UILU-ENG-73-2004

CIVIL ENGINEERING STUDIES
STRUCTURAL RESEARCH SERIES NO. 396

APPROXIMATE DYNAMIC RESPONSE OF LIGHT SECONDARY SYSTEMS

By
T. NAKHATA
N. M. NEWMARK
W. J. HALL

A Technical Report
on a Research Program Supported
by the Department of Civil Engineering
through John I. Parcel Funds

UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS
MARCH 1973

APPROXIMATE DYNAMIC RESPONSE OF LIGHT SECONDARY SYSTEMS

by

T. Nakhata

N. M. Newmark

and

W. J. Hall

A Technical Report

on a Research Program Supported

by the Department of Civil Engineering

through John I. Parcel Funds

UNIVERSITY OF ILLINOIS

URBANA, ILLINOIS

MARCH 1973

ACKNOWLEDGMENT

This report is based on the dissertation of T. Nakhata under the supervision of Dr. N. M. Newmark, Professor and Head of the Department of Civil Engineering, and Dr. W. J. Hall, Professor of Civil Engineering. The research was supported by the Department of Civil Engineering through the John I. Parcel funds.

Numerical results were obtained through the use of IBM 360/75 computer facilities of the Department of Computer Science, with funds for the computer time provided by the University Research Board.

TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

1.1 General

Secondary systems such as light appendages (penthouses, elevator housings, piping systems, etc.) in nuclear power facilities or other structures are usually designed after the primary system (building, nuclear reactor, etc.) design is completed. The secondary system is supported by connections to the primary structures. Under earthquake motion the base of the primary system tends to move with the ground, and the secondary system support motions follow the motion of the primary structure at the points of attachment.

In general, earthquake motion is relatively rapid, thus having great potential of causing severe damage to improperly designed structures. When the secondary system is considered stiff in comparison to the primary structure, it accelerates with the same motion as its support. However, for a very flexible secondary system, the differential motions between the masses and its base are large. Between these two limiting conditions the secondary system must be designed in a balanced fashion in order to survive the dynamic motion. Explicitly, the secondary system must be strong enough to resist the force acting on it as well as ductile enough to deform without collapse. The amount of strength and ductility is controlled by proper selection of the stiffness or flexibility of the system itself.

Several factors complicate the response analysis of the dynamical system; for example, the number of degrees of freedom required for reasonable modeling may be large, thus giving rise to problems of data processing, even in modern computing facilities. Another major factor is the lack of complete knowledge about the earthquake hazard for which these components should be designed.

The effects on the secondary system depend not only on the earthquake motion but also on the properties of the primary structure and on the properties of the secondary system itself. The most important properties are the energy absorption of each system and its support, its natural frequency and strength or resistance.

Due to the unpredictability of future earthquakes, current methods of seismic analysis often specify the design earthquake by a smooth response spectrum. This permits an analysis to be made of the primary system but provides no information for the design of the secondary systems.

Since the secondary system is generally a light system in comparison to the supporting structure, it is quite reasonable and convenient to have simple rules for the approximation of the strain and/or maximum motion response of the secondary system as a function of the separate parameters of the primary and secondary systems. If such rules can be developed and are applicable, they offer at least two advantages in the preliminary design stage.

1. The secondary system can be analyzed for any type of base motions of the primary system.

2. By considering the primary and secondary systems separately, computational difficulties of the combined system, due to different orders of magnitude of the masses and stiffness elements, are avoided.

1.2 Object and Scope

In recent years, considerable studies have been carried out towards the finding of approximate procedures which can be employed in the preliminary design of secondary systems. The procedures which have been suggested in many of the past studies involved decoupling of the primary and secondary systems. But in most cases this is not possible and interaction effects must be considered if reasonable results are to be achieved.

The main objective of this research involves the study of the dynamic response of the secondary system, whereby the interaction effects are taken into account.

In order to permit an extensive parameter variation, relatively simple analytical models (mass-spring systems) are used. In this study, both primary and secondary systems remain in the elastic range, with the primary system being a multiple-degree-of-freedom system subjected to base disturbances, while the secondary system ranges from a single-degree-of-freedom system to a multiple-degree-of-freedom system, but is attached at only one point to the primary structure. Attention is paid to the response of the secondary system, which is influenced to a major extent by the nature and type of the response of the primary element on which it is supported. The effect of damping upon response is also investigated in detail.

After various parameters of interest have been carefully studied, reasonable ranges of these parameters will be developed. The purpose of this part of the study is to arrive at approximate procedures which can be employed in the preliminary design of secondary systems and/or to estimate the adequacy of such systems, pending more elaborate analysis.

1.3 Review of Related Work

Various procedures employing the assumption of neglecting interaction effects between primary and secondary systems have been suggested to simplify the design of light secondary systems mounted on a responding structure subjected to earthquake or other dynamic motions.

Newmark (1) gave basic design criteria for very light subsystems. Based on the forced response of separate two-degree-of-freedom systems, one for each mode of the primary structure, Penzien and Chopra (2) presented an approximate method for a single-degree-of-freedom secondary system. Another simple design procedure has been suggested by Biggs and Roesset (3). Two assumptions were employed in developing this latter method. Firstly, the significant input to the equipment consists of a series of damped harmonics, each of which corresponds to one of the normal modes of the structure. Secondly, the most significant harmonic components of the earthquake motion with respect to the equipment are those which are in near resonance with the equipment. Sato (4) simulated the building-machine structure by an idealized two degree-of-freedom system, and characterized the response spectrum of a single-degree-of-freedom secondary system when the primary structure was subjected to earthquake motion.

A considerable amount of work has been carried out to study the response characteristics of a secondary system with two-end connections. Shibata et al. (5) treated each portion of a piping system as a simple beam and simulated the whole structure (piping and building) as a two-degree-of-freedom system. Sato and Suzuki (6) investigated the dynamic response characteristics of simple building-machine structures subjected to two seismic motions with certain time-lag intervals. Nakagawa et al. (7) employed the principle of superposition and developed a method of modal analysis of response of a structure subjected to two different earthquake input motions at its two supporting points.

Attempts have been made to characterize the response of a secondary system treated as a continuous beam. Included in this list are Watari et al. (8) who used a transfer matrix formulation, and Shimizu and

Shibata (9) who analyzed a piping system subjected to multirandom input.

Hart et al. (10) discussed several modal synthesis procedures for the dynamic analysis of large composite structural systems, and also gave schematic flow charts of the analysis procedure used in prominent methods.

Berkowitz (11) performed an analysis of a primary piping system by treating a reactor vessel and attached piping as a single coupled lumped mass model.

Kassawara (12) investigated earthquake response of multiply connected light secondary systems by spectrum techniques.

As indicated earlier, reliable results can be obtained only when the interaction between the primary and secondary systems is taken into account properly. The interaction is determined not only by the mass ratio, as postulated by many authors, but also by the damping coefficients of both primary and secondary systems. A sufficient condition for neglecting the interaction has been obtained by Caughey (13) who performed qualitative mathematical analysis of various approximate schemes.

The consideration of a single-degree-of-freedom primary system and a single-degree-of-freedom secondary system as a coupled two-degree-of-freedom system directly includes all interaction effects. Newmark et al. (14) indicated that the maximum amplification factor, even when the light secondary system was tuned to a frequency of the system on which it was supported, could not exceed the square root of the ratio of the effective masses* of the primary and secondary systems.

In a recent study by Newmark (15), involving a multiple-degree-of-freedom primary system and a single-degree-of-freedom secondary system, the amplification factors at resonance are shown to be affected by both

---

* The effective mass is defined in Chapter 2.

the damping factor and the effective mass ratio, but details of these combined influences have not yet been investigated.

Since the concept of effective mass ratio seems to be quite promising, it is used throughout the research conducted here.

1.4 Notation

Most of the symbols used in this report are defined when they first appear. However, a summary of frequently used symbols is also presented below for convenience.

k = spring constant factor

m = mass constant factor

K = stiffness matrix

M = diagonal mass matrix

$\bar{K}$ = stiffness matrix of the combined system

$\bar{M}$ = diagonal mass matrix of the combined system

p = first natural frequency of the secondary system

P = first natural frequency of the primary system

ω_{si} = i^{th} circular frequency of the secondary system

ω_{pi} = i^{th} circular frequency of the primary system

T = shortest natural period of the system considered

T_d = duration of the constant acceleration pulse

C_{pi} = i^{th} primary participation factor

C_{sj} = j^{th} secondary participation factor

$\bar{\alpha}_j$ = vector of the j^{th} mode shape of the secondary system

α_j = vector of the j^{th} mode shape of the secondary system normalized in such a way that the participation factor is equal to unity

$\bar{U}_i$ = vector of the i^{th} mode shape of the primary system

U_i = vector of the i^{th} mode shape of the primary system normalized in such a way that the participation factor is equal to unity

$\alpha_j(k)$ = k^{th} element of α_j

$U_i(k)$ = k^{th} element of U_i

u = modal displacement

s = modal spring distortion

S = secondary spring distortion

β = effective damping factor

E_{sj} = j^{th} effective secondary mass

E_{pi} = i^{th} effective primary mass

δ, γ = effective mass ratio

A_{ji} = amplification factor of the secondary system response due to the effects of the j^{th} mode of the secondary system and the i^{th} mode of the primary system

A.F. = total amplification factor of the secondary system response

X = vector of the absolute displacement of the combined system

Y = vector of the absolute ground displacement

Z = vector of the relative displacement of the combined system with respect to the ground

D = spectral displacement

V = pseudo-spectral velocity

A = pseudo-spectral acceleration

CHAPTER 2

METHOD OF ANALYSIS

2.1 Introductory Remarks

A sketch of a mass-spring secondary system mounted on a mass-spring primary system is illustrated in Fig. 2.1. Some of the reasons for not considering designing a structure with substructures as a single unit had already been stated in section 1.1. One additional reason is that sometimes this approach may lead to unreliable results because of the excessive number of degrees of freedom. Nevertheless, for comparison purposes, the solutions by this approach are regarded as exact solutions here.

The assumption of decoupling has been employed by many investigators in developing approximate procedures for the design of secondary systems mounted on a responding structure subjected to earthquake or other dynamic motions. Since this assumption (the same as neglecting interaction effects) is shown to be invalid in most cases (Ref. 13), a new approach which incorporates interaction effects should be investigated.

The concept of effective mass ratio* was first utilized by Newmark (14, 15). In Ref. 14, a mass-spring secondary system supported by a single mass-spring primary system was investigated. The spring distortions for the two modes of the combined system were considered to be added in numerical values. Hence, this approach gives the upper bound of the secondary system response. In Ref. 15, the approximate procedures for a single mass-spring secondary system mounted on a multiple-degree-of-

---

* The effective mass ratio is defined by Eq. (2.38). It is worth noting that when both primary and secondary systems are single-degree-of-freedom systems, the effective mass ratio is the same as the mass ratio.

freedom primary system had been presented. Heuristic relationships had also been given for a more complex secondary system with and without damping. Since this effective mass ratio approach considered the entire system as a single unit, therefore, interaction effects were included automatically.

In this report, the effective mass ratio concept is employed for the approximate analysis of more complex primary and secondary systems (multi-degree-of-freedom systems). Information concerning the exact analysis is provided in section 2.3.3, while the approximate analysis is discussed in section 2.3.4.

2.2 Details of the Study

2.2.1 Systems Considered

Before a real dynamic system can be analyzed it must be represented by a physical (or mathematical) model to define its masses, resistance, damping, strength and energy absorbing capacity. To obtain a basis for extrapolation, and at the same time to permit considering a number of important conditions, primary and secondary systems shown in Figs. 2.2 and 2.3, respectively, are investigated. Both primary and secondary systems are composed of lumped masses and linear springs. Combined systems formed by attaching the secondary to the primary systems are also illustrated in Figs. 2.4 and 2.5. In order to proceed with the description of the study presented here, it is first necessary to clarify what the primary, secondary and combined systems really represent.

The primary system, representing the building (or nuclear reactor, etc.) which provides support for the secondary system, is modeled as a singly supported shear-beam system (Fig. 2.2).* A three-degree-of-freedom nonuniform primary system is used. This particular system is carefully proportioned to enable the natural frequencies to be spread wide enough relative to each other.

Setback portions of a building or equipment located within the nuclear reactor building or a light appendage (penthouse, elevator housing, etc.) can be considered as a mass-spring secondary system supported by the primary structure. Figs. 2.3a and 2.3b illustrate a single-degree-of-freedom and a nonuniform two-degree-of-freedom secondary systems, respectively.

Natural frequencies and mode shapes of the primary and both secondary systems are listed in Tables 1a, 1b and 1c, respectively. The mode shapes have already been normalized such that the participation factors are unity (the purpose of this normalization will be clarified later on in this chapter).

By appropriate choice of the relative magnitude of the masses and spring constants, the frequencies of the systems considered can be adjusted to any desired values (See Appendix A). The three frequencies of the primary system are kept constant at 1, 2 and 3 hertz, respectively. Once the masses and spring constants of the primary system are chosen, the masses of the secondary system are defined by selecting the desired effective mass ratio (Appendix A). The fundamental frequency of the secondary system can now be defined through selection of the proper spring constants.

---

* This limitation is not significant. The theoretical approach is the same for more complex systems. The limitation does simplify the analysis and permits a wider range of parameter variation to be studied.

The entire structural system (building and appendage, etc.) can be represented by combinations of primary and secondary systems. In this study, the secondary system is limited, by being attached at only one point to the main structure. The purpose of this limitation is for simplicity, and to allow an extensive study to be made of the behavior of the secondary system. The supports for the secondary systems are individual masses of the shear-beam primary system. In order to study the effect of the support point upon the response of the secondary system, the location of the connection points will be varied. Figures 2.4 and 2.5 show four- and five-degree-of-freedom combined systems, respectively. The bases of these systems will be excited by the ground motion.

2.2.2 Parameters Studied

The physical parameters necessary for defining the system are the mass values and spring constants for the primary and secondary systems and the connectivity of the structure. For a specific primary or secondary system, frequency ratios remain constant and the only parameter necessary to define the system is the magnitude of one of the natural frequencies.

One of the objectives of this study is to observe the secondary spring distortion bounds as the physical parameters are varied.

One parameter most likely to affect the secondary spring distortion bounds is the effective mass ratio of the secondary to the primary systems. The secondary response chosen for observation is the maximum distortion in each of the secondary springs. This response is chosen because it provides a measure of the maximum strain at various portions of the system. The results of the study, then, are plots of $S/(V/p) = S/D = S/(A/p^2)$ vs $\log(p/P)$, where S = secondary spring distortion, D, V, and A are spectral displacement, velocity and acceleration for the ground motion applied to the primary system at frequency p, respectively, and p = fundamental frequency of the secondary system, P = fundamental frequency of the primary system.

The other significant parameter affecting the dynamic response of a system subjected to transient disturbances is the loss of energy involved in damping, hysteresis, or other mechanisms. The nature of the structure itself is not the only factor used in determining the energy absorbing capacity from damping. The type of joints or connections within the structure, the mechanism at the interface between the structure and its support, the level of stress or deformation permitted when the structure undergoes dynamic motion, etc., all these factors contribute significant portions to the total amount of damping.

Since damping is a very complicated matter, only limited information is available. Values for the design levels of damping for different types of structure are suggested in Refs. 15, 16 and 17. Discussion of various types of damping commonly employed in dynamic analysis of structural systems is available in Ref. 14.

By choosing damping* proportional to stiffness, the damping ratios are proportional to mode frequencies. Certain fractions of critical damping for the fundamental mode of vibration will be assigned to both primary and secondary systems. Using the relations of Eqs. (2.1) and (2.2), the required damping constant can be obtained.

$$R = \xi k \tag{2.1}$$

$$\xi = \frac{2\beta_1}{\omega_1} \tag{2.2}$$

---

* With this type of damping, the relative contributions of higher modes of vibration in the response are negligible. There is no intention to convey the thought that this type of damping is most appropriate for structural systems.

where

k = spring constant

ξ = a constant

R = damping constant

β_1 = fraction of critical damping for the fundamental mode of vibration

ω_1 = fundamental frequency

In summary, the following parameters are studied herein.

a) p/P, ratio of the fundamental frequency of the secondary system to the primary system. Selection of the fundamental frequency is sufficient to define all the frequencies and mode shapes of the system for which the relative magnitudes of masses and spring constants have been chosen.

b) γ, effective mass ratio. This parameter indicates the relative size of the secondary system with respect to the primary structure.

c) β, fraction of critical damping. This parameter indicates the effect of damping on the response of the system (see section 4.6.2).

d) $S/(V/p)$, amplification factor. This parameter indicates the relative magnitude of the secondary spring distortion to the response spectrum value.

2.2.3 Base Accelerograms Considered

2.2.3.1 Constant Acceleration Pulse of Specific Time Duration

A 0.5g acceleration of 0.155 sec. duration is used as the base accelerogram in this investigation. This ground motion gives a nearly constant velocity spectrum of approximately 30 in/sec from very low frequency up to about 2 cps; at that point a transition occurs. Beyond the transition zone (2-3 cps) the response spectrum is a constant acceleration of magnitude 1.0g (one way of calculating this response spectrum is discussed in Appendix B). Figures 2.6a and 2.6b show the pulse and the corresponding elastic response spectrum, respectively. With this selected base accelerogram, the spring distortion ratio will be computed for a combination of constant velocity and constant acceleration response spectra.

2.2.3.2 Earthquake Accelerogram

In order to substantiate the results obtained by the method developed herein, a dynamic analysis of a system subjected to an earthquake accelerogram was performed. The record used is the N21E component of the Taft, California record of 7/21/52. This accelerogram was selected because its response spectrum (Fig. 2.7) closely resembles a constant velocity spectrum in the intermediate frequency range. The record had been adjusted for base line position and the acceleration is assumed to have a linear variation between two consecutive acceleration time points. Zero ground acceleration was added at the end of the record to account for the free vibration response in the maximum response calculation. The time duration of this zero ground acceleration was arbitrarily taken as one-half of the longest natural period of the system considered.

2.3 Method of Analysis

2.3.1 General

The problem of the determination of the response of structures to prescribed transient forces in theory can be formulated and solved in very general terms, even for situations involving plastic deformation. Basic analytical methods are available in detail in Chapter 1 through 6 of Ref. 18. Due to the increasing effectiveness of high-speed digital computers, lengthy analyses of any system can be performed. However, this type of analysis is expensive; therefore, it is not generally suitable for preliminary design purposes.

In general, when modal analysis is used, the response of various modes can be combined by taking their absolute sum to obtain an upper bound, or by taking the square root of the sum of the squares of the modal responses to obtain the expected value (Ref. 19). For the approximate method developed herein, the absolute sum concept is employed.

A computer program was developed to perform all the necessary analysis. Given the physical parameters of the system (masses, spring constants, damping factor, connectivity, etc.), the program performs three main tasks:

a) Free vibration analysis of secondary, primary and combined systems -- Stiffness and mass matrices are generated and the eigen-value problem is solved, giving frequencies and mode shapes. The mode shapes are then normalized such that the participation factors are unity.

b) Time-history analysis of the combined system -- This is a dynamic analysis of an (m+n) degree-of-freedom system subjected to base excitation; where m and n denote the number of degree-of-freedom of the primary and secondary systems, respectively. The amplification factors of the secondary spring distortions are then plotted vs. the fundamental frequency ratios of the secondary to the primary systems.

c) Approximate analysis of the secondary system -- Secondary spring distortion bounds are computed by the approximate method developed later.

Besides the routine analysis mentioned above, this program is set to generate the response spectrum of the selected base accelerogram and then to plot the spectrum on tripartite logarithmic scales of maximum relative displacement, maximum pseudo-relative velocity and maximum pseudo-acceleration, against frequency of the system.

2.3.2 Equations of Motion (with Zero Damping)

In the following discussion, the subscript p will refer to the primary system and the subscript s to the secondary system. In Fig. 2.8, the absolute displacements of the primary and secondary masses are denoted X_p and X_s, respectively. The positive direction of displacements are shown in the same direction as the ground displacement.

The equations of motion of the complete structure are

$$\left[\begin{array}{c|c} M_p & 0 \\ \hline 0 & M_s \end{array}\right]\left[\begin{array}{c} \ddot{Z}_p \\ \hline \ddot{Z}_s \end{array}\right] + \left[\begin{array}{c|c} K_{pp} & K_{ps} \\ \hline K_{sp} & K_{ss} \end{array}\right]\left[\begin{array}{c} Z_p \\ \hline Z_s \end{array}\right] = - \left[\begin{array}{c|c} M_p & 0 \\ \hline 0 & M_s \end{array}\right]\left[\begin{array}{c} \ddot{Y} \end{array}\right] \tag{2.3}$$

or

$$\bar{M}\ \ddot{Z} + \bar{K}\ Z = -\ \bar{M}\ \ddot{Y} \tag{2.4}$$

where

$$\begin{aligned} Y &= \text{ground displacement} \\ Z = X - Y &= \text{relative displacement of the mass to the ground} \\ M &= \text{diagonal mass matrix} \\ K &= \text{stiffness matrix} \end{aligned} \tag{2.5}$$

$$\bar{M} = \left[\begin{array}{c|c} M_p & 0 \\ \hline 0 & M_s \end{array}\right] \quad ; \quad \bar{K} = \left[\begin{array}{c|c} K_{pp} & K_{ps} \\ \hline K_{sp} & K_{ss} \end{array}\right] \tag{2.6}$$

The partitioned stiffness matrix in Eq. (2.3) is a so-called "reduced" stiffness. This particular type of stiffness relates the forces and displacements in the direction in which information is required.

In Eq. (2.3), the dimension of the matrices M_p and K_{pp} are mxm, those of M_s and K_{ss} are nxn, that of K_{ps} is mxn, and that of K_{sp} is nxm, where m and n are the number of masses (i.e., number of degree-of-freedom) of the primary and secondary systems, respectively.

Equation (2.3) may be rewritten in a number of ways. It may be split up to reflect the primary and secondary systems as follows:

$$M_p \ddot{Z}_p + K_{pp} Z_p + K_{ps} Z_s = - M_p \ddot{Y} \tag{2.7}$$

and

$$M_s \ddot{Z}_s + K_{ss} Z_s + K_{sp} Z_p = - M_s \ddot{Y} \tag{2.8}$$

By limiting the secondary system to be attached at only one point to the primary structure, Eq. (2.8) may be rewritten as

$$M_s \ddot{\bar{X}}_s + K_{ss} \bar{X}_s = - M_s \ddot{X}_p(j) \tag{2.9}$$

or

$$M_s \ddot{X}_s + K_{ss} X_s = K_{ss} X_p(j) \tag{2.10}$$

where

$X_p(j)$ = absolute displacement of the j^{th} primary mass (support point of the secondary system)

$\bar{X}_s = X_s - X_p(j)$ = relative displacement of the secondary mass to the support point

2.3.3 Exact Analysis

2.3.3.1 Free Vibration Analysis

The equations of motion for free vibration are described by

$$M \ddot{X} + K X = 0 \tag{2.11}$$

For free vibration, the vector X is assumed to vary sinusoidally with time, $X = B \sin \omega t$, in which ω is the circular frequency, in radians per second. From Eq. (2.11)

$$KB = \lambda \ MB \tag{2.12}$$

where $\lambda = \omega^2$ is an eigen-value for free vibration problems, and B is the corresponding eigen-vector or mode shape. The solution to Eq. (2.12) can be found in standard texts in vibration theory (Refs. 20, 21, 22).

For most free vibration analyses of the combined system, the fundamental frequency of the secondary system was arbitrarily tuned to the fundamental frequency of the primary system (i.e., $\omega_{s1} = \omega_{p1}$).* As an example, mass and stiffness matrices of systems 2A, 2B and 2C for the resonant case and 1 percent effective mass ratio are given in Appendix A.

2.3.3.2 Time-History Analysis

Analytical solutions to Eq. (2.4) are well known (Refs. 20, 21, 22). In this investigation, numerical solutions are regarded as exact solutions in comparison to the solutions obtained by the approximate techniques. Equation (2.4) is integrated at discrete time intervals using a step-by-step procedure (Newmark's Beta-Method, Ref. 23). Beta = 0 is employed in this case. Before continuing the discussion, some notations should be introduced:

$$h_1 = 0.025T \tag{2.13}$$

---

* There is no intention to convey the thought that the tuning of fundamental frequencies, which is expected to produce the most resonance between the systems, will create the worst condition. It will be shown in Chapter 4 that the maximum amplification of the secondary response occurs off resonance, especially for high degrees of damping or large effective mass ratios.

$$h_2 = 0.025T_d \tag{2.14}$$

where

$$h_1, h_2 = \text{time interval}$$

$$T = \text{shortest natural period of the system considered}$$

$$T_d^{*} = \text{duration of the pulse}$$

During the pulse,** the time interval used is either h_1 or h_2, whichever is smaller. After the pulse, h_1 is employed. Regardless of the interval size, care is taken to insure that a response is computed at the very end of the pulse.

The integration has been carried long enough to obtain the maximum response (30 seconds for no damping, and decreasing in succession for higher damping values). Displacement responses are computed for the secondary masses and the corresponding maximum amplification factors are obtained.

2.3.4 Approximate Analysis

Although the symbolism has changed slightly, the analysis presented in this section follows the same procedure as originally discussed by Newmark (15). The analysis will be divided into two parts. The first part considers the case when one of the frequencies of the secondary system falls between two of the frequencies of the primary system. The second part deals with the case of tuning one of the frequencies of the secondary system to one of the frequencies of the primary system.

---

* For the 0.5g constant acceleration pulse of Fig. 2.6a, the duration is long enough to obtain 30 in/sec constant ground velocity.

** Refer to the pulse shown in Fig. 2.6a.

Before proceeding with the analysis, some useful relations should be introduced:

$$\omega_i^2 = \frac{A_i^T.K.A_i}{A_i^T.M.A_i} \tag{2.15}$$

where A_i = i^{th} mode shape.

When A_i is normalized so that the participation factor is unity, according to Eq. (2.16)

$$C_i = \frac{1.M.A_i}{A_i^T.M.A_i} \tag{2.16}$$

then, the normalized displacement of the i^{th} mode can be defined by B_i, where

$$B_i = C_i \; A_i \tag{2.17}$$

Since the participation factor is unity for the modal displacement B_i, one can obtain the following relation

$$1.M.B_i = B_i^T.M.B_i \tag{2.18}$$

By virtue of the orthogonality of the modal displacement B_i and B_j for mode i and mode j, it follows that

$$B_i^T.M.B_j = 0 \tag{2.19}$$

Several assumptions employed in the analyses described in the following two subsections are summarized below:

1) Only one frequency of the secondary system is equal to one of the frequencies of the primary system.

2) The significant input to the secondary system consists of a series of harmonic components with frequencies equal to the natural frequencies of the primary system.

3) The modal shapes of the secondary system remain the same,

except the magnitude has been enlarged by the result of the dynamic interaction between the primary and secondary systems.

4) The modal shapes of the combined system are the same as those for the primary system, with the addition of the amplified modal deflections of the added secondary masses.

2.3.4.1 Nonresonant Case

Let $\bar{U}_i$ = i^{th} mode shape of the primary system

$\bar{\alpha}_j$ = j^{th} mode shape of the secondary system

Then, with Eqs. (2.16) and (2.17)

$$U_i = C_{pi} \bar{U}_i \tag{2.20}$$

$$\alpha_j = C_{sj} \bar{\alpha}_j \tag{2.21}$$

where C_{pi} is the i^{th} primary participation factor, C_{sj} is the j^{th} secondary participation factor, U_i and α_j are the normalized mode shapes, respectively.

For modal solutions of Eq. (2.10), X_s is taken to be

$$X_s = \sum_{r=1}^{n} \alpha_r q_r(t) \tag{2.22}$$

where n = number of degree-of-freedom of the secondary system. Eq. (2.22) is substituted into Eq. (2.10), which is then premultiplied by α_j^T. The relations of Eqs. (2.15) and (2.19) are used to obtain

$$\ddot{q}_j(t) + \omega_{sj}^2 q_j(t) = \omega_{sj}^2 X_p(k) \tag{2.23}$$

Let us consider the harmonic components of the base input to the secondary system one at a time. By employing the assumptions given previously, $X_p(k)$ is taken to be

$$X_p(k) = U_i(k) \sin \omega_{pi} t \tag{2.24}$$

where $U_i(k)$ is the k^{th} component of U_i. Thus, Eq. (2.23) becomes

$$\ddot{q}_j(t) + \omega_{sj}^2 q_j(t) = \omega_{sj}^2 U_i(k) \sin \omega_{pi} t \tag{2.25}$$

The particular solution to Eq. (2.25) is a steady-state harmonic oscillation at the frequency of the disturbances (ω_{pi} in this case); therefore, the solution is assumed to be

$$q_j(t) = A_{ji} \sin \omega_{pi} t \tag{2.26}$$

where A_{ji} is the amplitude of oscillation. Substituting of Eq. (2.26) into Eq. (2.25) yields

$$-\omega_{pi}^2 A_{ji} + \omega_{sj}^2 A_{ji} = \omega_{sj}^2 U_i(k)$$

$$A_{ji} = \frac{U_i(k)}{1 - \omega_{pi}^2/\omega_{sj}^2} \tag{2.27}$$

In the light of Eqs. (2.22) and (2.26), A_{ji} can be interpreted as the amplification factor of the secondary system response due to the effects of the j^{th} secondary mode and the i^{th} primary mode.

2.3.4.2 Resonant Case ($\omega_{sj} = \omega_{pi}$)

It should be pointed out that Eq. (2.27) gives a very large value for A_{ji} when ω_{sj} is very close to ω_{pi}. When $\omega_{sj} = \omega_{pi}$, A_{ji} turns out to be infinity if there is no damping. Obviously, this is not the case. There should exist a reasonable upper bound to the amplification factor of Eq. (2.27), and therefore the situation needs further investigation when the secondary system is tuned to one of the frequencies of the primary system.

When one frequency of the secondary system falls on one of the frequencies of the primary system, the frequencies of all the modes of the combined system are shifted away in the same fashion, but there are two frequencies at or near the resonant frequency such that one of them is slightly above and one slightly below (Ref. 15). Both frequencies will be very close to the resonant frequency if the effective mass ratio of the secondary to the primary system is small.

Due to the fact that the participation factor of the primary modal displacement U_i and the secondary modal displacement α_j are unity, it follows that

$$1 . M_p . U_i = U_i^T . M_p . U_i \tag{2.28}$$

and

$$1 . M_s . \alpha_j = \alpha_j^T . M_s . \alpha_j \tag{2.29}$$

Based on the foregoing observations, let us consider the case of $\omega_{sj} = \omega_{pi}$, with $A^q_{ji} \alpha_j$ and $-A^r_{ji} \alpha_j$ representing the displacements of the secondary masses of the two modes (mode q and mode r), whose frequencies are close to the resonant frequency, and A^q_{ji} and $-A^r_{ji}$ are amplification factors of the secondary responses for mode q and mode r of the combined system, respectively.

Let us define also the displacements for mode q and mode r of the primary masses as precisely one-half those of the displacements of the i^{th} mode of the primary system.*

By forcing the participation factors of mode q and mode r to be equal to unity, the following results similar to that of Eq. (2.18)

---

* The displacements of the primary masses can be defined arbitrarily, but by using this definition, very good approximate results of the secondary responses are obtained.

are obtained:

$$\frac{1}{2}(1 \cdot M_p \cdot U_i) + A^q_{ji}(1 \cdot M_s \cdot \alpha_j) = \frac{1}{4}(U_i^T \cdot M_p \cdot U_i) + (A^q_{ji})^2(\alpha_j^T \cdot M_s \cdot \alpha_j) \tag{2.30}$$

$$\frac{1}{2}(1 \cdot M_p \cdot U_i) - A^r_{ji}(1 \cdot M_s \cdot \alpha_j) = \frac{1}{4}(U_i^T \cdot M_p \cdot U_i) + (A^r_{ji})^2(\alpha_j^T \cdot M_s \cdot \alpha_j) \tag{2.31}$$

Using the relations of Eqs. (2.28) and (2.29), Eq. (2.30) can be put into the form

$$(A^q_{ji})^2 - A^q_{ji} = \frac{1}{4}\frac{E_{pi}}{E_{sj}} \tag{2.32}$$

where

$$E_{pi} = i^{th} \text{ effective primary mass} = U_i^T \cdot M_p \cdot U_i \tag{2.33}$$

$$E_{sj} = j^{th} \text{ effective secondary mass} = \alpha_j^T \cdot M_s \cdot \alpha_j \tag{2.34}$$

Similarly,

$$(A^r_{ji})^2 + A^r_{ji} = \frac{1}{4}\frac{E_{pi}}{E_{sj}} \tag{2.35}$$

From Eqs. (2.32) and 2.35, we have immediately

$$A^q_{ji} = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{E_{pi}}{E_{sj}} + 1} \tag{2.36}$$

$$-A^r_{ji} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{E_{pi}}{E_{sj}} + 1} \tag{2.37}$$

Define $$\gamma_{ji} = \frac{E_{sj}}{E_{pi}} = \frac{j^{th} \text{ effective secondary mass}}{i^{th} \text{ effective primary mass}} \tag{2.38}$$

Since mode q and mode r are very close to each other, they are additive directly to yield the following result:

$$A_{ji} = |A^q_{ji}| + |A^r_{ji}| \cong \frac{1}{\sqrt{\gamma_{ji}}} \tag{2.39}$$

where A_{ji} = amplification factor of secondary response due to resonance of the j^{th} secondary mode and the i^{th} primary mode

2.4 Summary

Two important parameters which control the secondary spring distortions are the effective mass ratio, γ, as defined by Eq. (2.38), and the damping factor, β. Only the approximate analysis for the secondary system response without damping is developed in section 2.3.4. The combined effect of the two parameters will be discussed in Chapter 4.

In order to substantiate the validity and accuracy of the approximate technique, several analytical systems are studied extensively. Procedures for preparation and analyzing of these systems are summarized as follows:

(1) Select a primary system, arrange the magnitude of the masses and spring constants for a desired fundamental frequency.

(2) Select a secondary system, assign the relative magnitude of the masses and spring constants for desired frequency ratio.

(3) Solve the eigen-value problems of both primary and secondary systems to obtain the frequencies and mode shapes, then normalize the mode shapes so that the participation factors are unity.

(4) Specify the value of the effective mass ratio, then the secondary masses can now be obtained by the procedure described in Appendix A.

(5) Adjust the secondary spring constants in order to get the desired fundamental frequency (Appendix A).

(6) Select the damping factor and obtain the damping constants by Eqs. (2.1) and (2.2).

(7) Perform the analysis of the combined system by the procedures outlined in section 2.3.3.

(8) Repeat steps (6) and (7) for various damping factors.

(9) Repeat steps (5) through (8) for various fundamental frequencies of the secondary system.

(10) Repeat steps (4) through (9) for different effective mass ratios.

CHAPTER 3

DISCUSSION OF FREE VIBRATION SOLUTIONS

3.1 Introductory Remarks

In this chapter, attention is directed to the investigation of free vibration characteristics of the combined system when the effective mass ratio is varied. Several topics which may provide some insight into the behavior of the secondary system will be discussed. Section 3.2 provides the information of single-degree-of-freedom secondary system while section 3.3 is for multiple-degree-of-freedom secondary system.

Because most of the free vibration analysis is performed on the case of tuning the fundamental frequency of the secondary system to the fundamental frequency of the primary system; therefore, the discussion will be concentrated on this case. The attachment points of both one- and two-degree of-freedom systems will be varied to three positions. This is in order to provide some insight into the effects of the support points upon the secondary system response. For the single-degree-of-freedom secondary system, the effective mass ratio considered is 1 percent. The most extensive set of analyses is performed on the two-degree-of-freedom secondary system, where the effective mass ratio ranges from 1 to 20 percent.

Since the effective mass ratio plays an important role in this investigation, henceforth for convenience and simplicity, the notation γ will be used instead of γ_{11}, which is the ratio of the fundamental effective secondary mass to the fundamental effective primary mass.

3.2 SDF Secondary System

3.2.1 Discussion of Results

All significant information resulting from the analysis of systems 1A, 1B and 1C is listed in Tables 3.1, 3.2 and 3.3, respectively. The modal shapes presented have been normalized to obtain participation factors of unity.

From an analysis of the frequencies of the three combined systems, it can be seen that the fundamental frequencies are somewhat lower than the resonant frequencies, while the second natural frequencies are slightly higher. The deviations from the resonant frequencies in each case are less than 10 percent. The third and fourth frequencies are nearly the same as the second and third natural frequencies of the primary system, respectively. Explicitly, system 1A natural frequencies are within 5 percent of systems 1B and 1C, while system 1B is also within 5 percent of 1C.

Considering only the modal shapes of the primary masses, there are two important points that should be noted. First, although there exist some differences in the first and second modal shapes of the combined systems, the general shapes are nearly the same. The magnitudes of the displacements in each mode are about one-half those of the first mode of the primary system. Second, the third and fourth modal shapes are identical to the second and third modal shapes of the primary system, respectively.

For the secondary mass, the first two modes dominate the entire response. The spring distortions of the third and fourth modes are less than 5 percent of the first or second modes. This can be explained by the fact that the first two modes are generated by tuning of the fundamental frequencies of the primary and secondary systems, thus resulting in amplifying the corresponding modal responses. For this reason, the contributions of the third and fourth modes are usually negligible.

3.2.2 Comparison of Results

Approximate modal shapes of the combined systems are presented in Table 3.4. The secondary modal responses are computed by Eqs. (2.27), (2.36) and (2.37).

Examination of the primary responses indicates an accuracy of 2 percent or less for the third and fourth modes. Although the first and second modes are somewhat off, the sums of the two modes tabulated at the bottom of Tables 3.1, 3.2 and 3.3 are quite accurate. These results confirm the validity of the assumption that the presence of the secondary mass has very little or no influence upon the primary response.

Responses of the secondary masses are very well predicted by the approximate formulas. Generally, the approximate results are accurated to within 5 percent of the actual values. Based on comparison of solutions, the variation of support points does not have much influence upon the total response. Therefore, besides simplicity and good accuracy, one more advantage of using Eq. (2.39) is that only one calculation is enough to predict the response of various combined systems.

3.3 MDF Secondary System

3.3.1 General

The following discussion is based on the behavior of three combined systems: 2A, 2B and 2C. Necessary information is tabulated in Tables 3.5 through 3.7. The object of this section is to discuss phenomena concerning the effective ratio.

3.3.2 Effect of the Effective Mass Ratio upon Frequency Distribution

Additional information concerning the deviations from the resonant frequency of the first two modes is presented in Figs. 3.1 and 3.2. Examination of these results reveals the following four aspects. First, although the deviation from the resonant frequency varies from 3 percent when γ = 1% up to 30% for γ = 20%, the average of the first two frequencies is still very close to the resonant frequency. Second, the first two natural frequencies of system 2A deviate from the resonant frequency more than those of systems 2B and 2C, while those of system 2C have the lowest deviation. Third, the first two natural frequencies of all systems vary nearly linearly with $\sqrt{\gamma}$. The increasing of γ results in the decreasing and increasing of the fundamental and second natural frequencies, respectively. Fourth, remaining higher frequencies are very close to the corresponding original frequencies of the primary and secondary systems, no matter how large the effective mass ratio is.

Besides resonant cases, studies have also been made for many general cases. All results lead to the indication that the variation of the effective mass ratio does not change the frequency distribution pattern (See section 3.4) of the combined systems.

3.3.3 Effect of the Effective Mass Ratio upon Response

To have a better idea of the effect of the effective mass ratio upon modal responses of primary masses, additional plots (Figs. 3.3 - 3.5) have been prepared. These plots illustrate the variation of the first two modal shapes of the primary masses. Investigation of all existing information gives the following results:

1) The effect on individual mode increases with γ.

2) The discrepancies of the approximate values from the sum of

the first two modes are less than 10 percent, even for very large γ.

3) The variation of the support points has very little effect upon the sum of the first two modes.

4) The differences of displacements of higher modes of vibration between systems are increasing with γ.

5) The modal shape of system 2A, whose corresponding frequency is closed to the second natural frequency of the secondary system, is completely different from that of systems 2B and 2C. This result is also applied when comparison is made between 2C and 2B.

The consideration of only the first two modes indicates very good prediction of the secondary responses. Explicitly, the approximate values are accurate to within 10 percent, even for 20 percent effective mass ratio. However, the advantage of having modal quantities by approximate procedures no longer exists. The loss of this advantage is due to:

a) Only components of each modal displacements are computed by the approximate procedures.

b) Procedures for combining components to give distinct values of each mode have not yet been available.

Another interesting characteristic which should be pointed out involves the modal quantities for higher modes of vibration. These quantities are not small in comparison to that of the first or second mode. For example, using the notation of section 2.3, $A_3(5)/A_1(5)$ of system 2A when γ = 1% is about 1:6. When γ is up to 20%, this ratio is increased to 1:3. These results clearly give the idea that the effect of higher modes should not be negligible. The consideration of higher modes will make a drastic difference in total response, especially when γ is large.

From the foregoing fact, there remains the problem of combining components to give total amplification factor. The required procedure will be established in Chapter 4.

3.4 Observed Frequency Distribution Pattern of Combined Systems

One important property of the combined system which can reasonably be predicted is the frequency distribution. The frequencies of all the modes of the combined system will be either slightly higher or lower than the frequencies of the primary and secondary systems (Ref. 24). The patterns in which these frequencies follow depend on the magnitudes and distributions of frequencies of both original systems.

Figures 3.6 and 3.7 have been prepared from the observations of frequency distributions of all combined systems considered. From these two figures the following conclusions* can be drawn:

a) When one of the frequencies of either the secondary or primary system falls between two of the frequencies of the other system, then the frequency distribution of the combined system is as shown in Fig. 3.8a.

b) When resonance occurs, the frequencies of all the modes of the combined system are as shown in Fig. 3.8b. There are two frequencies near the resonant frequency such that one is slightly above and the other one is slightly below when the effective mass ratio is small.

---

* These same conclusions had already been presented by Newmark (15).

CHAPTER 4

DISCUSSION OF TIME-HISTORY SOLUTIONS

4.1 Introductory Remarks

The results of free-vibration analysis encourage the possibility of estimating the maximum secondary response, when the supporting structure is subjected to base disturbances, without going through the time-history analysis of the entire system. With the availability of Eqs. (2.27) and (2.39), components of total response, resulting from each secondary mode, can easily be computed. Formulas for total amplification factor will be developed for multi-degree-of-freedom subsystems.

Besides the effective mass ratio, damping also governs the response. The maximum amplification factor of the secondary system is limited by the quantity $1/2\beta$, where β is the effective damping factor.* Hence, if the effective mass ratio becomes small enough so that damping controls the maximum response, a further decrease in the effective mass ratio has no effects on the spring distortion bounds. The combined effects of the effective mass ratio and damping upon the response of the secondary system will be investigated in detail. Then, on the basis of observed behavior of the subsystems, empirical rules for the secondary system response with damping will be developed herein.

Before proceeding with the investigation, it is worthwhile explaining the idea of selecting the constant acceleration pulse as base input. Generally, the design response spectrum is approximated by three boundaries: the constant displacement, velocity and acceleration bounds. These bounds govern the low, intermediate and high frequency ranges,

---

* The effective damping factor is defined in section 4.6.2.

respectively. In most cases, dropping the consideration of constant displacement bound would not affect the overall result. For this reason, the use of the base accelerogram of Fig. 2.6a, which gives the response spectrum of Fig. 2.6b, serves the design analysis efficiently.

In following sections, time-history analyses of various combined systems are generated. To cover the entire primary frequency range, the fundamental secondary frequency is varied from 0.1 to 10 hertz. Both primary and secondary damping factors are kept equal through most sets of analyses except one. The damping factor range from 0.5 to 10 percent is considered. The effective mass ratio is 1 percent for SDF secondary systems and ranges from 1 up to 20 percent for MDF secondary systems.

4.2 Time-History Solutions of SDF Secondary Systems

Figures 4.1 through 4.3 are prepared from time-history analyses of systems 1A, 1B and 1C. Considering first the undamped case, the general characteristics of the secondary spring distortion bounds can be described as smoothly increasing to peak values at corresponding resonant conditions. The peaks formed at p/P = 1, 2 and 3, correspond to the tuning of the secondary frequency to the first, second and third natural frequencies of the primary system, respectively. It is clearly seen that the worst condition occurs when the fundamental frequencies of the supporting structure and subsystem are equal. Outside the resonant zone,* the secondary spring distortion bounds can be considered constant. For very flexible secondary systems (small p/P), the magnitudes of these bounds

* For SDF secondary systems, the resonant zone is considered spanning between p/P = 0.5 and 3.5.

are higher than those of very stiff secondary systems (large p/P).

The presence of damping not only results in the reduction of the amplification levels but also produces significant changes in the general characteristics of the bounds. With the type of damping considered in this study, the secondary responses are reduced faster at the higher frequency levels than at the lower one. Consequently, an increase in the damping factor, which gradually diminishes the second and third peaks, only flattens out the first peak.

Table 4.1 contains the maximum amplifications of the combined systems. For the undamped case, the amplifications are approximately equal in all systems. However, with damping, the amplifications are not. Explicitly, the highest amplification occurs in system 1A, while the lowest one belongs to 1C. These discrepancies also increase with the damping factors. Two reasons that might help explain this phenomenon are:

a) The support, which does not affect the undamped response, produces a greater influence on the overall response when damping is increased.

b) Since the primary and secondary damping constants are computed separately, when considering the entire system, the damping is no longer proportioned to the stiffness. Thus, it results in complex responses, each differing in phase as well as magnitude. Consequently, the support motions cannot be characterized by the existence of a fixed mode, as would be the case if the systems are undamped or if the damping is proportional (Ref. 20). This noted change in phase on the motion of the system is believed to be one source of reducing the amplification from the undamped case more in one system than in the other.

Another interesting characteristic is the location of the peak. It happens in some cases that the maxima do not occur at the resonant conditions. Observation of results indicates that there is no shifting of maximum location for system 1A, slightly off to the left* for 1B with a high damping factor, and further left for 1C.

4.3 MDF Secondary Systems

4.3.1 Effect of the Effective Mass Ratio upon Response

Figures 4.4 - 4.9 are used to illustrate the characteristics of both inner and outer secondary spring distortion bounds of systems 2A, 2B and 2C. Of particular interest are the following:

a) The general characteristics are similar in all systems. The only differences are in the magnitude of the amplification at corresponding resonant conditions.

b) The ratio of the maximum amplifications of the inner spring to the outer spring is nearly the same as the ratio of the fundamental modal distortions of the two springs of the secondary system.

c) For very high p/P, or in other words, when the secondary system is very stiff as compared to the primary structure, the acceleration of both secondary masses are approximately equal to that of the support. However, the support acceleration decreases with increasing γ. Consequently, the amplification has not been much affected by the variation of γ.

d) When the secondary system is quite flexible in comparison to the primary system (small p/P), the secondary masses do not move very much while the support is in motion. Hence, γ produces very little effect

---

* To the left means smaller p/P.

on the spring distortion.

e) The increase of γ results in the reduction of the amplification level, and at the same time flattening and shifting the location of the resonant zone. For a very large γ, there exists only one visible peak, and the location of the maximum amplification has been shifted from the resonant condition to a lower p/P. The larger the effective mass ratio, the further to the left (smaller p/P) is the location.

In order to demonstrate the effect of γ upon the maximum secondary responses, Figures 4.10 and 4.11 have been plotted. It can be seen that the maximum amplification approximately varies inversely as $\sqrt{\gamma}$ for all systems. This relationship explains why Eq. (2.39) gives very good predicted value.

4.3.2 Effect of the Effective Damping Factor upon Response

To emphasize the importance of β upon responses, the secondary spring distortion bounds have been replotted in Figs. 4.12 - 4.17. On the basis of observations made of the time-history solutions, several significant characteristics are recognized. First, general characteristics similar to those of SDF systems are obtained. The number of degrees-of-freedom do not affect the basic behavior. Second, in cases where there is a high degree of damping level or energy absorption, the variation of γ causes quite a change on the maximum amplification of system 2A, moderate change on 2B, and practically no change at all for 2C. However, for very low or high p/P, the variation of γ makes no difference in the magnitude of the amplification at any damping level. Third, the variation of β has more pronounced effects on a lighter secondary system than on a heavier one. Fourth, with the type of damping considered in this study, the dominance of the fundamental mode of the secondary system is still preserved.

Explicitly, the maximum inner spring amplification is about half of the outer spring for all damping levels.

Observations of the maximum amplification variation due to the effects of increasing β, (see Figs. 4.18 and 4.19), indicate the highest reduction in system 2C, and the lowest in 2A. Nevertheless, one can still conclude that the maximum amplification varies inversely with β for all systems.

4.4 Approximate Amplification Factor of Secondary System with Damping

In the light of preceding discussion concerning the relationship of β to the secondary responses, one can indicate the reduction in amplification due to damping by considering the amplification factor as

$$A_{ji} = \frac{1}{a\beta_j} \tag{4.1}$$

where a = constant

β_j = effective damping factor corresponding to the j^{th} secondary mode of vibration

The combined effects of β and γ upon the amplification of the secondary spring distortions are illustrated in Figs. 4.20 - 4.22. From these relationships it can be seen that the variation of γ has lesser effect on the amplification factor than the variation of β. However, they offer similar characteristics in that the rate of changing amplification is decreasing when either β or γ is increasing.

In any circumstances, the envelopes of the net amplification factors can be approximated as straight lines for all systems. The slopes of these envelopes are the steepest for 2A and reducing successively for 2B and 2C. The mathematical expression for this straight line envelope is

$$\frac{\beta}{\beta_o} + \frac{\sqrt{\gamma}}{\sqrt{\gamma_o}} = 1 \qquad (4.2)$$

where the subscript o referred to the values on the main axes.

Substitution of Eqs. (2.39) and (4.1) for $\sqrt{\gamma_o}$ and β_o into Eq. (4.2), and after some manipulations, the net combined amplification factor is obtained as

$$A_{ji} = \frac{1}{a\beta_j + \sqrt{\gamma_{ji}}} \qquad (4.3)$$

Considering the two limiting cases, it is now obvious that when β is very small, the maximum amplification is bounded by $1/\sqrt{\gamma}$. However, when γ becomes very small, the amplification is also limited by $1/2\beta$. From these two limiting cases and the observations of the time-history solutions of systems 2A, 2B and 2C, it is suggested that an appropriate value of a is 2 for $\beta/\sqrt{\gamma}$ either very small or very large; otherwise, a = 3 is recommended.*

4.5 Rules Used for Combination of Amplification Factors

Using the notation of section 2.3, assuming a secondary system with n masses and trying to be on the conservative side, amplification factors corresponding to secondary modes can be combined according to Eq. (4.4).

---

* $0.02 \geq \beta/\sqrt{\gamma}$; a = 2

$0.025 \leq \beta/\sqrt{\gamma} \leq 15$; a = 3

$\beta/\sqrt{\gamma} \geq 20$; a = 2

In range $0.02 < \beta/\sqrt{\gamma} < 0.025$, or $15 < \beta/\sqrt{\gamma} < 20$, interpolate "a" linearly.

$$A.F. = \sum_{j=1}^{n} A_{ji} \tag{4.4}$$

where $\quad A.F. = $ total amplification factor

For undamped vibration, A_{ji} is computed either from Eq. (2.27) or Eq. (2.39). In cases where there is damping, Eq. (4.3) is employed when $\omega_{sj} = \omega_{pi}$; when ω_{sj} is close to ω_{pi}, a modified amplification is required. By analogy with the undamped case, the following rule is established.

$$A_{ji} = \frac{b}{a\beta_j + \sqrt{\gamma_{ji}}} \tag{4.5}$$

where

$$b = \left| \frac{U_j(k)}{1 - \omega_{pi}^2/\omega_{sj}^2} \right| / \frac{1}{\sqrt{\gamma_{ji}}} \leq 1 \tag{4.6}$$

and

$$\beta_j = \frac{\beta\omega_{sj}}{\omega_{s1}} \tag{4.7}$$

To demonstrate the applicability of the approximate procedure, the following example is given.

<u>System 2A</u> $\gamma = 1\%$, $\beta = 0.5\%$

<u>First mode</u> $(\omega_{s1} = \omega_{p1})$

Eq. (4.3) gives $\quad A_{11} = \dfrac{1}{3 \times 0.005 + \sqrt{.01}}$

$= 8.7$

<u>Second mode</u>

Since the second natural frequency of the secondary system falls between the first and second natural frequencies of the primary system, the consideration of ω_{s2} and ω_{p2} will give the appropriate contribution to the total amplification factor. Eq. (2.38) gives

$$\gamma_{22} = 1.67\%$$

From Eq. (4.6)

$$b = \left| \frac{-0.6}{1 - 4/3} \right| / \frac{1}{\sqrt{0.0167}} = 0.232$$

From Eq. (4.7)

$$\beta_2 = \frac{0.5 \times 1.732 \times 2\pi}{2\pi} = 0.866\%$$

Then Eq. (4.5) gives

$$A_{22} = \frac{0.232}{3 \times 0.00866 + \sqrt{.0167}} = 1.48$$

Therefore,

$$A.F. = 8.7 + 1.48 = 10.18$$

For outer secondary spring,

$$A.F. = 10.18 \times 1 = 10.18$$

For inner secondary spring,

$$A.F. = 10.18 \times 0.5 = 5.09$$

Maximum amplification factors from the time-history analyses and approximate procedures are listed in Tables 4.1 through 4.5. Comparison of results will be made in the following section.

4.6 Comparison of Results

4.6.1 SDF Secondary System

When there is no damping, Eq. (2.39) predicts the total amplification factors very accurately for all three systems (1A, 1B and 1C). The approximate value is only 3 percent off on the underconservative side (see Table 4.1). This underconservatism comes from the fact that the approximate procedure considers only the first two modes of vibration. However, the discrepancy is so small that it is not worth complicating the procedure by taking account of the effect of higher modes of vibration.

In cases where there is damping or energy absorption, the total amplification factor is affected by the variation in the support of the secondary system. From the time-history analyses, it can be seen that system 1C is more sensitive to damping than the other two. Unfortunately, the consideration of the degree of sensitivity to damping will merely complicate the approximate analysis without giving any theoretical justification. For this reason, the approximate analysis described in sections 4.4 and 4.5 is considered applicable to all systems.

On this basis of comparison, it can be concluded that Eq. (4.3) is too conservative for system 1C, slightly underconservative for 1A, and very accurate for 1B with low damping factor. Explicitly, the discrepancies between the approximate and exact amplification factors of system 1A are less than 10 percent for all damping levels considered. For damping less than 1 percent, the approximate amplification factors of system 1B are accurate to within 2 percent of the actual values.

4.6.2 MDF Secondary System

Both undamped exact and approximate amplification factors of systems 2A, 2B and 2C are provided in Table 4.2. The approximate amplification factors are of great accuracy even though the effective mass of the subsystem is as large as 10 to 20 percent of that of the main system. For example, considering inner and outer spring distortions when γ = 20%, the approximate procedure overestimates the amplification factors of systems 2A and 2C by less than 10 percent, while slightly underestimates 2B. The reasons for achieving these excellent results are:

a) A_{11}, computed by Eq. (2.39), does take account of the relative mass of the subsystem.

b) A_{22}, computed by Eq. (2.27), although it does not consider the relative values of the effective primary and secondary masses, does take account of the relative motion of the support.

c) By adding A_{22} to A_{11}, the contributions to the total responses due to higher modes of vibration are included.

With the exception of system 2C, the procedure of section 4.4 estimates the amplification factors very accurately. From observations of results tabulated in Tables 4.3 and 4.4, it is apparent that the approximate values are in good agreement with the exact values for all damping levels and effective mass ratios considered (especially system 2A). Although the responses of system 2C are not as well predicted as those of the other systems, for the case of small damping and large γ, the approximate results are not too far off. For example, when γ = 20% and β = 0.5%, the approximate amplification factor is only 10 percent larger than the exact one.

All of the previous discussions are for the situation of equal damping in both primary and secondary systems. When the secondary damping factor is not equal to the primary damping factor, the effective damping factor is considered to be the average of the two values (Ref. 15). In order to substantiate this belief, the time-history analyses of system 2A with γ = 1%, β_p = 2% and β_s varying from 0 up to 10% were performed. The spring distortion bounds are plotted in Fig. 4.23 - 4.24, while the maximum amplification factors are recorded in Table 4.5. Although the approximate results are not conservative for any damping level, the differences between the exact and approximate values are less than 10 percent for the inner spring and less than 5 percent for the outer spring.

One additional set of time-history analyses was performed on system 2A with $\gamma = 1\%$. For this, the system was subjected to the earthquake record of Taft N21E. The effective damping factor considered in this case is 2 percent. Although the general characteristics of the secondary spring distortion bounds are not as smooth as the one subjected to the constant acceleration pulse (see Fig. 4.25), the maximum amplification factors are pretty well bounded by the predicted values. In fact, the approximate values are about 12 percent larger than the actual values for both inner and outer springs.

CHAPTER 5

CONCLUSIONS AND RECOMMENDATIONS

5.1 Summary and Conclusions

To accomplish the evaluation of the basic dynamic behavior of subsystems, free vibration and base excitation analyses were performed on several combined systems. Free vibration solutions were obtained by the usual analytical methods, while the time-history solutions were acquired by numerical integration.

Attempts have been made to develop simple rules for predicting the maximum amplification factor of the dynamic response of secondary systems when both primary and secondary systems do not have any damping. Also conducted in this investigation was the study of the effect of damping upon the secondary system responses. Then, on the basis of observed exact solutions and also by analogy to the undamped case, additional empirical rules were developed.

The approach used in obtaining the approximate procedure was discussed in Chapter 2. The information required by this method was the response spectrum for ground motion, the independent normalized mode shapes, and frequencies of the primary and secondary systems. Maximum amplification factors of secondary system responses were obtained for several combined systems using this approximate approach and corresponding time-history solutions, considering the constant base acceleration pulse. Results of these analyses indicate that, generally, a reasonable estimate of the time-history maximum secondary spring distortions can be obtained by this method.

On the basis of limited amount of information obtained in this investigation, the following specific conclusions can be drawn:

1) With the definition of Eq. (2.34), the effective mass of the subsystem can even be as large as 10 to 20 percent of that of the main system before the approximation becomes inaccurate, since the relationship takes account of the interaction between the systems.

2) For a light secondary system (i.e., $\gamma \leq 1\%$) without damping, whenever tuning occurs between primary and secondary systems, an increase in response usually results. The greatest increase occurs when dominant primary and secondary frequencies are tuned.

3) Without damping, the supports have very little influence on the secondary system responses. With damping, the supports provide a tremendous difference in the maximum amplification factor. However, this difference does decrease when γ is large.

4) The increasing of β reduces the amplification more effectively than the increasing of γ, but the increasing of γ shifts the location of the maximum amplification to a lower p/P condition than β.

Although the maximum amplification factors of the secondary responses do not necessarily occur at resonant condition, the approximate procedure still gives a very good estimate of the maximum amplification.

5.2 Recommendation for Further Study

Only secondary systems with one point of attachment to the supporting structure were investigated in this study. Modification of the approximate approach is needed if the secondary system considered is supported at more than one point. Since it is felt that the approach described herein provides a promising method that leads to a better understanding and a reasonable approximation to the behavior of a complex system, it therefore merits further investigation.

Further research should be carried out on the following categories:

a) Investigation of secondary systems with two points of attachment to the primary structure.

b) Investigation of secondary systems with more than two points of attachment to the primary structure.

c) Investigation of multi-degree-of-freedom secondary systems which have more than one frequency tuned to the frequencies of a multi-degree-of-freedom primary system.

