

Hitchin's Projectively Flat Connection and the Moduli Space of Higgs Bundles

John Benjamin McCarthy

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Abstract

In this thesis we investigate the geometric quantization of moduli spaces of vector bundles over compact Riemann surfaces. In particular we will recall the geometric quantization of the moduli space of stable holomorphic vector bundles carried out by Hitchin, and study the generalisation of this problem for the moduli space of stable holomorphic Higgs bundles. The geometric quantization of Higgs moduli spaces presents new difficulties, since these moduli spaces are non-compact. However, they come with natural \mathbb{C}^* actions, and this has implications for the geometric quantization: the quantum spaces for the Higgs moduli spaces split into finite-dimensional weight spaces for the \mathbb{C}^* action, which can be identified with spaces of sections of certain bundles over the compact stable bundle moduli space.

In the first part of this thesis, we review necessary background in differential geometry. Chapter 1 reviews the standard theory of connections on smooth vector bundles. Chapter 2 serves as an introduction to symplectic geometry, symplectic quotients, and their relationship to geometric invariant theory. Chapter 3 reviews the fundamental ideas in complex differential geometry, and in particular in Kähler geometry, as well as the basic theory of holomorphic vector bundles required later.

In the second part of this thesis, we introduce the moduli spaces of stable bundles and Higgs bundles, formally defining them as infinite-dimensional Kähler quotients. In Chapter 4 the stable bundle space is considered, and we review its important properties and interpretations. Chapter 5 concerns the Higgs bundle moduli space and the ways it generalises and compares to the stable bundle moduli space.

In the third and final part of this thesis, we recall the process of geometric quantization via Kähler polarisations, and apply it to the moduli space of Higgs bundles. In Chapter 6 we define geometric quantization, and state a theorem of Andersen on the existence of Hitchin connections for compact symplectic manifolds. In Chapter 7 we geometrically quantize the Higgs line bundle moduli space, and investigate the difficulties of generalising the techniques used to the higher rank spaces. In particular we construct a projectively flat Hitchin connection on the bundle of quantum spaces over Teichmüller space. As far as the author is aware, this is an original result.

Signed Statement

I certify that this work contains no material which has been accepted for the award of any other degree or diploma in my name in any university or other tertiary institution and, to the best of my knowledge and belief, contains no material previously published or written by another person, except where due reference has been made in the text. In addition, I certify that no part of this work will, in the future, be used in a submission in my name for any other degree or diploma in any university or other tertiary institution without the prior approval of the University of Adelaide and where applicable, any partner institution responsible for the joint award of this degree.

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Signed: Date:

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Part I

Geometry and Vector Bundles

Chapter 1

Connections on Vector Bundles

In this chapter we will recall the definition of a connection on a smooth vector bundle, its curvature, and some of its basic properties. Connections on vector bundles are the fundamental objects of study in gauge theory, and more broadly in many parts of differential geometry and geometric analysis. Connections, their curvatures, and their parallel transports will appear many times in this thesis. In Chapter 3 we will see the relation between connections and holomorphic structures on vector bundles. In Chapters 4 and 5 we will use these identifications to investigate moduli spaces of bundles over compact Riemann surfaces from a gauge-theoretic point of view, in the manner of Atiyah and Bott in [AB83], Donaldson in [Don83], and Hitchin in [Hit87]. In Chapters 6 and 7 we will be interested in constructing connections on certain vector bundles over Teichmüller space, where their parallel transport will provide a way of geometrically quantizing symplectic manifolds independently of a choice of Kähler polarisation.

The material of this chapter is standard, and more details can be found in any good reference about differential geometry or gauge theory. For a treatment of this material with a view towards complex geometry, as in this thesis, the book [WGP80] of Wells on complex differential geometry and the book [Kob87] of Kobayashi on complex and holomorphic vector bundles are recommended. For a more gauge theory-oriented treatment the text [DK90] of Donaldson and Kronheimer is more suitable. Indeed this book also includes discussion of many of the subtleties of moduli problems when viewed from a gauge-theoretic perspective.

1.1 Connections

Let $\pi : E \rightarrow M$ be a smooth (real or complex) vector bundle over a smooth manifold M . In our case we will always take E to be complex, unless otherwise stated. A section of E is a smooth map $s : M \rightarrow E$ such that $s(x) \in \pi^{-1}(x)$

for every $x \in M$. The set $\pi^{-1}(x)$ is called the fibre of E over x and is denoted E_x . Denote the vector space of smooth sections of E by $\Gamma(M, E)$. We will denote sections of $\bigwedge^k T^*M \otimes E$ by $\Omega^k(M, E)$, and in particular we will interchangeably use $\Omega^0(M, E)$ for $\Gamma(M, E)$. Since $\Omega^k(M, E) = \Omega^k(M) \otimes \Gamma(M, E)$, sections of $\Omega^k(M, E)$ are referred to as E -valued k -forms.

Given a section of a vector bundle, it is a natural question to ask how to differentiate this section. There is in general no canonical choice of how to differentiate sections of a vector bundle. Such a choice is called a *connection*.

Definition 1.1.1. A connection on a vector bundle $E \rightarrow M$ is a \mathbb{C} -linear map

$$\nabla : \Gamma(M, E) \rightarrow \Omega^1(M, E)$$

satisfying

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(M, E)$.

A connection does not define a section of the bundle $\text{Hom}(E, T^*M \otimes E)$, but we see that if ∇_1 and ∇_2 are connections on a bundle E , then $(\nabla_2 - \nabla_1)(fs) = f(\nabla_2 - \nabla_1)(s)$. A map on sections that is $C^\infty(M)$ -linear defines a unique bundle map $E \rightarrow T^*M \otimes E$, which may then be identified with a unique section of $\text{Hom}(E, T^*M \otimes E)$. Thus the difference of two connections is a section of $\text{Hom}(E, T^*M \otimes E)$.

On a trivial vector bundle $E = M \times \mathbb{C}^n$ there is a canonical choice of connection. Let $\{e_i\}$ denote the standard basis of \mathbb{C}^n , and also the corresponding standard frame of $M \times \mathbb{C}^n$. Then every section s looks like $s = s^i e_i$ for some smooth functions $s^i \in C^\infty(M)$. One may define the *trivial connection* $\nabla =: d$ by writing $ds := (ds^i) \otimes e_i$.

Proposition 1.1.2. Every vector bundle $E \rightarrow M$ admits a connection.

Proof. Let $\{U_\alpha\}$ be a trivialising open cover for E and let $\{\rho_\alpha\}$ be a partition of unity subordinate to this cover. Then on each $E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^n$ one has the trivial connection d_α . Define ∇ by

$$\nabla := \sum_{\alpha} \rho_\alpha d_\alpha.$$

Then it is straightforward to check ∇ satisfies the Leibniz rule because each d_α does. \square

Thus the space $\mathcal{A}(E)$ of connections is non-empty, and since the difference of two connections is an element of $\Gamma(M, \text{Hom}(E, T^*M \otimes E)) = \Omega^1(M, \text{End}(E))$, we have that $\mathcal{A}(E)$ is an affine space modelled on $\Omega^1(M, \text{End}(E))$.

Suppose we are on a trivialising open set U for a vector bundle E of rank k . Then since $E|_U \cong U \times \mathbb{C}^k$ is trivial, we have the trivial connection d defined

above. Given any other connection ∇ on E , it restricts to a connection on $E|_U$, and by the previous discussion the difference $\nabla - d$ lies in $\Omega^1(U, \text{End}(E|_U))$. But $E|_U \cong U \times \mathbb{C}^k$, so that $\text{End}(E|_U) \cong U \times \text{Mat}(k, \mathbb{C})$. Thus if we denote $A := \nabla - d$, then in terms of the local frame on U , A is a matrix of one-forms, called the *connection form of ∇ on U* .

If we interpret a local section $s \in \Gamma(U, E|_U)$ as a column vector with entries $s^i \in C^\infty(U)$ as determined by writing $s = s^i e_i$ for the local frame $\{e_i\}$, then $\nabla s = ds + As$. Here ds is the exterior derivative applied term-wise, and As is matrix multiplication, and we often write $\nabla = d + A$. Notice in particular that applying ∇ to the local frame, we have $\nabla e_i = A_i^j \otimes e_j$, where the one-form A_i^j is the (i, j) th component of the matrix A .

If U_α and U_β are a pair of trivialising open sets for E , then on $U_{\alpha\beta} := U_\alpha \cap U_\beta$ one can compare the local connection forms A_α and A_β . If $g_{\alpha\beta}$ is the transition function on $U_{\alpha\beta}$, then we have the compatibility condition

$$A_\beta = g_{\beta\alpha} A_\alpha g_{\beta\alpha}^{-1} + dg_{\beta\alpha} g_{\beta\alpha}^{-1}. \quad (\text{Eq. 1.1})$$

On the other hand, a system of matrix-valued one-forms $\{A_\alpha\}$ satisfying (Eq. 1.1) for a trivialisation $\{U_\alpha\}$ of a vector bundle E will define a connection ∇ , given locally by $\nabla = d + A_\alpha$.

We end with the definition of the covariant derivative – a directional derivative of sections. Given a connection ∇ and a vector field $X \in \Gamma(M, TM)$, we may contract ∇s with X for any $s \in \Gamma(M, E)$ to obtain another section.

Definition 1.1.3. *The covariant derivative of s in the direction of X is the contraction*

$$\nabla_X s \in \Gamma(M, E)$$

of ∇s with X .

On a coordinate chart $(U, (x^1, \dots, x^n))$ of the manifold M , define operators ∇_i on sections of E by

$$\nabla_i s := \nabla_{\frac{\partial}{\partial x^i}} s.$$

Note that the connection ∇ on U may be recovered by the operators ∇_i by the expression

$$\nabla s = dx^i \otimes \nabla_i s.$$

1.2 Curvature

Given a connection $\nabla : \Omega^0(M, E) \rightarrow \Omega^1(M, E)$, one may define an *exterior covariant derivative* $d_\nabla : \Omega^k(M, E) \rightarrow \Omega^{k+1}(M, E)$ by

$$d_\nabla(\omega \otimes s) := d\omega \otimes s + (-1)^k \omega \wedge \nabla s$$

on simple tensors, where $\omega \in \Omega^k(M)$ and $s \in \Gamma(M, E)$, and extending by linearity. Unlike for the exterior derivative, $d_{\nabla}^2 \neq 0$ in general, and the obstruction is called the *curvature*.

Definition 1.2.1. *The curvature of a connection ∇ on a vector bundle E is the map*

$$d_{\nabla} \circ \nabla : \Omega^0(M, E) \rightarrow \Omega^2(M, E).$$

It is straightforward to check that $d_{\nabla} \circ \nabla(fs) = fd_{\nabla} \circ \nabla(s)$, and so the curvature is actually an element of $\Omega^2(M, \text{End}(E))$, which is denoted F_{∇} . In a local trivialisation where $\nabla = d + A$, we have

$$d_{\nabla} \circ \nabla(e_i) = d_{\nabla}(A_i^j \otimes e_j) = dA_i^j \otimes e_j - A_i^j \wedge A_j^k \otimes e_k = (dA_i^j + A_k^j \wedge A_i^k) \otimes e_j.$$

Therefore if $A \wedge B$ denotes matrix multiplication with wedging of coefficients, the local form representing F_{∇} is given by $dA + A \wedge A$. A sanity check verifies that these local forms satisfy the conjugation condition to piece together on overlaps, with no extra terms dependent on transition functions, as was the case in (Eq. 1.1).

Since the curvature is a 2-form with values in $\text{End}(E)$, we may contract it with two vector fields X and Y . This provides the invariant formula

$$F_{\nabla}(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]}$$

for the curvature in terms of the covariant derivative. Notice that if X and Y are coordinate vector fields then the Lie bracket vanishes, and the expression reduces to the commutator

$$F_{\nabla}(\partial_i, \partial_j) = \nabla_i \nabla_j - \nabla_j \nabla_i.$$

In the study of connections, one is often concerned with picking a “best” connection in some sense. Viewing curvature as the obstruction to $d_{\nabla}^2 = 0$, the best connection would be one where the curvature vanishes.

Definition 1.2.2. *A connection is called flat if $F_{\nabla} = 0$.*

We will see in our discussion of characteristic classes that there are topological obstructions to the existence of flat connections. A substitute for flatness in cases where it is not possible would be for the curvature to be as small as possible. Precisely what is meant by this will be expanded upon in Chapter 4 when we investigate Yang-Mills theory in the context of stable bundles on a compact Riemann surface.

In addition to flatness, there is the condition of *projective flatness*. The bundle $\text{End}(E)$ always has a canonical non-vanishing section $\mathbf{1}$ corresponding to the identity endomorphism on E . Given any two-form $\omega \in \Omega^2(M)$, we see that $\omega \otimes \mathbf{1} \in \Omega^2(M, \text{End}(E))$.

Definition 1.2.3. A connection is called projectively flat if $F_\nabla = \omega \otimes \mathbf{1}$ for some two-form ω on M .

Note that upon passing to $\text{End}(E) \cong E^* \otimes E$, a projectively flat connection ∇ on E induces, in the sense of the following section, a genuinely flat connection on $\text{End}(E)$. The terminology “projectively flat” comes from the fact that the induced connection on the associated principal $\text{PGL}(n, \mathbb{C})$ -bundle over M is also flat, and indeed the flatness of this induced connection is often taken as the definition of projective flatness. The two notions are equivalent whenever the principal $\text{PGL}(n, \mathbb{C})$ -bundle arises from a vector bundle.

1.3 Induced Connections

Given connections ∇_1, ∇_2 on a pair of vector bundles E, F , there are canonical connections defined on the bundles $E \oplus F$, $E \otimes F$, E^* , $\text{Hom}(E, F) \cong E^* \otimes F$, $\det E = \bigwedge^{rk E} E$, and on any other associated bundles. Indeed for any functorial construction on vector spaces, and hence vector bundles, one may define an induced connection. One also has relations between the curvature of connections on E and F and the curvature of connections on their associated bundles.

For example, if $s = s_1 + s_2$ is a section of $E \oplus F$, then $\nabla(s) := \nabla_1(s_1) + \nabla_2(s_2) \in \Omega^1(M, E \oplus F)$. The curvature of this connection is given by $F_\nabla(s) = F_{\nabla_1}(s_1) + F_{\nabla_2}(s_2)$.

If $s = s_1 \otimes s_2$ is a simple section of $E \otimes F$, then one defines $\nabla(s) := \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2)$, where on the right we interpret the tensor product correctly, by moving the one-form part of $\nabla_2(s_2)$ to the front under the canonical isomorphism $E \otimes T^*M \otimes F \cong T^*M \otimes E \otimes F$. This is the unique combination of ∇_1 and ∇_2 on the tensor product that satisfies the Leibniz rule. The induced curvature is given by $F_\nabla = F_{\nabla_1} \otimes \mathbf{1} + \mathbf{1} \otimes F_{\nabla_2}$.

If ξ is a section of E^* , then we specify the dual connection ∇_1^* of ∇_1 by the expression

$$d\langle \xi, s \rangle = \langle \nabla_1^*(\xi), s \rangle + \langle \xi, \nabla_1(s) \rangle$$

where $\langle \cdot, \cdot \rangle$ is the natural pairing between sections of E^* and E . The induced curvature is given by $F_{\nabla_1^*} = -F_{\nabla_1}$, where the $\text{End}(E^*)$ -valued 2-form on the right is interpreted under the canonical isomorphism $\text{End}(E) \cong \text{End}(E^*)$ given by $E^* \otimes E \cong E \otimes E^*$.

1.4 Parallel Transport

Given a connection ∇ on a vector bundle $E \rightarrow M$, one obtains a way of identifying fibres of the vector bundle over different points. This motivates the name

connection.

Given two points p and q on M , and a path γ between them, there is a linear isomorphism $P_\gamma : E_p \rightarrow E_q$ defined by ∇ . In general the isomorphism depends on the choice of path γ , for otherwise one could trivialise the bundle $E \rightarrow M$. This isomorphism is called the *parallel transport* from E_p to E_q . We will now recall the existence of parallel transport maps.

Given a section s of E and a vector field $X \in \Gamma(M, TM)$, we can contract ∇s by X to obtain another section of E .

Note that the covariant derivative is $C^\infty(M)$ -linear in the X variable. We say that a section s is *covariantly constant*, or *horizontal*, if $\nabla_X s = 0$ for all vector fields X . Of course, this occurs if and only if $\nabla s = 0$ as a section of $\Omega^1(M, E)$.

Parallel transport of an element $\xi \in E_p$ along a path γ in M is given by finding a covariantly constant section s along γ with $s(p) = \xi$. Then $P_\gamma(\xi) = s(q)$. In order to find such a covariantly constant section, we need to introduce the covariant derivative along a curve.

Definition 1.4.1. Let $\gamma : (a, b) \rightarrow M$ be a curve, and suppose $s \in \Gamma(U, E)$ is a section of E defined on some open neighbourhood U of γ . Let X be a vector field defined on a (possibly distinct) neighbourhood of γ in M such that $X_{\gamma(t)} = \dot{\gamma}(t)$ for every $t \in (a, b)$. Define an operator

$$\frac{D}{dt}$$

on such sections $s \in \Gamma(U, E)$ by

$$\frac{Ds}{dt}(t) := \nabla_X s(\gamma(t))$$

for all $t \in (a, b)$. This is called the covariant derivative along the curve γ , and returns a section of E defined along the curve γ in M .

Note that the operator above does not depend on the choice of extension X of $\dot{\gamma}(t)$ to a neighbourhood of γ . Furthermore, such an extension always exists, and may be defined by using partitions of unity in local charts on M . Additionally, the covariant derivative of a section s along a curve γ depends only on the values of s along the curve. This means that one can consider sections defined *only* along the curve γ for the purpose of taking covariant derivatives along γ . We say that s is *horizontal along γ* if $\frac{Ds}{dt} = 0$.

If we wish to find a horizontal section along γ with initial value $\xi \in E_p$, then we need to solve the differential equation

$$\frac{Ds}{dt} = 0, \quad s(p) = \xi.$$

Suppose now that the image of γ in M lies entirely within a fixed trivialising set U of the vector bundle E , with local frame $\{e_i\}$ and local connection form A . Suppose that $s = s^i e_i$ in the local frame. Then we have

$$\frac{Ds}{dt} = \nabla_{\dot{\gamma}} s = (ds^i(\dot{\gamma}) + s^j A_j^i(\dot{\gamma})) \otimes e_i,$$

so the parallel transport equation becomes the system of differential equations

$$ds^i(\dot{\gamma}) + s^j A_j^i(\dot{\gamma}), \quad s(\gamma(0)) = \xi.$$

If one supposes that the set U is also a coordinate chart for M , then this becomes a genuine system of ordinary differential equations, for which the Picard-Lindelöf theorem guarantees existence and uniqueness of solutions. In particular there is a unique value $s(\gamma(1)) = s(q)$ for each choice of γ and $\xi \in E_p$.

The existence and uniqueness of solutions now solves the problem of restricting to a single trivialising chart U . If γ passes through multiple charts then splitting the path up into pieces each lying in a chart, with end points lying in intersections, means that the successive parallel transports all agree, giving a well-defined and unique end point.

Definition 1.4.2. *The parallel transport of $\xi \in E_p$ along γ by the connection ∇ is $P_\gamma(\xi) := s(\gamma(1)) = s(q) \in E_q$. The map P_γ is called the parallel transport map from E_p to E_q along γ defined by ∇ .*

We will now recall a number of important facts about parallel transport for a vector bundle $E \rightarrow M$ with connection ∇ . Firstly, the parallel transport maps P_γ are isomorphisms for each γ , with inverse given by P_{γ^-} , parallel transport along the same path in the opposite direction. Secondly, parallel transport may be concatenated along a pair of curves γ_1 and γ_2 with $\gamma_1(1) = \gamma_2(0)$. To do so, one simply parallel transports along γ_1 , and then along γ_2 . Note that this same process makes parallel transport well-defined along any piecewise smooth curve γ in M . Thirdly, it is possible to completely recover the connection ∇ from its parallel transport maps P_γ . Indeed we have the following formula, which can be proved by expanding in a local frame and using properties of integral curves.

Proposition 1.4.3. *Let ∇ be a connection on E , $s \in \Gamma(M, E)$ a section and $X \in \Gamma(M, TM)$ a vector field. Let $p \in M$ and suppose $\gamma : (-\varepsilon, \varepsilon) \rightarrow M$ is an integral curve for X near p . Then*

$$\nabla_X s(p) = \frac{d}{dt} (P_\gamma s(\gamma(t)))_{t=0}.$$

If one parallel transports around a loop γ based at a point p , one obtains a linear isomorphism $E_p \rightarrow E_p$. This is called the *holonomy* of ∇ along γ at p , denoted $\text{Hol}^\nabla(p, \gamma) \in \text{GL}(n, \mathbb{C})$. The set $\text{Hol}^\nabla(p)$ of linear maps given by choosing varying loops based at p is just called the holonomy of ∇ at p , and is a subgroup of $\text{GL}(n, \mathbb{C})$. For a path-connected manifold M , the subgroups $\text{Hol}^\nabla(p)$ and $\text{Hol}^\nabla(q)$ are conjugate in $\text{GL}(n, \mathbb{C})$ for any $p, q \in M$, and so one often simply talks of the *holonomy of ∇* as the subgroup of $\text{GL}(n, \mathbb{C})$ defined by taking $\text{Hol}^\nabla(p)$ for any $p \in M$.

There is a rich theory of holonomy for the Levi-Civita connection on Riemannian manifolds, but we will primarily be concerned with the relation between holonomy and flatness for general vector bundles. Let $\text{Hol}_0^\nabla(p)$ denote the subgroup of $\text{Hol}^\nabla(p)$ consisting of those loops at p that are null-homotopic. Note that if M is simply-connected then $\text{Hol}_0^\nabla(p) = \text{Hol}^\nabla(p)$. Here we have the following theorem.

Theorem 1.4.4. *If the connection ∇ is flat, then $\text{Hol}_0^\nabla(p)$ is trivial for all $p \in M$. Furthermore, if γ_1, γ_2 are any two homotopic loops based at p , then $\text{Hol}^\nabla(p, \gamma_1) = \text{Hol}^\nabla(p, \gamma_2)$.*

Notice that there is a well-defined homomorphism $\pi_1(M, p) \rightarrow \text{Hol}^\nabla(p) / \text{Hol}_0^\nabla(p)$ for each p . When $\text{Hol}_0^\nabla(p) = 0$ as is the case for flat connections by the above theorem, this is simply a homomorphism $\pi_1(M, p) \rightarrow \text{Hol}^\nabla(p) \subset \text{GL}(n, \mathbb{C})$. Thus the above theorem can be interpreted as saying that *flat connections give rise to representations of $\pi_1(M)$ into the structure group of the vector bundle E* .

In the case where ∇ is only projectively flat, we still have the following result.

Theorem 1.4.5. *If the connection ∇ is projectively flat, then $\text{Hol}_0^\nabla(p)$ is central in $\text{GL}(n, \mathbb{C})$. In particular, if γ_1 and γ_2 are homotopic loops based at p , then $\text{Hol}^\nabla(p, \gamma_1)$ is a constant multiple of $\text{Hol}^\nabla(p, \gamma_2)$, where the constant depends on γ_1 and γ_2 .*

In this case the group $\text{Hol}^\nabla(p) / \text{Hol}_0^\nabla(p)$ sits as a subgroup of $\text{PGL}(n, \mathbb{C}) = \text{GL}(n, \mathbb{C}) / \mathbb{C}^* \cdot \mathbf{1}$. Thus by the same arguments above, we see that *projectively flat connections give rise to representations of $\pi_1(M)$ into the projectivisation of the structure group of the vector bundle E* .

In fact, it is possible to go back and construct a (projectively) flat connection from a (projective) representation of the fundamental group. In the flat case, one simply considers the flat vector bundle with fibre \mathbb{C}^n associated to the universal cover, the natural principal $\pi_1(M)$ -bundle over M . In the projective case, one must ensure that the flat principal $\text{PGL}(n, \mathbb{C})$ -bundle obtained from the representation may be lifted to a projectively flat principal $\text{GL}(n, \mathbb{C})$ -bundle.

In Chapters 4 and 5 we will investigate flat and projectively flat connections on vector bundles over compact Riemann surfaces. By the above identification

between (projectively) flat connections and (projective) representations, we will obtain a wonderful interpretation of the theorems of Narasimhan and Seshadri, and Hitchin and Simpson about representations of the fundamental group and stable holomorphic vector bundles in terms of an infinite-dimensional Kempf-Ness theorem.

1.5 Characteristic Classes

In this section we will define the characteristic classes of a vector bundle in terms of the curvature of certain connections. First note that for a matrix F and an invertible matrix g , we have that $\det(F) = \det(gFg^{-1})$. If F is a local representative for the curvature of connection on a vector bundle, then on the overlap of trivialisations, F transforms as gFg^{-1} for a transition function g . Thus one can define globally $\det(F_\nabla)$ for a connection ∇ .

Definition 1.5.1 (Chern Classes). *Let $E \rightarrow M$ be a smooth complex vector bundle over a manifold M . Let ∇ be any connection on E . The k th Chern class $c_k(E)$ of E is defined to be the cohomology class of the coefficient of t^k in the expansion*

$$\det\left(I + \frac{it}{2\pi}F_\nabla\right) = \sum_k \tilde{c}_k(E)t^k.$$

That is, $c_k(E) = [\tilde{c}_k(E)] \in H^{2k}(M, \mathbb{C})$.

Of course, there are a number of details to be checked in this definition. Firstly one needs to show that the forms $\tilde{c}_k(E)$ are actually closed, so as to define cohomology classes. Additionally, one needs to show that these cohomology classes do not depend on the choice of connection. See [Kob87] for an account of these details, as well as more discussion on the definitions and properties of Chern classes. We will now state some of these properties.

Proposition 1.5.2 (Axioms of Chern Classes). *The Chern classes as defined above satisfy the axioms of Chern classes:*

1. *Existence: For each $k \in \mathbb{Z}_{\geq 0}$ there is a class $c_k(E) \in H^{2k}(M, \mathbb{C})$, and $c_0(E) = 1$. Write $c(E) := \sum_{k=0}^{\infty} c_k(E)$ for the total Chern class of E .*
2. *Naturality: Let $E \rightarrow M$ be a complex vector bundle and $f : N \rightarrow M$ a smooth map. Then $c_k(f^*E) = f^*c_k(E)$ for all k .*
3. *Whitney Sum: Let L_1, \dots, L_n be complex line bundles and $L_1 \oplus \dots \oplus L_n$ their Whitney sum. Then*

$$c(L_1 \oplus \dots \oplus L_n) = c(L_1) \cdots c(L_n).$$

4. *Normalization:* If L is the tautological line bundle over $\mathbb{C}\mathbb{P}^1$, then $-c_1(L)$ is the positive generator of $H^2(\mathbb{C}\mathbb{P}^1, \mathbb{Z}) \cong \mathbb{Z}$. That is, $c_1(L)[\mathbb{C}\mathbb{P}^1] = -1$.

Proposition 1.5.3 (Integrality). *The Chern classes $c_k(E)$ are all integral. That is, if $i_* : H^{2k}(M; \mathbb{Z}) \rightarrow H^{2k}(M; \mathbb{C})$ is the homomorphism induced by the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{C}$, then $c_k(E) \in i_*(H^{2k}(M; \mathbb{Z})) \subset H^{2k}(M; \mathbb{C})$ for all k .*

Proposition 1.5.4 (Formula for First Chern Class). *Let $E \rightarrow M$ be a smooth complex vector bundle over a manifold M , with connection ∇ . Then*

$$c_1(E) = \left[\text{tr} \left(\frac{i}{2\pi} F_\nabla \right) \right].$$

The formula for the first Chern class above is of course a simple consequence of Definition 1.5.1 and simple linear algebra.

Note that the above propositions show that there are topological obstructions to the existence of flat connections on a vector bundle. In particular if a connection ∇ on a vector bundle $E \rightarrow M$ is flat, then $F_\nabla = 0$ and so $c_1(E) = 0 \in H^2(M; \mathbb{Z})$ by the above formula for the first Chern class. Since there are many examples of vector bundles with the topological invariant $c_1(E) \neq 0$, flat connections may not always exist.

Proposition 1.5.5 (Universality of Chern Classes). *Any characteristic class $a(E)$ of smooth complex vector bundles over smooth manifolds is a polynomial in the Chern classes.*

By *characteristic class* here we mean a cohomology class $a(E)$ that may be defined for any complex vector bundle which satisfies a naturality condition analogous to that which is satisfied by the Chern classes. In particular $a(E)$ is a characteristic class if whenever $f : N \rightarrow M$ is smooth and $f^*E \rightarrow N$ denotes the pullback of $E \rightarrow M$ by f , we have $f^*a(E) = a(f^*E)$.

In addition to the Chern classes, there are a number of other characteristic classes of smooth vector bundles that we will be interested in. By the universality statement above, these classes are expressible in terms of the Chern classes.

Definition 1.5.6 (Chern Character). *Let $E \rightarrow M$ be a smooth complex vector bundle over a manifold M , and ∇ any connection on E . Then we define the Chern character $\text{Ch}(E)$ by*

$$\text{Ch}(E) = \left[\text{tr} \left(\exp \left(\frac{i}{2\pi} F_\nabla \right) \right) \right].$$

The $2k$ th degree component of $\text{Ch}(E)$ is called the k th Chern character of E , denoted $\text{Ch}_k(E)$. In particular,

$$\text{Ch}_k(E) = \left[\frac{1}{k!} \text{tr} \left(\left(\frac{i}{2\pi} \right)^k \underbrace{F_\nabla \wedge \cdots \wedge F_\nabla}_{k \text{ times}} \right) \right].$$

Definition 1.5.7 (Todd Class). *Let M be a smooth manifold and denote by $c_k(M)$ the Chern classes of TM . By the splitting principle, we may factor $\sum_{k=0}^{\infty} c_k(M)t^k = \prod(1 + \xi_i t)$ for some 2-forms ξ_i representing the Chern classes of line bundles L_i with TM and $\bigoplus_i L_i$ having the same characteristic classes. Define the Todd class denoted $\text{Td}(M)$ by*

$$\text{Td}(M) = \prod \frac{\xi_i}{1 - \exp(-\xi_i)}.$$

In particular we have

$$\text{Td}(M) = 1 + \frac{1}{2}c_1(M) + \frac{1}{12}(c_1(M)^2 + c_2(M)) + \dots .$$

Chapter 2

Symplectic Geometry

In this chapter we will review the basic constructions in symplectic geometry that will be relevant in later chapters. Symplectic manifolds will be a fundamental object of study in this thesis, for a number of reasons. In Chapter 3 we will investigate complex differential geometry in the context of Kähler manifolds, which are all symplectic. Some properties of Kähler manifolds, such as the existence of Kähler reduction, can be inferred from their underlying symplectic structure. In Chapters 4 and 5 we will use the theory of symplectic reduction to describe the moduli spaces of stable and Higgs bundles over compact Riemann surfaces. These moduli spaces are therefore naturally symplectic manifolds, and in Chapters 6 and 7 the symplectic structure will play a star role in the context of geometric quantization.

The topics discussed in this chapter may be found in any reference on symplectic geometry. In particular the notes [CdS06] of Cannas da Silva are a standard reference for this material, as well as the lecture notes [Hec13] of Heckman. In addition, this chapter will include a discussion on the relations between symplectic geometry and Mumford's Geometric Invariant Theory. Detailed references for this material include the notes [Hos12] of Hoskins and [Tho05] of Thomas.

2.1 Symplectic Vector Spaces

In order to define and discuss symplectic manifolds, it is necessary to develop the slightly non-standard theory of symplectic linear algebra. We will quickly recall the definitions and basic properties of symplectic vector spaces in this section.

Definition 2.1.1 (Symplectic Vector Space). *Let V be a finite-dimensional real vector space. A symplectic form ω on V is an element of $\bigwedge^2 V^*$ that is non-degenerate as a bilinear map $\omega : V \times V \rightarrow \mathbb{R}$. The pair (V, ω) is called a symplectic vector space.*

Example 2.1.2. Let $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ be a basis for \mathbb{R}^{2n} . Suppose the dual basis is given by $\{\epsilon^1, \dots, \epsilon^n, \xi^1, \dots, \xi^n\}$. Define a form $\omega : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ by

$$\omega := \sum_{i=1}^n \epsilon^i \wedge \xi^i.$$

The form ω is clearly antisymmetric and bilinear. With respect to the standard basis ω has the matrix

$$\omega = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}.$$

This matrix is clearly invertible, so ω is non-degenerate. The pair $(\mathbb{R}^{2n}, \omega)$ is called the standard symplectic vector space, and ω is called the standard symplectic form on \mathbb{R}^{2n} .

Consider \mathbb{R}^{2n} as \mathbb{C}^n and let $h : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ be the standard Hermitian form on \mathbb{C}^n . Then as a bilinear form h splits into real and imaginary parts $h = g - i\omega$. The symmetric bilinear form g is the standard inner product on \mathbb{R}^{2n} and the anti-symmetric bilinear form ω is the standard symplectic form on \mathbb{R}^{2n} .

Lemma 2.1.3. *Let (V, ω) be a symplectic vector space. Then there is a basis $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ of V such that $\omega(e_i, f_j) = \delta_{ij}$, $\omega(e_i, e_j) = 0$, and $\omega(f_i, f_j) = 0$. Such a basis is called a canonical basis for ω .*

Corollary 2.1.4. *Let (V, ω) be a symplectic vector space. Then $\dim V$ is even.*

Note that the block matrix for ω with respect to a canonical basis is

$$\omega = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}.$$

Definition 2.1.5. *Let $f : V \rightarrow W$ be a linear isomorphism of two symplectic vector spaces V and W . Then f is a symplectomorphism if $\omega_V(u, v) = \omega_W(f(u), f(v))$ for all $u, v \in V$. That is, $f^*\omega_W = \omega_V$.*

Thus Lemma 2.1.3 says that if (V, ω) is a $2n$ -dimensional symplectic vector space, then V is symplectomorphic to \mathbb{R}^{2n} with its standard symplectic form.

Definition 2.1.6. *Let (V, ω) be a symplectic vector space, and suppose $U \subset V$ is a subspace. Define U^\perp to be the subspace*

$$U^\perp := \{v \in V \mid \omega(u, v) = 0 \text{ for all } u \in U\}.$$

Then U^\perp is called the symplectic complement of U .

Lemma 2.1.7. *Let $U \subseteq V$ be a subspace of a symplectic vector space (V, ω) . Then*

1. $\dim U + \dim U^\perp = \dim V$.
2. $(U^\perp)^\perp = U$.
3. If $U \subseteq W$, then $W^\perp \subseteq U^\perp$.

Definition 2.1.8. *Let $U \subseteq V$ be a subspace. Then U is*

1. symplectic if $U \cap U^\perp = \{0\}$,
2. isotropic if $U \subseteq U^\perp$,
3. coisotropic if $U \supseteq U^\perp$, or
4. Lagrangian if $U = U^\perp$.

A subspace U is symplectic if and only if $\omega|_U$ is a symplectic form on U . Note that we also have $V = U \oplus U^\perp$ in this case.

A subspace U is isotropic if and only if $\omega|_U$ is the zero form. Note that since $\omega(v, v) = 0$ for any $v \in V$, one-dimensional subspaces of V are isotropic.

A subspace U is coisotropic if and only if U/U^\perp is a symplectic space with respect to the induced form defined by $\omega(u + U^\perp, v + U^\perp) := \omega|_U(u, v)$.

A subspace is Lagrangian if and only if it is isotropic and coisotropic.

Lemma 2.1.9. *Suppose $U \subset V$ is Lagrangian. Then $\dim U = \frac{1}{2} \dim V$.*

Proposition 2.1.10. *An antisymmetric bilinear form ω on an even-dimensional vector space V is non-degenerate if and only if $\omega^n \in \bigwedge^{2n} V^*$ is non-zero.*

Proof. (\implies) Suppose ω is non-degenerate. Let $\{e_i, f_i\}$ be a canonical basis for the symplectic vector space (V, ω) , and let $\{\epsilon^i, \xi^i\}$ be the dual basis. Then $\omega^n = n! \epsilon^1 \wedge \xi^1 \wedge \cdots \wedge \epsilon^n \wedge \xi^n$, which is non-zero.

(\impliedby) Suppose ω is degenerate. Then there is some $v \neq 0$ such that $\omega(v, w) = 0$ for all $w \in V$. Complete v to a basis $\{v, w_2, \dots, w_{2n}\}$ of V . Then $\omega^n(v, w_2, \dots, w_{2n}) = 0$. But if $\{\epsilon^i\}$ is the dual basis to $\{v, w_j\}$ then $\omega^n = k \epsilon^1 \wedge \cdots \wedge \epsilon^{2n}$ for some k . Since $v \wedge w_2 \wedge \cdots \wedge w_{2n}$ is the dual basis to $\epsilon^1 \wedge \cdots \wedge \epsilon^{2n}$ inside $\bigwedge^{2n} V^*$, one must have $k = 0$, so $\omega^n = 0$. \square

The form $d\text{vol} := \omega^n/n!$ is called the *symplectic volume form* on (V, ω) . The normalisation factor is to make the area of the unit parallelepiped in a canonical basis for ω equal to 1.

2.2 Symplectic Manifolds

Definition 2.2.1 (Symplectic Manifold). *Let M be a smooth manifold, and $\omega \in \Omega^2(M)$. Then ω is symplectic if ω is closed, and $\omega|_p : T_pM \times T_pM \rightarrow \mathbb{R}$ is symplectic. That is, ω is non-degenerate on each tangent space. The pair (M, ω) is called a symplectic manifold, and ω is the symplectic form.*

Note in particular that symplectic manifolds must be even-dimensional, since the dimension of the tangent space at each point is equal to the dimension of the manifold.

Example 2.2.2. Let $(x^1, \dots, x^n, y^1, \dots, y^n)$ be the standard coordinates on \mathbb{R}^{2n} . Then the form

$$\omega = \sum_{i=1}^n dx^i \wedge dy^i$$

gives \mathbb{R}^{2n} the structure of a symplectic manifold. This is the standard symplectic form on \mathbb{R}^{2n} . Notice that $\omega = -d\theta$ where

$$\theta := \sum_{i=1}^n y^i dx^i.$$

Example 2.2.3. Let (z^1, \dots, z^n) be the standard coordinates on \mathbb{C}^n . Then the form

$$\omega := \frac{i}{2} \sum_{i=1}^n dz^i \wedge d\bar{z}^i$$

gives \mathbb{C}^n a symplectic structure. This is the standard symplectic form on \mathbb{C}^n . Note that the normalisation factor means that in real coordinates where $dz^i = dx^i + idy^i$ and $d\bar{z}^i = dx^i - idy^i$ we have that ω is the standard real symplectic form on \mathbb{R}^{2n} .

