Generalised eta invariants, end-periodic manifolds, and their applications to positive scalar curvature

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Contents

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Dedication

For Kim.

Abstract

This thesis studies the applications of index theory to positive scalar curvature (PSC), in particular questions of existence and number of path components of the moduli space of PSC metrics. After Atiyah-Singer proved their legendary index theorem [AS63, AS68a, AS68b], many fruitful applications to positive scalar curvature were discovered, for instance Lichnerowicz's obstruction to positive scalar curvature metrics on spin manifolds with non-vanishing A-hat genus [Lic63]. The theorem of Lichnerowicz relies notably on the existence of a spin Dirac operator on any spin manifold—a self-adjoint, elliptic, first order differential operator having marvellous connections to the geometry of the underlying Riemannian manifold.

In 1975, Atiyah, Patodi and Singer [APS75a, APS75b, APS76] proved their index theorem for a Dirac operator D on a manifold Z with boundary $\partial Z = Y$. This takes a similar form to the Atiyah-Singer index theorem but notably has a correction term $\eta(A) = (\eta_A(0) + h)/2$, called the *eta invariant*, appearing for the boundary. The eta invariant is defined solely in terms of the spectrum of the Dirac operator A on the boundary, so is a spectral invariant. As it stands, this invariant is not at all robust; if one slightly perturbs the metric on Y then most likely one will produce a change in the eta invariant. A notable exception to this is conformal deformations, which leave the Dirac operator unchanged.

There is, however, a more robust invariant which can be procured from the eta invariant. If one twists the Dirac operator A on Y by two unitary representations $\sigma_1, \sigma_2 : \pi_1(Y) \to U(N)$ of the fundamental group of Y, one obtains two twisted Dirac operators A_1 and A_2 on Y which are locally isomorphic. Subtracting the eta invariants of these twisted Dirac operators yields a modified invariant $\rho(\sigma_1, \sigma_2; A)$, called the *rho invariant*. The rho invariant still isn't quite robust, but upon taking the *mod $\mathbb{Z}$ reduction* of the rho invariant, many striking invariance properties emerge. For example, writing $\pi = \pi_1(Y)$, the rho invariant descends to a well-defined map on geometric K-homology [HR10]:

$$\rho(\sigma_1, \sigma_2) : K_1(B\pi) \to \mathbb{R}/\mathbb{Z}.$$

The rho invariant can be used to further study the properties of PSC metrics on manifolds. Whereas the Atiyah-Singer theorem is mostly useful for even-dimensional manifolds, the APS index theorem allows one to obtain results for odd-dimensional manifolds by considering them as boundaries of even-dimensional manifolds. In particular, one can obtain obstructions to PSC [HR10], and study the number of path components of the moduli space of PSC metrics on a manifold [BG95].

More recently, Mrowka, Ruberman and Saveliev [MRS16] discovered and proved a new index theorem for manifolds with periodic ends. Roughly speaking, these are manifolds Z_∞ which have a compact piece Z, attached to which are one or more ends which repeat themselves periodically off to infinity. Such manifolds were first studied by Taubes [Tau87], who used them to prove (following work of Donaldson and Freedman) that $\mathbb{R}^4$ admits an uncountable family of mutually non-diffeomorphic smooth structures. The index theorem of MRS involves, like the APS index theorem, a correction term $\eta^{\mathrm{ep}}(D)$ appearing for the periodic ends.

The main contribution of this thesis is the development of a new analogue of geometric K-homology that is tailored to the setting of manifolds with periodic ends. The group is called $K_1^{\mathrm{ep}}(B\pi)$, and as in the APS case there is an analogous rho invariant descending to a well-defined map

$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2) : K_1^{\mathrm{ep}}(B\pi) \to \mathbb{R}/\mathbb{Z}.$$

Moreover, we establish a natural isomorphism $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$ which preserves rho invariants:

$$\begin{array}{ccc} K_1^{\mathrm{ep}}(B\pi) & \xleftrightarrow{\;\sim\;} & K_1(B\pi) \\ & {\scriptstyle \rho^{\mathrm{ep}}(\sigma_1,\sigma_2)} \searrow \quad \swarrow {\scriptstyle \rho(\sigma_1,\sigma_2)} & \\ & \mathbb{R}/\mathbb{Z} & \end{array}$$

The isomorphism can be used to transfer results on PSC found in [HR10] from the odd-dimensional case to the even-dimensional case. The analogous end-periodic bordism and PSC bordism groups are also introduced, and these are used to transfer results on the path components of the moduli space of PSC metrics found in [BG95] from odd dimensions to even dimensions, providing a conceptual framework for the methods of [MRS16].