REFERENCES

1. Newmark, N. M., "Notes on Shock Isolation Concepts", in Vibration in Civil Engineering, Butterworths, London, 1966, pp. 71-82.
2. Penzien, J., and A. K. Chopra, "Earthquake Response of Appendage on a Multi Story Building", Proceedings of the Third World Conference on Earthquake Engineering, New Zealand, Vol. 2, 1965, pp. 476-490.
3. Biggs, J. M., and J. M. Rosset, "Seismic Analysis of Equipment Mounted on a Massive Structure", in Seismic Design for Nuclear Power Plants, Hansen, R. J., Ed., M.I.T. Press, 1970, pp. 319-343.
4. Sato, H., "On the Response Spectrum of the Building-Machine Structure System to the Strong Motion Earthquake", Bulletin of J.S.M.E., Vol. 9, No. 36, 1966.
5. Shibata, H., H. Sato, and T. Shigeta, "Aseismic Design of Machine-Structure", Proceedings of the Third World Conference on Earthquake Engineering, Vol. 2, 1965, pp. 556-568.
6. Sato, H., and K. Suzuki, "On the Response of Building-Machine Structure Subjected to Two Different Seismic Forces", Journal of the Institute of Industrial Science, Tokyo University, Vol. 20, No. 9, 1969.
7. Nakagawa, K., H. Murata, and C. Caler, "Preliminary Study of Modal Analysis of Response of a Structure Subjected to Two Different Earthquake Motion at Its Two Supporting Points", Bulletin of the International Institute of Seismology and Earthquake Engineering, Vol. 12, No. 4, 1969.
8. Watari, A., S. Fujii, M. Iguchi, H. Sato, and T. Shigeta, "Aseismic Design of Piping Systems in Power and Chemical Engineering Plants", J.S.M.E. Semi-International Symposium, Tokyo, Japan, 1967.

9. Shimizu, N. and H. Shibata, "On Vibration Analysis Subjected to Multi-Random Input", Thesis for Master's Degree, University of Tokyo, Tokyo, Japan, 1968.

10. Hart, G. C., W. C. Hurty, and J. D. Collins, "A Survey of Modal Synthesis Methods", National Aeronautics and Space Administration, George C. Marshall Space Flight Center, Contract No. NAS8-26192.

11. Berkowitz, L., "Seismic Analysis of Primary Piping Systems for Nuclear Generating Stations", Reactor and Full Processing Technology, Vol. 12, No. 4, 1969.

12. Kassawara, R. P., "Earthquake Response of Light Multiple Degree-of-Freedom Secondary Systems by Spectrum Techniques", Doctoral Dissertation, University of Illinois, 1970.

13. Caughey, T. K., "Design of Subsystems in Large Structures", Technical Memorandum 33-484, National Aeronautics and Space Administration, June 1971.

14. Newmark, N. M., W. H. Walker, A. S. Veletsos, and R. J. Mosborg, "Design Procedures for Shock Isolation Systems of Underground Protective Structures", Vol. IV, Response of Two Degree-of-Freedom Elastic and Inelastic Systems, Report No. RTD-TDR-63-3096, Air Force Weapons Lab., New Mexico, 1965.

15. Newmark, N. M., "Earthquake Response Analysis of Reactor Structure", Structural Mechanics in Reactor Technology, Berlin, 20-24 September 1971.

16. Newmark, N. M., "Design Criteria for Nuclear Reactors Subjected to Earthquake Hazards", Proc. IAEA Panel on Aseismic Design and Testing of Nuclear Facilities, Japan Earthquake Engineering Promotion Society, Tokyo, 1969, pp. 90-113.

17. Newmark, N. M., and W. J. Hall, "Seismic Design Criteria for Nuclear Reactor Facilities", Proc. Fourth World Conference on Earthquake Engineering, Santiago, Chile, B-4, 1969, pp. 37-50.

18. Newmark, N. M., and E. H. Rosenblueth, "Fundamentals of Earthquake Engineering", Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

19. Goodman, L. E., E. Rosenblueth, and N. M. Newmark, "Aseismic Design of Firmly Founded Elastic Structures", Trans. ASCE, 120, 1955, pp. 782-802.

20. Hurty, W. C., and M. F. Rubinstein, "Dynamics of Structures", Prentice-Hall, Inc., 1964.

21. Thompson, W. T., "Vibration Theory and Applications", Prentice-Hall, Inc., 1965.

22. Meirovitch, L., "Analytical Methods in Vibrations", Macmillan, 1971.

23. Newmark, N. M., "A Method of Computation for Structural Dynamics", Journal of the Engineering Mechanics Division, Proc. ASCE, Vol. 85, No. EM3, July 1959.

24. Bisplinghoff, R. L., H. Ashley, and R. L. Halfman, "Aeroelasticity", Addison-Wesley Publishing Co., Inc., Reading, Mass., 1955.

Table 1.1 Modal Values u and s of Primary and Secondary Systems.

(a)

Primary system						
Mass number	Mode 1, Freq. = 1 cps		Mode 2, Freq. = 2 cps		Mode 3, Freq. = 3 cps	
	u	s	u	s	u	s
1	0.5	0.5	0.4	0.4	0.1	0.1
2	1.0	0.5	0.2	-0.2	-0.2	-0.3
3	1.5	0.5	-0.6	-0.8	0.1	0.3

(b)

SDF secondary system		
Mass number	Fundamental mode, Freq. = 1 cps	
	u	s
1	1.0	1.0

(c)

MDF secondary system				
Mass number	Mode 1, Freq. = 1 cps		Mode 2, Freq. = 1.732 cps	
	u	s	u	s
1	0.5	0.5	0.5	0.5
2	1.5	1.0	-0.5	-1.0

Table 3.1 Modal Values u and s, System 1A, Effective Mass Ratio = 1%

Effective mass ratio = 1.00%; Amplification factor = 10.00
Resonant frequency = 1.00 cycle/sec

Mode	Frequency	Response	Mass number 1	Mass number 2	Mass number 3	Mass number 4
1	0.9265	u	0.2318	0.4799	0.7703	5.4367
		s	0.2318	0.2482	0.2904	4.6664
2	1.0758	u	0.2744	0.5272	0.7276	-4.6221
		s	0.2744	0.2528	0.2005	-5.3497
3	2.0060	u	0.3943	0.1924	-0.5985	0.1979
		s	0.3943	-0.2019	-0.7908	0.7964
4	3.0008	u	0.0996	-0.1995	0.1005	-0.0126
		s	0.0996	-0.2991	0.3000	-0.1130
Max. displacement		Exact	0.5061	1.0071	1.4980	10.0588
		Approx.	0.5000	1.0000	1.5000	10.0000
Max. strain		Exact	0.5061	0.5010	0.4909	10.0161
		Approx.	0.5000	0.5000	0.5000	10.0000

Note: Exact is the sum of the absolute values of the first two modes.

Table 3.2 Modal Values u and s, System 1B, Effective Mass Ratio = 1%

Effective mass ratio = 1:00%; Amplification factor = 10.00

Resonant frequency = 1.00 cycle/sec

Mode	Frequency	Response	Mass number			
			1	2	3	4
1	0.9506	u	0.2606	0.5338	0.7640	5.5398
		s	0.2606	0.2732	0.2301	5.0060
2	1.0504	u	0.2388	0.4652	0.7358	-4.4980
		s	0.2388	0.2264	0.2706	-4.9631
3	2.0007	u	0.4015	0.2002	-0.5990	-0.0667
		s	0.4015	-0.2013	-0.7992	-0.2669
4	3.0034	u	0.0991	-0.1992	0.0993	0.0248
		s	0.0991	-0.2983	0.2985	0.2241
Max. displacement		Exact	0.4994	0.9990	1.4998	10.0378
		Approx.	0.5000	1.0000	1.5000	10.0000
Max. strain		Exact	0.4994	0.4996	0.5007	9.9691
		Approx.	0.5000	0.5000	0.5000	10.0000

Note: Exact is the sum of the absolute values of the first two modes.

Table 3.3 Modal Values u and s, System 1C, Effective Mass Ratio = 1%

Effective mass ratio = 1.00%; Amplification factor = 10.00 Resonant frequency = 1.00 cycle/sec						
Mode	Frequency	Response	Mass number			
			1	2	3	4
1	0.9745	u	0.2810	0.5292	0.7743	5.5899
		s	0.2810	0.2482	0.2451	5.3089
2	1.0245	u	0.2202	0.4699	0.7227	-4.4450
		s	0.2202	0.2497	0.2528	-4.6651
3	2.0027	u	0.3985	0.2011	-0.5970	-0.1324
		s	0.3985	-0.1974	-0.7981	-0.5309
4	3.0008	u	0.1004	-0.2001	0.1000	-0.0125
		s	0.1004	-0.3005	0.3001	-0.1129
Max. displacement		Exact	0.5011	0.9990	1.4970	10.0348
		Approx.	0.5000	1.0000	1.5000	10.0000
Max. strain		Exact	0.5011	0.4979	0.4980	9.9741
		Approx.	0.5000	0.5000	0.5000	10.0000
Note: Exact is the sum of the absolute values of the first two modes.						

Table 3.4 Approximate Modal Values u and s, Systems 1A, 1B and 1C, Effective Mass Ratio = 1%

Effective mass ratio = 1%; Amplification factor = 10 Resonant frequency = 1.0 cps							
Mode		Primary mass			Secondary mass		
		1	2	3	System 1A	System 1B	System 1C
1	u	0.25	0.5	0.75	5.52	5.52	5.52
	s	0.25	0.25	0.25	4.77	5.02	5.27
2	u	0.25	0.5	0.75	-4.52	-4.52	-4.52
	s	0.25	0.25	0.25	-5.27	-5.02	-4.77
3	u	0.4	0.2	-0.6	0.2	-0.0667	-0.133
	s	0.4	-0.2	-0.8	0.8	-0.2667	-0.533
4	u	0.1	-0.2	0.1	-0.0125	0.025	-0.0125
	s	0.1	-0.3	0.3	-0.1125	-0.225	-0.1125

Table 3.5a Modal Values u and s, System 2A

Effective mass ratio = 1.00%; Amplification factor = 10.00 Resonant frequency = 1.00 cycle/sec							
Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.9240	u	0.2379	0.4931	0.7931	3.2677	7.5860
		s	0.2379	0.2552	0.3000	2.4747	4.3183
2	1.0727	u	0.2602	0.5009	0.6940	-1.7410	-7.4746
		s	0.2602	0.2406	0.1931	-2.4350	-5.7335
3	1.7261	u	0.0399	0.0403	-0.0189	-1.3675	1.3867
		s	0.0399	0.0004	-0.0592	-1.3486	2.7542
4	2.0234	u	0.3629	0.1644	-0.5693	0.8724	-0.5044
		s	0.3629	-0.1986	-0.7336	1.4416	-1.3768
5	3.0020	u	0.0990	-0.1987	0.1011	-0.0315	0.0063
		s	0.0990	-0.2977	0.2998	-0.1327	0.0378
Max. displacement		Exact	0.4981	0.9940	1.4871	5.0088	15.0606
		Approx.	0.5000	1.0000	1.5000	5.0000	15.0000
Max. strain		Exact	0.4981	0.4959	0.4931	4.9097	10.0518
		Approx.	0.5000	0.5000	0.5000	5.0000	10.0000
Note: Exact is the sum of the absolute values of the first two modes.							

Table 3.5b Modal Values u and s, System 2A

Effective mass ratio = 2.00%; Amplification factor = 7.07 Resonant frequency = 1.00 cycle/sec							
Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.8926	u	0.2329	0.4895	0.8077	2.5362	5.4091
		s	0.2329	0.2566	0.3182	1.7285	2.8729
2	1.1008	u	0.2631	0.4983	0.6668	-1.0143	-5.2805
		s	0.2631	0.2352	0.1685	-1.6811	-4.2662
3	1.7209	u	0.0721	0.0735	-0.0326	-1.2642	1.2974
		s	0.0721	0.0014	-0.1060	-1.2317	2.5617
4	2.0458	u	0.3338	0.1360	-0.5443	0.7741	-0.4324
		s	0.3338	-0.1979	-0.6802	1.3184	-1.2066
5	3.0041	u	0.0981	-0.1973	0.1023	-0.0318	0.0063
		s	0.0981	-0.2954	0.2996	-0.1341	0.0382
Max. displacement		Exact	0.4960	0.9879	1.4746	3.5506	10.6896
		Approx.	0.5000	1.0000	1.5000	3.5355	10.6066
Max. strain		Exact	0.4960	0.4918	0.4867	3.4096	7.1390
		Approx.	0.5000	0.5000	0.5000	3.5355	7.0711
Note: Exact is the sum of the absolute values of the first two modes.							

Table 3.5c Modal Values u and s, System 2A

Effective mass ratio = 5.00%; Amplification factor = 4.47

Resonant frequency = 1.00 cycle/sec

Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.8313	u	0.2239	0.4824	0.8326	1.8888	3.5027
		s	0.2239	0.2585	0.3502	1.0562	1.6139
2	1.1522	u	0.2652	0.4870	0.6073	-0.3823	-3.3257
		s	0.2652	0.2218	0.1203	-0.9896	-2.9434
3	1.7089	u	0.1421	0.1477	-0.0567	-1.0517	1.1107
		s	0.1421	0.0057	-0.2044	-0.9950	2.1623
4	2.1087	u	0.2739	0.0758	-0.4889	0.5779	-0.2942
		s	0.2739	-0.1981	-0.5647	1.0668	-0.8720
5	3.0107	u	0.0949	-0.1929	0.1057	-0.0327	0.0065
		s	0.0949	-0.2878	0.2986	-0.1384	0.0392
Max. displacement		Exact	0.4891	0.9694	1.4399	2.2711	6.8284
		Approx.	0.5000	1.0000	1.5000	2.2361	6.7082
Max. strain		Exact	0.4891	0.4803	0.4705	2.0458	4.5573
		Approx.	0.5000	0.5000	0.5000	2.2361	4.4721

Note: Exact is the sum of the absolute values of the first two modes.

Table 3.5d Modal Values u and s, System 2A

Effective mass ratio = 10.00%; Amplification factor = 3.16

Resonant frequency = 1.00 cycle/sec

Mode	Frequency	Response	Mass number 1	2	3	4	5
1	0.7658	u	0.2153	0.4751	0.8555	1.5659	2.5710
		s	0.2153	0.2598	0.3803	0.7104	1.0051
2	1.2017	u	0.2610	0.4640	0.5351	-0.0866	-2.3232
		s	0.2610	0.2030	0.0710	-0.6217	-2.2365
3	1.6952	u	0.2157	0.2293	-0.0730	-0.8476	0.9255
		s	0.2157	0.0136	-0.3023	-0.7746	1.7730
4	2.2035	u	0.2190	0.0158	-0.4289	0.4024	-0.1799
		s	0.2190	-0.2031	-0.4448	0.8313	-0.5823
5	3.0233	u	0.0890	-0.1843	0.1114	-0.0340	0.0067
		s	0.0890	-0.2733	0.2957	-0.1454	0.0407
Max. displacement		Exact	0.4763	0.9391	1.3905	1.6525	4.8942
		Approx.	0.5000	1.0000	1.5000	1.5811	4.7434
Max. strain		Exact	0.4763	0.4629	0.4514	1.3321	3.2416
		Approx.	0.5000	0.5000	0.5000	1.5811	3.1623

Note: Exact is the sum of the absolute values of the first two modes.

Table 3.5e Modal Values u and s, System 2A

Effective mass ratio = 20.00%; Amplification factor = 2.24 Resonant frequency = 1.00 cycle/sec							
Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.6814	u	0.2061	0.4673	0.8813	1.3440	1.9465
		s	0.2061	0.2612	0.4140	0.4627	0.6026
2	1.2563	u	0.2440	0.4175	0.4350	0.0828	-1.5876
		s	0.2440	0.1735	0.0175	-0.3523	-1.6703
3	1.6787	u	0.3016	0.3290	-0.0798	-0.6362	0.7240
		s	0.3016	0.0274	-0.4088	-0.5565	1.3602
4	2.3662	u	0.1737	-0.0520	-0.3578	0.2454	-0.0898
		s	0.1737	-0.2257	-0.3058	0.6032	-0.3352
5	3.0562	u	0.0746	-0.1619	0.1212	-0.0359	0.0069
		s	0.0746	-0.2365	0.2831	-0.1572	0.0428
Max. displacement		Exact	0.4501	0.8848	1.3164	1.4267	3.5341
		Approx.	0.5000	1.0000	1.5000	1.1180	3.3541
Max. strain		Exact	0.4501	0.4347	0.4315	0.8149	2.2729
		Approx.	0.5000	0.5000	0.5000	1.1180	2.2361
Note: Exact is the sum of the absolute values of the first two modes.							

Table 3.6a Modal Values of u and s, System 2B

Effective mass ratio = 1.00%; Amplification factor = 10.00 Resonant frequency = 1.00 cycle/sec							
Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.9495	u	0.2663	0.5458	0.7803	3.1604	7.9207
		s	0.2663	0.2795	0.2345	2.6146	4.7603
2	1.0491	u	0.2320	0.4524	0.7145	-1.8922	-7.1055
		s	0.2320	0.2204	0.2621	-2.3445	-5.2133
3	1.7321	u	0.0000	0.0000	-0.0000	0.0000	-0.0000
		s	0.0000	0.0000	-0.0000	0.0000	-0.0000
4	2.0026	u	0.4038	0.1998	-0.5930	-0.3297	0.1970
		s	0.4038	-0.2040	-0.7928	-0.5294	0.5267
5	3.0079	u	0.0978	-0.1980	0.0982	0.0614	-0.0122
		s	0.0978	-0.2958	0.2962	0.2594	-0.0736
Max. displacement		Exact	0.4984	0.9982	1.4948	5.0526	15.0262
		Approx.	0.5000	1.0000	1.5000	5.0000	15.0000
Max. strain		Exact	0.4984	0.4998	0.4966	4.9592	9.9736
		Approx.	0.5000	0.5000	0.5000	5.0000	10.0000

Note: Exact is the sum of the absolute values of the first two modes.

Table 3.6b Modal Values u and s, System 2B

Effective mass ratio = 2.00%; Amplification factor = 7.07 Resonant frequency = 1.00 cycle/sec							
Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.9284	u	0.2726	0.5641	0.7915	2.4379	5.7307
		s	0.2726	0.2914	0.2274	1.8738	3.2928
2	1.0687	u	0.2242	0.4324	0.6982	-1.1721	-4.9128
		s	0.2242	0.2082	0.2658	-1.6045	-3.7407
3	1.7321	u	0.0000	0.0000	0.0000	-0.0000	0.0000
		s	0.0000	0.0000	0.0000	-0.0000	0.0000
4	2.0052	u	0.4075	0.1995	-0.5861	-0.3261	0.1940
		s	0.4075	-0.2080	-0.7856	-0.5255	0.5201
5	3.0158	u	0.0957	-0.1959	0.0964	0.0603	-0.0119
		s	0.0957	-0.2916	0.2924	0.2563	-0.0722
Max. displacement		Exact	0.4968	0.9965	1.4897	3.6100	10.6435
		Approx.	0.5000	1.0000	1.5000	3.5355	10.6066
Max. strain		Exact	0.4968	0.4997	0.4932	3.4783	7.0335
		Approx.	0.5000	0.5000	0.5000	3.5355	7.0711
Note: Exact is the sum of the absolute values of the first two modes.							

Table 3.6c Modal Values u and s, System 2B

Effective mass ratio = 5.00%; Amplification factor = 4.47

Resonant frequency = 1.00 cycle/sec

Mode	Frequency	Response	Mass number 1	2	3	4	5
1	0.8867	u	0.2842	0.5988	0.8115	1.8063	3.7959
		s	0.2842	0.3146	0.2127	1.2075	1.9896
2	1.1060	u	0.2080	0.3928	0.6632	-0.5480	-2.9703
		s	0.2080	0.1848	0.2704	-0.9407	-2.4223
3	1.7321	u	0.0000	0.0000	0.0000	-0.0000	0.0000
		s	0.0000	-0.0000	0.0000	-0.0000	0.0000
4	2.0128	u	0.4182	0.1984	-0.5660	-0.3155	0.1855
		s	0.4182	-0.2198	-0.7644	-0.5139	0.5010
5	3.0396	u	0.0896	-0.1899	0.0913	0.0572	-0.0111
		s	0.0896	-0.2795	0.2813	0.2471	-0.0683
Max. displacement		Exact	0.4922	0.9916	1.4747	2.3542	6.7662
		Approx.	0.5000	1.0000	1.5000	2.2361	6.7082
Max. strain		Exact	0.4922	0.4994	0.4831	2.1482	4.4120
		Approx.	0.5000	0.5000	0.5000	2.2361	4.4721

Note: Exact is the sum of the absolute values of the first two modes.

Table 3.6d Modal Values u and s, System 2B

Effective mass ratio = 10.00%; Amplification factor = 3.16 Resonant frequency = 1.00 cycle/sec							
Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.8405	u	0.2960	0.6354	0.8311	1.4977	2.8310
		s	0.2960	0.3394	0.1957	0.8623	1.3333
2	1.1450	u	0.1892	0.3490	0.6200	-0.2512	-1.9936
		s	0.1892	0.1598	0.2709	-0.6003	-1.7424
3	1.7321	u	0.0000	0.0000	-0.0000	0.0000	-0.0000
		s	0.0000	0.0000	-0.0000	0.0000	-0.0000
4	2.0246	u	0.4344	0.1957	-0.5344	-0.2988	0.1725
		s	0.4344	-0.2387	-0.7302	-0.4945	0.4713
5	3.0796	u	0.0804	-0.1802	0.0834	0.0523	-0.0098
		s	0.0804	-0.2605	0.2635	0.2325	-0.0621
Max. displacement		Exact	0.4852	0.9844	1.4511	1.7490	4.8246
		Approx.	0.5000	1.0000	1.5000	1.5811	4.7434
Max. strain		Exact	0.4852	0.4992	0.4666	1.4626	3.0757
		Approx.	0.5000	0.5000	0.5000	1.5811	3.1623

Note: Exact is the sum of the absolute values of the first two modes.

Table 3.6e Modal Values u and s, System 2B

Mode	Frequency	Response	Mass number 1	2	3	4	5
Effective mass ratio = 20.00%; Amplification factor = 2.24 Resonant frequency = 1.00 cycle/sec							
1	0.7777	u	0.3105	0.6823	0.8545	1.2905	2.1626
		s	0.3105	0.3718	0.1723	0.6083	0.8720
2	1.1936	u	0.1628	0.2910	0.5541	-0.0656	-1.3050
		s	0.1628	0.1282	0.2631	-0.3566	-1.2394
3	1.7321	u	0.0000	0.0000	-0.0000	0.0000	-0.0000
		s	0.0000	0.0000	-0.0000	0.0000	-0.0000
4	2.0453	u	0.4619	0.1886	-0.4781	-0.2687	0.1502
		s	0.4619	-0.2733	-0.6667	-0.4573	0.4189
5	3.1603	u	0.0649	-0.1618	0.0695	0.0437	-0.0077
		s	0.0649	-0.2267	0.2313	0.2056	-0.0515
Max. displacement		Exact	0.4732	0.9732	1.4086	1.3562	3.4676
		Approx.	0.5000	1.0000	1.5000	1.1180	3.3541
Max. strain		Exact	0.4732	0.5000	0.4354	0.9649	2.1114
		Approx.	0.5000	0.5000	0.5000	1.1180	2.2361

Note: Exact is the sum of the absolute values of the first two modes.

Table 3.7a Modal Values u and s, System 2C

Effective mass ratio = 1.00%; Amplification factor = 10.00 Resonant frequency = 1.00 cycle/sec							
Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.9742	u	0.2847	0.5359	0.7838	3.0060	8.1849
		s	0.2847	0.2511	0.2480	2.7212	5.1790
2	1.0242	u	0.2177	0.4642	0.7138	-2.0579	-6.8433
		s	0.2177	0.2465	0.2496	-2.2756	-4.7854
3	1.7258	u	0.0101	-0.0001	-0.0205	0.7037	-0.7140
		s	0.0101	-0.0103	-0.0204	0.6936	-1.4177
4	2.0104	u	0.3866	0.2004	-0.5770	-0.6203	0.3661
		s	0.3866	-0.1863	-0.7774	-1.0069	0.9863
5	3.0020	u	0.1008	-0.2003	0.0999	-0.0315	0.0063
		s	0.1008	-0.3011	0.3002	-0.1323	0.0377
Max. displacement		Exact	0.5024	1.0001	1.4976	5.0639	15.0282
		Approx.	0.5000	1.0000	1.5000	5.0000	15.0000
Max. strain		Exact	0.5024	0.4976	0.4975	4.9969	9.9643
		Approx.	0.5000	0.5000	0.5000	5.0000	10.0000
Note: Exact is the sum of the absolute values of the first two modes.							

Table 3.7b Modal Values u and s, System 2C

Effective mass ratio = 2.00%; Amplification factor = 7.07 Resonant frequency = 1.00 cycle/sec							
Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.9631	u	0.2997	0.5506	0.7970	2.2869	5.9930
		s	0.2997	0.2509	0.2464	1.9872	3.7061
2	1.0337	u	0.2051	0.4496	0.6983	-1.3388	-4.6535
		s	0.2051	0.2444	0.2487	-1.5439	-3.3147
3	1.7201	u	0.0185	-0.0005	-0.0378	0.6646	-0.6835
		s	0.0185	-0.0190	-0.0372	0.6461	-1.3480
4	2.0203	u	0.3751	0.2009	-0.5574	-0.5811	0.3376
		s	0.3751	-0.1741	-0.7583	-0.9561	0.9187
5	3.0040	u	0.1017	-0.2005	0.0999	-0.0316	0.0063
		s	0.1017	-0.3022	0.3004	-0.1333	0.0380
Max. displacement		Exact	0.5048	1.0001	1.4953	3.6256	10.6465
		Approx.	0.5000	1.0000	1.5000	3.5355	10.6066
Max. strain		Exact	0.5048	0.4953	0.4951	3.5311	7.0209
		Approx.	0.5000	0.5000	0.5000	3.5355	7.0711

Note: Exact is the sum of the absolute values of the first two modes.

Table 3.7c Modal Values u and s, System 2C

Effective mass ratio = 5.00%; Amplification factor = 4.47

Resonant frequency = 1.00 cycle/sec

Mode	Frequency	Response	Mass number 1	2	3	4	5
1	0.9405	u	0.3301	0.5792	0.8214	1.6628	4.0523
		s	0.3301	0.2492	0.2422	1.3327	2.3895
2	1.0516	u	0.1817	0.4211	0.6669	-0.7144	-2.7189
		s	0.1817	0.2393	0.2458	-0.8961	-2.0045
3	1.7049	u	0.0364	-0.0024	-0.0764	0.5756	-0.6138
		s	0.0364	-0.0387	-0.0740	0.5392	-1.1894
4	2.0475	u	0.3476	0.2033	-0.5115	-0.4918	0.2740
		s	0.3476	-0.1443	-0.7149	-0.8394	0.7657
5	3.0101	u	0.1042	-0.2013	0.0996	-0.0322	0.0064
		s	0.1042	-0.3054	0.3009	-0.1364	0.0386
Max. displacement		Exact	0.5118	1.0003	1.4883	2.3772	6.7712
		Approx.	0.5000	1.0000	1.5000	2.2361	6.7082
Max. strain		Exact	0.5118	0.4885	0.4880	2.2289	4.3940
		Approx.	0.5000	0.5000	0.5000	2.2361	4.4721

Note: Exact is the sum of the absolute values of the first two modes.

Table 3.7d Modal Values u and s, System 2C

Effective mass ratio = 10.00%; Amplification factor = 3.16 Resonant frequency = 1.00 cycle/sec							
Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.9141	u	0.3651	0.6104	0.8461	1.3633	3.0780
		s	0.3651	0.2454	0.2357	0.9983	1.7146
2	1.0703	u	0.1579	0.3902	0.6312	-0.4145	-1.7543
		s	0.1579	0.2323	0.2410	-0.5724	-1.3398
3	1.6847	u	0.0534	-0.0063	-0.1167	0.4798	-0.5378
		s	0.0534	-0.0596	-0.1104	0.4264	-1.0176
4	2.0874	u	0.3153	0.2080	-0.4597	-0.3954	0.2076
		s	0.3153	-0.1073	-0.6677	-0.7107	0.6030
5	3.0205	u	0.1084	-0.2023	0.0991	-0.0332	0.0065
		s	0.1084	-0.3106	0.3013	-0.1416	0.0398
Max. displacement		Exact	0.5230	1.0006	1.4773	1.7778	4.8323
		Approx.	0.5000	1.0000	1.5000	1.5811	4.7434
Max. strain		Exact	0.5230	0.4776	0.4767	1.5706	3.0545
		Approx.	0.5000	0.5000	0.5000	1.5811	3.1623
Note: Exact is the sum of the absolute values of the first two modes.							

Table 3.7e Modal Values u and s, System 2C

Effective mass ratio = 20.00%; Amplification factor = 2.24
Resonant frequency = 1.00 cycle/sec

Mode	Frequency	Response	Mass number				
			1	2	3	4	5
1	0.8758	u	0.4149	0.6518	0.8757	1.1691	2.3926
		s	0.4149	0.2369	0.2239	0.7542	1.2234
2	1.0942	u	0.1288	0.3492	0.5812	-0.2193	-1.0866
		s	0.1288	0.2205	0.2319	-0.3481	-0.8672
3	1.6548	u	0.0681	-0.0138	-0.1588	0.3710	-0.4493
		s	0.0681	-0.0819	-0.1449	0.3029	-0.8203
4	2.1538	u	0.2713	0.2161	-0.3956	-0.2856	0.1365
		s	0.2713	-0.0552	-0.6117	-0.5570	0.4221
5	3.0428	u	0.1169	-0.2033	0.0975	-0.0351	0.0068
		s	0.1169	-0.3203	0.3008	-0.1520	0.0419
Max. displacement		Exact	0.5436	1.0011	1.4569	1.3885	3.4791
		Approx.	0.5000	1.0000	1.5000	1.1180	3.3541
Max. strain		Exact	0.5436	0.4574	0.4558	1.1023	2.0907
		Approx.	0.5000	0.5000	0.5000	1.1180	2.2361

Note: Exact is the sum of the absolute values of the first two modes.

Table 4.1 Exact and Approximate Maximum Amplification Factors, Systems 1A, 1B and 1C, γ = 1%, β = 0, 0.5, 1, 2, 5 and 10%

System	Type of Solutions	Effective damping factor, β, (%)					
		0	0.5	1	2	5	10
1A	Exact	10.32	9.15	8.33	6.91	4.41	2.60
1B	Exact	10.21	8.74	7.60	5.87	3.33	1.91
1C	Exact	10.32	7.32	5.03	3.55	1.96	1.20
1A, 1B, 1C	Approx.	10.00	8.7	7.7	6.25	4.0	2.5

Table 4.2 Exact and Approximate Maximum Amplification Factors,
Systems 2A, 2B and 2C, β = 0%, γ = 1, 2, 5, 10 and 20%

Effective mass ratio, γ (%)	System	Inner spring				Outer spring			
		Approx.			Exact	Approx.			Exact
		A_{11}	A_{22}	A.F.	A.F.	A_{11}	A_{22}	A.F.	A.F.
1	2A	5.00	0.9	5.9	6.26	10.0	1.8	11.8	11.78
	2B	5.00	0.3	5.3	5.22	10.0	0.6	10.6	10.28
	2C	5.00	0.6	5.6	5.67	10.0	1.2	11.2	10.93
2	2A	3.54	0.9	4.54	4.45	7.07	1.8	8.87	8.91
	2B	3.54	0.3	3.84	3.79	7.07	0.6	7.67	8.13
	2C	3.54	0.6	4.14	4.32	7.07	1.2	8.27	7.90
5	2A	2.24	0.9	3.14	3.12	4.47	1.8	6.27	6.14
	2B	2.24	0.3	2.54	2.64	4.47	0.6	5.07	5.90
	2C	2.24	0.6	2.84	2.92	4.47	1.2	5.67	5.50
10	2A	1.58	0.9	2.48	2.22	3.16	1.8	4.96	4.89
	2B	1.58	0.3	1.88	2.11	3.16	0.6	3.76	4.27
	2C	1.58	0.6	2.18	2.09	3.16	1.2	4.36	4.15
20	2A	1.12	0.87	1.99	1.74	2.24	1.74	3.98	3.36
	2B	1.12	0.3	1.42	1.71	2.24	0.6	2.84	3.49
	2C	1.12	0.6	1.72	1.59	2.24	1.2	3.44	3.47

Table 4.3 Exact and Approximate Maximum Amplification Factors, Systems 2A, 2B and 2C, Inner Spring

Effective mass ratio, γ, (%)	Type	Effective damping factor, β, (%)								
		0.5			1			2		
		System			System			System		
		2A	2B	2C	2A	2B	2C	2A	2B	2C
1	Exact	5.22	4.37	3.70	4.60	3.80	2.59	3.64	2.94	1.77
	Approx.	5.09	4.60	4.85	4.49	4.06	4.28	3.63	3.29	3.46
2	Exact	3.68	3.21	3.11	3.39	2.91	2.43	2.92	2.42	1.67
	Approx.	3.99	3.46	3.72	3.62	3.15	3.39	3.06	2.68	2.87
5	Exact	2.79	2.27	2.41	2.57	2.06	2.04	2.21	1.81	1.51
	Approx.	2.92	2.37	2.65	2.74	2.23	2.48	2.43	1.99	2.21
10	Exact	1.78	1.82	1.74	1.63	1.68	1.54	1.55	1.44	1.24
	Approx.	2.38	1.82	2.10	2.25	1.71	1.98	2.05	1.57	1.81
20	Exact	1.52	1.40	1.36	1.48	1.28	1.25	1.41	1.09	1.08
	Approx.	1.94	1.38	1.68	1.84	1.33	1.60	1.72	1.24	1.50

Table 4.3 (continued)

Effective mass ratio, γ, (%)	Type	Effective damping factor, β, (%)					
		5			10		
		System			System		
		2A	2B	2C	2A	2B	2C
1	Exact	2.28	1.68	1.00	1.34	0.98	0.96
	Approx.	2.30	2.10	2.20	1.43	1.31	1.37
2	Exact	2.00	1.53	0.97	1.22	0.94	0.76
	Approx.	2.08	1.84	1.97	1.37	1.21	1.29
5	Exact	1.52	1.28	0.92	1.17	0.94	0.76
	Approx.	1.81	1.50	1.66	1.28	1.06	1.17
10	Exact	1.35	1.04	0.85	1.12	0.92	0.75
	Approx.	1.63	1.26	1.44	1.21	0.95	1.08
20	Exact	1.24	1.01	0.83	1.03	0.89	0.75
	Approx.	1.44	1.05	1.25	1.13	0.83	0.99

Table 4.4 Exact and Approximate Maximum Amplification Factors, Systems 2A, 2B and 2C, Outer Spring

Effective mass ratio, γ, (%)	Type	Effective damping factor, β, (%)								
		0.5			1			2		
		System			System			System		
		2A	2B	2C	2A	2B	2C	2A	2B	2C
1	Exact	9.66	8.75	7.42	8.79	7.59	5.37	7.38	5.89	3.81
	Approx.	10.18	9.20	9.69	8.98	8.12	8.55	7.25	6.58	6.92
2	Exact	7.61	6.62	6.05	6.95	5.91	4.97	5.81	4.95	3.63
	Approx.	7.97	6.92	7.44	7.24	6.30	6.77	6.11	5.35	5.73
5	Exact	5.68	5.12	4.55	5.30	4.46	3.96	4.65	3.88	3.10
	Approx.	5.84	4.74	5.29	5.47	4.45	4.96	4.85	3.97	4.41
10	Exact	4.37	3.95	3.49	4.17	3.72	3.07	3.81	3.31	2.58
	Approx.	4.76	3.63	4.19	4.49	3.42	3.95	4.09	3.14	3.61
20	Exact	3.18	3.18	3.03	3.04	3.05	2.68	2.79	2.80	2.24
	Approx.	3.87	2.77	3.35	3.68	2.65	3.20	3.44	2.48	2.99

Table 4.4 (continued)

Effective mass ratio, γ, (%)	Type	Effective damping factor, β, (%)					
		5			10		
		System			System		
		2A	2B	2C	2A	2B	2C
1	Exact	4.46	3.40	2.04	2.72	1.98	1.34
	Approx.	4.60	4.20	4.40	2.86	2.62	2.74
2	Exact	4.13	3.14	2.01	2.63	1.96	1.34
	Approx.	4.17	3.68	3.93	2.73	2.42	2.58
5	Exact	3.51	2.83	1.91	2.36	1.88	1.33
	Approx.	3.62	2.99	3.31	2.55	2.12	2.34
10	Exact	2.94	2.53	1.85	2.00	1.76	1.31
	Approx.	3.25	2.51	2.88	2.42	1.89	2.15
20	Exact	2.18	2.20	1.79	1.51	1.55	1.27
	Approx.	2.87	2.09	2.50	2.25	1.65	1.97

Table 4.5 Exact and Approximate Maximum Amplification Factors, System 2A, γ = 1%, Inner and Outer Secondary Springs, β_p = 2%, β_s = 0, 0.5, 1, 2, 5 and 10%

Type of solution	Type of spring	Effective damping factor, β (%)					
		1	1.25	1.5	2	3.5	6
Exact	Inner	4.85	4.48	4.16	3.64	2.77	2.14
	Outer	9.4	8.79	8.25	7.38	5.57	4.13
Approx.	Inner	4.49	4.24	4.01	3.63	2.82	2.05
	Outer	8.98	8.47	8.02	7.25	5.63	4.10

