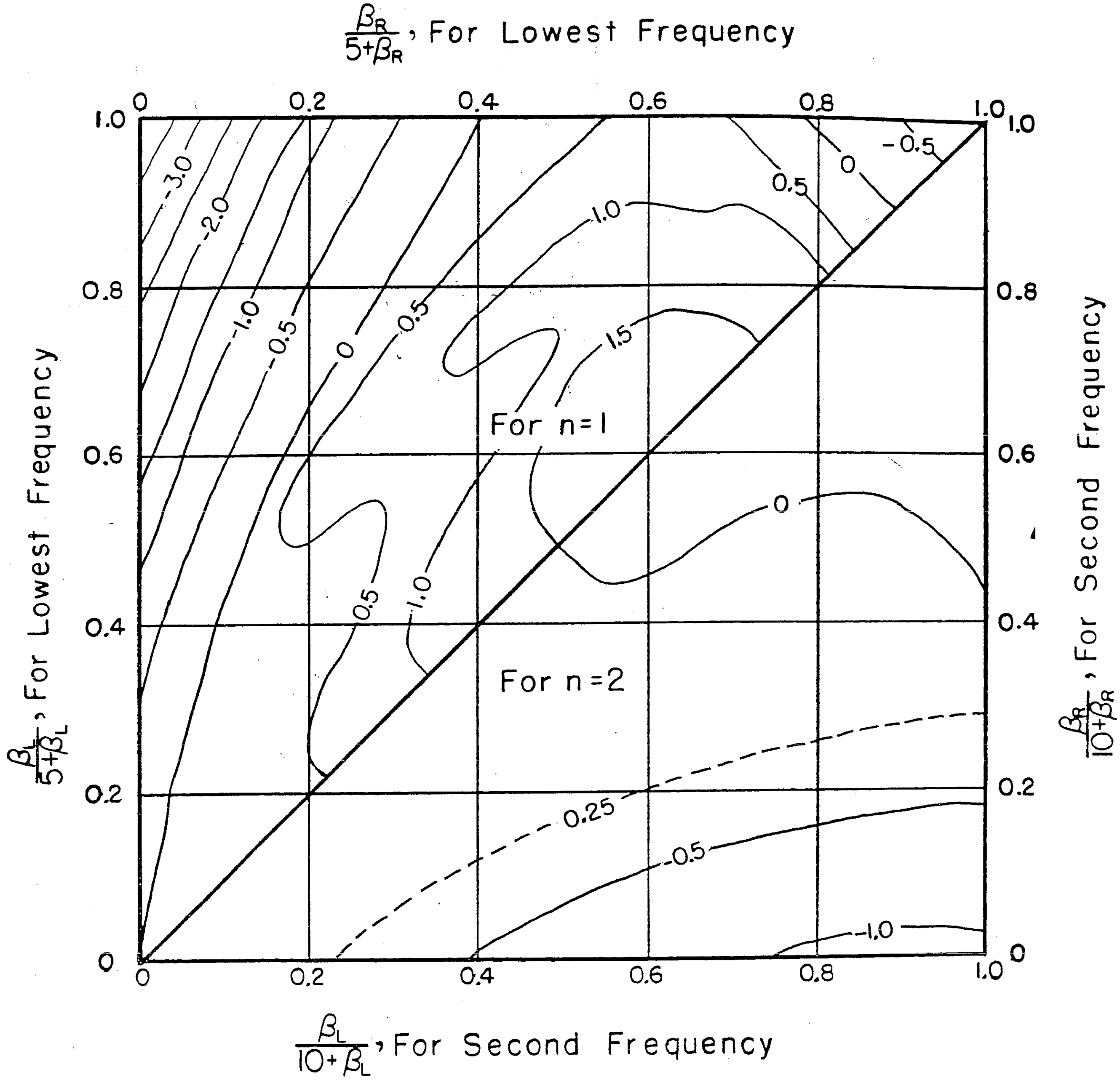

FIG.1. CONTOUR PLOTS OF PERCENTAGE ERROR IN NATURAL FREQUENCY AS DETERMINED BY EQUATION (4)