These results are collected in a preprint with my supervisor V. Mathai [HM17], which constitutes the final chapter of this thesis. The beginning of the thesis is dedicated to establishing the relevant background theory needed to understand the statement and proof of the MRS index theorem, and the applications to positive scalar curvature.

Structure of the thesis

The main body of the thesis has six chapters, of total length 118 pages. The first five chapters consist of background material—here I have expended much effort in attempting to carefully exposit facts and proofs which I found difficult to understand or locate in the literature. Where possible I have tried to work out proofs independently, rather than relying on references. Many of these are routine, but there are a few that I am pleased with, particularly in Chapter 4. The final Chapter 6 contains the new work of this thesis, in the form of an unpublished paper with my supervisor. After this are three appendices of total length 34 pages. These are mostly background material that anyone with basic knowledge of the field could comfortably ignore. Nevertheless, I felt it important to include these for the sake of completeness, and also to address any questions of notation or convention the reader might have.

Chapter 1 gives a brief overview of the basic definitions and facts concerning end-periodic manifolds. This stemmed from questions I had while reading the paper [MRS16], concerning the differential topology involved in constructing such manifolds.

Chapter 2 introduces the basic notions of positive scalar curvature, including fundamental examples and results such as the PSC surgery theory of Gromov-Lawson [GL80] and Schoen-Yau [SY79b]. I have carefully detailed how a continuous path in the moduli space $\mathcal{M}^+(M)$ of PSC metrics may be represented up to homotopy by a smooth path in the space $\mathcal{R}^+(M)$ of PSC metrics, assuming only the Ebin slice theorem [Ebi70]. This seems to be something which is well known in the literature, but for which I had trouble locating a precise reference. The proof of Proposition 2.2.11 was worked out independently, following a remark in [RS01] which says that sharper estimates may be obtained than those in [GL80]. I have also interpreted path components of the moduli space of PSC metrics as 'truly distinct' PSC metrics, as is explained in the introduction to the chapter.

Chapter 3 is a short survey of eta invariants and the Atiyah-Patodi-Singer index theorem. An overview of the original proof of the APS index theorem in [APS75a] is given. We also introduce rho invariants, and briefly cover the applications of these to the moduli space of PSC metrics from [BG95].

Chapter 4 details the analysis needed to understand the framework of the end-periodic index theorem. There is much new in the way of exposition here. The proofs of 2. $\Rightarrow$ 3. and 1. $\Rightarrow$ 2. in Proposition 4.1.7 were worked out independently—this includes the formulation and proof of Lemma 4.1.6. The Fourier-Laplace transform is motivated by considering it as a kind of smoothly varying family of Fourier transforms on the circle—this interpretation yields a quick proof that the Fourier-Laplace transform is an isometry on L^2 spaces—see Lemma 4.2.5 and Proposition 4.2.6. Corollary 4.2.16 spells out a fact stated in [MRS16], that the signature operator on an end-periodic manifold is never Fredholm. The proof of Lemma 4.2.20 gives details on something which Taubes says in [Tau87] is a standard argument. Section 4.2.3 spells out what the change of index formula (upon changing the weight) means in the cylindrical case.

Chapter 5 covers the end-periodic index theorem of Mrowka, Ruberman and Saveliev [MRS16]. We begin by describing how the theorem reduces to the APS index theorem in both the Fredholm and non-Fredholm cases. We then cover some aspects of the proof, and describe how the theorem was applied to the study of PSC in [MRS16].

Chapter 6 contains the unpublished paper [HM17] with my supervisor V. Mathai, preceded by a few cautionary words on changes in notation and convention.

Appendix A is a short description of the Fréchet topology on the space of sections $\Gamma(E)$, for the reader in need of a reminder. This is only used in Chapter 2 for discussing the space $\mathcal{R}^+(M)$ of PSC metrics.

Appendix B gives a summary of Clifford algebras, Dirac operators and the Atiyah-Singer index theorem, including a sample obstruction to positive scalar curvature. This was originally intended to be the third chapter of this thesis, with proofs included. It quickly grew into a mega-chapter of over 50 pages, so in order to keep the thesis to a reasonable length the proofs were removed, and the chapter relegated to the appendices. The most important results for the thesis are probably Proposition B.4.7 and the facts in Section B.4.4, for which the proofs have been left in.

Appendix C is a summary of heat kernels for Dirac operators on manifolds of bounded geometry. This too was meant to be a chapter in the thesis, and likewise it has been stripped naked of its proofs and banished to the appendices in the name of brevity. A rough outline of the proof of existence and asymptotic expansion for heat kernels on manifolds of bounded geometry is included.