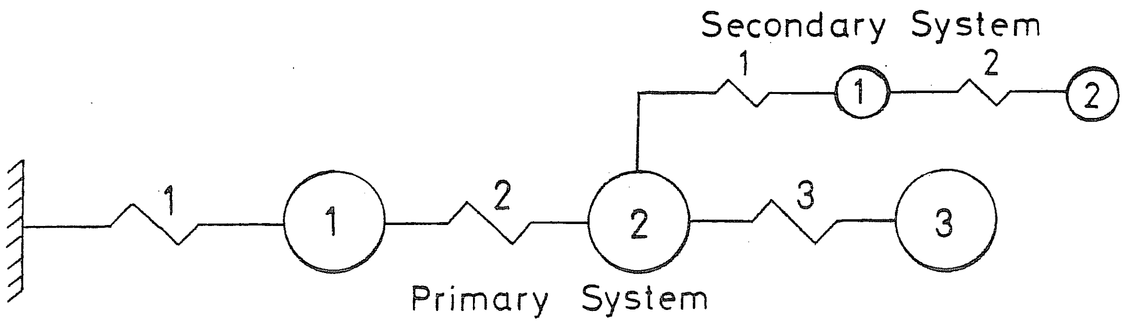


FIG. 2.1 PRIMARY AND SECONDARY SYSTEM

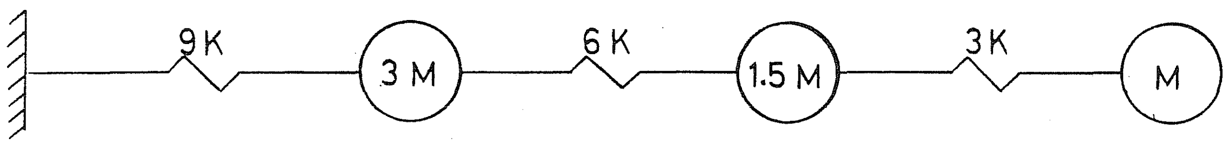


FIG. 2.2 PRIMARY SYSTEM CONSIDERED

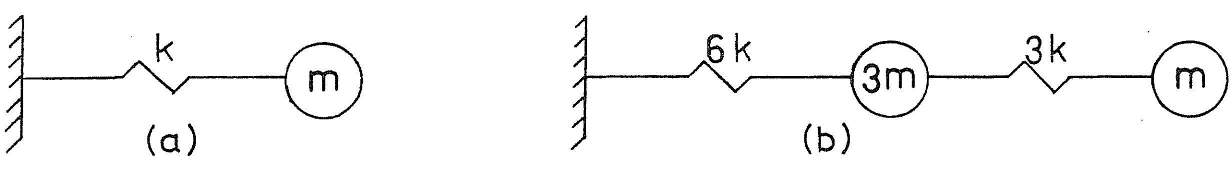


FIG. 2.3 SECONDARY SYSTEMS CONSIDERED

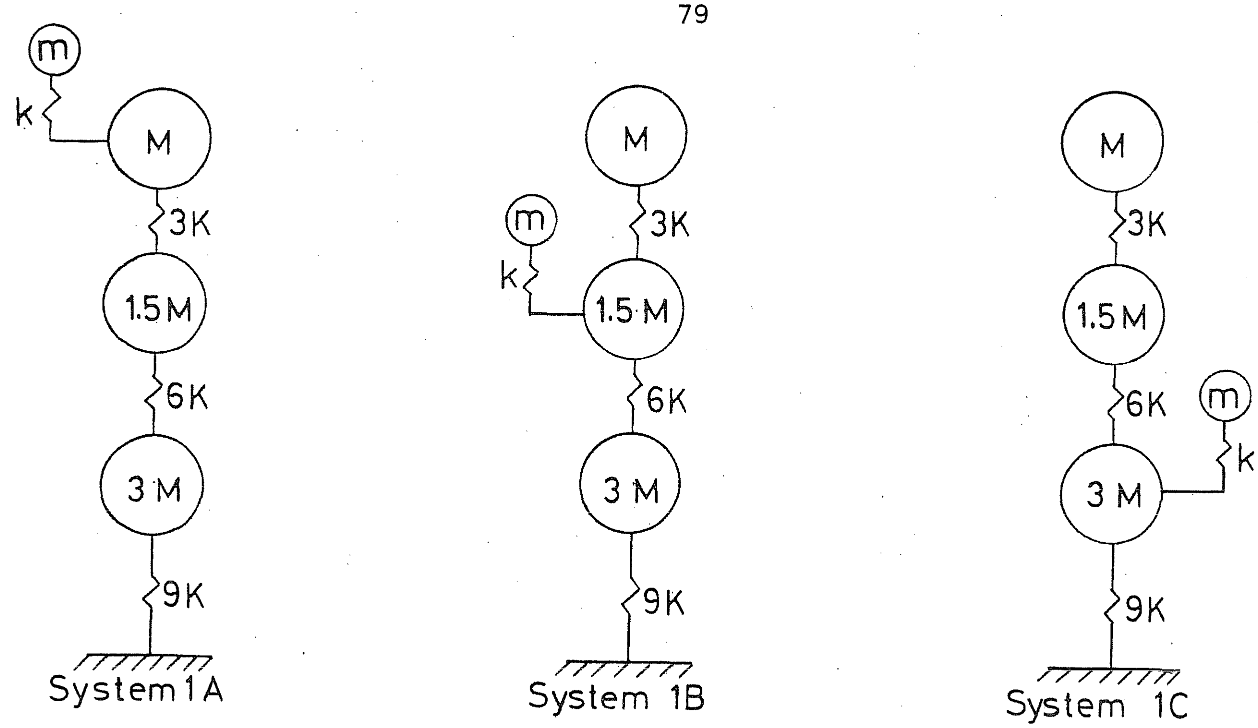


FIG. 2.4 FOUR DEGREE OF FREEDOM COMBINED SYSTEMS CONSIDERED

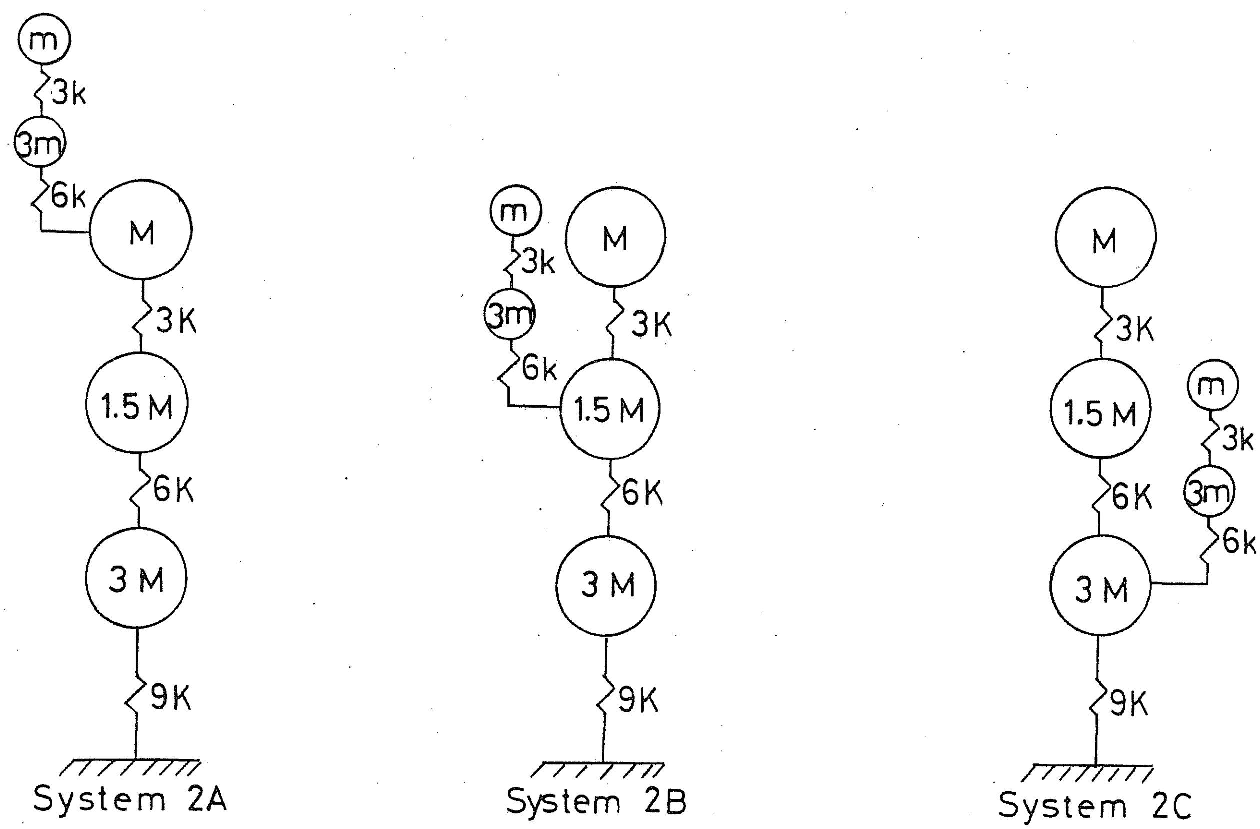


FIG. 2.5 FIVE DEGREE OF FREEDOM COMBINED SYSTEMS CONSIDERED

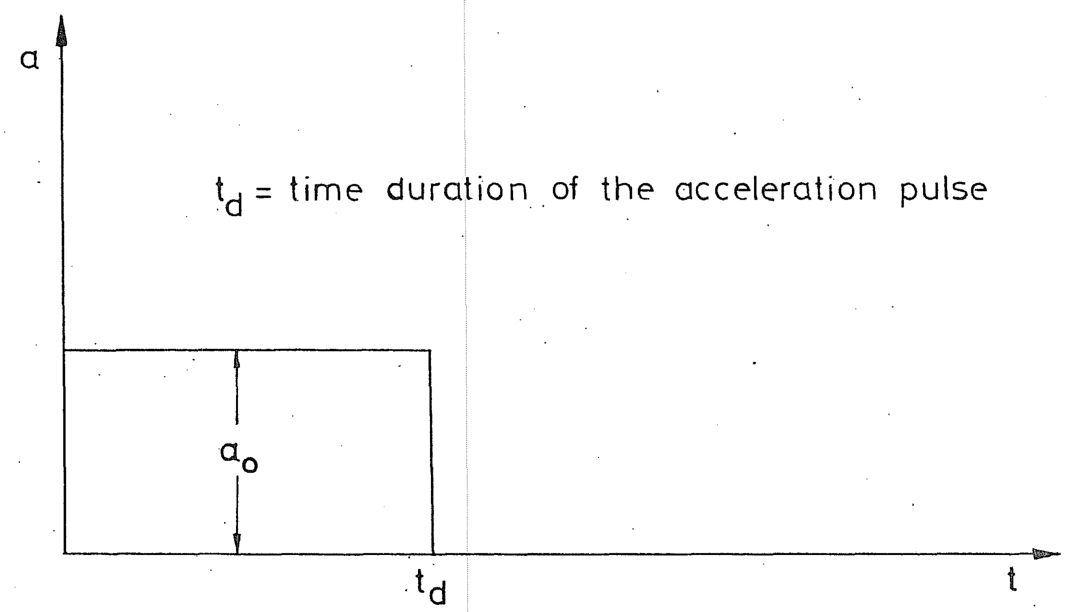


FIG. 2.6a CONSTANT BASE ACCELERATION PULSE

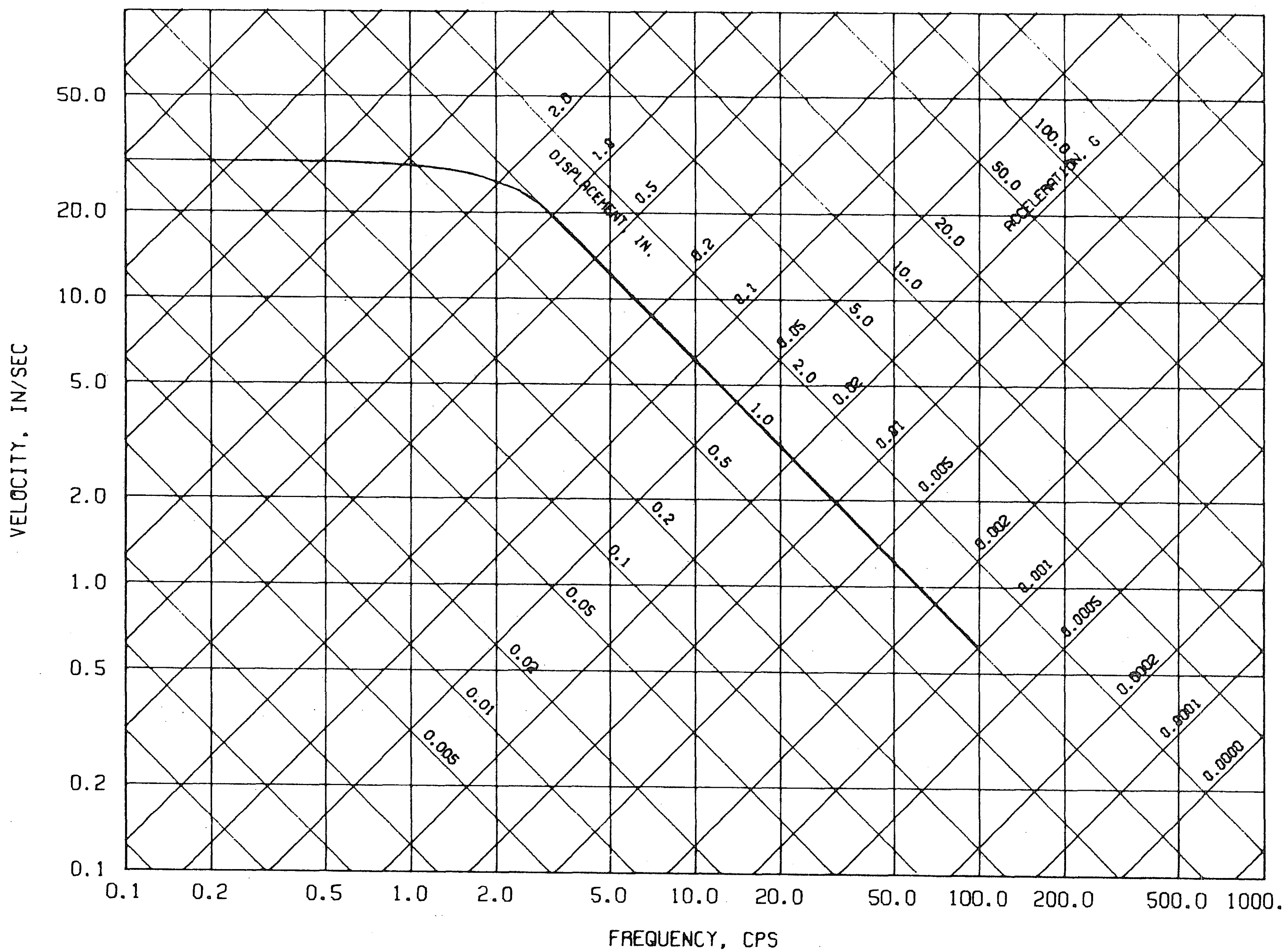


FIG. 2.6b RESPONSE SPECTRUM, 0.5 G CONSTANT ACCELERATION PULSE, TIME DURATION = 0.155 SEC, NO DAMPING

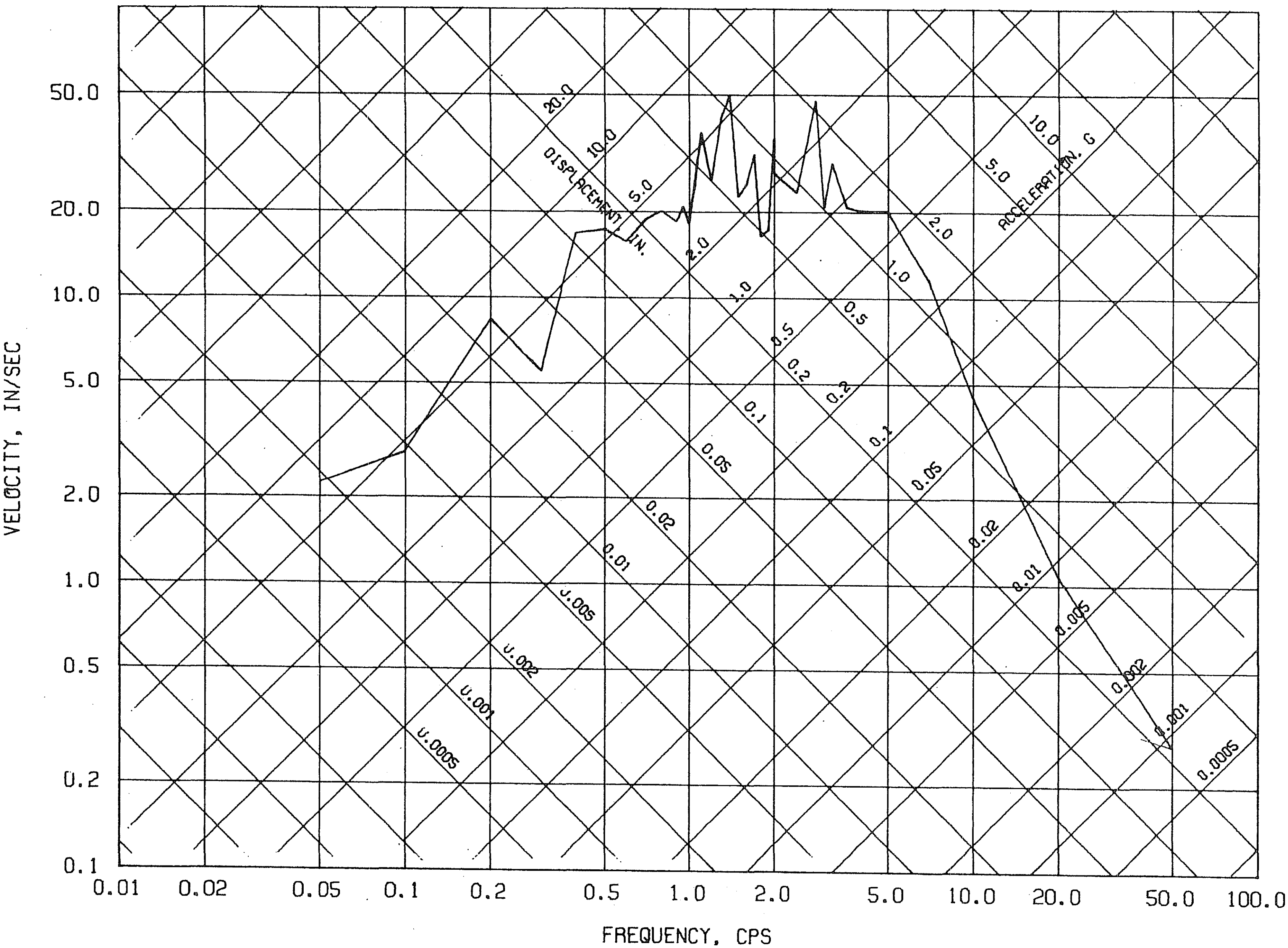


FIG. 2.7 RESPONSE SPECTRUM, ELASTIC SYSTEM, TAFT N21E, 1952 EARTHQUAKE, NO DAMPING

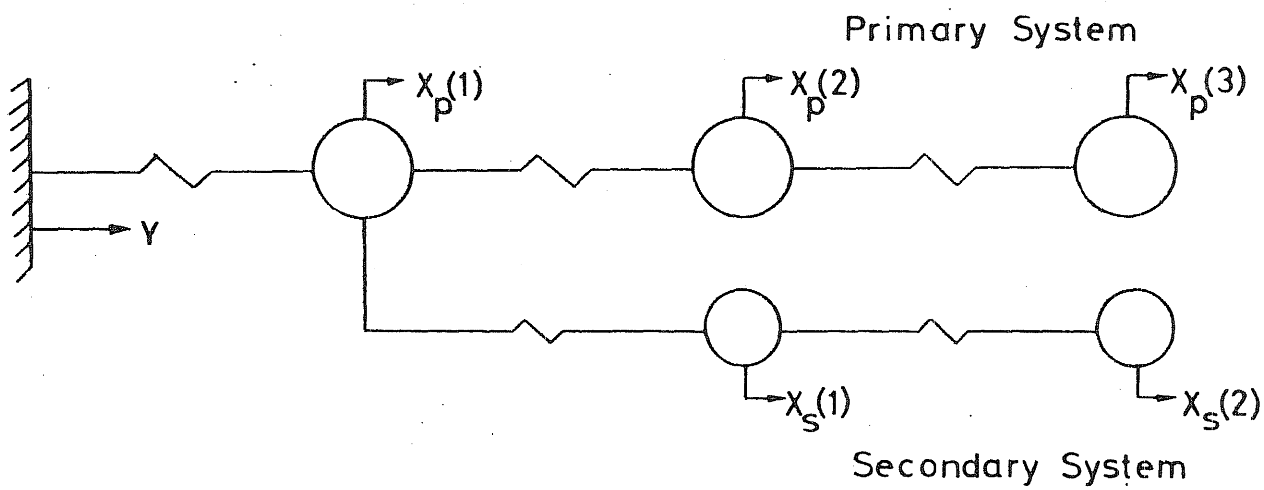


FIG. 2.8 ILLUSTRATION OF INPUT TO THE COMBINED SYSTEM

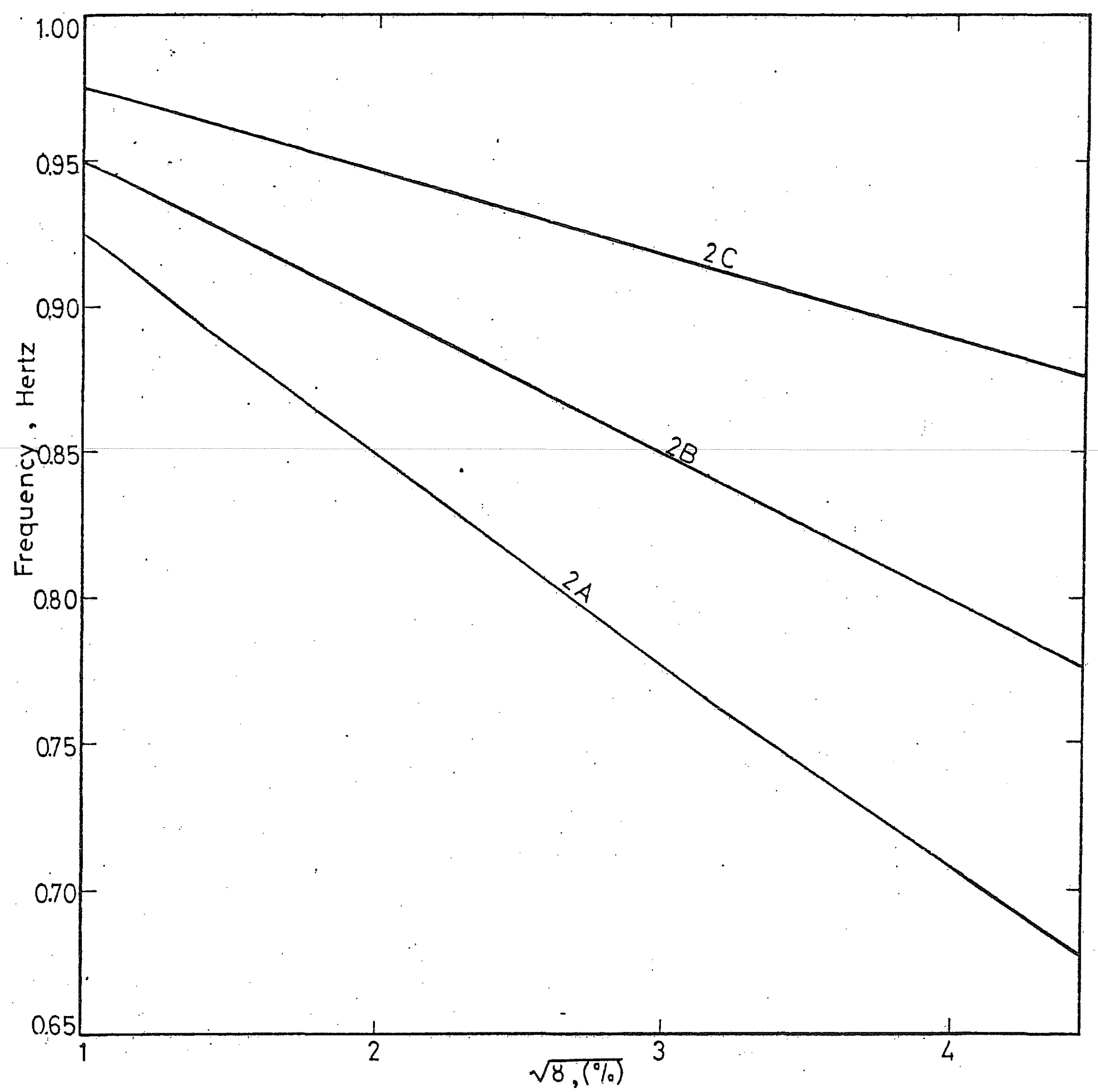


FIG. 3.1 FUNDAMENTAL FREQUENCY VARIATIONS OF SYSTEMS 2A, 2B AND 2C

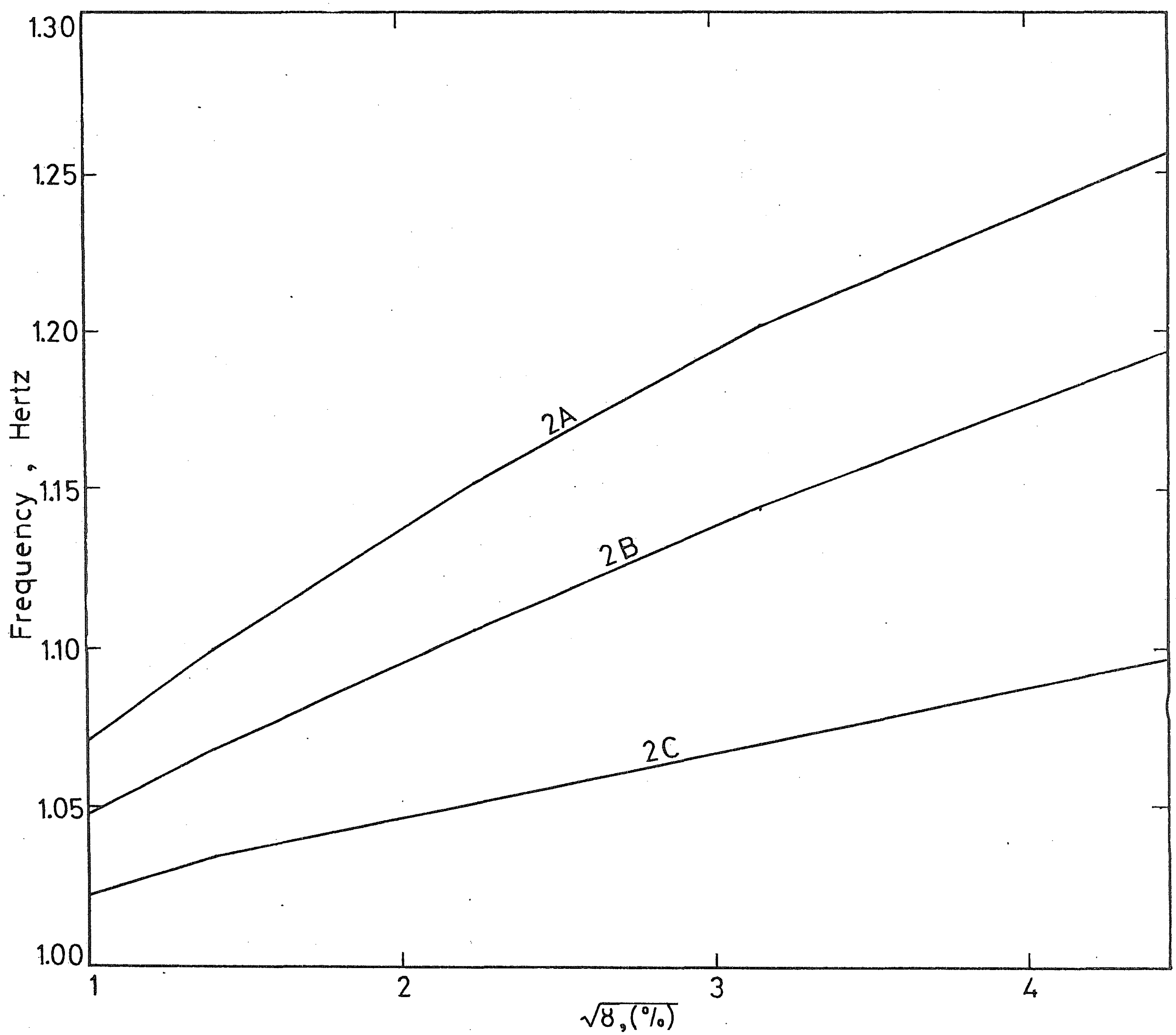


FIG. 3.2 SECOND NATURAL FREQUENCY VARIATIONS OF SYSTEMS 2A, 2B AND 2C

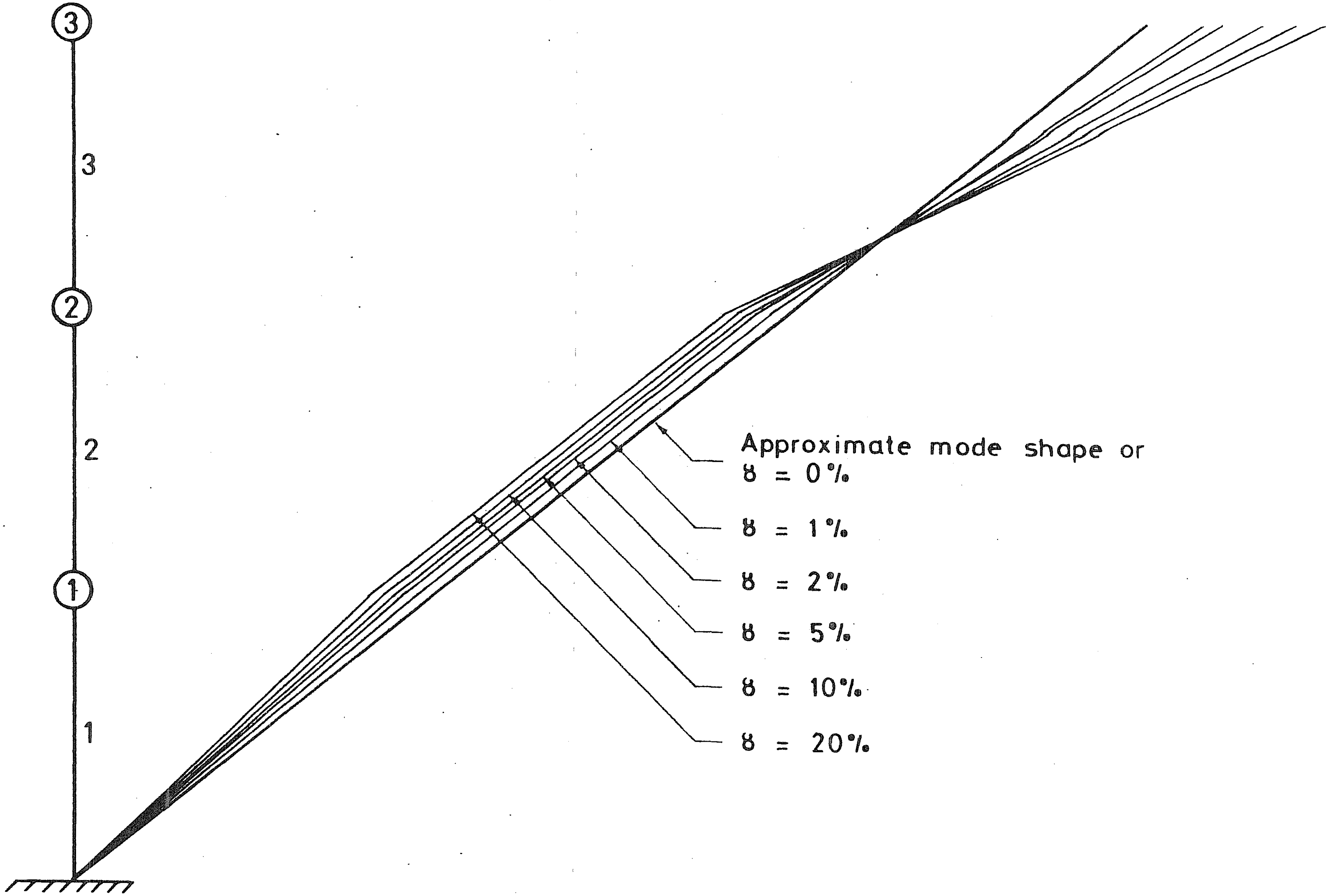


FIG. 3.3a FUNDAMENTAL MODE SHAPE VARIATIONS OF SYSTEM 2A, PRIMARY MASSES ONLY

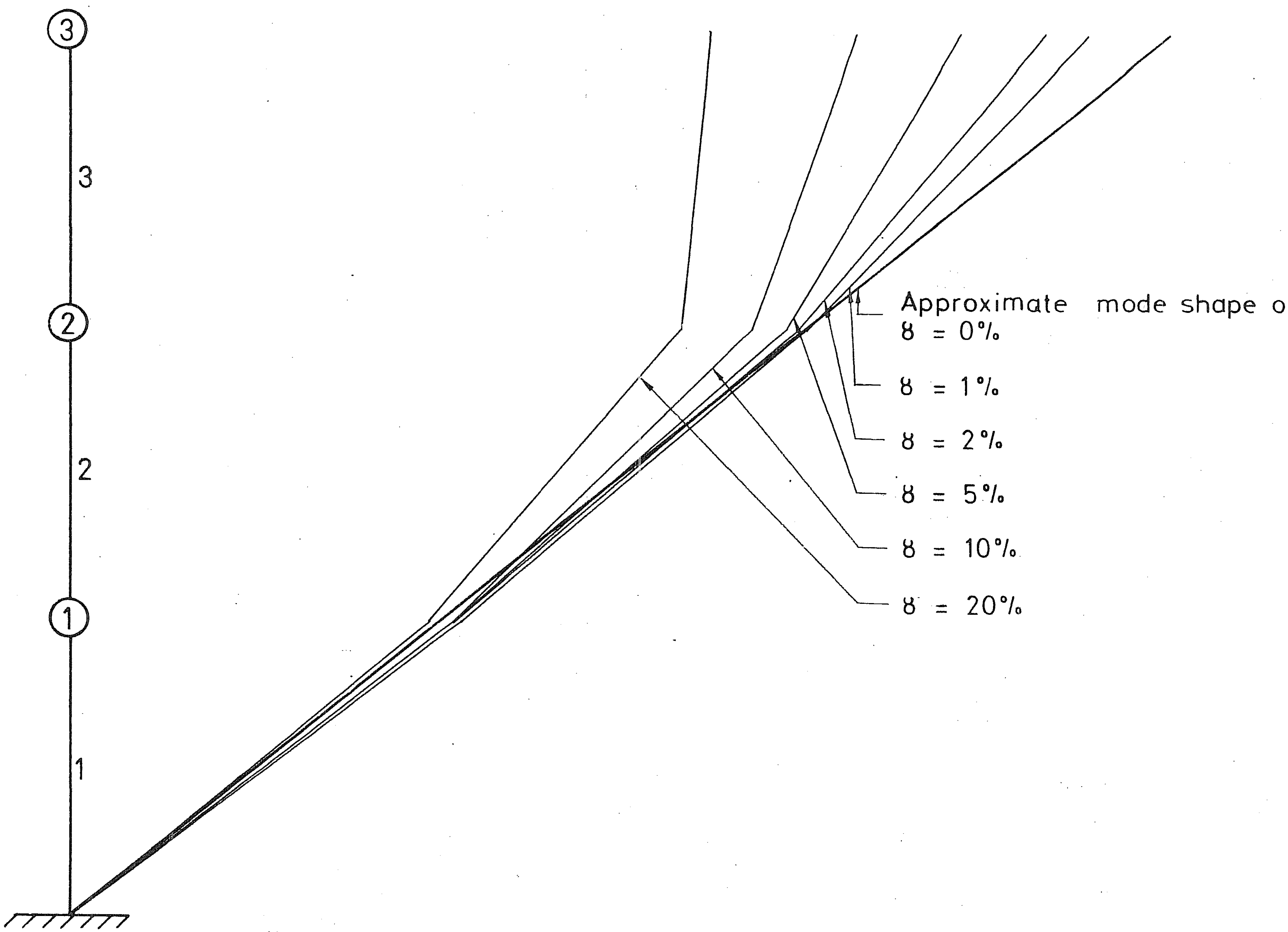


FIG. 3.3b SECOND MODE SHAPE VARIATIONS OF SYSTEM 2A, PRIMARY MASSES ONLY

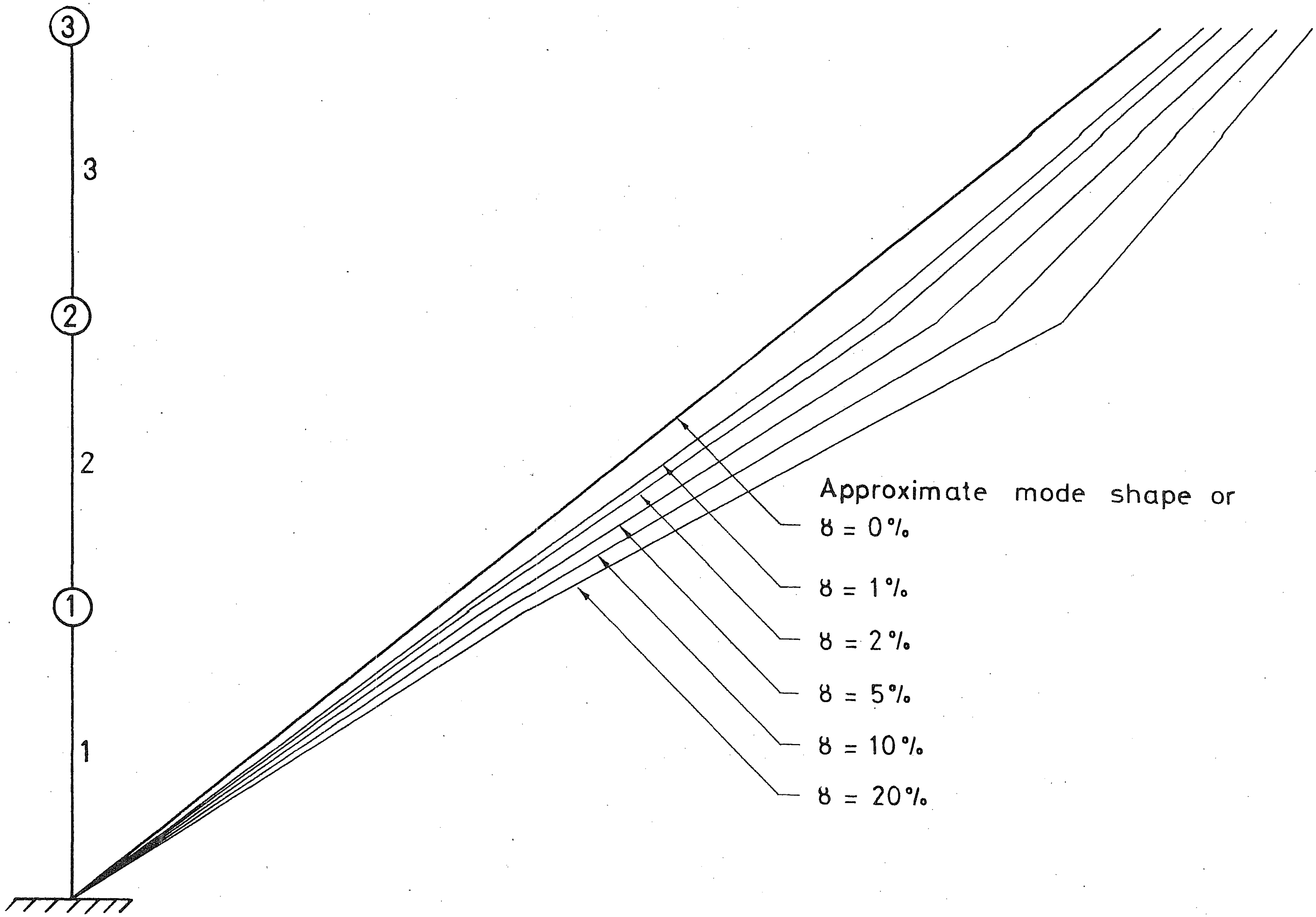


FIG. 3.4a FUNDAMENTAL MODE SHAPE VARIATIONS OF SYSTEM 2B, PRIMARY MASSES ONLY

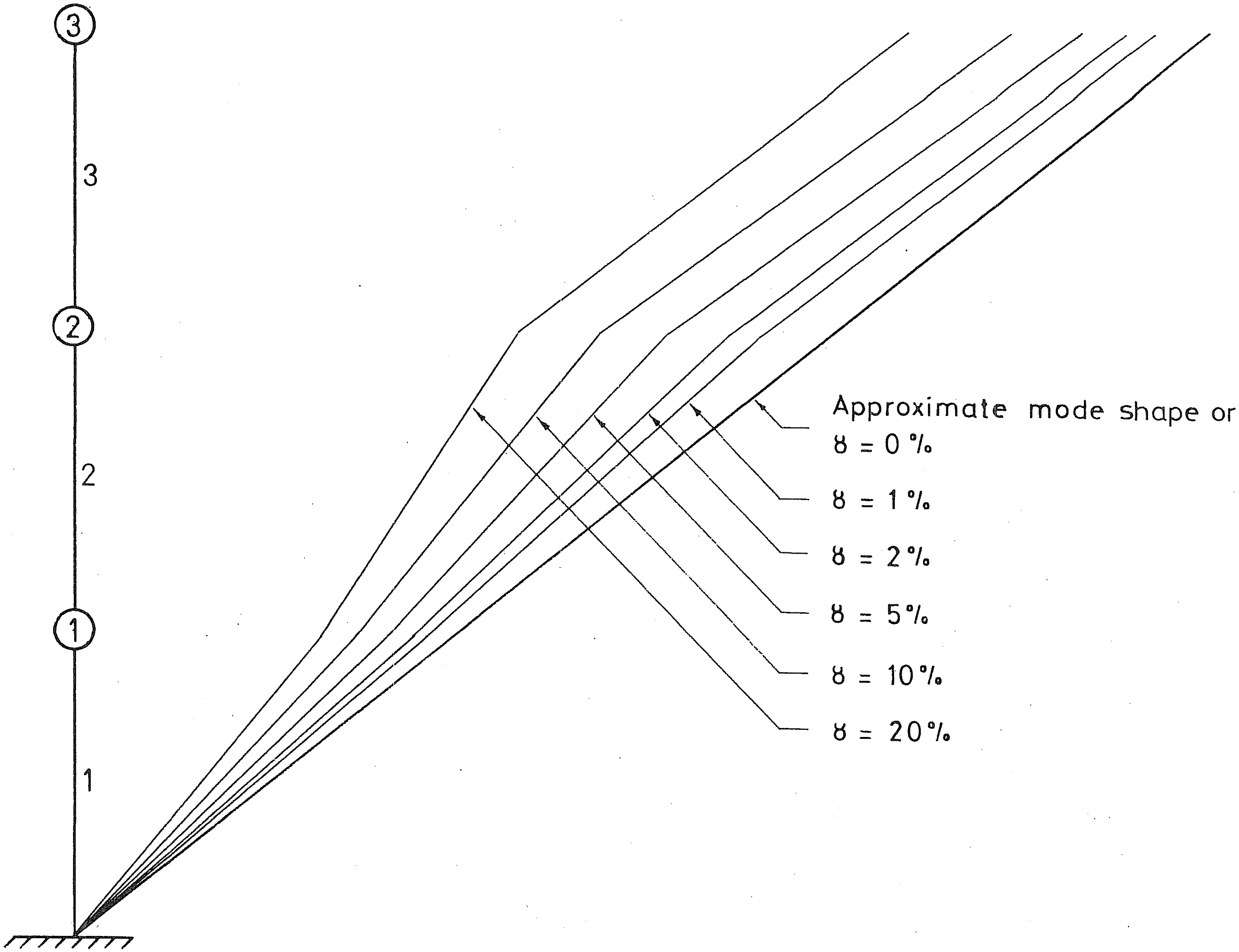


FIG. 3.4b SECOND MODE SHAPE VARIATIONS OF SYSTEM 2B, PRIMARY MASSES ONLY

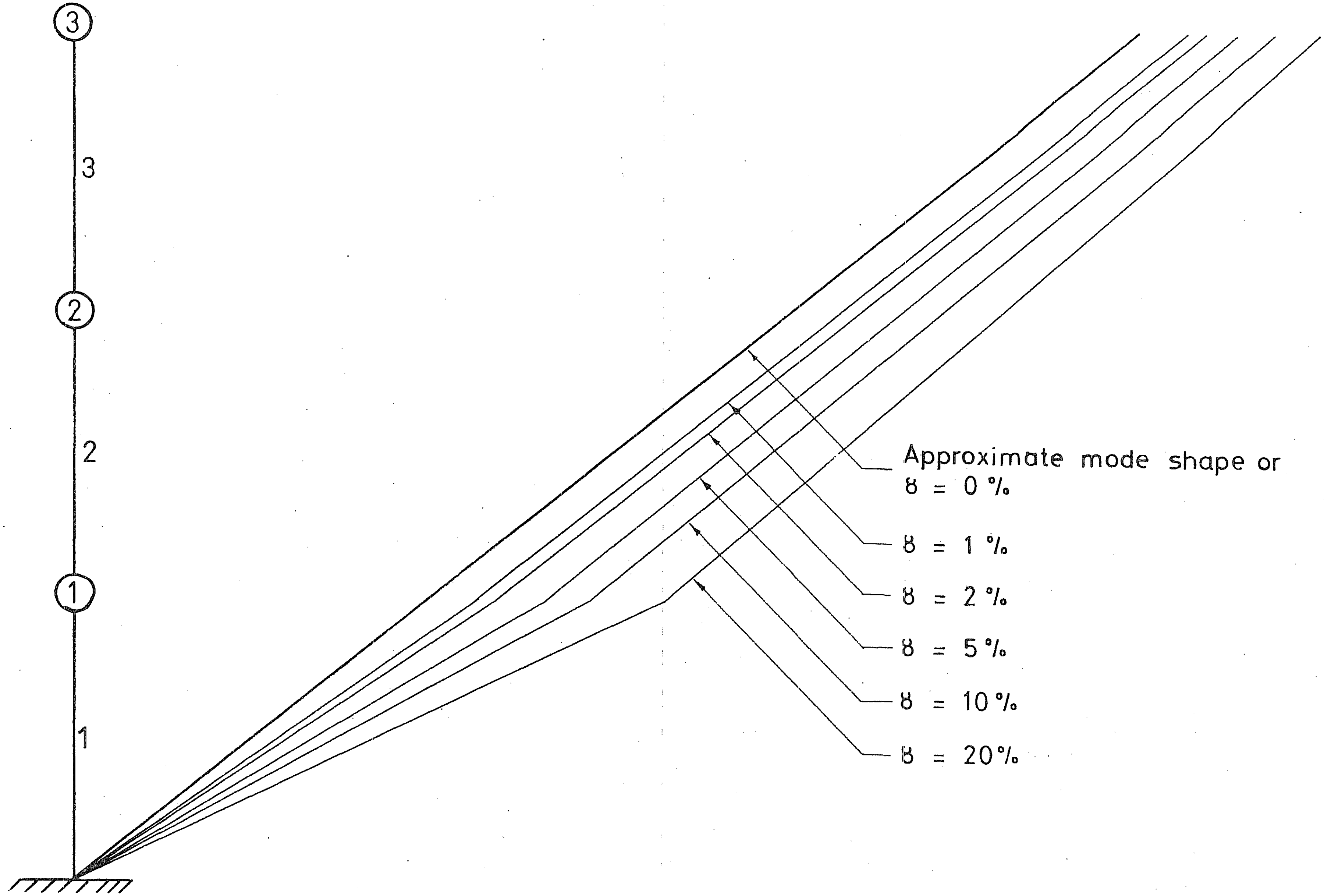


FIG. 3.5a FUNDAMENTAL MODE SHAPE VARIATIONS OF SYSTEM 2C, PRIMARY MASSES ONLY

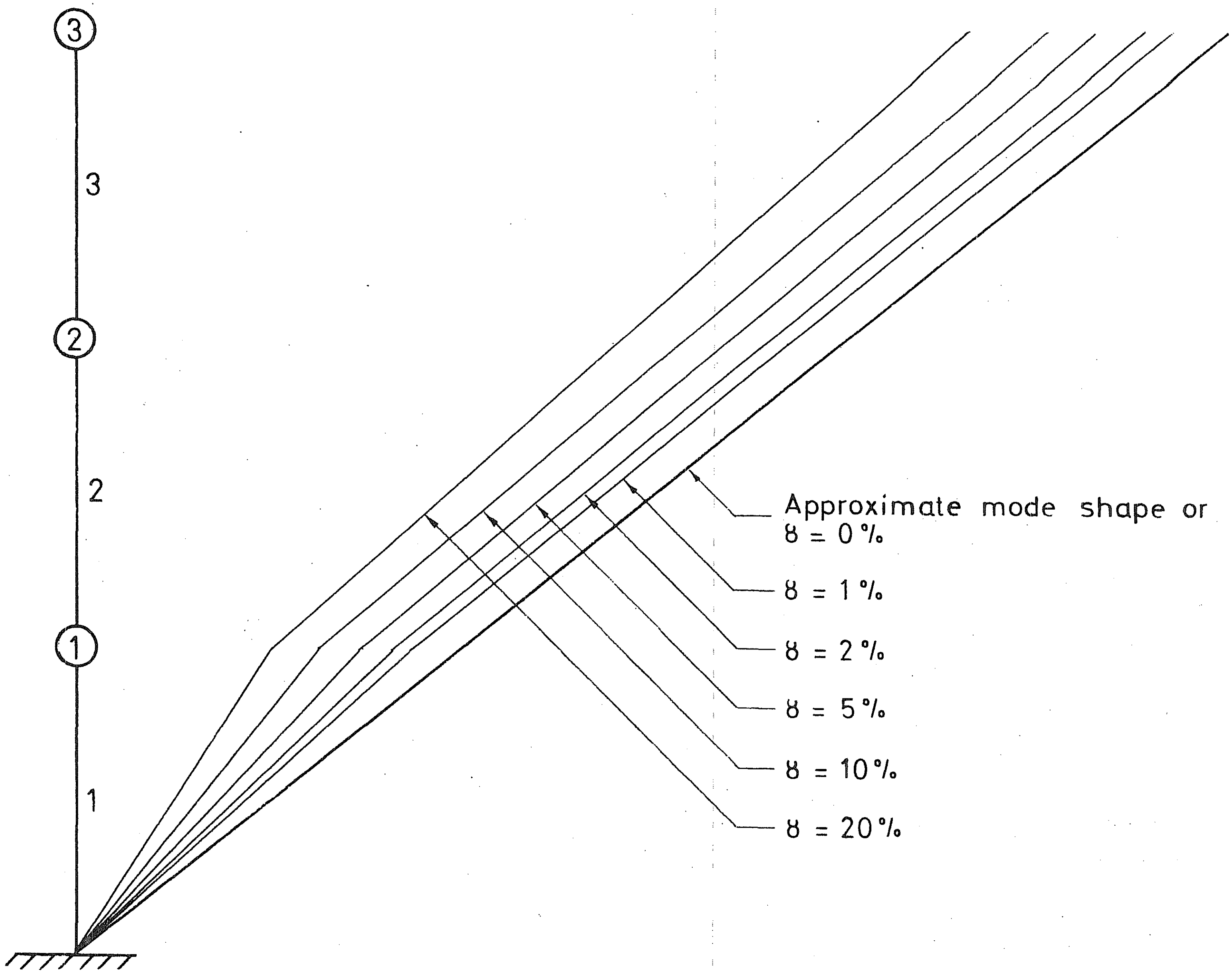


FIG. 3.5b SECOND MODE SHAPE VARIATIONS OF SYSTEM 2C, PRIMARY MASSES ONLY

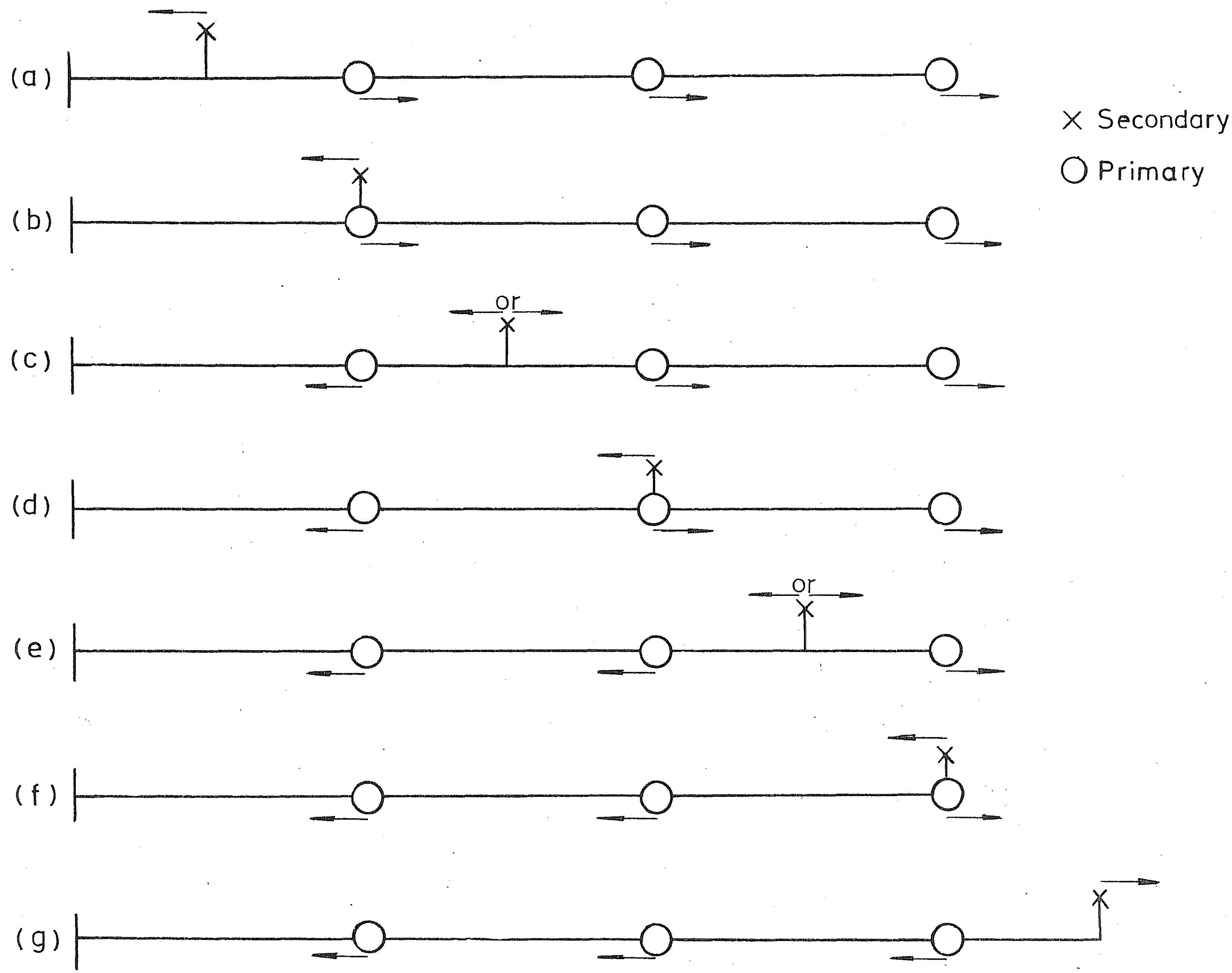


FIG. 3.6 OBSERVED FREQUENCY DISTRIBUTIONS OF SYSTEMS 1A, 1B AND 1C

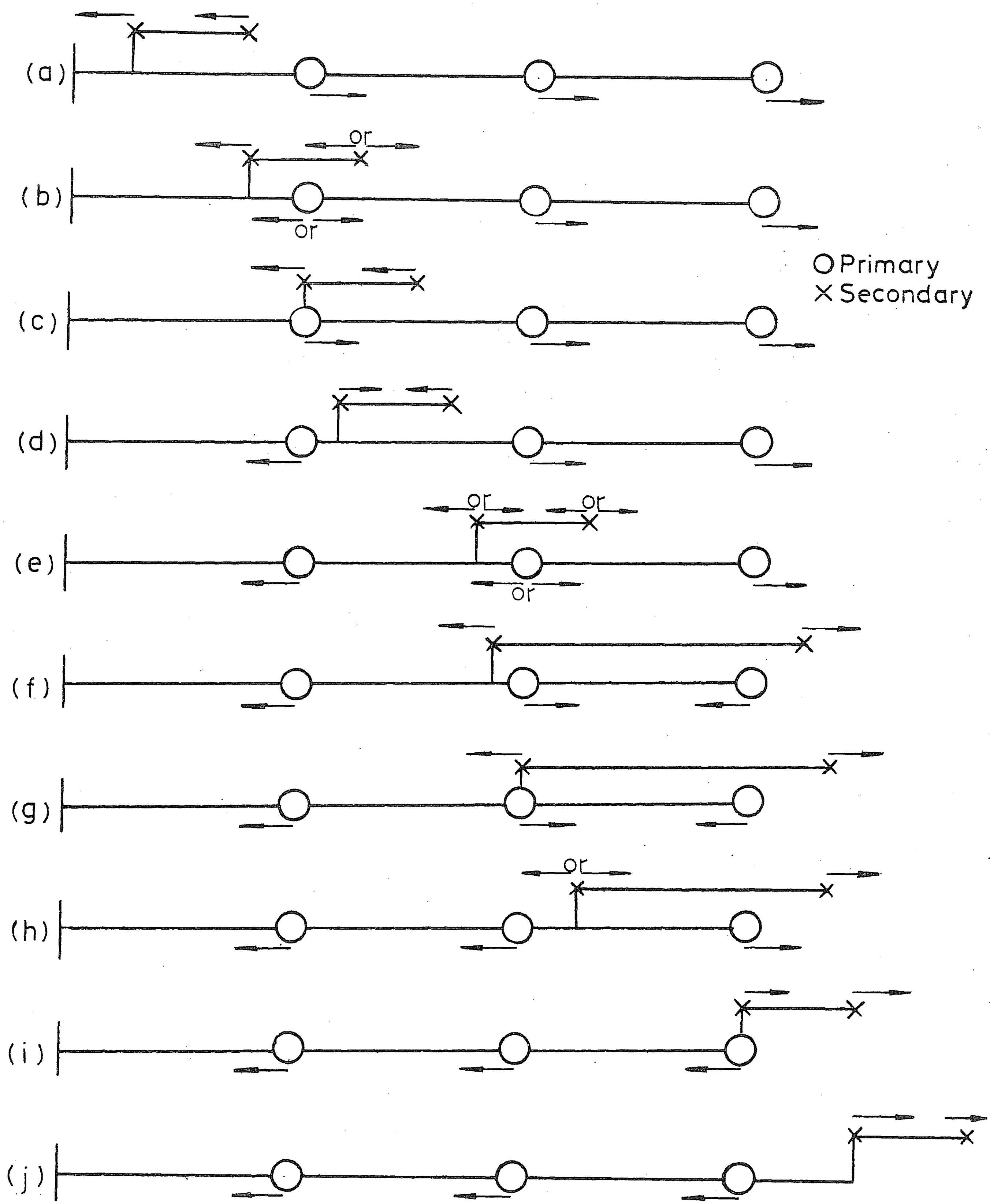


FIG. 3.7 OBSERVED FREQUENCY DISTRIBUTIONS OF SYSTEMS 2A, 2B AND 2C

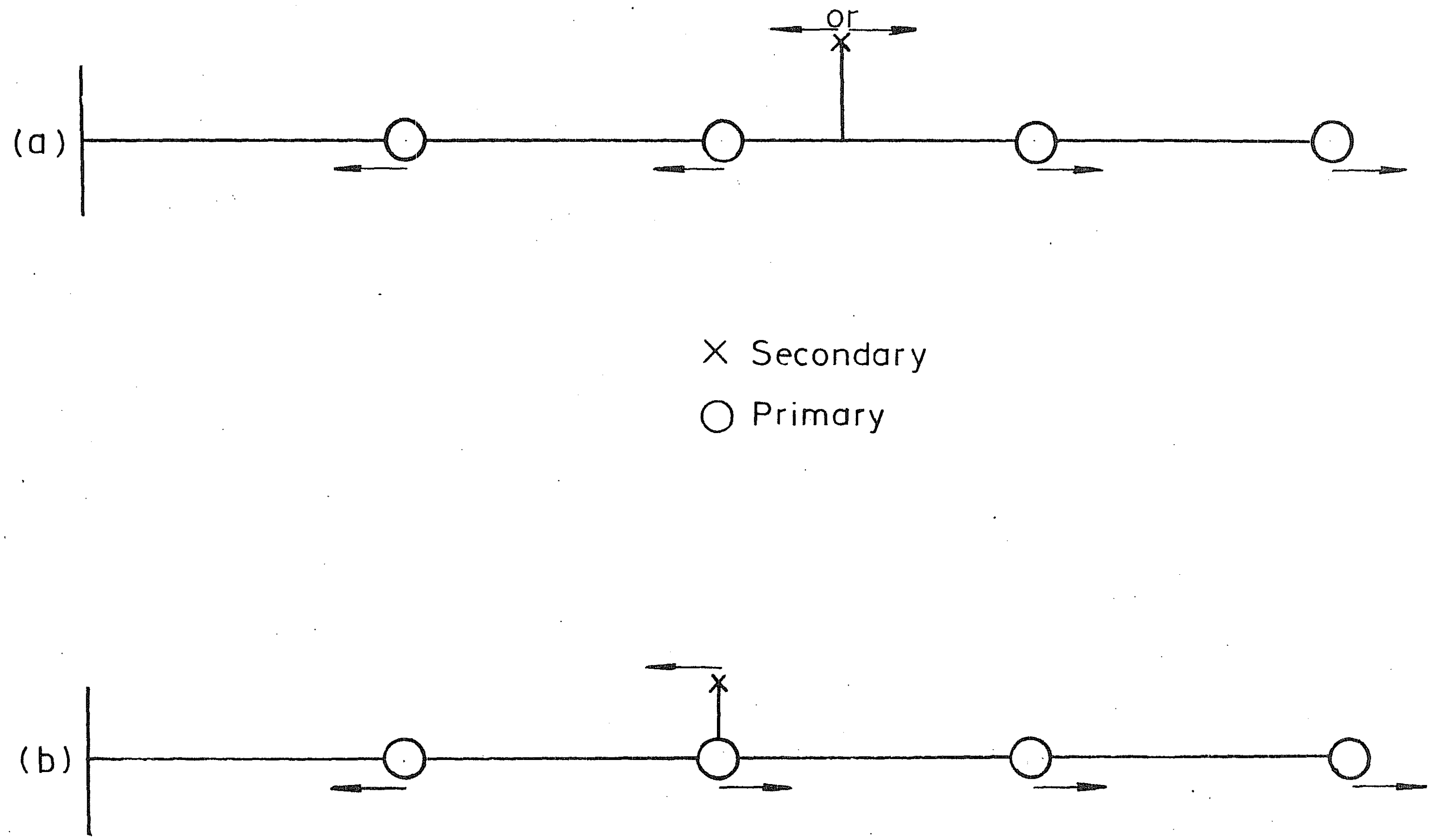


FIG. 3.8 FREQUENCY DISTRIBUTION OF COMBINED SYSTEM

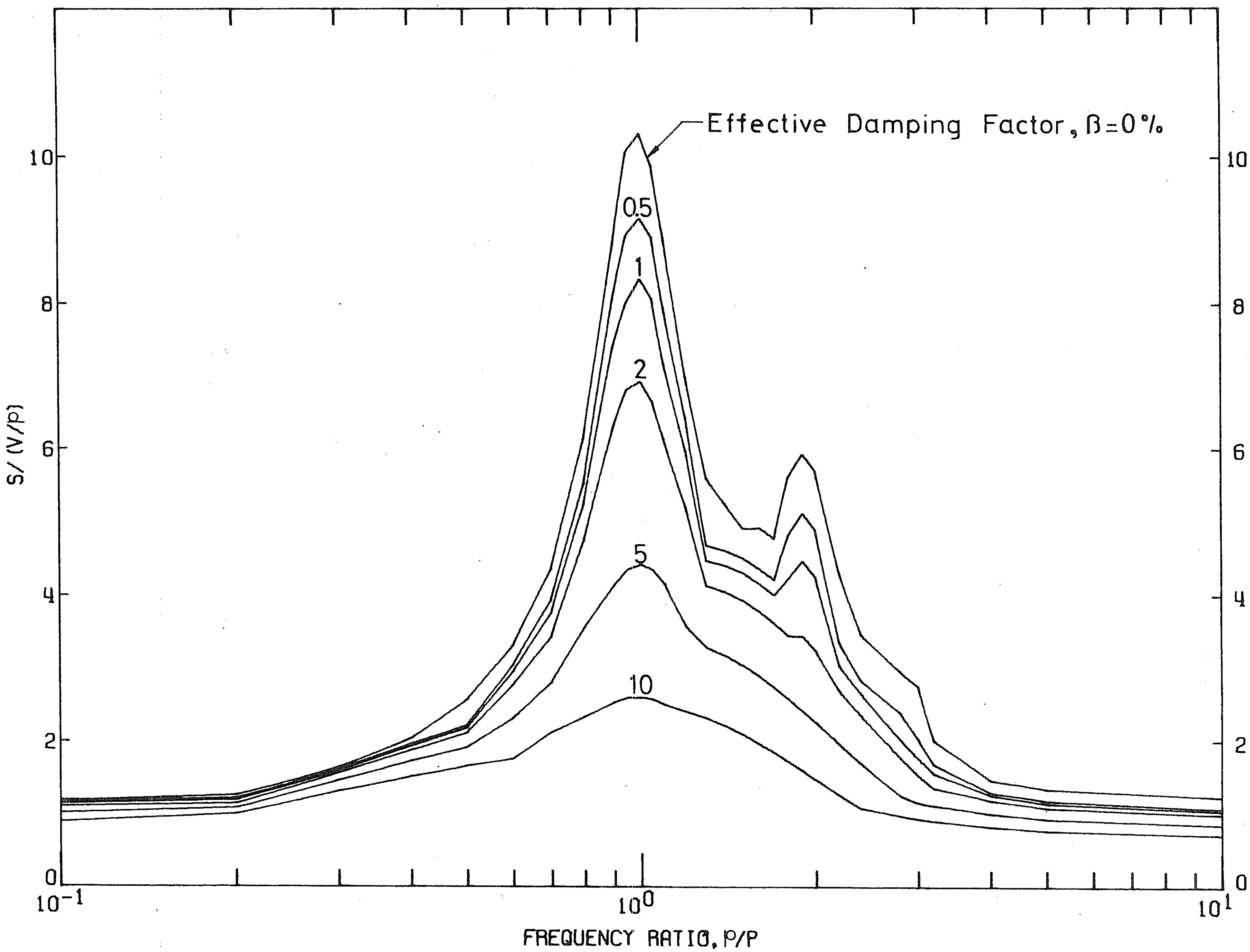


FIG. 4.1 SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 1A, EFFECTIVE MASS RATIO = 1 %, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

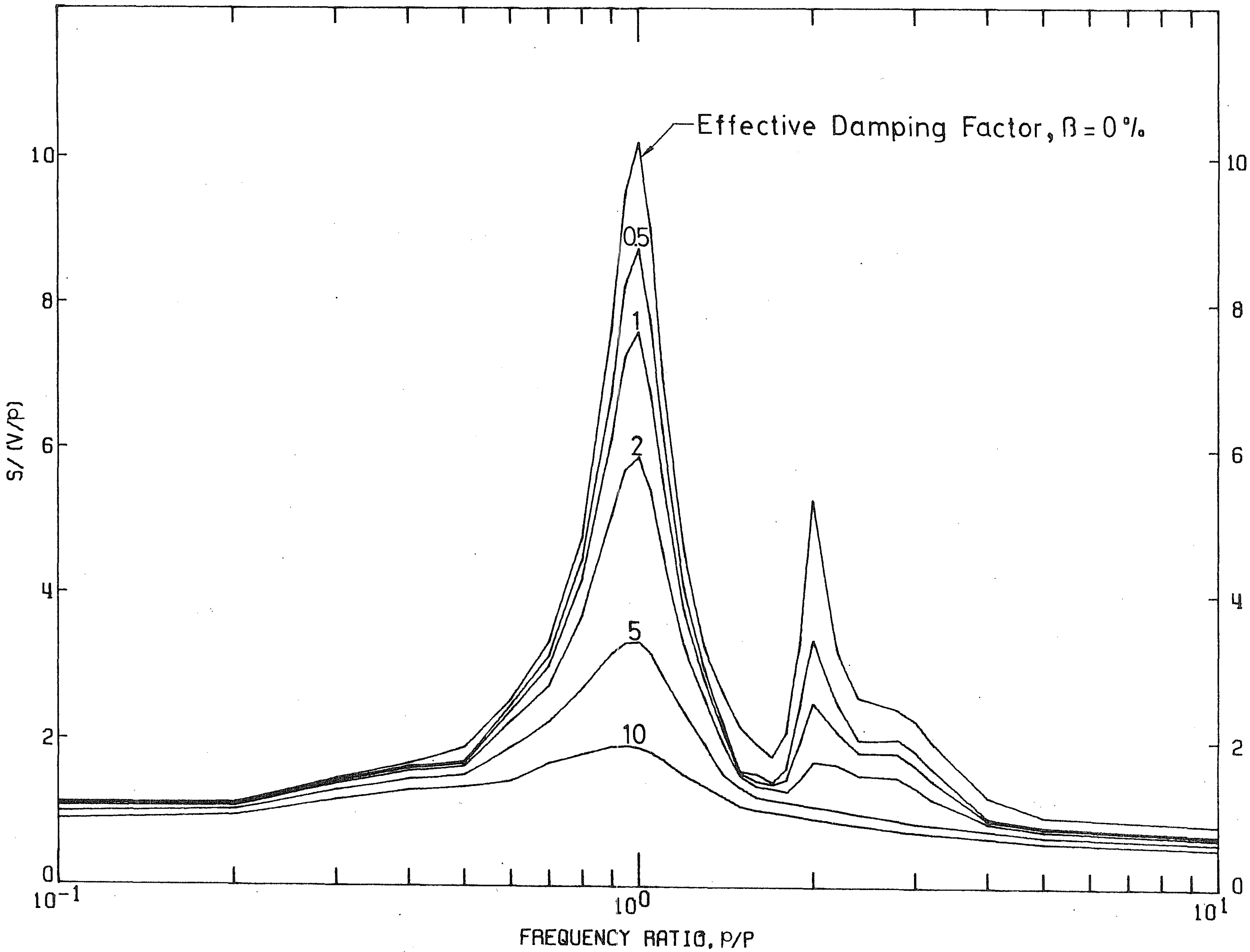


FIG. 4.2 SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 1B, EFFECTIVE MASS RATIO = 1 %, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

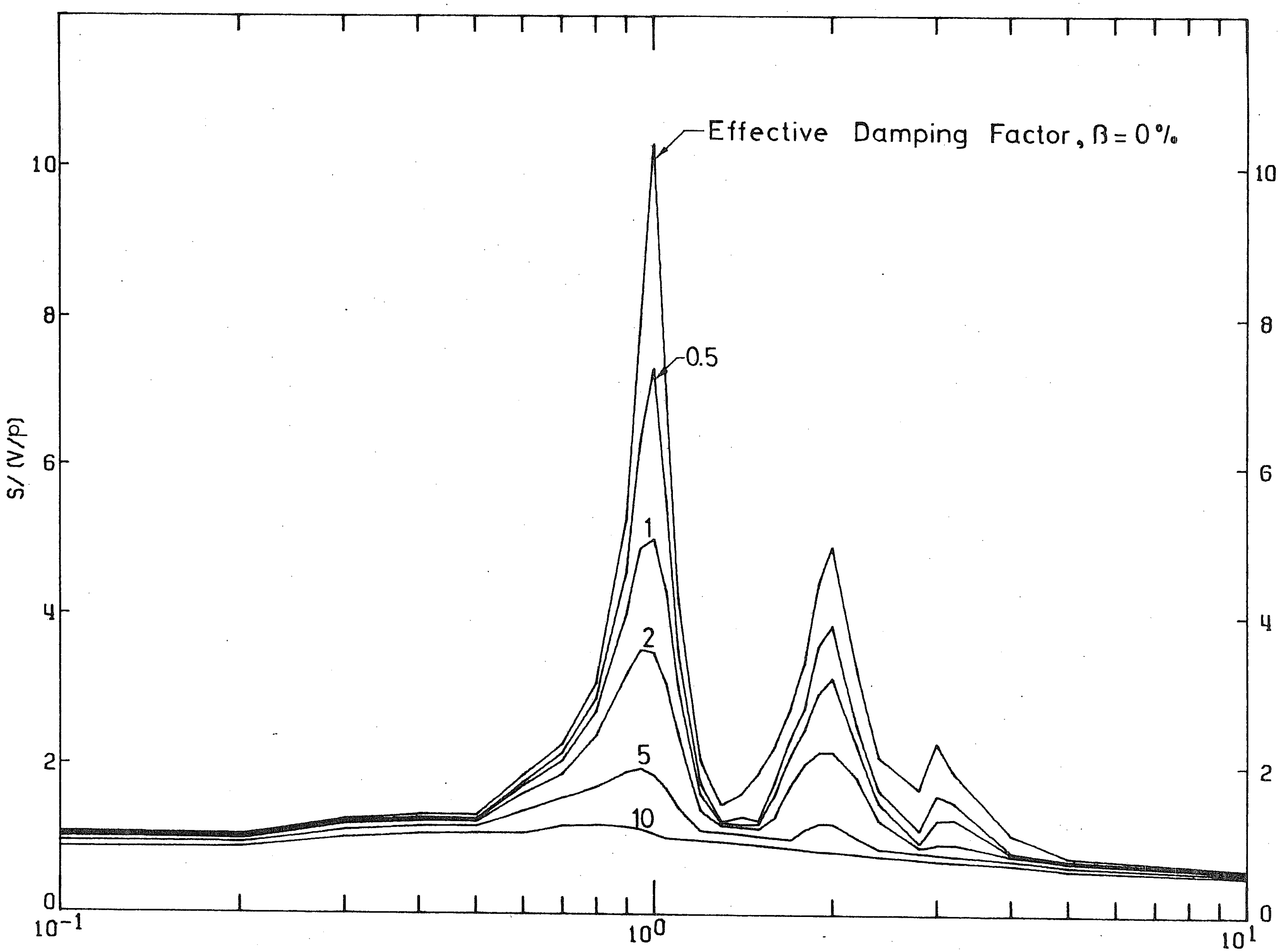


FIG. 4.3 SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 1C, EFFECTIVE MASS RATIO = 1 %, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