Let (M, ω) be a symplectic manifold of dimension $2n$. Since ω is everywhere non-degenerate, it defines a volume form $d\text{vol} := \omega^n/n!$. In particular, M is orientable. As was mentioned earlier, M must also be even dimensional.

Suppose further that M is compact. Then the closed form ω defines a non-zero cohomology class in $H^2(M, \mathbb{R})$. The volume form $d\text{vol}$ also defines a non-zero cohomology class in $H^{2n}(M, \mathbb{R})$. Since $d\text{vol} = \omega^n/n!$, we must have $\omega^k \neq 0$ for every k . In particular $H^{2k}(M, \mathbb{R})$ must be non-trivial for all $k = 1, \dots, n$.

This gives us a collection of simple obstructions to the existence of a symplectic structure on a smooth manifold. A manifold that fails to satisfy those properties mentioned above cannot admit any symplectic structure.

Example 2.2.4. The Lie group $U(2) \cong S^3 \times S^1$ is compact, even-dimensional, and orientable. However, by the Künneth formula, the second degree cohomology of $U(2)$ is zero. Thus $U(2)$ does not admit any symplectic structures.

Similarly to the linear case, we can define a notion of equivalence of symplectic manifolds.

Definition 2.2.5 (Symplectomorphism). *Let (M, ω) and (N, η) be symplectic manifolds. Let $f : M \rightarrow N$ be a diffeomorphism. Then f is a symplectomorphism if $f^*\eta = \omega$.*

Equivalently, f is a symplectomorphism if $df|_{T_p M}$ is a symplectomorphism of symplectic vector spaces, for each $p \in M$.

A submanifold $N \subset M$ of a symplectic manifold has each of the properties symplectic, isotropic, coisotropic, and Lagrangian if and only if its tangent spaces $T_p N \subset T_p M$ have these properties, for all $p \in N$. In particular note that a Lagrangian manifold $L \subset M$ has dimension half that of M , a symplectic submanifold $N \subset M$ has even dimension, and a Lagrangian submanifold is both isotropic and coisotropic.

2.3 Phase Space

In the previous section, we saw that the standard symplectic form ω on \mathbb{R}^{2n} may be written as $-d\theta$ for a one-form θ . In this section we will expand on this observation, to show that every cotangent bundle is a symplectic manifold.

Let M be a smooth manifold, and $(U, (x^1, \dots, x^n))$ be a coordinate chart. Let $\varphi : T^*M|_U \rightarrow U \times \mathbb{R}^n$ be the trivialisation, and $\phi : U \rightarrow \mathbb{R}^n$ be the chart. Then one can define coordinates $\psi := (\phi, \mathbf{1}) \circ \varphi$.

Given a point $p \in T^*M|_U$ with $\phi(\pi(p)) = (x^1, \dots, x^n)$, one has $\psi(p) = (x^1, \dots, x^n, y_1, \dots, y_n)$ for some $y_1, \dots, y_n \in \mathbb{R}$. Indeed if $\{dx^i\}$ is the local frame for the cotangent bundle over U , then $p = \sum_i y_i dx^i \in T_{\pi(p)}^*M$.

Definition 2.3.1 (Canonical Coordinates). *The coordinates $(x^1, \dots, x^n, y_1, \dots, y_n)$ defined above on $T^*M|_U$ are called the canonical coordinates on T^*M .*

Let M be a smooth manifold. Let $N := T^*M$. Define a one-form θ on T^*M as follows. Let $\pi : N \rightarrow M$ be the standard projection, and consider $d\pi : TN \rightarrow TM$. Let $n \in N$. Then if $\pi(n) = q$, we may interpret n as a linear functional $n : T_q M \rightarrow \mathbb{R}$ on the tangent space at q . Define $\theta|_n := n \circ d\pi|_{T_q N}$. That is, given a covector $n \in T_q^*M$ and a tangent vector v in $T_n(T^*M)$, project v to a tangent vector $d\pi(v)$ in $T_q M$, and then evaluate $d\pi(v)$ on the covector n , to obtain a real number.

Definition 2.3.2 (Tautological One-Form). *The differential form θ on $N = T^*M$ is called the tautological one-form of M .*

The tautological one-form may also be described as the unique one-form on T^*M with the following property.

Proposition 2.3.3. *The tautological one-form on a manifold M satisfies $\xi^*\theta = \xi$ for every $\xi \in \Omega^1(M)$.*

Proof. Let $\xi \in \Omega^1(M)$ be a one-form. Then we may interpret ξ as a section $\xi : M \rightarrow T^*M$. Then $\xi^*\theta = \xi$. To see this, let $p \in M$ and $v \in T_pM$. Then

$$\begin{aligned} \xi^*\theta|_p(v) &= \theta_{\xi|_p}(d\xi(v)) \\ &= \xi|_p \circ d\pi \circ d\xi(v) \\ &= \xi|_p \circ d(\pi \circ \xi)(v) \\ &= \xi|_p(v). \end{aligned}$$

□

If one considers \mathbb{R}^{2n} as $T^*\mathbb{R}^n$, with the standard global coordinates on \mathbb{R}^n , then the one-form θ described in Example 2.2.2 is precisely the tautological one-form on \mathbb{R}^n .

Definition 2.3.4 (Canonical Symplectic Form). *Let $\omega := -d\theta$. Then ω is called the canonical symplectic form.*

The form ω is clearly symplectic, because in canonical coordinates

$$\omega = \sum_{i=1}^n dx^i \wedge dy_i.$$

We remark that the zero section of the cotangent bundle $N = T^*M$ is an embedded Lagrangian submanifold with respect to this symplectic form.

Any smooth manifold produces a symplectic manifold. The cotangent bundle $N = T^*M$ equipped with its canonical symplectic form is often called a phase space. This terminology arises from physics, where M is usually the configuration space of a classical system, whose points describe positions of all the particles in the system. Then the cotangent vectors on this configuration space are the momentums of the particles. A point in the cotangent space thus completely describes the state of the system, and so is called the phase space. The dynamics of the classical system can be naturally interpreted in terms of the canonical symplectic form.

2.3.1 Darboux's Theorem

A general Riemannian manifold has non-trivial local moduli, given by the Riemannian curvature tensor. In striking contrast, in symplectic geometry all symplectic manifolds of the same dimension are locally symplectomorphic. This is the celebrated theorem of Darboux about the existence of canonical local coordinates on a symplectic manifold. In particular this theorem shows that every symplectic manifold locally looks like $(\mathbb{R}^{2n}, \omega_{std})$ where ω_{std} is the standard symplectic form on \mathbb{R}^{2n} .

Theorem 2.3.5 (Darboux's Theorem). *Let (M, ω) be a symplectic manifold of dimension $2n$, and let $p \in M$. By the Poincaré lemma there is a neighbourhood U of p and a one-form θ such that $\omega = -d\theta$. On this neighbourhood U there are coordinates $\varphi : U \rightarrow \mathbb{R}^{2n}$ given by $\varphi = (q^1, \dots, q^n, p_1, \dots, p_n)$ such that in these coordinates,*

$$\theta = \sum_{i=1}^n p_i dq^i.$$

In particular we have

$$\omega = -d\theta = \sum_{i=1}^n dq^i \wedge dp_i$$

so that $\omega = \varphi^* \omega_{std}$.

This theorem may be proved using the elegant argument of Moser, now known as the Moser trick, to construct a diffeomorphism $\phi : M \rightarrow M$ that takes one differential form α to $\phi^* \alpha = \beta$. The idea of the Moser trick is to construct vector fields X_t on M depending on some $t \in \mathbb{R}$ such that the flow diffeomorphisms ϕ_t of X_t (given by evaluating the flows of each X_t at time 1) have the property that $\alpha = \phi_t^* \beta_t$. Here β_t is a carefully chosen family of differential forms such that $\beta_0 = \alpha$ and $\beta_1 = \beta$. The construction of the family of forms β_t and the vector fields X_t turns out to involve solving a simple differential equation, and this trick finds many applications.

Moser originally used this trick to prove that if α, β are two volume forms on a compact manifold M , then there is a diffeomorphism ϕ taking α to β if and only if their integrals over M agree. In addition to its applicability to Darboux's theorem, the Moser trick may also be used to prove the tubular neighbourhood theorem of Weinstein.

Theorem 2.3.6 (Weinstein's Lagrangian Neighbourhood Theorem). *Let $L \subset (M, \omega)$ be a Lagrangian submanifold of a symplectic manifold (M, ω) . Then there exists a neighbourhood U of L and a symplectomorphism $U \rightarrow U'$ where U' is a neighbourhood of the zero section in T^*L , with U' inheriting the canonical symplectic structure from T^*L .*

We noted earlier that the zero section of T^*M is an embedded Lagrangian submanifold of T^*M , and the neighbourhood theorem of Weinstein shows that this example is in some sense universal: all Lagrangian submanifolds look like the zero sections of their cotangent bundles.

For more details about the Moser trick and the proof of Darboux's and Weinstein's theorems, see the notes [CdS06] of da Silva.

2.4 Hamiltonian Vector Fields and the Poisson Bracket

Let (M, ω) be a symplectic manifold and $f \in C^\infty(M)$ be a smooth function on M . Then $df \in \Omega^1(M)$ is a one-form.

Let $X \in \Gamma(M, TM)$. Let i_X be the contraction operator $i_X : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ defined by

$$i_X \eta(X_2, \dots, X_k) := \eta(X, X_2, \dots, X_k)$$

for $\eta \in \Omega^k(M)$ and $X_2, \dots, X_k \in \Gamma(M, TM)$, and $i_X|_{\Omega^0(M)} := 0$.

Then we have $i_X \omega \in \Omega^1(M)$ for our symplectic form ω and for any $X \in \Gamma(M, TM)$. Since ω is non-degenerate, it defines an isomorphism $\Gamma(M, TM) \rightarrow \Omega^1(M)$ given by $X \mapsto i_X \omega$. But then given $f \in C^\infty(M)$, there exists a smooth vector field $X_f \in \Gamma(M, TM)$ such that $i_{X_f} \omega = -df$ (why we require a negative sign will become clear later).

Definition 2.4.1. *Let (M, ω) be a symplectic manifold and let $f \in C^\infty(M)$. Then the unique vector field $X_f \in \Gamma(M, TM)$ such that*

$$i_{X_f} \omega = -df$$

is called the Hamiltonian vector field associated to f .

Note that not every vector field $X \in \Gamma(M, TM)$ is Hamiltonian, because not every one-form η is necessarily exact. If $H^1(M, \mathbb{R}) = 0$ however (for example if M is simply connected), then every vector field which preserves ω is the Hamiltonian vector field of some function. Notice the condition that X preserves ω implies that $d(i_X \omega) = 0$, and therefore $H^1(M, \mathbb{R}) = 0$ implies that $i_X \omega = -df$ for some function f .

A natural question one might ask is whether the Hamiltonian vector fields are closed under standard operations on vector fields. Clearly the Hamiltonian vector fields form a vector subspace of $\Gamma(M, TM)$ by linearity of d . What about the commutator $[X_f, X_g]$ of two Hamiltonian vector fields?

First we recall a useful formula of Cartan.

Theorem 2.4.2 (Cartan's Magic Formula). *Let $X \in \Gamma(M, TM)$ and $\omega \in \Omega^k(M)$. Then*

$$\mathcal{L}_X \omega = i_X d\omega + d(i_X \omega).$$

Lemma 2.4.3. *Let $f, g \in C^\infty(M)$ and let $h := \omega(X_f, X_g)$. Then $[X_f, X_g] = X_h$.*

Proof. Using Cartan's magic formula, we have

$$\mathcal{L}_{X_f}(g) = i_{X_f} dg = i_{X_f}(-i_{X_g} \omega) = \omega(X_f, X_g).$$

Then

$$\begin{aligned} d(\omega(X_f, X_g)) &= d(\mathcal{L}_{X_f} g) \\ &= \mathcal{L}_{X_f}(dg) \\ &= -\mathcal{L}_{X_f}(i_{X_g} \omega). \end{aligned}$$

Now $\mathcal{L}_U(i_V \alpha) = i_{[U, V]} \alpha + i_V \mathcal{L}_U(\alpha)$, and by Cartan's magic formula if α is closed and $i_U \alpha$ is closed, then $\mathcal{L}_U \alpha = 0$. Since $i_{X_f} \omega = -df$ is exact, it is closed, so we have

$$d(\omega(X_f, X_g)) = -i_{[X_f, X_g]} \omega$$

as desired. \square

Therefore the Hamiltonian vector fields form a subalgebra of the space of vector fields on M with respect to the Lie bracket of vector fields. By the non-degeneracy of ω , we can transport this Lie bracket backwards to the space $C^\infty(M)$ of smooth functions on M .

Definition 2.4.4. *Let $f, g \in C^\infty(M)$ and suppose $X_f, X_g \in \Gamma(M, TM)$ are their corresponding Hamiltonian vector fields. Define the Poisson bracket of f and g to be the smooth function*

$$\{f, g\} := \omega(X_f, X_g).$$

By Cartan's magic formula we can write the Poisson bracket in several equivalent ways:

$$\mathcal{L}_{X_f}(g) = i_{X_f} dg = i_{X_f}(-i_{X_g} \omega) = \omega(X_f, X_g) = \{f, g\}.$$

From the definition of the Poisson bracket, we have that $\{f, g\} = -\{g, f\}$ for $f, g \in C^\infty(M)$, and from the expression

$$X_{\{f, g\}} = [X_f, X_g]$$

we also observe that $\{f, g\}$ satisfies the Jacobi identity, and therefore gives $C^\infty(M)$ the structure of a Lie algebra.

In Darboux coordinates $\{q^i, p_i\}$ we have

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right).$$

2.5 Moment Maps and Symplectic Reduction

Given a symplectic manifold (M, ω) , it is a natural question to ask when we can quotient by the action of a Lie group G to obtain another symplectic manifold.

Suppose that G acts on (M, ω) by symplectomorphisms, otherwise known as the action being *symplectic*. Explicitly, the maps $x \mapsto g \cdot x$ are symplectomorphisms of (M, ω) for all $g \in G$.

Definition 2.5.1. *The action of G on (M, ω) is said to be Hamiltonian if for every $X \in \mathfrak{g}$, the Lie algebra of G , the induced vector field \tilde{X} on (M, ω) is Hamiltonian. That is, the one-form $-i_{\tilde{X}}\omega$ is exact and so there exists $H_X \in C^\infty(M)$ such that $\tilde{X} = X_{H_X}$. This choice of H_X is unique up to a constant.*

Note that the map $X \mapsto \tilde{X}$ is a Lie algebra *anti-homomorphism* from \mathfrak{g} to $\Gamma(M, TM)$, with \tilde{X} defined by

$$\tilde{X} \Big|_p := \frac{d}{dt}(\exp(tX) \cdot p)_{t=0}.$$

When the action of G on (M, ω) is Hamiltonian, one may define a map $\mu : M \rightarrow \mathfrak{g}^*$ as follows. Let $X \in \mathfrak{g}$ and H_X be its corresponding smooth function on M . Denote by μ_X the smooth function $x \mapsto \langle \mu(x), X \rangle$ for some yet undetermined $\mu : M \rightarrow \mathfrak{g}^*$. Then define $\mu_X := H_X$ for each $X \in \mathfrak{g}$. This assignment is linear, and determines μ . Notice that this map has the property that

$$i_{\tilde{X}}\omega = -d\mu_X,$$

which follows immediately from the definition of Hamiltonian vector fields. A map satisfying this property is known as a *moment map*.

Definition 2.5.2. *A moment map for the Hamiltonian action of G on (M, ω) is a map $\mu : M \rightarrow \mathfrak{g}^*$ satisfying*

$$d\langle \mu, X \rangle = -i_{\tilde{X}}\omega$$

for every $X \in \mathfrak{g}$, which is linear in the sense that $X \mapsto \mu_X := \langle \mu, X \rangle$ (the so-called comoment map) is a linear map from \mathfrak{g} to $C^\infty(M)$.

There is some ambiguity in defining this moment map μ , in so far as one may add a constant to $\langle \mu, X \rangle$ for each linearly independent $X \in \mathfrak{g}$. In addition to the existence of a moment map $\mu : M \rightarrow \mathfrak{g}^*$, one often requires further that μ is equivariant with respect to the coadjoint action ad^* on \mathfrak{g}^* . This action is given by $\langle \text{ad}^*(g)\xi, X \rangle := \langle \xi, \text{ad}(g^{-1})X \rangle$, where we use g^{-1} to ensure that this is a left action of G on \mathfrak{g}^* .

The condition that μ is G -equivariant is therefore that

$$\langle \mu(g \cdot x), X \rangle = \langle \mu(x), \text{ad}(g^{-1})X \rangle$$

for all $X \in \mathfrak{g}$, and $g \in G$. When G is connected we can check this equivariance on the level of Lie algebras, and an equivalent condition is that $\{\mu_X, \mu_Y\} = -\mu_{[X, Y]}$ for all $X, Y \in \mathfrak{g}$. Indeed this means that the map $X \mapsto \mu_X$ is a Lie algebra anti-homomorphism from \mathfrak{g} to $C^\infty(M)$. If we had instead chosen that Hamiltonian vector fields satisfy $i_{X_f}\omega = df$ the corresponding equivariance condition would be that $X \mapsto \mu_X$ is a Lie algebra *homomorphism*, but at the cost of the map $f \mapsto X_f$ from smooth functions to Hamiltonian vector fields then becoming a Lie algebra anti-homomorphism.

Theorem 2.5.3 (Marsden-Weinstein). *Suppose G is a compact Lie group with a Hamiltonian action on (M, ω) having associated equivariant moment map μ . Suppose that $\xi \in \mathfrak{g}^*$ is a regular value of this action, and that ξ is central in the sense that it is fixed by the coadjoint action. Since μ is G -equivariant, $\mu^{-1}(\xi)$ is a G -invariant subset of M . Suppose the action of G on M is such that the quotient space $\tilde{M} := \mu^{-1}(\xi)/G$ is a manifold (for example if G acts freely). Then if $i : \mu^{-1}(\xi) \hookrightarrow M$ is the inclusion and $\pi : M \rightarrow \tilde{M}$ is the projection:*

1. *The form $i^*\omega$ is basic for the action of G on \tilde{M} , and therefore descends to give a unique closed and non-degenerate form $\tilde{\omega}$ on \tilde{M} such that $i^*\omega = \pi^*\tilde{\omega}$.*
2. *With respect to $\tilde{\omega}$, $(\tilde{M}, \tilde{\omega})$ therefore inherits the structure of a symplectic manifold from (M, ω) , called the symplectic reduction of (M, ω) by G at $\xi \in \mathfrak{g}^*$, with dimension $\dim \tilde{M} = \dim M - 2 \dim G$.*

We remark that it suffices to chose $\xi = 0$ in the above construction, because if ξ is some central non-zero element of \mathfrak{g}^* then $\mu' := \mu - \xi$ is also an equivariant moment map for the action of G on (M, ω) . Later we will see that symplectic reduction also transfers several other structures from M to \tilde{M} , including Kähler and hyper-Kähler structures.

The symplectic reduction technique can be used to construct many well-known examples of symplectic manifolds. For example, the action of $U(1)$ on $\mathbb{C}^{n+1} \setminus \{0\}$ is Hamiltonian with respect to the standard symplectic form inherited from \mathbb{C}^{n+1} . The Lie algebra of $U(1)$ and its dual can be identified with \mathbb{R} , and the corresponding equivariant moment map is given by $\mu(z^0, \dots, z^n) := \frac{1}{2} \sum_{i=0}^n |z^i|^2 - \frac{1}{2}$. The symplectic reduction $\mathbb{C}\mathbb{P}^n := \mu^{-1}(0)/U(1) = S^{2n+1}/S^1$ then inherits a symplectic form ω_{FS} known as the Fubini-Study form on $\mathbb{C}\mathbb{P}^n$. Later we will see that this is in fact an example of a *Kähler* reduction, and the form ω_{FS} is the standard Kähler form on projective space $\mathbb{C}\mathbb{P}^n$.

Other examples of symplectic manifolds constructed in this way include many examples of moduli spaces, such as the moduli spaces of stable holomorphic or stable Higgs bundles on a compact Riemann surface, as we will see later.

2.6 The Kempf-Ness Theorem

2.6.1 Geometric Invariant Theory

In this section we will quickly recall the ideas of Geometric Invariant Theory, first expounded by Mumford in 1965 based on ideas from Hilbert's classical invariant theory. Mumford's original book was expanded by Fogarty and Kirwan in [MFK94]. The exposition of this section closely follows the notes [Hos12] of Hoskins. We will state theorems without proofs, all of which can be found in these notes.

Given an algebraic variety X and a linear algebraic group G , or more generally a scheme X and a group scheme G , Geometric Invariant Theory gives the conditions under which a quotient of X by G exists, and when it is also a scheme or variety with the nice properties one might desire a quotient to have. In our case we will simply take X to be a complex algebraic variety, although most definitions and theorems hold more generally.

To clarify what is meant by nice properties above, we have the following characterisation of quotients.

Definition 2.6.1 (Categorical Quotient). *A categorical quotient of X by G is a G -invariant morphism $\varphi : X \rightarrow Y$ such that given any other G -invariant morphism $f : X \rightarrow Z$, there exists a unique $\tilde{f} : Y \rightarrow Z$ such that the following diagram commutes:*

$$\begin{array}{ccc} X & & \\ \downarrow \varphi & \searrow f & \\ Y & \xrightarrow{\tilde{f}} & Z \end{array}$$

Definition 2.6.2 (Good Quotient). *A good quotient of X by G is a variety Y and a morphism $\varphi : X \rightarrow Y$ such that*

1. φ is constant on orbits of G in X ,
2. φ is surjective,
3. whenever $U \subset Y$ is open, the morphism $\mathcal{O}_Y(U) \rightarrow \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism onto the G -invariant functions $\mathcal{O}_X(\varphi^{-1}(U))^G$,

4. whenever $W \subset X$ is a G -invariant closed subset of X , its image $\varphi(W)$ is closed in Y ,
5. whenever W_1 and W_2 are disjoint G -invariant closed subsets of X , $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint, and
6. φ is affine, in the sense that the preimage of every affine open subset of Y is affine in X .

Definition 2.6.3 (Geometric Quotient). *A good quotient $\varphi : X \rightarrow Y$ is said to be a geometric quotient if the preimage of every point in Y is a single orbit of G in X . In this case Y is the genuine topological quotient of X by G (at least in the analytic topology).*

It is not difficult to show that a good quotient is also a categorical quotient, and is therefore unique up to isomorphism. Furthermore, good quotients have the following extra properties:

Proposition 2.6.4. *Let $\varphi : X \rightarrow Y$ be a good quotient. Then*

1. $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$ if and only if $\varphi(x_1) = \varphi(x_2)$ in Y ,
2. for each $y \in Y$, there is a unique closed orbit contained in $\varphi^{-1}(y)$. In particular if every orbit of G in X is closed, then φ must be a geometric quotient, and
3. if $U \subset Y$ is open then $\varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \rightarrow U$ is also a good quotient (and geometric whenever φ is).

The goal of Geometric Invariant Theory is to construct *good* quotients of algebraic varieties by linear algebraic groups. To do so, observe that if a linear algebraic group G acts on an algebraic variety X , then it also acts on $A(X)$ the coordinate ring of X , by $g \cdot f(x) := f(g^{-1} \cdot x)$. Therefore one may consider the ring of invariants $A(X)^G$ consisting of those coordinate functions that are constant on the orbits of G .

Definition 2.6.5 (Affine GIT Quotient). *The affine GIT quotient of an affine algebraic variety X by the action of a linear algebraic group G is the affine algebraic variety*

$$X//G := \text{Spec } A(X)^G$$

with the morphism $\varphi : X \rightarrow X//G$ induced by the inclusion $A(X)^G \hookrightarrow A(X)$.

In the case of projective algebraic varieties, $A(X)$ is now the homogeneous coordinate ring, which is graded so that $A(X) = \bigoplus_{d \geq 0} A(X)_d$. Denote by $A(X)_+$ the ideal $A(X)_+ := \bigoplus_{d > 0} A(X)_d$.

Definition 2.6.6 (Projective GIT Quotient). *The projective GIT quotient of a projective algebraic variety X by the action of a linear algebraic group G is the projective algebraic variety*

$$X//G := \text{Proj } A(X)^G$$

with the morphism again induced by the inclusion $A(X)^G \hookrightarrow A(X)$.

We remark that in the case of projective algebraic varieties, the action of G on X is required to be *linear* in the sense that if $X \subset \mathbb{C}\mathbb{P}^n$ for some n , then the action of G on X is the restriction of some linear action of G on $\mathbb{C}\mathbb{P}^n$ descending from a linear action of G on \mathbb{C}^{n+1} . In particular this means that the ring $A(X)^G$ of invariants is graded, so that $\text{Proj } A(X)^G$ makes sense. This also means that the projective GIT quotient $X//G$ depends on the particular embedding of X into $\mathbb{C}\mathbb{P}^n$.

We also remark that the morphism $\varphi : X \rightarrow X//G$ is undefined on the null cone $N_{A(X)^G} := \{x \in X \mid f(x) = 0 \text{ for all } f \in A(X)^G_+\}$. Here $A(X)^G_+$ denotes the intersection of $A(X)_+$ with $A(X)^G$. The morphism φ is well-defined on $X \setminus N_{A(X)^G}$, and later we will see that this set has a more explicit description as the set of *semi-stable points* of X .

Further, note that the constructions Spec and Proj (which we take to only return the *maximal* ideals, so as to obtain precisely the algebraic variety rather than its associated scheme, which includes all prime ideals) will only return nice algebraic varieties if the ring of invariants $A(X)^G$ is finitely-generated. In general this need not be the case, but for a nice class of algebraic groups, the so-called *reductive* groups, this holds by a theorem of Nagata.

Theorem 2.6.7 (Nagata). *Let G be a reductive group acting on a complex algebraic variety X . Then $A(X)^G$ is finitely-generated.*

Examples of reductive groups include the complexification of compact connected Lie groups, including $\text{GL}(n, \mathbb{C})$ and $\text{SL}(n, \mathbb{C})$. The fundamental result of Geometric Invariant Theory is the following theorem of Mumford.

Theorem 2.6.8 (Mumford). *The GIT quotient of an algebraic variety X by a reductive group G is a good quotient. In the projective case, the GIT quotient is a good quotient after restricting φ to the set $X \setminus N_{A(X)^G}$ where it is defined.*

2.6.2 Stability

It is not always the case that the GIT quotient of the previous section is geometric. In addition to the construction of GIT quotients, Mumford isolated certain *stability criterion* on elements of the variety X that allow one to obtain a geometric quotient after restricting to an open subset. First we will consider the case where X is affine.

Definition 2.6.9 (Stability). *Let X be an affine algebraic variety acted upon by a reductive group G . We say a point $x \in X$ is stable if its orbit is closed in X , and $\dim G \cdot x = \dim G$, so that $\dim G_x = 0$ where G_x is the stabiliser of x in G . Denote by X^s the set of stable points.*

One can show that X^s is an open subset of X , and therefore the good quotient $\varphi : X \rightarrow X//G$ restricts to X^s . Denote by $(X//G)^s$ the image $\varphi(X^s)$.

Theorem 2.6.10. *The restricted quotient $\varphi^s : X^s \rightarrow (X//G)^s$ is a geometric quotient for the action of G on X^s , so $(X//G)^s = X^s/G$.*

In the case of projective algebraic varieties, the picture is slightly more subtle. If $X \subset \mathbb{C}\mathbb{P}^n$ is projective, then one may consider its affine cone $\tilde{X} \subset \mathbb{C}^{n+1}$, which is the union of $\pi^{-1}(X)$ with 0 under the projection $\mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{\pi} \mathbb{C}\mathbb{P}^n$. The action of G on X lifts to the affine cone \tilde{X} .

Definition 2.6.11 (Stability and Semi-Stability). *Suppose that X is a projective algebraic variety acted upon by a reductive group G . Let $x \in X$ and suppose \tilde{x} is any point in \tilde{X} lying over x . Then*

- x is stable if and only if $\dim G_{\tilde{x}} = 0$ and $G \cdot \tilde{x}$ is closed in \tilde{X} .
- x is semi-stable if and only if $\overline{G \cdot \tilde{x}}$ does not contain $0 \in \mathbb{C}^{n+1}$.

Denote by X^s the set of stable points of X , and by X^{ss} the set of semi-stable points. Note that $X^s \subseteq X^{ss}$.

We remark that the set $X^{ss} = X \setminus N_{A(X)G}$ where $N_{A(X)G}$ is the null cone discussed in the previous section, and this is in fact an alternate definition of the set of semi-stable points. Then we have the following theorem.

Theorem 2.6.12. *Let X be a projective algebraic variety and G a reductive group acting linearly on X . Then the GIT quotient space $X//G$ may be identified with the good quotient $\varphi : X^{ss} \rightarrow X^{ss}/G$ into the projective variety X^{ss}/G . Furthermore, restricting φ to $X^s \subseteq X^{ss}$ gives a geometric quotient $\varphi^s : X^s \rightarrow (X//G)^s = X^s/G$. X^s/G is an open subset of $X//G$ and is therefore quasi-projective.*

In the projective case we can give a slightly more explicit description of the GIT quotient $X//G$ as a set.

Definition 2.6.13 (Polystability and S-Equivalence). *A semistable point $x \in X$ is said to be polystable if its orbit is closed in X^{ss} . Note that a polystable point with zero-dimensional stabiliser is therefore stable. We say two semistable points $x_1, x_2 \in X^{ss}$ are S-equivalent if $\overline{G \cdot x_1} \cap \overline{G \cdot x_2} \neq \emptyset$. Denote by X^{ps} the set of polystable points in X^{ss} .*

Using polystability we obtain the following description of $X//G$.

Theorem 2.6.14. *Let x_1 and x_2 be semistable points in X . Then $\varphi(x_1) = \varphi(x_2)$ if and only if x_1 is S -equivalent to x_2 . In particular, there is a bijection of sets*

$$X//G \cong X^{ps}/G.$$

The above theorem indicates that the GIT quotient $X//G$ may be viewed as the regular quotient X^{ss}/G where we then identify two points if they are S -equivalent.

2.6.3 Symplectic Quotients and the Kempf-Ness Theorem

In this section we state the Kempf-Ness theorem as it is found in the notes [Hos12]. This theorem gives an explicit relation between symplectic reductions of complex projective manifolds and Geometric Invariant Theory.

Recall that if G is a connected compact Lie group, then $G_{\mathbb{C}}$ the complexification is reductive. Further, recall that complex projective space $\mathbb{C}\mathbb{P}^n$ has a canonical symplectic form inherited from its description as a symplectic reduction, the Fubini-Study form. In the statement of the Kempf-Ness theorem we will use the fact that if X is a smooth complex submanifold of $\mathbb{C}\mathbb{P}^n$, such as if X is complex projective, then the Fubini-Study form ω_{FS} restricts to give a symplectic form on X .

Theorem 2.6.15 (Kempf-Ness Theorem). *Let G be a compact connected Lie group with reductive complexification $G_{\mathbb{C}}$. Suppose $G_{\mathbb{C}}$ acts linearly on a smooth complex projective variety $X \subset \mathbb{C}\mathbb{P}^n$, and suppose further that the restricted action of G is Hamiltonian for the symplectic manifold X . Suppose $\mu : X \rightarrow \mathfrak{g}^*$ is the associated equivariant moment map given by the restriction of the standard moment map $\hat{\mu} : \mathbb{C}\mathbb{P}^n \rightarrow \mathfrak{g}^*$. Then:*

1. $G_{\mathbb{C}} \cdot \mu^{-1}(0) = X^{ps}$,
2. if $x \in X$ is polystable, then $G_{\mathbb{C}} \cdot x$ meets $\mu^{-1}(0)$ in a single G orbit,
3. a point $x \in X$ is semi-stable if and only if $\overline{G_{\mathbb{C}} \cdot x} \cap \mu^{-1}(0) \neq \emptyset$, and
4. 0 is a regular value of μ if and only if $X^{ss} = X^s$.

Using this theorem, we can relate the GIT quotient of X by $G_{\mathbb{C}}$ to the symplectic reduction of X by G .

Corollary 2.6.16. *The inclusion $\mu^{-1}(0) \hookrightarrow X^{ss}$ induces a homeomorphism*

$$\mu^{-1}(0)/G \rightarrow X//G_{\mathbb{C}}.$$

Proof. First note that the Kempf-Ness theorem implies that $\mu^{-1}(0)$ is in fact a subset of X^{ss} . Theorem 2.6.14 implies that $X//G_{\mathbb{C}}$ as a set is X^{ss} with S-equivalent points identified. Furthermore this set is isomorphic to $X^{ps}/G_{\mathbb{C}}$. By part 2 of the Kempf-Ness theorem, every polystable orbit meets $\mu^{-1}(0)$ in a unique G -orbit. This gives a bijection between $X^{ps}/G_{\mathbb{C}}$ and $\mu^{-1}(0)/G$. Note that $\mu^{-1}(0)$ is a closed subspace of X , which is compact, and therefore $\mu^{-1}(0)/G$ is also compact. But then the inclusion $\mu^{-1}(0) \hookrightarrow X^{ss}$ induces a continuous bijection from a compact space to a Hausdorff space, which is therefore a homeomorphism. \square

The continuous inverse $X//G_{\mathbb{C}} \rightarrow \mu^{-1}(0)/G$ may be found by following the gradient flow of the norm squared of the moment map with respect to some G -invariant norm on \mathfrak{g}^* . Indeed one statement of the Kempf-Ness theorem is that a point is stable if and only if this norm attains its minimum somewhere on the orbit of the point.

We will investigate an infinite-dimensional version of the Kempf-Ness theorem for the moduli space of stable holomorphic vector bundles over a compact Riemann surface in Chapter 4. There too the norm of the moment map will appear as the Yang-Mills functional for connections on a smooth complex vector bundle. The minima (or critical points) of this norm are the so-called Yang-Mills connections, and the Narasimhan-Seshadri theorem states that such connections correspond to stable holomorphic structures on the underlying smooth vector bundle.

Chapter 3

Complex Differential Geometry

In this chapter we will recall many important details about the geometry of complex manifolds and holomorphic vector bundles, with a particular focus on Kähler manifolds. Of particular importance is the theory of almost-complex structures on a smooth manifold, the integrability of which to a genuine complex structure is described by the famous Newlander-Nirenberg theorem. Additionally we will be concerned with the theory of holomorphic vector bundles on complex manifolds, and the alternate description of holomorphic structures on vector bundles in terms of Dolbeault operators. This is a separate application of the Newlander-Nirenberg theorem, and we will present a linearized proof of the theorem in the case of holomorphic vector bundles over compact Riemann surfaces provided by Atiyah and Bott in [AB83].

In addition to these topics, we will recall the definition of Dolbeault cohomology and its relation to the sheaf cohomology of the sheaves of holomorphic sections of holomorphic vector bundles, and state the major theorems for the cohomology groups of holomorphic vector bundles over compact complex manifolds.

The results in this chapter are standard and can be found in the first chapter of the excellent Principles of Algebraic Geometry [GH78] by Griffiths and Harris, as well as in [WGP80] by Wells and [Kob87] by Kobayashi.

3.1 Almost Complex Structures

Suppose M is a complex manifold of complex dimension n , and let (U, φ) be a local holomorphic chart for M , with $\varphi = (z^1, \dots, z^n) : U \rightarrow \mathbb{C}^n$. Then each $z^i = x^i + iy^i$ for real coordinates $(x^1, \dots, x^n, y^1, \dots, y^n)$, and one has distinguished complex-valued one-forms $dz^i = dx^i + idy^i$ and $d\bar{z}^i = dx^i - idy^i$ on U . Dual to

these one-forms are the complex-valued vector fields

$$\frac{\partial}{\partial z^i} = \frac{1}{2} \left(\frac{\partial}{\partial x^i} - i \frac{\partial}{\partial y^i} \right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^i} = \frac{1}{2} \left(\frac{\partial}{\partial x^i} + i \frac{\partial}{\partial y^i} \right),$$

and taken all together these vector fields form a frame for $TU_{\mathbb{C}} := TU \otimes \mathbb{C}$. These local vector fields also define a splitting of $TU_{\mathbb{C}}$ into two subbundles defined by

$$T^{1,0}U := \text{Span} \left\{ \frac{\partial}{\partial z^i} \right\} \quad \text{and} \quad T^{0,1}U := \text{Span} \left\{ \frac{\partial}{\partial \bar{z}^i} \right\}.$$

This local direct-sum splitting of the tangent bundle is invariant under change of holomorphic coordinates z^i , and therefore defines a splitting $TM_{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$. The bundle $T^{1,0}M$ is called the *holomorphic tangent bundle*, a holomorphic vector bundle whose holomorphic sections correspond to the holomorphic vector fields on X , and $T^{0,1}M$ is called the *anti-holomorphic tangent bundle*.

Remark 3.1.1. The holomorphic tangent bundle $T^{1,0}M$ is isomorphic as a real vector bundle of rank $2n$ to the regular tangent bundle TM of M considered as a smooth manifold. The real isomorphism is given simply by the composition $TM \hookrightarrow TM_{\mathbb{C}} \xrightarrow{\text{pr}_{1,0}} T^{1,0}M$. We also remark that there is a real isomorphism $T^{1,0}M \rightarrow T^{0,1}M$ given by conjugation.

Similarly the one-forms dz^i and $d\bar{z}^i$ give a splitting $T^*M_{\mathbb{C}} = T_{1,0}^*M \oplus T_{0,1}^*M$ into the *holomorphic and anti-holomorphic cotangent bundles*.

On the vector bundle $TM_{\mathbb{C}}$ there is a certain endomorphism $I : TM_{\mathbb{C}} \rightarrow TM_{\mathbb{C}}$ defined by

$$\frac{\partial}{\partial z^i} \mapsto i \frac{\partial}{\partial z^i} \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^i} \mapsto -i \frac{\partial}{\partial \bar{z}^i}.$$

In terms of the real coordinates I is defined by

$$\frac{\partial}{\partial x^i} \mapsto \frac{\partial}{\partial y^i} \quad \text{and} \quad \frac{\partial}{\partial y^i} \mapsto -\frac{\partial}{\partial x^i}$$

for each i . Notice that $I^2 = -\mathbf{1}$ on $TM_{\mathbb{C}}$ and that the expression for I in the real coordinates makes sense just on the real tangent bundle TM . Moreover, by construction the holomorphic tangent bundle is just the $+i$ -eigenspace of I acting on $TM_{\mathbb{C}}$ and the anti-holomorphic tangent bundle is the $-i$ -eigenspace. Such an endomorphism of TM is called an *almost-complex structure* on M . As can be seen in the above definition of I for a complex manifold M , this endomorphism encodes the ability to multiply real tangent vectors by i on the underlying real smooth manifold. In particular $I|_p$ turns T_pM into a complex vector space for every $p \in M$, giving TM the structure of a smooth complex vector bundle over the smooth manifold M .

Definition 3.1.2 (Almost-Complex Structure). *Let M be a $2n$ -dimensional smooth manifold. Then an endomorphism $I : TM \rightarrow TM$ such that $I^2 = -\mathbf{1}$ is called an almost-complex structure on M .*

The above discussion shows that if M has a complex structure, then it automatically inherits an almost-complex structure. The converse is not always true. If an almost-complex structure I on a smooth manifold M is genuinely the almost-complex structure of some complex structure on M , then we say the almost-complex structure is *integrable*. The famous theorem of Newlander and Nirenberg gives necessary and sufficient conditions for an almost-complex structure to be integrable.

Definition 3.1.3 (Nijenhuis Tensor). *Let $A : TM \rightarrow TM$ be an endomorphism of the tangent bundle of M , and $X, Y \in \Gamma(M, TM)$ any two smooth vector fields. Define a tensor N_A by*

$$N_A(X, Y) := -A^2[X, Y] + A([AX, Y] + [X, AY]) - [AX, AY].$$

Then N_A is called the Nijenhuis tensor of A .

Theorem 3.1.4 (Newlander-Nirenberg Theorem). *An almost-complex structure I on a smooth $2n$ -dimensional manifold M is integrable if and only if $N_I = 0$, and the compatible complex structure is unique.*

Using the Newlander-Nirenberg theorem, one may essentially define a complex manifold as a smooth manifold with an integrable almost-complex structure, and we will follow along these lines in Chapter 6 when we discuss Kähler polarisations of symplectic manifolds.

The proof of the Newlander-Nirenberg theorem is difficult in general, but may be simplified slightly by assuming that M has a real-analytic structure. Later in this chapter we will give a proof of the Newlander-Nirenberg theorem for holomorphic structures on complex vector bundles over Riemann surfaces found by Atiyah and Bott in [AB83], where the linear setting and low dimension of the base significantly simplify the proof. The full proof may be found in the textbook [Hor73] by Hörmander on analysis in several complex variables.

Note finally that not all smooth $2n$ -manifolds admit an almost-complex structure. For example, the only spheres that admit almost-complex structures are S^2 and S^6 , where the structure on S^2 is integrable with $S^2 \cong \mathbb{C}\mathbb{P}^1$. The almost-complex structure on S^6 is inherited from its description as the unit imaginary octonions, and it is known that this almost-complex structure is not integrable. It is a major open problem whether or not S^6 admits any other almost-complex structures and whether or not they are integrable.

3.2 Complex Differential Forms

In the previous section we saw that any complex manifold M comes with a canonical splitting $T^*M_{\mathbb{C}} = T_{1,0}^*M \oplus T_{0,1}^*M$ of its complexified cotangent bundle. This splitting has ramifications for the structure of the algebra of complex-valued differential forms on M .

Let $\Omega^k(M)$ denote the complex-valued differential k -forms on M . Then the direct-sum splitting of $T^*M_{\mathbb{C}}$ induces a splitting $\Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$ where $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$ denote the smooth sections of the holomorphic and anti-holomorphic cotangent bundles respectively. After taking exterior powers of the complexified cotangent bundle, observe that we have further splittings

$$\bigwedge^k T^*M_{\mathbb{C}} = \bigoplus_{p+q=k} \left(\bigwedge^p T_{1,0}^*M \oplus \bigwedge^q T_{0,1}^*M \right)$$

for each k , and that these splittings induce splittings on sections given by

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$$

where now $\Omega^{p,q}(M)$ denotes the space of smooth sections of the direct sum of the p th exterior power of $T_{1,0}^*M$ and the q th exterior power of $T_{0,1}^*M$.

Notice that in local coordinates $T_{1,0}^*M$ and $T_{0,1}^*M$ have frames given by the dz^i and the $d\bar{z}^j$ respectively. Therefore forms of type (p, q) are differential forms that in local coordinates consist of p forms of type dz^i and q forms of type $d\bar{z}^j$.

To see what happens to the exterior derivative d under this splitting, note that for $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$ we may consider the operators

$$\pi_{p',q'} \circ d : \Omega^k(M) \rightarrow \Omega^{p',q'}(M)$$

where $\pi_{p',q'}$ is the projection $\Omega^{k+1}(M) \rightarrow \Omega^{p',q'}(M)$ and $p' + q' = k + 1$. If we first restrict d to $\Omega^{p,q}(M)$ where $p + q = k$, then we see that $(p', q') = (p + 1, q), (p, q + 1), (p + 2, q - 1)$, or $(p - 1, q + 2)$. Indeed neither p nor q will drop by more than 1 degree in general, which can be seen by analysing the projection in local coordinates. Furthermore, if the splitting of $T^*M_{\mathbb{C}}$ comes from an integrable almost-complex structure as is the case when M itself is complex, then neither p or q will drop a degree, so we see that $(p', q') = (p + 1, q)$ or $(p', q') = (p, q + 1)$.

In the case where $k = 1$ this gives a splitting of $d : \Omega^1(M) \rightarrow \Omega^1(M)$ into two operators $d = \partial + \bar{\partial}$ where ∂ increases the first bidegree and $\bar{\partial}$ the second. The projections sending forms of type (p, q) to those of type $(p + 1, q)$ and $(p, q + 1)$ are in fact the extensions of ∂ and $\bar{\partial}$ to the full complex $\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$. In local coordinates we have

$$\bar{\partial}(f_{I,J} dz^I \wedge d\bar{z}^J) := \frac{\partial f_{I,J}}{\partial d\bar{z}^i} d\bar{z}^i \wedge dz^I \wedge d\bar{z}^J$$

and similarly for ∂ .

Remark 3.2.1. One possible alternate definition of integrability of almost-complex structures is that the ∂ and $\bar{\partial}$ operators, which may be defined for any almost-complex structure, actually do sum up $\partial + \bar{\partial} = d$. Another definition is that forms of type (p, q) do not have any components of type $(p - 1, q + 2)$ or $(p + 2, q - 1)$ after applying d to them. See [WGP80, Ch. 1] for more details.

Due to its ability to encode holomorphic information on the manifold M , the operator $\bar{\partial}$ holds a privileged position, similarly to the famous Cauchy-Riemann operator $\frac{\partial}{\partial \bar{z}}$ in complex analysis in one variable.

Definition 3.2.2 (Cauchy-Riemann Operator). *The operator $\bar{\partial} : \Omega^{p,q}(M) \rightarrow \Omega^{p,q+1}(M)$ is called the Cauchy-Riemann operator or Dolbeault operator on M , and is defined purely in terms of the complex structure on M .*

The Cauchy-Riemann operator has the property that a function $f \in \Omega^0(M)$ is holomorphic if and only if $\bar{\partial}f = 0$, and that a form ω in $\Omega^{p,0}(M)$ is holomorphic if and only if $\bar{\partial}\omega = 0$. Moreover, the condition $d^2 = 0$ means that $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} + \bar{\partial}\partial = 0$. Thus $(\Omega^{p,\bullet}(M), \bar{\partial})$ forms a cochain complex for each p . We will discuss the cohomology of this cochain complex later in this chapter, as well considering the case where it is augmented by a holomorphic vector bundle.

We also have the analogy of the Poincaré lemma for this operator.

Lemma 3.2.3 (Dolbeault's Lemma). *Let P be a (possibly unbounded) open polydisc in \mathbb{C}^n . If $\alpha \in \Omega^{p,q}(B)$ with $\bar{\partial}\alpha = 0$ then there exists $\beta \in \Omega^{p,q-1}(B)$ with $\bar{\partial}\beta = \alpha$.*

3.3 Holomorphic Vector Bundles

Let $E \rightarrow M$ be a smooth complex vector bundle over a complex manifold M . Then $E \rightarrow M$ is *holomorphic* if the total space E may be equipped with the structure of a complex manifold such that the projection map $\pi : E \rightarrow M$ is holomorphic. We will refer to E together with its holomorphic structure by \mathcal{E} . If \mathcal{E} is holomorphic then there is a trivialisation $\{U_\alpha, \varphi_\alpha\}$ of E with the $\varphi_\alpha : E|_{U_\alpha} \rightarrow U_\alpha \times \mathbb{C}^k$ biholomorphisms, where k is the rank of E .

This is equivalent to the condition that E is a smooth complex vector bundle and admits a local trivialisation $\{U_\alpha\}$ such that the transition functions $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow \text{GL}(n, \mathbb{C})$ are holomorphic.

We have already seen one example of a holomorphic vector bundle over any complex manifold, the holomorphic tangent bundle. Indeed suppose $\{U_\alpha, \psi_\alpha\}$ is a holomorphic atlas for a complex manifold M . Then the transition functions

$\psi_\alpha \circ \psi_\beta^{-1}$ are biholomorphisms, and so the derivatives $g_{\alpha\beta} := D(\psi_\alpha \circ \psi_\beta^{-1})$ define holomorphic maps $g_{\alpha\beta} : U_{\alpha\beta} \rightarrow \text{GL}(n, \mathbb{C})$. The associated vector bundle is the holomorphic tangent bundle $T^{1,0}M \rightarrow M$, hence the name holomorphic tangent bundle.

Dual to the holomorphic tangent bundle $T^{1,0}M$ is the holomorphic cotangent bundle $T_{1,0}^*M$. Just as in the case of smooth vector bundles, all the tensor operations on holomorphic bundles are well-defined, and indeed this gives a wealth of examples of holomorphic vector bundles attached to any complex manifold. For example, if E and F are holomorphic vector bundles, so are $E \oplus F$, $E \otimes F$, E^* , $\bigwedge^k E$, $\bigotimes^k E$, $S^m E$, $\text{Hom}(E, F)$, and $\text{End}(E)$. We point out the so-called *canonical bundle*.

Definition 3.3.1 (Canonical Bundle). *Let M be a complex manifold of complex dimension n . The holomorphic line bundle $K := \bigwedge^n T_{1,0}^*M$ is called the canonical bundle of M . Its holomorphic sections are the holomorphic volume forms on M .*

On a complex manifold of dimension 1, a Riemann surface, note that $K = T_{1,0}^*M$ is just the holomorphic cotangent bundle itself.

3.3.1 Dolbeault Operators

We saw earlier that a complex manifold M comes equipped with an operator $\bar{\partial}$ on differential forms, the Cauchy-Riemann or Dolbeault operator. This operator has the special property that a function f is holomorphic if and only if $\bar{\partial}f = 0$. It turns out that one can encode the holomorphic structure of holomorphic vector bundles using a similar operator.

Let U be a trivialising open set for a smooth complex vector bundle E , and let $\{e_i\}$ be the corresponding local frame. Then a local section $s : U \rightarrow E|_U$ can be viewed as a vector $s = s^i e_i$ where the s^i are smooth functions $s^i : U \rightarrow \mathbb{C}$. In this local frame, there is a natural differential operator $\bar{\partial} : \Gamma(U, E|_U) \rightarrow \Omega^{0,1}(U, E|_U)$ acting on such local sections by $\bar{\partial}(s) := \bar{\partial}(s^i) \otimes e_i$. This differential operator has the property that a local section $s \in \Gamma(U, E|_U)$ is holomorphic if and only if $\bar{\partial}s = 0$.

As in the case of connections, these local $\bar{\partial}$ operators do not *necessarily* glue together to form a global differential operator on sections of the vector bundle E . In the case of connections, this occurs precisely when there is a trivialisation of E with *constant* transition functions. The $\bar{\partial}$ analogue of such constant transition functions are holomorphic transition functions, so we see that if E is equipped with a holomorphic structure, then there exists a global differential operator denoted $\bar{\partial}_E : \Omega^0(M, E) \rightarrow \Omega^{0,1}(M, E)$ with the property that $s \in \Omega^0(M, E)$ is holomorphic if and only if $\bar{\partial}_E(s) = 0$.

It is natural to ask whether, given such a differential operator, one can go back and obtain a holomorphic structure on E . Note that any such differential operator must necessarily satisfy $\bar{\partial}_E^2 = 0$ and a $\bar{\partial}$ -Leibniz rule. Thus we make a definition:

Definition 3.3.2 (Dolbeault Operator). *A map $\bar{\partial}_E : \Omega^0(M, E) \rightarrow \Omega^{0,1}(M, E)$ on sections of a smooth complex vector bundle E over a complex manifold M is called a Dolbeault operator if it is \mathbb{C} -linear, satisfies the Leibniz rule*

$$\bar{\partial}_E(fs) = \bar{\partial}(f) \otimes s + f\bar{\partial}_E(s)$$

and satisfies $\bar{\partial}_E^2 = 0$.

It turns out that these properties are also sufficient.

Theorem 3.3.3. *A smooth complex vector bundle $E \rightarrow M$ is holomorphic if and only if there is a Dolbeault operator $\bar{\partial}_E$ on $E \rightarrow M$.*

The standard proof of this fact uses the Newlander-Nirenberg theorem to show a certain explicit almost complex structure on E is integrable. However, on a Riemann surface Atiyah and Bott, in [AB83, p. 555], proved this result without the Newlander-Nirenberg theorem. We present this proof here.

Proof of Theorem 3.3.3 when M is a compact Riemann surface. It suffices to take a trivialising set U for E and construct a local frame $\{f_i\}$ with $\bar{\partial}_E(f_i) = 0$ for all i . Let $\{e_i\}$ be an arbitrary frame for U . Then in this frame we have $\bar{\partial}_E(e_i) = \theta_i^j \otimes e_j$ for some matrix θ of $(0, 1)$ -forms.

Let $f = (f_i^j)$ be a matrix of smooth functions on U . Then we have

$$\bar{\partial}_E(f_i^j e_j) = (\bar{\partial}(f_i^j) + f_i^k \theta_k^j) \otimes e_j.$$

Thus it would suffice to solve the matrix equation

$$f^{-1}\bar{\partial}_E f + \theta = 0$$

for the matrix of smooth functions f .

Assume first that the space M is S^2 , and the bundle E is trivial. Then let $T : L_2^2 \rightarrow L_1^2$ be the non-linear operator defined by $T(f) := f^{-1}\bar{\partial}_E f$. We have

$$\begin{aligned} T(1 + \varepsilon) &= (1 + \varepsilon)^{-1}\bar{\partial}_E(1 + \varepsilon) \\ &= (1 - \varepsilon + O(\varepsilon^2))\bar{\partial}_E \varepsilon \\ &= \bar{\partial}_E \varepsilon - \varepsilon \bar{\partial}_E \varepsilon + O(\varepsilon^2)\bar{\partial}_E \varepsilon. \end{aligned}$$

Letting $\|\varepsilon\| \rightarrow 0$ we see that the differential of T at $f = 1$ is the elliptic operator $\bar{\partial}_E$.

Now this operator $\bar{\partial}_E$ is surjective on S^2 , because we can use the Cauchy integral formula to recover a function from its derivatives here, and by Liouville's theorem the kernel of $\bar{\partial}_E$ is just the constant matrices.

Thus the implicit function theorem for Banach spaces says that $T(f) = -\theta$ has a unique (in some neighbourhood of $1 \in L_2^2$) solution $f \in L_2^2$ orthogonal to the constant matrices, whenever θ is close enough to zero in L_1^2 . Since $\bar{\partial}_E$ is elliptic we also have that f is smooth whenever θ is.

To transfer this analysis from S^2 to a trivialising set on M , define a radially symmetric cutoff function $\rho : z \mapsto \rho'(|z|)$ on $S^2 = \mathbb{C} \cup \{\infty\}$, where

$$\rho'(x) := \begin{cases} 1 & x \leq \delta/2 \\ 1 - \frac{2x-\delta}{\delta} & \delta/2 \leq x \leq \delta \\ 0 & x > \delta \end{cases}$$

for some $\delta > 0$. Choose a trivialising open set for E that is biholomorphic to some open ball containing the support of ρ . Clearly we have $\rho \in L_1^2$ and one can compute that $\|\rho\|_1 \leq 2\sqrt{\pi}$ regardless of the choice of δ .

Now if we define $\phi = \rho\theta$ then we can estimate over the region $\{z \mid |z| \leq \delta\}$

$$\begin{aligned} \|\phi\|_1^2 &= \|\rho\theta\|^2 + \|d\rho\theta + \rho d\theta\|^2 \\ &\leq \sup |\rho|^2 \|\theta\|^2 + \|d\rho\theta\|^2 + \|\rho d\theta\|^2 \\ &\leq \|\theta\|^2 + \sup |\theta|^2 \|d\rho\|^2 + \sup |\rho|^2 \|d\theta\|^2 \\ &\leq 2\|\theta\|_1^2 + C \sup |\theta|^2. \end{aligned}$$

for some constant C independent of δ .

By a judicious choice of gauge transformation, we can choose a frame in which $\theta(0) = 0$ and $d\theta(0) = 0$. In this frame, a particular choice of δ can make these norms arbitrarily close to zero, and hence one can create a ϕ close enough to zero to guarantee the existence of an $f \in L_2^2$ satisfying $f^{-1}\bar{\partial}_E f + \phi = 0$. By restricting to the ball where $|z| < \delta/2$, we have $\phi = \theta$ so elliptic regularity implies the solution f is smooth, and therefore (after possibly shrinking the trivialising open set) we obtain a local holomorphic frame for the vector bundle E . \square

3.3.2 Chern Connections

Given a fixed Hermitian metric on a holomorphic vector bundle, we can obtain a unique connection that is compatible with both the metric and the holomorphic structure.

Theorem 3.3.4. *Let (\mathcal{E}, h) be a holomorphic Hermitian vector bundle with Dolbeault operator $\bar{\partial}_E$. Then there is a unique connection ∇ on \mathcal{E} such that ∇ is compatible with h and $\nabla^{0,1} = \bar{\partial}_E$. This connection is called the Chern connection.*

Proof. Let U be a trivialising open set for \mathcal{E} and let $\{e_i\}$ be a holomorphic frame on U . Then in this frame we obtain a Hermitian matrix H with $H_i^j = h(e_i, e_j)$. Now if ∇ is a connection satisfying the conditions above then since $\nabla^{0,1} = \bar{\partial}_E$, $\nabla e_i = A_i^j \otimes e_j$ for some matrix of $(1,0)$ -forms A_i^j . In this frame we have

$$dH_i^j = h(\nabla e_i, e_j) + h(e_i, \nabla e_j).$$

Thus

$$dH_i^j = A_i^k H_k^j + \bar{A}_j^k H_i^k = A_i^k H_k^j + \bar{A}_k^j H_i^k,$$

or in matrix notation $dH = HA + \bar{A}^T H$. Since A is of type $(1,0)$ we see that $\partial H = HA$, or in matrix notation $A = H^{-1} \partial H$. This determines the connection ∇ uniquely, so what remains is a proof of existence.

Let $\{U_\alpha\}$ be a trivialisaton of \mathcal{E} with holomorphic transition functions $g_{\alpha\beta}$. Then we have $H_\alpha = \bar{g}_{\alpha\beta}^T H_\beta g_{\alpha\beta}$ and hence

$$\begin{aligned} H_\alpha^{-1} \partial H_\alpha &= (\bar{g}_{\alpha\beta}^T H_\beta g_{\alpha\beta})^{-1} \partial (\bar{g}_{\alpha\beta}^T H_\beta g_{\alpha\beta}) \\ &= g_{\alpha\beta}^{-1} H_\beta^{-1} \partial H_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1} d g_{\alpha\beta} \end{aligned}$$

because $\partial g_{\alpha\beta} = d g_{\alpha\beta}$ and $\partial \bar{g}_{\alpha\beta} = 0$.

Thus the local forms $H^{-1} \partial H$ satisfy the compatibility condition for connection forms, and one obtains a unique global connection satisfying the two conditions. \square

The Chern connection satisfies $F_\nabla^{0,2} = 0$ since $\bar{\partial}_E^2 = 0$, and by compatibility with the Hermitian metric we also have $F_\nabla^{2,0} = 0$, so the curvature of ∇ is of type $(1,1)$.

3.3.3 Holomorphic Sub-Bundles

Given a holomorphic vector bundle $\pi : \mathcal{E} \rightarrow M$, a smooth sub-bundle $F \subset \mathcal{E}$ is a holomorphic sub-bundle if $\pi|_F : F \rightarrow M$ gives F the structure of a holomorphic vector bundle \mathcal{F} . The inclusion map $i : \mathcal{F} \hookrightarrow \mathcal{E}$ is therefore a holomorphic map, and we have a short exact sequence of holomorphic vector bundles given by

$$0 \longrightarrow \mathcal{F} \xrightarrow{i} \mathcal{E} \longrightarrow \mathcal{E}/\mathcal{F} \longrightarrow 0.$$

Smoothly this short exact sequence splits, so that $E \cong F \oplus E/F$ as smooth complex vector bundles. To see this, one may for example take a Hermitian metric on E , and then $E/F \cong F^\perp$ the orthogonal complement of F in E .

Holomorphically, the short exact sequence may not necessarily split. Indeed, suppose that $\bar{\partial}_E$ is the Dolbeault operator for the holomorphic structure on \mathcal{E} .

Then the Dolbeault operator $\bar{\partial}_F$ for the holomorphic structure on \mathcal{F} is simply given by $\text{pr}_F \circ \bar{\partial}_E$ where pr_F is the projection onto F under the smooth splitting $E \cong F \oplus F^\perp$. With respect to this smooth splitting therefore, $\bar{\partial}_E$ has the form

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_F & \beta \\ 0 & \bar{\partial}_{E/F} \end{pmatrix},$$

where $\beta : E/F \rightarrow F \otimes T_{0,1}^*M$ is called the *second fundamental form* of F in E , and $\bar{\partial}_{E/F}$ is the Dolbeault operator for the holomorphic vector bundle $F^\perp \cong \mathcal{E}/\mathcal{F}$. Indeed $\beta \in \Omega^{0,1}(M, \text{Hom}(E/F, F))$ and if $\beta = 0$ then the splitting $E \cong F \oplus E/F$ is holomorphic. By Atiyah's interpretation of extensions of bundles [Ati57a], the short exact sequence splits if and only if β is $\bar{\partial}$ -cohomologous to zero.

We will further interpret this splitting in terms of the associated Chern connections for \mathcal{E} and \mathcal{F} . Indeed fix the Hermitian metric h on \mathcal{E} , and let $\nabla_{\mathcal{E}}$ be the Chern connection such that $\nabla_{\mathcal{E}}^{0,1} = \bar{\partial}_E$. Then because \mathcal{F} is a holomorphic sub-bundle, $\nabla_{\mathcal{F}} = \text{pr}_F \circ \nabla_{\mathcal{E}}$. This shows the Chern connection $\nabla_{\mathcal{E}}$ is of the form

$$\nabla_{\mathcal{E}} = \begin{pmatrix} \nabla_{\mathcal{F}} & \beta \\ -\beta^* & \nabla_{\mathcal{E}/\mathcal{F}} \end{pmatrix}.$$

Finally, this induces a splitting of the curvature $F_{\nabla_{\mathcal{E}}}$ as

$$F_{\nabla_{\mathcal{E}}} = \begin{pmatrix} F_{\nabla_{\mathcal{F}}} - \beta \wedge \beta^* & d_{\nabla_{\text{Hom}(E/F, F)}} \beta \\ -d_{\nabla_{\text{Hom}(F, E/F)}} \beta^* & F_{\nabla_{\mathcal{E}/\mathcal{F}}} - \beta^* \wedge \beta \end{pmatrix}.$$

3.3.4 Holomorphic Bundles on a Riemann Surface

Let us now restrict to the case where M is a compact Riemann surface. Here we have that $\Omega^{0,2}(M) = 0$ since M has complex dimension one, so any operator satisfying the $\bar{\partial}$ -Leibniz rule is automatically a Dolbeault operator. In particular, if E is a smooth complex vector bundle over M and ∇ is a connection on E , then $\nabla^{0,1}$ is a Dolbeault operator. If we fix a Hermitian metric h on E and require that the connection ∇ is compatible with this metric, then we see that ∇ is actually the Chern connection with respect to the holomorphic structure induced by $\nabla^{0,1}$. Hence:

Theorem 3.3.5. *Let (E, h) be a fixed smooth complex Hermitian vector bundle on a Riemann surface M . Then there is a bijection between unitary connections on (E, h) and holomorphic structures on (E, h) , given by taking a connection ∇ to $\nabla^{0,1}$, or by taking a Dolbeault operator $\bar{\partial}_E$ to its Chern connection.*

The set $\mathcal{A}(E, h)$ of all unitary connections on (E, h) is an affine space modelled on $\Omega^1(M, \text{ad } P)$ where P is the $U(n)$ -principal bundle of unitary frames in (E, h) ,

and $\text{ad } P$ is the associated Lie algebra bundle. In a similar way, the set $\text{Dol}(E)$ of Dolbeault operators (or holomorphic structures) on E is an affine space modelled on $\Omega^{0,1}(M, \text{End}(E))$.

The space $\mathcal{A}(E, h)$ has a natural action of the *unitary* gauge group \mathcal{G} , whilst the space of Dolbeault operators is naturally acted upon by the *complex* gauge group $\mathcal{G}_{\mathbb{C}}$. Under the bijection $\mathcal{A}(E, h) \cong \text{Dol}(E)$ one can pull back this $\mathcal{G}_{\mathbb{C}}$ action to $\mathcal{A}(E, h)$. Here we have a remarkable result.

Theorem 3.3.6. *The $\mathcal{G}_{\mathbb{C}}$ action on $\mathcal{A}(E, h)$ extends the \mathcal{G} action. In particular the $\mathcal{G}_{\mathbb{C}}$ orbits inside $\mathcal{A}(E, h)$ break up into a disjoint union of \mathcal{G} orbits.*

Proof. The action of $\mathcal{G}_{\mathbb{C}}$ on $\mathcal{A}(E, h)$ can be computed to be

$$u \cdot \nabla = \nabla - [(\nabla^{0,1}u)u^{-1} - ((\nabla^{0,1}u)u^{-1})^*]$$

where here we are using the extension of $\nabla^{0,1}$ to $\text{End}(E)$.

If $u^{-1} = u^*$ then $u \in \mathcal{G}$, and in this case

$$\begin{aligned} u \cdot \nabla &= \nabla - (\nabla^{0,1}u)u^{-1} + u(\nabla^{0,1}u)^* \\ &= \nabla - (\nabla^{0,1}u)u^{-1} + u\nabla^{1,0}u^* \\ &= \nabla - (\nabla^{0,1}u)u^{-1} - uu^*(\nabla^{1,0}u)u^* \\ &= \nabla - (\nabla u)u^{-1}, \end{aligned}$$

which is the expression for the action of \mathcal{G} on $\mathcal{A}(E, h)$. □

3.4 Dolbeault Cohomology

We have seen that if M is a complex manifold, then

$$(\Omega^{p,\bullet}(M), \bar{\partial})$$

is a cochain complex for each $p \in \mathbb{Z}_{\geq 0}$. The cohomology of this complex is an important holomorphic invariant of the complex manifold M .

Definition 3.4.1 (Dolbeault Cohomology). *The (p, q) th Dolbeault cohomology group of M is defined to be*

$$H_{\bar{\partial}}^{p,q}(M) := H^q((\Omega^{p,\bullet}(M), \bar{\partial})).$$

Notice that we could have used the cochain complex with differential ∂ , but since conjugation gives a real isomorphism $\Omega^{p,q}(M) \rightarrow \Omega^{q,p}(M)$ interchanging $\bar{\partial}$ and ∂ , we have an isomorphism $H_{\bar{\partial}}^{p,q}(M) \cong H_{\partial}^{q,p}(M)$ for all p, q . Thus no information is lost by concerning ourselves only with Dolbeault cohomology. Furthermore

the Dolbeault cohomology groups are where many obstructions to holomorphic constructions on the complex manifold M naturally live.

In addition to the regular Dolbeault cohomology of M , the fact that Dolbeault operators on holomorphic vector bundles square to zero means we can augment the the (p, q) -forms by sections of a holomorphic vector bundle \mathcal{E} to obtain a new complex.

$$0 \longrightarrow \Omega^{p,0}(M, E) \xrightarrow{\bar{\partial}_E} \Omega^{p,1}(M, E) \xrightarrow{\bar{\partial}_E} \Omega^{p,2}(M, E) \longrightarrow \dots$$

Definition 3.4.2. *The (p, q) th Dolbeault cohomology group of M with coefficients in the holomorphic vector bundle \mathcal{E} is defined to be*

$$H_{\bar{\partial}_E}^{p,q}(M) := H^q((\Omega^{p,\bullet}(M, E), \bar{\partial}_E)).$$

The above construction of Dolbeault cohomology and Dolbeault cohomology with coefficients in a holomorphic vector bundle closely mirrors the de Rham cohomology (and de Rham cohomology with coefficients in a flat vector bundle) of smooth manifolds. Indeed for de Rham cohomology one has the well-known de Rham isomorphism theorem, giving an alternate description of the de Rham cohomology groups in terms of the cohomology of the constant \mathbb{R} sheaf on the smooth manifold. In the complex case there is an analogous result.

Denote by \mathcal{E} the sheaf of holomorphic sections of the holomorphic vector bundle \mathcal{E} , and $\Omega^p(\mathcal{E})$ the sheaf of holomorphic $(p, 0)$ -form valued-sections of \mathcal{E} . Consider the sheaf cohomology groups $H^q(M, \Omega^p(\mathcal{E}))$ for each q .

Theorem 3.4.3 (Dolbeault's Theorem). *There is an isomorphism*

$$H_{\bar{\partial}_E}^{p,q}(M) \cong H^q(M, \Omega^p(\mathcal{E}))$$

for all (p, q) .

The proof of Dolbeault's theorem is standard sheaf theory, and follows from the existence of a fine resolution of the sheaf $\Omega^p(\mathcal{E})$ given by

$$0 \longrightarrow \Omega^p(\mathcal{E}) \hookrightarrow \Omega^{p,0}(E) \xrightarrow{\bar{\partial}_E} \Omega^{p,1}(E) \xrightarrow{\bar{\partial}_E} \dots,$$

where $\Omega^{p,q}(E)$ denotes the sheaf of smooth (p, q) -forms with values in the underlying smooth vector bundle E . The cohomology of this resolution is Dolbeault cohomology with coefficients in \mathcal{E} , which is isomorphic to the sheaf cohomology of the sheaf being resolved. See Griffiths and Harris [GH78] for more details.

3.5 Kähler Manifolds

By the existence of partitions of unity, every smooth manifold M admits a Riemannian metric g . In particular this is the case if M is a complex manifold. However, a priori there is no relation between the complex structure and the Riemannian structure of M . The condition that the metric g is compatible with the complex structure of M is called the *Kähler condition*, and a manifold satisfying the Kähler condition is called a *Kähler manifold*.

To isolate the Kähler condition, let us identify the complex structure on M with the corresponding integrable almost-complex structure $I : TM \rightarrow TM$. We say that the metric g is compatible with I if

$$g(IX, IY) = g(X, Y)$$

for all smooth vector fields $X, Y \in \Gamma(M, TM)$. Since $I^2 = -\mathbf{1}$, when the metric is compatible with I we may define an anti-symmetric bilinear form ω by

$$\omega(X, Y) := g(X, IY).$$

The Kähler condition is that ω is closed.

Definition 3.5.1 (Kähler manifold). *A complex manifold M with Riemannian metric g is Kähler if g is compatible with the associated integrable almost-complex structure I , and the associated 2-form ω is closed. Then ω is called the Kähler form of M and $h = g + i\omega$ is called the Kähler metric.*

To show that the closedness of ω is the natural condition that signifies the compatibility of g and I , we have the following list of conditions equivalent to the Kähler condition.

Proposition 3.5.2. *Let M be a complex manifold with Riemannian metric g , associated integrable almost-complex structure I , and Levi-Civita connection ∇ . Then the following are equivalent:*

1. $h = g + i\omega$ is a Kähler metric,
2. $d\omega = 0$,
3. $\nabla I = 0$,
4. the Chern connection of the Hermitian metric $h := g + i\omega$ on $TM_{\mathbb{C}}$ agrees with the Levi-Civita connection ∇ ,
5. in terms of local holomorphic coordinates, we have

$$\frac{\partial g_{j\bar{k}}}{\partial z^l} = \frac{\partial g_{l\bar{k}}}{\partial z^j},$$

6. for each point $p \in M$ there is a smooth real function F in a neighbourhood of p such that $\omega = i\partial\bar{\partial}F$, and F is called the Kähler potential, and
7. for each point $p \in M$ there are holomorphic coordinates centred at p such that $g(z) = \mathbf{1} + O(|z|^2)$.

As a consequence of the above equivalences, one finds that Kähler manifolds enjoy many nice properties not necessarily shared by general complex manifolds. For example, the intimate relationship between the Riemannian and complex structures of a Kähler manifold imply relations between the d and $\bar{\partial}$ Laplacians. We will see the consequences of this in the section on Hodge theory. For more details about Kähler geometry see [WGP80], [GH78], or the excellent lecture notes [Bal06] of Ballmann.

Although the nice properties of Kähler manifolds imply restrictions on which complex manifolds can be Kähler, there are still many examples of Kähler manifolds in practice. For example, we have the following.

Proposition 3.5.3. *Any complex submanifold of a Kähler manifold is Kähler.*

Proof. If ω is a Kähler form on M and $i : N \hookrightarrow M$ is a complex submanifold, then $i^*\omega$ is Kähler with respect to the induced integrable almost-complex structure I_N on N . This induced almost-complex structure is given by $I_N := \text{pr}_{TN} \circ I$ where pr_{TN} is orthogonal projection onto $TN \subset TM$ with respect to the Riemannian metric g on M . \square

Immediately we conclude that any complex submanifold of \mathbb{C}^n is Kähler. Furthermore, the Fubini-Study form ω_{FS} on $\mathbb{C}\mathbb{P}^n$ obtained via symplectic reduction is a Kähler form. Indeed $\mathbb{C}\mathbb{P}^n$ has a Kähler metric (the Fubini-Study metric) given in a standard local coordinate chart by

$$h_{i\bar{j}} := \frac{(1 + |z|^2)\delta_{i\bar{j}} - \bar{z}^i z^j}{(1 + |z|^2)^2}.$$

The imaginary part of h is ω_{FS} . Combining these two facts we have:

Proposition 3.5.4. *Any complex projective manifold (equivalently, any non-singular projective complex analytic variety) is Kähler.*

The converse of this result is not true. That is, it is not necessarily the case that a compact Kähler manifold is projective. Precisely when this occurs is the content of Kodaira's embedding theorem: If the Kähler form is integral, known as a *Hodge form*, then there is an embedding.

Theorem 3.5.5 (Kodaira's Embedding Theorem). *A compact complex manifold can be embedded as a complex submanifold of $\mathbb{C}\mathbb{P}^n$ if and only if it has a Hodge form.*

3.6 Kähler Reduction

In this section we will expand slightly on the idea of symplectic reduction discussed in Chapter 2 in the setting of complex manifolds. Suppose then that a symplectic manifold (M, ω) is actually Kähler, with integrable almost-complex structure I . Further, suppose a Lie group G has a Hamiltonian action on (M, ω, I) with corresponding equivariant moment map $\mu : M \rightarrow \mathfrak{g}^*$.

The symplectic reduction of (M, ω) by G is the space

$$\tilde{M} := \mu^{-1}(0)/G.$$

Let $V \subset T_x M$ be the subspace defined of those $v \in T_x M$ that occur as tangent vectors in a vector field \tilde{X} associated to some $X \in \mathfrak{g}$ under the G action. The equivariance of μ implies that $\mu^{-1}(0)$ is symplectically orthogonal to G orbits, and this means that

$$T_{[x]}\tilde{M} = V^\perp/V$$

where V^\perp is the symplectic complement. But since $\omega = g \cdot I$, $V^\perp = (I \cdot V)^{\perp_g}$ where \perp_g denotes the Riemannian orthogonal complement with respect to g . Thus

$$T_{[x]}\tilde{M} = (I \cdot V)^{\perp_g}/V \cong (I \cdot V)^{\perp_g} \cap V^{\perp_g}.$$

This implies that $T_{[x]}\tilde{M}$ is I -invariant when considered as a subspace of $T_x M$ in this way, so the endomorphism I descends to \tilde{M} .

Now, if the action of G on (M, ω) is isometric, then I on the quotient \tilde{M} will be compatible with the induced Riemannian metric g and the induced symplectic form ω . One also therefore obtains integrability of the induced almost-complex structure, so we obtain:

Theorem 3.6.1. *Let (M, ω, I) be a Kähler manifold with an isometric, Hamiltonian action by a Lie group G . Suppose $\mu : M \rightarrow \mathfrak{g}^*$ is the corresponding equivariant moment map. Then whenever the quotient $\tilde{M} := \mu^{-1}(0)/G$ is a manifold, it inherits the structure of a Kähler manifold from M . The manifold \tilde{M} is called the Kähler reduction of M by G .*

Notice that the action of $U(1)$ on $\mathbb{C}^{n+1} \setminus \{0\}$ given by multiplication is an isometry of $\mathbb{C}^{n+1} \setminus \{0\}$. Thus the Fubini-Study form constructed via the symplectic reduction of $\mathbb{C}^{n+1} \setminus \{0\}$ by $U(1)$ is in fact a Kähler form on the quotient $\mathbb{C}\mathbb{P}^n$.

3.7 Hodge Theory

3.7.1 Hodge Theory in Riemannian Geometry

We will first review the Hodge theory of compact oriented Riemannian manifolds. Let M be such a manifold of dimension n . Then M comes equipped with a Hodge

star operator $\star : \Omega^*(M) \rightarrow \Omega^*(M)$ which takes k -forms to $(n - k)$ -forms. If $d\text{vol}$ is the Riemannian volume form on M then the Hodge star can be uniquely characterised by requiring

$$\alpha \wedge \star\beta = \langle \alpha, \beta \rangle d\text{vol}$$

for k -forms $\alpha, \beta \in \Omega^k(M)$, where $\langle \cdot, \cdot \rangle$ is the fibrewise inner product on the bundle of k -forms induced by the Riemannian metric on M .