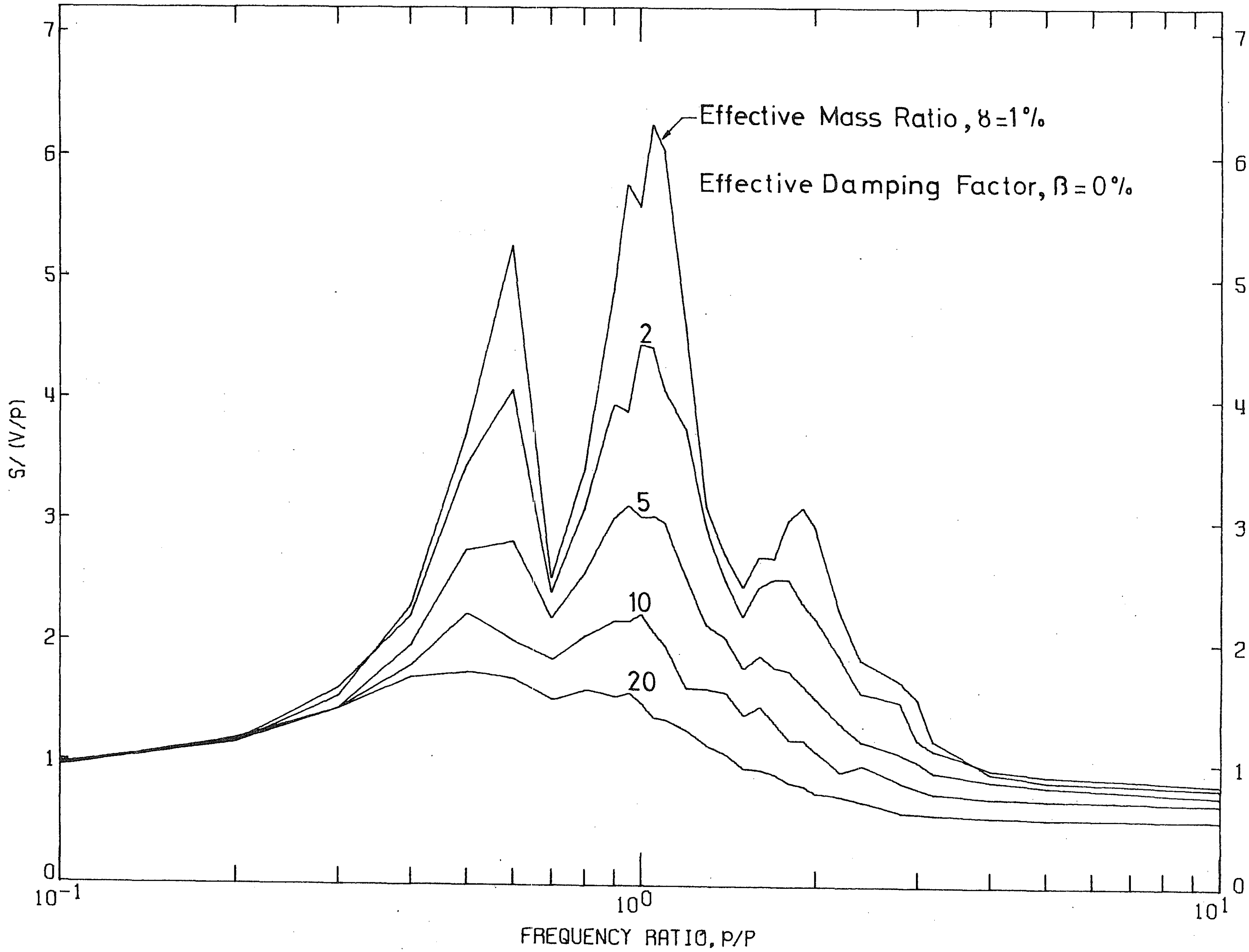


FIG. 4.4a INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

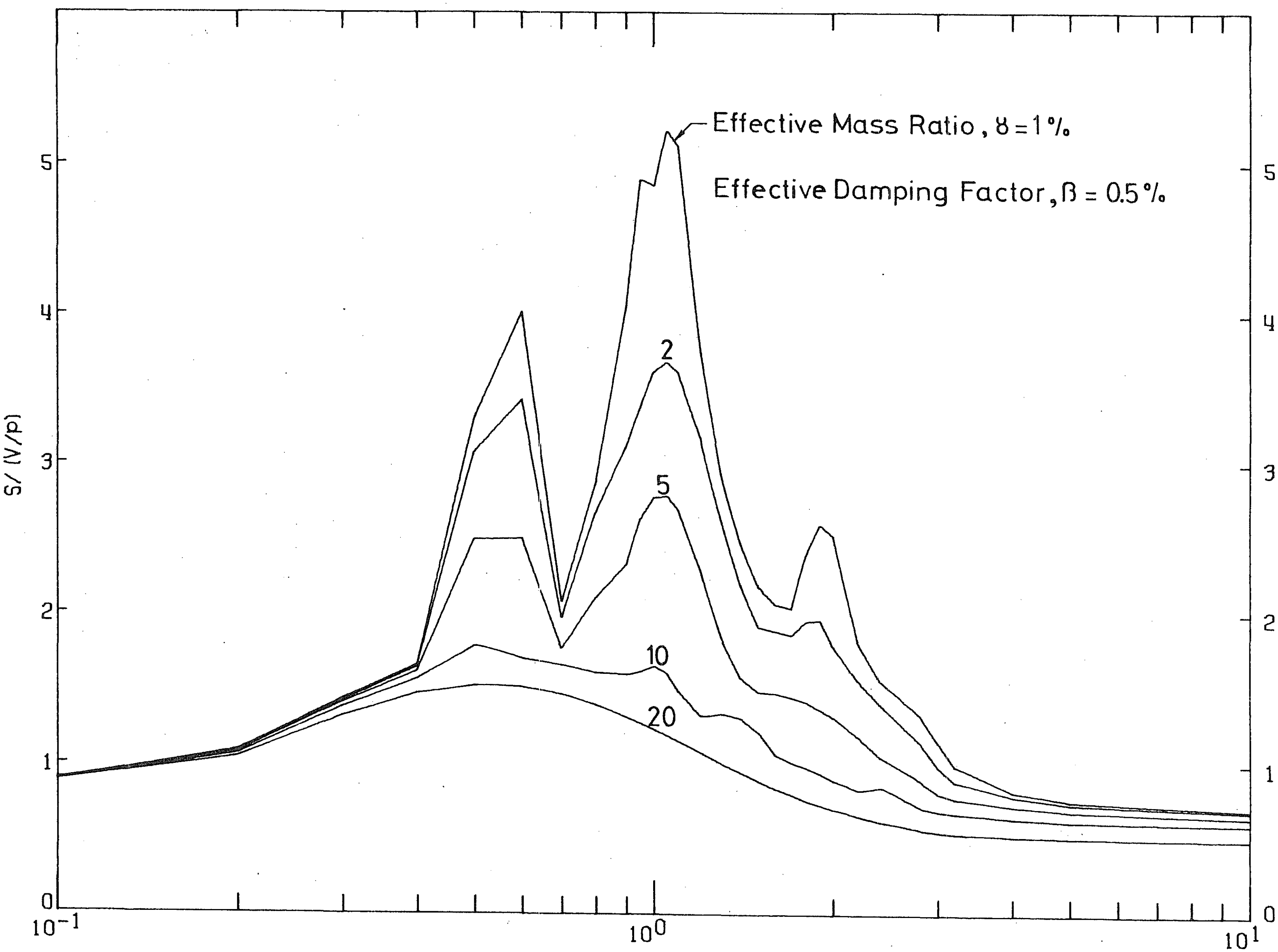


FIG. 4.4b INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

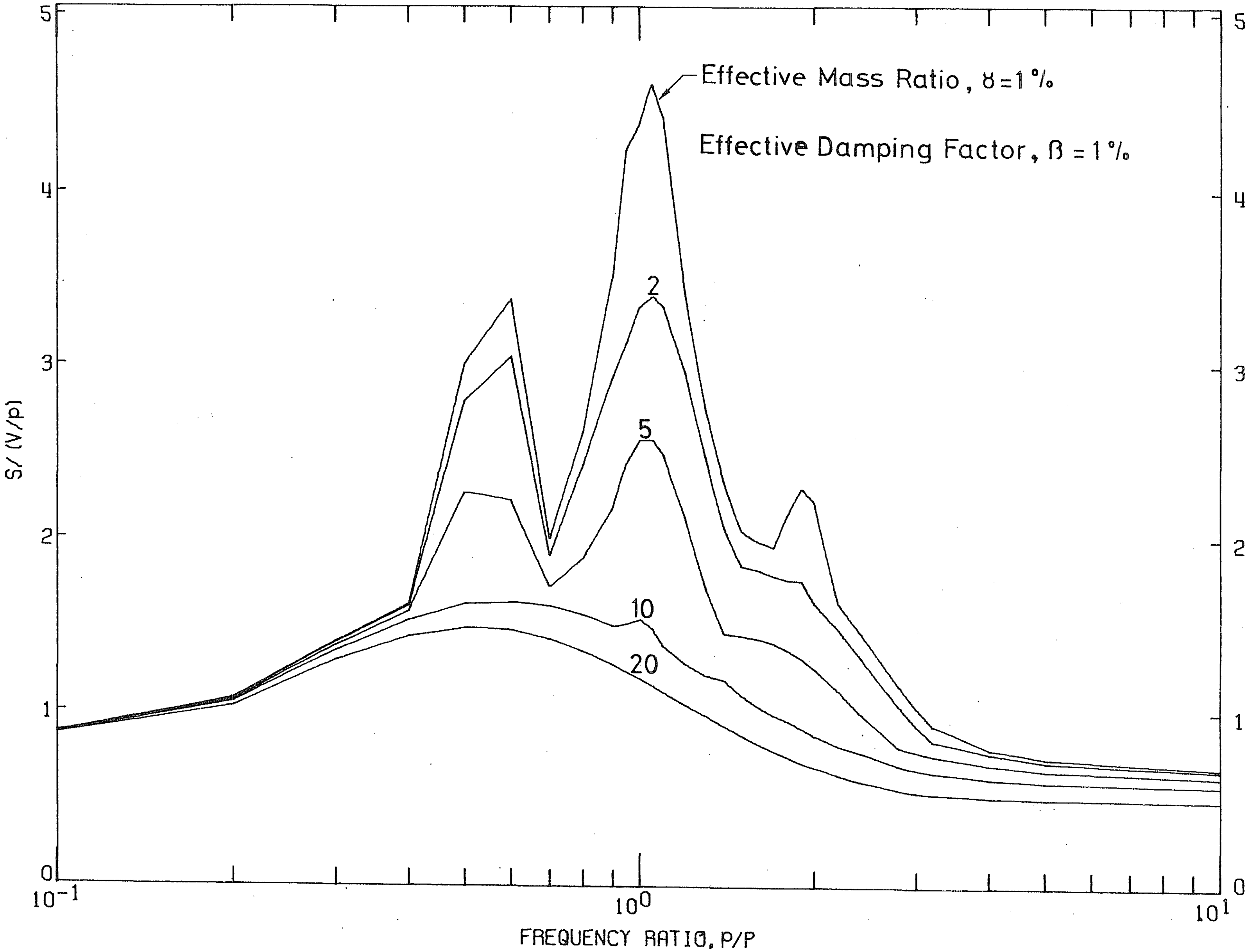


FIG. 4.4c INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

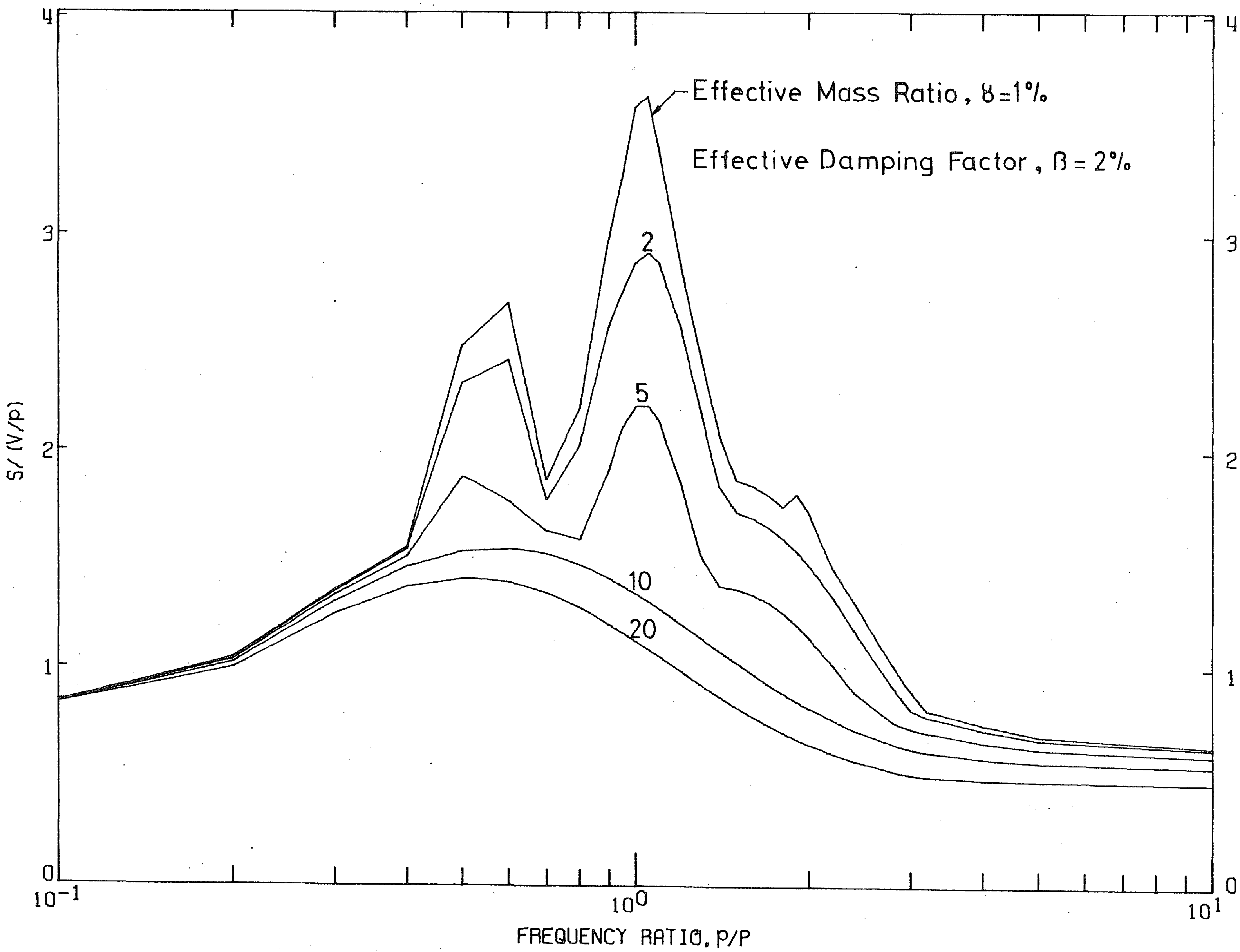


FIG. 4.4d INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

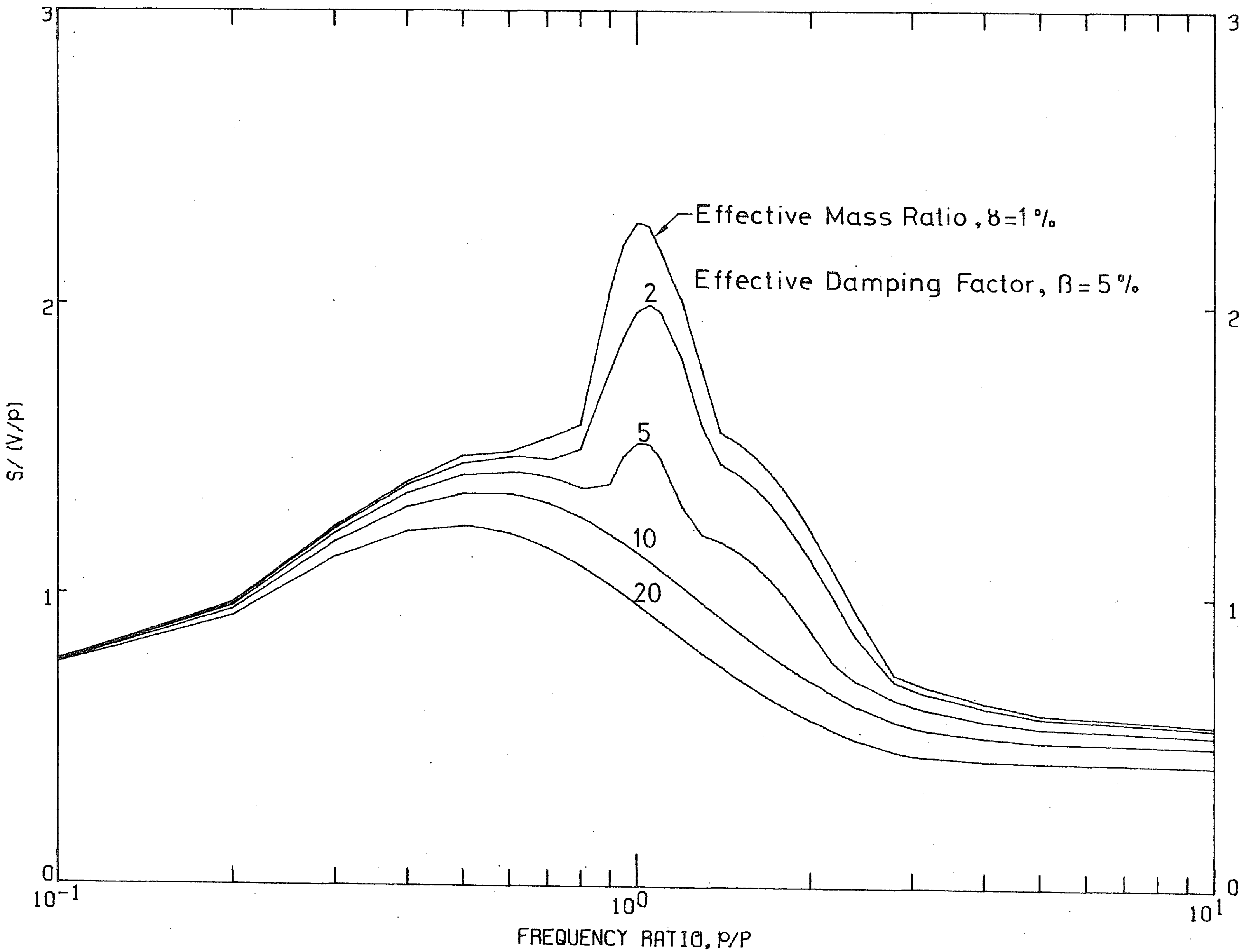


FIG. 4.4e INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

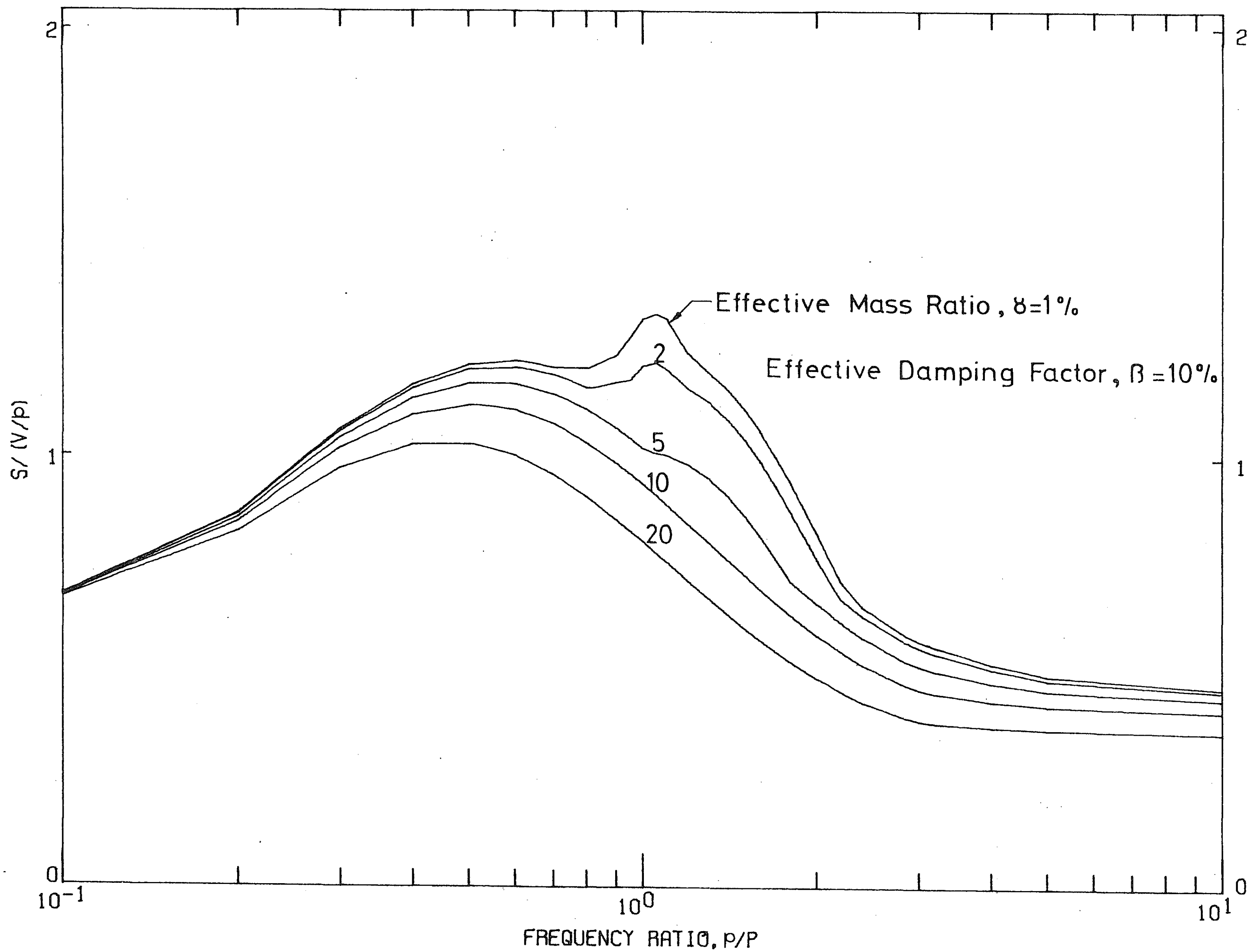


FIG. 4.4f INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

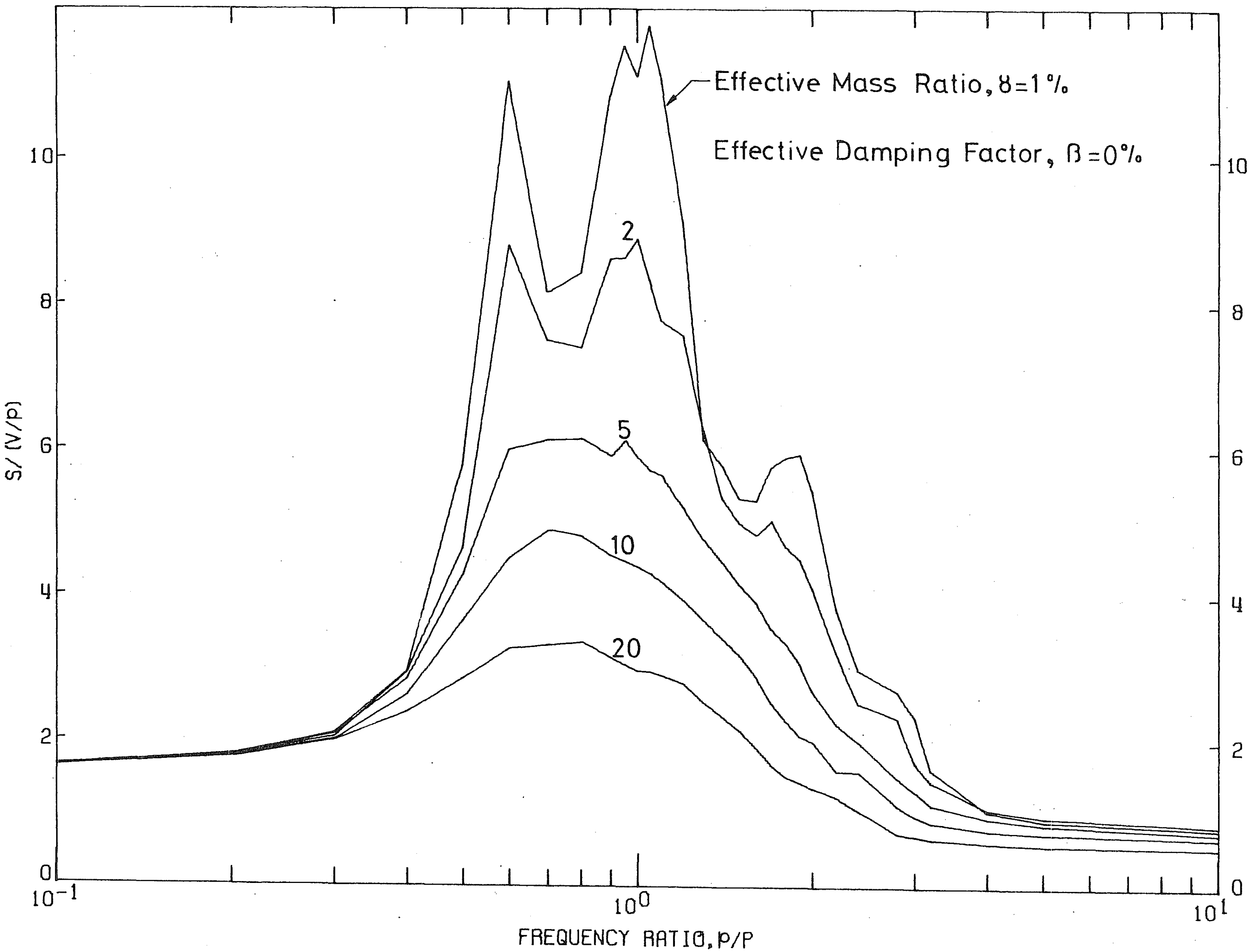


FIG. 4.5a OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

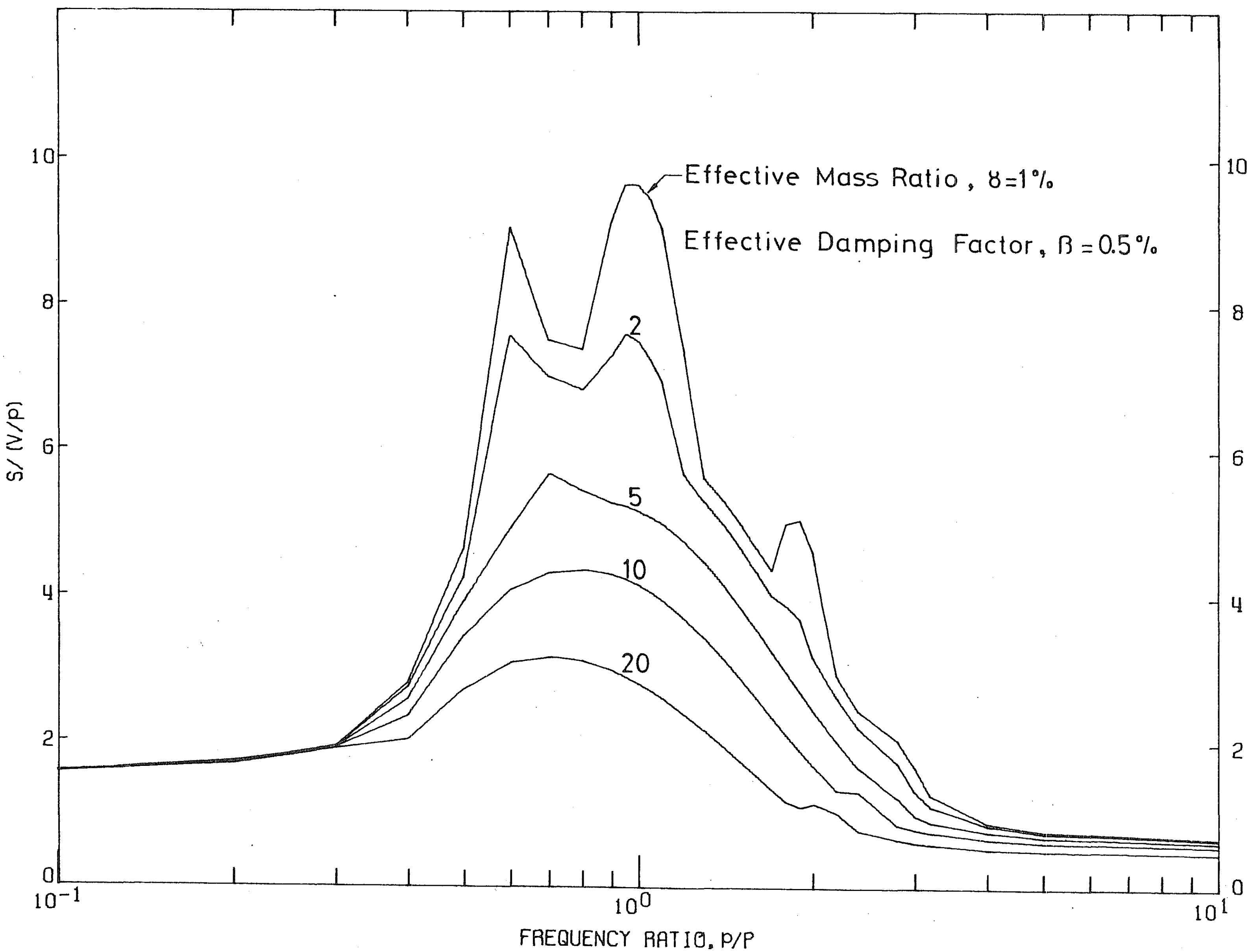


FIG. 4.5b OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

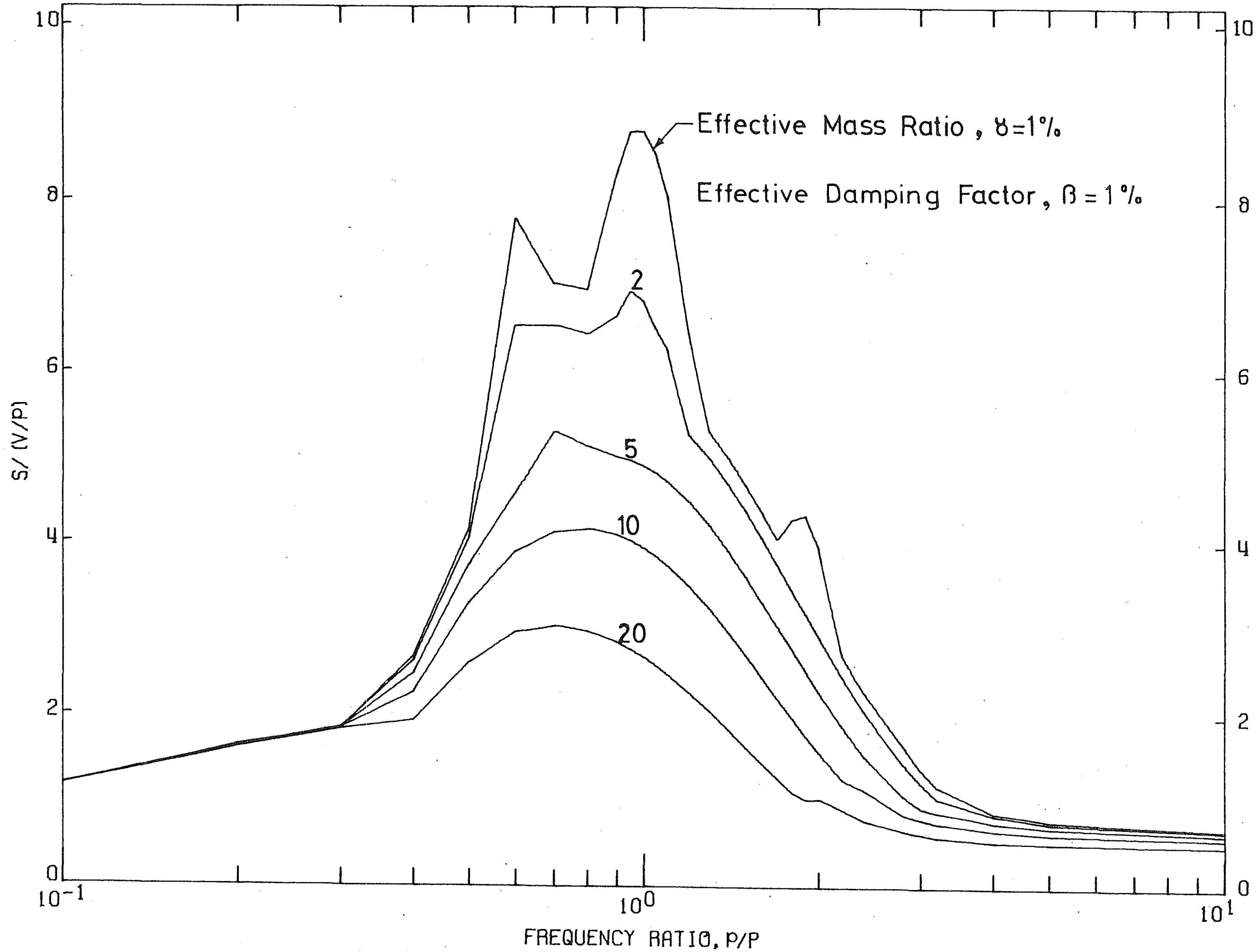


FIG. 4.5c OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

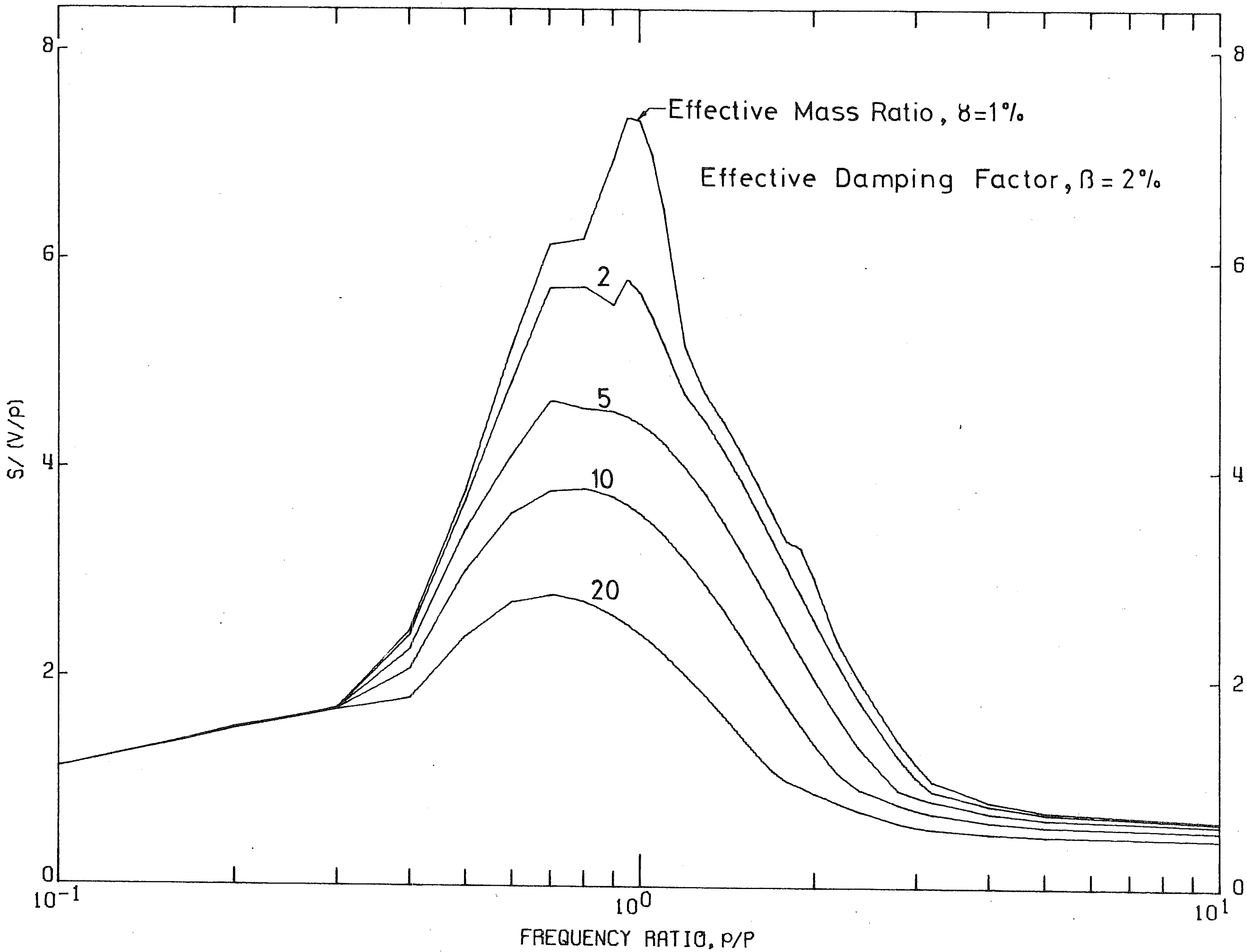


FIG. 4.5d OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

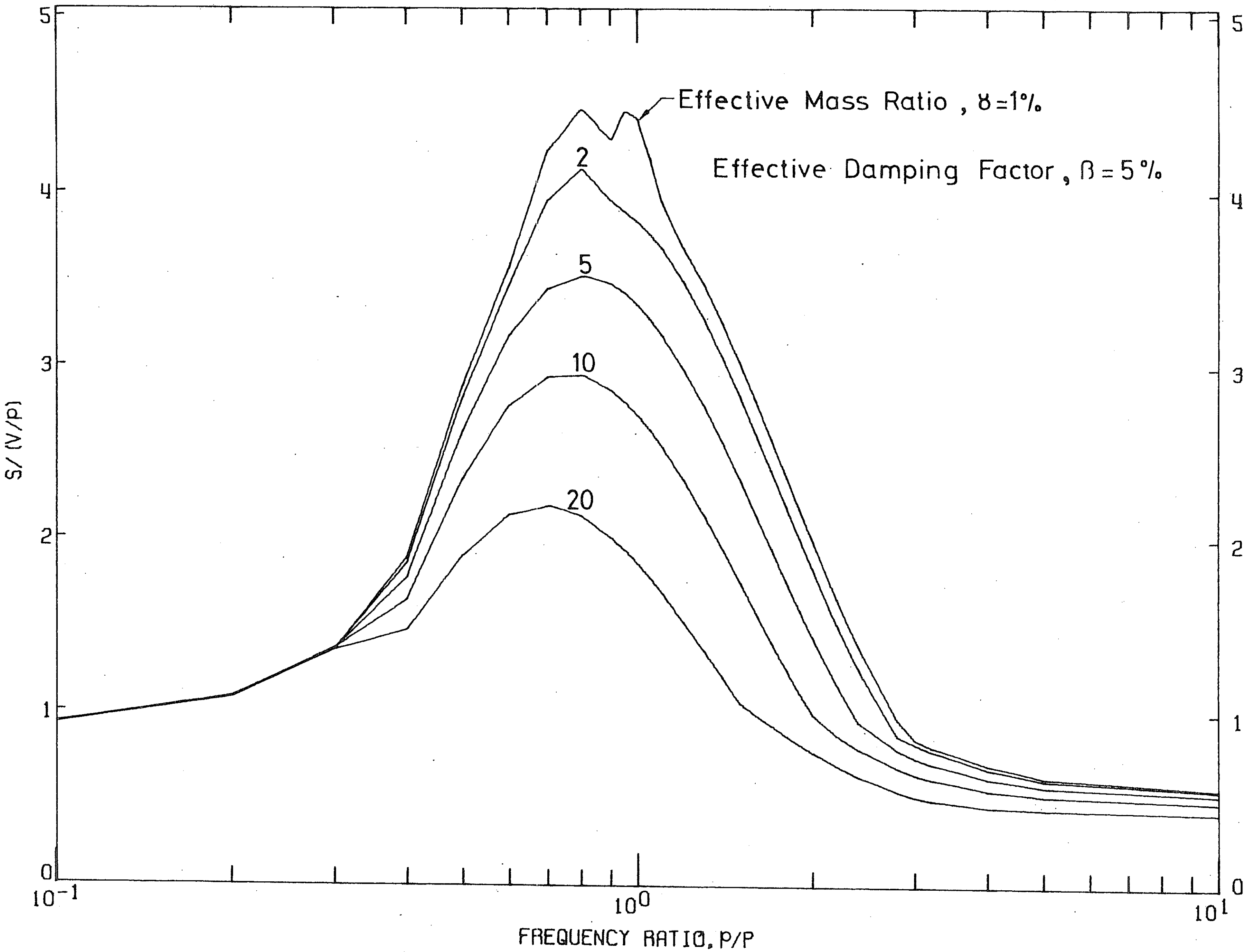


FIG. 4.5e OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

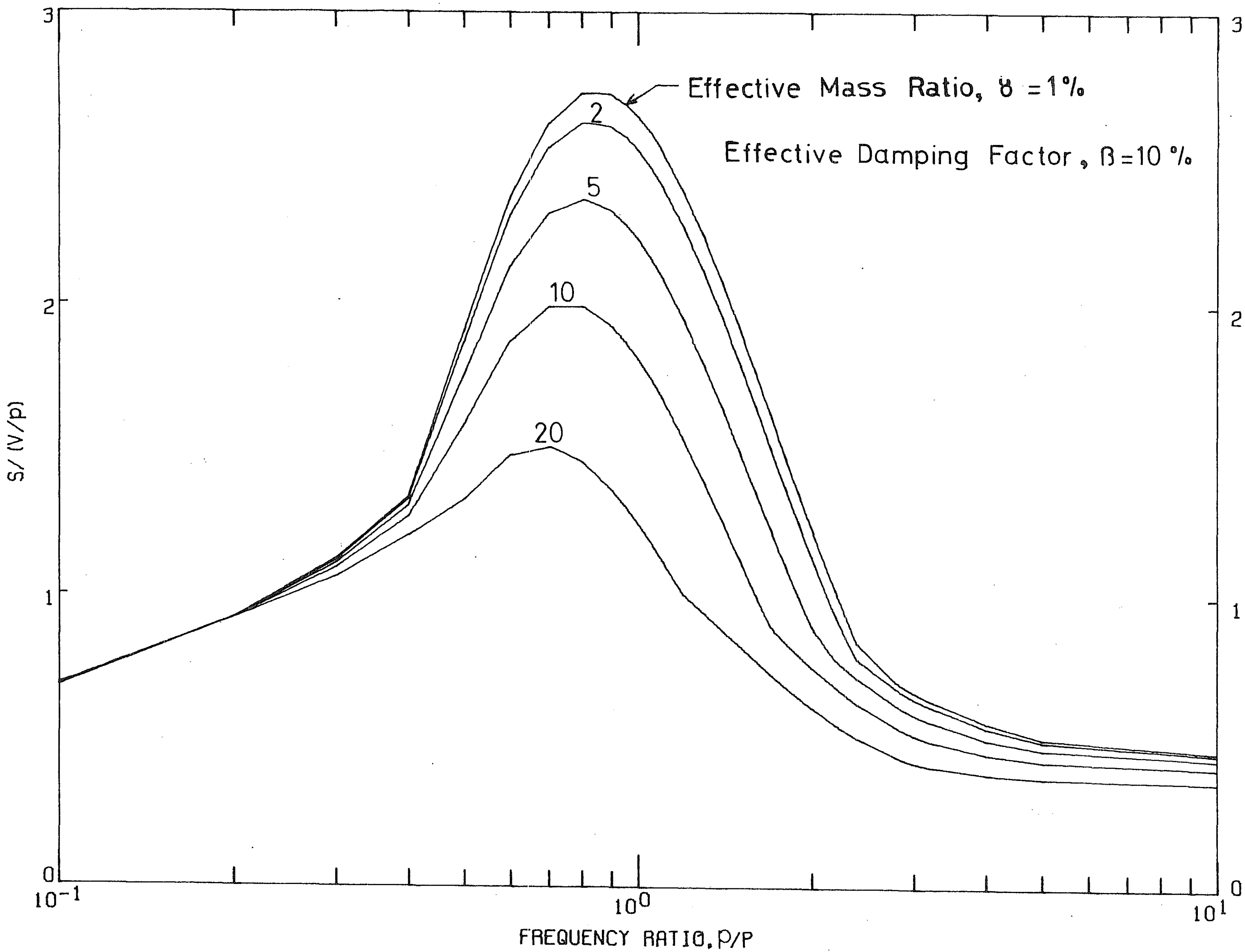


FIG. 4.5f OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

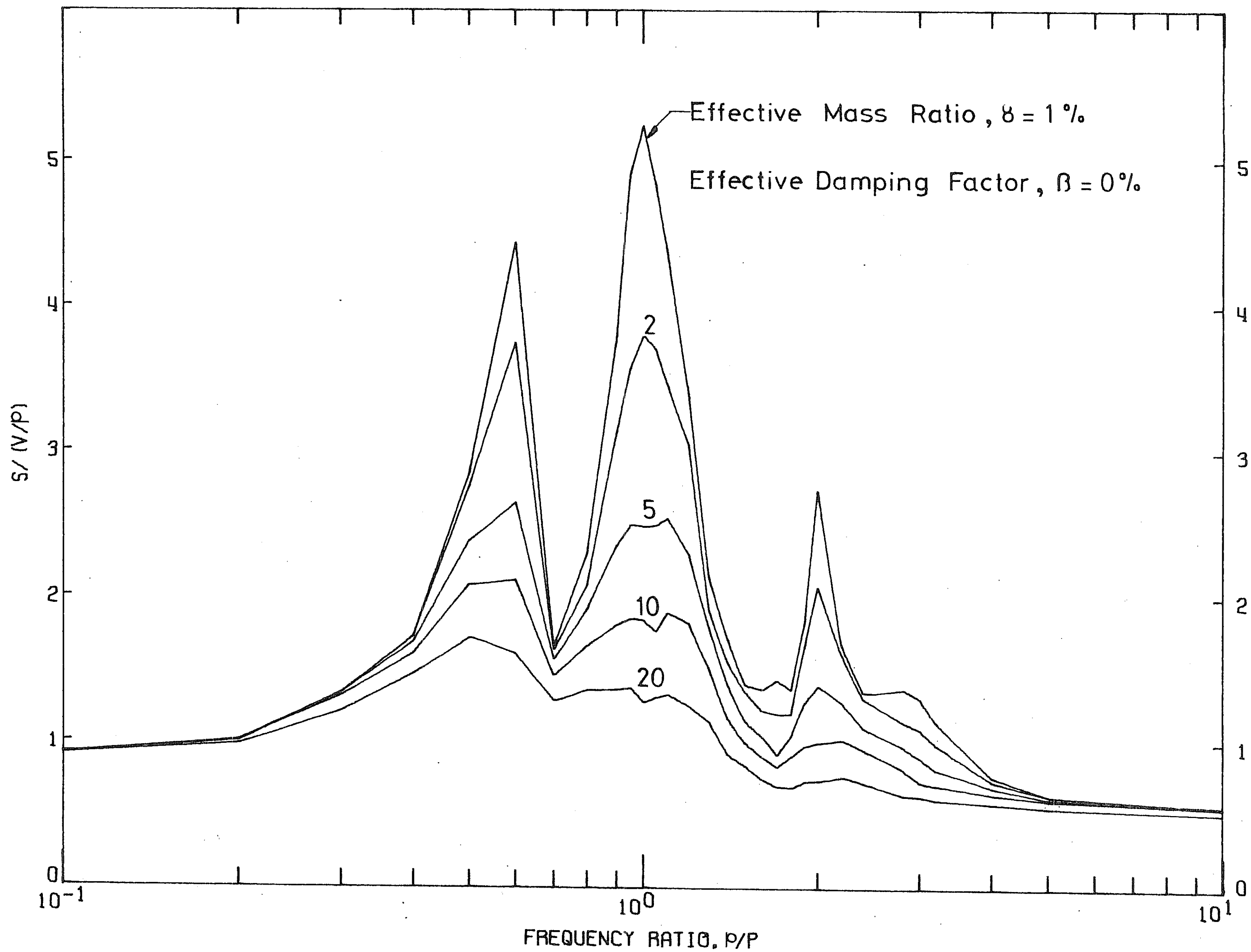


FIG. 4.6a INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

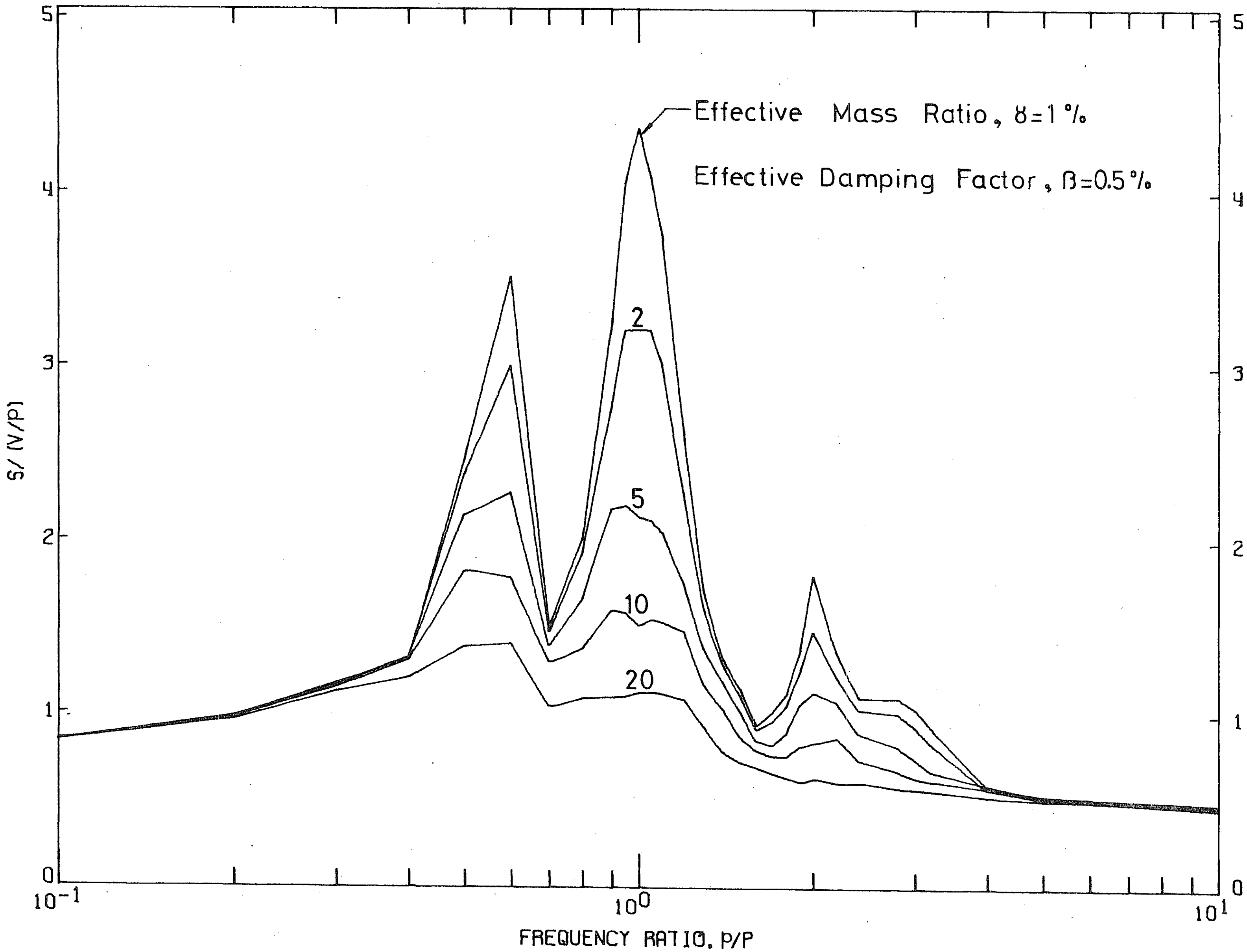


FIG. 4.6b INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

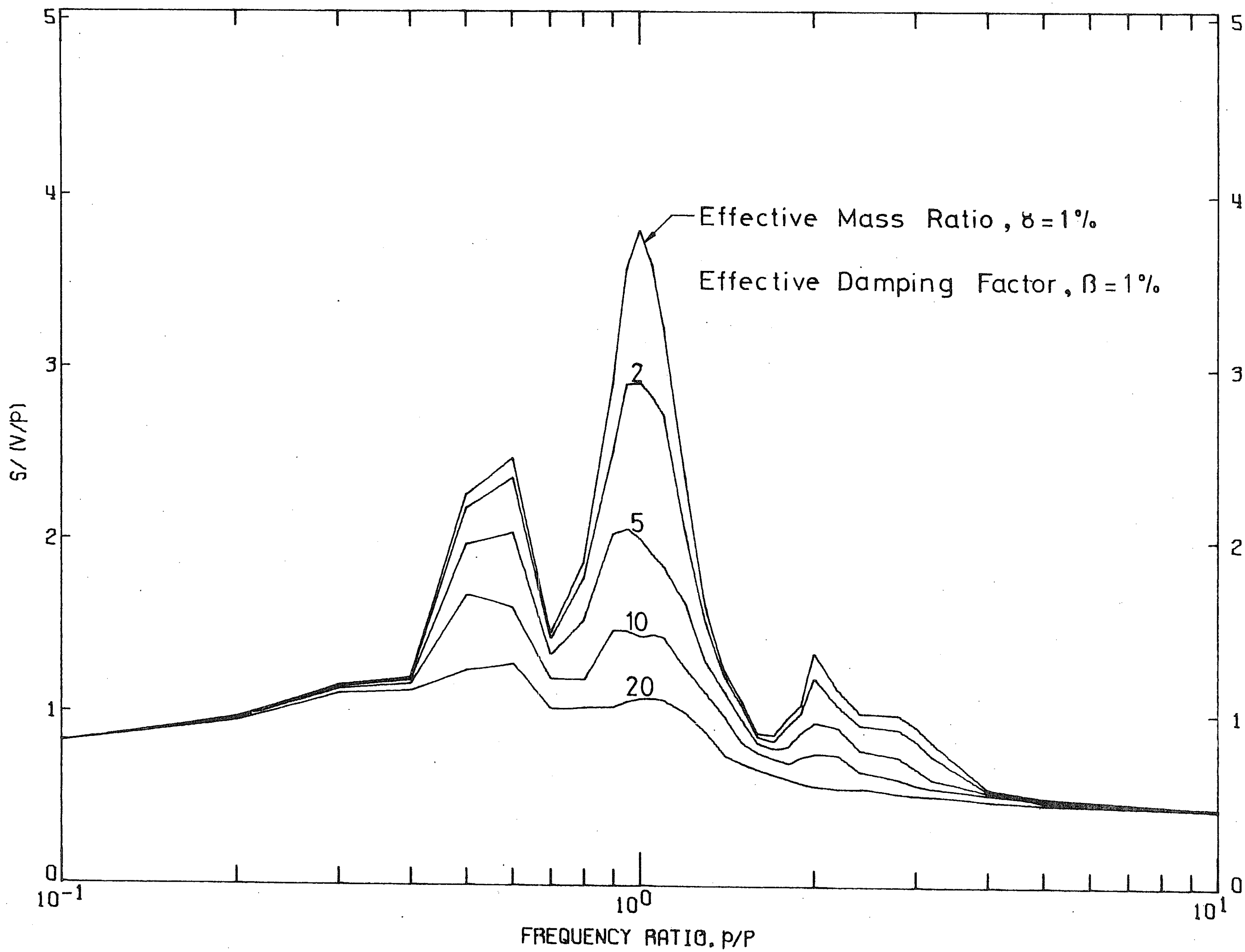


FIG. 4.6 c INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

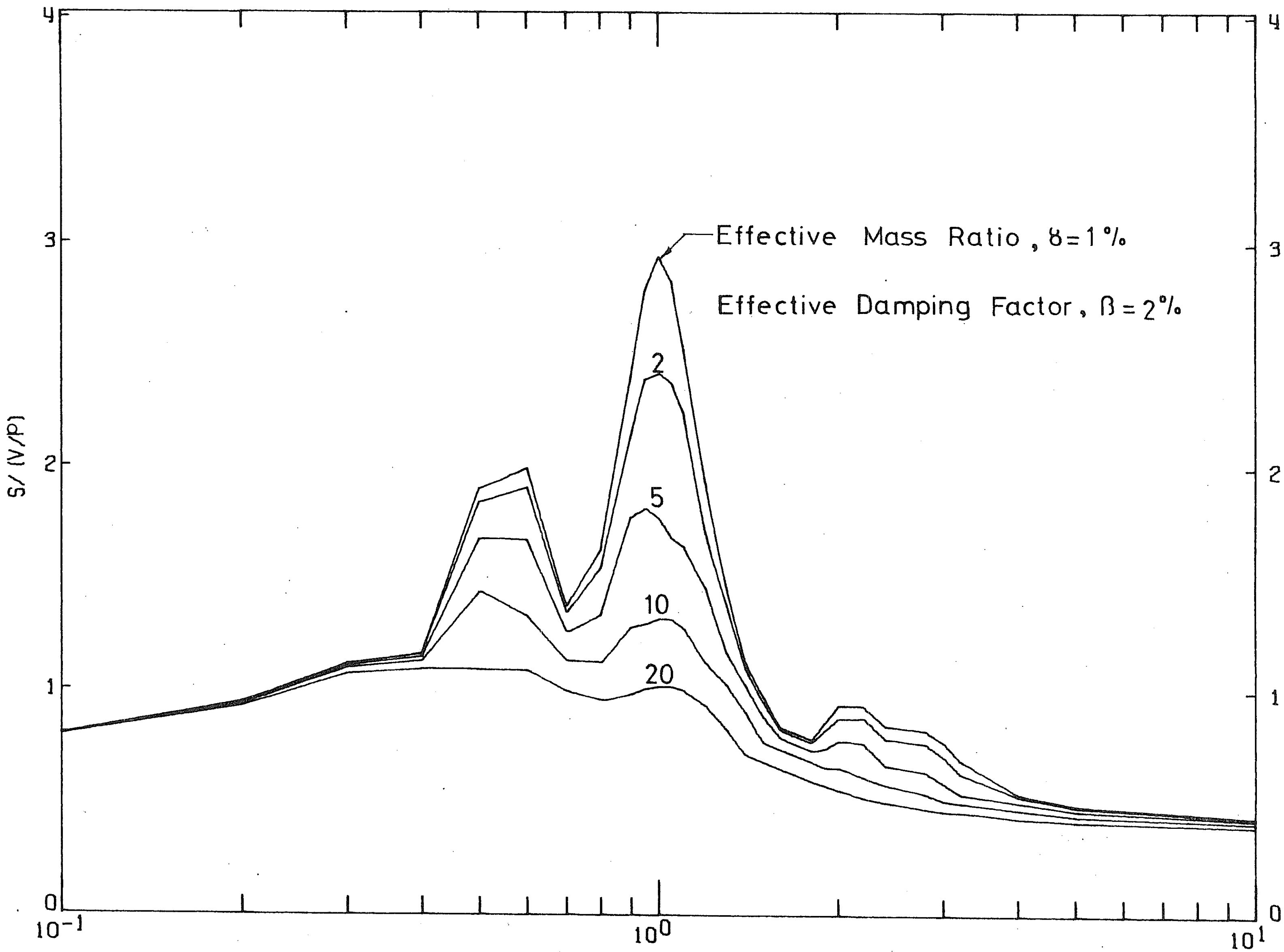


FIG. 4.6d INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

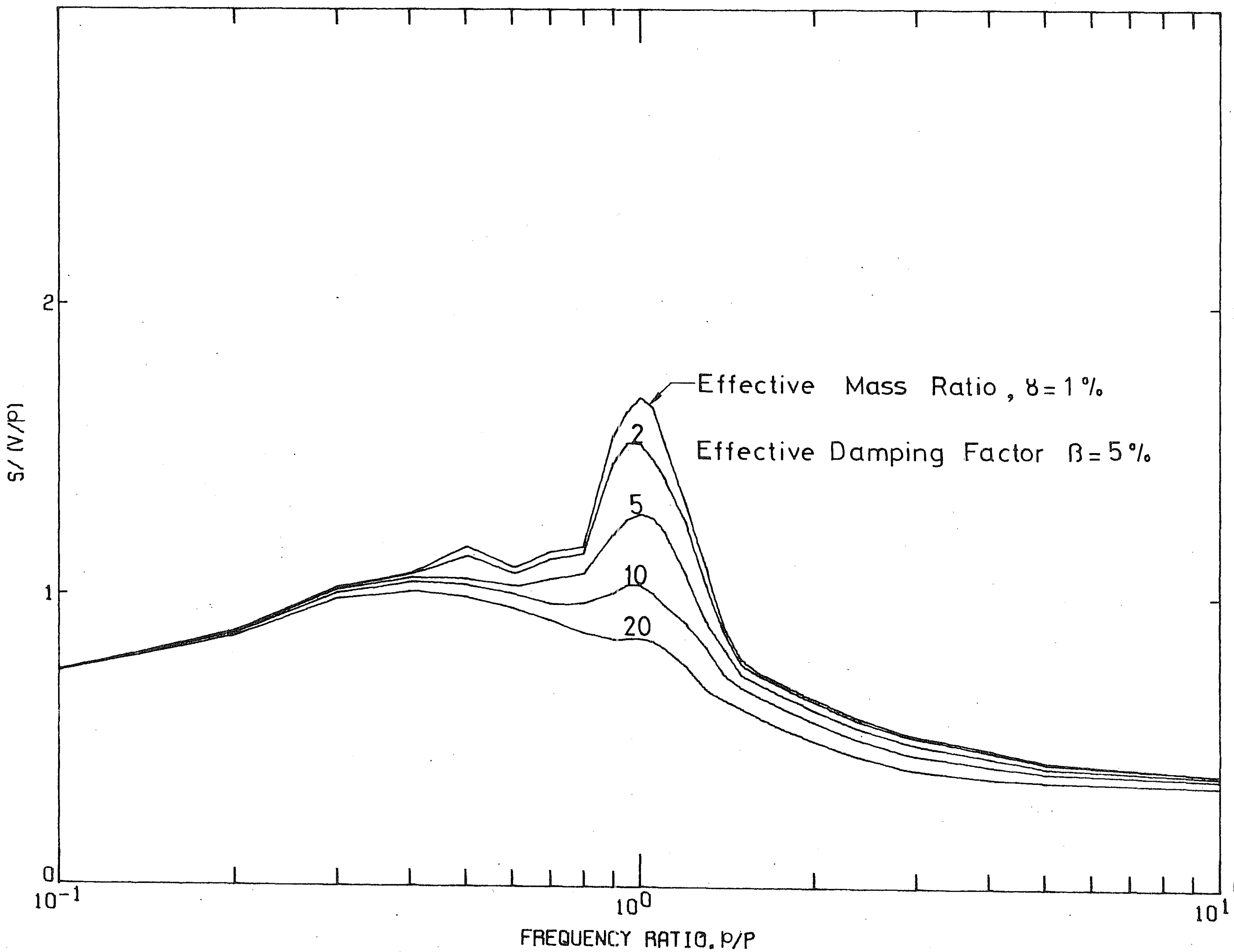


FIG. 4.6 e INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

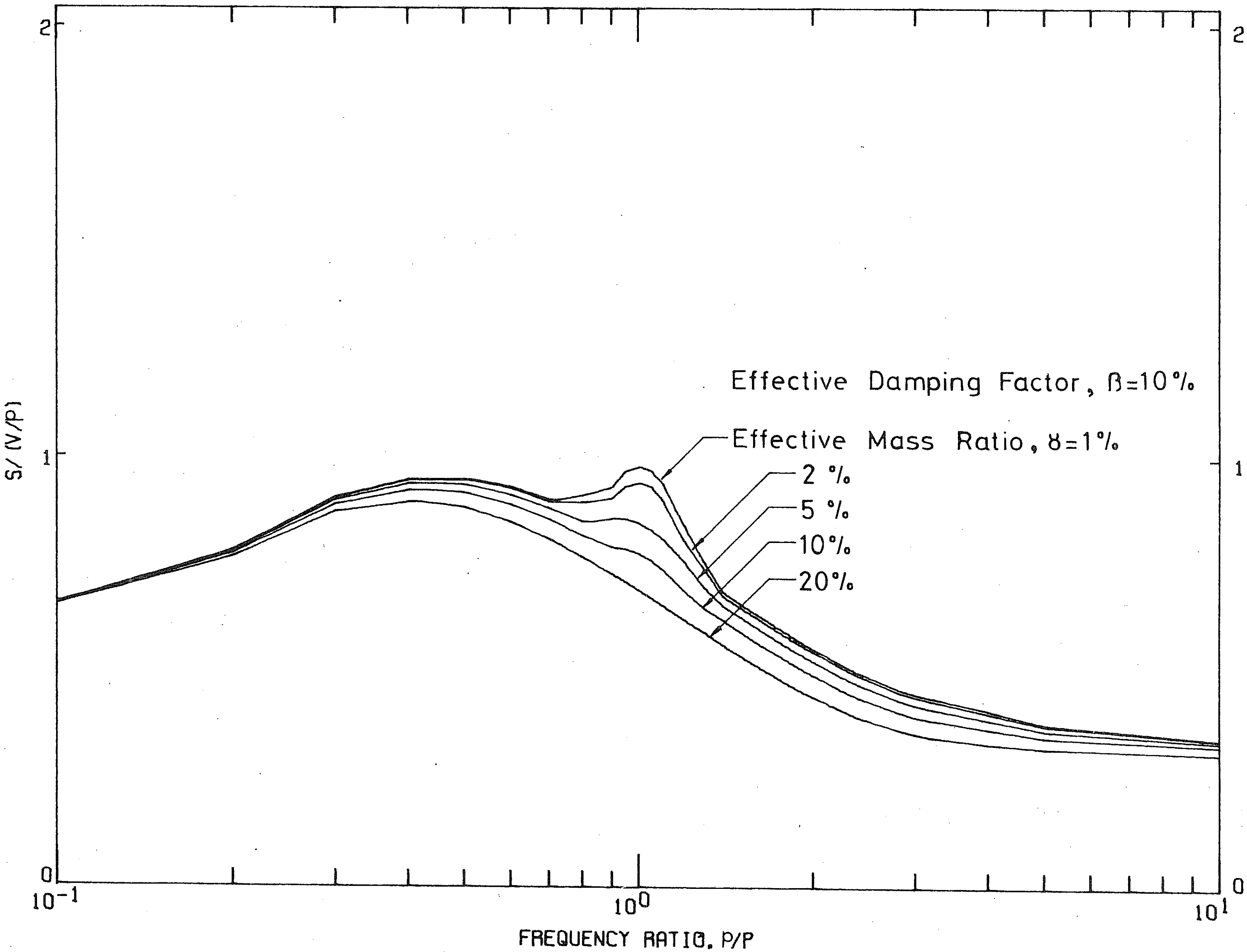


FIG. 4.6f INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