Using this Hodge star (or otherwise), one can define an L^2 -inner product on the vector spaces $\Omega^k(M)$ for each k , by defining

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \alpha \wedge \star\beta = \int_M \langle \alpha, \beta \rangle d\text{vol}.$$

With respect to these inner products, one obtains a formal adjoint $d^* : \Omega^k(M) \rightarrow \Omega^{k-1}(M)$ of the exterior derivative $d : \Omega^k(M) \rightarrow \Omega^{k+1}(M)$, and this can be characterised in terms of the Hodge star by

$$d^* = (-1)^{n(k-1)+1} \star d \star.$$

The sign here can be determined by observing that $\star\star = (-1)^{k(n-k)}$ and $d^* = (-1)^k \star^{-1} d \star$.

Using the adjoint d^* , we now define the d -Laplacian $\Delta : \Omega^k(M) \rightarrow \Omega^k(M)$ by

$$\Delta := dd^* + d^*d.$$

The operator Δ is an elliptic operator on the complex $\Omega^\bullet(M)$, and a form $\alpha \in \Omega^k(M)$ is called *harmonic* if $\Delta(\alpha) = 0$. A form is harmonic if and only if $d\alpha = d^*\alpha = 0$, so if we define $\mathcal{H}^k(M) := \ker(\Delta : \Omega^k(M) \rightarrow \Omega^k(M))$ then there is a well-defined map $\mathcal{H}^k(M) \rightarrow H^k(M, \mathbb{R})$. The Hodge theorem can be stated as follows:

Theorem 3.7.1 (Hodge Theorem). *There is an isomorphism*

$$\mathcal{H}^k(M) \cong H^k(M, \mathbb{R})$$

induced by the inclusion of harmonic forms into $\Omega^k(M)$. Furthermore, there is a decomposition

$$\Omega^k(M) = \mathcal{H}^k(M) \oplus d(\Omega^{k-1}(M)) \oplus d^*(\Omega^{k+1}(M)) = \mathcal{H}^k(M) \oplus \Delta(\Omega^k(M))$$

which is orthogonal with respect to the L^2 -inner product on $\Omega^k(M)$.

For a proof of the Hodge theorem using the general theory of elliptic complexes see [WGP80], and for a proof using analysis on tori see [GH78].

This fundamental theorem has several important consequences for the de Rham cohomology of compact smooth manifolds. Firstly, since the operator Δ is elliptic, in particular it is Fredholm. Fredholm operators are characterised by the finite-dimensionality of their kernels and cokernels, hence:

Corollary 3.7.2. *The de Rham cohomology of a compact oriented smooth manifold is finite-dimensional.*

Now note that since $\star\Delta = \Delta\star$, if a form α is harmonic, then so is $\star\alpha$. If we introduce a bilinear pairing

$$([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta$$

on $H^k(M, \mathbb{R}) \times H^{n-k}(M, \mathbb{R})$ (which is well-defined by Stokes' theorem), then we have:

Corollary 3.7.3 (Poincaré Duality). *The bilinear pairing above is non-degenerate, and therefore gives a linear isomorphism*

$$H^k(M, \mathbb{R}) \cong H^{n-k}(M, \mathbb{R})^*.$$

In particular, $\dim H^k(M, \mathbb{R}) = \dim H^{n-k}(M, \mathbb{R})$.

Proof. Let $[\alpha]$ be a class in $H^k(M, \mathbb{R})$, and suppose α is the unique harmonic representative of this class. Then $[\star\alpha]$ is a class in $H^{n-k}(M, \mathbb{R})$, and

$$([\alpha], [\star\alpha]) = \int_M \alpha \wedge \star\alpha = \langle \alpha, \alpha \rangle_{L^2} \geq 0,$$

with equality if and only if $[\alpha] = 0$. In particular if $[\alpha]$ is such that $([\alpha], [\beta]) = 0$ for all $[\beta]$ then $[\alpha] = 0$. \square

The above Hodge theory works identically when working with complex differential forms and complex de Rham cohomology $H^*(M, \mathbb{C})$.

3.7.2 Hodge Theory for Complex Manifolds

Let M be a compact complex manifold of complex dimension n . Then in particular M is oriented and smooth of dimension $2n$, and may be equipped with a Riemannian metric. We will now investigate Hodge theory in this setting.

Note that the Hodge star operator acts on the Dolbeault complex as $\star : \Omega^{p,q}(M) \rightarrow \Omega^{n-q, n-p}(M)$. Since complex conjugation interchanges the bidegrees, if one defines $\bar{\star}\alpha := \star\bar{\alpha}$ then $\bar{\star} : \Omega^{p,q}(M) \rightarrow \Omega^{n-p, n-q}(M)$ so one can obtain a Hermitian L^2 -inner product on the vector spaces $\Omega^{p,q}(M)$ by defining

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \alpha \wedge \bar{\star}\beta.$$

Furthermore, suppose one augments the forms of type (p, q) by tensoring with sections of a holomorphic vector bundle \mathcal{E} . If \mathcal{E} has a Hermitian metric h , then this provides a conjugate-linear isomorphism $\mathcal{E} \cong \mathcal{E}^*$, denoted τ say. Define $\bar{\star}$ on $\Omega^{p,q}(M, \mathcal{E})$ by $\bar{\star}_{\mathcal{E}}(\omega \otimes s) := \star \bar{\omega} \otimes \tau(s)$. Contracting sections of \mathcal{E} and \mathcal{E}^* inside the integral we can therefore obtain an L^2 -inner product on $\Omega^{p,q}(M, \mathcal{E})$ by the same formula

$$\langle \alpha, \beta \rangle_{L^2} := \int_M \alpha \wedge \bar{\star}_{\mathcal{E}} \beta.$$

Using such L^2 -inner products, one may define adjoints $\bar{\partial}^*$, ∂^* , and $\bar{\partial}_E^*$ for the operators $\bar{\partial}, \partial$ on $\Omega^{p,q}(M)$ and for $\bar{\partial}_E$ on $\Omega^{p,q}(M, E)$. In particular one may define Laplacians

$$\square := \partial \partial^* + \partial^* \partial, \quad \bar{\square} := \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}$$

on $\Omega^{p,q}(M)$ and

$$\bar{\square}_E := \bar{\partial}_E \bar{\partial}_E^* + \bar{\partial}_E^* \bar{\partial}_E$$

on $\Omega^{p,q}(M, \mathcal{E})$.

Let $\mathcal{H}_{\bar{\partial}}^{p,q}$ and $\mathcal{H}_{\bar{\partial}_E}^{p,q}$ denote the \square and $\bar{\square}$ -harmonic forms of type (p, q) respectively. With this set up Hodge theory now says the following.

Theorem 3.7.4 (Hodge Theorem for Dolbeault Cohomology). *There are isomorphisms*

$$\mathcal{H}_{\bar{\partial}}^{p,q} \cong H_{\bar{\partial}}^{p,q}(M), \quad \mathcal{H}_{\bar{\partial}_E}^{p,q} \cong H_{\bar{\partial}_E}^{p,q}(M)$$

for each bidegree (p, q) . Furthermore there are corresponding decompositions of the forms of type (p, q) analogous to the Riemannian Hodge decomposition.

As a corollary we see that the Dolbeault cohomology of a compact complex manifold is finite-dimensional. Furthermore, if we denote by $\mathcal{H}_{\bar{\partial}_E}^{p,q}$ the space of $\bar{\square}_E$ -harmonic sections in $\Omega^{p,q}(M, \mathcal{E})$ then we have:

Theorem 3.7.5 (Hodge Theorem for Dolbeault Cohomology with Coefficients). *There are isomorphisms*

$$\mathcal{H}_{\bar{\partial}_E}^{p,q} \cong H_{\bar{\partial}_E}^{p,q}(M)$$

for every bidegree (p, q) . Furthermore there is a corresponding decomposition of sections analogous to the Riemannian Hodge decomposition.

The analogue of Poincaré duality for Hodge theory on compact complex manifolds is *Serre duality*. Notice that in the construction of the L^2 -inner product we used the Hermitian inner product on \mathcal{E} in the pairing. This pairing gives a conjugate-linear isomorphism $\mathcal{E} \cong \mathcal{E}^*$, or in other words a complex-linear isomorphism $\bar{\mathcal{E}} \cong \mathcal{E}^*$.

Theorem 3.7.6 (Serre Duality). *Let M be a compact complex manifold, and $\mathcal{E} \rightarrow M$ be a holomorphic vector bundle. Then there is a complex linear isomorphism*

$$H_{\bar{\partial}_E}^{p,q}(M, \mathcal{E}) \cong H_{\bar{\partial}_{E^*}}^{n-p, n-q}(M, \mathcal{E}^*)^*.$$

Proof. Let $\alpha \in \mathcal{H}_{\bar{\partial}_E}^{p,q}$. Then $\bar{\kappa}_E \alpha \in \mathcal{H}_{\bar{\partial}_{E^*}}^{n-p, n-q}$ and

$$(\alpha, \bar{\kappa}_E \alpha) = \langle \alpha, \alpha \rangle_{L^2} \geq 0,$$

with equality if and only if $\alpha = 0$. Thus the complex bilinear pairing between these two spaces is non-degenerate. \square

To obtain the statement of Serre duality in its more familiar form, we use Dolbeault's theorem:

$$H^q(M, \mathcal{E}) \cong H^{n-q}(M, \mathcal{E}^* \otimes K)^*$$

Here we have restricted to the case $p = 0$, identifying the Dolbeault cohomology groups with their sheaf-theoretic counterparts, and observing that $\Omega^n(\mathcal{E}^*)$ consists of the holomorphic sections of $\mathcal{E}^* \otimes K$ where K is the canonical bundle of M .

3.7.3 Hodge Theory for Kähler Manifolds

In general, there is no explicit relationship between the Dolbeault cohomology groups $H_{\bar{\partial}}^{p,q}(M)$ and the de Rham cohomology groups $H^k(M, \mathbb{C})$ on a complex manifold M . Indeed the Dolbeault cohomology groups depend on the choice of complex structure on M , and two diffeomorphic complex manifolds that are not bi-holomorphic need not have the same Dolbeault cohomology groups, despite having the same de Rham cohomology.

However, on a Kähler manifold, the strong relationship between the Riemannian structure of M and the complex structure has consequences for the Dolbeault and de Rham groups.

Theorem 3.7.7. *On a Kähler manifold, the Laplacians \square and $\bar{\square}$ are real, and furthermore*

$$\Delta = 2\square = 2\bar{\square}.$$

For a proof of this fact, see [WGP80]. As a consequence, if a form α is Δ -harmonic, so that $\alpha \in \mathcal{H}^k$, then α is also $\bar{\square}$ -harmonic. Thus under the direct sum decomposition $\alpha = \sum_{p+q=k} \alpha^{p,q}$ of α into bidegrees, we have each $\alpha^{p,q}$ is the unique $\bar{\square}$ -harmonic representative of a class in $H_{\bar{\partial}}^{p,q}(M)$. We immediately obtain:

Theorem 3.7.8 (Hodge Decomposition Theorem). *Let M be a compact Kähler manifold. Then*

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H_{\bar{\partial}}^{p,q}(M).$$

Since any Kähler manifold is symplectic, we already have some topological obstructions to the existence of Kähler metrics on compact complex manifolds. In particular de Rham cohomology must not vanish in even degrees. The Hodge decomposition gives further obstructions to the existence of Kähler metrics. Notably, $h^{p,q} = h^{q,p}$ on any complex manifold due to complex conjugation. Since the de Rham cohomology decomposes as a direct sum, this implies that $\dim H^1(M, \mathbb{C}) = h^{1,0} + h^{0,1}$ is even. Indeed a Kähler manifold must have even Betti numbers in all odd degrees.

In fact the Kähler condition also implies further cohomological restrictions on M , stemming from the fact that $[\omega \wedge -]$ is an isomorphism $H_{\bar{\partial}}^{p,q}(M) \rightarrow H_{\bar{\partial}}^{p+1,q+1}(M)$ for all (p, q) . This fact combined with the representation theory of $\mathfrak{sl}(2, \mathbb{C})$ allows one to prove the Lefschetz decomposition of the cohomology of a compact Kähler manifold. For more details on the cohomological implications of the Kähler condition see [WGP80] and [Bal06].

3.8 Major Theorems

In this section we will collect some important theorems about holomorphic vector bundles over complex manifolds. In order to state several of these theorems, it will be convenient to introduce the holomorphic Euler characteristic of a holomorphic vector bundle.

Definition 3.8.1. *The holomorphic Euler characteristic $\chi(\mathcal{E})$ of a holomorphic vector bundle $\mathcal{E} \rightarrow M$ over a complex manifold M of dimension n is*

$$\chi(\mathcal{E}) := \sum_{i=0}^n (-1)^i \dim H^i(M, \mathcal{E}),$$

provided the cohomology groups $H^i(M, \mathcal{E})$ are all finite-dimensional.

We also use the definition that $\deg(\mathcal{E}) = \int_{\Sigma_g} c_1(\mathcal{E})$ for holomorphic vector bundles over a surface.

Theorem 3.8.2 (Riemann-Roch). *Let $\mathcal{L} \rightarrow \Sigma_g$ be a holomorphic line bundle over a compact Riemann surface Σ_g . Then*

$$\chi(\mathcal{L}) = \deg(\mathcal{L}) - g + 1.$$

Theorem 3.8.3 (Riemann-Roch for Vector Bundles). *Let $\mathcal{E} \rightarrow \Sigma_g$ be a holomorphic vector bundle of rank n over a compact Riemann surface Σ_g . Then*

$$\chi(\mathcal{E}) = \deg(\mathcal{E}) + n(1 - g).$$

Theorem 3.8.4 (Kodaira Vanishing Theorem). *Let $\mathcal{L} \rightarrow M$ be a holomorphic line bundle over a compact complex manifold M . Then if $\mathcal{L} \otimes K^*$ is positive, in the sense that $c_1(\mathcal{L} \otimes K^*)$ is represented by a Kähler form on M , then $H^i(M, \mathcal{L}) = 0$ for $i > 0$.*

Theorem 3.8.5 (Hirzebruch-Riemann-Roch). *Let $\mathcal{E} \rightarrow M$ be a holomorphic vector bundle over a compact complex manifold M . Then*

$$\chi(\mathcal{E}) = \int_M \text{Ch}(\mathcal{E}) \text{Td}(M).$$

Note of course that the Hirzebruch-Riemann-Roch theorem implies the regular Riemann-Roch theorem. In the case of a vector bundle over a compact Riemann surface we have $\text{Ch}(\mathcal{E}) = \text{tr}(\exp(c_1(\mathcal{E}))) = n + c_1(\mathcal{E})$ and $\text{Td}(\Sigma_g) = 1 + \frac{1}{2}c_1(\Sigma_g)$. Then using the fact that $\int_{\Sigma_g} c_1(\mathcal{E}) = \deg(\mathcal{E})$ and $\int_{\Sigma_g} c_1(\Sigma_g) = 2 - 2g$ we obtain

$$\chi(\mathcal{E}) = \int_{\Sigma_g} \left(\frac{n}{2}c_1(\Sigma_g) + c_1(\mathcal{E}) \right) = \deg(\mathcal{E}) + n(1 - g).$$

Part II

Moduli Spaces of Bundles

Chapter 4

Stable Bundles over Riemann Surfaces

In this chapter we will investigate the classification of vector bundles over a Riemann surface. In the case of smooth complex vector bundles, the classification is discrete, with such bundles being determined by their rank and degree, two numbers taking values in the positive integers and integers respectively. When holomorphic structures are added however, smoothly isomorphic vector bundles become distinct. In place of a discrete classification of holomorphic bundles, which no longer exists, one must pass to a *moduli space*. That is, an algebraic variety or manifold that classifies holomorphic bundles up to a suitable notion of equivalence.

The classification of holomorphic bundles on particular compact Riemann surfaces was first carried out by Grothendieck in [Gro57] for the case of $\mathbb{C}P^1$, where it was shown that holomorphic vector bundles are simply direct sums of line bundles. The case of elliptic curves was then investigated by Atiyah in [Ati57b], and in [Mum63] Mumford began the study of genus greater than one by showing that, after restricting to a sub-class of *stable* bundles, the moduli space $\mathcal{N}^{n,d}$ of stable holomorphic vector bundles (of fixed rank n and degree d) has the structure of a quasi-projective algebraic variety. The properties of this space were famously illuminated by Narasimhan and Seshadri in [NS65], who showed that a holomorphic bundle is stable precisely if it arises from a projective irreducible unitary representation of the fundamental group of the surface. In [New67] Newstead used this description to begin the investigation of the topology of the spaces $\mathcal{N}^{n,d}$, computing the Betti numbers of $\mathcal{N}^{2,1}$. Later using number-theoretic techniques, Harder and Narasimhan applied the then recently proved Weil conjectures to compute the Betti numbers for arbitrary n and d in [HN75] by counting points over finite fields.

In the early 1980's, the work of Donaldson, and of Atiyah and Bott gave the study of these moduli spaces a new gauge-theoretic flavour. In [Don83] Donaldson framed the result of Narasimhan and Seshadri in a new light, identifying

representations of the fundamental group with projectively flat connections on the underlying smooth complex vector bundle. Using recently proven results of Uhlenbeck, Donaldson showed that stable holomorphic vector bundles on a Riemann surface corresponded to solutions of the Yang-Mills equations. Atiyah and Bott expanded on this theme significantly in [AB83], combining techniques of Morse theory, differential and algebraic geometry, and gauge theory to study the topology and geometry of $\mathcal{N}^{n,d}$. In particular the description of stable bundles in terms of solutions to the Yang-Mills equations gives rise to a description of $\mathcal{N}^{n,d}$ as an infinite-dimensional Kähler reduction, and the theorem of Narasimhan and Seshadri, as proved by Donaldson, may be interpreted as a kind of infinite-dimensional Kempf-Ness theorem for this quotient.

We will start out by classifying holomorphic line bundles on a Riemann surface. After showing that the smooth vector bundles on a Riemann surface are classified by their rank and degree, both discrete, we will give an account of the construction of the moduli space $\mathcal{N}^{n,d}$ for $n > 1$ in terms of Kähler reduction. An understanding of $\mathcal{N}^{n,d}$ will serve as the base case for understanding the geometric quantization of the moduli space of $\mathcal{M}^{n,d}$ of stable *Higgs* bundles over a surface.

4.1 Classification of Line Bundles on a Riemann Surface

4.1.1 Smooth Case

Let $L \rightarrow \Sigma$ be a smooth complex line bundle on a Riemann surface Σ_g of genus g . Let $C^\infty(\mathbb{C})$ (resp. $C^\infty(\mathbb{C}^*)$) denote the sheaf of smooth \mathbb{C} -valued (resp. \mathbb{C}^* -valued) functions on Σ_g . Since $\mathrm{GL}(1, \mathbb{C}) \cong \mathbb{C}^*$, smooth complex line bundles are classified up to isomorphism by their representative in $H^1(\Sigma, C^\infty(\mathbb{C}^*))$.

Let $\underline{\mathbb{Z}}$ denote the constant \mathbb{Z} -valued sheaf on Σ_g . Consider the short exact sequence sheaf sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow C^\infty(\mathbb{C}) \xrightarrow{\exp} C^\infty(\mathbb{C}^*) \longrightarrow 0,$$

where $\exp : C^\infty(\mathbb{C}) \rightarrow C^\infty(\mathbb{C}^*)$ sends f to $\exp(2\pi i f)$.

This induces a long exact sequence in sheaf cohomology, which takes the form

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^0(\Sigma_g, \underline{\mathbb{Z}}) & \longrightarrow & H^0(\Sigma_g, C^\infty(\mathbb{C})) & \longrightarrow & H^0(\Sigma_g, C^\infty(\mathbb{C}^*)) \\
 & & & & \longleftarrow & & \longleftarrow \\
 & & H^1(\Sigma_g, \underline{\mathbb{Z}}) & \longrightarrow & H^1(\Sigma_g, C^\infty(\mathbb{C})) & \longrightarrow & H^1(\Sigma_g, C^\infty(\mathbb{C}^*)) \\
 & & & & \longleftarrow & & \longleftarrow \\
 & & H^2(\Sigma_g, \underline{\mathbb{Z}}) & \longrightarrow & H^2(\Sigma_g, C^\infty(\mathbb{C})) & \longrightarrow & H^2(\Sigma_g, C^\infty(\mathbb{C}^*)) \longrightarrow 0.
 \end{array}$$

Now $C^\infty(\mathbb{C})$ is a fine sheaf, so $H^i(\Sigma, C^\infty(\mathbb{C})) = 0$ for all $i > 0$. Thus we obtain the short exact sequence

$$0 \longrightarrow H^1(\Sigma_g, C^\infty(\mathbb{C}^*)) \longrightarrow H^2(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow 0.$$

But $H^2(\Sigma_g, \underline{\mathbb{Z}}) \cong \mathbb{Z}$, from which we conclude:

Theorem 4.1.1. *Isomorphism classes of smooth complex line bundles over a Riemann surface are in bijection with the integers.*

Label the map $H^1(\Sigma_g, C^\infty(\mathbb{C})) \rightarrow \mathbb{Z}$ by deg . The image of a line bundle $L \in H^1(\Sigma_g, C^\infty(\mathbb{C}^*))$ under deg is called the *degree* of the line bundle. Thus the theorem above states that smooth complex line bundles on a surface are classified by their degree.

Furthermore, the above isomorphism is one of *groups*. The group structure on $H^1(\Sigma_g, C^\infty(\mathbb{C}^*))$ is by multiplication of transition functions on overlaps $U_{\alpha\beta}$. In the case of line bundles, this is the same as the tensor product of transition functions, as per the construction of the tensor product bundle, from which we conclude:

Corollary 4.1.2.

$$(\{\text{Isomorphism classes of smooth complex line bundles on } \Sigma_g\}, \otimes) \cong (\mathbb{Z}, +)$$

as groups.

Remark 4.1.3. The degree of a smooth complex line bundle here may be interpreted as the integral of $c_1(L)$ over the surface Σ_g . Further it can be identified with the number of zeros (counted with multiplicity) of a section of L which intersects the zero-section transversally. For proofs of these facts see Chapters 1 and 2 of [BT95].

4.1.2 Holomorphic Case

Consider the exponential sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \hookrightarrow \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \longrightarrow 0.$$

This induces a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(\Sigma_g, \underline{\mathbb{Z}}) & \longrightarrow & H^0(\Sigma_g, \mathcal{O}) & \longrightarrow & H^0(\Sigma_g, \mathcal{O}^*) \\ & & & & & & \downarrow \\ & & & & & & H^1(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow H^1(\Sigma_g, \mathcal{O}) \longrightarrow H^1(\Sigma_g, \mathcal{O}^*) \\ & & & & & & \downarrow \\ & & & & & & H^2(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow H^2(\Sigma_g, \mathcal{O}) \longrightarrow H^2(\Sigma_g, \mathcal{O}^*) \longrightarrow 0. \end{array}$$

Since Σ_g has complex dimension one, the group $H^2(\Sigma_g, \mathcal{O}) \cong H_{\bar{\partial}}^{0,2}(\Sigma_g) = 0$. Since Σ_g is compact, global holomorphic sections in \mathcal{O} or \mathcal{O}^* are constant. Thus $H^0(\Sigma_g, \mathcal{O}) \cong \mathbb{C}$ and $H^0(\Sigma_g, \mathcal{O}^*) \cong \mathbb{C}^*$. But then taking a logarithm, the map $H^0(\Sigma_g, \mathcal{O}) \rightarrow H^0(\Sigma_g, \mathcal{O}^*)$ is surjective. Since the sequence is exact, we must have $H^0(\Sigma_g, \mathcal{O}^*) \rightarrow H^1(\Sigma_g, \underline{\mathbb{Z}})$ has full kernel. In particular, the image of this map is zero, so we get a reduction of our sequence to

$$0 \longrightarrow H^1(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow H^1(\Sigma_g, \mathcal{O}) \longrightarrow H^1(\Sigma_g, \mathcal{O}^*) \longrightarrow H^2(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow 0.$$

By the first isomorphism theorem, the image of $H^1(\Sigma_g, \mathcal{O}) \rightarrow H^1(\Sigma_g, \mathcal{O}^*)$ is isomorphic to the quotient $H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbb{Z})$, so we have a reduction to the short exact sequence

$$0 \longrightarrow H^1(\Sigma, \mathcal{O})/H^1(\Sigma, \mathbb{Z}) \longrightarrow H^1(\Sigma, \mathcal{O}^*) \longrightarrow H^2(\Sigma, \underline{\mathbb{Z}}) \longrightarrow 0.$$

Now $H^1(\Sigma, \underline{\mathbb{Z}}) \cong \mathbb{Z}^{2g}$, and $H^1(\Sigma, \mathcal{O}) \cong \mathbb{C}^g$, and we also know that $H^2(\Sigma, \underline{\mathbb{Z}}) \cong \mathbb{Z}$. Thus we obtain the exact sequence

$$0 \longrightarrow \mathbb{C}^g/\mathbb{Z}^{2g} \longrightarrow H^1(\Sigma, \mathcal{O}^*) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

Remark 4.1.4. Note that the map $H^1(\Sigma_g, \underline{\mathbb{Z}}) \rightarrow H^1(\Sigma_g, \mathcal{O})$ embeds \mathbb{Z}^{2g} inside \mathbb{C}^g as a lattice. Indeed if one considers the homomorphism of short exact sequences from $\underline{\mathbb{Z}} \rightarrow \underline{\mathbb{C}} \rightarrow \underline{\mathbb{C}^*}$ to $\underline{\mathbb{Z}} \rightarrow \mathcal{O} \rightarrow \mathcal{O}^*$ then the corresponding triangle in the long exact sequence is

$$\begin{array}{ccc}
 H^1(\Sigma_g, \mathbb{Z}) & \longrightarrow & H^1(\Sigma_g, \mathbb{C}) \\
 & \searrow & \downarrow \\
 & & H^1(\Sigma_g, \mathcal{O})
 \end{array}$$

which commutes, where the vertical projection is given by the projection

$$H^1(\Sigma_g, \mathbb{C}) \rightarrow H_{\bar{\partial}}^{0,1}(\Sigma_g)$$

from the Hodge decomposition $H^1(\Sigma_g, \mathbb{C}) = H_{\bar{\partial}}^{1,0}(\Sigma_g) \oplus H_{\bar{\partial}}^{0,1}(\Sigma_g)$. Since the diagonal map is injective, and the horizontal map is $\mathbb{Z}^{2g} \rightarrow \mathbb{Z}^{2g} \otimes_{\mathbb{Z}} \mathbb{C}$, it must be the case that the diagonal map embeds \mathbb{Z}^g as a lattice in $\mathbb{C}^g = H^1(\Sigma_g, \mathcal{O})$.

Let $L \in H^1(\Sigma_g, \mathcal{O}^*)$ be a line bundle. Then the image under the map $H^1(\Sigma_g, \mathcal{O}^*) \rightarrow \mathbb{Z}$ is again called the degree $\deg(L)$ of L . To show that this is the same notion of degree as before, consider the homomorphism of short exact sequences of sheaves given by

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & \underline{\mathbb{Z}} & \hookrightarrow & \mathcal{O} & \xrightarrow{\text{exp}} & \mathcal{O}^* & \longrightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \underline{\mathbb{Z}} & \hookrightarrow & C^\infty(\mathbb{C}) & \xrightarrow{\text{exp}} & C^\infty(\mathbb{C}^*) & \longrightarrow & 0.
 \end{array}$$

This induces a homomorphism of the corresponding long exact cohomology sequences. The relevant part of this homomorphism of sequences is the following commutative diagram:

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & H^1(\Sigma_g, \mathcal{O}) & \longrightarrow & H^1(\Sigma_g, \mathcal{O}^*) & \longrightarrow & H^2(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \parallel \\
 \dots & \longrightarrow & 0 & \longrightarrow & H^1(\Sigma_g, C^\infty(\mathbb{C}^*)) & \xrightarrow{\cong} & H^2(\Sigma_g, \underline{\mathbb{Z}}) \longrightarrow 0.
 \end{array}$$

This last square shows that two holomorphic line bundles have the same smooth degree if and only if they have the same holomorphic degree. Thus one simply denotes the degree of a line bundle by $\deg(L)$, regardless of whether L is being considered as smooth or holomorphic.

It follows from the above diagram that the group $\text{Jac}(\Sigma)$ of all line bundles of degree zero is isomorphic to the torus

$$\text{Jac}(\Sigma_g) \cong \mathbb{C}^g / \mathbb{Z}^{2g}.$$

It follows that the set $\text{Pic}(\Sigma)$ of all holomorphic line bundles over Σ up to isomorphism is isomorphic as a group to $\text{Jac}(\Sigma_g) \times \mathbb{Z}$, a countable disjoint union of $2g$ -tori. In particular if $\text{Pic}_d(\Sigma)$ denotes the set of all holomorphic line bundles of a fixed degree d , then $\text{Pic}_d(\Sigma) \cong \text{Jac}(\Sigma_g) \cong \mathbb{C}^g / \mathbb{Z}^{2g}$ for all $d \in \mathbb{Z}$.

4.1.3 Flat Case

The group $H^1(\Sigma_g, \mathbb{C}^*)$ classifies *flat* complex line bundles on a Riemann surface Σ_g . These are the smooth complex line bundles that admit a trivialisation with constant transition functions. Equivalently, flat line bundles are those admitting a flat connection.

There exists an isomorphism from the sheaf cohomology groups $H^i(\Sigma_g, \underline{G})$ for some Abelian group G with the singular cohomology groups $H^i(\Sigma_g, G)$. By the universal coefficients theorem, $H^i(\Sigma_g, G) \cong H^i(\Sigma_g, \mathbb{Z}) \otimes_{\mathbb{Z}} G$. Thus we have $H^1(\Sigma_g, \mathbb{C}^*) \cong H^1(\Sigma_g, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^{2g}$.

A flat complex line bundle gives rise to a representation of the fundamental group. Namely, if one takes a flat connection on such a line bundle, then its holonomy depends only on the homotopy classes of loops, and defines a homomorphism $\pi_1(\Sigma_g) \rightarrow \mathbb{C}^*$. Since \mathbb{C}^* is Abelian and $\pi_1(\Sigma_g)$ is generated by $2g$ loops,

$$\text{Hom}(\pi_1(\Sigma_g), \mathbb{C}^*) / \mathbb{C}^* = (\mathbb{C}^*)^{2g} \cong H^1(\Sigma_g, \mathbb{C}^*).$$

One may carry out the same discussion in the case of flat *unitary* line bundles. Here one obtains a torus $U(1)^{2g}$ classifying such line bundles, which may be identified with the corresponding space of representations of $\pi_1(\Sigma_g)$. Since the torus $U(1)^{2g}$ is of the same dimension as the Jacobian, it is natural to ask if there is an identification between them.

Suppose \mathcal{L} is some holomorphic line bundle of degree zero, with a fixed Hermitian metric h . Fix a smooth trivialisation of \mathcal{L} . Then h is some positive real function on Σ_g , and therefore $h = e^f$ for some f . The associated Chern connection has curvature

$$F_{\nabla} = \bar{\partial}(h^{-1}\partial h) = \bar{\partial}\partial f.$$

If $a \in \Omega^1(\Sigma_g)$, then $\nabla + a$ will be unitary whenever a is purely imaginary. It will be flat when

$$\partial\bar{\partial}f = da.$$

By Hodge theory for a compact Riemann surface, a two-form $\partial\bar{\partial}h$ is d -exact, and when h is real the form a such that $da = \partial\bar{\partial}h$ is purely imaginary. Thus one may modify ∇ to obtain a flat unitary connection $\nabla + a$ on L . Under the identification of flat connections with representations of the fundamental group, this gives isomorphisms

$$\text{Jac}(\Sigma_g) \cong \text{Hom}(\pi_1(\Sigma_g), U(1)) \cong U(1)^{2g}.$$

This result is essentially the theorem of Narasimhan and Seshadri in the case of holomorphic line bundles, and also serves as the base case in the inductive proof of the theorem given by Donaldson. We will discuss the general statement of this theorem in Section 4.4.

The corresponding result for the case of flat *complex* line bundles requires the introduction of holomorphic Higgs bundles, objects we will see in Chapter 5.

4.2 Classification of Smooth Vector Bundles

Before describing the stable bundle moduli space $\mathcal{N}^{n,d}$, we should justify the process of discretizing the classification problem by the two integers n and d . In the following section we will show that these two integers classify a vector bundle over a Riemann surface smoothly. Since any two biholomorphic vector bundles are also smoothly (and topologically) isomorphic, there is no loss in restricting to studying biholomorphic bundles of a fixed topological type. This also serves the purpose of reducing the full classification process into more manageable bites.

Definition 4.2.1. *Let $E \rightarrow \Sigma_g$ be a smooth (or holomorphic) complex vector bundle of rank n on a Riemann surface Σ_g . The degree of E , denoted $\deg(E)$, is defined to be the degree of the smooth (or holomorphic) line bundle $\wedge^n E$.*

Lemma 4.2.2. *Let $E \rightarrow \Sigma_g$ be a smooth complex vector bundle of rank n on a Riemann surface Σ_g of genus g . Then if $s \in \Gamma(\Sigma_g, E)$ is a non-vanishing section, $E \cong E' \oplus (\mathbb{C} \times \Sigma_g)$ where E' has rank $n - 1$.*

Proof. Since s is non-zero, one can define a sub-bundle $I := \text{Span}\{s\}$. This sub-bundle admits a global non-vanishing section (namely s) so $I \cong \mathbb{C} \times \Sigma_g$. Take any metric on E , and let $E' := I^\perp$. Then E' has rank $n - 1$ and $E \cong E' \oplus (\mathbb{C} \times \Sigma_g)$. \square

Lemma 4.2.3. *Let $E \rightarrow \Sigma_g$ be a smooth complex vector bundle of rank n on a Riemann surface Σ_g . If $n > 1$ there exists a non-vanishing section.*

Proof. Let $s \in \Gamma(\Sigma_g, E)$ be any section. Then s defines a smooth 2-dimensional submanifold of the smooth $(2n + 2)$ -dimensional manifold E . The zero section defines another smooth 2-dimensional submanifold. Since $n > 1$, these two 2-dimensional submanifolds meet inside a space of dimension at least 5. In particular the section s may be smoothly homotoped to a perturbed section s' that no longer intersects the zero section. \square

Theorem 4.2.4. *Smooth complex vector bundles on a Riemann surface Σ_g are classified up to isomorphism by their rank and degree.*

Proof. Let $E \rightarrow \Sigma_g$ be a smooth complex vector bundle of rank n . If $n = 1$ then E is classified by its degree.

Suppose $n > 1$. Then by Lemma 4.2.3 E has a non-vanishing section s . By Lemma 4.2.2 then $E \cong E' \oplus (\mathbb{C} \times \Sigma_g)$ where E' has rank $n - 1$. Repeating this

process a total of $n - 1$ times we obtain $E \cong L \oplus (\mathbb{C}^{n-1} \times \Sigma_g)$ for some smooth complex line bundle L of some degree $d := \deg(L)$.

We claim the pair (n, d) classifies E up to isomorphism. First we will show d does not depend on the particular choice of deconstruction of E . In particular we have $d = \deg(E)$.

To see this, note that $\bigwedge^n E \cong \bigwedge^n (L \oplus (\mathbb{C}^{n-1} \times \Sigma_g))$. Furthermore, note that the isomorphism $\bigwedge(V \oplus W) \cong \bigwedge(V) \otimes \bigwedge(W)$ of exterior algebras of vector spaces (or of exterior algebra bundles of vector bundles) is graded in the sense that

$$\bigwedge^n (L \oplus (\mathbb{C}^{n-1} \times \Sigma_g)) \cong \bigoplus_{p+q=n} \left(\bigwedge^p L \right) \otimes \left(\bigwedge^q (\mathbb{C}^{n-1} \times \Sigma_g) \right).$$

Since L is rank one, any higher exterior powers of L are the zero vector bundle, so we must have

$$\bigwedge^n E \cong L \otimes \bigwedge^{n-1} (\mathbb{C}^{n-1} \times \Sigma_g) \cong L \otimes (\mathbb{C} \times \Sigma_g) \cong L.$$

But then $\deg(E) = \deg(L) = d$.

Now suppose $F \rightarrow \Sigma_g$ is another vector bundle of rank n with $\deg(F) = \deg(E)$. Then we may similarly deconstruct F as $F \cong L' \oplus (\mathbb{C}^{n-1} \times \Sigma_g)$ for some line bundle L' of degree $\deg(E)$. Then since they have the same degrees, we have an isomorphism $L \cong L'$ of line bundles. Using the identity on the trivial part, we then obtain an isomorphism $E \cong F$. Clearly we also have that if $E \cong F$ then $\deg(E) = \deg(F)$ and the two bundles have the same rank, so we conclude $E \cong F$ if and only if they have the same rank and degree. \square

4.3 Stability of Vector Bundles

In this section we will define stability for holomorphic vector bundles over a compact Riemann surface. This is initially distinct from the definitions given in Section 2.6.2, however we will comment on the relation between the two notions. More details about the results of this section can be found in the original paper [NS65] of Narasimhan and Seshadri, or a more modern treatment such as the lecture notes [Sch13] of Schaffhauser.

Definition 4.3.1 (Slope). *Let $E \rightarrow \Sigma_g$ be a vector bundle over a compact Riemann surface Σ_g . The slope of E is defined to be the rational number*

$$\mu(E) := \frac{\deg(E)}{\text{rk}(E)}.$$

Definition 4.3.2 (Stability). *A holomorphic vector bundle $\mathcal{E} \rightarrow \Sigma_g$ over a compact Riemann surface Σ_g is said to be stable (resp. semi-stable) if, for every proper non-zero holomorphic subbundle $\mathcal{F} \subset \mathcal{E}$, we have*

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \quad (\text{resp. } \leq).$$

This is often referred to as *slope-stability*, but we will simply refer to slope-(semi-)stable bundles as (semi-)stable. Note that every holomorphic line bundle over Σ_g is automatically stable.

Lemma 4.3.3. *If $\text{rk}(\mathcal{E})$ and $\text{deg}(\mathcal{E})$ are coprime, then semi-stability of \mathcal{E} implies stability of \mathcal{E} .*

Proof. Suppose \mathcal{E} is semi-stable with rank and degree coprime. Let \mathcal{F} be any proper holomorphic sub-bundle of \mathcal{E} , and suppose

$$\frac{\text{deg}(\mathcal{F})}{\text{rk}(\mathcal{F})} = \frac{\text{deg}(\mathcal{E})}{\text{rk}(\mathcal{E})}.$$

Since the rank and degree of \mathcal{E} are coprime, the fraction on the right is in simplest form. But $0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$ so the fraction on the left would represent the rational number $\mu(\mathcal{E})$ in strictly simpler form. This is a contradiction, so $\mu(\mathcal{F}) < \mu(\mathcal{E})$. \square

Now we have a series of lemmas about the existence and slopes of certain sub-bundles of stable bundles. Proofs of these lemmas are in the notes [Sch13].