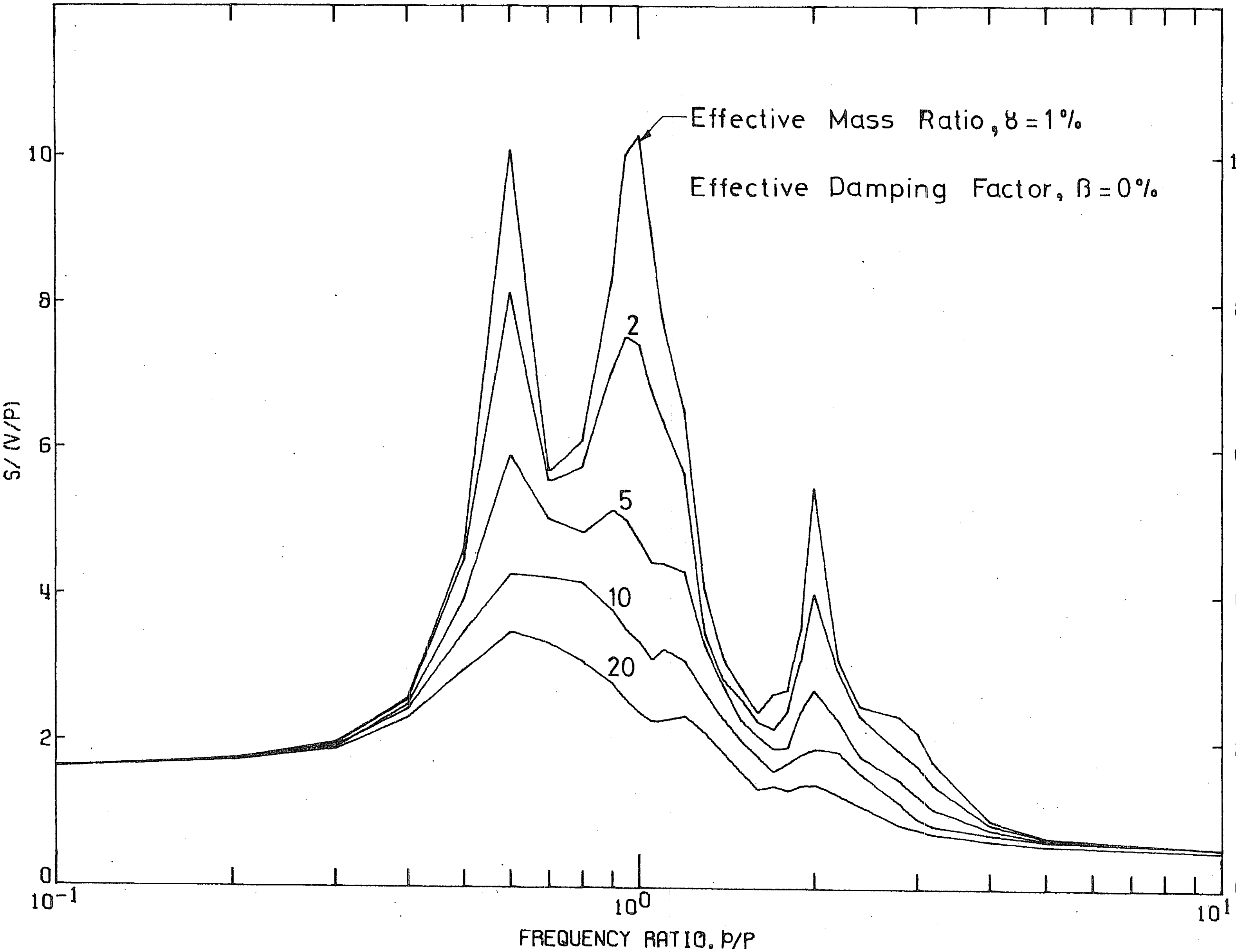


FIG. 4.7a OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

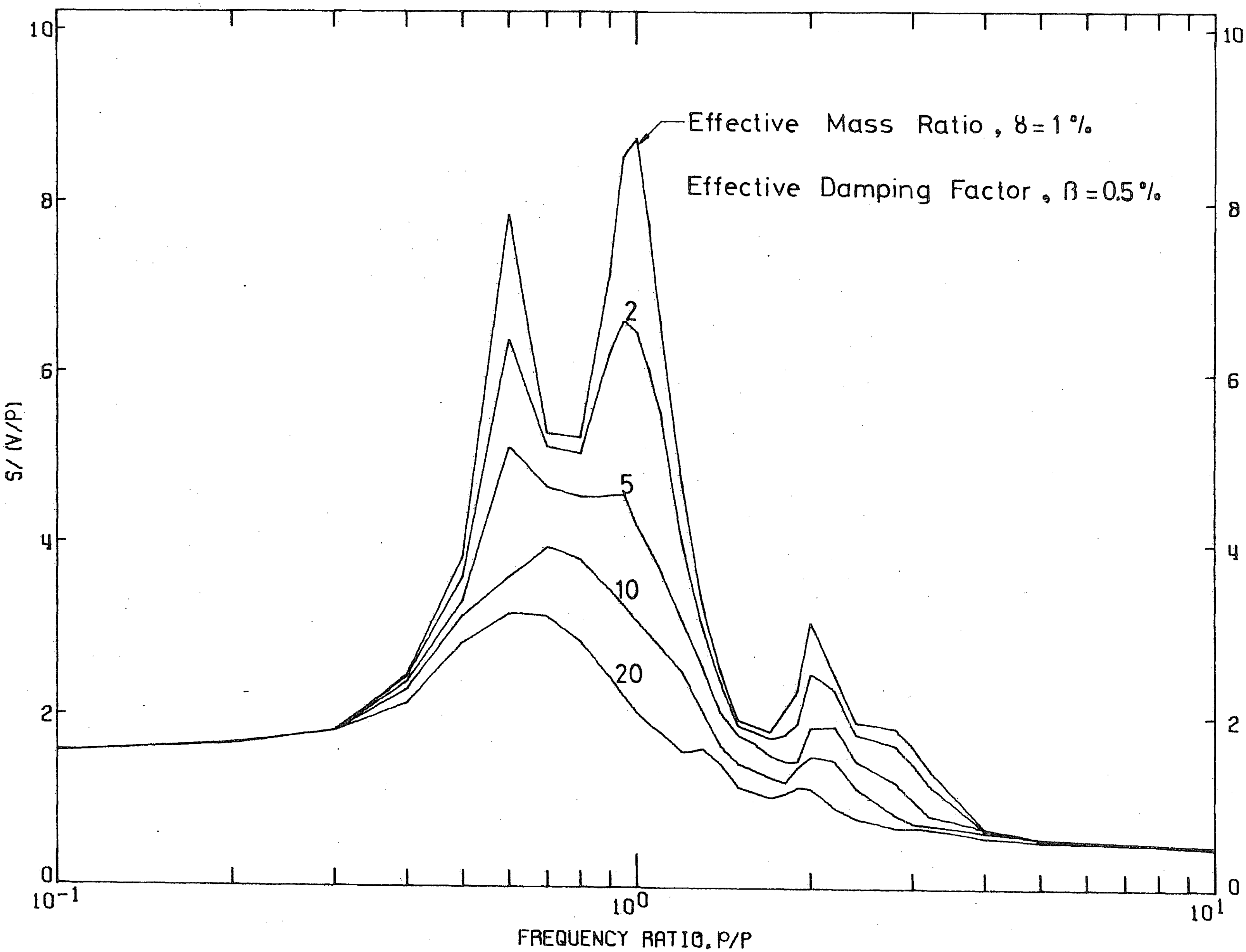


FIG. 4.7b OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

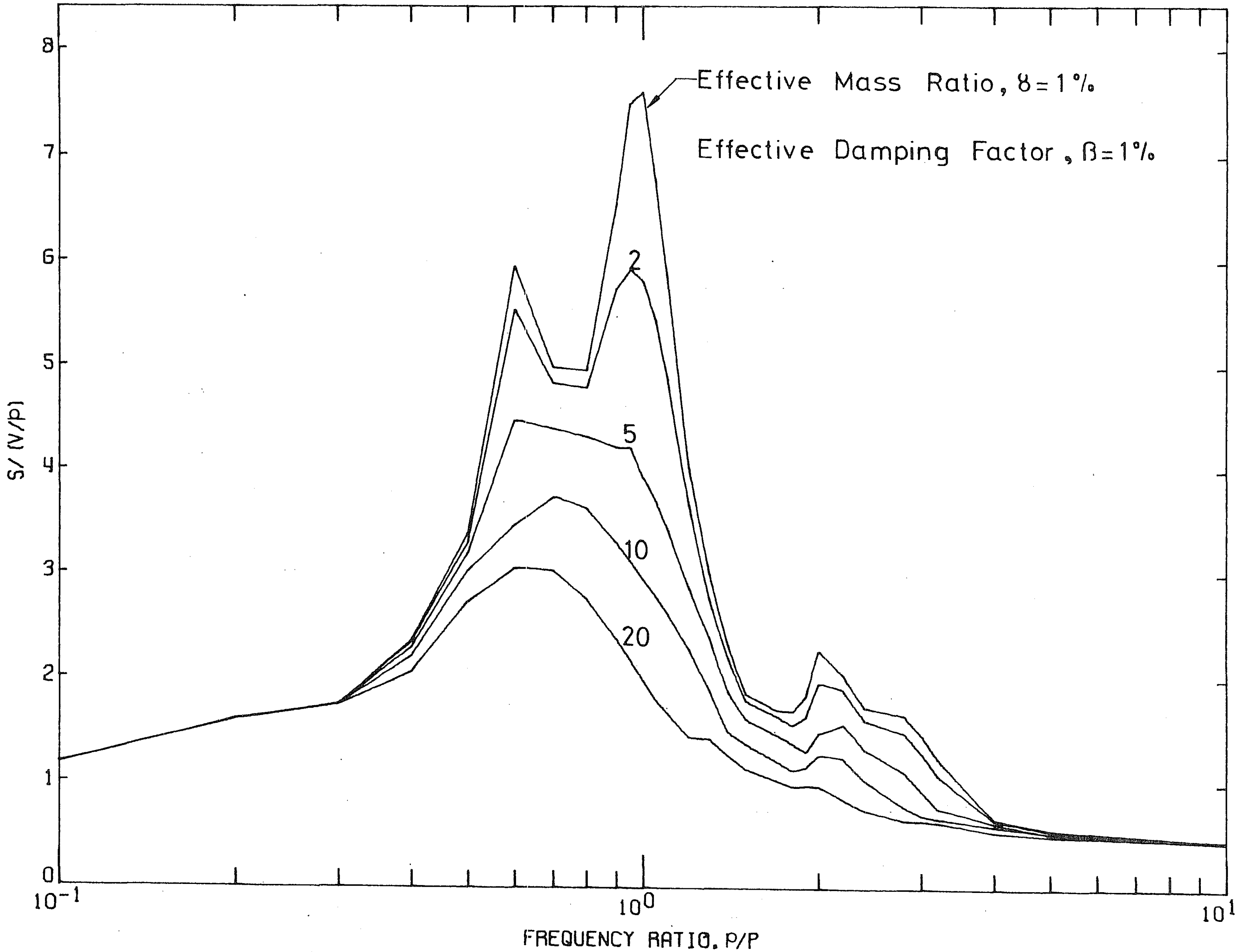


FIG. 4.7c OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

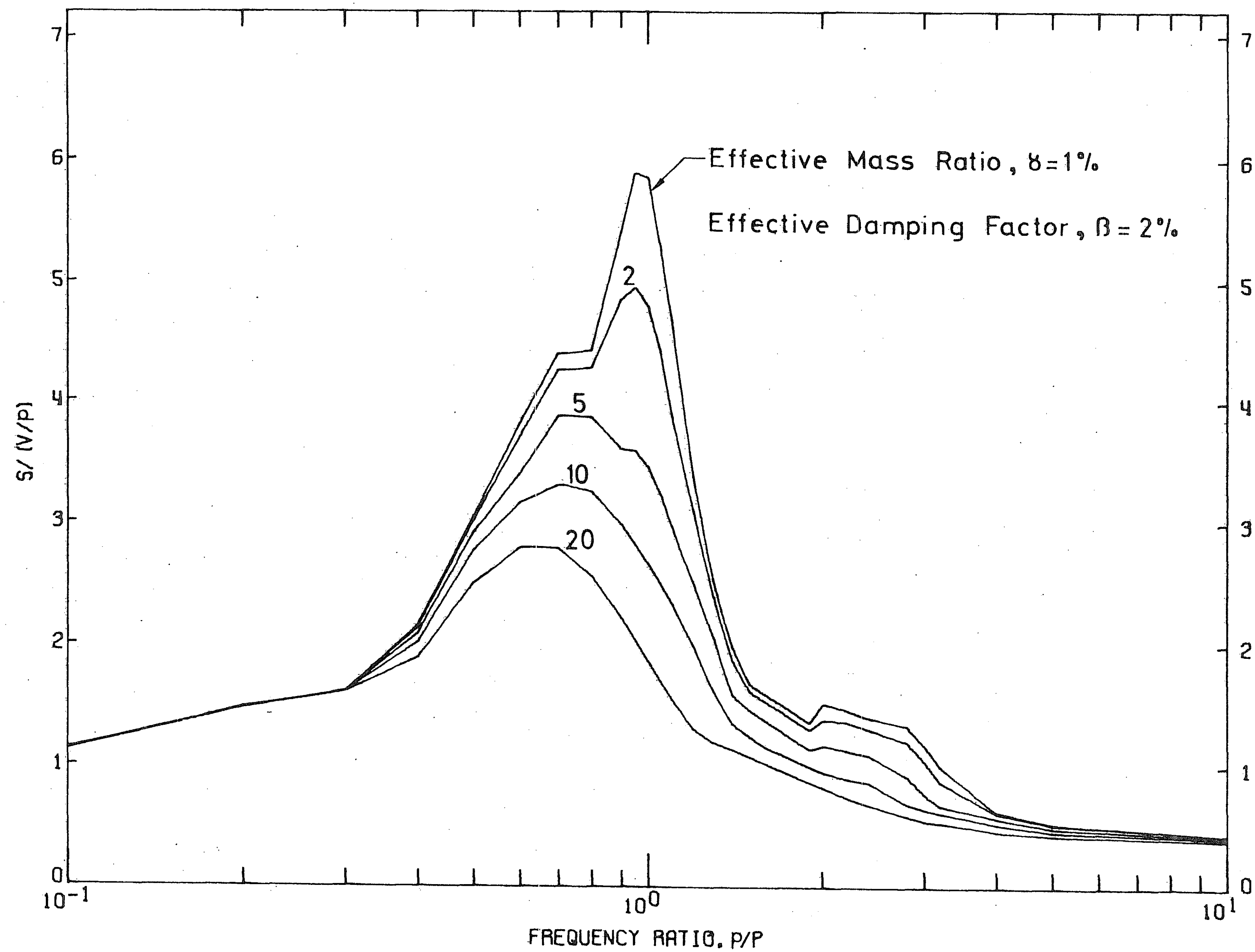


FIG. 4.7d OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

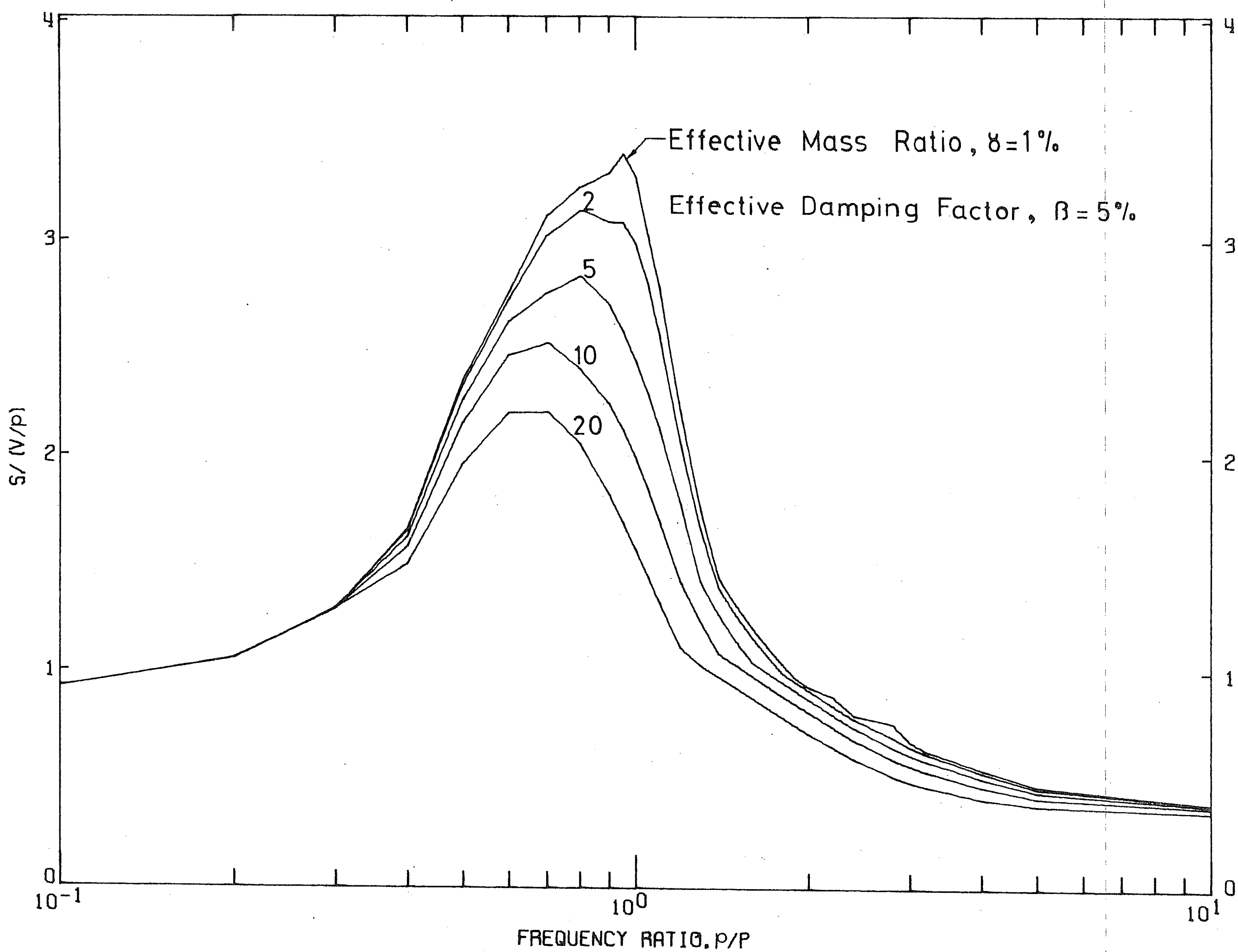


FIG. 4.7e OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

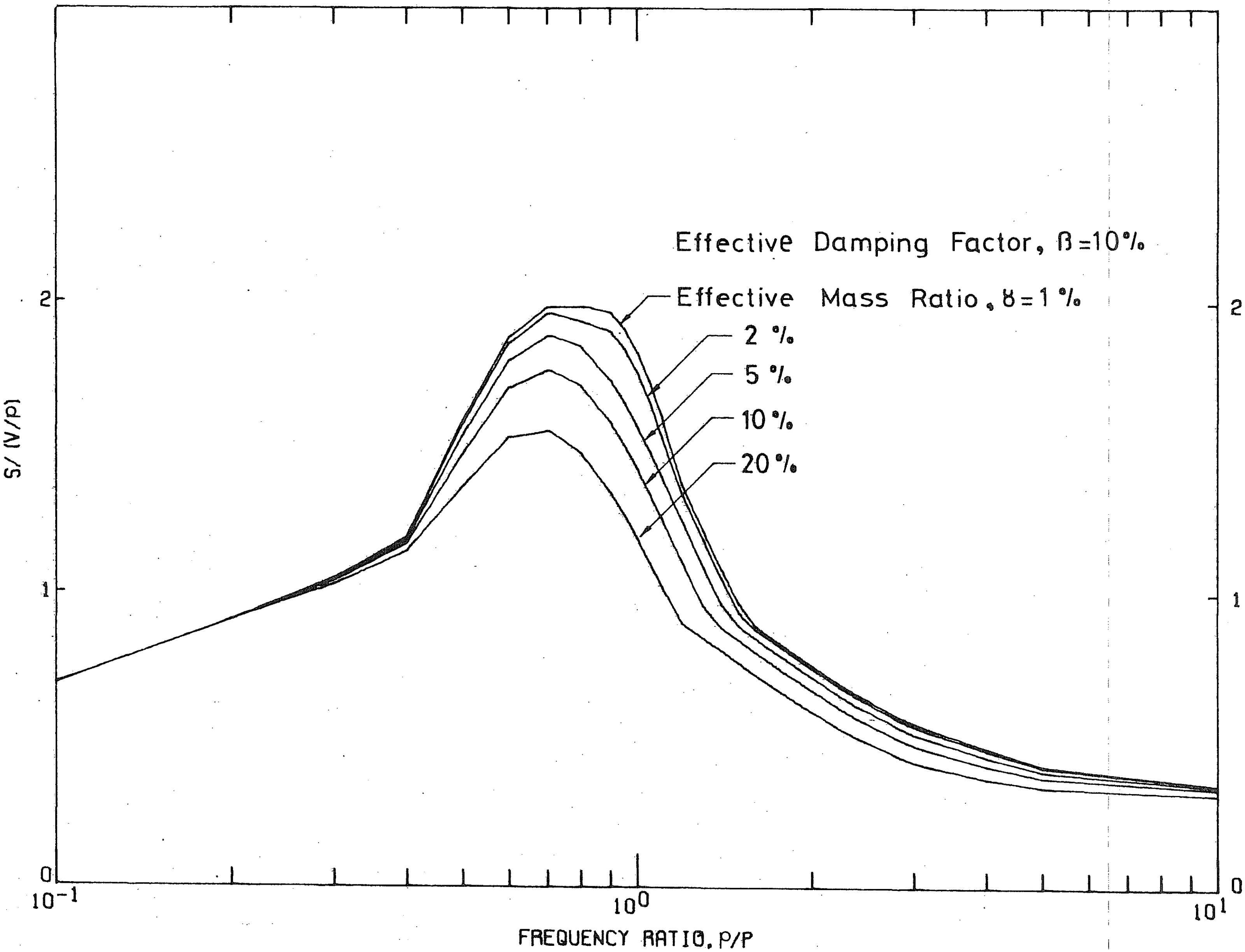


FIG. 4.7f OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

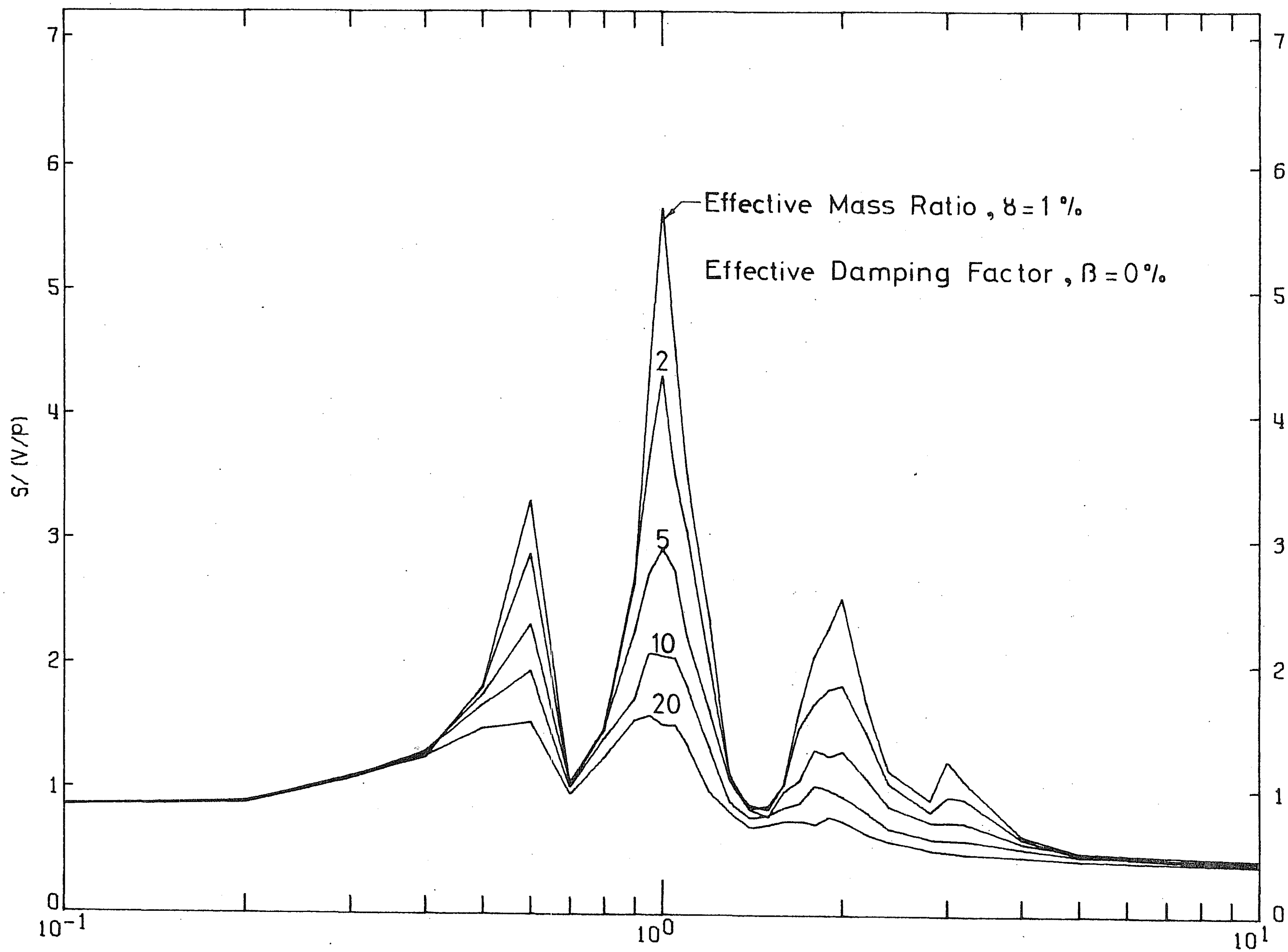


FIG. 4.8a INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

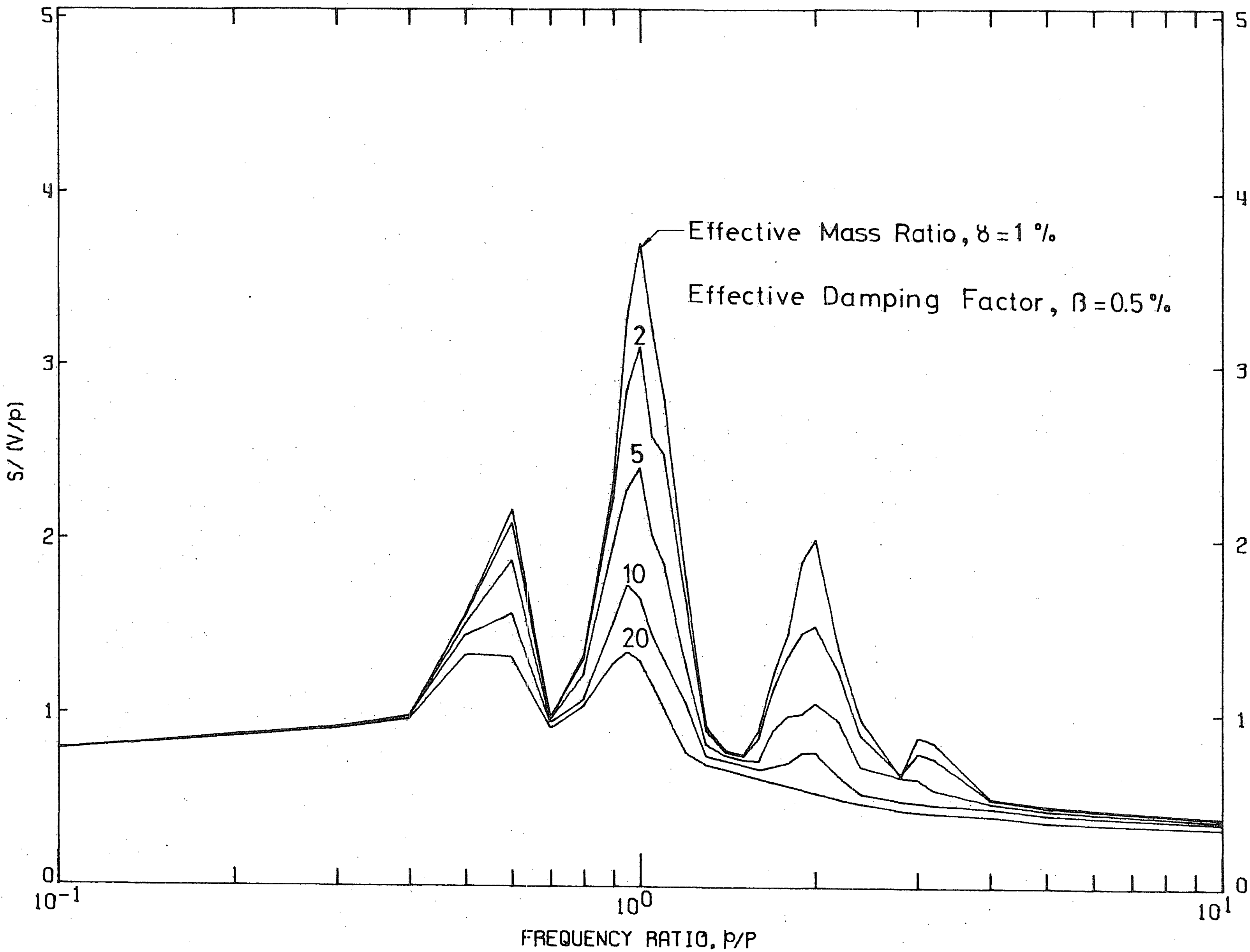


FIG. 4.8b INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

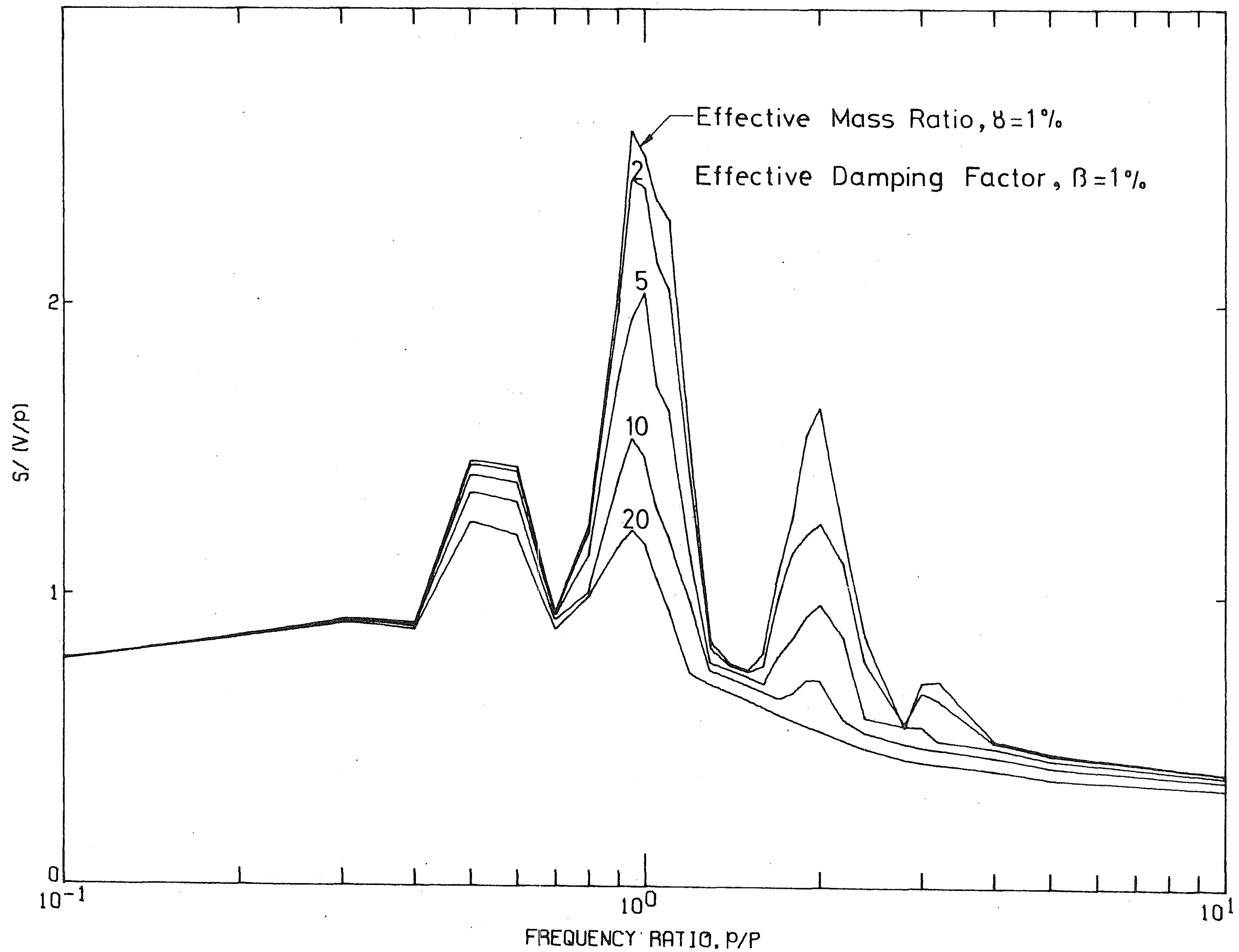


FIG. 4.8c INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

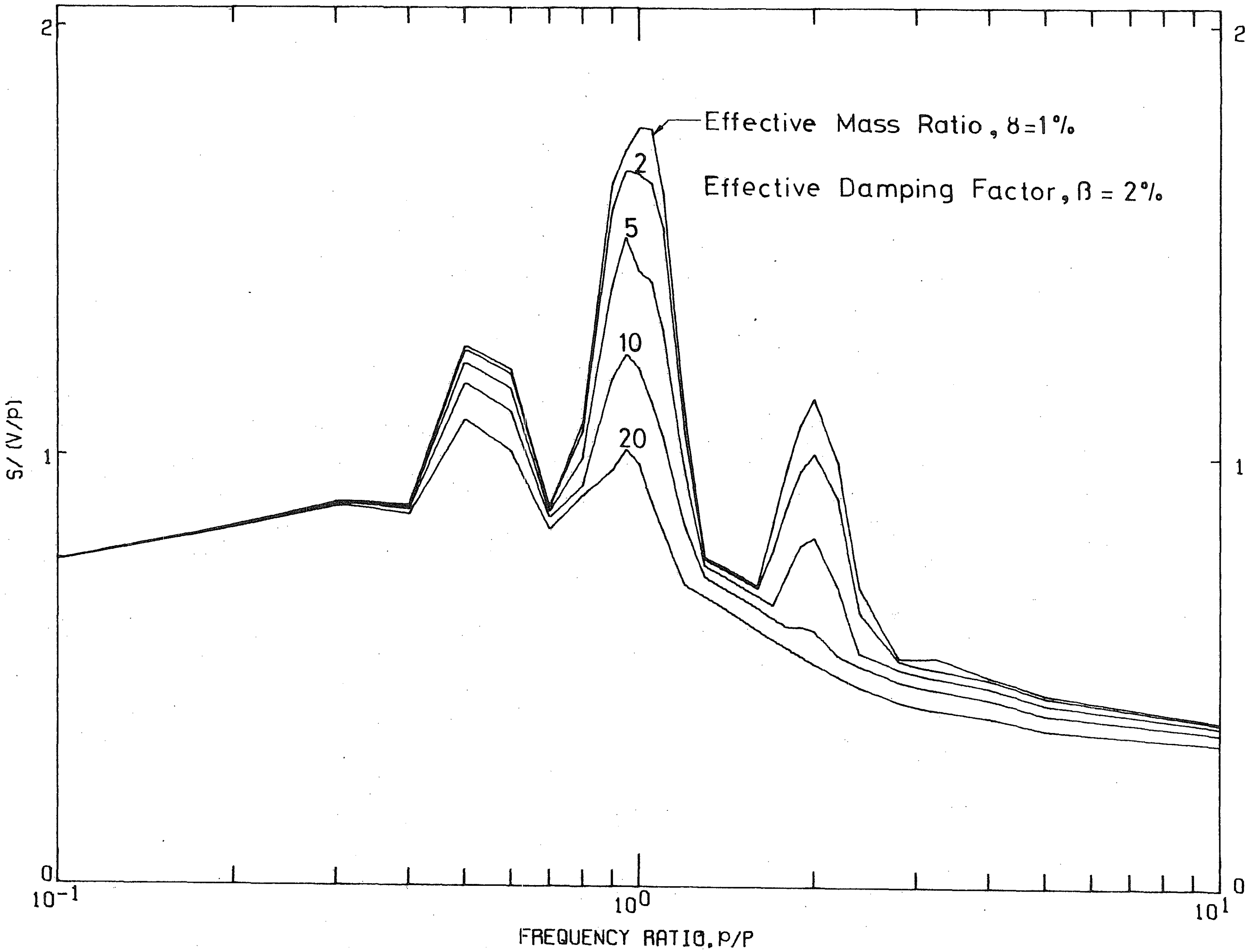


FIG. 4.8d INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

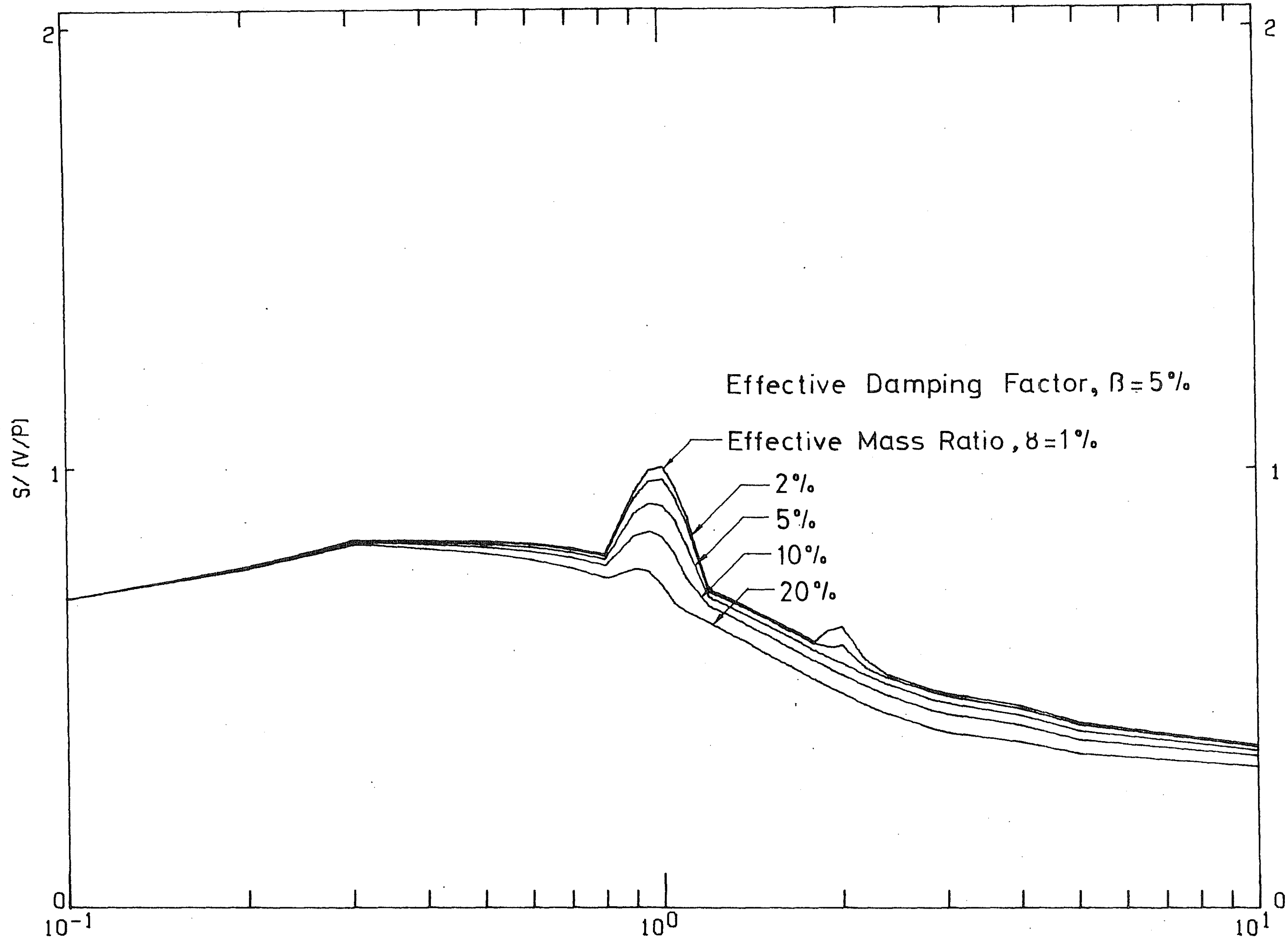


FIG. 4.8e INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

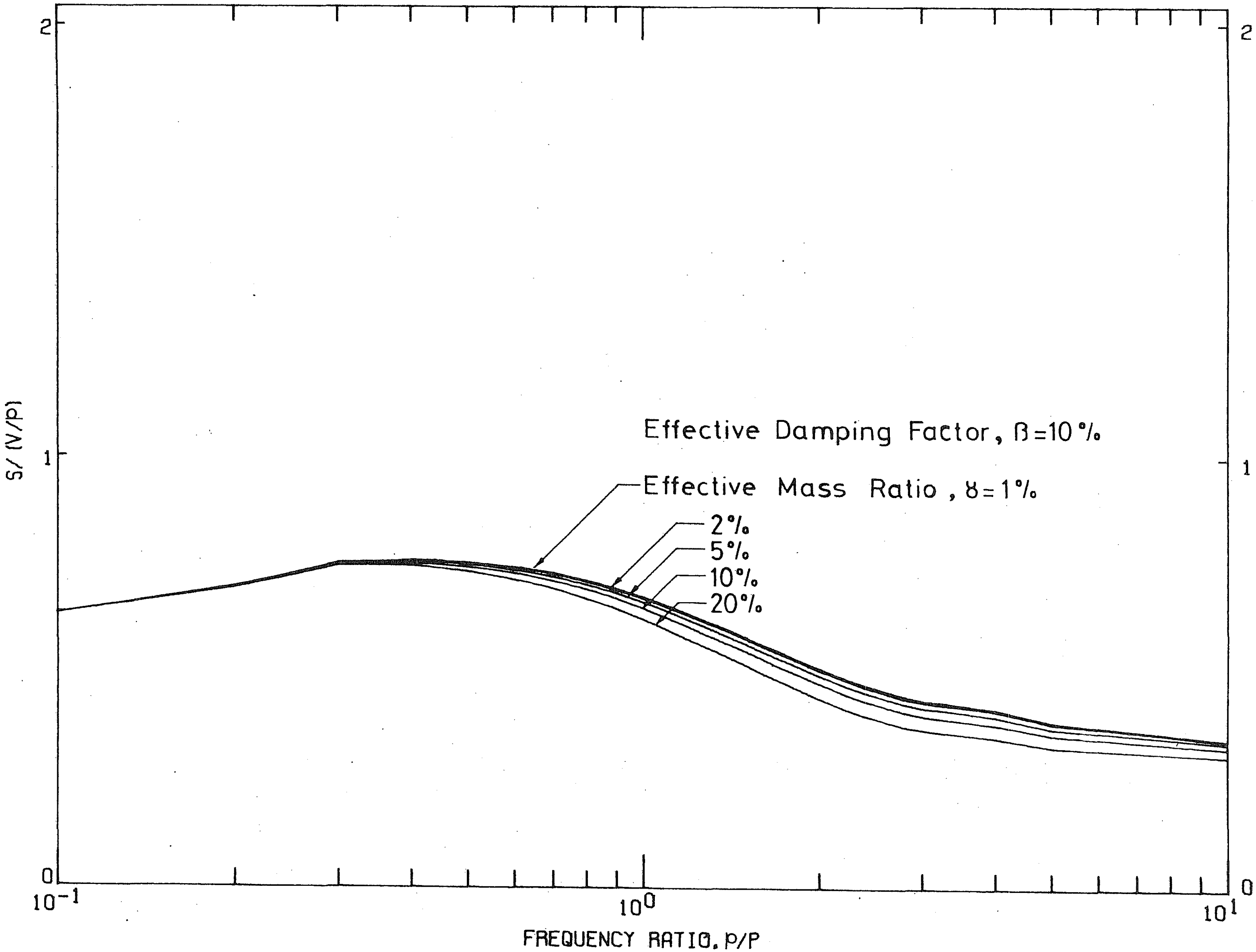


FIG. 4.8f INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

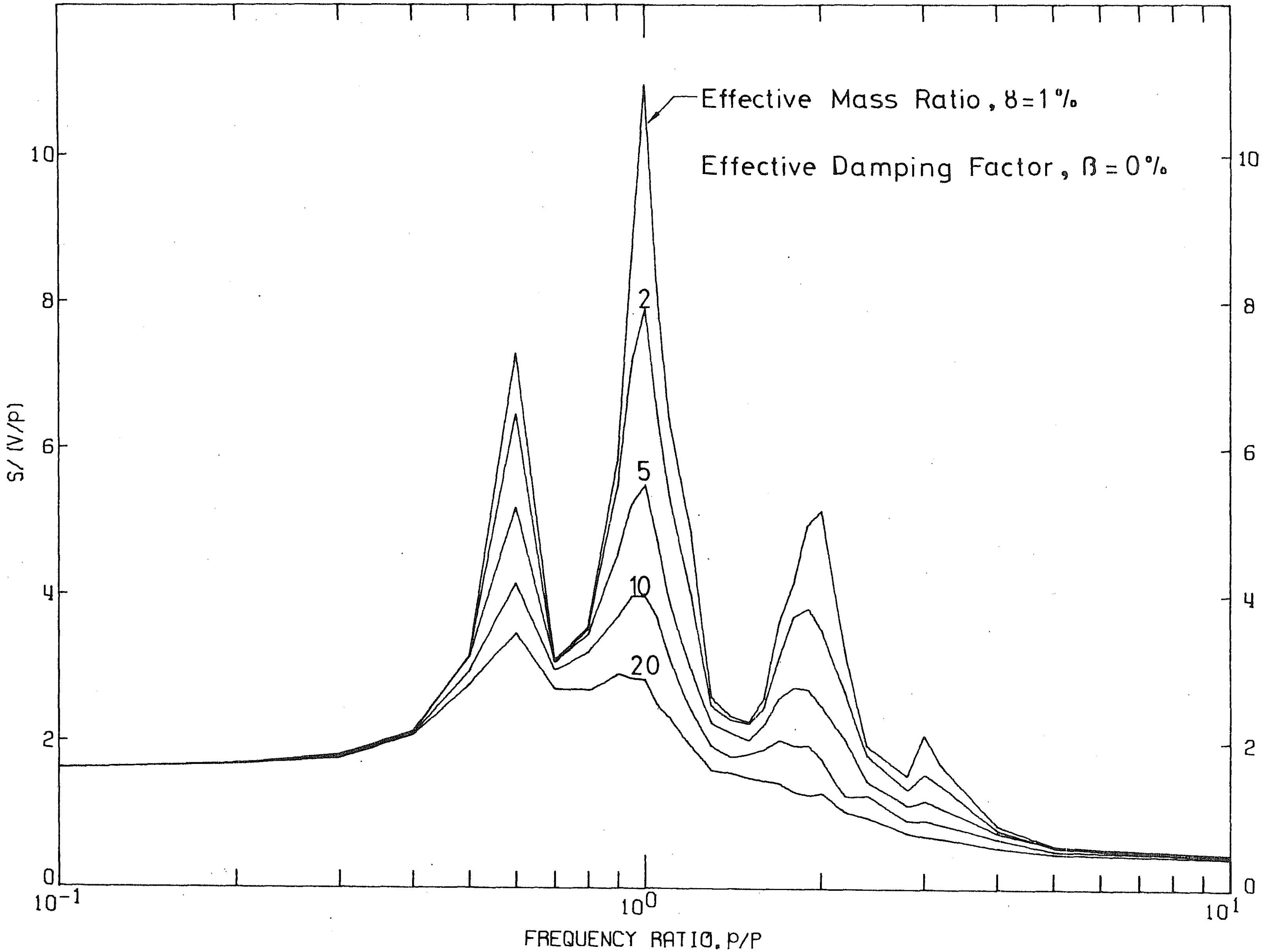


FIG. 4.9a OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

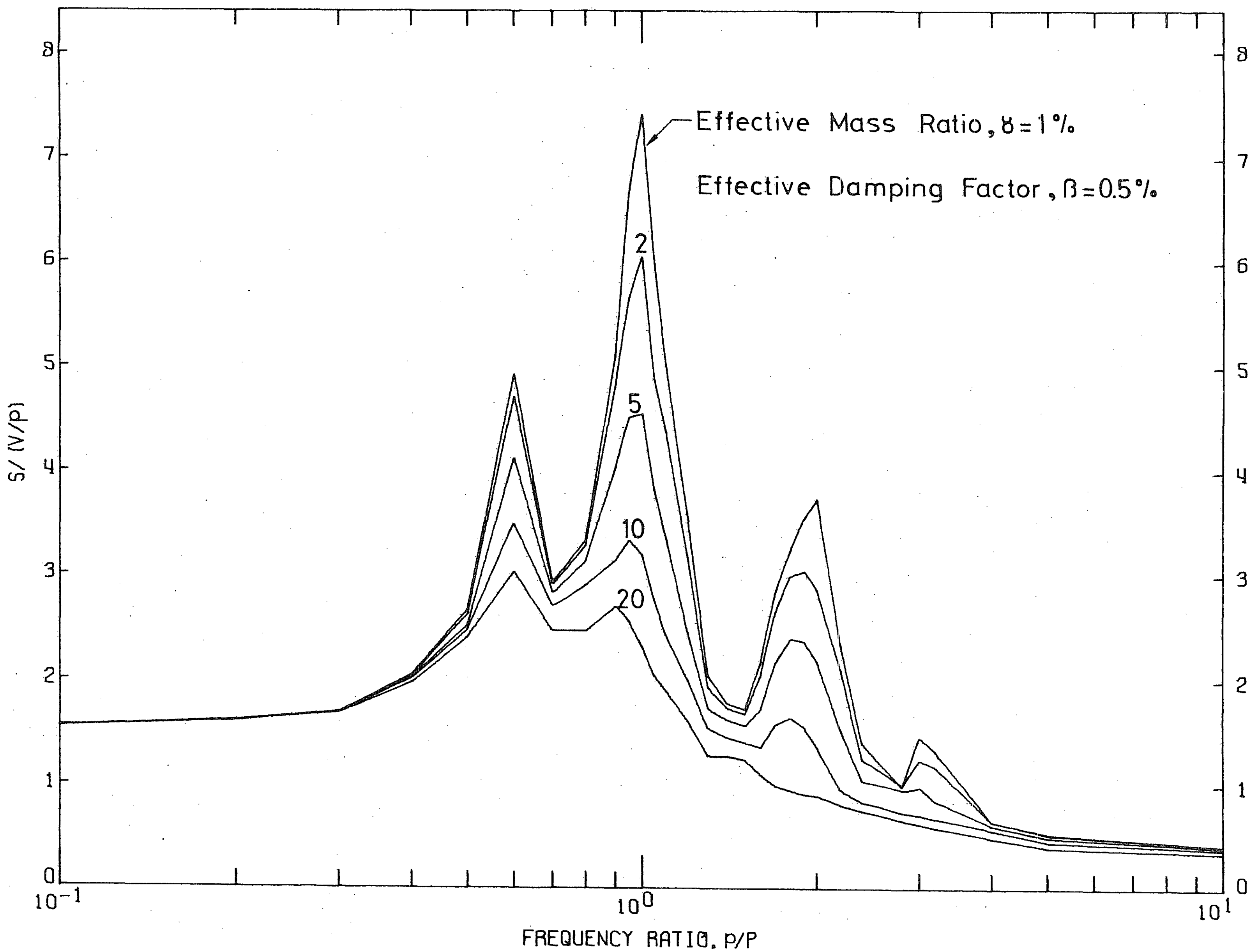


FIG. 4.9b OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

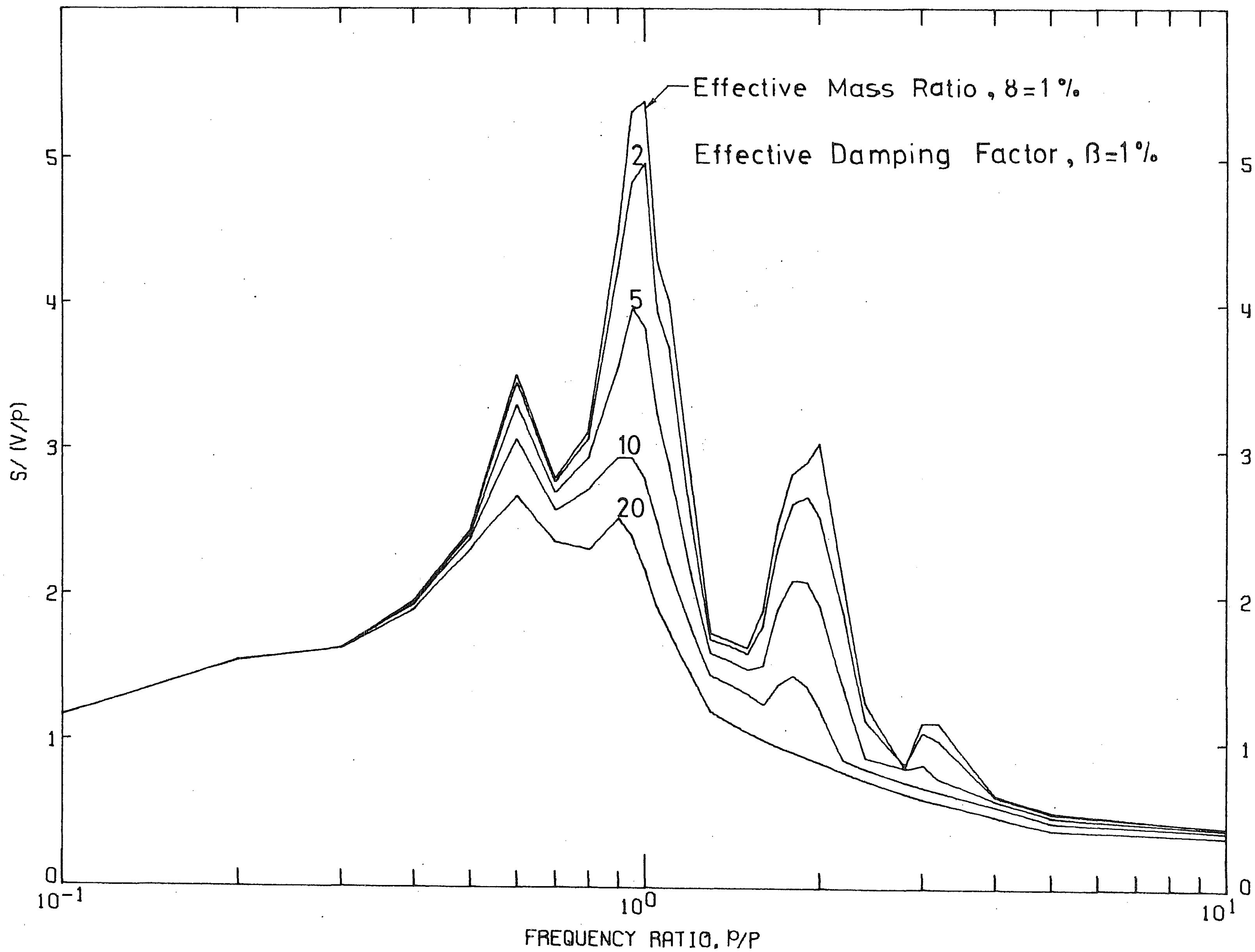


FIG. 4.9c OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

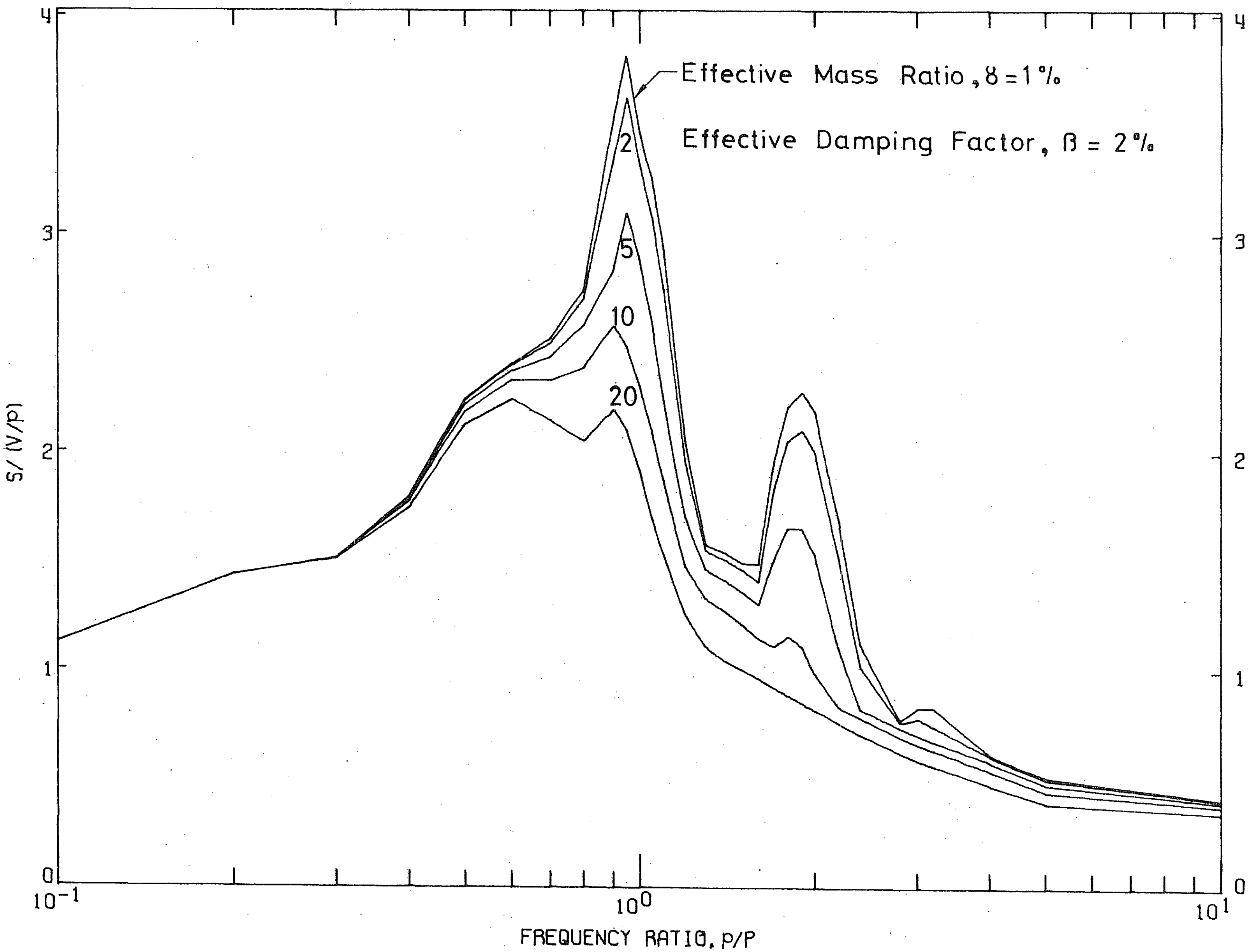


FIG. 4.9d OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

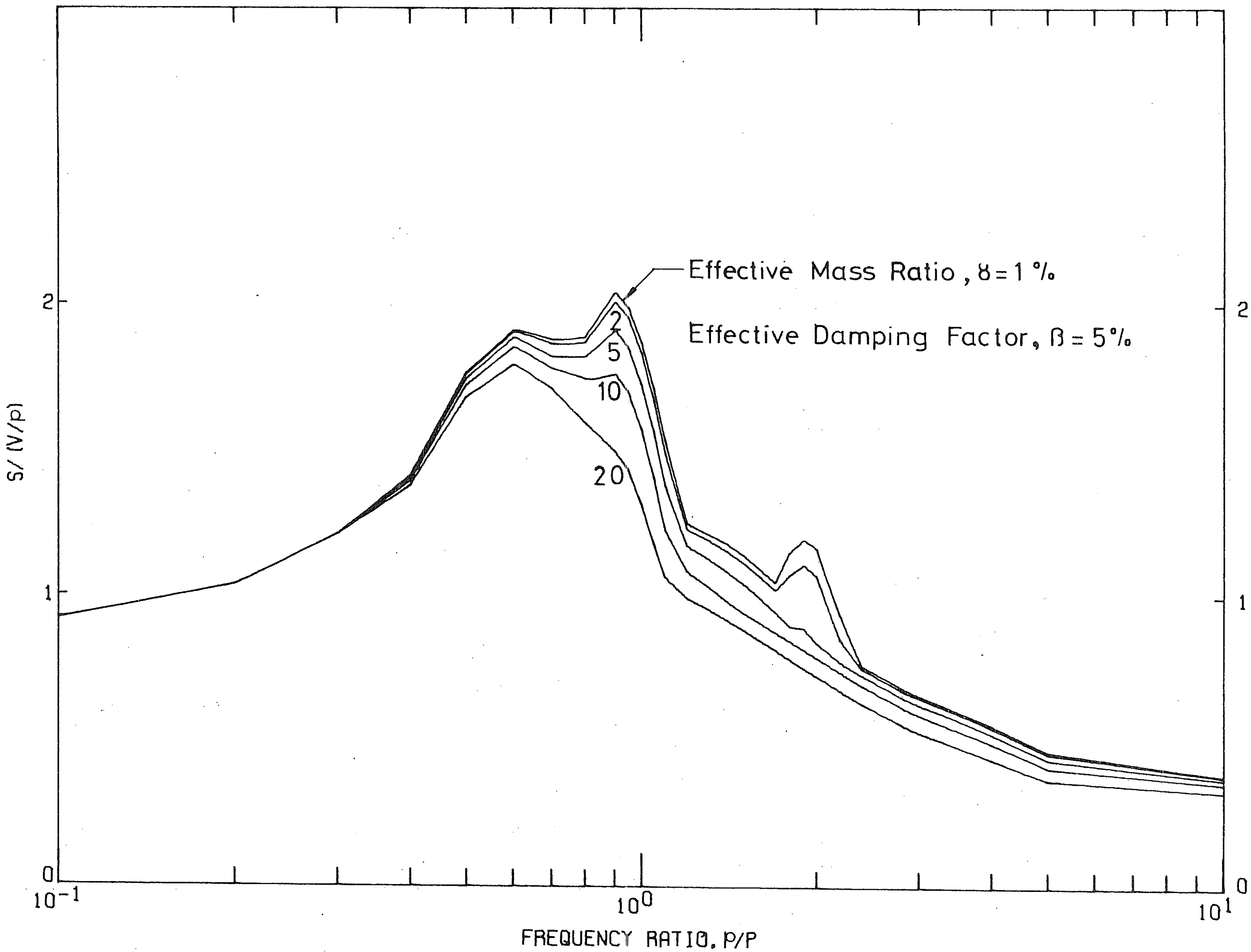


FIG. 4.9e OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

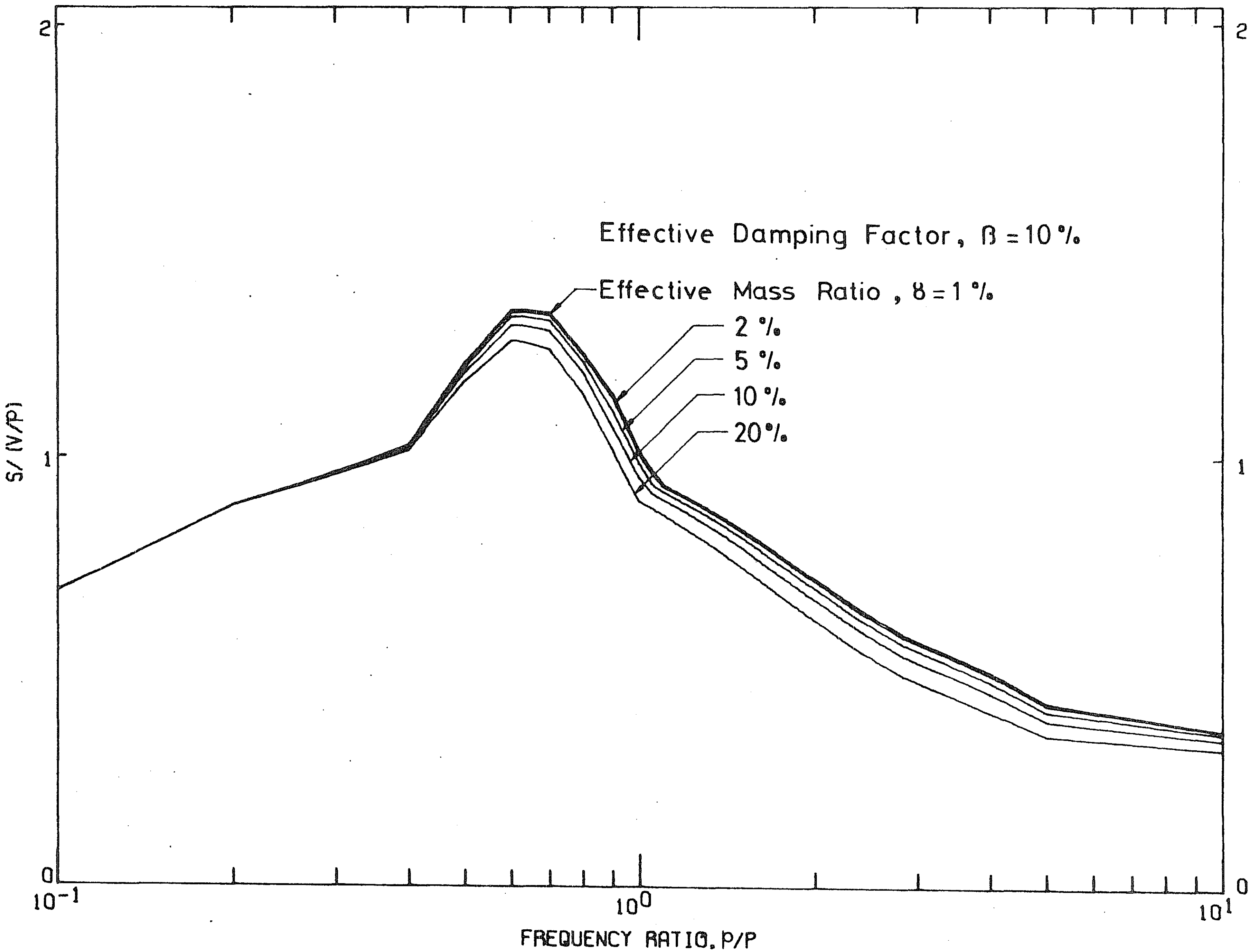


FIG. 4.9f OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE MASS RATIO = 1, 2, 5, 10 AND 20 %

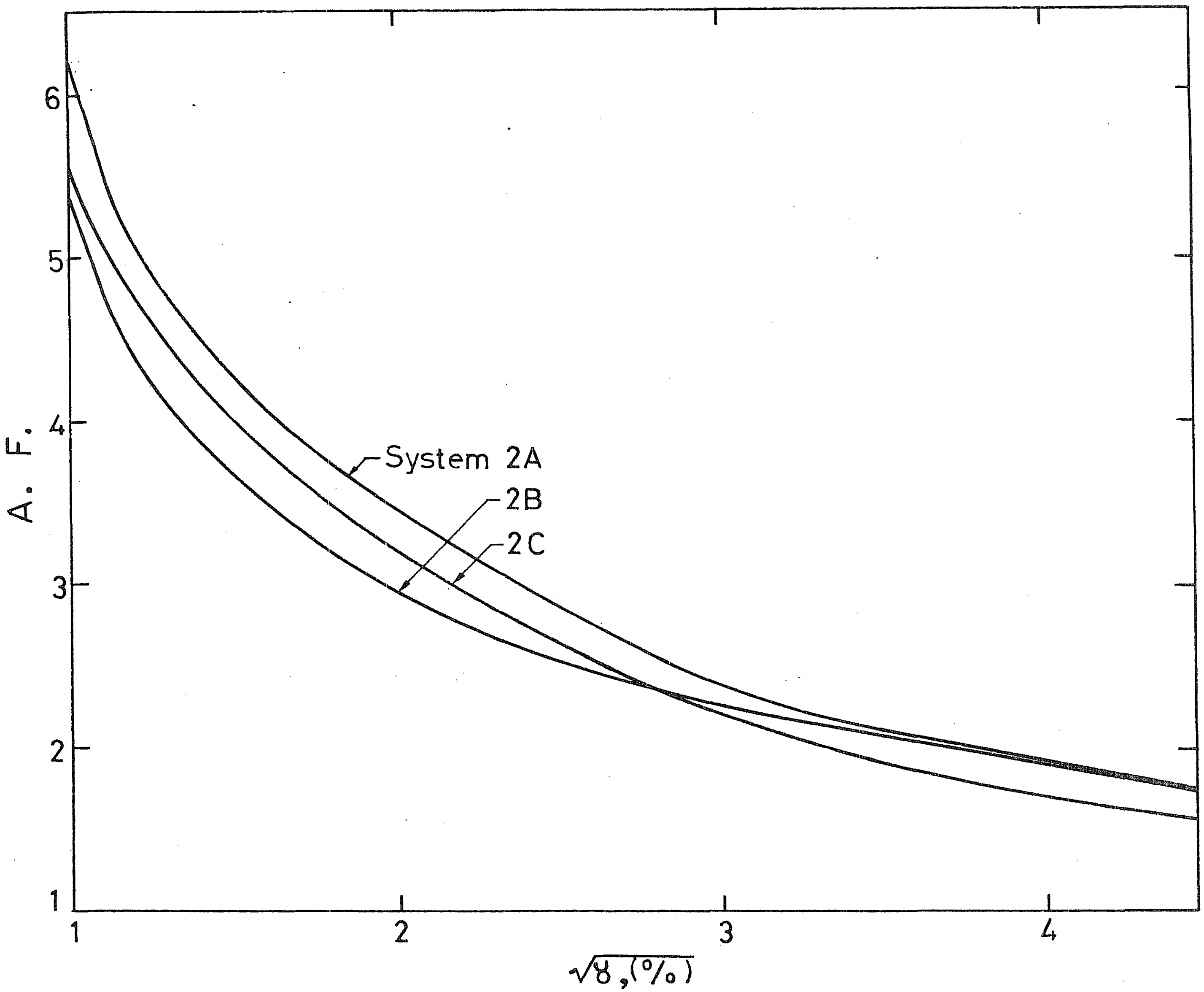


FIG. 4.10 RELATIONSHIPS BETWEEN A.F. AND $\sqrt{\gamma}$, β = 0 %, INNER SPRING, SYSTEMS 2A, 2B AND 2C

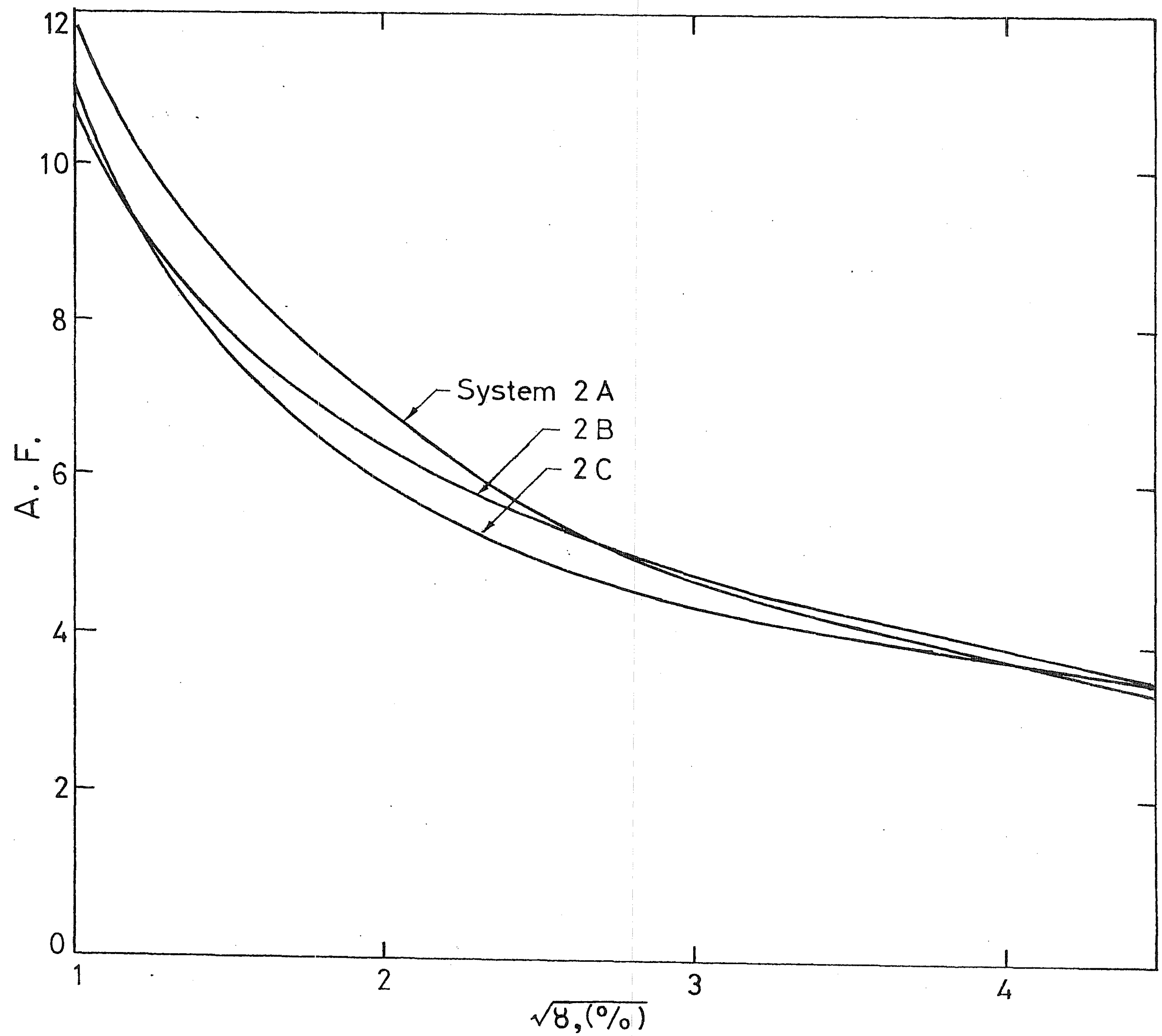


FIG. 4.11 RELATIONSHIPS BETWEEN A.F. AND $\sqrt{\delta}$, β = 0 %, OUTER SPRING, SYSTEMS 2A, 2B AND 2C

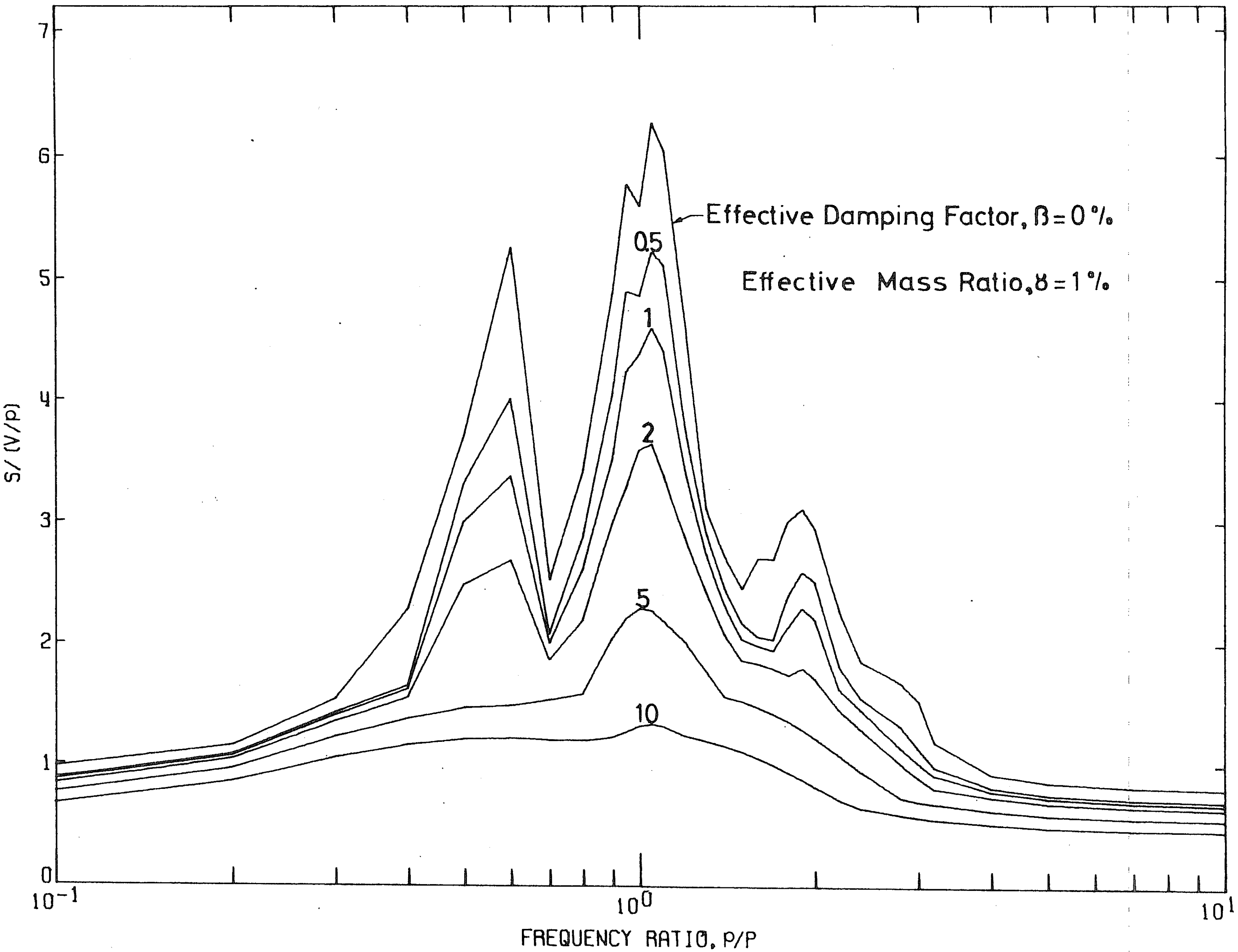


FIG. 4.12a INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

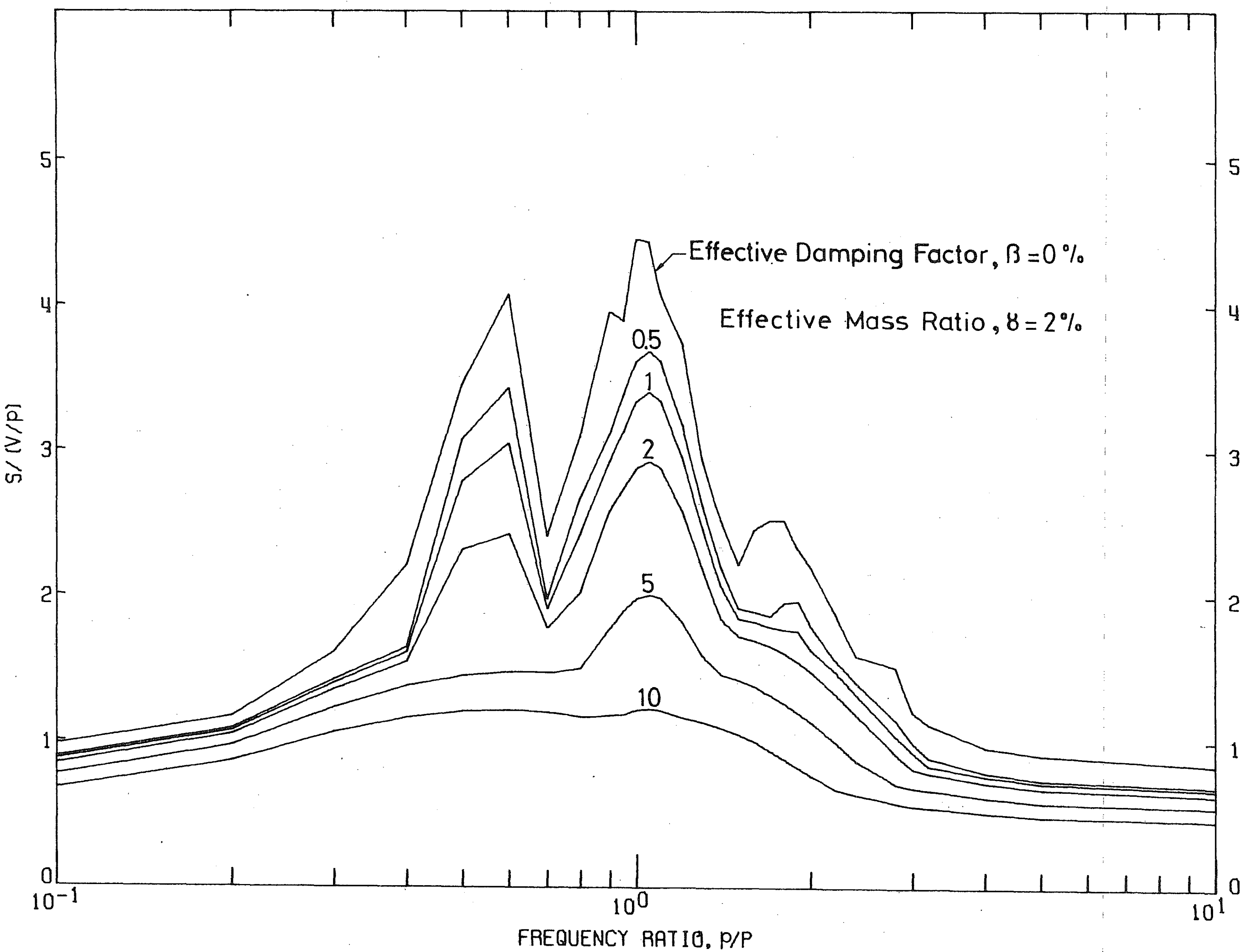


FIG. 4.12b INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

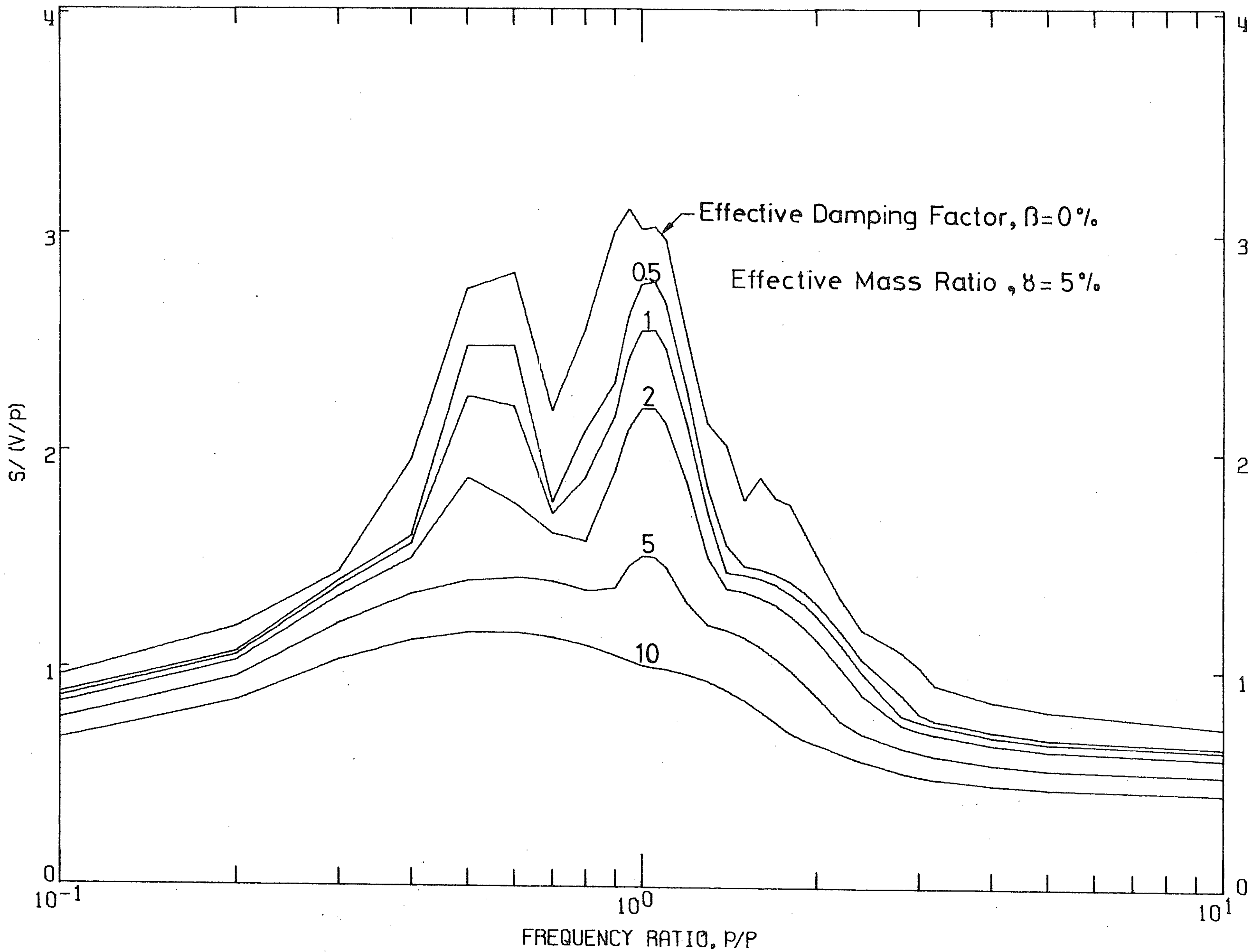


FIG. 4.12c INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

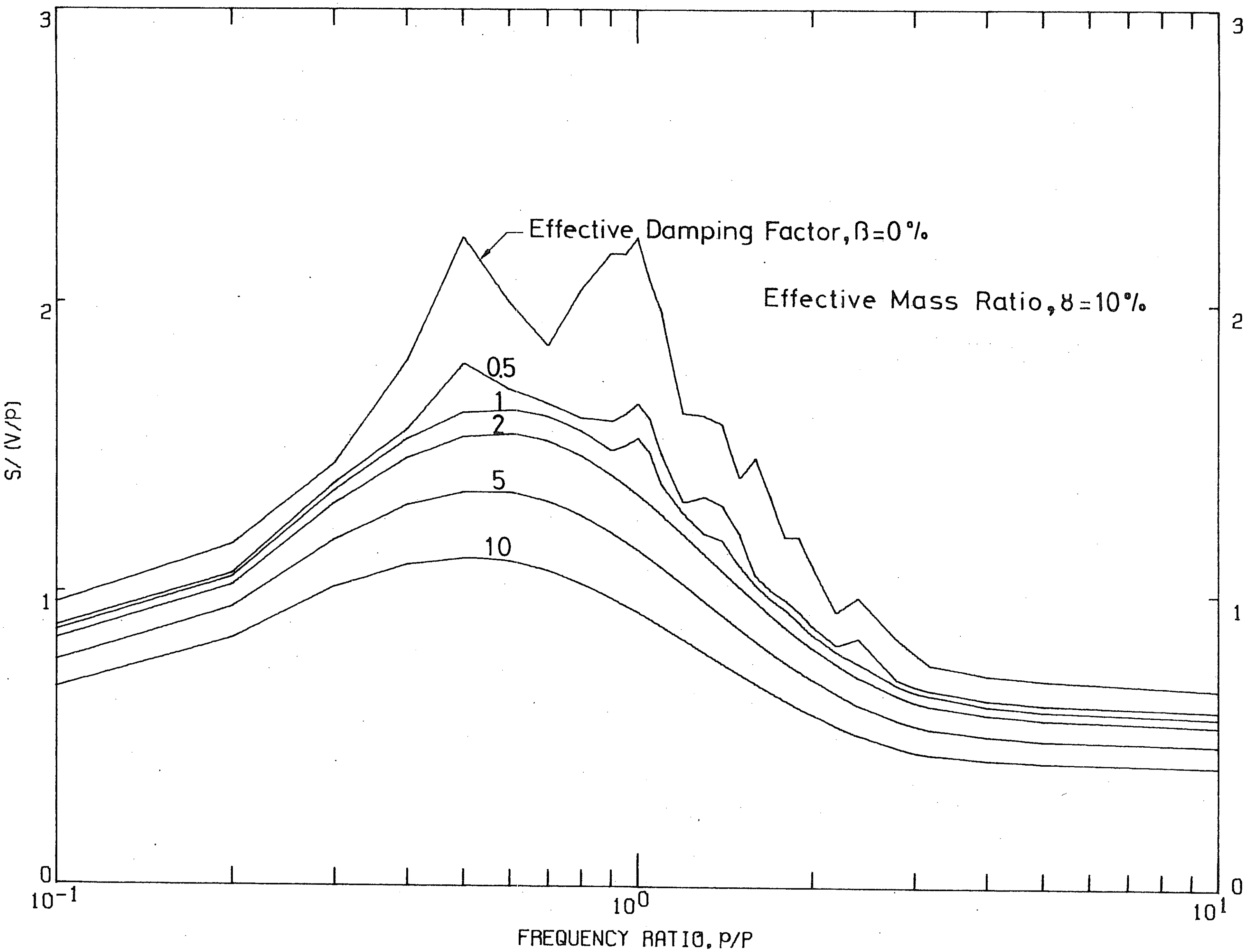


FIG. 4.12d INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

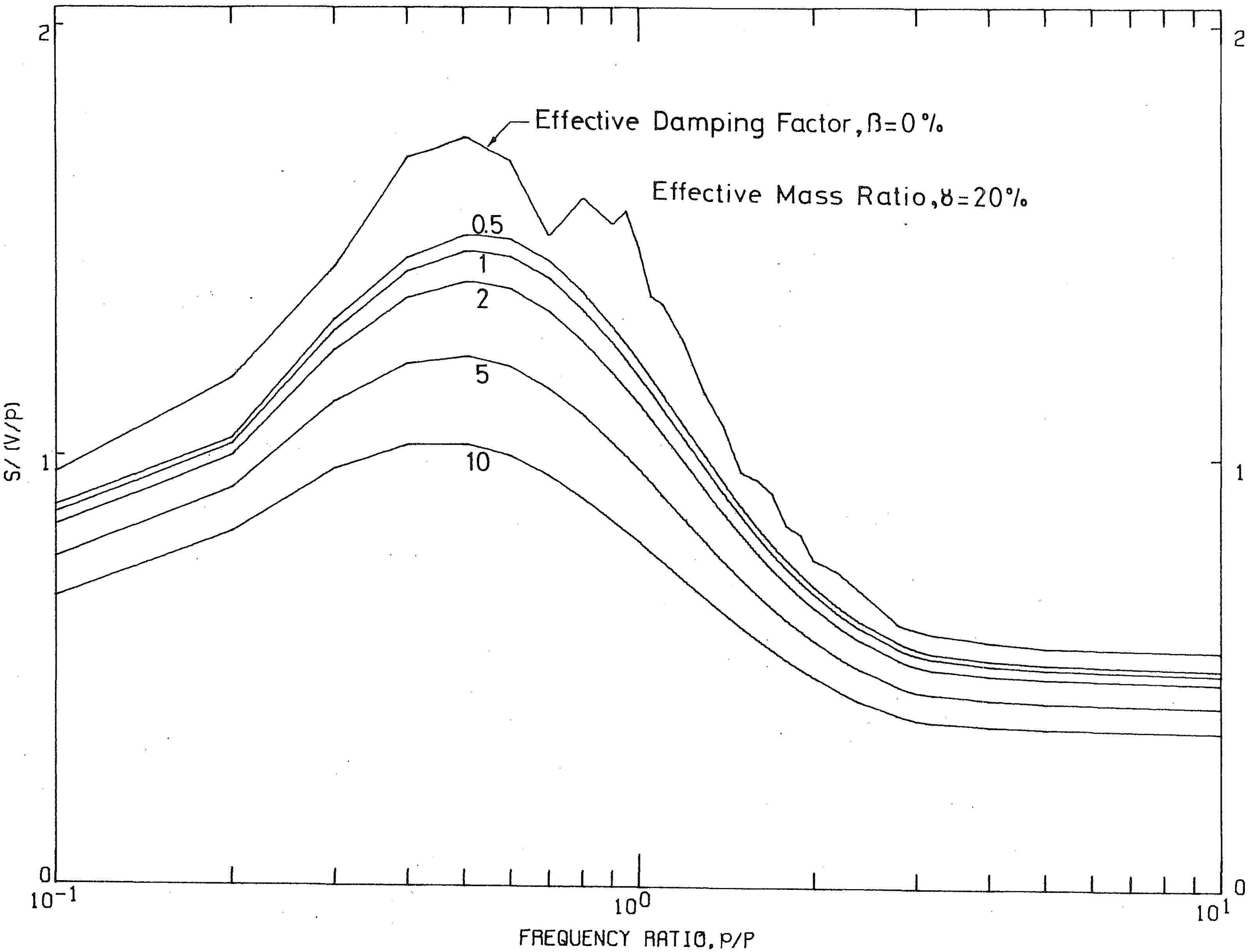


FIG. 4.12e INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

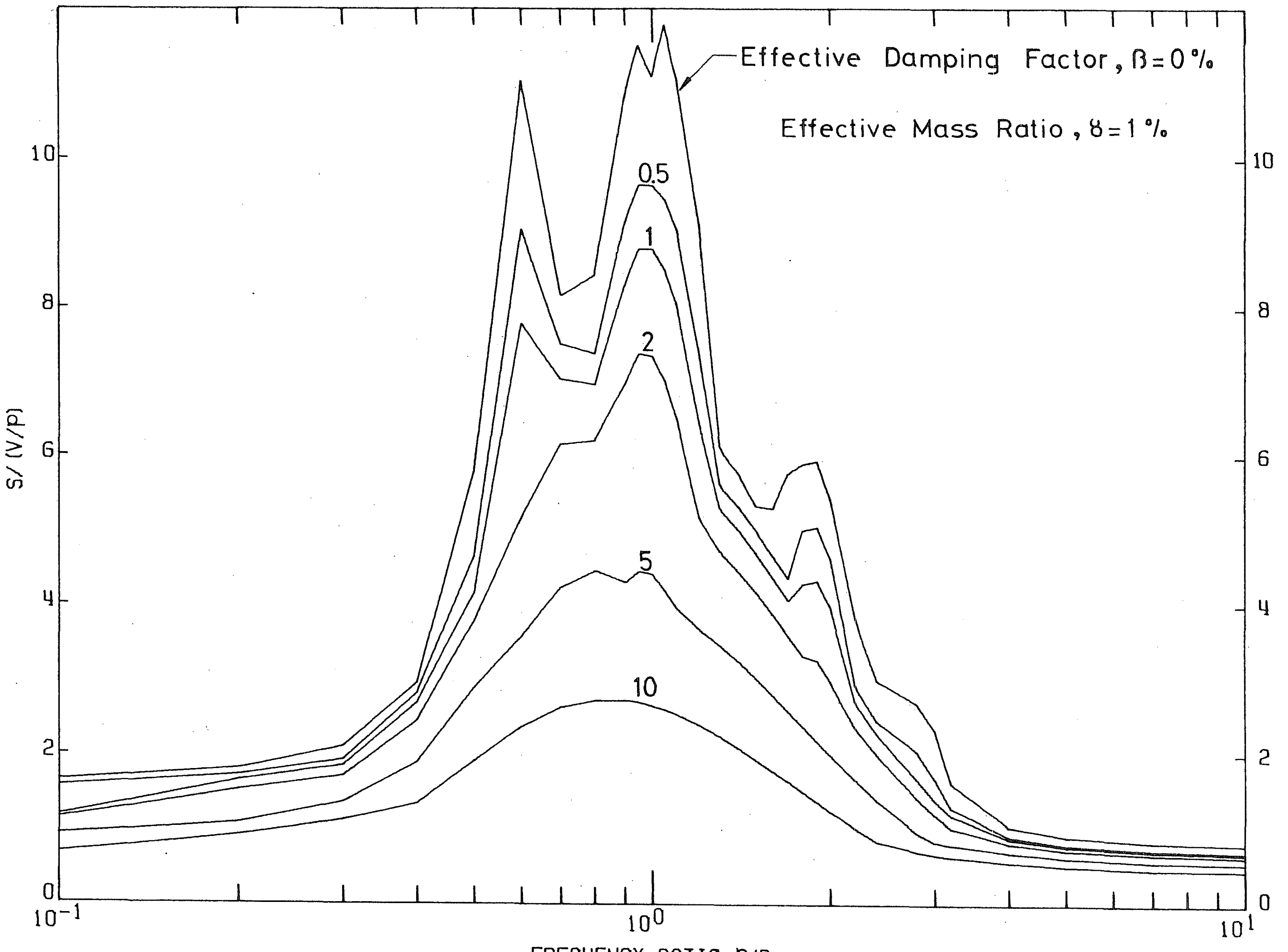


FIG. 4.13a OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

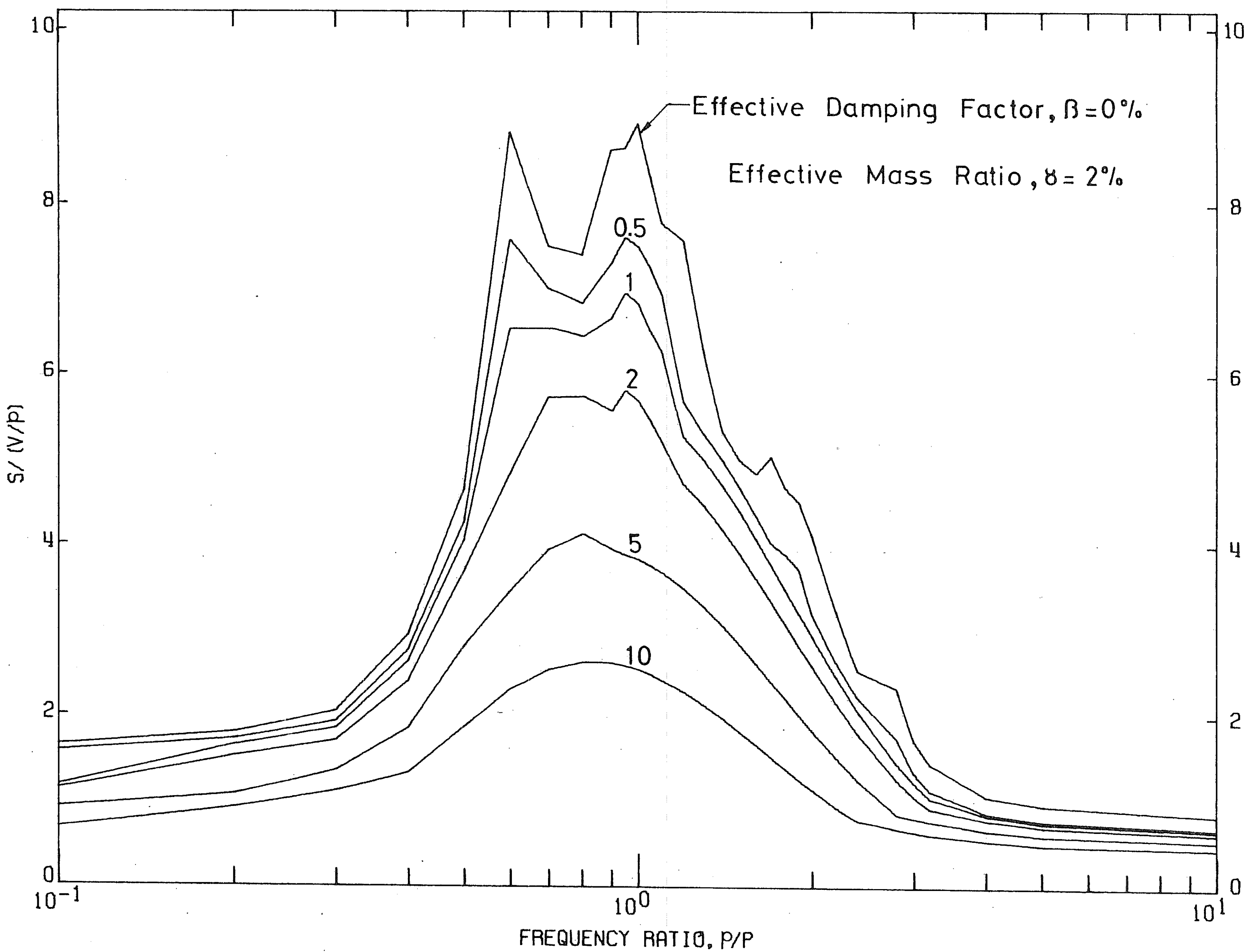


FIG. 4.13b OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

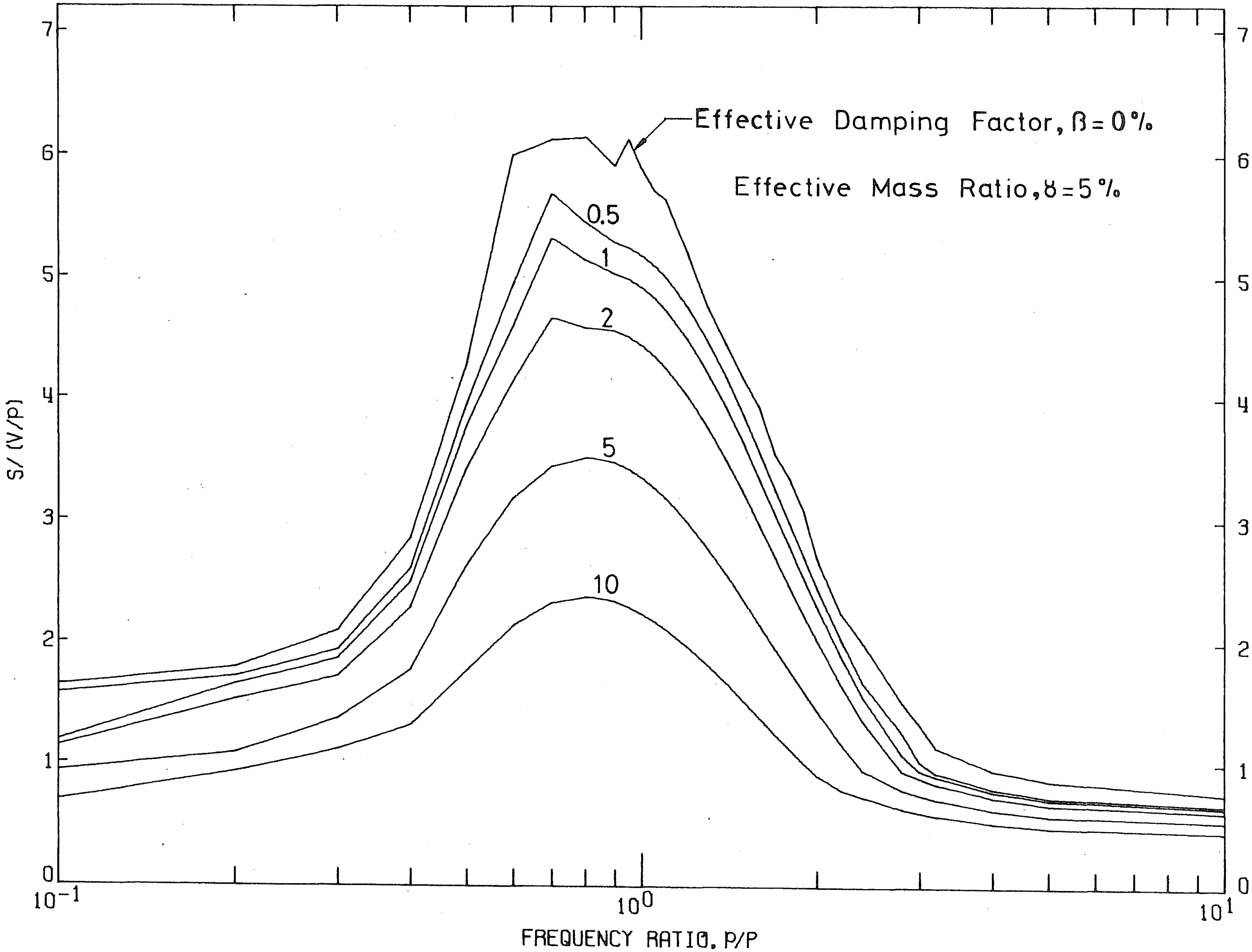


FIG. 4.13c OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

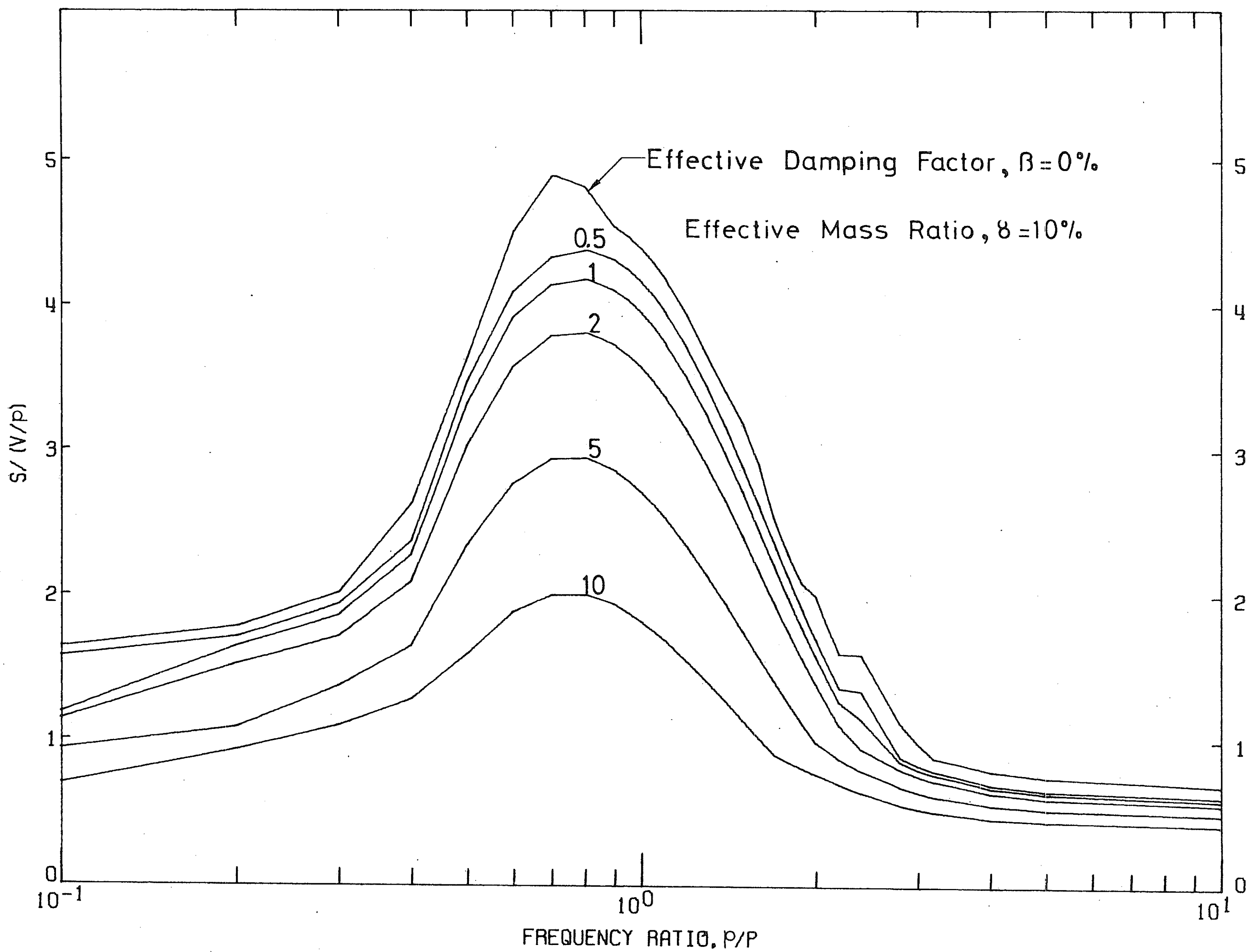


FIG. 4.13d OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

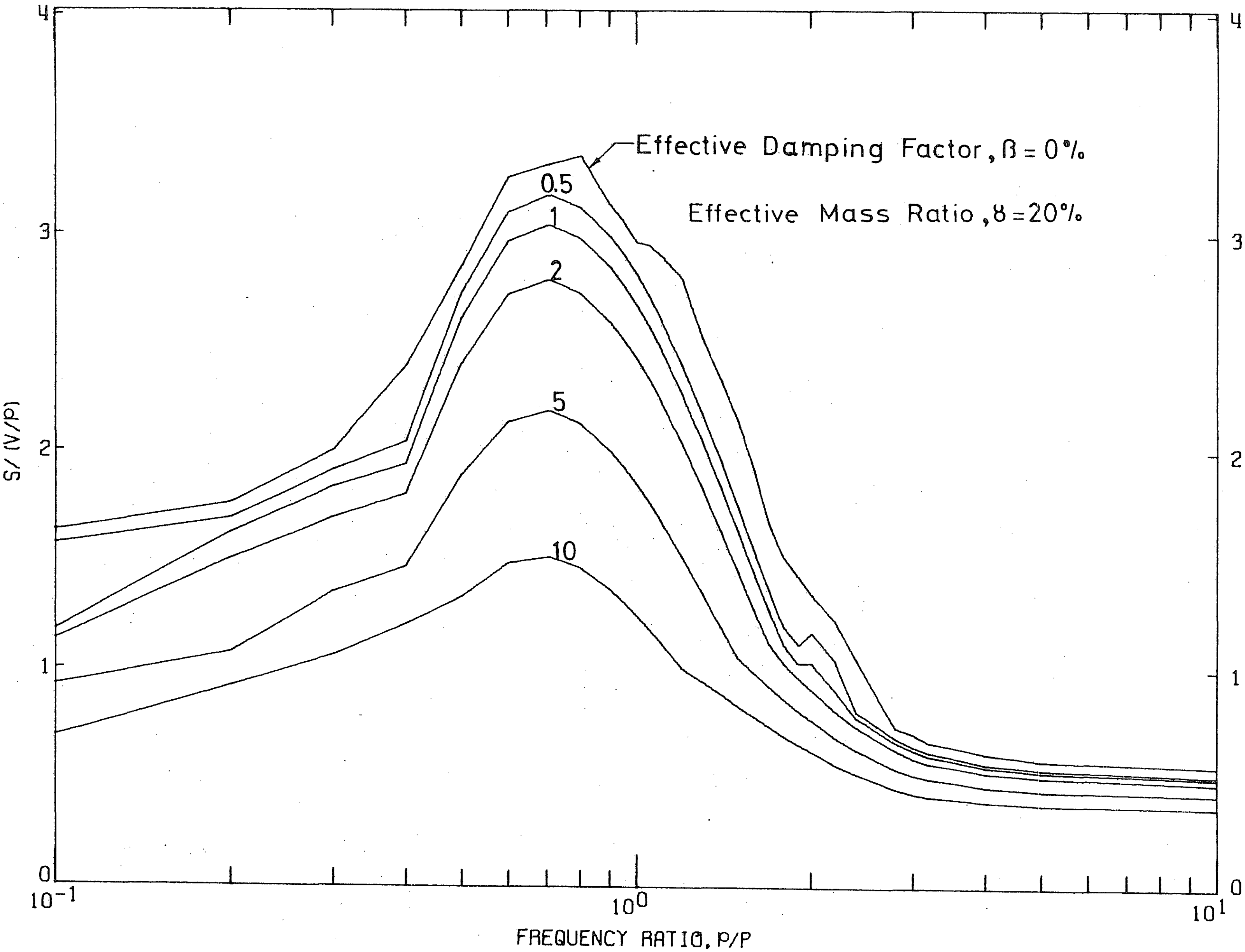


FIG. 4.13e OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

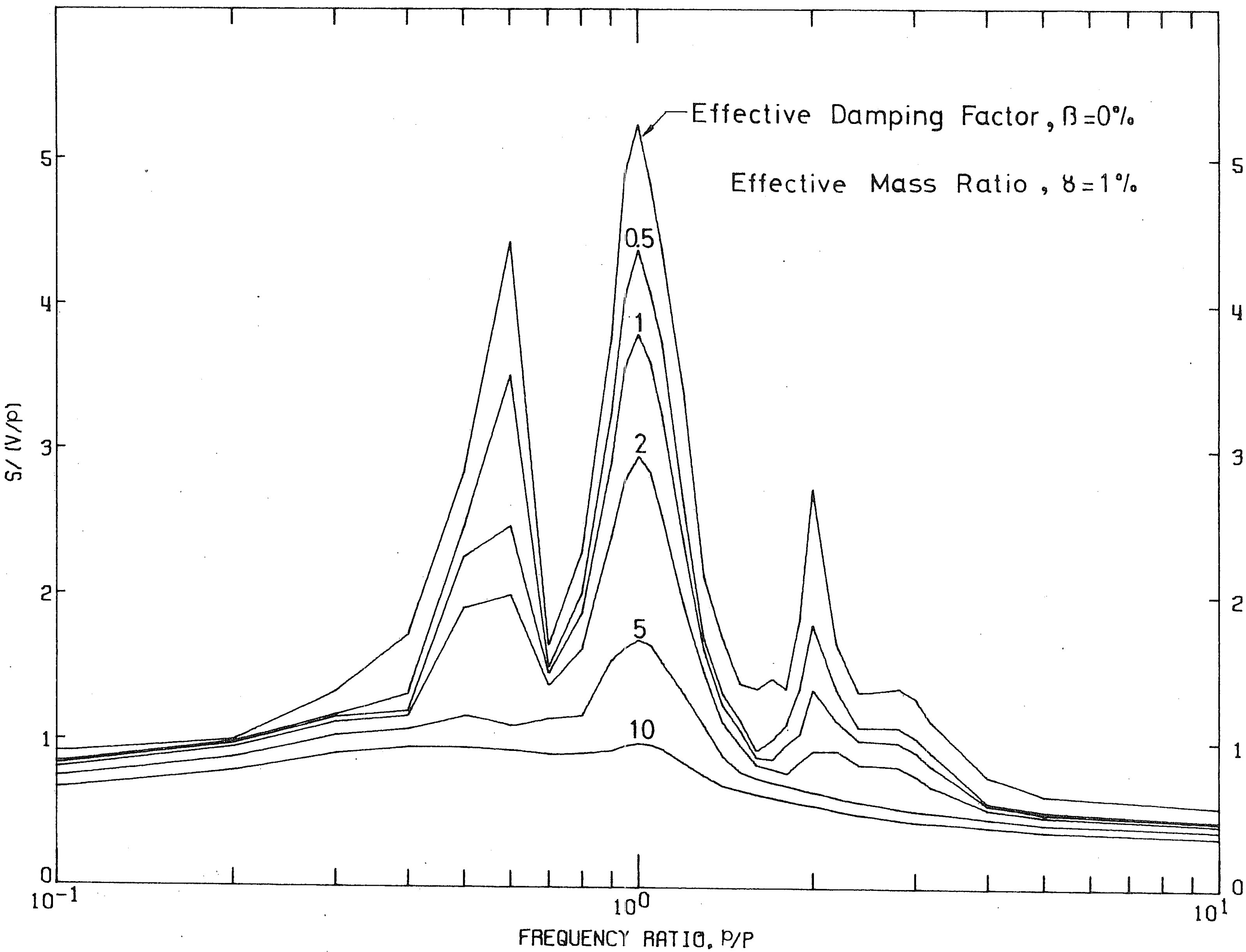


FIG. 4.14a INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

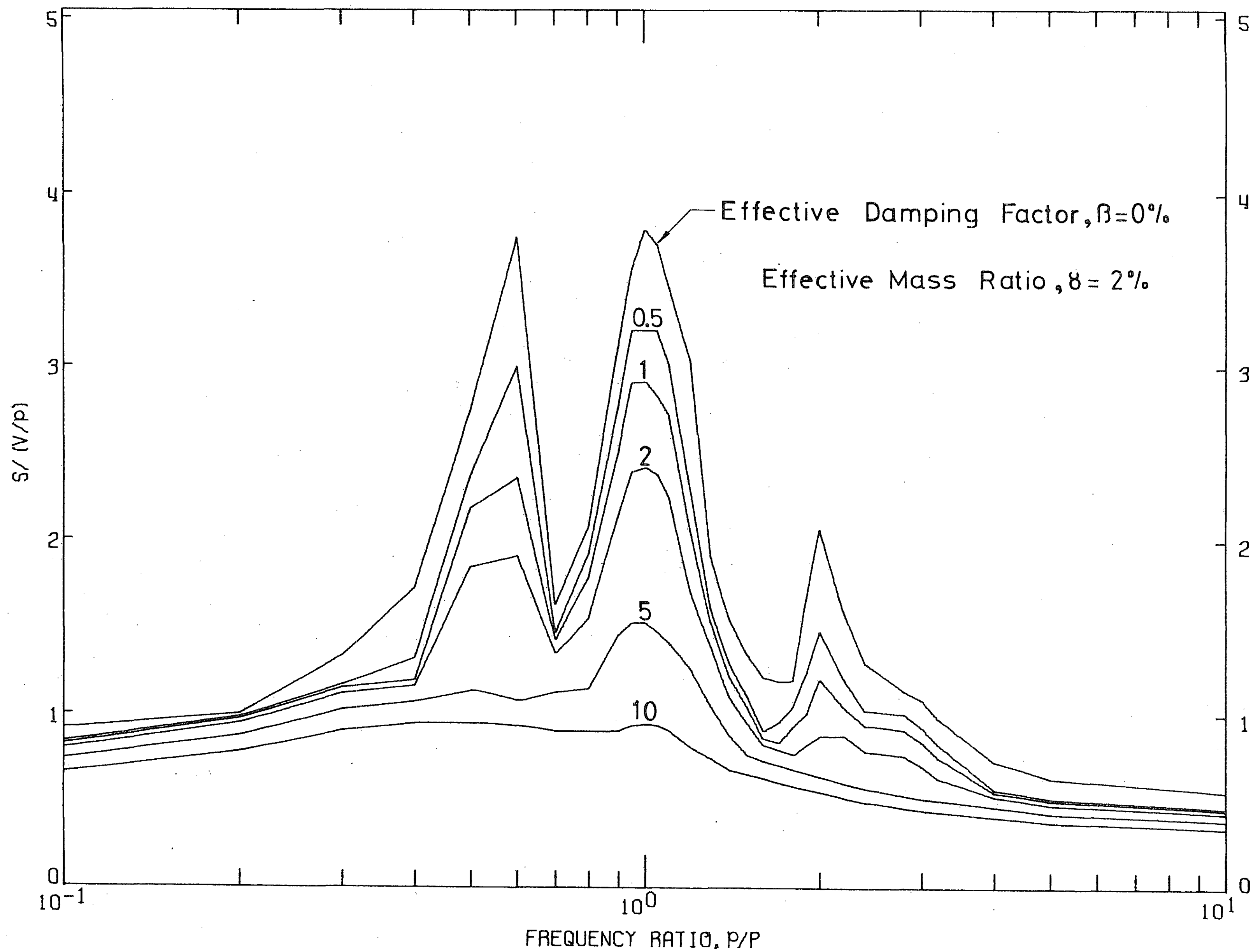


FIG. 4.14b INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

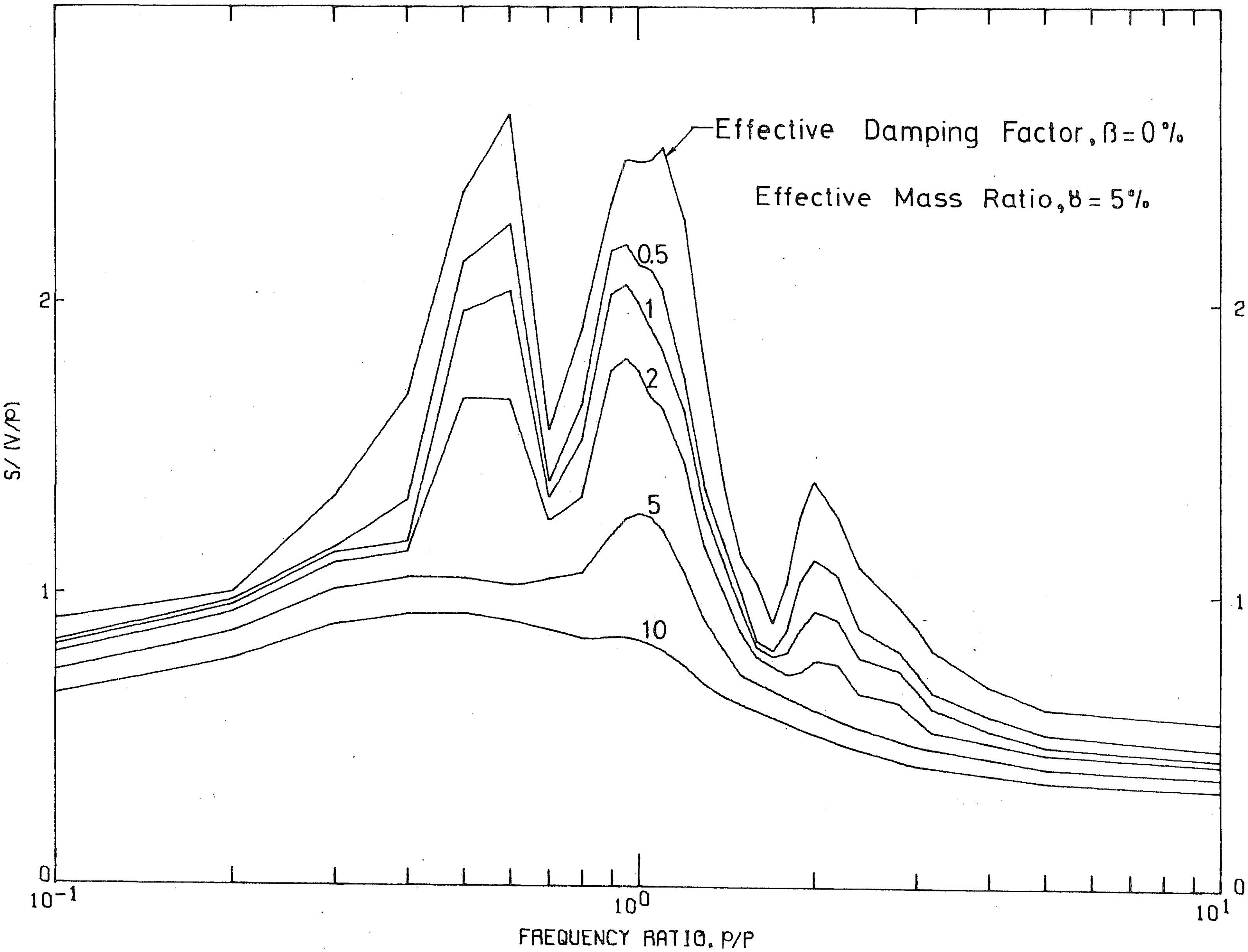


FIG. 4.14c INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

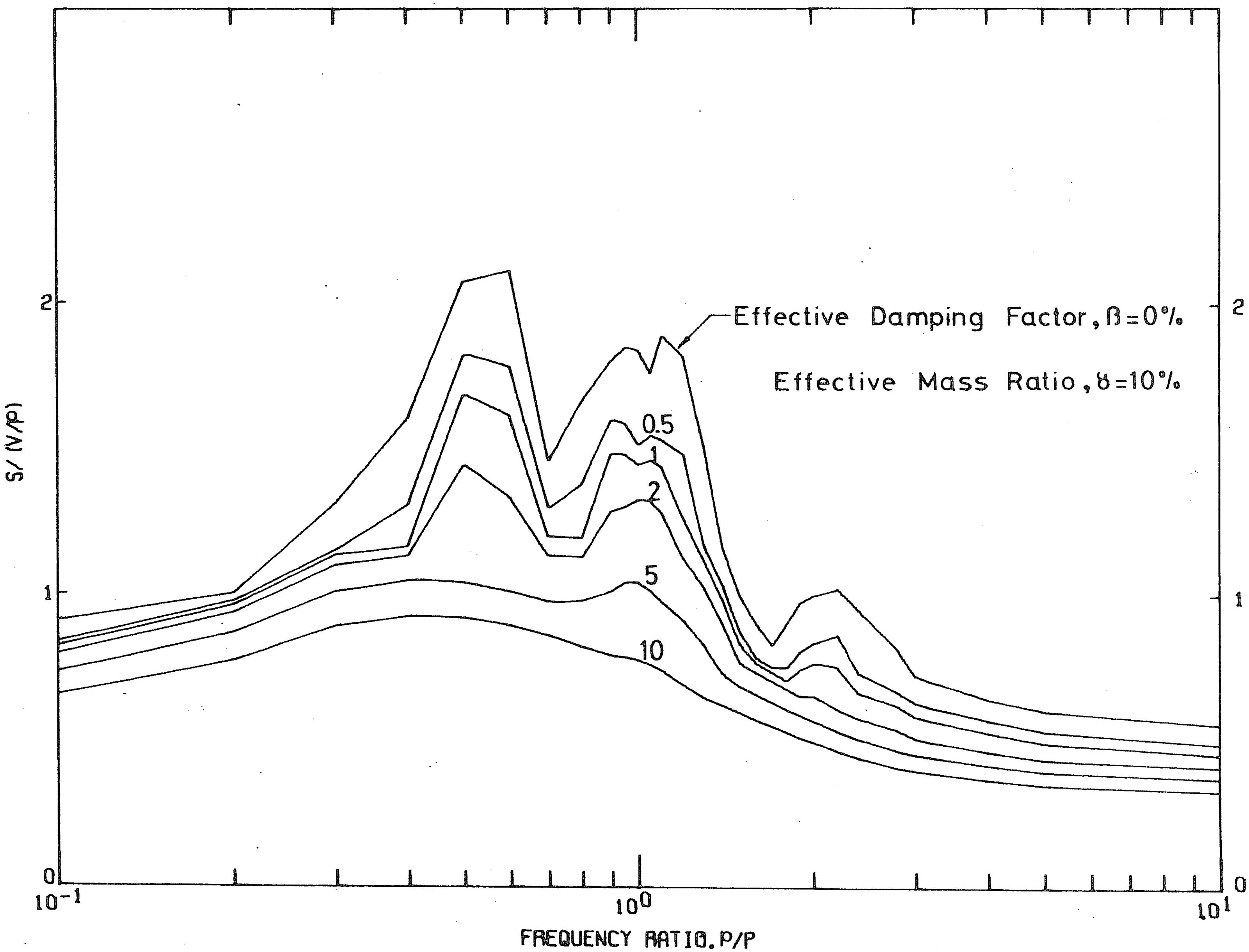


FIG. 4.14d INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