Lemma 4.3.4. *If $u : \mathcal{E} \rightarrow \mathcal{F}$ is a non-zero homomorphism of vector bundles over Σ_g , then $\mu(\mathcal{E}/\ker(u)) \leq \mu(\text{im}(u))$, and equality occurs if and only if $\mathcal{E}/\ker(u) \cong \text{im}(u)$.*

Lemma 4.3.5. *Let \mathcal{E} and \mathcal{F} be semi-stable with $\mu(\mathcal{E}) > \mu(\mathcal{F})$. Then any homomorphism $u : \mathcal{E} \rightarrow \mathcal{F}$ is zero.*

Lemma 4.3.6. *Let $u : \mathcal{E} \rightarrow \mathcal{F}$ be a non-zero homomorphism of semi-stable bundles of the same slope. Then $\ker(u)$ and $\text{im}(u)$ are semi-stable of slope $\mu(\mathcal{E}) = \mu(\mathcal{F})$, and $\mathcal{E}/\ker(u) \cong \text{im}(u)$.*

Using these lemmas, we can determine the automorphism groups of stable holomorphic vector bundles.

Definition 4.3.7 (Simple Bundle). *A holomorphic bundle $\mathcal{E} \rightarrow \Sigma_g$ is simple if $\dim H^0(\Sigma_g, \text{End}(\mathcal{E})) = 1$.*

Lemma 4.3.8. *A non-zero endomorphism $u : \mathcal{E} \rightarrow \mathcal{E}$ of a stable bundle is an isomorphism.*

Proof. Suppose u is such an endomorphism. Then $\ker(u) \neq \mathcal{E}$ by assumption. Since \mathcal{E} is stable and therefore semi-stable, by Lemma 4.3.6 either $\ker u$ is trivial or has slope $\mu(E)$. But $\ker(u) \subset \mathcal{E}$, so it must be the case that $\ker(u) = 0$. The natural map $\mathcal{E} = \mathcal{E}/\ker(u) \rightarrow \text{im}(u)$ is therefore an isomorphism. But $\text{im}(u)$ is a non-zero holomorphic sub-bundle of \mathcal{E} of slope $\mu(E)$, and is therefore equal to \mathcal{E} . \square

Proposition 4.3.9. *A stable bundle $\mathcal{E} \rightarrow \Sigma_g$ is simple.*

Proof. Let $u : \mathcal{E} \rightarrow \mathcal{E}$ be a non-zero endomorphism of \mathcal{E} . Let $x \in \Sigma_g$ be any point and let $\lambda \in \mathbb{C}^*$ be any eigenvalue of $u_x : \mathcal{E}_x \rightarrow \mathcal{E}_x$. Define $\tilde{u} : \mathcal{E} \rightarrow \mathcal{E}$ by $\tilde{u} := u - \lambda \mathbf{1}$. Then \tilde{u} is not an isomorphism because it has non-trivial kernel at $x \in \Sigma_g$, and by Lemma 4.3.8 it is therefore zero. But then $u = \lambda \mathbf{1}$ for some $\lambda \in \mathbb{C}^*$.

Thus $H^0(\Sigma_g, \text{End}(\mathcal{E})) = \mathbb{C}$ and $\text{Aut}(\mathcal{E}) \cong \mathbb{C}^*$. \square

Corollary 4.3.10. *A stable bundle \mathcal{E} is indecomposable.*

Proof. Suppose \mathcal{E} was a direct sum $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$. Then $\mathbb{C}^* \times \mathbb{C}^* \subseteq \text{Aut}(\mathcal{E}_1 \oplus \mathcal{E}_2)$, but \mathcal{E} has automorphisms \mathbb{C}^* . This is a contradiction, so \mathcal{E} is not a direct sum of sub-bundles. \square

Theorem 4.3.11 (Jordan-Hölder Filtration). *Any semi-stable holomorphic bundle $\mathcal{E} \rightarrow \Sigma_g$ admits a filtration by holomorphic sub-bundles*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_k = \mathcal{E}$$

called the Jordan-Hölder filtration, such that each quotient $\mathcal{E}_i/\mathcal{E}_{i-1}$ is stable, and $\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu(\mathcal{E})$ for all i . Moreover, any two Jordan-Hölder filtrations $\{\mathcal{E}_i\}$ and $\{\mathcal{E}'_i\}$ are equivalent in the sense that

$$\mathcal{E}_i/\mathcal{E}_{i-1} \cong \mathcal{E}'_i/\mathcal{E}'_{i-1}$$

for all i , and therefore the associated graded objects

$$\text{gr}(\mathcal{E}) := \bigoplus_i \mathcal{E}_i/\mathcal{E}_{i-1}$$

and

$$\text{gr}(\mathcal{E}') := \bigoplus_i \mathcal{E}'_i/\mathcal{E}'_{i-1}$$

are isomorphic.

Definition 4.3.12 (Polystable). *A holomorphic bundle $\mathcal{E} \rightarrow \Sigma_g$ is polystable if it is a direct sum of stable bundles of the same slope.*

The Jordan-Hölder filtration therefore canonically (up to isomorphism) associates a polystable bundle to any semi-stable bundle - its associated graded bundle.

Definition 4.3.13 (S-Equivalence). *Two semi-stable holomorphic bundles $\mathcal{E}, \mathcal{F} \rightarrow \Sigma_g$ are S-equivalent if $\text{gr}(\mathcal{E}) \cong \text{gr} \mathcal{F}$.*

Notice that the terminology of this section exactly mirrors that of Section 2.6.2. Indeed in the space of all holomorphic vector bundles (which may be identified with $\text{Dol}(E)$), the stable, semi-stable, and polystable objects in the sense of Geometric Invariant Theory may be identified with the conditions of stability, semi-stability, and polystability for holomorphic vector bundles defined here. This was identified first by Mumford in [Mum63]. See [Sai09] for more details on the relation of slope-stability and stability in the sense of GIT.

Remark 4.3.14. There is a more general filtration available for *any* holomorphic bundle $\mathcal{E} \rightarrow \Sigma_g$, the so-called *Harder-Narasimhan filtration*, first discovered by Harder and Narasimhan in [HN75]. In this filtration the successive quotients are semi-stable of strictly increasing slope, and two holomorphic bundles with the same Harder-Narasimhan filtration (up to a suitable equivalence) are said to be of the same Harder-Narasimhan type. These types give a filtration of the full space $\text{Dol}(E)$ of holomorphic bundles (of fixed rank and degree) for which the largest open stratum is precisely the semi-stable bundles. The properties of this filtration were investigated by Atiyah and Bott in [AB83]. In the next section we will introduce the Yang-Mills functional, and the observation of Atiyah and Bott was that the Harder-Narasimhan strata equivariantly deformation retract onto the Morse strata of the Yang-Mills functional (which is an equivariantly perfect Morse function on $\text{Dol}(E) \cong \mathcal{A}(E, h)$), allowing one to determine the topology of the moduli space of stable bundles using equivariant cohomology. The equivalence of these stratifications was proved by Daskalopoulos in [Das92].

4.4 The Stable Bundle Moduli Space

Since a smooth vector bundle over a Riemann surface Σ_g is classified by its rank and degree, one can break up the classification of holomorphic vector bundles over Σ_g by these discrete parameters. A convenient way to do this is to consider a fixed smooth complex vector bundle $E \rightarrow \Sigma_g$ of rank $\text{rk}(E) = n$ and degree $\text{deg}(E) = d$, and consider the possible holomorphic structures on this bundle, for example by considering all possible Dolbeault operators on E . This is without loss of generality, since if E and E' are two smooth complex vector bundles of equal rank and degree, and \mathcal{E} is a holomorphic structure on E , then pulling the holomorphic trivialisation of E through the smooth isomorphism $E \cong E'$ induces

a holomorphic trivialisation on E' with \mathcal{E}' biholomorphic to \mathcal{E} , and every such trivialisation may be obtained in this way.

We have seen that holomorphic structures on a smooth complex vector bundle $E \rightarrow \Sigma_g$ over a compact Riemann surface are given by Dolbeault operators up to complex gauge equivalence. In particular, the set of all holomorphic vector bundles on Σ_g up to isomorphism may be identified with the quotient $\text{Dol}(E)/\mathcal{G}_{\mathbb{C}}$. In fact, this quotient is also a topological space, but being a quotient of an infinite-dimensional space by an infinite-dimensional topological group, it fails to have any nice properties. For example, it is not even Hausdorff. The condition of stability discussed above gives a way of removing the Dolbeault operators in $\text{Dol}(E)$ that prevent this quotient from having a nice topology. In particular, if $\text{Dol}(E)^s$ and $\text{Dol}(E)^{ss}$ denote the subsets of $\text{Dol}(E)$ corresponding to those Dolbeault operators inducing stable and semi-stable holomorphic structures respectively, then $\text{Dol}(E)^s/\mathcal{G}_{\mathbb{C}}$ has the structure of a non-singular quasi-projective complex algebraic variety, and $\text{Dol}(E)^{ss}/\mathcal{G}_{\mathbb{C}}$ is the (possibly singular) projective complex algebraic variety in which $\text{Dol}(E)^s/\mathcal{G}_{\mathbb{C}}$ sits as an open subset. When $(n, d) = 1$, $\text{Dol}(E)^s = \text{Dol}(E)^{ss}$ and we have that the set of all stable holomorphic vector bundles up to isomorphism is a compact projective Kähler manifold.

In the following we will assume that the rank and degree are coprime, so that the moduli space is smooth, and every semi-stable bundle is stable. Furthermore, we will discuss only the formal process of constructing the moduli space. In practice one must complete the affine spaces into infinite-dimensional Banach manifolds, for example by considering L_1^2 -connections, and consider L_2^2 -gauge transformations acting upon them. Due to the work of Atiyah and Bott in [AB83], it is known that every such gauge-equivalence class contains a smooth representative, and two such smooth representatives can always be gauge transformed by smooth gauge transformations. This allows us to safely ignore these technical analytic details for the purpose of this formal discussion.

4.4.1 $\mathcal{N}^{n,d}$ as a Kähler Quotient

To describe this quotient in terms of a Kähler reduction, we will use the identification $\text{Dol}(E) \cong \mathcal{A}(E, h)$ between Dolbeault operators on E and unitary connections on E with respect to a fixed auxiliary Hermitian metric h . The construction of the moduli space will turn out to be independent of the choice of metric h , because any two metrics are related by a complex gauge transformation.

Let P denote the principal $U(n)$ -bundle associated to (E, h) , and denote by $\text{ad } P$ the associated Lie algebra bundle with fibre $\mathfrak{u}(n)$. Then as an affine space $\mathcal{A}(E, h)$ is modelled on the vector space $\Omega^1(\Sigma_g, \text{ad } P)$, and so at any particular connection ∇ , we have $T_{\nabla}\mathcal{A}(E, h) = \Omega^1(\Sigma_g, \text{ad } P)$. Using this fact, we can write down an expression for a symplectic form on $\mathcal{A}(E, h)$.

The Lie algebra $\mathfrak{u}(n)$ comes equipped with a positive-definite bilinear pairing $(X, Y) \mapsto -\operatorname{tr}(XY)$, defining a map $\mathfrak{u}(n) \otimes \mathfrak{u}(n) \rightarrow \mathbb{C}$. Using this pairing and the wedge product of forms, we can define an anti-symmetric pairing ω on $\Omega^1(\Sigma_g, \operatorname{ad} P)$ by

$$\omega(\alpha, \beta) := - \int_{\Sigma_g} \operatorname{tr}(\alpha \wedge \beta),$$

where $\alpha, \beta \in \Omega^1(\Sigma_g, \operatorname{ad} P)$, and by $\alpha \wedge \beta$ we mean the $\operatorname{ad} P \otimes \operatorname{ad} P$ -valued 2-form given by wedging the form part. This integral requires a choice of orientation on the Riemann surface Σ_g , and we will require that this orientation is such that $\operatorname{vol}(\Sigma_g) = 1$, and comes from a fixed Riemannian metric on Σ_g .

Proposition 4.4.1. *The form ω defined above is a symplectic form on $\mathcal{A}(E, h)$.*

Proof. Since it has no dependence on the choice of $\nabla \in \mathcal{A}(E, h)$, this 2-form is closed on $\mathcal{A}(E, h)$. To see that ω is non-degenerate, consider the form $\star\alpha$ defined by taking the Hodge star of the form component of α , as defined by the Riemannian metric and volume form on Σ_g . Then

$$\omega(\alpha, \star\alpha) = - \int_{\Sigma_g} \operatorname{tr}(\alpha \wedge \star\alpha).$$

Since the trace pairing is non-degenerate, and $\alpha \wedge \star\alpha = \|\alpha\|^2 d\operatorname{vol}$ for $\alpha \in \Omega^1(\Sigma_g)$, we have $\omega(\alpha, \star\alpha) = 0$ if and only if $\alpha = 0$. \square

The use of \star here is not by accident. The pairing defined by

$$g(\alpha, \beta) := - \int_{\Sigma_g} \operatorname{tr}(\alpha \wedge \star\beta)$$

is a positive-definite inner product on $\mathcal{A}(E, h)$. Indeed $g(\alpha, \beta) = \omega(\alpha, \star\beta)$, and since $\star^2 = -1$ on $\Omega^1(\Sigma_g, \operatorname{ad} P)$, the Hodge star operator on Σ_g actually defines an integrable almost complex structure on $\mathcal{A}(E, h)$, $I = \star$, and with respect to the induced complex structure $\mathcal{A}(E, h)$ is Kähler with Kähler metric g .

The next step in the Kähler reduction of $\mathcal{A}(E, h)$ is to identify a Hamiltonian group action and corresponding moment map. As the space of unitary connections on (E, h) , $\mathcal{A}(E, h)$ comes equipped naturally with the action of the *unitary* gauge group \mathcal{G} . Now as a group $\mathcal{G} = \Gamma(\Sigma_g, \operatorname{Ad} P)$, where $\operatorname{Ad} P$ is the $U(n)$ -bundle associated to P by the conjugation action of $U(n)$ on itself. In this sense we see that the Lie algebra of \mathcal{G} is $\mathfrak{g} := \Gamma(\Sigma_g, \operatorname{ad} P) = \Omega^0(\Sigma_g, \operatorname{ad} P)$. Formally we can identify the dual \mathfrak{g}^* of this Lie algebra with $\Omega^2(\Sigma_g, \operatorname{ad} P)$, justified by the natural bilinear pairing between these two spaces.

Although \mathfrak{g}^* is not the dual of \mathfrak{g} as a vector space, which would contain for example distributional sections, formally it allows one to identify a moment map

for the action of \mathcal{G} . In particular, Atiyah and Bott in [AB83] recognised that the curvature map $F : \mathcal{A}(E, h) \mapsto \Omega^2(\Sigma_g, \text{ad } P)$ is a candidate for such a formal moment map.

Proposition 4.4.2 (Atiyah and Bott [AB83]). *The curvature map $F : \mathcal{A}(E, h) \mapsto \Omega^2(\Sigma_g, \text{ad } P)$ sending ∇ to F_∇ is a moment map, and the action of \mathcal{G} on $\mathcal{A}(E, h)$ is Hamiltonian.*

With respect to this moment map, we may then define a quotient space given by

$$N := F^{-1}(-2\pi i \mu(E) \star \mathbf{1}_E) / \mathcal{G}.$$

In the setting where $\text{rk}(E)$ and $\text{deg}(E)$ are coprime, this quotient space N will be a manifold, and the symplectic structure on $\mathcal{A}(E, h)$ will induce a symplectic structure on N . Furthermore, the complex structure on $\mathcal{A}(E, h)$ will induce a Kähler structure on N . In this setting, the celebrated theorem of Narasimhan and Seshadri may be interpreted as follows.

Theorem 4.4.3 (Narasimhan-Seshadri [NS65], Donaldson [Don83]). *The symplectic quotient N is diffeomorphic to the GIT quotient*

$$\mathcal{N}^{n,d} := \text{Dol}(E) // \mathcal{G}_{\mathbb{C}} = \text{Dol}(E)^s / \mathcal{G}_{\mathbb{C}},$$

the moduli space of stable holomorphic vector bundles over Σ_g of rank n and degree d .

Note that the second equality above does not necessarily hold when n and d are not coprime.

Remark 4.4.4. As mentioned in Section 2.6.3, one may interpret a Kempf-Ness theorem in terms of the corresponding norm-squared of the moment map. In this setting where the moment map is curvature, the norm squared is

$$\|F_\nabla\|^2 = \int_{\Sigma_g} \text{tr}(\star F_\nabla \wedge F_\nabla).$$

This expression is denoted by $\text{YM}(\nabla)$, and the map $\nabla \mapsto \text{YM}(\nabla)$ is the famous *Yang-Mills functional* of physical origins. The associated Euler-Lagrange equation for the critical points of the Yang-Mills functional is

$$d_\nabla^* F_\nabla = 0$$

which may be equivalently written as

$$d_\nabla(\star F_\nabla) = 0$$

by the expression $d_{\nabla}^* = \pm \star d_{\nabla} \star$. Solutions to these equations, the *Yang-Mills equations*, are called Yang-Mills connections, and $\star F_{\nabla}$ must be given by a constant central element of the Lie algebra $\mathfrak{u}(n)$ which is therefore diagonal. If we require the connection to be irreducible, then these diagonal entries must be equal (otherwise one could decompose the bundle into eigenspaces invariant under the connection), and since $\int_M \text{tr}(F_{\nabla}) = -2\pi i c_1(E)$, each of these eigenvalues must be $-2\pi i \mu(E)$. But this is precisely the central element that defined the symplectic quotient N . Therefore the Narasimhan-Seshadri theorem states that *(irreducible) Yang-Mills connections correspond to stable holomorphic vector bundles*.

More details on the intimate relationship between stability and the Yang-Mills equations can be found in Atiyah and Bott's [AB83].

4.4.2 Unitary Representations of the Fundamental Group

In order to complete the picture of the Narasimhan-Seshadri theorem as originally explicated, one should identify the space N of flat connections with its associated representation variety. Denote by $\hat{\pi}_1(\Sigma_g)$ the universal central extension of $\pi_1(\Sigma_g)$ given by

$$1 \longrightarrow \mathbb{Z} \longrightarrow \hat{\pi}_1(\Sigma_g) \longrightarrow \pi_1(\Sigma_g) \longrightarrow 1.$$

Explicitly there is a presentation

$$\hat{\pi}_1(\Sigma_g) = \langle a_1, \dots, a_g, b_1, \dots, b_g, c \mid [a_i, c] = [b_i, c] = e, \prod_{i=1}^g [a_i, b_i] = c \rangle.$$

Suppose $(E, h) \rightarrow \Sigma_g$ is a fixed Hermitian vector bundle of rank n and degree d . Then projectively flat unitary connections on E may be identified with representations of $\hat{\pi}_1(\Sigma_g)$ into $U(n)$ sending $c \mapsto \exp(2\pi i \frac{d}{n}) \mathbf{1} \in Z(U(n))$. If two such connections are gauge equivalent, then the associated representations are conjugate, and the Narasimhan-Seshadri theorem gives an isomorphism

$$\text{Hom}_d(\hat{\pi}_1(\Sigma_g), U(n)) / U(n) \cong N \cong \mathcal{N}^{n,d}$$

where the subscript indicates the condition on the image of c in $Z(U(n))$, and $U(n)$ acts on representations by conjugation.

4.4.3 Stable Bundles with Fixed Determinant

It is possible to modify the construction above to obtain a slightly smaller but more workable moduli space of bundles. Let (E, h) again be a Hermitian vector bundle over Σ_g of fixed rank n and degree d . Let $\mathcal{L} \rightarrow \Sigma_g$ denote a fixed holomorphic line

bundle over Σ_g with degree d . Denote by $\widehat{\text{Dol}}(E)_{\mathcal{L}}$ the affine space of Dolbeault operators on (E, h) with the property that the induced holomorphic structure on $\det E$ is biholomorphic to \mathcal{L} . Note that this is possible since $\deg(\det E) = \deg(\mathcal{L}) = d$.

The complex gauge group $\mathcal{G}_{\mathbb{C}}$ acts on $\widehat{\text{Dol}}(E)_{\mathcal{L}}$ and one may define a moduli space

$$\widehat{\mathcal{N}}_{\mathcal{L}}^{n,d} := \widehat{\text{Dol}}(E)_{\mathcal{L}} / \mathcal{G}_{\mathbb{C}}$$

of stable holomorphic vector bundles of rank n and degree d with fixed determinant line bundle \mathcal{L} . If one chose a different line bundle \mathcal{L}' of degree d , then tensoring \mathcal{E} by $\mathcal{L}^* \otimes \mathcal{L}'$ induces a biholomorphism of moduli spaces, and so we drop the dependence of $\widehat{\mathcal{N}}^{n,d}$ on \mathcal{L} .

Remark 4.4.5. In fact, tensoring with a fixed holomorphic line bundle \mathcal{L} induces an isomorphism between the moduli spaces $\mathcal{N}^{n,d}$ and $\mathcal{N}^{n,d+n \deg(\mathcal{L})}$ for any n and d , and similarly for the moduli spaces with fixed determinants. Thus the moduli space depends only on the pair $(n, d \bmod n)$.

The moduli space of stable bundles with fixed determinant may be explicitly related to the full moduli space as follows. Denote by $\text{Jac}_d(\Sigma_g)$ the Jacobian treated as the moduli space of line bundles of degree d over Σ_g . Then there is a natural map

$$\det : \mathcal{N}^{n,d} \rightarrow \text{Jac}_d(\Sigma_g)$$

given by taking the determinant bundle. Furthermore there is an action of $\text{Jac}_0(\Sigma_g) = \text{Jac}(\Sigma_g)$ on $\mathcal{N}^{n,d}$ given by tensor product. Since the determinant of $\mathcal{E} \otimes \mathcal{L}$ is $\det \mathcal{E} \otimes \mathcal{L}^n$ for a rank n bundle \mathcal{E} , if one allows $\text{Jac}_0(\Sigma_g)$ to act on $\text{Jac}_d(\Sigma_g)$ by n th power tensor product, the map \det is equivariant.

In particular, if $\sigma_n : \text{Jac}_0(\Sigma_g) \rightarrow \text{Jac}_0(\Sigma_g)$ denotes this n th power map, then

$$\mathcal{N}^{n,d} = (\widehat{\mathcal{N}}^{n,d} \times \text{Jac}_d(\Sigma_g)) / \ker \sigma_n$$

and since $\ker \sigma_n$ is finite, $\mathcal{N}^{n,d}$ is a finite covering of $\widehat{\mathcal{N}}^{n,d} \times \text{Jac}_d(\Sigma_g)$.

Just as in the case of the stable bundle moduli space, there is a Narasimhan-Seshadri theorem for this moduli space with fixed determinant structure. Note that the centre of $\text{SU}(n)$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ and consists of the n th roots of unity. Then the condition that $c \mapsto \exp(2\pi i \frac{d}{n}) \mathbf{1}$ makes sense for representations of $\widehat{\pi}_1(\Sigma_g) \rightarrow \text{SU}(n)$. Indeed the Narasimhan-Seshadri theorem states that there is a homeomorphism

$$\text{Hom}_d(\widehat{\pi}_1(\Sigma_g), \text{SU}(n)) / \text{SU}(n) \cong \widehat{\mathcal{N}}^{n,d}.$$

4.5 Properties of $\mathcal{N}^{n,d}$ and $\hat{\mathcal{N}}^{n,d}$

In this section we will collect a number of useful facts about the moduli spaces $\mathcal{N}^{n,d}$ and $\hat{\mathcal{N}}^{n,d}$. In particular we will discuss the existence of a certain line bundle \mathcal{L} on $\mathcal{N}^{n,d}$ isolated by Quillen in [Qui85], which turns out to be prequantum for the symplectic form ω on $\mathcal{N}^{n,d}$ obtained through viewing it as a Kähler quotient. In addition, we will discuss the existence of a certain holomorphic symmetric tensor G on $\mathcal{N}^{n,d}$.

4.5.1 Dimension

It is a remarkable fact that the quotient space $\mathcal{N}^{n,d}$ is finite-dimensional. The dimension of $\mathcal{N}^{n,d}$ may be deduced from the Atiyah-Singer index theorem as follows.

The space $\text{Dol}(E)$ of Dolbeault operators on E is an affine space modelled on $\Omega^{0,1}(\Sigma_g, \text{End}(E))$, and the stability condition is open, so the tangent space to $\text{Dol}(E)^s$ at any $\bar{\partial}_E$ is also $\Omega^{0,1}(\Sigma_g, \text{End}(E))$. Now the tangent space to $\mathcal{N}^{n,d}$ at $[\bar{\partial}_E]$ is given by the quotient

$$T_{[\bar{\partial}_E]}\mathcal{N}^{n,d} = \frac{T_{\bar{\partial}_E}\text{Dol}(E)^s}{\ker(d\pi : T_{\bar{\partial}_E}\text{Dol}(E)^s \rightarrow T_{[\bar{\partial}_E]}\mathcal{N}^{n,d})}$$

where $\pi : \text{Dol}(E)^s \rightarrow \mathcal{N}^{n,d}$ is the quotient map. We can determine $\ker d\pi$ as follows. Let $\alpha \in \Omega^{0,1}(\Sigma_g, \text{End}(E))$ be a tangent vector and let $\bar{\partial}_E(t) := \bar{\partial}_E + t\alpha$ be a curve through $\bar{\partial}_E$ for small time t . Then $\alpha \in \ker d\pi$ precisely when this curve arises from a one-parameter family of complex gauge transformations $g_t \in \mathcal{G}_{\mathbb{C}}$, such that

$$\bar{\partial}_E + t\alpha = g_t \bar{\partial}_E g_t^{-1}.$$

When $t \neq 0$ this implies

$$\alpha = \frac{g_t \bar{\partial}_E g_t^{-1} - g_0 \bar{\partial}_E g_0^{-1}}{t}$$

noting that $g_0 = \mathbf{1}$, and so

$$\alpha = \lim_{t \rightarrow 0} \frac{g_t \bar{\partial}_E g_t^{-1} - g_0 \bar{\partial}_E g_0^{-1}}{t} = \partial_t (g_t \bar{\partial}_E g_t^{-1})_{t=0}.$$

But then $\alpha = \bar{\partial}_E(\partial_t (g_t^{-1})_{t=0}) = \bar{\partial}_E(a)$ for some $a \in T_e \mathcal{G}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}} = \Omega^0(\Sigma_g, \text{End}(E))$. Thus we have concluded $\alpha \in \ker d\pi$ precisely when $\alpha \in \text{im}(\bar{\partial}_E : \Omega^0(\Sigma_g, \text{End}(E)) \rightarrow \Omega^{0,1}(\Sigma_g, \text{End}(E)))$. Using the fact that any $(0,1)$ -form on a Riemann surface is $\bar{\partial}$ -closed, we conclude that the tangent space to $\mathcal{N}^{n,d}$ at a point $[\bar{\partial}_E]$ is isomorphic to $H_{\bar{\partial}_E}^{0,1}(\Sigma_g, \text{End}(E))$. Keeping this in mind, we can now prove:

Proposition 4.5.1. *The complex dimension of $\mathcal{N}^{n,d}$ is $1 + n^2(g - 1)$.*

Proof. By the above discussion, we need to compute the dimension of

$$H_{\bar{\partial}_E}^{0,1}(\Sigma_g, \text{End}(E)) \cong H^1(\Sigma_g, \text{End}(\mathcal{E})),$$

where $\text{End}(\mathcal{E})$ denotes the space of holomorphic sections of $\text{End}(E)$ with respect to the holomorphic structure induced by $\bar{\partial}_E$.

In this setting the Atiyah-Singer index theorem reduces to the Hirzebruch-Riemann-Roch theorem, and we have

$$\dim H^0(\Sigma_g, \text{End}(E)) - \dim H^1(\Sigma_g, \text{End}(E)) = \int_{\Sigma_g} \text{Ch}(\text{End}(E)) \text{Td}(\Sigma_g).$$

We have $\text{Ch}(\text{End}(E)) = \text{Ch}(E^*) \text{Ch}(E) = (n - c_1(E))(n + c_1(E))$ and $\text{Td}(\Sigma_g) = 1 + c_1(\Sigma_g)/2$ so

$$\int_{\Sigma_g} \frac{n^2 c_1(\Sigma_g)}{2} = n^2(g - 1).$$

In addition we saw in Section 4.3 that stable bundles are simple. Since we have taken $\bar{\partial}_E \in \text{Dol}(E)^s$, this implies $\dim H^0(\Sigma_g, \text{End}(E)) = 1$ and we obtain the dimension formula. \square

Note that in the case of holomorphic line bundles, we obtain $\dim \mathcal{N}^{1,d} = g$, which is as expected because $\mathcal{N}^{1,d} \cong \text{Jac}(\Sigma_g)$ is a complex torus of dimension g .

By the expression $\mathcal{N}^{n,d} \cong (\hat{\mathcal{N}}^{n,d} \times \text{Jac}_d(\Sigma_g)) / \ker \sigma_n$ for the moduli space with fixed determinant, and noting that $\ker \sigma_n$ is finite, we obtain:

Proposition 4.5.2. *The complex dimension of $\hat{\mathcal{N}}^{n,d}$ is $1 + n^2(g - 1) - g = (n^2 - 1)(g - 1)$.*

4.5.2 Quillen Determinant Bundle

The moduli space of stable bundles on a compact Riemann surface comes equipped with a distinguished line bundle \mathcal{L} called the determinant line bundle of $\mathcal{N}^{n,d}$. The properties of this line bundle were first investigated by Quillen in [Qui85] in the context of determinants of Dolbeault operators. It has since appeared in the context of Chern-Simons theory, for example in [Wit89], and plays an important role in the geometric quantization of both the moduli space of stable and Higgs bundles on a Riemann surface, being a prequantum line bundle for the Atiyah-Bott symplectic form.

In Quillen's investigation of the determinant bundle, a complex structure on Σ_g is fixed, and one considers the space of Dolbeault operators $\text{Dol}(E)$ for the fixed smooth complex vector bundle over Σ_g .

Definition 4.5.3. Given a Dolbeault operator $\bar{\partial}_E$ on $E \rightarrow \Sigma_g$, define the associated zeta function as follows. Let $\bar{\square}_{\bar{\partial}_E}$ denote the Laplacian associated to $\bar{\partial}_E$, and A the set of eigenvalues of $\bar{\square}_{\bar{\partial}_E}$. Then

$$\zeta_{\bar{\partial}_E}(s) := \sum_{\lambda \in A} \lambda^{-s}$$

for $\operatorname{Re}(s) > 1$. Analytically continue this to a meromorphic function of s with non-zero derivative at $s = 0$.

Let $f(\bar{\partial}_E) := \exp(-\zeta'_{\bar{\partial}_E}(0))$ be a function on $\operatorname{Dol}(E)$. If Λ denotes taking the top exterior power of a finite-dimensional vector space, then Quillen proved the following theorem:

Theorem 4.5.4 (Quillen [Qui85]). *There is a holomorphic line bundle \mathcal{L} over $\operatorname{Dol}(E)$ with fibre given by $\mathcal{L}_{\bar{\partial}_E} = \Lambda(\ker \bar{\partial}_E)^* \otimes \Lambda(\operatorname{coker} \bar{\partial}_E)$ at $\bar{\partial}_E \in \operatorname{Dol}(E)$. Furthermore, this holomorphic line bundle comes with a Hermitian metric induced by those on $\ker \bar{\partial}_E$ and $\operatorname{coker} \bar{\partial}_E$, after multiplying by the function f , and with respect to these structures, the Chern connection has curvature equal to $-2\pi i \omega$ where ω is the Atiyah-Bott symplectic form defined on $\operatorname{Dol}(E)$.*

Note that this fibre is well-defined because every $\bar{\partial}_E \in \operatorname{Dol}(E)$ is elliptic and hence Fredholm. The introduction of the zeta function on $\operatorname{Dol}(E)$ is to counteract the jumping of the Hermitian inner product on $\mathcal{L}_{\bar{\partial}_E}$ induced by the jumping of the dimensions of $\ker \bar{\partial}_E$ and $\operatorname{coker} \bar{\partial}_E$.

The holomorphic line bundle, Hermitian metric, and associated connection are all $\mathcal{G}_{\mathbb{C}}$ -invariant. Since a stable bundle is simple, the stabiliser of the $\mathcal{G}_{\mathbb{C}}$ action on any fibre $\mathcal{L}_{\bar{\partial}_E}$ is \mathbb{C}^* . This acts by scalar multiplication on $\ker \bar{\partial}_E$ and $\operatorname{coker} \bar{\partial}_E$ and hence with weight $-\operatorname{ind}(\bar{\partial}_E) = -d - n(1 - g)$. This weight is the same for any fibre, so after modifying the $\mathcal{G}_{\mathbb{C}}$ action on \mathcal{L} by a character of \mathbb{C}^* the holomorphic Hermitian line bundle descends to the quotient $\mathcal{N}^{n,d}$, where we also denote it by \mathcal{L} .

Although this construction depends on a choice of complex structure on Σ_g and hence $\mathcal{N}^{n,d}$, the moduli space may be constructed as a symplectic manifold without choosing a complex structure, via the symplectic reduction described in Section 4.4.1. In [RSW89] it was shown, at least in the case of $\operatorname{SU}(2)$ bundles, that a line bundle L with Hermitian metric and connection may be constructed without choosing a complex structure on Σ_g . This line bundle was constructed using techniques from Chern-Simons theory for a 3-manifold M with $\partial M = \Sigma_g$.

It was also shown that after choosing a complex structure on Σ_g and hence $\mathcal{N}^{n,d}$, the induced holomorphic structure on L made it biholomorphic to the

Quillen determinant bundle. This construction is key, since it justifies the approach to the geometric quantization of $\mathcal{N}^{n,d}$ where the spaces of holomorphic sections $H^0(\mathcal{N}^{n,d}, \mathcal{L})$ for various choices of complex structure on Σ_g are viewed as subspaces of the infinite-dimensional vector space $\Gamma(\mathcal{N}^{n,d}, L)$ of smooth sections.

Furthermore, the constructions of Quillen and others of the determinant bundle readily extend to the case of the stable bundle moduli space with fixed determinant $\hat{\mathcal{N}}^{n,d}$ with its associated Atiyah-Bott symplectic form.

4.5.3 Topology

In this section we note some useful results about the topology of the stable bundle moduli space. These properties will show that $\hat{\mathcal{N}}^{n,d}$ satisfies the topological conditions required for the existence of a Hitchin connection in the sense of Andersen.

Theorem 4.5.5 (Atiyah and Bott [AB83]). *The moduli space $\hat{\mathcal{N}}^{n,d}$ is simply-connected.*

Theorem 4.5.6 (Ramanan [Ram73], Atiyah and Bott [AB83]). *The cohomology $H^2(\hat{\mathcal{N}}^{n,d}, \mathbb{Z})$ is isomorphic to \mathbb{Z} , and is generated by $\frac{1}{2}c_1(\hat{\mathcal{N}}^{n,d})$.*

As a result of this second theorem, one may also show that $\text{Pic}(\hat{\mathcal{N}}^{n,d}) \cong \mathbb{Z}$. Indeed the Quillen determinant line bundle for the moduli space of stable bundles with fixed determinant may be identified with the positive generator of $\text{Pic}(\hat{\mathcal{N}}^{n,d})$.

Chapter 5

Higgs Bundles over Riemann Surfaces

In this chapter we will investigate a generalisation of the moduli space of stable bundles, the moduli space of *Higgs* bundles over a compact Riemann surface, denoted $\mathcal{M}^{n,d}$. This object was first introduced by Hitchin in [Hit87], and has its origins, similarly to the moduli space of stable bundles, in the Yang-Mills equations.

Hitchin's foundational paper explored the case of rank 2 Higgs bundles of odd degree with a fixed holomorphic structure on the determinant completely. In particular it was shown that the moduli space of such objects is a smooth manifold of dimension $12g - 12$, and remarkably the space additionally has the structure of not just a Kähler manifold, but a *hyper-Kähler* manifold, and may be obtained via a hyper-Kähler reduction, analogously to the case of the stable bundle moduli space. Using Morse-theoretic techniques similar to Atiyah and Bott for the stable bundle setting, Hitchin computed the Betti numbers of the moduli space, and following a similar argument to the one Donaldson used in [Don83], proved a Higgs bundle version of the Narasimhan and Seshadri theorem, identifying stable Higgs bundles with projective irreducible reductive *complex* representations of the fundamental group of the surface. In a series of papers starting with [Sim88], Simpson then expanded upon the treatment of Hitchin to describe the Higgs moduli space not just for arbitrary rank and degree, but also for Higgs bundles over higher-dimensional complex manifolds.

In addition to the rich differential-geometric analysis of the moduli space of Higgs bundles given by Hitchin and others, it was shown by Nitsure in [Nit91] that this space has an algebraic description as a GIT quotient, and in particular that $\mathcal{M}^{n,d}$ is a non-singular (when $(n, d) = 1$) quasi-projective variety.

We will start by defining the notion of a Higgs bundle, and stability of these objects, before classifying the Higgs line bundles (all of which are of course stable)

on a compact Riemann surface. We will then describe the construction of the moduli space $\mathcal{M}^{n,d}$ for higher rank through the use of a Kähler quotient, and discuss how this space may also be constructed as a hyper-Kähler quotient. We will conclude by discussing some interesting properties of the Higgs moduli space.

5.1 Higgs Bundles

Definition 5.1.1. A Higgs Bundle on a compact Riemann surface Σ_g is a pair (\mathcal{E}, Φ) where \mathcal{E} is a holomorphic vector bundle over Σ_g and $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes K$ is an $\text{End}(\mathcal{E})$ -valued holomorphic 1-form on Σ_g . That is, $\Phi \in H^0(\Sigma_g, \text{End}(\mathcal{E}) \otimes K)$.

Note that we can consider an equivalent formulation of the concept by passing to Dolbeault operators. Fix a smooth complex vector bundle E of some rank and degree. Then we may define a Higgs pair to be a pair $(\bar{\partial}_E, \Phi)$ where $\bar{\partial}_E$ is a Dolbeault operator on E , and $\Phi \in \Gamma(\Sigma_g, \text{End}(E) \otimes K)$ satisfies $\bar{\partial}_E(\Phi) = 0$, where here we have extended $\bar{\partial}_E$ to this tensor product using the complex structure on Σ_g .