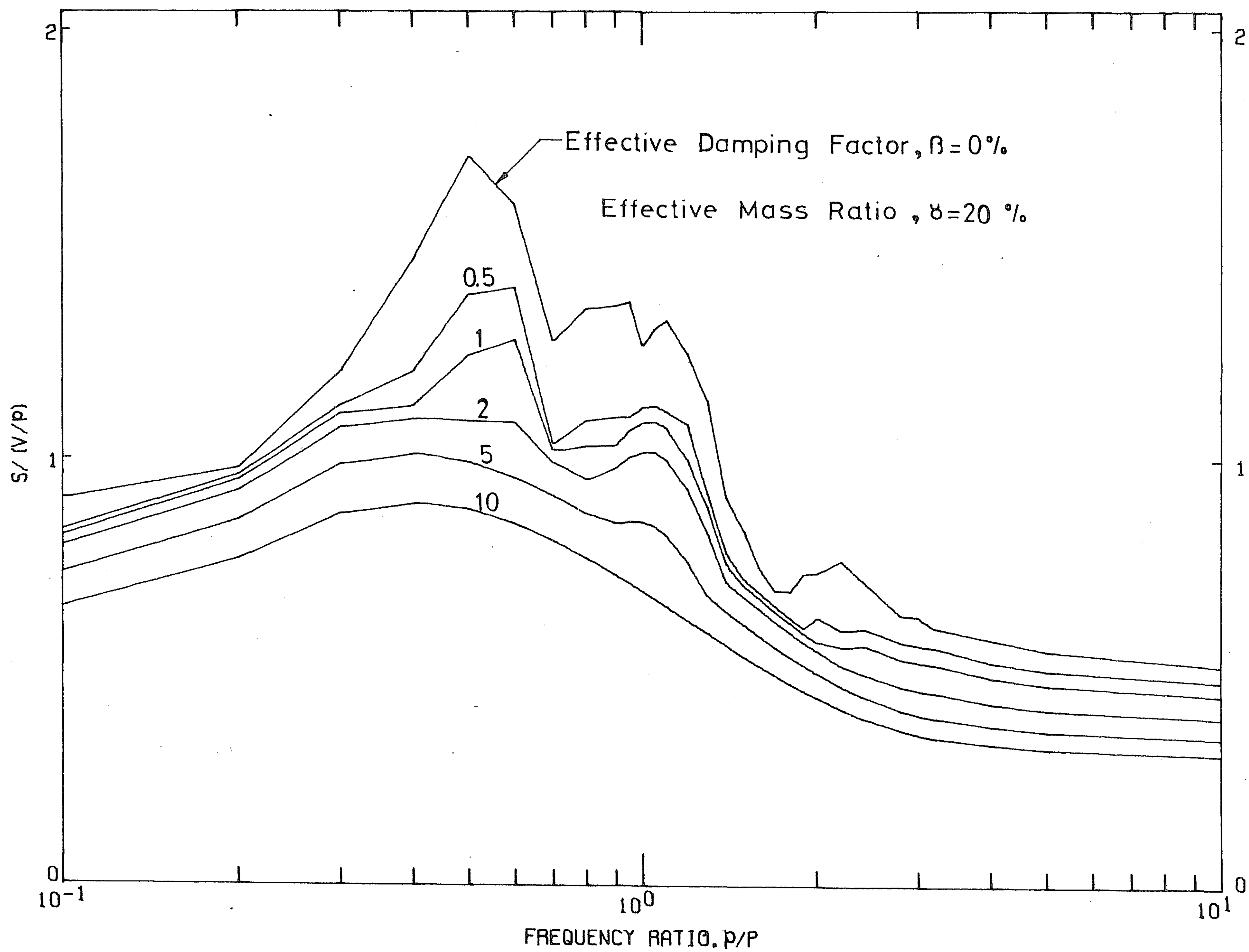


FIG. 4.14e INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

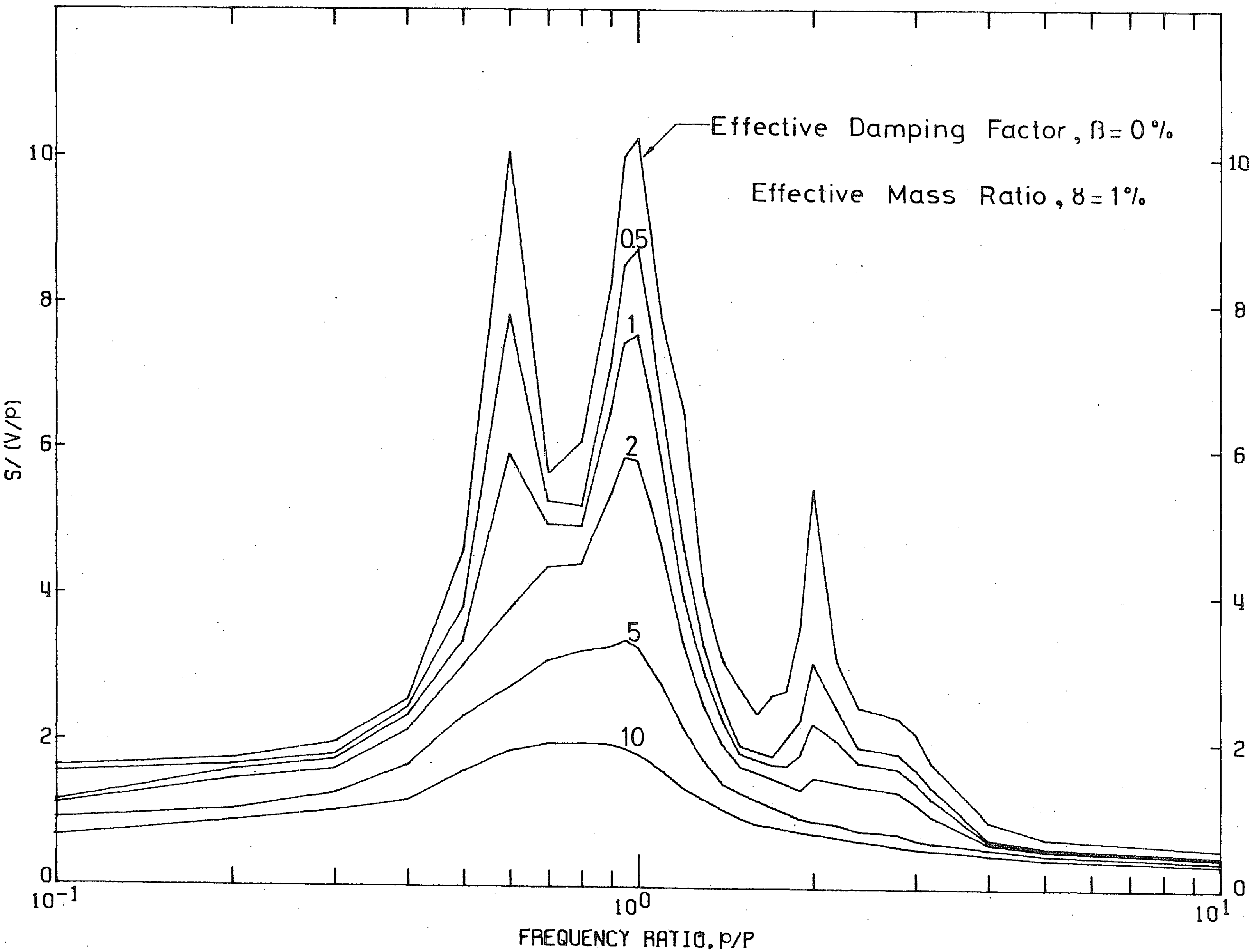


FIG. 4.15a OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

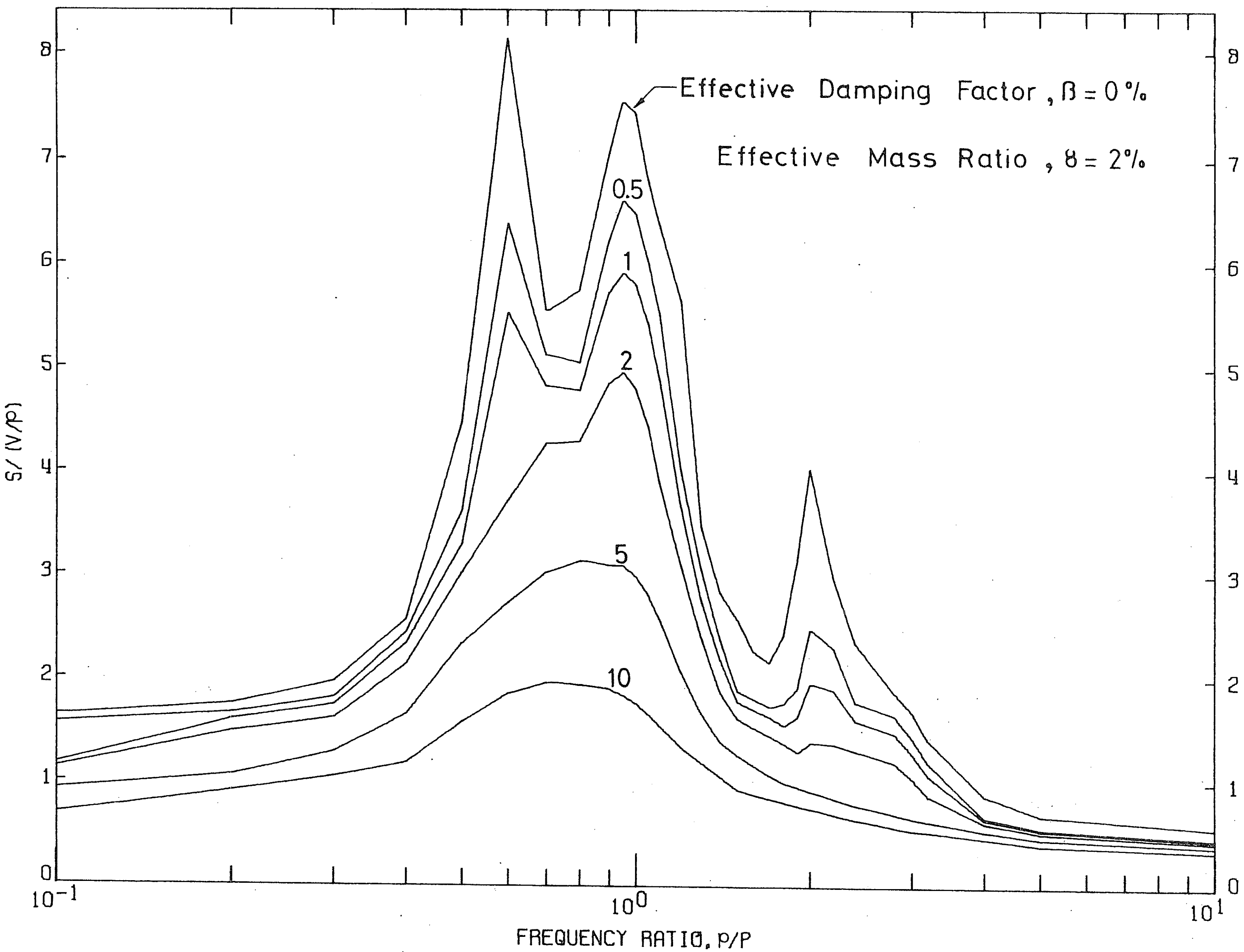


FIG. 4.15b OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

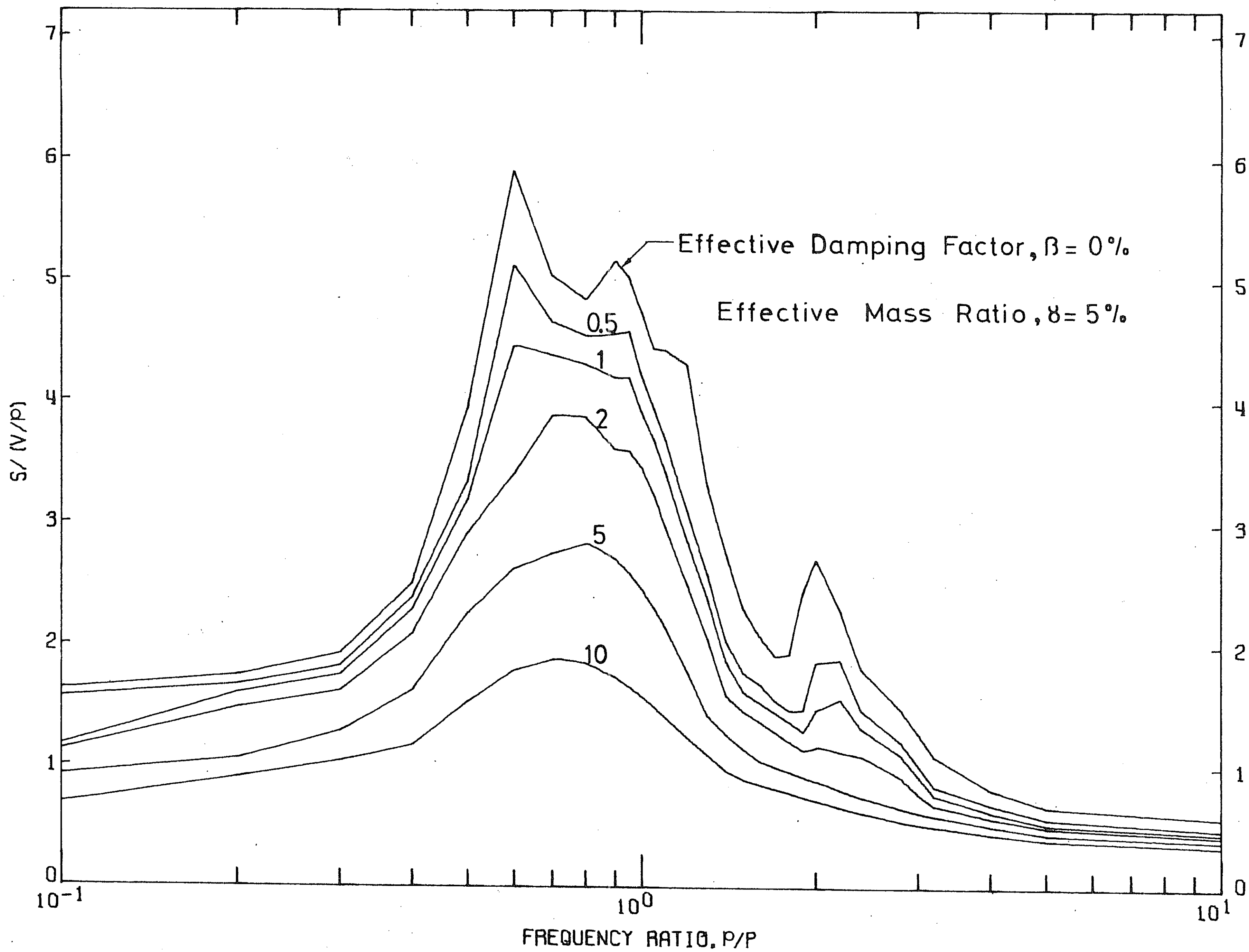


FIG. 4.15c OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

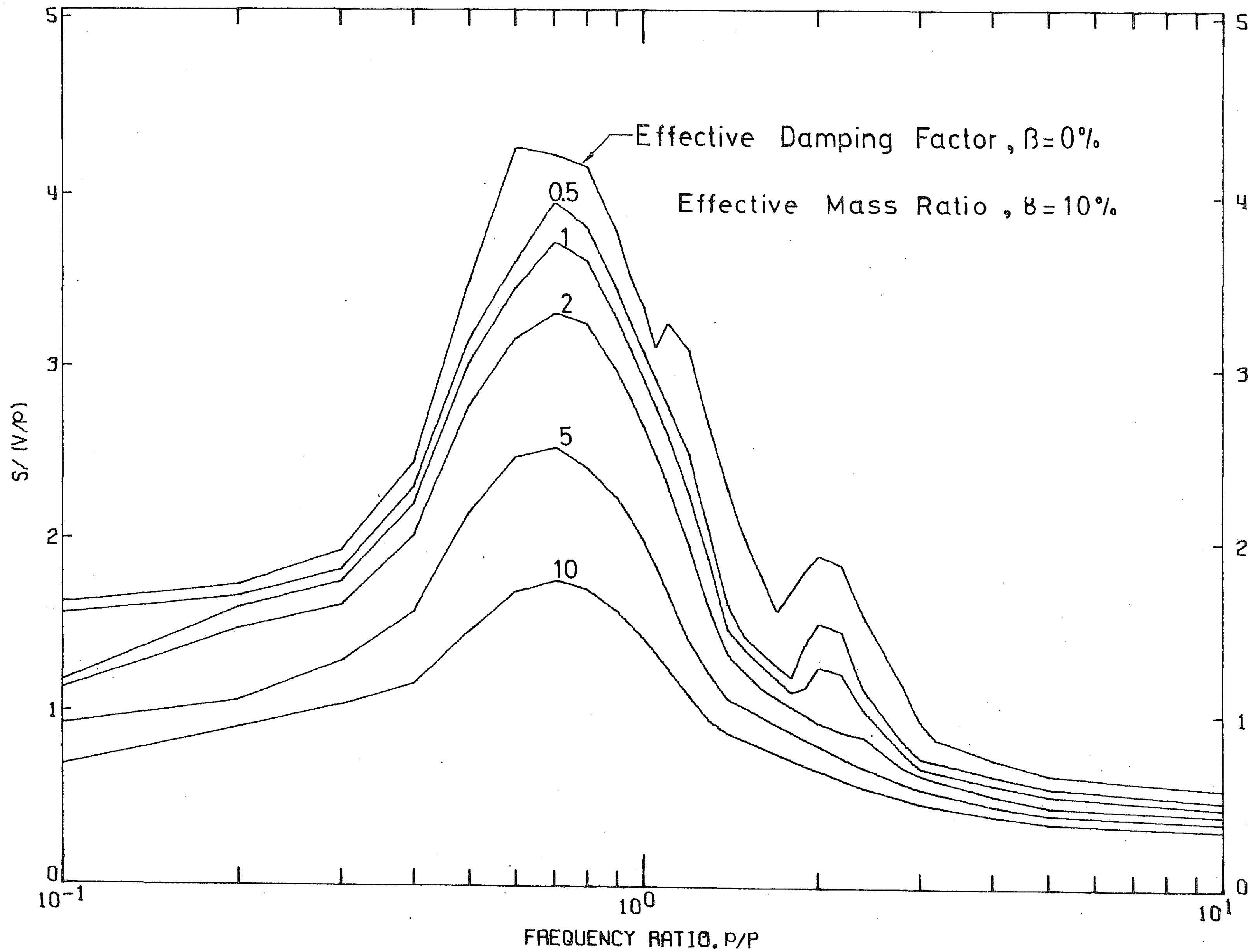


FIG. 4.15d OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

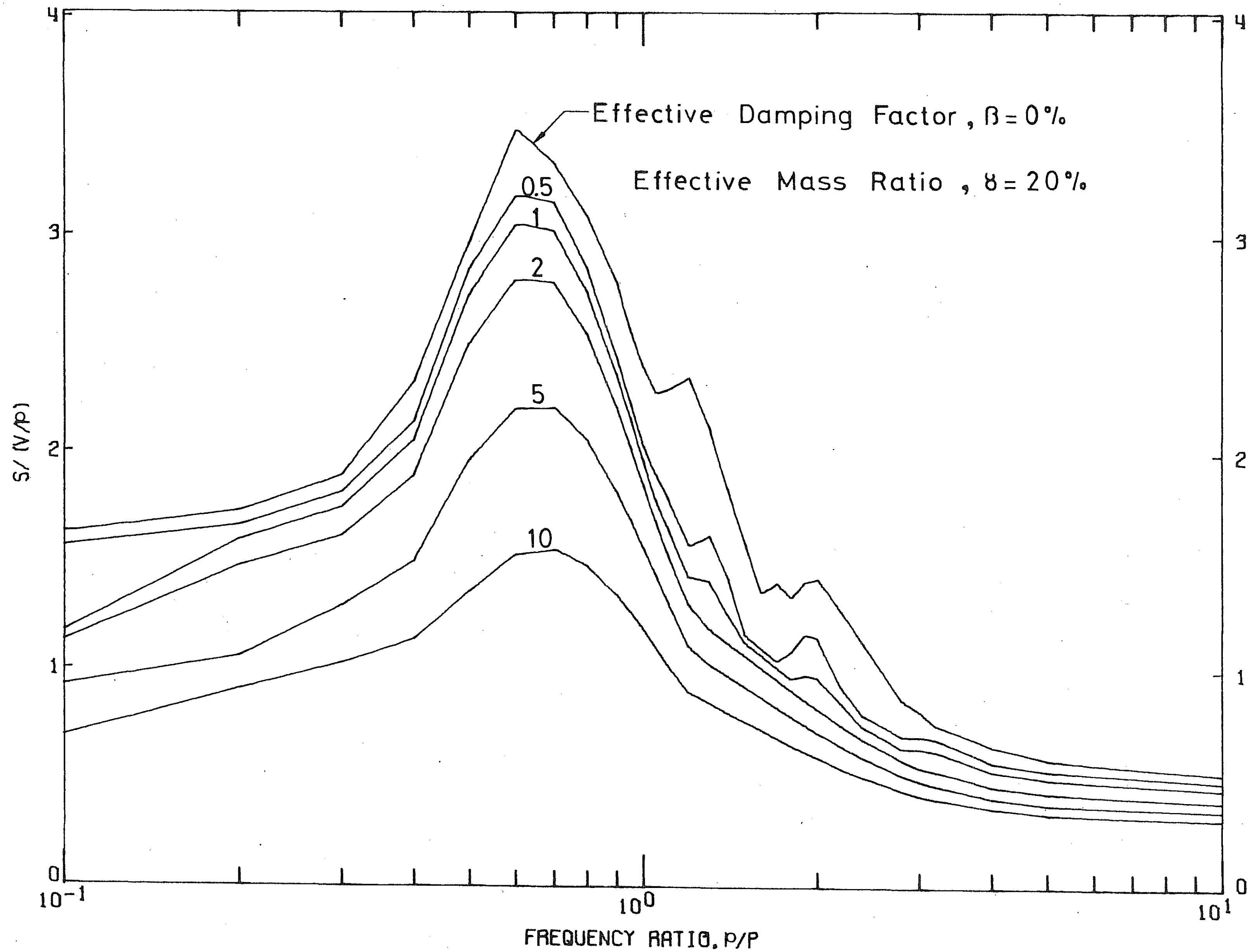


FIG. 4.15e OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2B, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

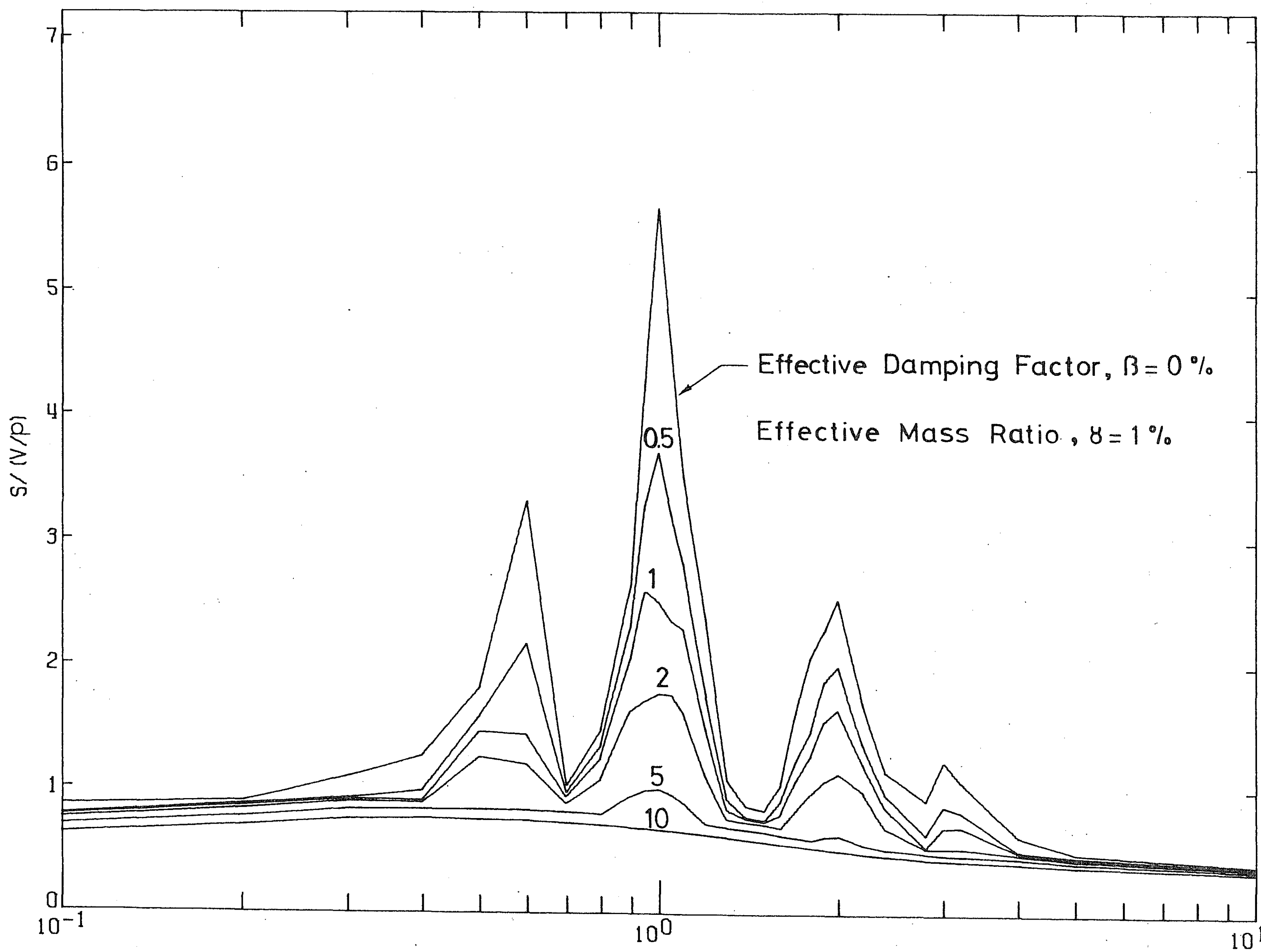


FIG. 4.16a INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

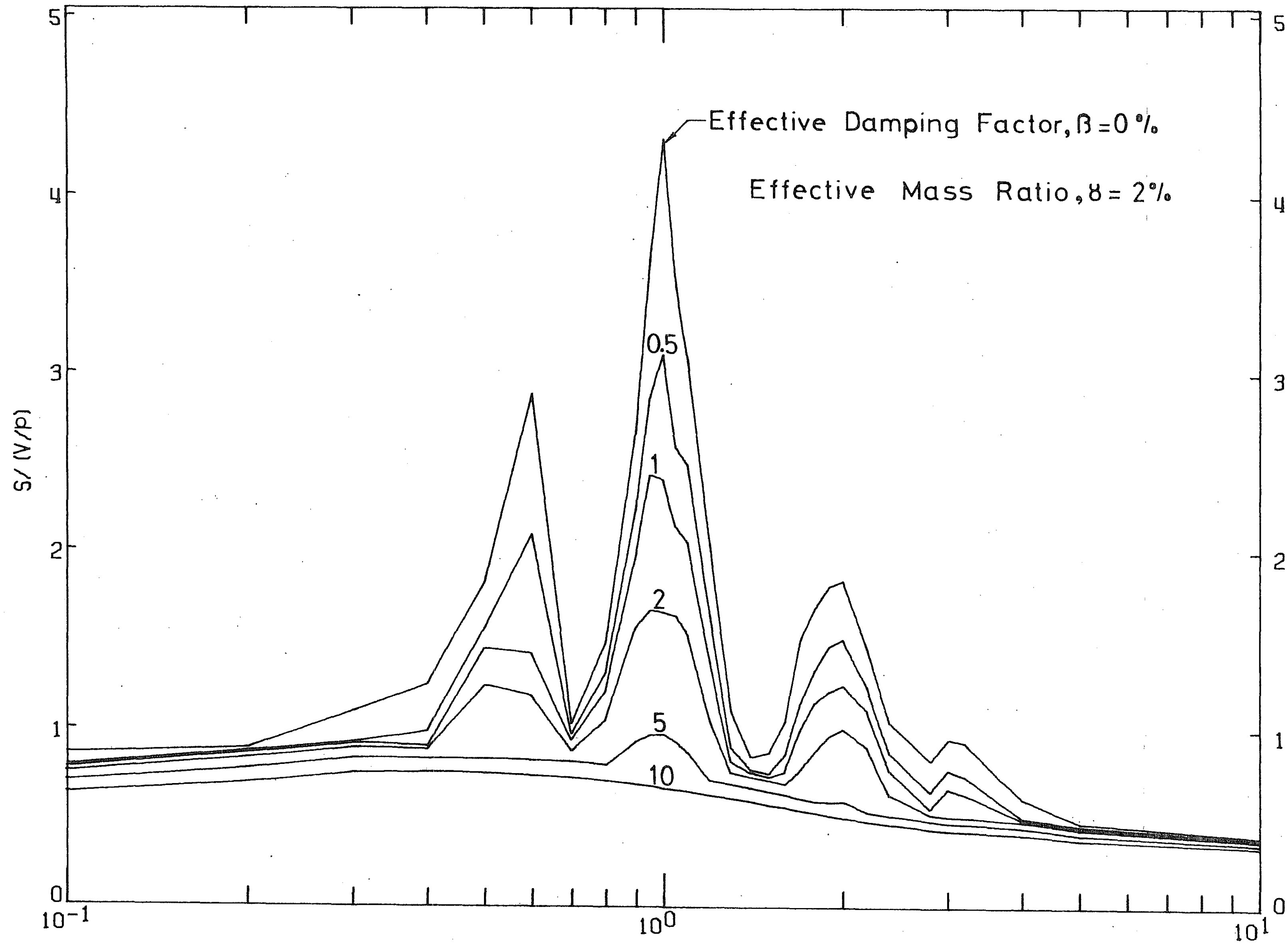


FIG. 4.16b INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

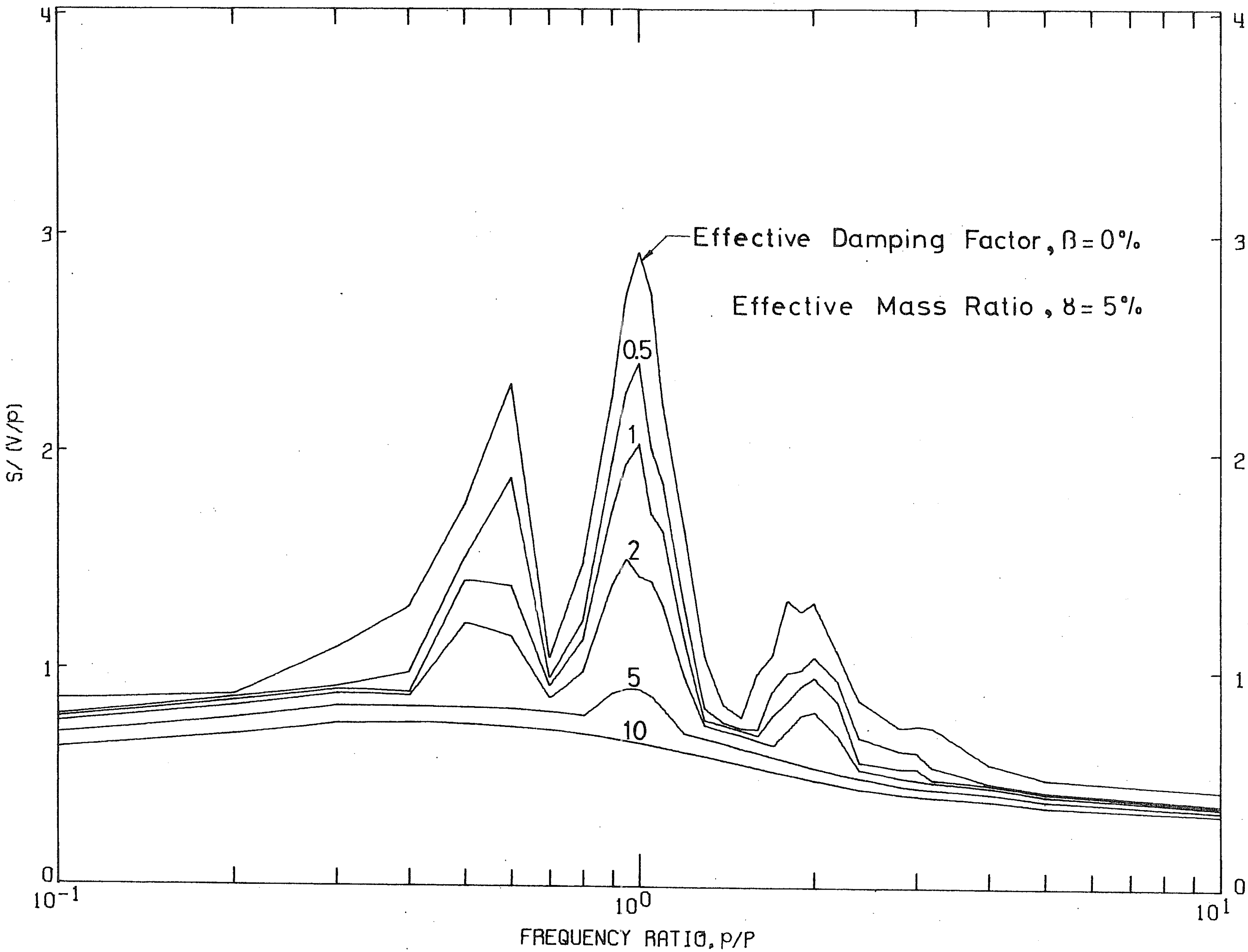


FIG. 4.16c INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

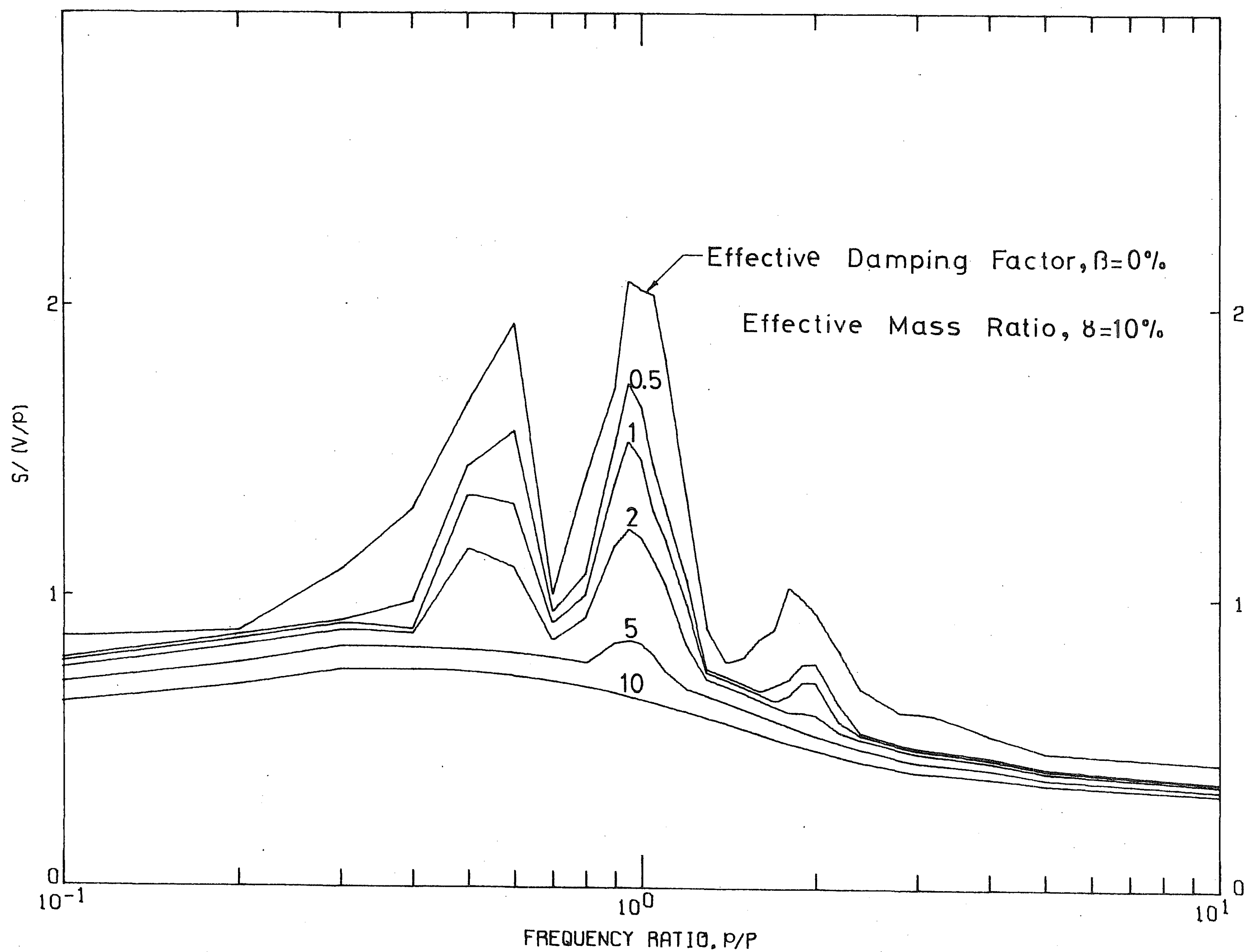


FIG. 4.16d INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

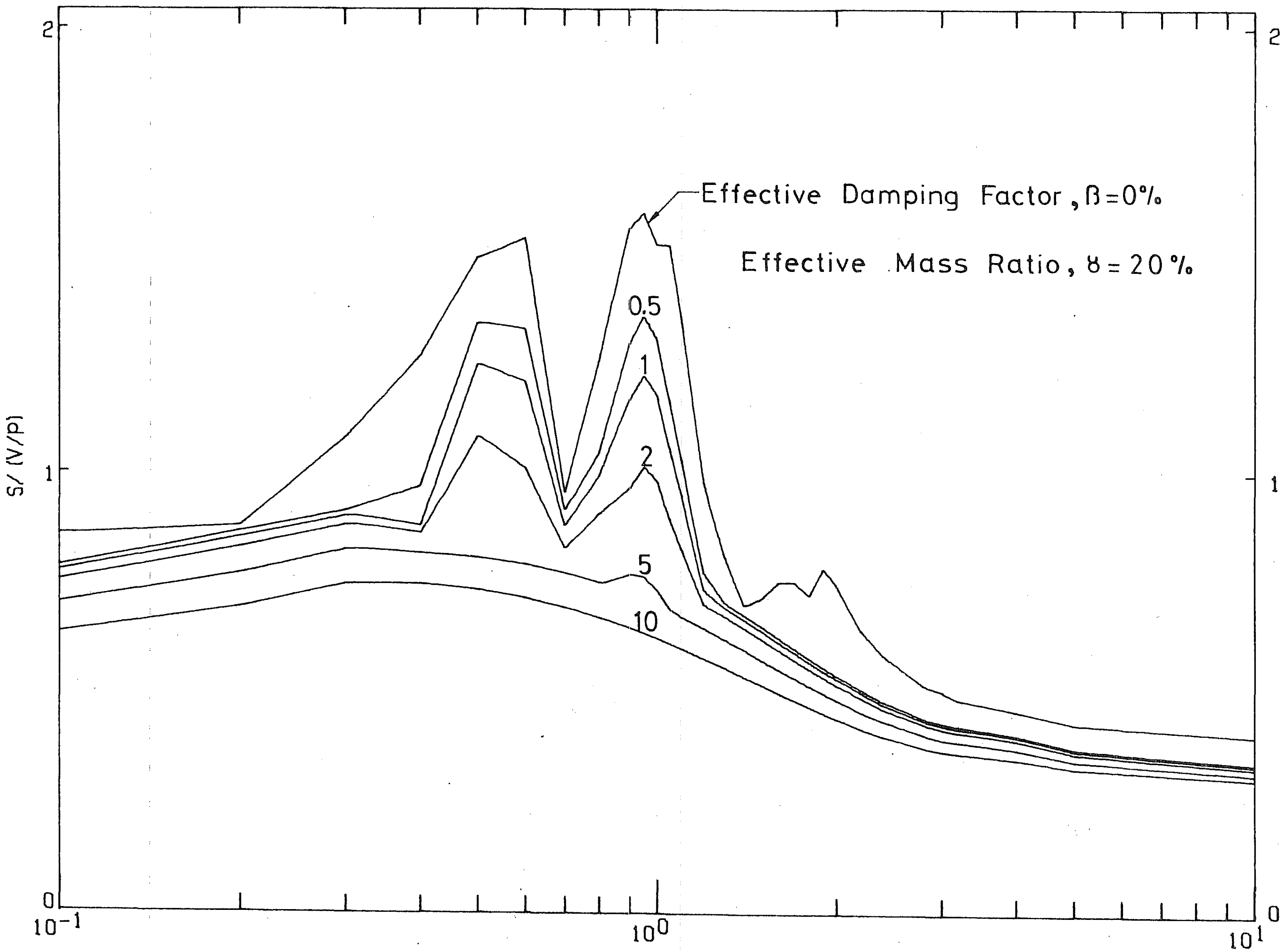


FIG. 4.16e INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

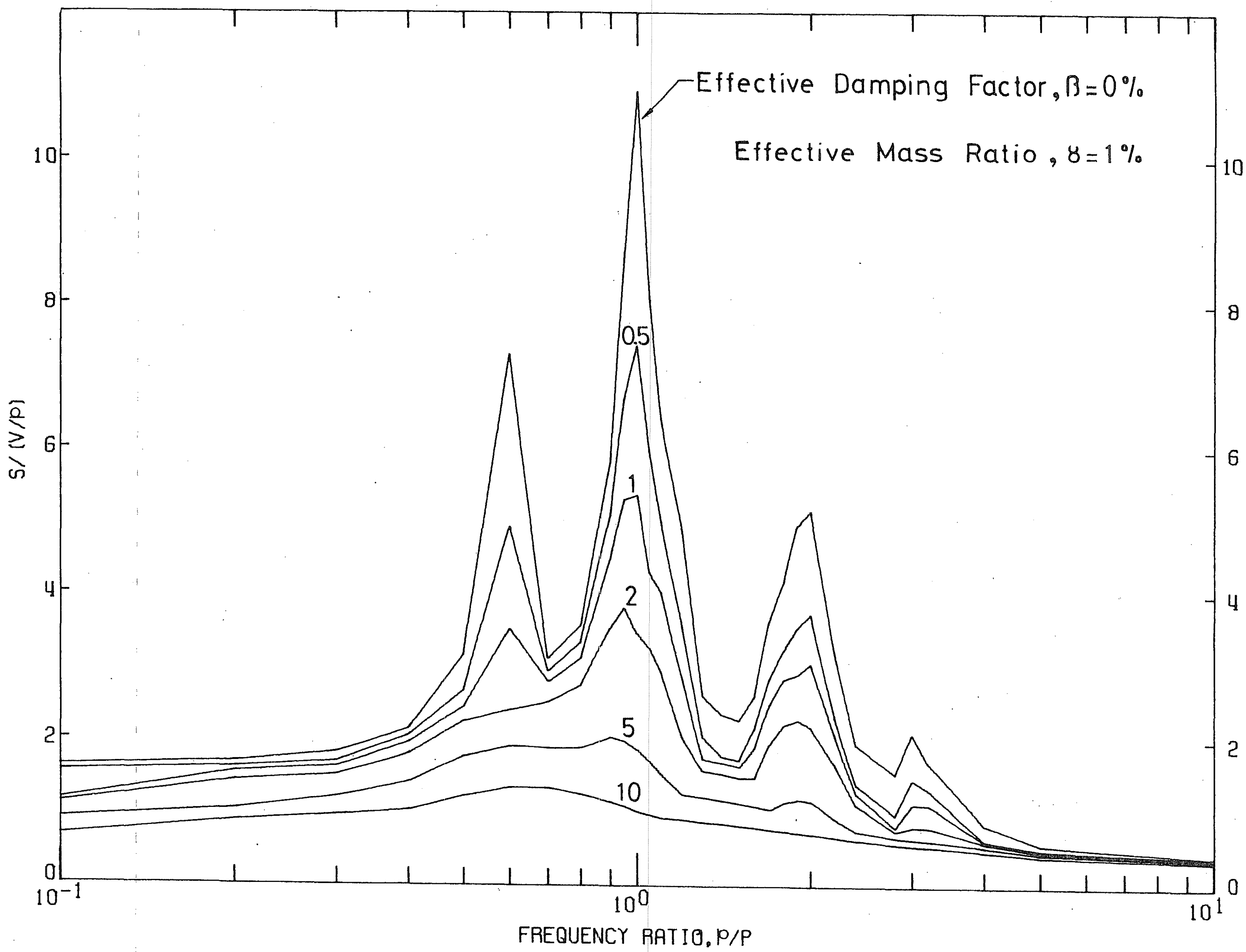


FIG. 4.17a OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

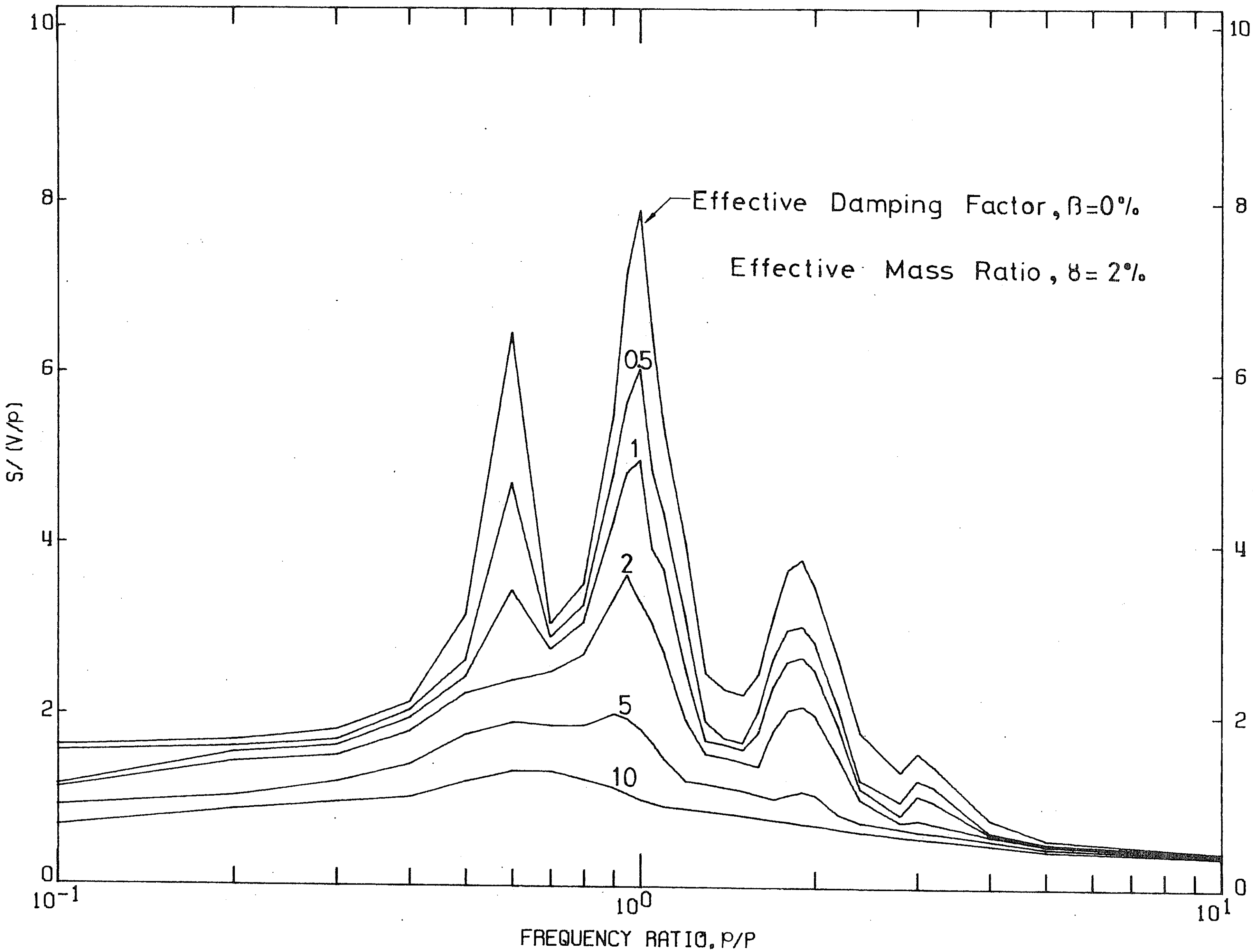


FIG. 4.17b OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

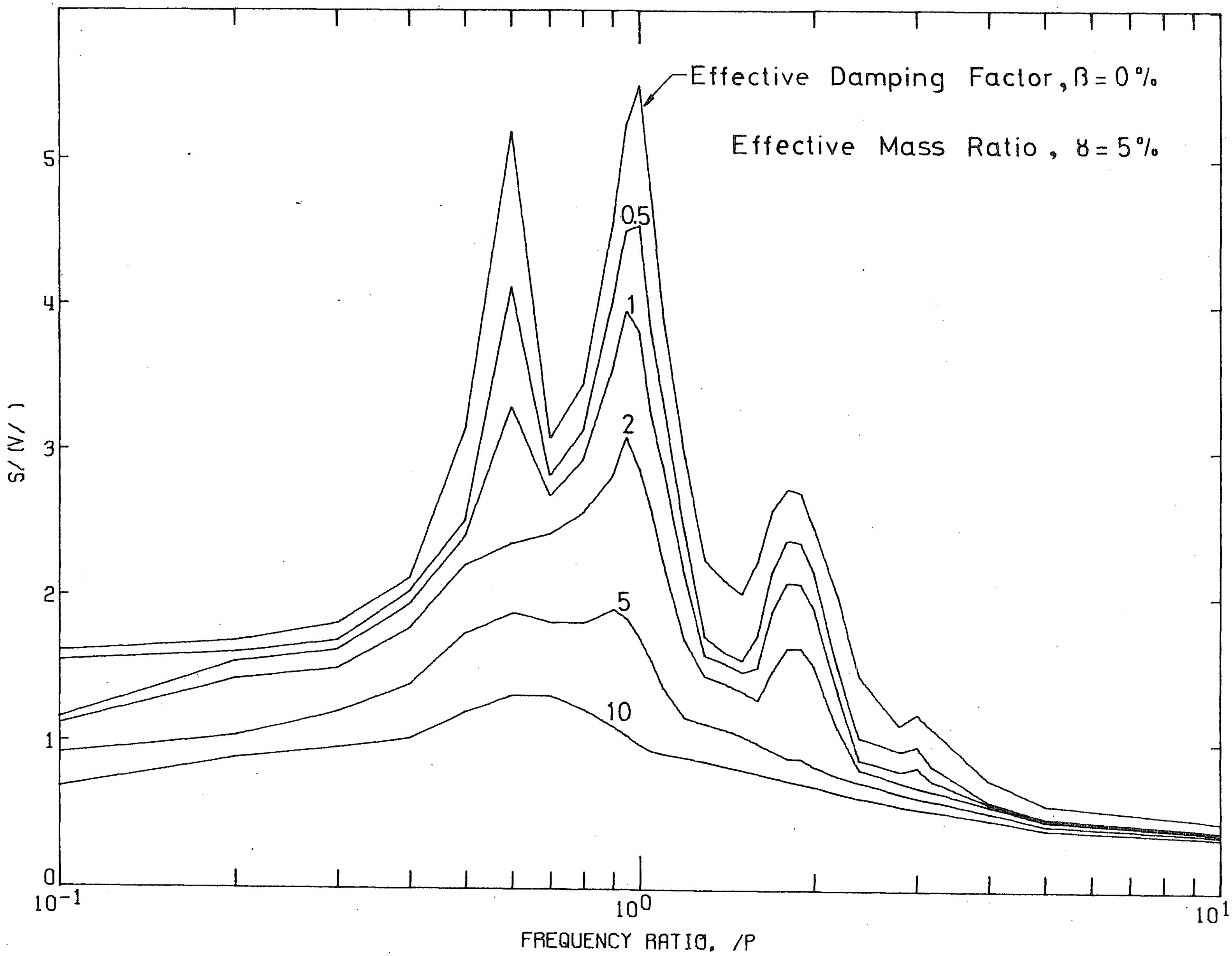


FIG. 4.17c OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

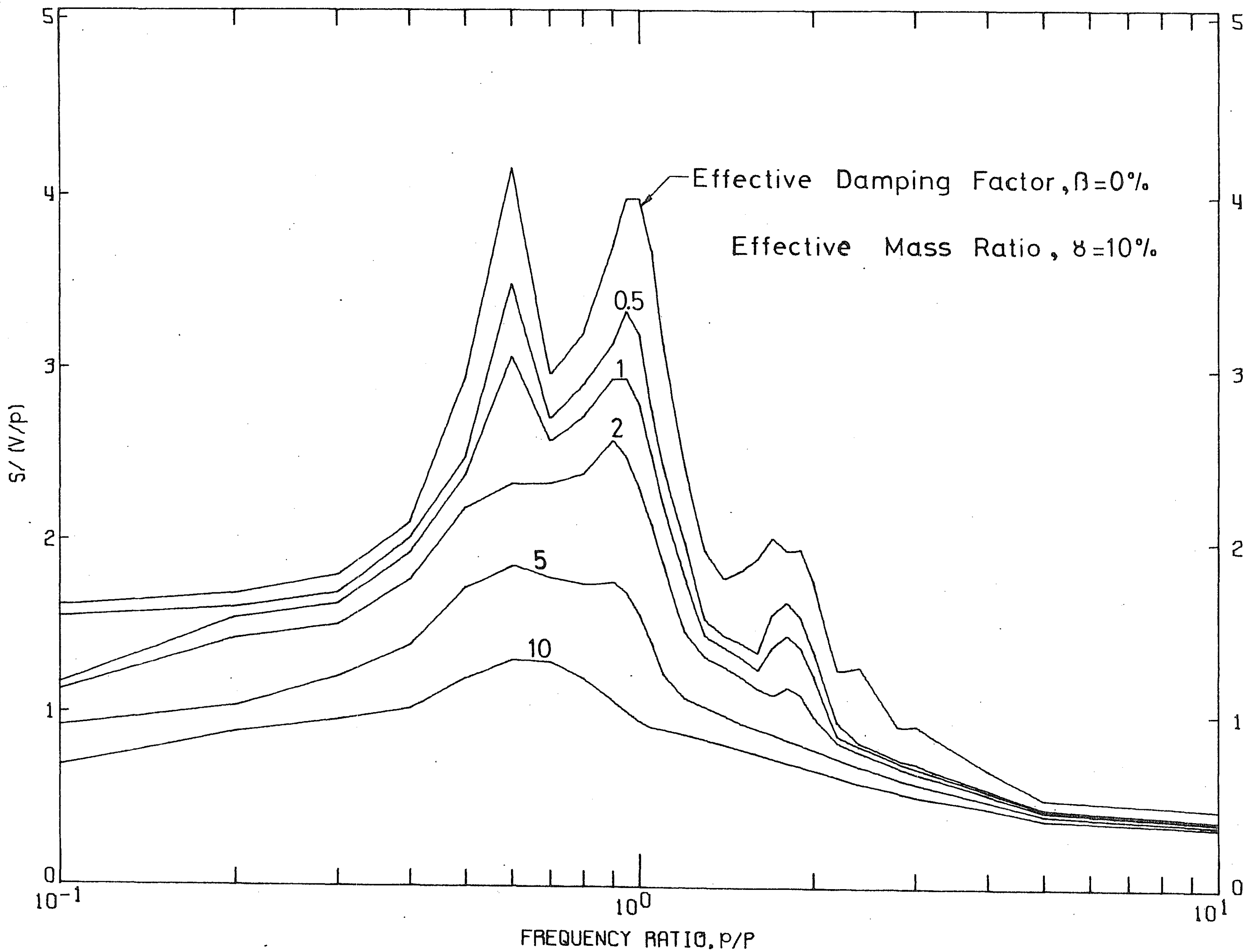


FIG. 4.17d OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

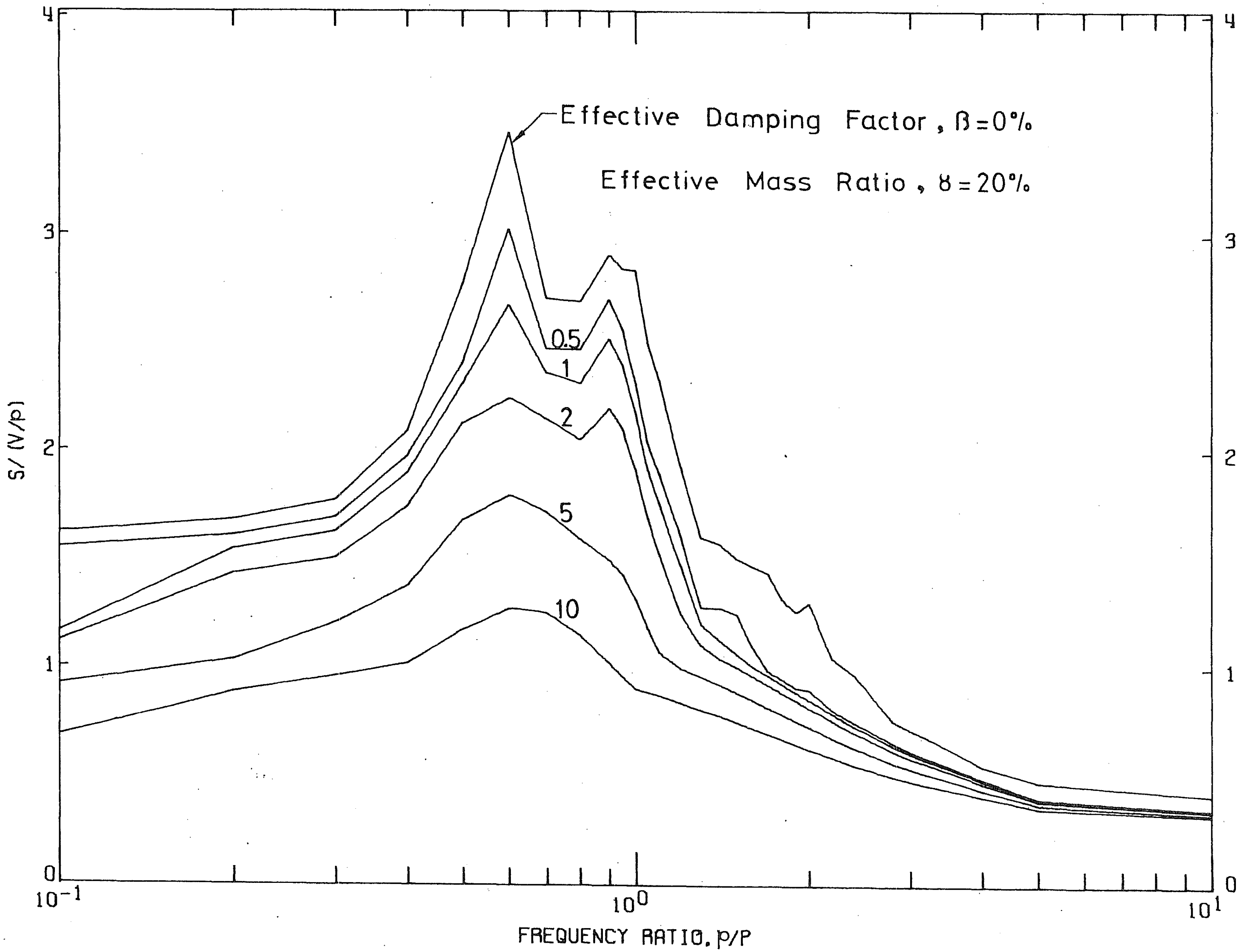


FIG. 4.17e OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2C, EFFECTIVE DAMPING FACTOR = 0, 0.5, 1, 2, 5 AND 10 %