With respect to this formulation, we can phrase the equivalence of Higgs pairs as follows: A pair $(\bar{\partial}_E, \Phi)$ is equivalent to $(\bar{\partial}'_E, \Phi')$ if there exists some complex gauge transformation $g \in \mathcal{G}_{\mathbb{C}}$ such that $g(\bar{\partial}_E, \Phi)g^{-1} = (\bar{\partial}'_E, \Phi')$, where $g(\bar{\partial}_E, \Phi)g^{-1} := (g \circ \bar{\partial}_E \circ g^{-1}, g \circ \Phi \circ g^{-1})$.

Just as in the case of holomorphic bundles, there is a notation of slope-stability for Higgs bundles.

Definition 5.1.2. A Higgs bundle (\mathcal{E}, Φ) is stable (resp. semi-stable) if, for all proper non-zero holomorphic sub-bundles \mathcal{F} such that $\Phi(\mathcal{F}) \subseteq \mathcal{F} \otimes K$, one has

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \quad (\text{resp. } \leq).$$

Polystability is also defined similarly:

Definition 5.1.3. A Higgs bundle is polystable if it decomposes as a direct sum of Higgs bundles of the same slope. The Higgs field is therefore in block-diagonal form.

Every stable bundle gives rise to a stable Higgs bundle when paired with a Higgs field. However, there are more stable Higgs bundles than stable vector bundles, as the following example shows.

Example 5.1.4. Suppose $g > 1$, and fix a square root $K^{1/2}$ of the canonical bundle $K \rightarrow \Sigma_g$, and let $\mathcal{E} := K^{1/2} \oplus K^{-1/2}$. Then $\mu(\mathcal{E}) = 0$.

Endomorphisms of \mathcal{E} split up into parts, and $\text{Hom}(K^{1/2}, K^{-1/2}) \cong K^{-1}$. Since the Higgs field lives in endomorphisms tensored by K , the $\text{Hom}(K^{1/2}, K^{-1/2}) \otimes K$

component of the Higgs fields has a canonical section $\mathbf{1}$ of $K^{-1} \otimes K$. Define therefore a Higgs field

$$\Phi := \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}.$$

This is clearly holomorphic, and so (\mathcal{E}, Φ) is a Higgs pair.

The only Φ -invariant sub-bundle is $K^{-1/2}$, which has negative degree when $g > 1$, so (\mathcal{E}, Φ) is a stable Higgs bundle. However, \mathcal{E} is clearly not stable as a holomorphic vector bundle, since it contains the sub-bundle $K^{1/2}$, which is of positive slope.

Just as in the case of stable bundles, stability for Higgs bundles has consequences for the existence of homomorphisms. As a sample, we have the following proposition.

Proposition 5.1.5. *Suppose $f : (\mathcal{E}, \Phi) \rightarrow (\mathcal{E}, \Phi)$ is an endomorphism of a stable Higgs bundle such that $\Phi f = f\Phi$. Then $f = 0$ or f is an isomorphism.*

Proof. Suppose $\ker(f)$ is non-trivial. Then it is Φ -invariant, so $\mu(\ker(f)) < \mu(\mathcal{E})$. Suppose that $\text{im}(f) \neq \mathcal{E}$. Then $\text{im}(f)$ is also Φ -invariant, so $\mu(\text{im}(f)) < \mu(\mathcal{E})$. But we also have a short exact sequence

$$0 \rightarrow \ker(f) \rightarrow \mathcal{E} \rightarrow \mathcal{E}/\ker(f) \rightarrow 0$$

which implies one of $\ker(f)$ or $\mathcal{E}/\ker(f)$ has slope greater than \mathcal{E} . But $\mu(\mathcal{E}/\ker(f)) \leq \mu(\text{im}(f)) < \mu(\mathcal{E})$, so this is a contradiction. Thus $f = 0$ or f is an isomorphism. \square

5.2 Classification of Higgs Line Bundles on a Riemann Surface

Let (\mathcal{L}, Φ) be a Higgs line bundle on a compact Riemann surface. Since $\text{End}(\mathcal{L}) = \mathcal{O}$, the Higgs field Φ is simply a holomorphic 1-form on Σ_g . Furthermore, since the complex gauge group $\mathcal{G}_{\mathbb{C}} \cong \text{Maps}(\Sigma_g, \mathbb{C}^*)$ is Abelian, the conjugation action on Φ is trivial. In particular we have that $\mathcal{M}^{1,d} \cong \mathcal{N}^{1,d} \times H^0(\Sigma_g, K)$, where K is the canonical bundle of Σ_g . At a point $[\bar{\partial}_E] \in \mathcal{N}^{n,d}$ the tangent space is given by $H^1(\Sigma_g, \text{End}(\mathcal{L})) \cong H^1(\Sigma_g, \mathcal{O})$. Serre-Duality says that this tangent space is canonically dual to $H^0(\Sigma_g, K)$ for every $[\bar{\partial}_L]$, and furthermore since $\mathcal{N}^{1,d}$ is a torus, its cotangent bundle is trivial and may also be identified $T^*\mathcal{N}^{1,d} \cong \mathcal{N}^{1,d} \times H^0(\Sigma_g, K)$.

Thus we conclude that the moduli space of Higgs line bundles on a surface is given by $\mathcal{M}^{1,d} \cong T^*\mathcal{N}^{1,d} \cong T^*\text{Jac}(\Sigma_g)$ for all d .

5.3 The Stable Higgs Bundle Moduli Space

In this section we will describe the Higgs bundle moduli space in a way mirroring the definition of the stable bundle moduli space. As in the case of the stable bundle space, the Higgs bundle moduli space is smooth when the rank and degree of the underlying bundle are coprime, and we will always take this assumption to be satisfied. Further, we note that one should consider L_1^2 connections, L_1^2 Higgs fields, and L_2^2 gauge transformations in order to make the analytical details of the construction of the Higgs moduli space rigorous. For the same reasons as in the previous chapter, we will ignore these technical details in place of a formal discussion on the definition of $\mathcal{M}^{n,d}$.

5.3.1 $\mathcal{M}^{n,d}$ as a Kähler Quotient

The construction of the moduli space of Higgs bundles is analogous to the case of the stable bundle moduli space. We will use the description of a Higgs bundle (\mathcal{E}, Φ) as a Higgs pair $(\bar{\partial}_E, \Phi)$ where $\bar{\partial}_E$ is a Dolbeault operator on a fixed smooth complex line bundle $E \rightarrow \Sigma_g$, and $\Phi : E \rightarrow E \otimes K$ is such that $\bar{\partial}_E(\Phi) = 0$. If we consider the infinite-dimensional affine space

$$\text{Dol}(E) \times \Omega^{1,0}(\Sigma_g, \text{End}(E))$$

then the Higgs pairs are the points of the subset

$$\mathcal{B} := \{(\bar{\partial}_E, \Phi) \mid \bar{\partial}_E(\Phi) = 0\} \subseteq \text{Dol}(E) \times \Omega^{1,0}(\Sigma_g, \text{End}(E)).$$

It turns out that \mathcal{B} is in fact an infinite-dimensional orbifold, and the action of $\mathcal{G}_{\mathbb{C}}$ on $\text{Dol}(E) \times \Omega^{1,0}(\Sigma_g, \text{End}(E))$ defined by conjugation restricts to \mathcal{B} . If we fix a Hermitian metric h on the bundle E , then the identification $\mathcal{A}(E, h) \cong \text{Dol}(E)$ allows us to view a Higgs pair as a pair (∇, Φ) such that $\nabla^{0,1}\Phi = 0$, where ∇ is a unitary connection on (E, h) . Therefore \mathcal{B} sits inside $\mathcal{A}(E, h) \times \Omega^{1,0}(\Sigma_g, \text{End}(E))$ also.

Now the vector space $\Omega^{1,0}(\Sigma_g, \text{End}(E))$ comes equipped with an anti-symmetric non-degenerate bilinear form η defined by

$$\eta(\alpha, \beta) := -i \int_{\Sigma_g} \text{tr}(\alpha \wedge \beta^*).$$

Combined with the form ω on $\mathcal{A}(E, h)$ discovered by Atiyah and Bott, one obtains a symplectic structure on the product $\mathcal{A}(E, h) \times \Omega^{1,0}(\Sigma_g, \text{End}(E))$, and it turns out that the subset \mathcal{B} is an infinite-dimensional symplectic orbifold with respect to the induced symplectic structure.

In addition to the symplectic structure on $\mathcal{A}(E, h)$ we also encountered an integrable almost-complex structure given by the Hodge star operator \star . Fortunately, the vector space $\Omega^{1,0}(\Sigma_g, \text{End}(E))$ is already a complex vector space, and therefore has an almost-complex structure induced by multiplication by i . Combining these almost-complex structures we obtain an integrable almost-complex structure on the product, which descends to make \mathcal{B} an infinite-dimensional Kähler manifold, at least where it is non-singular.

The next step is to identify a moment map for the action of \mathcal{G} on \mathcal{B} . This was done by Hitchin.

Proposition 5.3.1 (Hitchin [Hit87]). *The map $\Phi \mapsto [\Phi, \Phi^*]$ is a moment map for the action of \mathcal{G} on $\Omega^{1,0}(\Sigma_g, \text{End}(E))$.*

As we have already observed, $\nabla \mapsto F_\nabla$ is a moment map for the action of \mathcal{G} on $\mathcal{A}(E, h)$, and combining these two maps gives a moment map $\mu : \mathcal{A}(E, h) \times \Omega^{1,0}(\Sigma_g, \text{End}(E)) \rightarrow \mathfrak{g}^* \cong \Omega^2(\Sigma_g, \text{End}(E))$ defined by

$$\mu(\nabla, \Phi) := F_\nabla + [\Phi, \Phi^*].$$

Since \mathcal{B} is \mathcal{G} -invariant, this moment map μ descends to give a moment map on \mathcal{B} , say $\tilde{\mu}$, and using this moment map on the infinite-dimensional symplectic orbifold \mathcal{B} we can define the symplectic reduction

$$M := \mu^{-1}(-2\pi i \mu(E) \star \mathbf{1}) / \mathcal{G}.$$

Theorem 5.3.2 (Hitchin [Hit87], Simpson [Sim88]). *The space M is a finite-dimensional Kähler manifold when n and d are coprime.*

Just as the Narasimhan-Seshadri theorem could be interpreted as a kind of infinite-dimensional Kempf-Ness theorem for the moduli space of stable bundles on a Riemann surface, the generalisation of this theorem to the case of Higgs bundles may be viewed as Kempf-Ness theorem for the Higgs moduli space. This theorem was first proved by Hitchin for the case of rank 2 Higgs bundles of degree 1 with fixed determinant bundle, and was subsequently expanded to the general case by Simpson, who also generalised the result to higher dimensions, building on the work of Donaldson, Uhlenbeck, and Yau, in the proof of the Hitchin-Kobayashi correspondence

This correspondence between stable Higgs bundles and irreducible solutions to the so-called self-duality equations is often called the non-Abelian Hodge theorem (for Riemann surfaces).

Theorem 5.3.3 (Non-Abelian Hodge Theorem; Hitchin [Hit87], Simpson [Sim88]). *The space M is diffeomorphic to the quotient*

$$\mathcal{M}^{n,d} := \mathcal{B} // \mathcal{G}_{\mathbb{C}}.$$

Remark 5.3.4. Similarly to the case of the stable bundle moduli space, there is an interpretation of the symplectic quotient defining the Higgs moduli space in terms of a functional and its associated Euler-Lagrange equations. Here the norm-squared of the moment map $\|F_{\nabla} + [\Phi, \Phi^*]\|^2$ is denoted $\text{YMH}(\nabla)$, the so-called *Yang-Mills-Higgs functional*, and the associated Euler-Lagrange equations are given by

$$d_{\nabla} \star (F_{\nabla} + [\Phi, \Phi^*]) = 0, \quad [\Phi, \star(F_{\nabla} + [\Phi, \Phi^*])] = 0.$$

The critical points of the Yang-Mills-Higgs functional that satisfy the holomorphicity condition $d_{\nabla}^{0,1} \Phi = 0$ are precisely the stable Higgs bundles, and we have another interpretation of the non-Abelian Hodge theorem: Irreducible solutions (∇, Φ) of the Hitchin self-duality equations

$$\begin{cases} \star(F_{\nabla} + [\Phi, \Phi^*]) = -2\pi i \mu(E) \\ d_{\nabla}^{0,1} \Phi = 0 \end{cases}$$

correspond to stable Higgs bundles.

For more details on the Hitchin equations and their implications for the existence of Higgs bundles, we refer to Hitchin's original paper [Hit87]. For an accessible proof of the non-Abelian Hodge theorem in this setting, see the notes [Wen16] by Wentworth.

5.3.2 Complex Representations of the Fundamental Group

One may again reinterpret this theorem by identifying M with a corresponding space of *complex* representations of the fundamental group. In this case

$$\mathcal{M}^{n,d} \cong M \cong \text{Hom}_d^+(\hat{\pi}_1(\Sigma_g), \text{GL}(n, \mathbb{C})) / \text{GL}(n, \mathbb{C}),$$

where the superscript $+$ indicates that one restricts to *reductive* representations. The second diffeomorphism is in fact a biholomorphism of complex manifolds.

In the case of the Narasimhan-Seshadri theorem, the identification of the symplectic quotient N with the stable bundle moduli space $\mathcal{N}^{n,d}$ induces a complex structure on N . In fact this is precisely the Kähler structure N obtains from the complex structure \star . However in the case of Higgs bundles, the Kähler structure J on M induced by its identification with the representation space is *not* isomorphic to that of $\mathcal{M}^{n,d}$. If we denote the complex structure on $\mathcal{M}^{n,d}$ as a Kähler quotient by I , then it turns out that $IJ = -JI$ on $\mathcal{M}^{n,d}$. Thus if one defines a third almost-complex structure $K := IJ$, then K is integrable and $IJK = -\mathbf{1}$, so that with respect to the triple (I, J, K) , $\mathcal{M}^{n,d}$ inherits the structure of a *hyper-Kähler* manifold.

Remark 5.3.5. One may describe the Higgs moduli space via a hyper-Kähler reduction process analogous to the symplectic reduction and Kähler reduction processes that have been discussed previously. In this case the moment map is valued in $\mathfrak{g}^* \otimes \mathbb{R}^3$ and combines both the holomorphic and differential-geometric descriptions of Higgs bundles. For more details on this construction in general see Hitchin's [Hit92], and for the specific case of the Higgs moduli space refer to the appendix by García-Prada of [WGP80].

5.3.3 Higgs Bundles with Fixed Determinant

One may define for Higgs bundles an analogue of the moduli space $\hat{\mathcal{N}}^{n,d}$ of stable holomorphic vector bundles with fixed determinant, denoted $\hat{\mathcal{M}}^{n,d}$. In this case the Higgs fields will take values in the *trace-free* endomorphisms of the corresponding vector bundle, and all other constructions then carry through analogously.

In the statement of the non-Abelian Hodge theorem for this restricted case, one simply replaces $\mathrm{GL}(n, \mathbb{C})$ with $\mathrm{SL}(n, \mathbb{C})$, just as in the case of stable bundles where $\mathrm{U}(n)$ was replaced by $\mathrm{SU}(n)$. Explicitly, there is a diffeomorphism

$$\hat{\mathcal{M}}^{n,d} \cong \mathrm{Hom}_d^+(\hat{\pi}_1(\Sigma_g), \mathrm{SL}(n, \mathbb{C})) / \mathrm{SL}(n, \mathbb{C}).$$

Just as in the case of the full moduli space, $\hat{\mathcal{M}}^{n,d}$ inherits two different complex structures which anti-commute via its two descriptions as a GIT quotient and as a space of complex representations. It is again a hyper-Kähler manifold, and the natural projection

$$\mathcal{M}^{n,d} \rightarrow \mathrm{Jac}_d(\Sigma_g)$$

allows one to easily relate the full Higgs moduli space to the space with fixed determinant.

5.4 Properties of $\mathcal{M}^{n,d}$

5.4.1 Dimension

Define an operator $D := \bar{\partial}_E + \Phi$. Then there is a deformation complex $C(\bar{\partial}_E, \Phi)$ that describes the tangent structure to the moduli space, given by

$$\begin{aligned} C(\bar{\partial}_E, \Phi) \quad : \quad 0 \rightarrow \Omega^0(\Sigma_g, \mathrm{End}(E)) \xrightarrow{D} \Omega^{1,0}(\Sigma_g, \mathrm{End}(E)) \oplus \Omega^{0,1}(\Sigma_g, \mathrm{End}(E)) \\ \xrightarrow{D} \Omega^{1,1}(\Sigma_g, \mathrm{End}(E)) \rightarrow 0 \end{aligned}$$

This complex is obtained by differentiating the condition $\bar{\partial}_E \Phi = 0$ and the $\mathcal{G}_{\mathbb{C}}$ -action.

The tangent space to $\mathcal{M}^{n,d}$ at a point $[\bar{\partial}_E, \Phi]$ corresponding to a stable Higgs bundle is isomorphic to $H^1(C(\bar{\partial}_E, \Phi))$, which has dimension $2n^2(g-1)+2$. Observe that the real dimension is a multiple of 4, which agrees with the result that $\mathcal{M}^{n,d}$ admits a hyper-Kähler structure, and that $\dim \mathcal{M}^{n,d} = 2 \dim \mathcal{N}^{n,d}$. This was already seen in the case of Higgs line bundles, where we observed that $\mathcal{M}^{1,d} \cong T^*\mathcal{N}^{1,d}$.

Notice that $H^0(C(\bar{\partial}_E, \Phi))$ consists of those endomorphisms of $(\bar{\partial}_E, \Phi)$ that commute with Φ and are holomorphic. These are precisely the endomorphisms of the Higgs bundle respecting the Higgs field. We therefore define simplicity for Higgs bundles in terms of this cohomology group:

Definition 5.4.1. *A Higgs bundle (\mathcal{E}, Φ) is simple if $\dim H^0(C(\bar{\partial}_E, \Phi)) = 1$.*

Using similar arguments as in the case of stable bundles, Proposition 5.1.5 therefore says that all stable Higgs bundles are simple.

5.4.2 \mathbb{C}^* Action

The moduli space of stable Higgs bundles on a Riemann surface comes equipped with an action of the group \mathbb{C}^* , which is not present in the case of stable bundles with a compact structure group. The presence of the \mathbb{C}^* action has its origins in the Higgs field Φ of a Higgs bundle (\mathcal{E}, Φ) . Given such a pair, $(\mathcal{E}, t\Phi)$ defines another Higgs bundle, for any $t \in \mathbb{C}^*$, and this amounts to an action of \mathbb{C}^* on \mathcal{B} .

Since $\mathbb{C}^*\mathbf{1}$ lies in the centre of $\mathrm{GL}(n, \mathbb{C})$, this action commutes with the action of \mathcal{G} and $\mathcal{G}_{\mathbb{C}}$ on \mathcal{B} . In particular it descends to the quotient $\mathcal{M}^{n,d}$.

5.4.3 Relation to $T^*\mathcal{N}^{n,d}$

Recall that the tangent space to $\mathcal{N}^{n,d}$ at a point $[\bar{\partial}_E]$ is given by the Dolbeault cohomology $H_{\bar{\partial}_E}^{0,1}(\Sigma_g, \mathrm{End}(E))$. Serre-duality gives isomorphisms

$$H_{\bar{\partial}_E}^{0,1}(\Sigma_g, \mathrm{End}(E)) \cong H_{\bar{\partial}_E}^{1,0}(\Sigma_g, \mathrm{End}(E))^* \cong H^0(\Sigma_g, \mathrm{End}(\mathcal{E}) \otimes K)^*$$

where K is the canonical bundle of the Riemann surface Σ_g , and \mathcal{E} is the holomorphic structure on E with $\bar{\partial}_E$. In particular, the cotangent space to $\mathcal{N}^{n,d}$ at $[\bar{\partial}_E]$ consists of Higgs fields compatible with the holomorphic structure $[\bar{\partial}_E]$.

Given a point (\mathcal{E}, Φ) in this cotangent space, notice that the pair is a stable Higgs bundle because \mathcal{E} is already stable. Therefore one obtains an inclusion

$$T^*\mathcal{N}^{n,d} \hookrightarrow \mathcal{M}^{n,d}.$$

In the case of Higgs line bundles, as we have already seen, there is no stability condition, so the inclusion is actually an isomorphism.

In general the inclusion is not surjective however. Example 5.1.4 is a stable Higgs bundle whose underlying holomorphic vector bundle is not itself stable.

Fortunately, the Higgs bundles not arising via this inclusion are rare, in the sense that the inclusion is open and dense. In general it is known that, at least when $g > 1$ and excluding the case of $\mathcal{M}^{2,2}$, the complement of $T^*\mathcal{N}^{n,d}$ in $\mathcal{M}^{n,d}$ has codimension greater than one. This was shown by Hitchin in [Hit87] for the case of $\hat{\mathcal{M}}^{2,d}$ for d odd, and in general in [Hit90].

All of these results apply readily to the case of moduli spaces with fixed determinant structure also.

5.4.4 Determinant Bundle

In this section we will recall a generalisation of the Quillen determinant line bundle discussed in Section 4.5.2 to the moduli space of Higgs bundles. This will serve as the prequantum line bundle for the quantization of the Higgs moduli space we will investigate in Chapter 7.

In its description as a hyper-Kähler manifold, $\mathcal{M}^{n,d}$ comes equipped with three integrable almost-complex structures. Recall that I is the complex structure on $\mathcal{M}^{n,d}$ obtained as a Kähler quotient, whilst J comes from its more representation-theoretic description, and K is the combination $K = IJ$. The Kähler form for $\mathcal{M}^{n,d}$ with respect to I was identified by Hitchin in [Hit87] to be $\omega_I(X, Y) := g(X, IY)$ where g is the natural Kähler metric on $\mathcal{M}^{n,d}$ obtained by combining Hermitian L^2 -inner products on $\text{Dol}(E)$ and $\Omega^{1,0}(\Sigma_g, \text{End}(E))$ previously mentioned. Restricted to $\mathcal{N}^{n,d} \subset \mathcal{M}^{n,d}$ this becomes the Kähler structure on $\mathcal{N}^{n,d}$ encountered in Chapter 4.

A prequantum line bundle for the Kähler structure I has its origins in complex Chern-Simons gauge theory. An explicit Chern-Simons cocycle for this holomorphic line bundle is written down, for example, in [AG14] by Andersen and Gammelgaard. Alternatively, the prequantum line bundle may be described via a Quillen determinant construction, for example as was done by Dey in [Dey17].

In particular, one considers the holomorphic line bundle \mathcal{L}_H over the product $\text{Dol}(E) \times \Omega^{1,0}(\Sigma_g, \text{End}(E))$ with fibres given by the determinant lines of the associated Dolbeault operators over each point. This holomorphic line bundle can be shown to descend to \mathcal{B} and $\mathcal{M}^{n,d}$ using similar arguments to the case of $\mathcal{N}^{n,d}$. It then manifests that $\mathcal{L}_H|_{T^*\mathcal{N}^{n,d}} \cong \pi^*\mathcal{L}$ where \mathcal{L} is the original Quillen determinant bundle. In fact since the complement of $T^*\mathcal{N}^{n,d}$ in $\mathcal{M}^{n,d}$ has codimension greater than or equal to two in general, one may think of \mathcal{L}_H as an extension of $\pi^*\mathcal{L}$ across this complement. After a modification to the Hermitian metric defined by Quillen, this line bundle can be shown to be prequantum for the symplectic form ω_I , for example as performed by Dey in [Dey17].

The natural \mathbb{C}^* action on the Higgs moduli space lifts to the line bundle \mathcal{L}_H , and passes then to the space $H^0(\mathcal{M}^{n,d}, \mathcal{L}_H)$ of global holomorphic sections. This \mathbb{C}^* action will be made explicit in the case of $\mathcal{M}^{1,d} \cong T^*\text{Jac}(\Sigma_g)$ in Chapter 7.

Part III

Quantization of Moduli Spaces

Chapter 6

Geometric Quantization

In this chapter we will collect some results about the geometric quantization of symplectic manifolds via Kähler polarisations, with a view towards quantizing moduli spaces of bundles on Riemann surfaces. Geometric quantization in this sense was first carried out independently by Kostant in [Kos70] and Souriau in [Sou69].

The problem of quantizing moduli spaces of bundles is of both mathematical and physical interest. The moduli space of stable holomorphic vector bundles on a compact Riemann surface appears as the phase space of classical Chern-Simons theory with compact gauge group in 2+1 dimensions. The quantization of this topological quantum field theory therefore constitutes quantizing the moduli space, which is naturally a symplectic manifold as we have seen in Chapter 4. Using geometric quantization, this problem was first investigated by Witten in [Wit89], and again with Axelrod and Della Pietra in [ADPW91]. As we will see, the geometric quantization of this moduli space depends on a choice of complex structure on the compact Riemann surface, and the independence of the quantization on this choice manifests as a projectively flat connection on the bundle of quantum spaces over the parameter space. Axelrod, Della Pietra, and Witten isolated this connection and proved it was projectively flat.

Approaching the problem from a differential-geometric point of view, Hitchin in [Hit90] constructed the same connection, now known as the Hitchin connection, and also proved it was projectively flat. Further, Hitchin determined an explicit expression for the associated parallel transport operator used to canonically identify the quantum spaces for various choices of complex structures. This differential-geometric perspective has since been expanded by Andersen and others (for example in [And06]) as a way of geometrically quantizing general symplectic manifolds.

We will begin by recalling the problem of canonically quantizing a classical system, noting that this process fails due to the existence of a number of no-go

theorems. After defining geometric quantization via Kähler polarisations, we will phrase the dependence of the quantization on a choice of Kähler polarisation in terms of families. Following the formalism of Hitchin in [Hit90] and Andersen in [And06], we will then define Hitchin connections for arbitrary prequantizable symplectic manifolds, before finishing by recalling Andersen’s construction of a Hitchin connection in the case of compact prequantizable symplectic manifolds (satisfying certain topological conditions) in [And06]. When specialised to $\mathcal{N}^{n,d}$, this construction turns out to provide the same connection as discovered by Hitchin and Axelrod, Della Pietra, and Witten.

6.1 Quantization

In Dirac’s original work on quantum mechanics (see [Dir30]), he described a process of “canonically” quantizing a classical system. Formally this process involved replacing Poisson brackets by commutators (depending on a parameter \hbar) such that when taking the limit $\hbar \rightarrow 0$ one recovers the original classical system. In mathematical terminology, one must look for a map Q sending a function f on the classical phase space to its quantum operator Q_f acting on a Hilbert space, such that

$$i\hbar Q_{\{f,g\}} = [Q_f, Q_g]. \quad (\text{Eq. 6.1})$$

Unfortunately, under reasonable (and physically motivated) conditions on the quantization map Q , such an assignment is impossible. In particular, consider the simplest phase space $\mathbb{R}^2 = \{(x, p) \mid x, p \in \mathbb{R}\}$. If we require $Q_1 = \mathbf{1}$, that Q sends the classical position and momentum operators (x and p) to their quantum counterparts ($\psi \mapsto x\psi$ and $\psi \mapsto -i\hbar\partial_x\psi$), and that Q is “linear” over polynomials, then Groenewold proved the following no-go theorem:

Theorem 6.1.1 (Groenewold [Gro46]). *It is not possible to find a map Q satisfying the above conditions as well as the commutator relation of Dirac for all polynomials f and g .*

By “linear” over polynomials, precisely one means that Q should satisfy the Weyl quantization laws, which are completely determined by the expression

$$(ax + bp)^n = (aQ(x) + bQ(p))^n.$$

This is the unique linearity condition on polynomials of degree less than or equal to two that leads to a consistent quantization. A consequence of these laws is, for example, that $Q(xp) = \frac{1}{2}(Q(x)Q(p) + Q(p)Q(x))$.

We can see the failure of the existence of such a map Q in Groenewold’s theorem by passing to polynomials of degree less than or equal to three. There is a classical

equality

$$\frac{1}{9}\{x^3, p^3\} = \frac{1}{3}\{x^2p, p^2x\}.$$

By applying the operator Q to either side, and using the commutator relation (Eq. 6.1) and the fact that $[Q(x), Q(p)] = i\hbar\mathbf{1}$ we obtain on the left

$$Q(x)^2Q(p)^2 - 2i\hbar Q(x)Q(p) - \frac{2}{3}\hbar^2\mathbf{1}$$

and on the right

$$Q(x)^2Q(p)^2 - 2i\hbar Q(x)Q(p) - \frac{1}{3}\hbar^2\mathbf{1}.$$

This is clearly inconsistent unless $\hbar = 0$.

Remark 6.1.2. In Groenewold's work the convention that $[Q_f, Q_g] = -i\hbar Q_{\{f,g\}}$ is used. This is opposite to Dirac's, and thus the negative of the above expressions are obtained. The expressions found are (in Dirac's convention)

$$\frac{1}{2}(Q(x)^2Q(p)^2 + Q(p)^2Q(x)^2) + \frac{1}{3}\hbar^2\mathbf{1}$$

on the left and

$$\frac{1}{2}(Q(x)^2Q(p)^2 + Q(p)^2Q(x)^2) + \frac{2}{3}\hbar^2\mathbf{1}.$$

These can be checked to be equivalent using (Eq. 6.1).

We may formulate the conditions one would like Q to satisfy as axioms in the following way:

1. $Q_x\psi = x\psi$ and $Q_p\psi = -i\hbar\partial_x\psi$,
2. $f \mapsto Q_f$ is linear,
3. $i\hbar Q_{\{f,g\}} = [Q_f, Q_g]$,
4. $Q_{g \circ f} = g(Q_f)$.

One can show as in [AE05] that *any three* of these axioms are inconsistent, at least without some modifications. In the next section we will discuss geometric quantization, one possible remedy for these inconsistencies. Geometric quantization amounts to restricting the space of observables f to exclude counterexamples such as the one above.

6.2 Geometric Quantization

In this section we will introduce geometric quantization for symplectic manifolds in terms of Kähler quantization.

To that end, let (M, ω) be a symplectic manifold. Recall that ω induces a Lie bracket, the Poisson bracket, on the vector space $C^\infty(M)$ of smooth functions on M , defined by $\{f, g\} := \omega(X_f, X_g)$ where X_f denotes the Hamiltonian vector field associated to f . The functions $f \in C^\infty(M)$ are to be interpreted as the classical observables for the classical system described by M .

By a quantization of (M, ω) we will mean a Hilbert space \mathcal{H} and a map

$$Q : \mathfrak{g} \rightarrow \text{Op}(\mathcal{H})$$

where $\mathfrak{g} \subseteq C^\infty(M)$ is a Lie subalgebra of $C^\infty(M)$ under the Poisson bracket, and $\text{Op}(\mathcal{H})$ denotes the self-adjoint operators defined on a dense subset of \mathcal{H} . In addition, Q must satisfy the following axioms:

1. The assignment $f \mapsto Q_f$ is linear.
2. If 1_M denotes the constant function with value 1 on M , then $Q_{1_M} = \mathbf{1}$.
3. $[Q_f, Q_g] = i\hbar Q_{\{f, g\}}$.
4. Given two symplectic manifolds (M, ω) and (M', ω') and a symplectomorphism $F : (M, \omega) \rightarrow (M', \omega')$ then $Q_{f' \circ F}$ and $Q'_{f'}$ should be conjugate by a unitary operator from \mathcal{H} to \mathcal{H}' (or on dense subsets of these).
5. One recovers the Schrödinger representation for \mathbb{R}^{2n} with the standard symplectic structure.

In practice it is not possible to achieve all of these axioms while still requiring that Q is defined on all of $C^\infty(M)$. However Kostant in [Kos70] and Souriau in [Sou69] developed a technique for achieving these axioms, provided we are allowed to restrict the domain of the map Q to some subalgebra of $C^\infty(M)$. We now describe that method. In fact, we will not be so concerned with the properties of the map Q from now on, but instead with the construction of the space \mathcal{H} , which we will refer to as the geometric quantization of (M, ω) , and will only mention the definition of Q in passing.

Definition 6.2.1 (Prequantum Line Bundle). *Let (M, ω) be a symplectic manifold. A prequantum line bundle on (M, ω) is a Hermitian line bundle (L, h) with unitary connection ∇ such that $F_\nabla = -2\pi i \omega$. If (M, ω) admits such a prequantum line bundle, we say M is prequantizable.*

Notice that $c_1(L) := [\frac{i}{2\pi}F_\nabla]$, and so this definition implies that $[\omega] = c_1(L)$ is an *integral* symplectic form. This is a topological condition on the symplectic manifold (M, ω) that may not always be satisfied, as the lattice $H^2(M; \mathbb{Z}) \hookrightarrow H_{dR}^2(M)$ may not hit any classes containing non-degenerate representatives.

The prequantum line bundle (L, h, ∇) provides a Hilbert space \mathcal{H} by taking the L^2 -completion of the space $\Gamma(M, L)$ of smooth sections. Given a function $f \in C^\infty(M)$ we can define an operator Q_f on $\Gamma(M, L)$ by

$$Q_f = f - i\hbar\nabla_{X_f}$$

which extends to \mathcal{H} . As previously stated, with respect to this map Q and space \mathcal{H} , the axioms are not satisfied. Kostant and Souriau's idea was to restrict to a nice subalgebra of $C^\infty(M)$ where Q does satisfy these axioms, by taking a *polarisation* of (M, ω) . We will focus on so-called Kähler polarisations. In the following we will define a Kähler polarisation in language suitable for generalisation to other types of polarisations, before providing an alternative characterisation of the condition that will be more useful for our goals.

Definition 6.2.2 (Kähler Polarisation). *Let (M, ω) be a symplectic manifold. A Kähler polarisation of (M, ω) is a subbundle $\mathcal{P} \subset TM_{\mathbb{C}}$ of the complexified tangent bundle $TM_{\mathbb{C}}$ such that*

1. \mathcal{P} is Lagrangian,
2. \mathcal{P} is involutive, $[\mathcal{P}, \mathcal{P}] \subseteq \mathcal{P}$,
3. $\mathcal{P} \cap \overline{\mathcal{P}} = 0$, and
4. the bilinear form on \mathcal{P} defined by $g(X, Y) := \omega(X, iY)$ is positive-definite.

Note that the first and third conditions imply that $TM_{\mathbb{C}} \cong \mathcal{P} \oplus \overline{\mathcal{P}}$. Therefore, given such a Kähler polarisation \mathcal{P} , one can define an almost-complex structure I on M by defining $I(v) = iv$ for $v \in \mathcal{P}$ and $I(v) = -iv$ for $v \in \overline{\mathcal{P}}$. Now involutivity of \mathcal{P} implies, by the Newlander-Nirenberg theorem, that the almost-complex structure I is integrable, and hence induces a complex structure on (M, ω) . The last condition implies that (M, ω, I) is a Kähler manifold, with Riemannian metric $g(X, Y) := \omega(X, IY)$ and Kähler form ω . With respect to this complex structure, $\mathcal{P} = T^{1,0}M$ and $\overline{\mathcal{P}} = T^{0,1}M$.

Conversely, given a Kähler manifold M , choosing $\mathcal{P} = T^{1,0}M$ delivers a Kähler polarisation, and the constructed almost-complex structure agrees with the existing almost-complex structure from the complex structure on M . Thus we could have defined a Kähler polarisation as a choice of integrable almost-complex structure I on M such that the triple (M, ω, I) is Kähler, and this is the characterisation we will use henceforth.

Now given a prequantizable symplectic manifold (M, ω) and a Kähler polarisation I of (M, ω) , observe that I acts on $(0, 1)$ -forms by $-i$. Take a prequantum line bundle (L, h, ∇) on (M, ω) and define operators

$$\nabla^{1,0} = \frac{1}{2}(\mathbf{1} - iI)\nabla, \quad \nabla^{0,1} = \frac{1}{2}(\mathbf{1} + iI)\nabla.$$

Then $\nabla = \nabla^{1,0} + \nabla^{0,1}$, and since I acts on $(0, 1)$ -forms by $-i$, $\nabla^{0,1}(s) \in \Omega^{0,1}(M, L)$ for all $s \in \Gamma(M, L)$. Furthermore, $(\nabla^{0,1})^2 = F_{\nabla}^{0,2} = 0$ since $F_{\nabla} = -2\pi i\omega$ which is a multiple of the Kähler form ω and hence of type $(1, 1)$. Thus $\nabla^{0,1}$ is a Dolbeault operator on the smooth complex line bundle L , and induces a holomorphic structure \mathcal{L}_I .

Definition 6.2.3 (Geometric Quantization). *The geometric quantization \mathcal{H}_I of the prequantizable symplectic manifold (M, ω) with respect to the prequantum line bundle (L, h, ∇) and Kähler polarisation I is the vector space $\mathcal{H}_I := H^0(M, \mathcal{L}_I)$ of global holomorphic sections of \mathcal{L}_I over M .*

Further, the geometric quantization at level k is the geometric quantization obtained after replacing L by L^k and ω by $k\omega$. Denote this by \mathcal{H}_I^k .

For completeness, the subalgebra of observables \mathfrak{g} that one must restrict to is given by

$$\mathfrak{g} := \{f \in C^\infty(M) \mid [X, X_f] \in \mathcal{P} \text{ for all } X \in \mathcal{P}\}.$$

The shortcoming of geometric quantization is that this algebra \mathfrak{g} is often very small, or zero. For more discussion see [Bla11] and [Woo97].

6.3 Families of Kähler Polarisation

The geometric quantization defined in the previous section depended on a choice of Kähler polarisation I for the symplectic manifold (M, ω) . Our goal is now to address this dependence. To do this we will consider *families* of polarisations.

A smooth family of Kähler polarisations will be a manifold \mathcal{T} and a map $I : \mathcal{T} \rightarrow \Gamma(M, \text{End}(TM))$ sending each $\sigma \in \mathcal{T}$ to a Kähler polarisation $I_\sigma : TM \rightarrow TM$, which is smooth in the sense that it defines a smooth section of the following pullback bundle,

$$\begin{array}{ccc} \text{pr}_2^* \text{End}(TM) & \longrightarrow & \text{End}(TM) \\ \downarrow & & \downarrow \pi \\ \mathcal{T} \times M & \xrightarrow{\text{pr}_2} & M. \end{array}$$

Equivalently, I is smooth if the corresponding map $\mathcal{T} \times TM \rightarrow TM$ is smooth. We have already seen in Chapter 4 one example of a family of Kähler polarizations for the moduli space $\mathcal{N}^{n,d}$ of stable holomorphic vector bundles. Here \mathcal{T} is the Teichmüller space of a compact Riemann surface Σ_g , and the map $I : \mathcal{T} \rightarrow \Gamma(\mathcal{N}^{n,d}, \text{End}(T\mathcal{N}^{n,d}))$ sends a conformal class of metrics on Σ_g to its corresponding Hodge star operator \star , which we saw gives an integrable almost-complex structure on $\mathcal{N}^{n,d}$. Indeed essentially the same map describes a family of Kähler polarizations for the Higgs moduli space $\mathcal{M}^{n,d}$, as we saw in Chapter 5. Later we will consider the case where \mathcal{T} is the Siegel upper-half space parametrising the distinct complex structures on a complex Abelian variety.

Given a smooth family of Kähler polarizations of a prequantized compact symplectic manifold (M, ω) , we obtain a set $\mathcal{H}^k := \bigsqcup_{\sigma \in \mathcal{T}} \mathcal{H}_\sigma^k$ with a natural projection $\pi : \mathcal{H}^k \rightarrow \mathcal{T}$. The fibres of this projection are the vector spaces $\mathcal{H}_\sigma^k := \mathcal{H}_{I_\sigma}^k$, which are finite-dimensional when M is compact. If these fibres were all of the same dimension, the disjoint union \mathcal{H}^k would be a candidate for a smooth vector bundle over \mathcal{T} . Fortunately, an argument presented by Hitchin in [Hit90] shows that provided we choose k to be large enough, then these fibres do in fact all have the same dimension.

Theorem 6.3.1. *Let (L, h, ∇) be a prequantum line bundle over a compact symplectic manifold (M, ω) with Kähler polarisation I . Then the dimension $\dim \mathcal{H}_I^k$ of the level k geometric quantization of (M, ω) is independent of I for k sufficiently large.*

Proof. The Hirzebruch-Riemann-Roch theorem states that

$$\chi(\mathcal{L}_I) = \int_M \text{Ch}(L) \text{Td}(M),$$

where the right-hand side depends only on the underlying topological line bundle L , but the left-hand side is the holomorphic Euler characteristic

$$\chi(\mathcal{L}_I) := \sum_{i=0}^n (-1)^i \dim H^i(M, \mathcal{L}_I).$$

Suppose we replace L by its tensor power L^k for some $k \in \mathbb{Z}_{\geq 0}$. Then with the induced Hermitian metric and connection, L^k is prequantum for $k\omega$ on M . The Kodaira vanishing theorem asserts that $H^i(M, \mathcal{L}_I^k) = 0$ for $i > 0$ whenever $\mathcal{L}_I^k \otimes K^*$ is a *positive* line bundle, that is, when $c_1(\mathcal{L}_I^k \otimes K^*)$ is represented by a Kähler form on M . But $c_1(\mathcal{L}_I^k \otimes K^*) = kc_1(L) - c_1(K) = k[\omega] - c_1(K)$. Since M is compact, we may take k sufficiently large such that this class will still be represented by a Kähler form, since ω is Kähler. Thus for k sufficiently large,

$\chi(\mathcal{L}_I^k) = \dim H^0(M, \mathcal{L}_I^k)$ and by the Hirzebruch-Riemann-Roch theorem, at least in the compact case, $\dim H^0(M, \mathcal{L}_I^k) = \dim \mathcal{H}_I^k$ does not depend on the choice of Kähler polarisation I on M . \square

One can now apply the deformation theory of Kodaira to show that the vector spaces \mathcal{H}_σ^k do form a vector bundle. In particular, associated to the family of Kähler polarisations there is a smooth family of elliptic operators $\nabla_\sigma^{0,1}$ on the smooth family of smooth vector bundles $L^k \rightarrow M_\sigma$ over the family of complex manifolds $M_\sigma \rightarrow \mathcal{T}$. In this setting elliptic regularity implies that the spaces $H^0(M, \mathcal{L}_\sigma^k) = \ker(\nabla_\sigma^{0,1} : \Gamma(M, L^k) \rightarrow \Omega^1(M, L^k))$ form a smooth vector bundle over \mathcal{T} .

6.4 Hitchin Connections

So far we have defined the geometric quantization of (M, ω) , and showed that, at least when M is compact, we can find a vector bundle \mathcal{H}^k over \mathcal{T} , a parameter space of Kähler polarisations, whose fibres are the quantum Hilbert spaces \mathcal{H}_σ^k from our geometric quantization.

As the next step in addressing the dependence of \mathcal{H}_σ^k on $\sigma \in \mathcal{T}$, Axelrod, Della Pietra, and Witten in [ADPW91] and Hitchin in [Hit90] had the idea of finding a projectively flat connection on the bundle \mathcal{H}^k . The parallel transport of the connection would identify the spaces \mathcal{H}_σ^k for various $\sigma \in \mathcal{T}$, and this identification would be canonical in the sense that the induced maps on $\mathbb{P}(H^0(M, \mathcal{L}_\sigma^k))$ for various σ would be independent of the paths chosen, at least when \mathcal{T} is simply-connected.

6.4.1 Definition

In [And06] Andersen gave a definition of a Hitchin connection for a general symplectic manifold, that we present here. For each $\sigma \in \mathcal{T}$, we obtain a vector space \mathcal{H}_σ^k of holomorphic sections of \mathcal{L}_σ^k . Each such vector space sits as a subspace of the infinite-dimensional vector space $\Gamma(M, L^k)$, and therefore we obtain a subset $\mathcal{H}^k \subset \Gamma(M, L^k) \times \mathcal{T}$ of the trivial bundle $\Gamma(M, L^k) \times \mathcal{T}$, with fibre over each $\sigma \in \mathcal{T}$ given by \mathcal{H}_σ^k .

As we observed earlier, this subset is actually a smooth subbundle of the trivial bundle $\Gamma(M, L^k) \times \mathcal{T}$, and therefore we may specify connections on \mathcal{H}^k by defining a connection on the trivial bundle that preserves the fibres.

Definition 6.4.1 (Hitchin Connection). *A Hitchin connection on the bundle \mathcal{H}^k is a connection $\nabla = d + u$ on the infinite-dimensional trivial bundle $\Gamma(M, L^k) \times \mathcal{T} \rightarrow \mathcal{T}$ such that*

1. ∇ preserves the fibres \mathcal{H}_σ^k ,
2. u takes values in differential operators on sections of L^k , and
3. the induced connection on the bundle \mathcal{H}^k is projectively flat.

To say that ∇ preserves the fibres \mathcal{H}_σ^k means that if s is some local section of \mathcal{H}^k inside $\Gamma(M, L^k) \times \mathcal{T}$ and V is a vector field on \mathcal{T} , then

$$(\nabla_V s)_\sigma \in \mathcal{H}_\sigma^k$$

for all σ where s is defined. In terms of the Dolbeault operator $\nabla_\sigma^{0,1}$ this can be rephrased as

$$\nabla_\sigma^{0,1}(\nabla_V s)_\sigma = 0.$$

The following lemma gives a useful characterisation of this condition which will be used later. Given a family of Kähler polarisations $I : \mathcal{T} \rightarrow \Gamma(M, \text{End}(TM))$ we may differentiate this family in the direction of a vector field V on \mathcal{T} . Following the notation of [And06], write this derivative as $V[I]$. Then we have the following.

Lemma 6.4.2. *A Hitchin connection ∇ preserves the fibres \mathcal{H}_σ^k if and only if for any local section s of \mathcal{H}^k defined on a neighbourhood of $\sigma \in \mathcal{T}$, and any vector field V on \mathcal{T} , we have*

$$\frac{i}{2} V[I]_\sigma \nabla_\sigma^{1,0} s_\sigma = \nabla_\sigma^{0,1}(u_\sigma(V_\sigma)(s_\sigma)).$$

Proof. Note that the differential equation

$$\nabla_\sigma^{0,1}(\nabla_V s)_\sigma = 0$$

may be rewritten as

$$\nabla_\sigma^{0,1}(ds)_\sigma(V_\sigma) + \nabla_\sigma^{0,1}u_\sigma(V_\sigma)(s_\sigma) = 0.$$

On the other hand since s is a local section of \mathcal{H}^k , it satisfies the differential equation

$$\nabla_\sigma^{0,1}s_\sigma = 0$$

for all σ . This may be rewritten as

$$\frac{i}{2}(\mathbf{1} + iI_\sigma)\nabla s_\sigma = 0.$$

We can differentiate this along the vector field V , to obtain

$$\frac{i}{2} V[I]_\sigma \nabla s_\sigma + \nabla_\sigma^{0,1}(u_\sigma(V_\sigma)(s_\sigma)) = 0.$$

Subtracting the two expressions and using the fact that $\nabla s_\sigma = \nabla_\sigma^{1,0} s_\sigma + \nabla_\sigma^{0,1} s_\sigma = \nabla_\sigma^{1,0} s_\sigma$, we obtain

$$\frac{i}{2} V[I]_\sigma \nabla_\sigma^{1,0} s_\sigma = \nabla_\sigma^{0,1} (u_\sigma(V_\sigma)(s_\sigma)).$$

□

Here we use the notation $V[I]$ for the derivative of the family I in the direction of the vector field V on \mathcal{T} .

6.4.2 The Symmetric Tensor G

In order to write down a Hitchin connection, we will also need to know about certain symmetric tensors $G_\sigma \in \Gamma(M, S^2 T_\sigma^{1,0} M)$. Using these symmetric tensors we will write down the expression for a Hitchin connection for compact symplectic manifolds satisfying certain assumptions. Full details of this result can be found in [And06].

At a point $\sigma \in \mathcal{T}$ we have that $V[I]_\sigma : TM \rightarrow TM$ is again an endomorphism of TM . Differentiating the identity $I_\sigma^2 = -\mathbf{1}$ in the direction of V , we obtain the relation

$$V[I]_\sigma I_\sigma + I_\sigma V[I]_\sigma = 0$$

for all $\sigma \in \mathcal{T}$.

This relation implies that after extending I and $V[I]$ to $TM_{\mathbb{C}}$, the endomorphism $V[I]_\sigma$ interchanges the type of a vector field $X \in \Gamma(M, TM)$. In particular, if $I_\sigma X = iX$ so that X is of type $(1, 0)$, then $I_\sigma V[I]_\sigma X = -iV[I]_\sigma X$ and similarly for when $I_\sigma X = -iX$. Thus with respect to the splitting

$$TM_{\mathbb{C}} = T_\sigma^{1,0} \oplus T_\sigma^{0,1}$$

we have

$$V[I]_\sigma \in \Gamma(M, \text{Hom}(T_\sigma^{1,0}, T_\sigma^{0,1}) \oplus \text{Hom}(T_\sigma^{0,1}, T_\sigma^{1,0})).$$

Write the two components of this direct sum decomposition as

$$V[I]_\sigma = V[I]_\sigma'' + V[I]_\sigma'$$

where $V[I]_\sigma'' : T_\sigma^{1,0} \rightarrow T_\sigma^{0,1}$ and $V[I]_\sigma' : T_\sigma^{0,1} \rightarrow T_\sigma^{1,0}$. Now $V[I]_\sigma$ is a section of $\text{End}(TM_{\mathbb{C}}) = T^*M_{\mathbb{C}} \otimes TM_{\mathbb{C}}$, so there exists some bivector field $\tilde{G}(V)$ depending on V that we may contract with ω in such a way that

$$\tilde{G}(V) \cdot \omega = V[I].$$

The non-degeneracy of the symplectic form ω means that this tensor $\tilde{G}(V)$ is unique. Additionally, associated to the family I and the symplectic form ω we

have a family of Kähler metrics $g = \omega \cdot I$ obtained by contraction of ω with I in the second input. For each $\sigma \in \mathcal{T}$, $g_\sigma = \omega \cdot I_\sigma$ is symmetric, and since ω is independent of σ , if we differentiate in the direction of V we again obtain a symmetric tensor

$$V[g] = \omega \cdot V[I] = \omega \cdot \tilde{G}(V) \cdot \omega.$$

The antisymmetry of ω and the symmetry of $V[g]$ implies that $\tilde{G}(V)$ is a *symmetric* bivector field. Furthermore since ω is of type $(1, 1)$ with respect to I_σ and $V[I]_\sigma$ interchanges type, $V[g]_\sigma$ contains terms only of type $(2, 0)$ and $(0, 2)$:

$$V[g]_\sigma \in \Gamma(M, S^2 T_\sigma^{1,0*} \oplus S^2 T_\sigma^{0,1*}).$$

But then $V[g] = \omega \cdot \tilde{G}(V) \cdot \omega$ implies that $\tilde{G}(V)$ must also only contain terms of type $(2, 0)$ and $(0, 2)$, so we actually have

$$\tilde{G}(V) \in \Gamma(M, S^2 T_\sigma^{1,0} \oplus S^2 T_\sigma^{0,1}).$$

Write $\tilde{G}(V) = G(V) + \bar{G}(V)$ where $G(V)$ is of type $(2, 0)$ and $\bar{G}(V)$ is of type $(0, 2)$. Then $G(V)$ is the symmetric tensor that we are interested in.

6.4.3 A Hitchin Connection for Compact Symplectic Manifolds

In this section we will state Andersen's construction of a Hitchin connection for compact symplectic manifolds. In the special case of $\mathcal{N}^{n,d}$ the connection will agree with those constructed by Hitchin in [Hit90] and Axelrod, Della Pietra, and Witten in [ADPW91].

In order to state the result of Andersen it is necessary to make two more assumptions on the family of Kähler polarisations I . Firstly, we must assume that the family I of Kähler polarisations is *holomorphic* in the following sense.

Definition 6.4.3. *A family I of Kähler polarisations is holomorphic if the parameter space \mathcal{T} is a complex manifold with complex structure J , and for all vector fields $V = V' + V''$ on \mathcal{T} we have*

$$V'[I] = V[I]' \text{ and } V''[I] = V[I]''.$$

We remark that this holomorphicity condition implies that $G(V') = G(V)$. Further, we remark that a family of Kähler polarisations is holomorphic if and only if the corresponding almost-complex structure on $\mathcal{T} \times M$ given by $\tilde{I}(V \oplus X) = JV \oplus IX$ where J is the integrable almost-complex structure on \mathcal{T} , and $V \oplus X$ is a vector field on $\mathcal{T} \times M$, is integrable. This motivates the terminology.

The second assumption we must make is that the family I is *rigid*.

Definition 6.4.4. A family of Kähler polarisations is rigid if, for each $\bar{\partial}$ operator $\bar{\partial}_\sigma$ on M induced by I_σ , we have

$$\bar{\partial}_\sigma(G(V))_\sigma = 0$$

for all vector fields V on \mathcal{T} and for all $\sigma \in \mathcal{T}$.

Remark 6.4.5. For a rigid holomorphic family of Kähler polarisations, $V''[I]\nabla_\sigma^{1,0}s = 0$, and so we only need to consider holomorphic vector fields on \mathcal{T} for the purposes of defining the Hitchin connection and checking it preserves fibres.

In addition to these assumptions, we will also need to define a certain differential operator depending on $G(V)$. Denote by Δ_G the operator given by the composition

$$\begin{aligned} \Delta_G : \Gamma(M, L^k) &\xrightarrow{\nabla_\sigma^{1,0}} \Gamma(M, T_\sigma^{1,0*} \otimes L^k) \xrightarrow{G \otimes \mathbf{1}} \Gamma(M, T_\sigma^{1,0} \otimes L^k) \\ &\xrightarrow{\nabla_\sigma^{1,0} \otimes \mathbf{1} + \mathbf{1} \otimes \nabla_\sigma^{1,0}} \Gamma(M, T_\sigma^{1,0*} \otimes T_\sigma^{1,0} \otimes L^k) \xrightarrow{\text{tr}} \Gamma(M, L^k). \end{aligned}$$

Now we can define the connection form u . On a compact Kähler manifold (M, ω) we can define a certain smooth real-valued function F , the *Ricci potential*, given by

$$\text{Ric} = \text{Ric}^H + 2i\partial\bar{\partial}F$$

where Ric is the Ricci form given by contracting the Ricci tensor with I , and Ric^H is its harmonic part. For each $\lambda, k \in \mathbb{Z}$ such that $2k \neq -\lambda$, let $u(V)$ be given by

$$u(V) := \frac{1}{4\pi(2k + \lambda)} (\Delta_{G(V)} - \nabla_{G(V)dF} - \lambda V'[F]) + V'[F] \quad (\text{Eq. 6.2})$$

Theorem 6.4.6 (Andersen [And06]). *Let (M, ω) be a prequantized compact symplectic manifold with the property that $c_1(M) = \lambda[\omega]$ for some $\lambda \in \mathbb{Z}$. Suppose $H^1(M, \mathbb{R}) = 0$, and that $I : \mathcal{T} \rightarrow \Gamma(M, \text{End}(TM))$ is a rigid holomorphic family of Kähler polarisations of (M, ω) , and let $k \in \mathbb{Z}$ be such that $2k + \lambda \neq 0$. Then u defined above satisfies the conditions of Lemma 6.4.2 and preserves the holomorphic subspaces of $\Gamma(M, L^k)$.*

As was discussed in Section 4.5.3, the topological assumptions above are satisfied for the moduli space $\hat{\mathcal{N}}^{n,d}$ of stable holomorphic bundles over a surface with fixed determinant. In this setting the connection of Andersen is precisely the connection constructed by Hitchin, and agrees with the connection constructed by Axelrod, Della Pietra, and Witten, and is therefore projectively flat by their arguments. This gives:

Theorem 6.4.7 (Andersen [And06]). *The connection defined by (Eq. 6.2) on the moduli space $\hat{\mathcal{N}}^{n,d}$ of stable holomorphic bundles over a surface with fixed determinant is a Hitchin connection in the sense of Definition 6.4.1.*

A purely differential-geometric argument for the projective flatness of the connection over any symplectic manifold is given by Andersen in [AG14].

Chapter 7

Quantization of the Higgs Bundle Moduli Space

In this chapter we will investigate the geometric quantization of the moduli space of Higgs bundles over a Riemann surface. In particular we will construct a projectively flat Hitchin connection for the case of Higgs line bundles, as well as investigate the existence of such a connection that preserves the weight spaces of the \mathbb{C}^* action for higher rank.

We will proceed by first investigating the geometric quantization of the stable bundle moduli space carried out by Hitchin in [Hit90]. Hitchin utilised cohomological techniques to isolate the connection, and then used properties of the Hitchin system in order to prove projective flatness of this connection. The techniques for the quantization of the stable bundle moduli space turn out to not be applicable in the case of the Higgs moduli space, as was noted by Hitchin himself in [Hit91].

In order to get around this issue, we will take the explicit expression for the second-order differential operator u defining the Hitchin connection and modify this for the case of the Higgs bundle moduli space. The rank 1 case falls under the problem of geometrically quantizing the cotangent bundle of an Abelian variety, the case of the Abelian variety itself being covered by Hitchin, as well as more explicitly by Blaavand in [Bla11].

The problem of geometrically quantizing the Higgs moduli space has also been approached by Witten in [Wit91] from the perspective of physics, utilising the interpretation of the Higgs moduli space as the configuration space for *complex* 3-dimensional Chern-Simons theory. Here a real polarisation is used in order to construct a Hilbert space of polarised sections, and Witten found a projectively flat connection for the bundle of spaces of polarised sections over Teichmüller space. The same approach was used by Andersen in [AG14], where a family of projectively flat connections depending on a parameter $t \in \mathbb{C}$ was constructed using purely differential-geometric techniques. It is not clear how the \mathbb{C}^* action on the spaces

of polarised sections interacts with the projectively flat connections constructed in this way. In particular, it is not known whether or not the projectively flat connections preserve the weight spaces of the \mathbb{C}^* action.

In the case of Kähler polarisations, one considers the infinite-dimensional vector space of holomorphic sections of a determinant bundle over the Higgs moduli space. Although not a Hilbert space, this vector space is the right space to consider in the quantization of $\mathcal{M}^{n,d}$. In particular it may be interpreted as a space of generalised theta functions, for which the dimension of the graded pieces is of interest in conformal field theory. For example, an equivariant Verlinde formula was proven to compute the dimensions of these graded pieces independently by Andersen, Gukov, and Pei in [AGP16] and Halpern-Leistner in [HL16]. The explicit relationship between the space of holomorphic sections and the corresponding L^2 sections on $\mathcal{N}^{n,d}$ should mirror the situation for comparing different types of polarisations of \mathbb{R}^{2n} , but has not been explicitly determined.

7.1 Hitchin's Techniques

In [Hit90] Hitchin used cohomological techniques to construct the Hitchin connection for the moduli space of stable bundles over a compact Riemann surface. In this section we will briefly review these techniques, and then investigate their applicability in the case of the Higgs bundles moduli space.

7.1.1 Dolbeault Hypercohomology

In this section we will define Dolbeault hypercohomology in the form applicable to the construction of the Hitchin connection for the stable bundle moduli spaces. First, introduce vector bundles $\mathcal{D}^i(\mathcal{L})$ whose holomorphic sections are the holomorphic linear differential operators on sections of \mathcal{L} of order i . These may be identified as the vector bundles $\mathcal{D}^i(\mathcal{L}) := \text{Hom}(\mathcal{J}^i(\mathcal{L}), \mathcal{L})$ where $\mathcal{J}^i(\mathcal{L})$ denotes the vector bundle of i -jets of holomorphic sections of \mathcal{L} .

Next define spaces $A^p := \Omega^{0,p}(M, \mathcal{D}^1(\mathcal{L}^k)) \oplus \Omega^{0,p-1}(M, \mathcal{L}^k)$ of cocycles and a differential map $d_s : A^p \rightarrow A^{p+1}$ for each holomorphic section $s \in H^0(M, \mathcal{L}^k)$ given by

$$d_s(D, u) := (\bar{\partial}D, \bar{\partial}u + (-1)^{p-1}Ds).$$

Lemma 7.1.1. *The differential d_s squares to 0.*

Proof. We have the equality

$$\begin{aligned}
d_s(d_s(D, u)) &= d_s(\bar{\partial}D, \bar{\partial}u + (-1)^{p-1}Ds) \\
&= (0, \bar{\partial}(\bar{\partial}u + (-1)^{p-1}Ds) + (-1)^p\bar{\partial}Ds) \\
&= (0, (-1)^p(-\bar{\partial}(Ds) + (\bar{\partial}D)s)) \\
&= 0,
\end{aligned}$$

where in the last step we have used that $\bar{\partial}s = 0$ since s is holomorphic. \square

By the lemma, we obtain a cochain complex, and may take its cohomology.

Definition 7.1.2. *The p th cohomology of the complex (A^\bullet, d_s) is called the p th Dolbeault hypercohomology of $\mathcal{D}^1(\mathcal{L}^k)$ with respect to s , and is denoted*

$$\mathbb{H}_s^p(M, \mathcal{D}^1(\mathcal{L}^k)).$$

7.1.2 Existence of a Connection

The Dolbeault hypercohomology defined in the previous section was used by Hitchin in [Hit90] and (for the case of Abelian varieties) by Welters in [Wel83] to capture the existence of a Hitchin connection on compact symplectic manifolds.

To see how this complex captures the existence of a Hitchin connection, observe that if one takes the element $iV[I]\nabla^{1,0} \in \Omega^{0,1}(M, \mathcal{D}^1(\mathcal{L}^k))$ then some $u(V)(s)$ satisfying $d_s(iV[I]\nabla^{1,0}, u(V)(s)) = 0$ provides a solution to the differential equation given by Lemma 6.4.2. In particular if one can choose $u(V)(s)$ in a smoothly varying way as V and s vary such that u is linear in s , the \mathcal{L}^k -valued form u would define a Hitchin connection on the bundle of vector spaces \mathcal{H}^k over the compact symplectic manifold M . We then have the following theorem.

Theorem 7.1.3 (Hitchin [Hit90]). *Suppose that $[\omega] : H^0(M, T_\sigma^{1,0}) \rightarrow H^1(M, \mathcal{O})$ is an isomorphism for each $\sigma \in \mathcal{T}$, and that for each $s \in H^0(M, \mathcal{L}^k)$ and V a vector field on \mathcal{T} , there exists a smoothly varying cohomology class $A(V, s) \in \mathbb{H}_s^1(M, \mathcal{D}^1(\mathcal{L}^k))$ depending linearly on s such that $-i\rho(A(V, s)) = [V]$. Then A defines a Hitchin connection on the bundle of spaces \mathcal{H}^k over \mathcal{T} .*

Here $\rho(A(V, s))$ is the symbol map induced by the short exact sequence

$$0 \longrightarrow \mathcal{D}^{i-1}(\mathcal{L}^k) \longrightarrow \mathcal{D}^i(\mathcal{L}^k) \xrightarrow{\rho} S^i T_\sigma^{1,0} \longrightarrow 0,$$

where $S^i T$ denotes the i th symmetric power of the holomorphic tangent bundle $T_\sigma^{1,0}$ and $\rho : \mathcal{D}^i(\mathcal{L}^k) \rightarrow S^i T_\sigma^{1,0}$ is the symbol map on differential operators. Applying this map ρ gives a homomorphism $\rho : \mathbb{H}_s^1(M, \mathcal{D}^1(M, \mathcal{L}^k)) \rightarrow H^1(M, T)$ and

$\rho(V[I]\nabla^{1,0}, u) = [V]$, the Kodaira-Spencer deformation class corresponding to the infinitesimal deformation V of the complex structure on M .

The proof of the theorem proceeds by considering the short exact sequence above for $i = 1$, which becomes

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{D}^1(\mathcal{L}^k) \longrightarrow T^{1,0} \longrightarrow 0.$$

By Atiyah's work in [Ati57a] on extensions of bundles, this extension of $T^{1,0}$ by \mathcal{O} defines a class in $H^1(M, T^{1,0*})$, which is a multiple of the first Chern class $k[\omega]$ of \mathcal{L}^k . In the corresponding long exact sequence we have

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0(M, \mathcal{O}) & \longrightarrow & H^0(M, \mathcal{D}^1(\mathcal{L}^k)) & \longrightarrow & H^0(M, T^{1,0}) \\ & & & & \text{[}\omega\text{]} & & \\ & & \longleftarrow & & \longleftarrow & & \longleftarrow \\ & & H^1(M, \mathcal{O}) & \longrightarrow & H^1(M, \mathcal{D}^1(\mathcal{L}^k)) & \xrightarrow{\rho} & H^1(M, T^{1,0}) \\ & & & & & & \\ & & \longleftarrow & & \longleftarrow & & \longleftarrow \\ & & H^2(M, \mathcal{O}) & \longrightarrow & \dots & & \end{array}$$

The assumption that $[\omega]$ was an isomorphism implies that the symbol map $\rho : H^1(M, \mathcal{D}^1(\mathcal{L}^k)) \rightarrow H^1(M, T^{1,0})$ is injective. Injectivity of the symbol map implies the existence of a pair (D, u) which is d_s -closed. Namely, given an $A(V, s)$ as in the statement of the theorem, represented by some pair $(D, w) \in A^1$, we have $-i\rho(D)$ is cohomologous to $[V]$ in $\Omega^{0,1}(M, T^{1,0})$. By injectivity of the map ρ , this implies that there exists $P \in \Omega^0(M, \mathcal{D}^1(\mathcal{L}^k))$ such that $D - iV[I]\nabla^{1,0} = \bar{\partial}P$. Then $u(V)(s) := Ps + w$ gives a solution to the differential equation in Lemma 6.4.2.

Additionally, the assumption that $[\omega]$ is an isomorphism also implies that $H^0(M, \mathcal{O}) \cong H^0(M, \mathcal{D}^1(\mathcal{L}^k))$. But then if we take two solutions u_1 and u_2 with $(iV[I]\nabla^{1,0}, u_1)$ cohomologous to $(iV[I]\nabla^{1,0}, u_2)$ in $\mathbb{H}_s^1(M, \mathcal{D}^1(\mathcal{L}^k))$, one has

$$(0, u_1 - u_2) = d_s Q$$

for some Q in $A^0 = \Omega^0(M, \mathcal{D}^1(\mathcal{L}^k))$. Since the first term is zero, this simply means $\bar{\partial}Q = 0$ so that Q is a global holomorphic function on M . When M is compact as assumed, $Q \in \mathbb{C}$ is a constant, so that the class $A(V, s)$ defines a connection on \mathcal{H}^k up to a constant multiple of s , and therefore descends to give a well-defined connection on $\mathbb{P}(\mathcal{H}^k)$.

In the case of the Higgs bundle moduli space, $\mathcal{M}^{n,d}$ is not compact, instead containing $T^*\mathcal{N}^{n,d}$ as an open dense subset. Thus in this setting, even after \mathcal{L}^k is replaced by \mathcal{L}_H^k , the class $A(V, s)$ would be unable to capture $u(V)(s)$ up to a constant, but instead only up to some non-zero holomorphic function on $\mathcal{M}^{n,d}$.

Additionally, it is not immediate that the assumptions that $[\omega]$ is an isomorphism and that the classes $A(V, s)$ exist are even satisfied in the case of the Higgs moduli space.

This indicates that the cohomological techniques of Hitchin used in [Hit90] to construct a Hitchin connection are not applicable for the case of the Higgs moduli space. This was noted by Hitchin in [Hit91] where he reviewed the paper [ADPW91] of Axelrod, Della Pietra and Witten.

7.2 Explicit Connection

In the previous section we saw that the cohomological techniques of Hitchin are not applicable when working with moduli spaces of Higgs bundles over a compact Riemann surface. However, in [Hit90] an explicit expression for the connection form was obtained. In fact this expression is precisely that given by the differential operator (Eq. 6.2), as was noted previously.

Expressed in local coordinates, the connection form is defined by

$$u(V)(s) = \frac{1}{4\pi(2k + \lambda)} (\nabla_i(G^{ij}\nabla_j(s)) - 2G^{ij}\frac{\partial F}{\partial z^i}\nabla_j(s) + ikf_G s)$$

where f_G is such that

$$\bar{\partial}f_G = 2iG^{ij}\omega_{j\bar{k}}\frac{\partial F}{\partial z^i} + (\nabla_i G^{ij})\omega_{j\bar{k}}.$$

Such a function exists due to the vanishing of $H^1(\hat{\mathcal{N}}^{n,d}, \mathcal{O})$.

In the following sections, we will see that, at least in the case of $n = 1$, one may modify this connection form u to obtain a connection preserving the holomorphic subspaces on the corresponding trivial bundle $\Gamma(\mathcal{M}^{n,d}, L_H^k) \times \mathcal{T} \rightarrow \mathcal{T}$ for the Higgs bundle moduli space. Indeed when $n = 1$ the stable bundle moduli space $\mathcal{N}^{1,d}$ over the compact Riemann surface Σ_g is the Jacobian $\text{Jac}(\Sigma_g)$. This is a complex torus, with a naturally flat metric. Thus the Ricci potential F vanishes, as does the constant λ .

This provides a simple expression for the Hitchin connection in this case, simply

$$u(V)(s) := \frac{1}{8\pi k}\nabla_i(G^{ij}\nabla_j(s)).$$

Indeed this connection form actually defines a Hitchin connection for any complex Abelian variety when the parameter space \mathcal{T} is taken to be the Siegel upper-half plane, and the curvature may be explicitly computed as in [Bla11]. We will take on this point of view in the following section.

7.3 For the Cotangent Bundle of Abelian Varieties

The Higgs line bundle moduli space is $\mathcal{M}^{1,d} \cong T^*\text{Jac}(\Sigma_g)$, which is the cotangent bundle to the complex Abelian variety $\text{Jac}(\Sigma_g)$. Upon this moduli space is a \mathbb{C}^* action, and we will be concerned with finding a connection which preserves this \mathbb{C}^* action. The existence of such a connection would indicate that the \mathbb{C}^* action on the Higgs moduli space is a *quantizable feature*, and is, in the sense of geometric quantization, independent of the Kähler polarisation.

We will now phrase the problem in terms of the geometric quantization of the cotangent bundle to a complex Abelian variety. Let $M = \mathbb{R}^{2g}/\mathbb{Z}^{2g}$ be the standard $2g$ -torus. Then a choice of complex structure on M is given by a choice of *period matrix* Ω . This is a $g \times 2g$ matrix $\Omega = (A, B)$ sending the basis \mathbb{Z}^{2g} to $A\mathbb{Z}^g + B\mathbb{Z}^g$. The corresponding complex structure is defined by $M_\Omega = \mathbb{C}^g/(\mathbb{Z}^g + B\mathbb{Z}^g)$. When the Abelian variety is principally polarised, we may always choose this matrix to be of the form $\Omega = (\mathbf{1}, Z)$ for some $g \times g$ matrix $Z \in \mathcal{T}$ where $\mathcal{T} := \{Z \in \text{Mat}(n, \mathbb{C}) \mid Z = Z^\top, \text{Im}(Z) > 0\}$ is the *Siegel upper-half space*. Then we define a complex structure on M by taking coordinates $z^i = x^i + Z_j^i y^j$ and $\bar{z}^i = x^i + \bar{Z}_j^i y^j$, giving a diffeomorphism

$$M \cong \mathbb{C}^g/(\mathbb{Z}^g + Z\mathbb{Z}^g).$$

Definition 7.3.1 (Complex Abelian Variety). *A principally polarised complex Abelian variety of dimension g is a complex torus*

$$\mathbb{C}^g/(\mathbb{Z}^g + Z\mathbb{Z}^g)$$

for some $Z \in \mathcal{T}$.

Remark 7.3.2. It can be shown that every complex Abelian variety is a connected compact complex Lie group, and in fact that such Lie groups are precisely the complex tori. The condition that a complex torus comes from a quotient by a lattice $\mathbb{Z}^g + Z\mathbb{Z}^g$ for some $Z \in \mathcal{T}$ implies that the torus is a projective algebraic variety.

Remark 7.3.3. Although we are only restricting to the case of principally polarised Abelian varieties here, this causes no trouble since every Jacobian of a compact Riemann surface is principally polarised, and indeed every complex Abelian variety admits a finite covering by a principally polarised Abelian variety.

We also have a diffeomorphism $M \times \mathbb{R}^{2g} \cong \mathbb{C}^g/(\mathbb{Z}^g + Z\mathbb{Z}^g) \times \mathbb{C}^g$ given by the same map on coordinates in the second factor, and since we have natural isomorphisms $T^*M \cong M \times \mathbb{R}^{2g}$ and $T^*(\mathbb{C}^g/(\mathbb{Z}^g + Z\mathbb{Z}^g)) \cong \mathbb{C}^g/(\mathbb{Z}^g + \Omega\mathbb{Z}^g) \times \mathbb{C}^g$, we get a complex structure on T^*M .

Let ω be the standard symplectic form on M descending from the standard symplectic form on \mathbb{R}^{2g} . Let $L \rightarrow M$ be a prequantum line bundle for this symplectic form. Such a prequantum line bundle may be described explicitly.

Let $Z \in \mathcal{T}$, and $\{\lambda_1, \dots, \lambda_{2g}\}$ an integral basis of $\mathbb{Z}^g + Z\mathbb{Z}^g$ such that in the dual coordinates $x = (x^1, \dots, x^{2g})$ we have

$$\omega = \sum_{i=1}^g dx^i \wedge dx^{i+g}.$$

Then multipliers for the line bundle $L \rightarrow M$ on the universal cover \mathbb{C}^g are given by

$$e_{\lambda_i}(z) = 1, \quad e_{\lambda_{i+g}}(z) = \exp(-2\pi i k z_i - \pi i k Z_{ii})$$

for $i = 1, \dots, g$. More details on this construction can be found in [GH78, Ch. 2 §6].

Given a choice of $\Omega \in \mathcal{T}$, we obtain a Kähler polarisation of M with respect to ω , and space of holomorphic sections $\mathcal{H}_Z^k := H^0(M, \mathcal{L}_Z^k)$ at some level k is the space of *theta functions*. In the case where $g = 1$ and $k = 1$, this space is one-dimensional and is generated by a section of \mathcal{L} given by the *Jacobi theta function* on the universal cover:

$$\vartheta(z; \tau) := \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z},$$

where $\tau \in \mathbb{H} = \mathcal{T}$.

Fix $Z \in \mathcal{T}$ to obtain an isomorphism $\phi_Z : T^*M \rightarrow M \times \mathbb{C}^g$. Then we have an action of \mathbb{C}^* on T^*M given by pulling back multiplication in the second factor through this isomorphism.

Consider now the line bundle $L_H := \pi^*L \rightarrow T^*M$ where $\pi : T^*M \rightarrow M$ is the projection. We will assume for now that L_H is also prequantum for some symplectic form on T^*M , and that a choice of $Z \in \mathcal{T}$ is a Kähler polarisation of T^*M with respect to this symplectic form. Then again we can consider the spaces \mathcal{H}_Z^k of holomorphic sections of L_H^k . For any given Ω , the \mathbb{C}^* action on T^*M lifts to \mathcal{L}_H^k , and then to \mathcal{H}_Z^k . Thus we obtain a decomposition

$$\mathcal{H}_{H,Z}^k = \widehat{\bigoplus}_{m \in \mathbb{Z}} (\mathcal{H}_Z^k)^m$$

where $(\mathcal{H}_Z^k)^m$ are the weight spaces of the \mathbb{C}^* action, with $\lambda \cdot s = \lambda^m s$ for $s \in (\mathcal{H}_Z^k)^m$. Note these spaces are empty for $m < 0$. We will remark on the exact nature of this direct sum momentarily.

Lemma 7.3.4. *For each $Z \in \mathcal{T}$ and $m \in \mathbb{Z}$, we have $(\mathcal{H}_Z^k)^m \cong H^0(M, \mathcal{L}^k \otimes S^m T_Z^{1,0} M)$.*

Proof. Let $\mathbb{P}(T^*M)$ denote the projectivisation of the vector bundle T^*M , given by $\mathbb{P}(T^*M) = (T^*M \setminus \{0\})/\mathbb{C}^*$, and let $p : \mathbb{P}(T^*M) \rightarrow M$ be the corresponding projection. Then

$$\begin{aligned} (\mathcal{H}_Z^k)^m &= H^0(T^*M, \mathcal{L}_H^k)^m \\ &\cong H^0(\mathbb{P}(T^*M), p^* \mathcal{L}^k \otimes \mathcal{O}(m)) \\ &\cong H^0(M, p_*(p^* \mathcal{L}^k \otimes \mathcal{O}(m))) \\ &\cong H^0(M, \mathcal{L}^k \otimes S^m T_{\Omega}^{1,0} M). \end{aligned}$$

□

Now $H^0(T^*M, \mathcal{L}_H^k) \cong H^0(M \times \mathbb{C}^g, \text{pr}_1^* \mathcal{L}^k) \cong H^0(M, \mathcal{L}^k) \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{C}^g)$, and through this tensor product the \mathbb{C}^* action on $H^0(T^*M, \mathcal{L}_H^k)$ is given by $\lambda \cdot s \otimes f(z) = s \otimes f(\lambda z)$. Thus the weight spaces are given by $H^0(M, \mathcal{L}^k) \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{C}^g)^m$, where $\mathcal{O}(\mathbb{C}^g)^m$ consists of those holomorphic functions on \mathbb{C}^g with the property that $f(\lambda z) = \lambda^m f(z)$ for all $z \in \mathbb{C}$. Such functions are necessarily homogenous polynomials in the coordinates on \mathbb{C}^g , as can be seen by expanding in a Taylor polynomial around $0 \in \mathbb{C}^g$. This representation of weight m sections will be useful later.