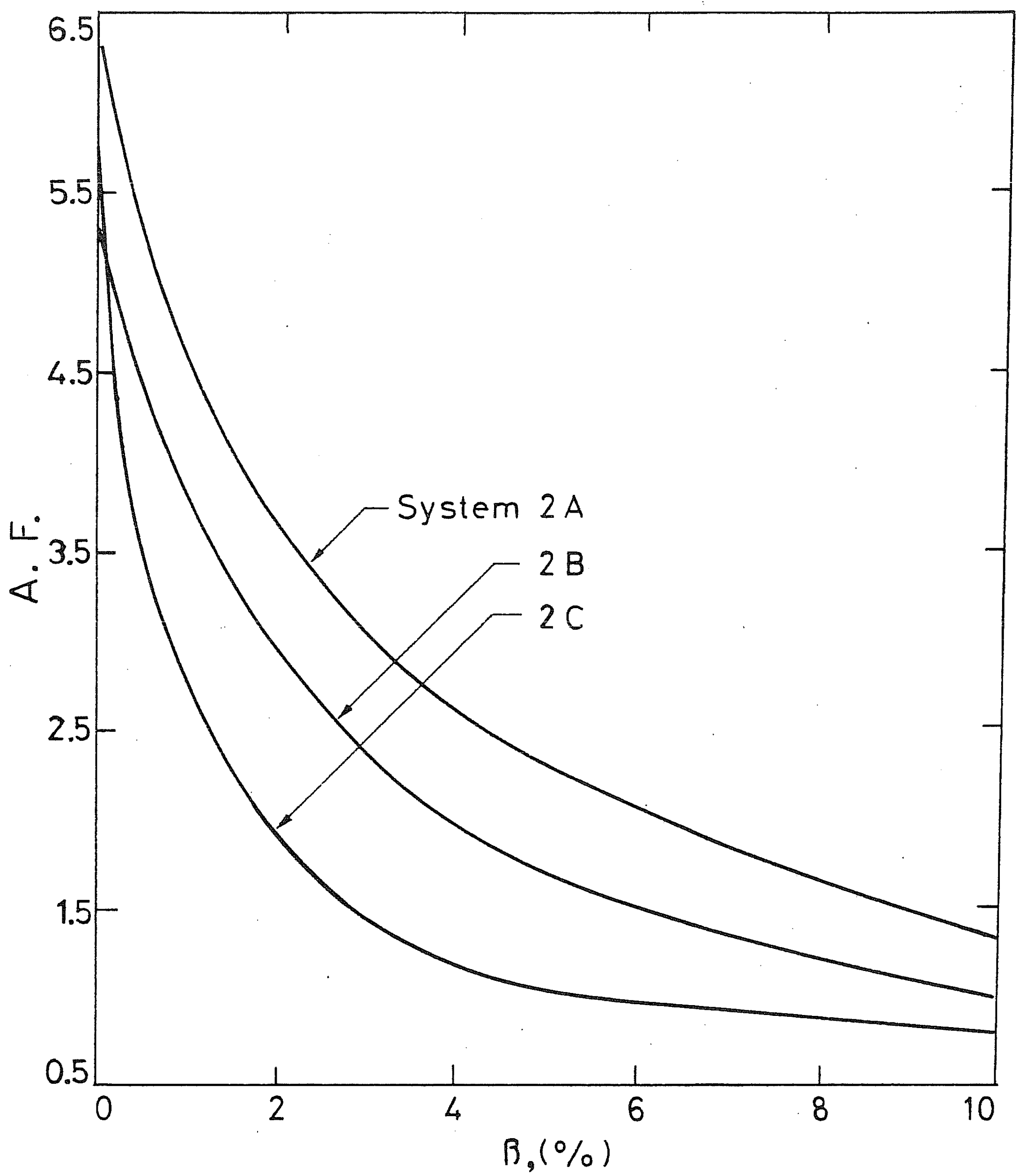


FIG. 4.18 RELATIONSHIPS BETWEEN A.F. AND ß, γ = 1 %, SYSTEMS 2A, 2B AND 2C, INNER SPRING

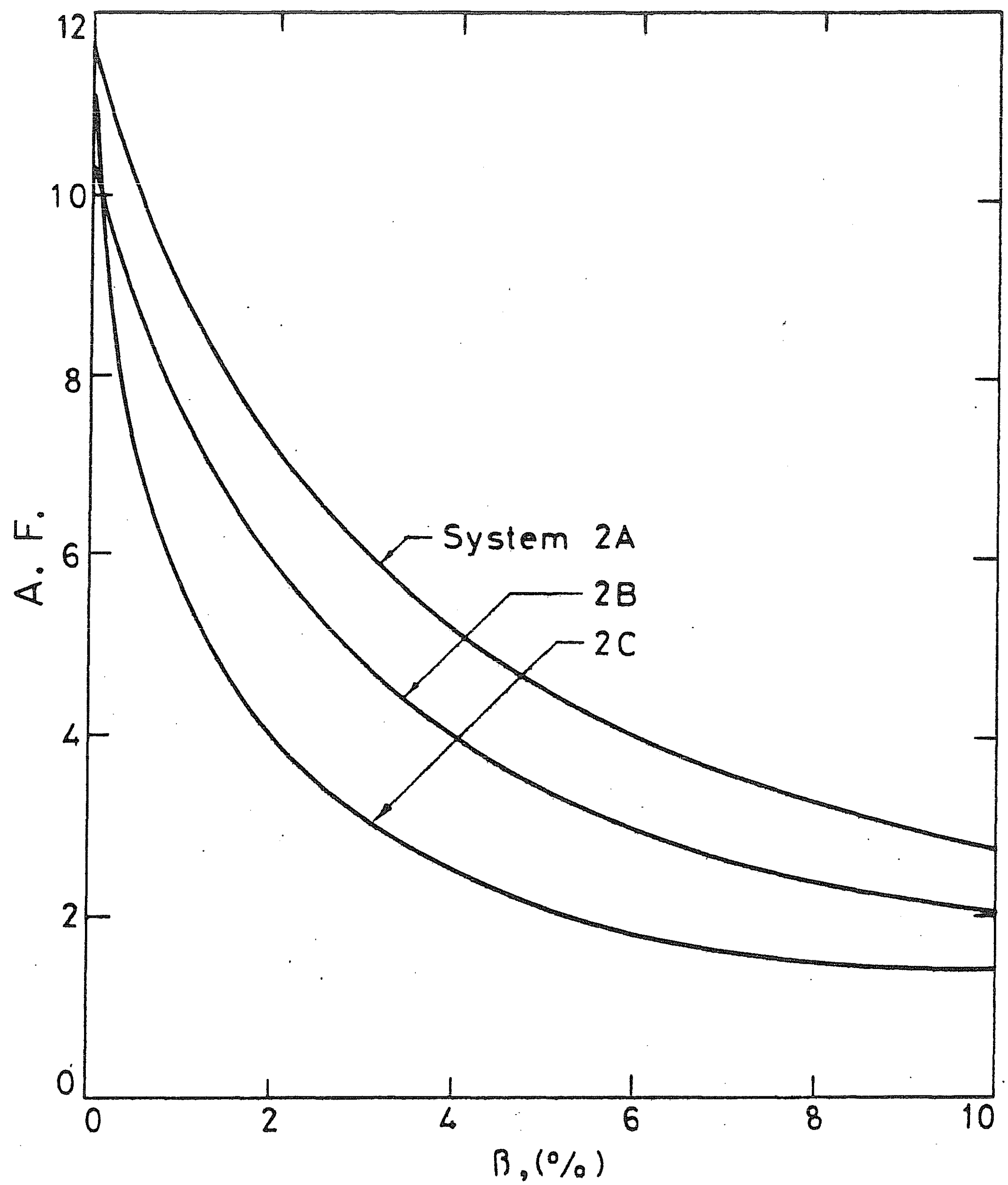


FIG. 4.19 RELATIONSHIPS BETWEEN A.F. AND β, γ = 1 %, SYSTEMS 2A, 2B AND 2C, OUTER SPRING

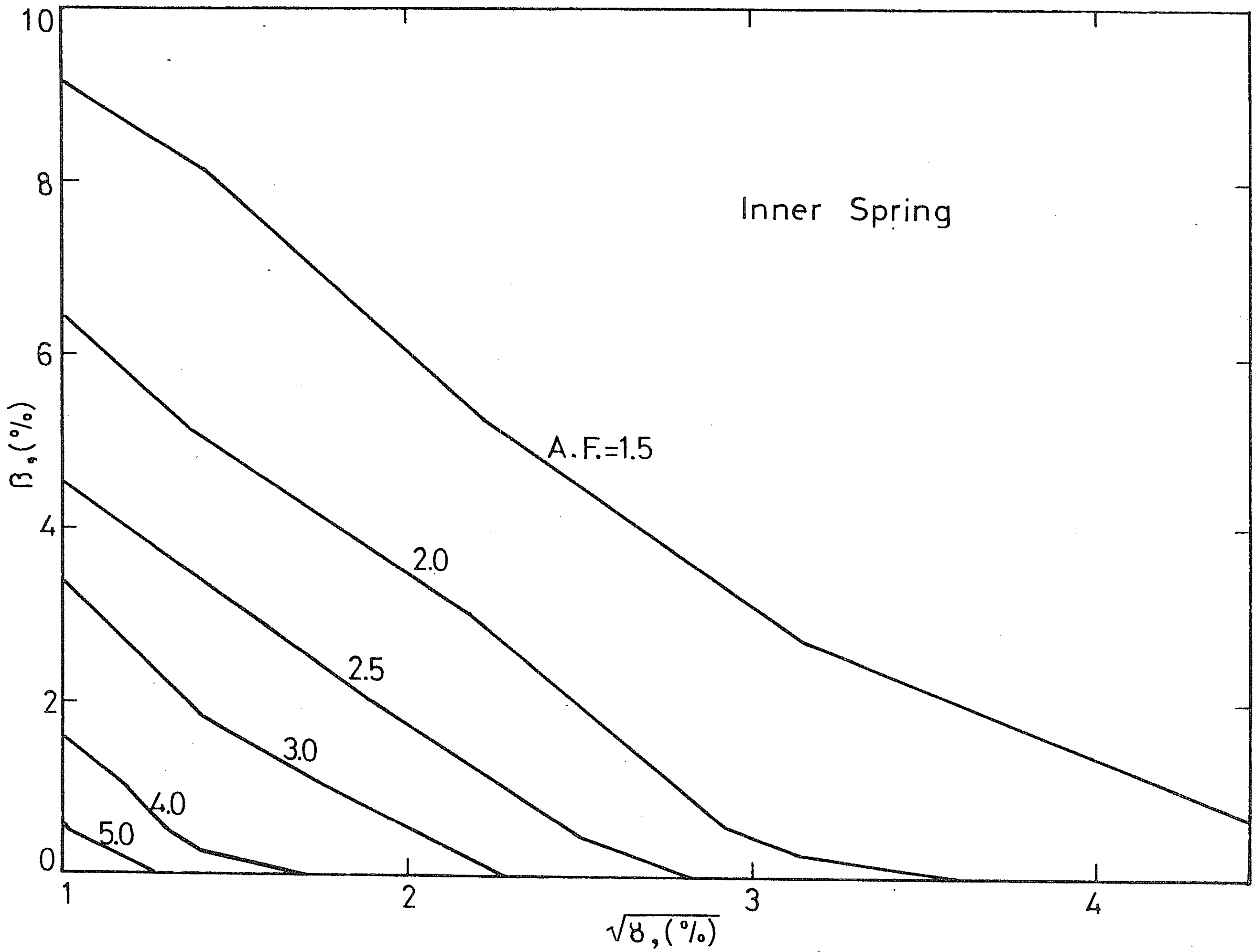


FIG. 4.20a AMPLIFICATION FACTOR ENVELOPES, SYSTEM 2A

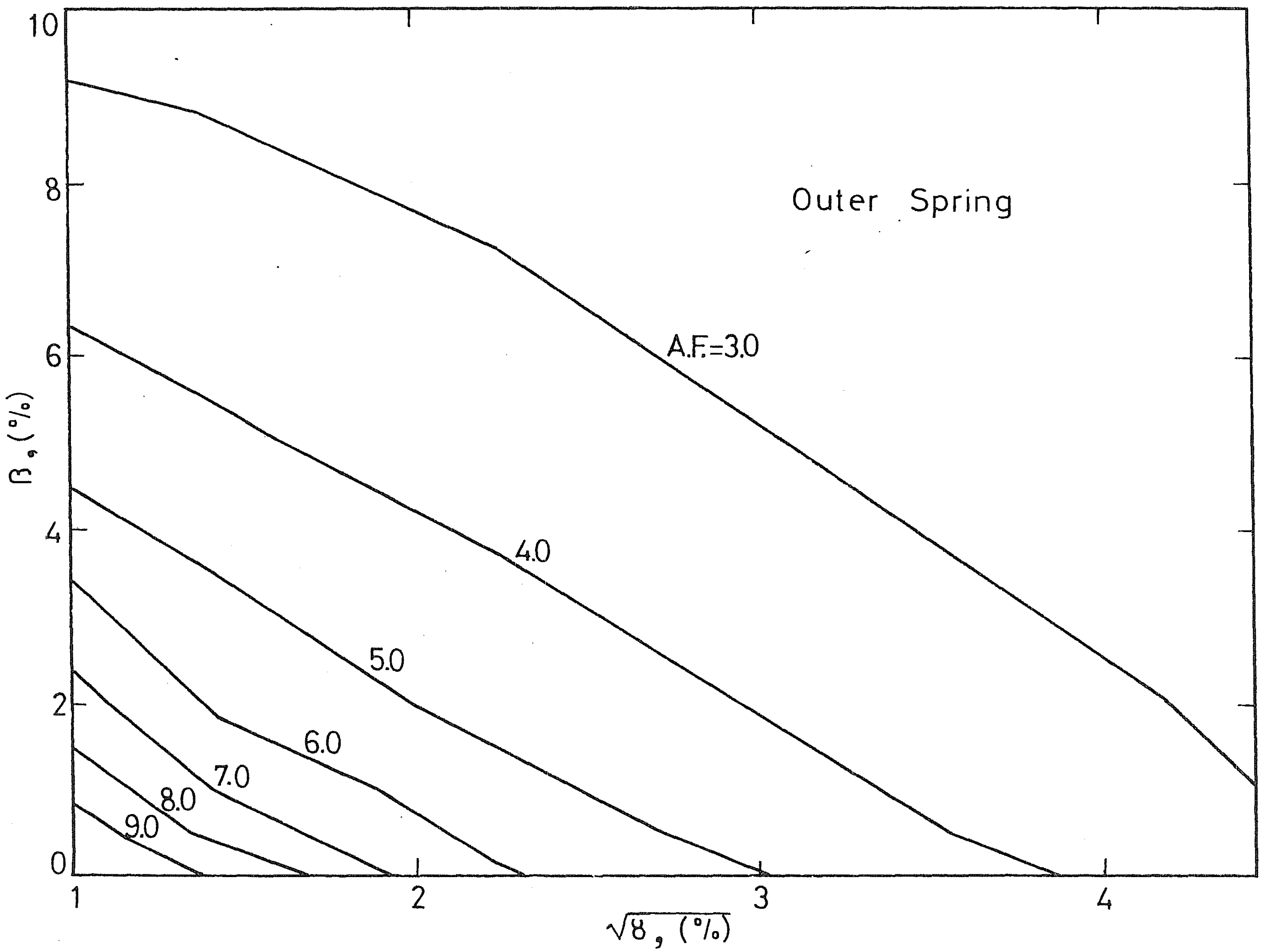


FIG. 4.20b AMPLIFICATION FACTOR ENVELOPES, SYSTEM 2A

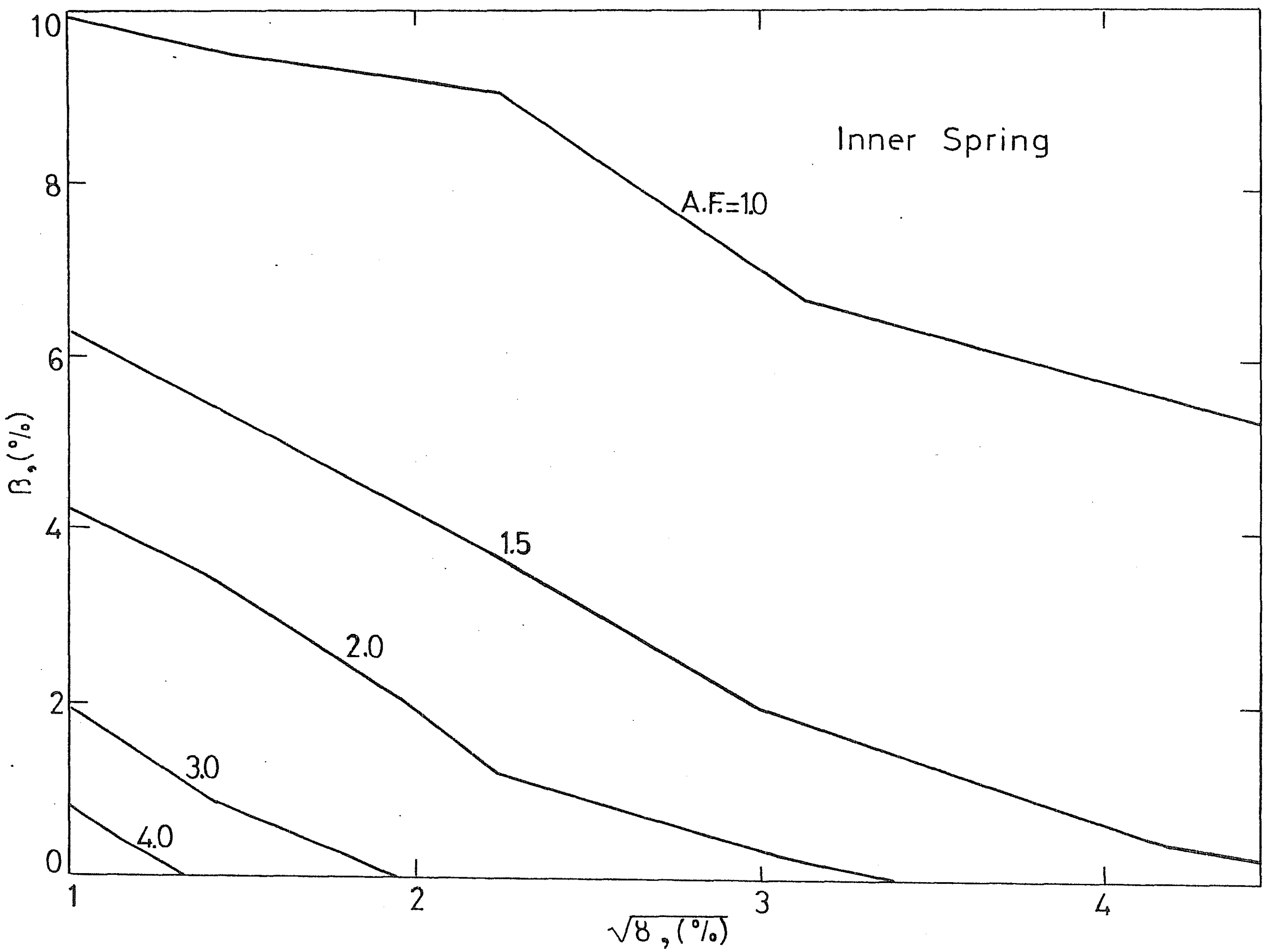


FIG. 4.21a AMPLIFICATION FACTOR ENVELOPES, SYSTEM 2B

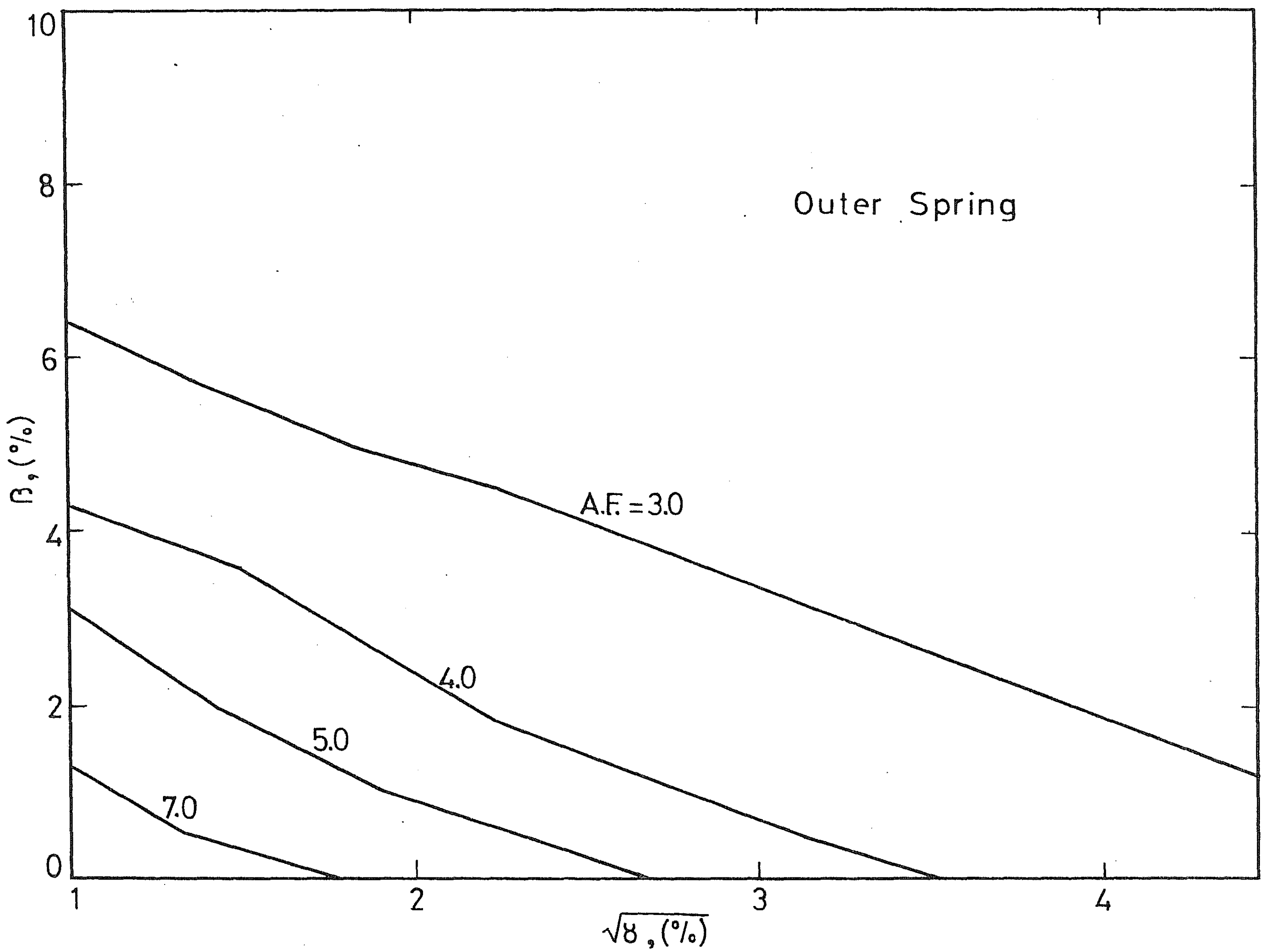


FIG. 4.21b AMPLIFICATION FACTOR ENVELOPES, SYSTEM 2B

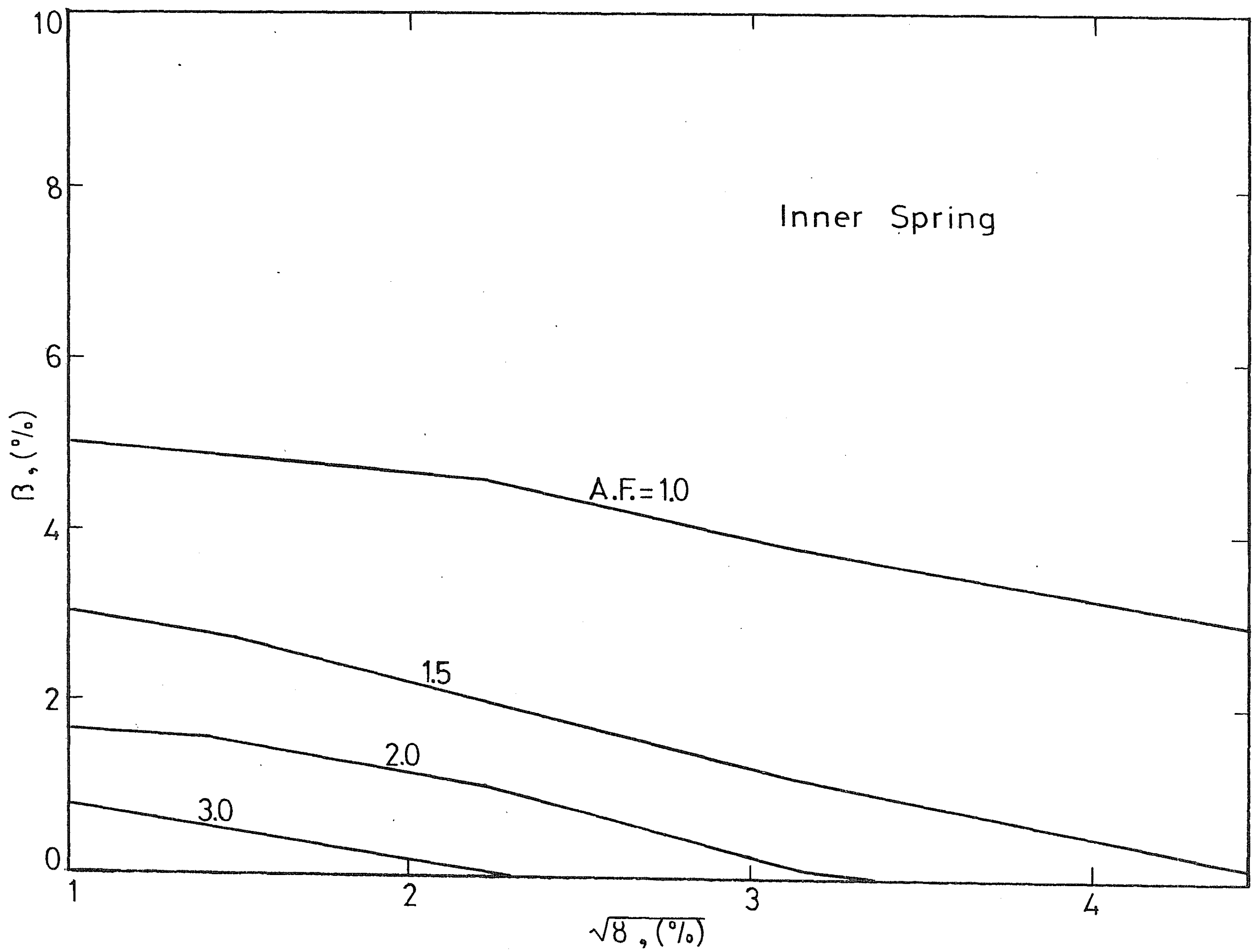


FIG. 4.22a AMPLIFICATION FACTOR ENVELOPES, SYSTEM 2C

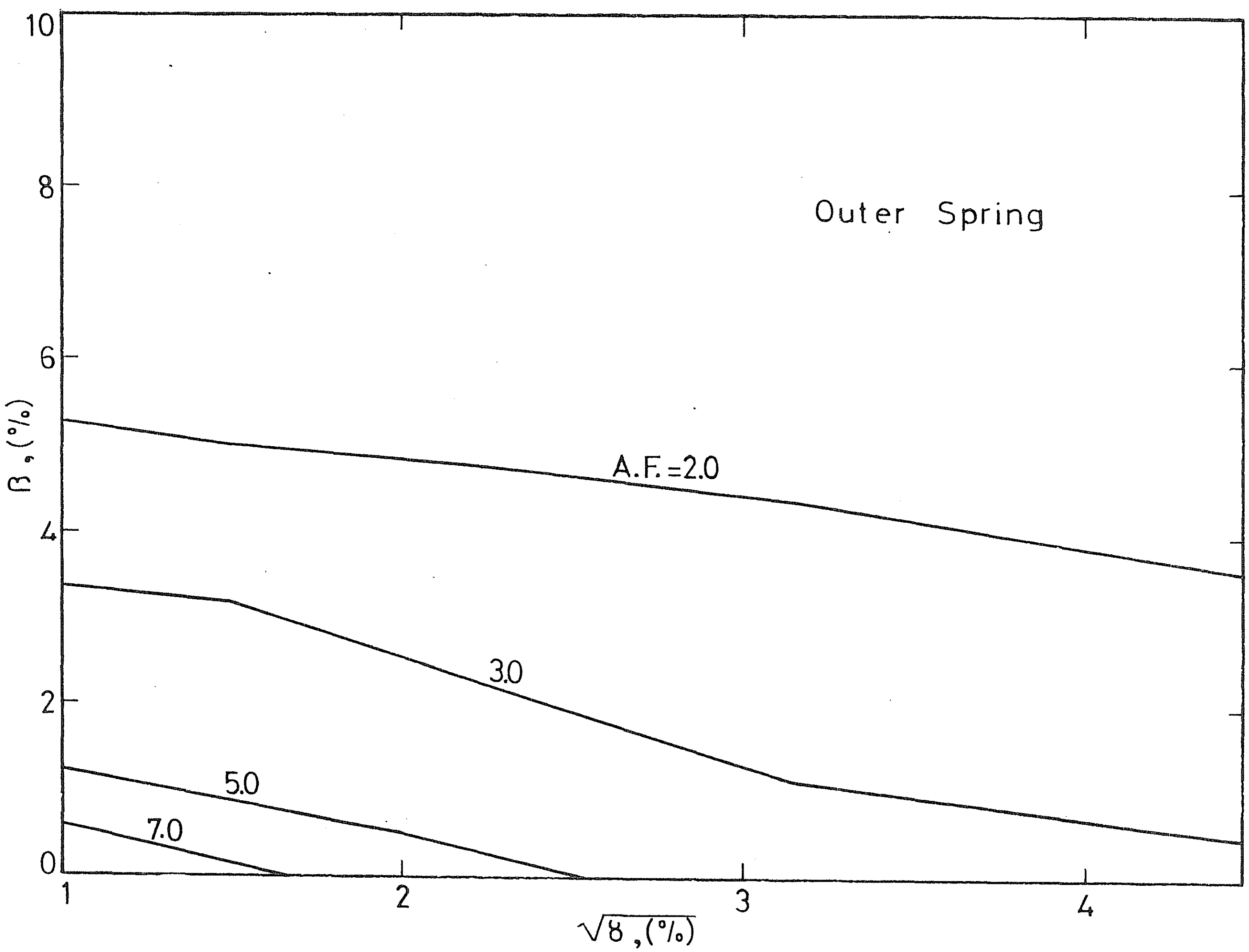


FIG. 4.22b AMPLIFICATION FACTOR ENVELOPES, SYSTEM 2C

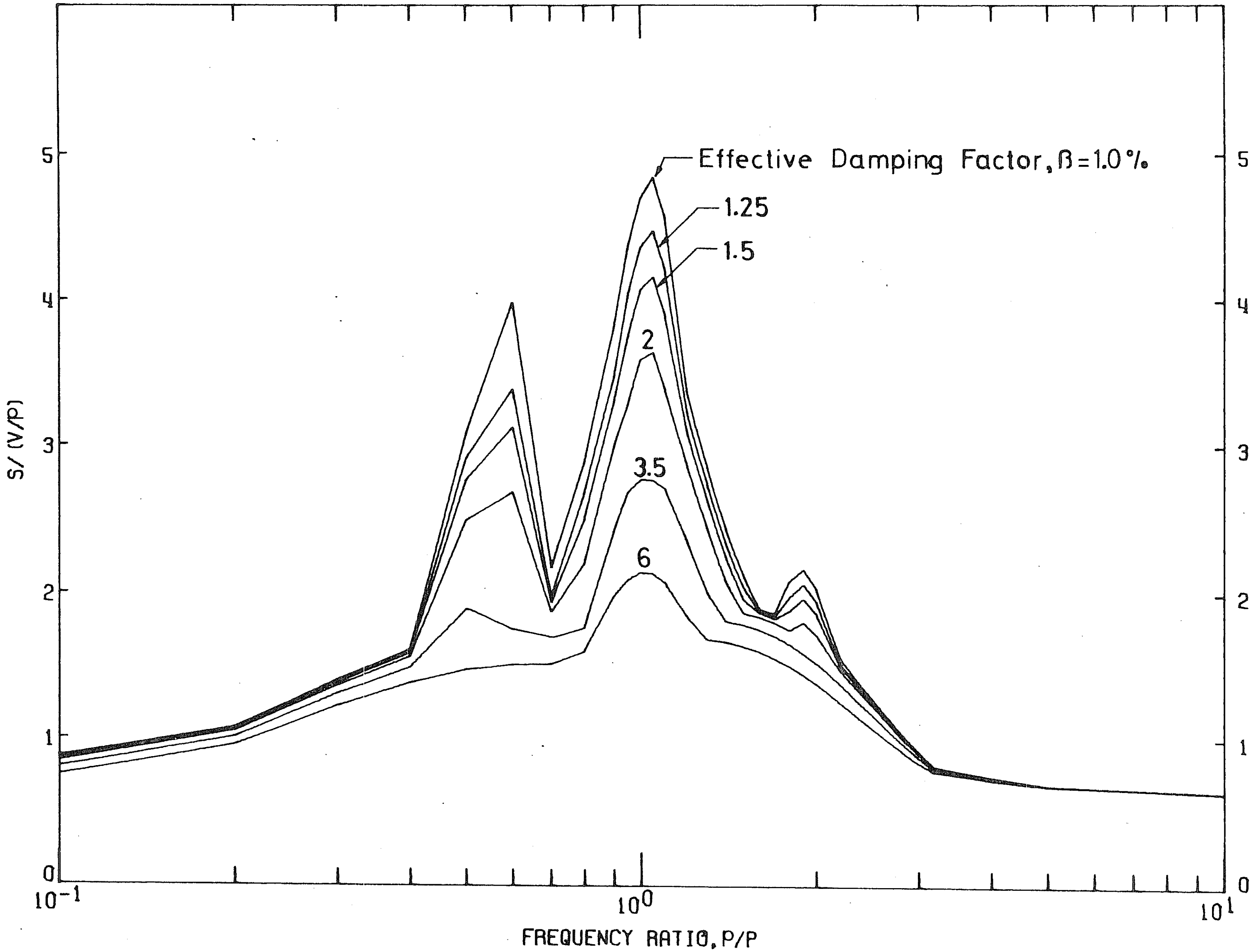


FIG. 4.23 INNER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1 %, β_p = 2 %, β_s = 0, 0.5, 1, 2, 5 AND 10 %

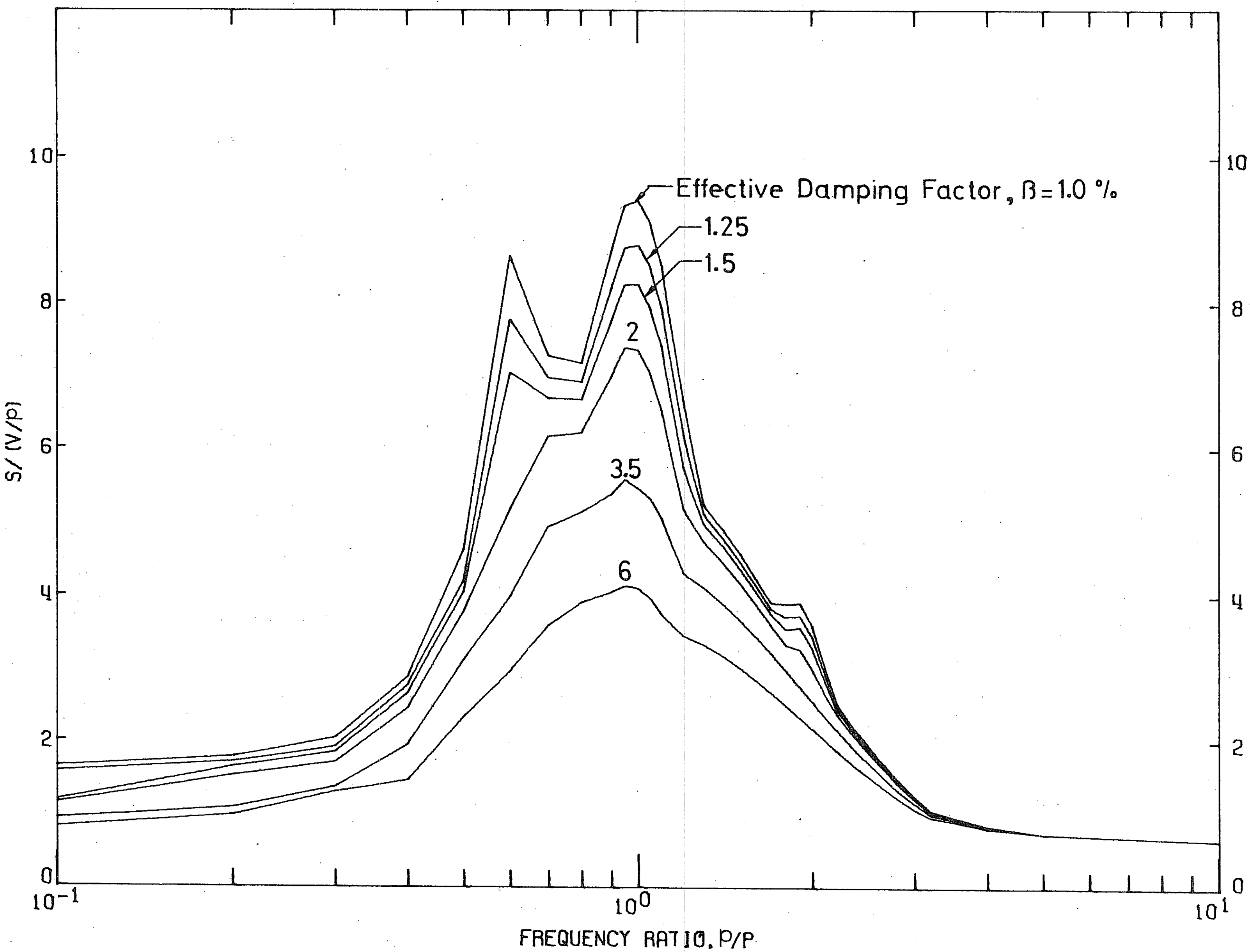


FIG. 4.24 OUTER SECONDARY SPRING DISTORTION BOUNDS, SYSTEM 2A, EFFECTIVE MASS RATIO = 1 %, β_p = 2 %, β_s = 0, 0.5, 1, 2, 5 AND 10 %

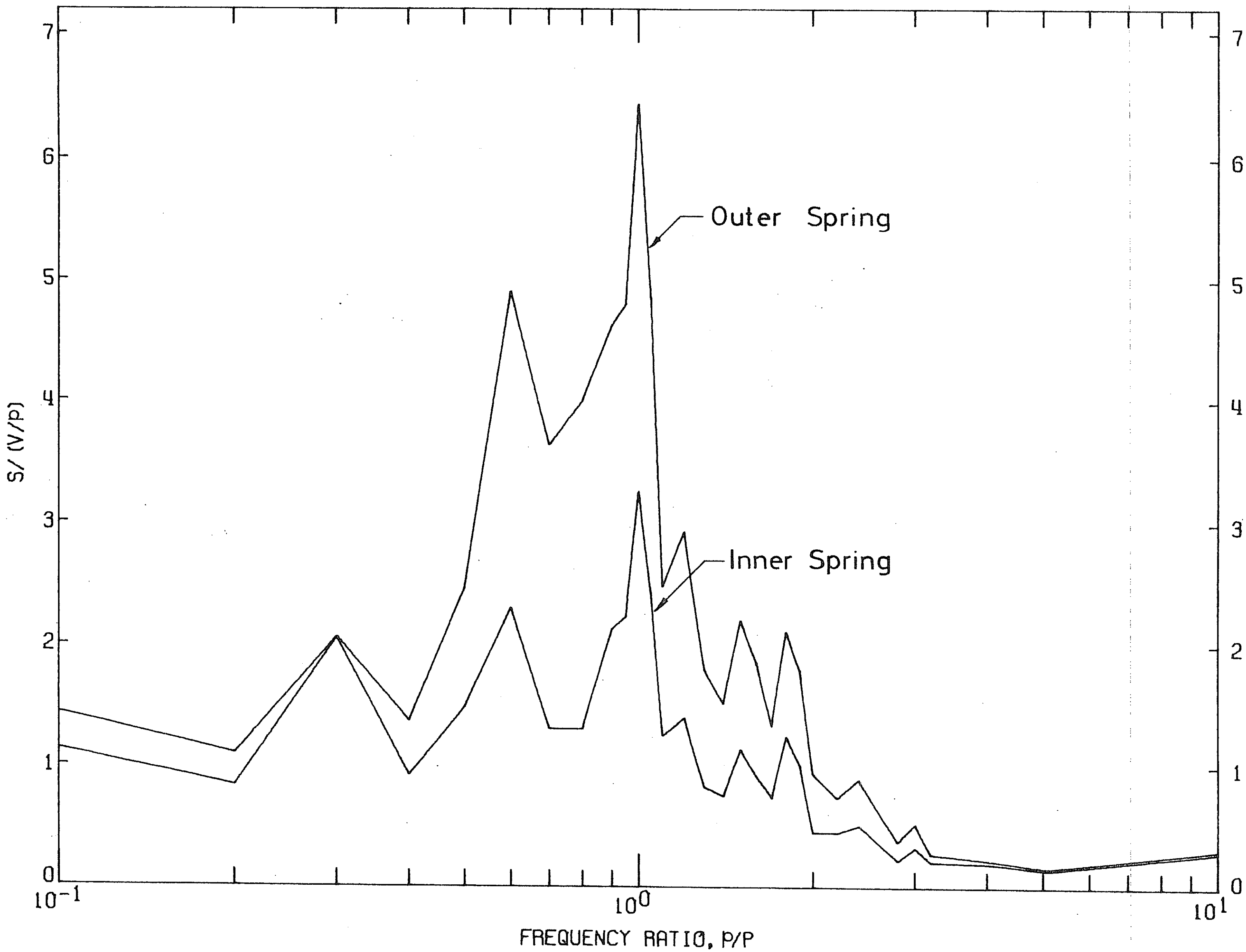


FIG. 4.25 SECONDARY SPRING DISTORTIONS, TAFT N21E, 1952, SYSTEM 2A, EFFECTIVE MASS RATIO = 1 %, EFFECTIVE DAMPING FACTOR = 2 %

APPENDIX A

DESIRED FREQUENCY AND EFFECTIVE MASS RATIO

For linearly elastic, positive definite systems treated in this study, when the relative values of the mass and spring constant had been selected, the eigen-value λ_r may be obtained as the ratios of two quadratic forms as noted in Eq. (A.1):

$$\lambda_r = \frac{U_r^T.K.U_r}{U_r^T.M.U_r} , \quad r = 1,2,----,n \tag{A.1}$$

where U_r are corresponding eigen-vectors.

When the mass factors (m, M) of the secondary and primary systems had been assigned specific values, the effective mass can be obtained from Eq. (2.33) or (2.34). The effective mass ratio can now be computed by Eq. (2.38). For example, if M = m = 1 kip-sec^2/inch, then Eq. (2.33) gives $E_{p1} = 4.5$ and Eq. (2.34) gives $E_{s1} = 3$ kip-sec^2/inch. Consequently, Eq. (2.38) yields $\gamma_{11} = 2/3$.

In order to have a desired effective mass ratio with respect to the fundamental frequency of both primary and secondary systems, m must be substituted by $\bar{m}$, which is obtained as follows:

$$(\gamma_{11})_{desired} = \frac{\alpha_1^T.\bar{M}_s.\alpha_1}{U_1^T.M_p.U_1} \tag{A.2}$$

From the fact that

$$\frac{\bar{m}}{m} = \frac{\alpha_1^T.\bar{M}_s.\alpha_1}{\alpha_1^T.M_s.\alpha_1} \tag{A.3}$$

Eq. (A.2) can be put into the form

$$(\gamma_{11})_{desired} = \frac{\bar{m}}{m} \frac{\alpha_1^T \cdot M_s \cdot \alpha_1}{U_1^T \cdot M_p \cdot U_1}$$

which yields

$$\bar{m} = \frac{3}{2}(\gamma_{11})_{desired} \tag{A.4}$$

Once the proper magnitude of the masses has been decided, one of the frequencies of the system can be adjusted to any desired value by using the spring constant factor obtained from Eq. (A.5):

$$\bar{k} = F_r k \tag{A.5}$$

where

$$F_r = \frac{(\omega_r)^2_{desired}}{\lambda_r} \frac{\bar{m}}{m} \tag{A.6}$$

$$\bar{k},\ k = \text{spring constant factor}$$

As an example, let the secondary masses be designated as 3m and m, and the spring constants as 6k and 1.5k. If the values of k = 1 kip/inch is employed, relation (A.1) yields λ_1 = 1 radian/sec. Then any value of of ω_{s1} may be obtained by using the spring constant factor $\bar{k}$ instead of k according to Eq. (A.7):

$$\bar{k} = (\omega^2_{s1})_{desired}\ \bar{m} \tag{A.7}$$

Mass and stiffness matrices of systems 2A, 2B and 2C for the case of $\omega_{p1} = \omega_{s1} = 2\pi$ radian/sec and γ_{11} = 1 percent are presented in Tables A.1 and A.2.

Table A.1 Mass Matrix of Systems 2A, 2B and 2C.

$$
\begin{bmatrix}
3.000 & & & & \\
& 1.500 & & & \\
& & 1.000 & & \\
& & & 0.045 & \\
& & & & 0.015
\end{bmatrix}
$$

Table A.2a Stiffness Matrix of System 2A.

$$
\begin{bmatrix}
592.175 & -236.87 & & & \\
-236.87 & 355.305 & -118.435 & & \\
& -118.435 & 121.988 & -3.553 & \\
& & -3.553 & 4.441 & -0.888 \\
& & & -0.888 & 0.888
\end{bmatrix}
$$

Table A.2b Stiffness Matrix of System 2B

$$
\begin{bmatrix}
592.175 & -236.87 & & & \\
-236.87 & 358.858 & -118.435 & -3.553 & \\
 & -118.435 & 118.435 & & \\
 & -3.553 & & 4.441 & -0.888 \\
 & & & -0.888 & 0.888
\end{bmatrix}
$$

Table A.2c Stiffness Matrix of System 2C

$$
\begin{bmatrix}
595.728 & -236.87 & & -3.553 & \\
-236.87 & 355.305 & -118.435 & & \\
 & -118.435 & 118.435 & & \\
-3.553 & & & 4.441 & -0.888 \\
 & & & -0.888 & 0.888
\end{bmatrix}
$$

Presented below are numerical calculations of some stiffness elements tabulated in Table A.2.

Element K_{11} of system 2C

$$K_{11} = 9K + 6K + 6\bar{k}$$

$$= (2\pi)^2(9\times1 + 6\times1 + 6\times\frac{3}{2}\times0.01)$$

$$= 595.728$$

Element K_{22} of system 2B

$$K_{22} = 6K + 3K + 6\bar{k}$$

$$= (2\pi)^2(6\times1 + 3\times1 + 6\times\frac{3}{2}\times0.01)$$

$$= 358.858$$

APPENDIX B

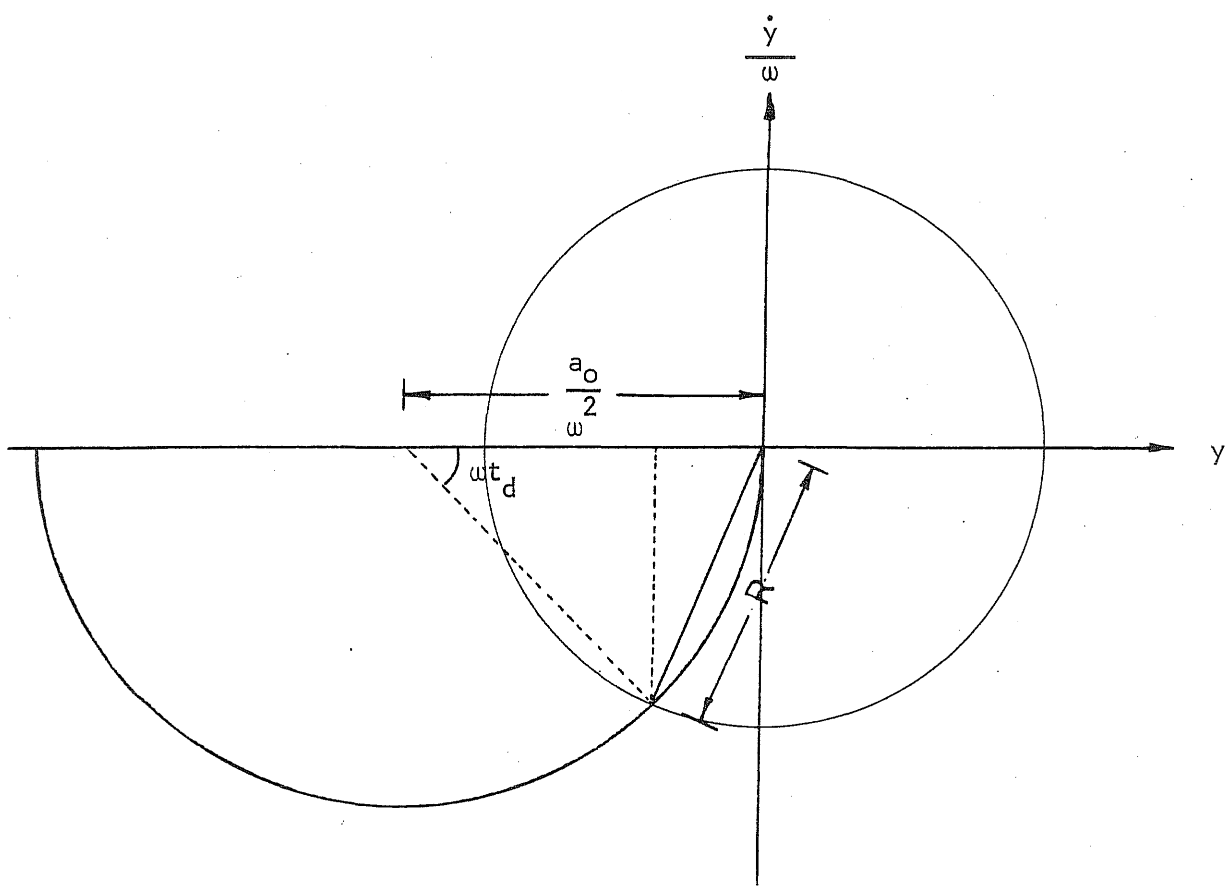
RESPONSE SPECTRUM OF THE CONSTANT ACCELERATION PULSE

The gyrogram may be used to evaluate the response spectrum of the step function accelerogram of Fig. 2.6a. The procedure employed in such a construction is described briefly below. Using the spectrum terminology,

$$V = \omega D$$

and

$$A = \omega^2 D$$



Gyrogram of the Constant Acceleration Pulse

Case $\omega t_d < \pi$

$$D = R$$

$$R^2 = \left\{ \frac{a_o}{\omega^2} \sin \omega t_d \right\}^2 + \left\{ \frac{a_o}{\omega^2} (1 - \cos \omega t_d) \right\}^2$$

$$= \left\{ \frac{a_o}{\omega^2} \right\}^2 \left\{ \sin^2 \omega t_d + 1 + \cos^2 \omega t_d - 2 \cos \omega t_d \right\}$$

$$D = R = \frac{a_o}{\omega^2} \sqrt{2(1 - \cos \omega t_d)} = \frac{2a_o}{\omega^2} \sin \frac{\omega t_d}{2}$$

$$V = \frac{2a_o}{\omega} \sin \frac{\omega t_d}{2} \tag{B.1}$$

For small ωt_d,

$$V = \frac{2a_o}{\omega} \cdot \frac{\omega t_d}{2} = a_o t_d \tag{B.2}$$

Case $\omega t_d \geq \pi$

$$D = R = \frac{2a_o}{\omega^2} \tag{B.3}$$

$$A = 2a_o$$

Equations (B.1), (B.2) and (B.3) completely describe the characteristics of the response spectrum shown in Fig. 2.6b. For very small ωt_d, Eq. (B.2) gives a constant velocity spectrum of equal magnitude to the constant base velocity. Equation (B.1) governs the transition zone, and for large ωt_d, Eq. (B.3) gives a constant acceleration spectrum of twice the magnitude of the input pulse.