Remark 7.3.5. By considering the representation of elements of $(\mathcal{H}_Z^k)^m$ as homogenous polynomials tensored by theta functions, we see that the direct sum decomposition

$$\mathcal{H}_{H,Z}^k = \widehat{\bigoplus}_{m \in \mathbb{Z}} (\mathcal{H}_Z^k)^m$$

is in the following sense: a collection (s^m) , with $s^m \in (\mathcal{H}_Z^k)^m$, defines an element in $\mathcal{H}_{H,Z}^k$ precisely if the corresponding homogenous polynomials sum to a convergent power series with infinite radius of convergence on \mathbb{C}^g .

In order to geometrically quantize T^*M , we will construct separately Hitchin connections for the holomorphic vector bundles $\mathcal{L}^k \otimes S^m T^{1,0} M$ for each $m \in \mathbb{Z}_{\geq 0}$, and show that these combine to give a projectively flat Hitchin connection for \bar{L}_H^k over T^*M . By construction this connection will necessarily preserve the \mathbb{C}^* action on the spaces \mathcal{H}_H^k .

Now we may modify the Hitchin connection to apply to this case. Take the trivial connection on $TM \cong M \times \mathbb{R}^{2g}$ and, for each $\Omega \in \mathcal{T}$, pull back this connection via the smooth bundle isomorphism $\phi_{\Omega} : T^{1,0} M \rightarrow TM$. Then we have a smooth vector bundle $L^k \otimes S^m TM$ with connection $\tilde{\nabla}$, such that taking an almost-complex structure I associated to some $\Omega \in \mathcal{T}$, we obtain the holomorphic structure $\mathcal{L}_I^k \otimes S^m T^{1,0} M$ when defining $\tilde{\nabla}^{0,1} = \frac{i}{2}(\mathbf{1} + iI)\tilde{\nabla}$.

Consider now the infinite-dimensional trivial bundle $\Gamma(T^*M, L_H^k) \times \mathcal{T} \rightarrow \mathcal{T}$. Then the spaces $\mathcal{H}_{H,Z}^k$ sit inside the fibres of this trivial bundle as infinite-dimensional

subspaces, and furthermore the weight spaces $(\mathcal{H}_Z^k)^m$ sit inside as *finite-dimensional* subspaces.

Elliptic regularity implies that the finite-dimensional collection of subspaces $(\mathcal{H}_Z^k)^m$ sit inside as a subbundle for each m , and indeed once we have constructed a connection for the infinite-dimensional bundle \mathcal{H}_Z^k , it too will sit inside $\Gamma(T^*M, L_H^k) \times \mathcal{T}$ as an infinite-dimensional subbundle.

To that end, fix an $m \in \mathbb{Z}_{\geq 0}$, and define a one-form $\tilde{u} \in \Omega^1(\mathcal{T}, D^1(M, L^k \otimes S^m TM))$ as follows. At $\Omega \in \mathcal{T}$, let V be a tangent vector to \mathcal{T} and $s \otimes A \in \Gamma(M, L^k \otimes S^m TM)$, and define, using local coordinates on M ,

$$\tilde{u}(V)(s \otimes A) := \frac{1}{8\pi k} \tilde{\nabla}_i(G^{ij} \tilde{\nabla}_j(s \otimes A)). \quad (\text{Eq. 7.1})$$

Theorem 7.3.6. *The one-form \tilde{u} defines a connection ∇^m on the bundle $(\mathcal{H}_H^k)^m$ of weight m weight-spaces of \mathcal{H}_H^k over \mathcal{T} .*

Proof. First note that the connection $\tilde{\nabla}$ on $L^k \otimes S^m TM$ has curvature $F_{\tilde{\nabla}} = -2\pi i \omega \otimes \mathbf{1}$, since the connection on $S^m TM$ is trivial, and hence flat. Also recall that the holomorphic vector fields on an Abelian variety are covariantly constant, and since G is a holomorphic symmetric $(2, 0)$ -tensor, $\tilde{\nabla}_i G^{ij} = 0$. Additionally since ω is Kähler on M , it is covariantly constant, and $\tilde{\nabla}_i \omega_{j\bar{k}} = 0$.

Now take a section $s \otimes A \in H^0(M, \mathcal{L}^k \otimes S^m T^{1,0}M) \subset \Gamma(M, L^k \otimes S^m TM)$, i.e. such that $\tilde{\nabla}_{\bar{k}}(s \otimes A) = 0$. Computing locally, we have

$$\begin{aligned} \tilde{\nabla}^{0,1} \tilde{u}(V)(s \otimes A) &= \frac{1}{8\pi k} \tilde{\nabla}_{\bar{k}} \tilde{\nabla}_i(G^{ij} \tilde{\nabla}_j(s \otimes A)) \otimes d\bar{z}^k \\ &= \frac{1}{8\pi k} \left[\tilde{\nabla}_i \tilde{\nabla}_{\bar{k}}(G^{ij} \tilde{\nabla}_j(s \otimes A)) - 2\pi i k \omega_{\bar{k}i} G^{ij} \tilde{\nabla}_j(s \otimes A) \right] \otimes d\bar{z}^k \\ &= \frac{1}{8\pi k} \left[\tilde{\nabla}_i G^{ij} \tilde{\nabla}_{\bar{k}} \tilde{\nabla}_j(s \otimes A) + 2\pi i k \omega_{i\bar{k}} G^{ij} \tilde{\nabla}_j(s \otimes A) \right] \otimes d\bar{z}^k \\ &= \frac{1}{8\pi k} \left[\tilde{\nabla}_i G^{ij} (\tilde{\nabla}_j \tilde{\nabla}_{\bar{k}}(s \otimes A) - 2\pi i k \omega_{\bar{k}j} s \otimes A) \right. \\ &\quad \left. + 2\pi i k \omega_{i\bar{k}} G^{ij} \tilde{\nabla}_j(s \otimes A) \right] \otimes d\bar{z}^k \\ &= \frac{2\pi i k}{8\pi k} \left[G^{ij} \tilde{\nabla}_i(\omega_{j\bar{k}} s \otimes A) + \omega_{i\bar{k}} G^{ij} \tilde{\nabla}_j(s \otimes A) \right] \otimes d\bar{z}^k \\ &= \frac{i}{2} G^{ij} \omega_{j\bar{k}} \tilde{\nabla}_i(s \otimes A) \otimes d\bar{z}^k. \end{aligned}$$

On the other hand, we have $V[I] = G^{ij} \omega_{j\bar{k}} \frac{\partial}{\partial z^i} \otimes d\bar{z}^k$, and so

$$\frac{i}{2} V[I] \tilde{\nabla}^{1,0}(s \otimes A) = \frac{i}{2} G^{ij} \omega_{j\bar{k}} \tilde{\nabla}_i(s \otimes A) \otimes d\bar{z}^k.$$

But then

$$\frac{i}{2}V[I]\tilde{\nabla}^{1,0}(s \otimes A) = \tilde{\nabla}^{0,1}\tilde{u}(V)(s \otimes A)$$

and by Lemma 6.4.2, $\tilde{u} \in \Omega^1(\mathcal{T}, D^2(M, L^k))$ defines a connection on the subbundle \mathcal{H}^k of $\Gamma(M, L^k) \times \mathcal{T} \rightarrow \mathcal{T}$. □

7.4 Connection on \mathcal{H}_H^k

In this section we will investigate whether the connections ∇^m sum up to give a genuine connection on the infinite-dimensional bundle \mathcal{H}_H^k over \mathcal{T} . Fibrewise, we know that $\mathcal{H}_{H,Z}^k$ is a direct sum

$$\mathcal{H}_{H,Z}^k = \widehat{\bigoplus}_{m \in \mathbb{Z}_{\geq 0}} (\mathcal{H}_{H,Z}^k)^m.$$

To understand how the connections sum to give a connection on this infinite-dimensional bundle, we need to examine the direct sum given.

Elements of $\mathcal{H}_{H,Z}^k$ are elements of $H^0(T^*M, \mathcal{L}_{H,Z}^k) \cong H^0(M, \mathcal{L}_Z^k) \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{C}^g)$. The space $H^0(M, \mathcal{L}_Z^k)$ is finite-dimensional, with dimension k^g , and has a basis of theta functions defined on the universal cover \mathbb{C}^g of M by

$$\vartheta_{\alpha}^{(k)}(z; Z) := \sum_{n \in \mathbb{Z}^g} \exp(\pi i k(n + \alpha) \cdot Z(n + \alpha) + 2\pi i k(n + \alpha) \cdot z); \alpha \in \frac{1}{k}\mathbb{Z}^g / \mathbb{Z}^g.$$

Thus under this isomorphism an element $s \in \mathcal{H}_{H,Z}^k$ may be written

$$s = \sum_{\alpha \in \frac{1}{k}\mathbb{Z}^g / \mathbb{Z}^g} \vartheta_{\alpha}^{(k)} \otimes f_{\alpha}$$

for some collection of entire holomorphic functions $f_{\alpha} \in \mathcal{O}(\mathbb{C}^g)$. To take the weight m part s^m of s means simply to expand the f_{α} about 0 in a power series and then take the degree m homogenous terms f_{α}^m (and view these as sections of $S^m T^{1,0}M$) to obtain

$$s^m = \sum_{\alpha \in \frac{1}{k}\mathbb{Z}^g / \mathbb{Z}^g} \vartheta_{\alpha}^{(k)} \otimes f_{\alpha}^m.$$

Conversely, given sections $s^m \in (\mathcal{H}_{H,Z}^k)^m$ for each $m \in \mathbb{Z}_{\geq 0}$, these will define a section in the direct sum $\mathcal{H}_{H,Z}^k$ precisely when the corresponding f_{α}^m sum to give a power series around $0 \in \mathbb{C}^g$ with infinite radius of convergence, which is precisely an entire holomorphic function on \mathbb{C}^g .

Instead of the connection ∇ on \mathcal{H}_H^k , we will go directly to its corresponding parallel transport operator, and show that this is well-defined as a map between fibres of \mathcal{H}_H^k . On each finite-dimensional weight m subbundle we have a projectively flat connection ∇^m , which defines a parallel transport operator P^m . Since each ∇^m is projectively flat, the linear isomorphism $P_\gamma^m : (\mathcal{H}_{H,\gamma(0)}^k)^m \rightarrow (\mathcal{H}_{H,\gamma(1)}^k)^m$ is independent of the choice of path γ up to a constant. Critically, this is the *same* constant independent of m .

Now take a section $s_0 \in \mathcal{H}_{H,Z}^k$ and let γ be a path from Ω to some Ω' . Then define a parallel transport operator $P_\gamma : \mathcal{H}_{H,Z}^k \rightarrow \mathcal{H}_{H,Z'}^k$ by

$$P_\gamma(s_0) := \sum_{m \in \mathbb{Z}_{\geq 0}} P_\gamma^m(s_0^m)$$

where s_0^m is the weight m part of s_0 . By construction of the connections ∇^m , each section $P_\gamma^m(s_0^m)$ is again of weight m , so it suffices to check that the function parts of these sections sum up to give holomorphic functions on \mathbb{C}^g .

Let $s_0^m := \sum_{\alpha \in \frac{1}{k}\mathbb{Z}^g/\mathbb{Z}^g; |J|=m} f_J \vartheta_\alpha^{(k)} \otimes z^J$ for some constants $f_J \in \mathbb{C}$. The parallel transport P_γ^m is the solution to the differential equation

$$\frac{ds^m(t)}{dt} = -\tilde{u}\left(\frac{\partial}{\partial t}\right)(s^m(t)); \quad s^m(0) = s_0^m.$$

Now since $\tilde{\nabla} := \nabla \otimes \mathbf{1} + \mathbf{1} \otimes \phi_\Omega^* d$ and the coefficients of the basis sections z^J (actually $\frac{\partial^{|J|}}{\partial z^J}$) are constant, we see that

$$\begin{aligned} \tilde{u}\left(\frac{\partial}{\partial t}\right)(f_J(t) \vartheta_{\alpha,t}^{(k)} \otimes z_t^J) &= \frac{1}{8\pi k} \tilde{\nabla}_i (G_t^{ij} \tilde{\nabla}_j (f_J(t) \vartheta_{\alpha,t}^{(k)} \otimes z_t^J)) \otimes d\bar{z}^k \\ &= \left(\frac{1}{8\pi k} \nabla_i (G_t^{ij} \nabla_j (f_J(t) \vartheta_{\alpha,t}^{(k)}))\right) \otimes z_t^J \otimes d\bar{z}^k \\ &= u\left(\frac{\partial}{\partial t}(f_J(t) \vartheta_{\alpha,t}^{(k)})\right) \otimes z_t^J, \end{aligned}$$

where u is the one-form in the original Hitchin connection for the compact space M . In the context of the parallel transport equation, this means that if Q_γ is the parallel transport for the regular Hitchin connection, then

$$P_\gamma^m(s_0^m) = \sum_{\alpha \in \frac{1}{k}\mathbb{Z}^g/\mathbb{Z}^g; |J|=m} Q_\gamma(f_J \vartheta_\alpha^{(k)}) \otimes z^J.$$

From this expression we see immediately that P_γ is well-defined as a map $\mathcal{H}_{H,\Omega}^k \rightarrow \mathcal{H}_{H,\Omega'}^k$, because if we write $Q_\gamma(\vartheta_{\alpha,0}^{(k)}) = Q_\alpha^\beta \vartheta_{\beta,1}^{(k)}$ for some constants $Q_\alpha^\beta \in \mathbb{C}$, then

$$P_\gamma(s) = \sum_{m \in \mathbb{Z}_{\geq 0}} \sum_{\alpha \in \frac{1}{k}\mathbb{Z}^g/\mathbb{Z}^g; |J|=m} \vartheta_{\beta,1}^{(k)} \otimes Q_\alpha^\beta f_J z_1^J$$

and so the power series on the right is simply a linear combination of convergent power series, and hence itself everywhere convergent, defining a holomorphic function on \mathbb{C}^g .

Furthermore, if we vary the path to some η also from Ω to Ω' , as linear maps Q_γ and Q_η agree up to a constant depending on γ and η , and therefore P_γ and P_η agree up to this same constant. In particular, since this constant is the same as for Q , the parallel transport of a projectively flat connection, we can conclude using the fact that \mathcal{T} is simply-connected that:

Theorem 7.4.1. *The connection on \mathcal{H}_H^k described by the parallel transport P is projectively flat, and therefore gives a well-defined diffeomorphism between $\mathbb{P}(\mathcal{H}_{H,Z}^k)$ and $\mathbb{P}(\mathcal{H}_{H,Z'}^k)$ for any $Z, Z' \in \mathcal{T}$. In particular, the connection is a Hitchin connection in the sense of Definition 6.4.1.*

7.4.1 Curvature of the Connection

A more direct approach to showing the connection is projectively flat is taken by Blaavand in [Bla11], where the curvature of the Hitchin connection for M an Abelian variety is computed directly, and is manifestly a constant multiple of the identity. For each m , essentially the same computations carry through to show that the connections ∇^m are projectively flat. In this section we will recall this computation for the case of the cotangent bundle.

Recall that the principally polarised Abelian variety M_Z is defined by some $Z \in \mathcal{T}$, the Siegel upper-half space, such that $M_Z = \mathbb{C}^g / (\mathbb{Z}^g + Z\mathbb{Z}^g)$. We have $Z = X + iY$ for some real matrices X and Y , and in the dz basis the symplectic form on M_Z may be written

$$\omega = \frac{i}{2} w_{ij} dz^i \wedge d\bar{z}^j$$

where $W = (w_{ij})$ is easily computed to be Y^{-1} .

Recall that there is a global frame $\frac{\partial}{\partial Z_i^j}$ for the holomorphic tangent bundle of \mathcal{T} . Since $G(V) \cdot \omega = V'[I]$, in order to compute the Hitchin connection explicitly, one must determine the tensors $G(\frac{\partial}{\partial Z_i^j})$ for the complex coordinates Z_i^j on \mathcal{T} . To do this, one must take the derivative of the map $I(Z)$ with respect to Z_i^j , and compare with the contraction with ω .

Lemma 7.4.2. *When $Z = X + iY$ for real matrices X and Y we have*

$$I(Z) = \begin{pmatrix} -Y^{-1}X & -(Y + XY^{-1}X) \\ Y^{-1} & XY^{-1} \end{pmatrix}.$$

Proof. Recall that $dz^i = dx^i + Z_j^i dy^j$ and $d\bar{z}^i = dx^i + \bar{Z}_i^j dy^j$. Using the relations

$$dz^i \left(\frac{\partial}{\partial z^j} \right) = \delta_j^i, \quad dz^i \left(\frac{\partial}{\partial z^j} \right) = dz^i \left(\frac{\partial}{\partial \bar{z}^j} \right) = 0, \quad d\bar{z}^i \left(\frac{\partial}{\partial \bar{z}^j} \right) = \delta_j^i$$

and the corresponding relations for the coordinates x and y , it is easily computed that

$$\frac{\partial}{\partial z} = \frac{i}{2}(Y^{-1}X - i\mathbf{1}) \frac{\partial}{\partial x} + \frac{1}{2i}Y^{-1} \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2i}(Y^{-1}X + i\mathbf{1}) \frac{\partial}{\partial x} + \frac{i}{2}Y^{-1} \frac{\partial}{\partial y}.$$

Using that $I(Z) \frac{\partial}{\partial z^i} = i \frac{\partial}{\partial z^i}$ and $I(Z) \frac{\partial}{\partial \bar{z}^j} = -i \frac{\partial}{\partial \bar{z}^j}$ one can substitute in the real coordinate expressions above and solve for $I(Z)$ to obtain the result. \square

Since $I(Z)$ is expressed in a basis independent of the $\frac{\partial}{\partial Z_i^j}$, if one converts the coefficients back into functions of Z_i^j the Z_i^j -derivative may be taken. To convert the coefficients back, one can use the familiar formulas $X = \frac{1}{2}(Z + \bar{Z})$ and $Y = \frac{1}{2i}(Z - \bar{Z})$. We now make a simplification also made in [Bla11], and assume that the matrix Z is *normal*, which since Z is symmetric is simply the condition that $[Z, \bar{Z}] = 0$. This has the consequence that X, Y, Z, \bar{Z} , and Δ_i^j will all commute.

Lemma 7.4.3. *We have expressions*

$$\frac{\partial I(Z)}{\partial Z_i^j} \left(\frac{\partial}{\partial z} \right) = 0, \quad \frac{\partial I(Z)}{\partial Z_i^j} \left(\frac{\partial}{\partial \bar{z}} \right) = -Y^{-1} \Delta_i^j \frac{\partial}{\partial z}$$

for the derivatives of $I(Z)$, where Δ_i^j is the matrix with a 1 in the (i, j) and (j, i) positions and zeroes elsewhere.

Proof. Recall that

$$\frac{\partial A^{-1}}{\partial a_i^j} = -A^{-1} \frac{\partial A}{\partial a_i^j} A^{-1} = -A^{-1} \Delta_i^j A^{-1}$$

for a symmetric invertible matrix $A = (a_i^j)$. Thus we can use $Y = \frac{1}{2i}(Z - \bar{Z})$ to obtain

$$\frac{\partial Y^{-1}}{\partial Z_i^j} = -\frac{1}{2i} Y^{-1} \Delta_i^j Y^{-1}.$$

A short computation using the facts that $\frac{\partial X}{\partial Z_i^j} = \frac{1}{2} \Delta_i^j$ and $\frac{\partial Y}{\partial Z_i^j} = \frac{1}{2i} \Delta_i^j$ shows that

$$\frac{\partial I(Z)}{\partial Z_i^j} = \frac{1}{2i} Y^{-1} \Delta_i^j Y^{-1} \begin{pmatrix} \bar{Z} & \bar{Z}^2 \\ -\mathbf{1} & -\bar{Z} \end{pmatrix}. \quad (\text{Eq. 7.2})$$

But then using the expressions for $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ from Lemma 7.4.2, we see

$$\frac{\partial I(Z)}{\partial Z_i^j} \left(\frac{\partial}{\partial z} \right) = \frac{i}{2} Y^{-1} \bar{Z} \frac{\partial I(Z)}{\partial Z_i^j} \left(\frac{\partial}{\partial x} \right) + \frac{1}{2i} Y^{-1} \frac{\partial I(Z)}{\partial Z_i^j} \left(\frac{\partial}{\partial y} \right)$$

which clearly vanishes by comparison with (Eq. 7.2). A similar calculation shows the result for $\frac{\partial}{\partial \bar{z}}$. \square

Corollary 7.4.4. *Written in index notation,*

$$\frac{\partial I(Z)}{\partial Z_i^j} = \begin{cases} -w_{ki} \frac{\partial}{\partial z^j} \otimes d\bar{z}^k - w_{kj} \frac{\partial}{\partial z^i} \otimes d\bar{z}^k, & i \neq j \\ -w_{ki} \frac{\partial}{\partial z^i} \otimes d\bar{z}^k, & i = j, \end{cases}$$

and

$$G\left(\frac{\partial}{\partial Z_i^j}\right) = \begin{cases} 2i \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^j} + 2i \frac{\partial}{\partial z^j} \otimes \frac{\partial}{\partial z^i}, & i \neq j \\ 2i \frac{\partial}{\partial z^i} \otimes \frac{\partial}{\partial z^i}, & i = j. \end{cases}$$

Proof. This follows immediately from $(w_{ij}) = W = Y^{-1}$ and $G\left(\frac{\partial}{\partial Z_i^j}\right) \cdot \omega = \frac{\partial I(Z)}{\partial Z_i^j}$. \square

Remark 7.4.5. The family of Kähler polarisations described by choices of $Z \in \mathcal{T}$ is rigid in the sense of Definition 6.4.4, since $G\left(\frac{\partial}{\partial Z_i^j}\right) = 0$ and by the above computation we clearly see $G\left(\frac{\partial}{\partial Z_i^j}\right)$ is holomorphic.

In order to now compute the curvature of the connection, we note that the invariant expression of the operator Δ_G given in Section 6.4.3 is given in local coordinates by the expression (Eq. 7.1), when the connection used to define Δ_G is the tensor product connection on $L^k \otimes S^m TM$. But then we have

$$\Delta_{G\left(\frac{\partial}{\partial Z_i^j}\right)} = \begin{cases} 2i \tilde{\nabla}_i \tilde{\nabla}_j + 2i \tilde{\nabla}_j \tilde{\nabla}_i, & i \neq j \\ 2i \tilde{\nabla}_i \tilde{\nabla}_i, & i = j \end{cases}$$

and the Hitchin connection is explicitly

$$\nabla_{\frac{\partial}{\partial Z_i^j}}^m = \begin{cases} \frac{\partial}{\partial Z_i^j} + \frac{i}{4\pi k} \tilde{\nabla}_i \tilde{\nabla}_j + \frac{i}{4\pi k} \tilde{\nabla}_j \tilde{\nabla}_i, & i \neq j \\ \frac{\partial}{\partial Z_i^j} + \frac{i}{4\pi k} \tilde{\nabla}_i \tilde{\nabla}_i, & i = j, \end{cases}$$

$$\nabla_{\frac{\partial}{\partial \bar{Z}_i^j}}^m = \frac{\partial}{\partial \bar{Z}_i^j}.$$

Equipped with an explicit expression for the connection, we can now determine its curvature. As can be seen, this agrees with the expressions found by Blaavand in [Bla11] for the unmodified situation on an Abelian variety.

Proposition 7.4.6. *The connection ∇^m is projectively flat, and the coefficients of the curvature F_{∇^m} are given by*

$$F_{\nabla^m}\left(\frac{\partial}{\partial \bar{Z}_k^l}, \frac{\partial}{\partial Z_i^j}\right) = \begin{cases} \frac{1}{4}(w_{ki}w_{lj} + w_{li}w_{kj}) \otimes \mathbf{1}_{S^m TM}, & i \neq j, k \neq l \\ \frac{1}{4}w_{ki}w_{kj} \otimes \mathbf{1}_{S^m TM}, & i \neq j, k = l \\ \frac{1}{4}w_{ki}w_{li} \otimes \mathbf{1}_{S^m TM}, & i = j, k \neq l \\ \frac{1}{8}(w_{ki})^2 \otimes \mathbf{1}_{S^m TM}, & i = j, k = l, \end{cases}$$

at least when Z is normal.

Proof. Following [Bla11], we note that in order to compute F_{∇^m} it suffices to determine the commutator

$$\left[\frac{\partial}{\partial \bar{Z}_k^l}, \tilde{\nabla}_i \right].$$

This is because the curvature is a $(1, 1)$ -form and is determined by

$$F_{\nabla^m}\left(\frac{\partial}{\partial \bar{Z}_k^l}, \frac{\partial}{\partial Z_i^j}\right) = \left[\frac{\partial}{\partial \bar{Z}_k^l}, \nabla_{\frac{\partial}{\partial Z_i^j}}^m \right],$$

and substituting in the expression for $\nabla_{\frac{\partial}{\partial Z_i^j}}^m$ this reduces to computing

$$\left[\frac{\partial}{\partial \bar{Z}_k^l}, \tilde{\nabla}_i \tilde{\nabla}_j + \tilde{\nabla}_j \tilde{\nabla}_i \right]$$

in the case where $i \neq j$, and similarly for $i = j$. But these commutators are given by

$$\left[\frac{\partial}{\partial \bar{Z}_k^l}, \tilde{\nabla}_i \tilde{\nabla}_j \right] = \left[\frac{\partial}{\partial \bar{Z}_k^l}, \tilde{\nabla}_i \right] \tilde{\nabla}_j + \tilde{\nabla}_i \left[\frac{\partial}{\partial \bar{Z}_k^l}, \tilde{\nabla}_j \right]$$

and similarly for $i = j$.

To determine this commutator, observe that in local coordinates we have a potential

$$\alpha = -\pi i \sum_{i=1}^g x^i dy^i - y^i dx^i$$

for the form $-2\pi i \omega = -2\pi i \sum_{i=1}^g dx^i \wedge dy^i$. But then the connection $\tilde{\nabla}$ on $L^k \otimes S^m TM$ is given in local coordinates by

$$\tilde{\nabla} = d + k\alpha \otimes \mathbf{1}_{S^m TM},$$

because

$$F_{\tilde{\nabla}} = d(k\alpha \otimes \mathbf{1}_{S^m TM}) = -2\pi i k \omega \otimes \mathbf{1}_{S^m TM}$$

and also because this is the local expression for the tensor product connection $\tilde{\nabla}$ in terms of the original connection ∇ on L^k .

Converting to complex coordinates we have

$$\alpha = \frac{\pi}{2} \sum_{i,j=1}^g w_{ij} z^i d\bar{z}^j - \bar{z}^j dz^j$$

and so

$$\tilde{\nabla}_i = \frac{\partial}{\partial z^i} - \frac{k\pi}{2} \sum_{j=1}^g w_{ij} \bar{z}^j \otimes \mathbf{1}_{S^m T^{1,0} M_Z}, \quad \tilde{\nabla}_{\bar{i}} = \frac{\partial}{\partial \bar{z}^i} + \frac{k\pi}{2} \sum_{j=1}^g w_{ij} z^j \otimes \mathbf{1}_{S^m T^{1,0} M_Z}.$$

In tensor notation this is

$$\tilde{\nabla}_{\frac{\partial}{\partial z}} = \frac{\partial}{\partial z} - k\pi i (Z - \bar{Z})^{-1} \bar{z} \otimes \mathbf{1}_{S^m T^{1,0} M_Z}, \quad \tilde{\nabla}_{\frac{\partial}{\partial \bar{z}}} = \frac{\partial}{\partial \bar{z}} + k\pi i (Z - \bar{Z})^{-1} z \otimes \mathbf{1}_{S^m T^{1,0} M_Z}.$$

By changing to x, y -coordinates, and differentiating we obtain

$$\frac{\partial}{\partial \bar{Z}_k^l} \tilde{\nabla}_{\frac{\partial}{\partial z}} = \frac{i}{2} Y^{-1} \Delta_k^l \tilde{\nabla}_{\frac{\partial}{\partial \bar{z}}}$$

which in index notation is

$$\frac{\partial}{\partial \bar{Z}_k^l} \tilde{\nabla}_i = \begin{cases} \frac{i}{2} w_{ki} \tilde{\nabla}_{\bar{l}} + \frac{i}{2} w_{li} \tilde{\nabla}_{\bar{k}} & k \neq l \\ \frac{i}{2} w_{ki} \tilde{\nabla}_{\bar{k}} & k = l. \end{cases}$$

By the local expression for $\tilde{\nabla}$ we also see that

$$[\tilde{\nabla}_{\bar{k}}, \tilde{\nabla}_i] = -k\pi w_{ki} \otimes \mathbf{1}_{S^m T^{1,0} M_Z},$$

and since sections of $\mathcal{H}^k \rightarrow \mathcal{T}$ consist of holomorphic sections of $\mathcal{L}_Z^k \otimes S^m T^{1,0} M_Z$ on each fibre, any terms with a $\tilde{\nabla}_{\bar{k}}$ on the right vanish. Furthermore, the bundle \mathcal{H}^k is in fact holomorphic, and so we may assume that any terms with $\frac{\partial}{\partial \bar{Z}_k^l}$ on the right vanish. This second assumption is not strictly necessary and we would still obtain a 2-form for the curvature if it was ignored, but in the setting of Abelian varieties it simplifies the situation.

Combining all the above assumptions we can then compute the curvature. Let us perform for example the case where $i = j$ and $k = l$ as a sample. We have

$$F_{\nabla^m} \left(\frac{\partial}{\partial \bar{Z}_k^l}, \frac{\partial}{\partial Z_i^j} \right) = \frac{i}{4\pi k} \left[\frac{\partial}{\partial \bar{Z}_k^l}, \tilde{\nabla}_i \tilde{\nabla}_i \right]$$

$$\begin{aligned}
&= \frac{i}{4\pi k} \left(\left[\frac{\partial}{\partial \bar{Z}_k}, \tilde{\nabla}_i \right] \tilde{\nabla}_i + \tilde{\nabla}_i \left[\frac{\partial}{\partial \bar{Z}_k}, \tilde{\nabla}_i \right] \right) \\
&= \frac{i}{4\pi k} \left(\frac{i}{2} w_{ki} \tilde{\nabla}_{\bar{k}} \tilde{\nabla}_i + \frac{i}{2} w_{ki} \tilde{\nabla}_i \tilde{\nabla}_{\bar{k}} \right) \\
&= -\frac{1}{8\pi k} w_{ki} \left[\tilde{\nabla}_{\bar{k}}, \tilde{\nabla}_i \right] \\
&= \frac{1}{8} (w_{ki})^2 \otimes \mathbf{1}.
\end{aligned}$$

The other coefficients may be similarly computed. \square

Notice that at no point were we forced to use the fact that our connection had been coupled to the trivial bundle $S^m TM$. As such the coefficients of the curvature of ∇^m are independent of the weight m , and thus the direct sum connection ∇ is also projectively flat in the correct sense.

We also remark that this computation assumed that Z was normal. This contributed to the simplicity for the expression for G and subsequently the Hitchin connection and its curvature. Although one would still obtain a constant multiple of the identity endomorphism in the case where Z was not normal, the calculations are more complicated.

7.5 Higher Rank

Consider now the \mathbb{C}^* action on the space $H^0(\mathcal{M}^{n,d}, \mathcal{L}_H^k)$ which is the geometric quantization of the Higgs moduli space for higher rank $n > 1$. Firstly, when $g > 1$ and excluding the case of $\mathcal{M}^{2,2}$, Hitchin showed in [Hit90] that $T^*\mathcal{N}^{n,d} \subseteq \mathcal{M}^{n,d}$ and the codimension of the complement is greater than 1. Since $\mathcal{L}_H^k|_{T^*\mathcal{N}^{n,d}} = \pi^*\mathcal{L}^k$, we have

$$\begin{aligned}
H^0(\mathcal{M}^{n,d}, \mathcal{L}_H^k) &\cong H^0(T^*\mathcal{N}^{n,d}, \pi^*\mathcal{L}^k) \\
&\cong H^0(T^*\mathcal{N}^{n,d} \setminus \{0\}, \pi^*\mathcal{L}^k).
\end{aligned}$$

The action of \mathbb{C}^* is compatible with these isomorphism, and by the exact same argument as in Lemma 7.3.4 we have that, for all $m \in \mathbb{Z}$,

$$H^0(\mathcal{M}^{n,d}, \mathcal{L}_H^k)^m \cong H^0(\mathcal{N}^{n,d}, \mathcal{L}^k \otimes S^m T^{1,0} \mathcal{N}^{n,d}).$$

In particular, we may attempt the same program of trying to find a projectively flat connection \tilde{u} on the bundles of m th weight spaces over \mathcal{T} , where \mathcal{T} now is Teichmüller space for the underlying Riemann surface Σ_g .

In the case of Abelian varieties corresponding to $\mathcal{N}^{1,d}$, the vector bundles $S^m T^{1,0} \mathcal{N}^{1,d}$ were trivial, so we could deform the connection on \mathcal{L}^k by tensoring

with the trivial connection. For $n > 1$ these vector bundles are not trivial, and the best we can do is consider the Levi-Civita connection ∇^g on $\mathcal{N}^{n,d}$.

Denote by $\tilde{\nabla}$ the connection on $L^k \otimes S^m T\mathcal{N}^{n,d}$ given by tensoring ∇ on \mathcal{L}^k and the connection induced on $S^m T\mathcal{N}^{n,d}$ by ∇^g on $T\mathcal{N}^{n,d}$. For each choice of $\sigma \in \mathcal{T}$, we obtain the corresponding holomorphic structure making this smooth bundle isomorphic to $\mathcal{L}^k \otimes S^m T^{1,0}\mathcal{N}^{n,d}$.

Now define a general second-order differential operator on sections of $L^k \otimes S^m T\mathcal{N}^{n,d}$ in local coordinates by

$$\tilde{u}(V) := a^{ij} \tilde{\nabla}_i \tilde{\nabla}_j + b^i \tilde{\nabla}_i + c$$

for some smooth endomorphisms a^{ij} , b^i , and c , depending on the choice of V , a tangent vector to \mathcal{T} at some $\sigma \in \mathcal{T}$. By Lemma 6.4.2 this second order differential operator will preserve the subspaces of holomorphic sections precisely when

$$[\tilde{\nabla}_\sigma^{0,1}, \tilde{u}(V)] = \frac{i}{2} V[I] \tilde{\nabla}_\sigma^{1,0} \quad (\text{Eq. 7.3})$$

as operators on holomorphic sections $s \in (\mathcal{H}_{H,\sigma}^k)^m$.

In order to find a choice of a^{ij} , b^i and c , one should compute both sides of this expression and compare terms of different orders. On the left-hand side of (Eq. 7.3) we have

$$\tilde{\nabla}_{\bar{k}}(a^{ij} \tilde{\nabla}_i \tilde{\nabla}_j + b^i \tilde{\nabla}_i + c) = \tilde{\nabla}_{\bar{k}}(a^{ij}) \tilde{\nabla}_i \tilde{\nabla}_j + a^{ij} \tilde{\nabla}_{\bar{k}} \tilde{\nabla}_i \tilde{\nabla}_j + \tilde{\nabla}_{\bar{k}}(b^i) \tilde{\nabla}_i + b^i \tilde{\nabla}_{\bar{k}} \tilde{\nabla}_i + \tilde{\nabla}_{\bar{k}} c.$$

Immediately we see that the a^{ij} must be holomorphic, since the right-hand side of (Eq. 7.3) contains no second order terms. Computing the first order terms on the left-hand side we have

$$\begin{aligned} a^{ij} \tilde{\nabla}_{\bar{k}} \tilde{\nabla}_i \tilde{\nabla}_j + \tilde{\nabla}_{\bar{k}}(b^i) \tilde{\nabla}_i &= a^{ij} (\tilde{\nabla}_i \tilde{\nabla}_{\bar{k}} - 2\pi i k \omega_{\bar{k}i} \otimes \mathbf{1} + \mathbf{1} \otimes \hat{F}_{\bar{k}i}) \tilde{\nabla}_j + \tilde{\nabla}_{\bar{k}}(b^i) \tilde{\nabla}_i \\ &= a^{ij} (\tilde{\nabla}_i (-2\pi i k \omega_{\bar{k}j} \otimes \mathbf{1} + \mathbf{1} \otimes F_{\bar{k}j}) \\ &\quad - (2\pi i k \omega_{\bar{k}i} \otimes \mathbf{1} + \mathbf{1} \otimes \hat{F}_{\bar{k}i}) \tilde{\nabla}_j + \tilde{\nabla}_{\bar{k}}(b^i) \tilde{\nabla}_i. \end{aligned}$$

Here $F_{\bar{k}i}$ denotes the curvature endomorphism on $S^m T$ given by $X \mapsto F(\partial_{\bar{k}}, \partial_i)X$ where F is the curvature induced by the Levi-Civita connection, and \hat{F} is the corresponding curvature endomorphism on $S^m T \otimes T^*$.

On the right-hand side of (Eq. 7.3) we have the first order term $\frac{i}{2} G^{ij} \omega_{j\bar{k}} \tilde{\nabla}_i$ where G is the familiar holomorphic symmetric tensor discussed previously.

There is no obvious choice of a holomorphic endomorphisms a^{ij} that will cancel out the endomorphisms that occur in the expression on the left-hand side of (Eq. 7.3). Indeed the only such holomorphic tensors known are combinations of the

holomorphic symmetric tensor G^{ij} , and the identity endomorphism. A straightforward computation shows that no combination of these tensors can counteract the smoothly varying endomorphism \hat{F} .

In the case of Hitchin's connection for $\mathcal{N}^{n,d}$, the computation above reduces to a trace over the curvature endomorphism \hat{F} on T^* . In this case one can use the fact that the Ricci form is in the same cohomology class as $\lambda\omega$ to write down the functions a^{ij} and b^i in terms of the Ricci potential. No such trace occurs when L^k is replaced by the vector bundle $L^k \otimes S^m T\mathcal{N}^{n,d}$.

The situation cannot be improved by passing to higher order operators \tilde{u} , and it is likely that it is not possible to specify such a holomorphic endomorphism at all. This suggests that for $n > 1$ one is obstructed from constructing a Hitchin connection on $\mathcal{M}^{n,d}$ that respects the natural \mathbb{C}^* actions for varying choices of complex structures in \mathcal{T} .

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