Chapter 1

Manifolds with periodic ends

In this brief preliminary chapter, we introduce the main objects we will be working with later in the thesis. These are *manifolds with periodic ends*, or *end-periodic manifolds* for short. Roughly, an end-periodic manifold consists of compact piece Z, attached to which are one or more non-compact pieces (ends) that repeat themselves periodically off to infinity; see Figure 1.1. These manifolds were first introduced by Taubes [Tau87] in his work on gauge theory. Previously, the existence of an exotic $\mathbb{R}^4$ was deduced from work of Freedman [Fre82] and Donaldson [Don83]. Using the exotic $\mathbb{R}^4$, Taubes proved that $\mathbb{R}^4$ admits an uncountable family of mutually non-diffeomorphic smooth structures—a truly remarkable result. The proof required the analysis of the elliptic anti-self-dual de Rham complex in Yang-Mills gauge theory over an end-periodic manifold M. Hence Taubes was led to pioneer analysis on end-periodic manifolds; we will see some of his work in Chapter 4.

More recent studies on end-periodic manifolds include the work of Mazzeo, Pollack and Uhlenbeck [MPU96], Miller [Mil06], and the series of papers:

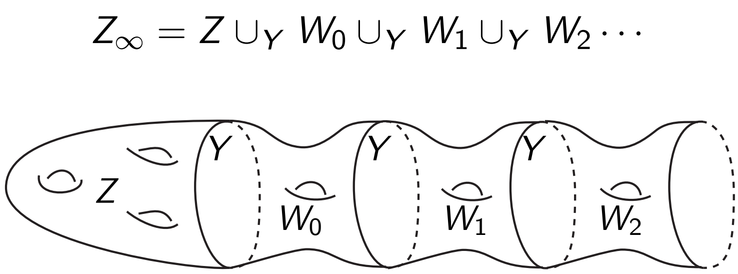

Figure 1.1: An end-periodic manifold with one end.

[RS07] (Ruberman and Saveliev), [MRS11, MRS14, MRS16] (Mrowka, Ruberman and Saveliev), and the preprint [LRS17] (Lin, Ruberman and Saveliev). In [MRS16], the authors prove an index theorem for end-periodic Dirac operators on end-periodic manifolds and apply their result to the study of positive scalar curvature on compact manifolds. It is their theorem on which the new work in this thesis is based. Our ultimate goal is to study their theorem in the setting of geometric K-homology, following Higson and Roe [HR10], and the applications of the theorem to the study of positive scalar curvature.

The original definition of an end-periodic manifold given by Taubes was rather geometric in nature, involving gluing a pre-existing open manifold end to end in order to construct the non-compact pieces of an end. In our exposition we take a different but equivalent route, beginning with a compact manifold X and cutting this open using Poincaré duality to form the fundamental segment of the periodic end. This is the definition used in [MRS16], for example. While this method is technically more demanding than the definition given by Taubes, it pays off immeasurably later on in studying the new end-periodic K-homology and bordism theories, and will provide us a smoother path to the main definitions and theorems.

This chapter begins with a brief recollection of standard results in differential topology: Poincaré duality and the de Rham isomorphism theorem. Following this is a series of technical lemmas that are fundamental to the chosen definition of an end-periodic manifold. After this preparation comes the definition of an end-periodic manifold. We end by describing the types of objects that we will consider on end-periodic manifolds, namely *end-periodic objects*—those which repeat themselves periodically over the periodic end.

The definition of an end-periodic manifold being a differentio-topological notion, this chapter has a strong differential topology flavour; the reader will readily find the relevant background material in books such as [BT82], [Lee13], or [War83]. The material on algebraic topology such as the singular cohomology groups and the universal coefficient theorem may also be found in [Hat02].

1.1 Poincaré duality and the de Rham isomorphism

We begin by recalling the statement of Poincaré duality: If X is a compact oriented n-dimensional manifold then there exists a natural group isomorphism $H^i(X,\mathbb{Z}) \simeq H_{n-i}(X,\mathbb{Z})$ for each i. This isomorphism depends on the choice of orientation; it sends the fundamental homology class $[X] \in$

$H_n(X, \mathbb{Z})$ to the cohomology class $1 \in H^0(X, \mathbb{Z})$. In de Rham cohomology, Poincaré duality takes the form of a very explicit non-degenerate pairing $H^i_{dR}(X) \times H^{n-i}_{dR}(X) \to \mathbb{R}$:

$$([\omega], [\eta]) \mapsto \int_X \omega \wedge \eta, \tag{1.1}$$

where ω and η are closed forms and the square brackets indicate their classes in de Rham cohomology. If $[\omega] \in H^i_{dR}(X)$ then we say a compact oriented submanifold $Y \subset X$ of dimension $n - i$ is *Poincaré dual to* $[\omega]$ if

$$\int_Y \eta = \int_X \omega \wedge \eta$$

for any closed $(n - i)$-form η. On the left hand side we write η for what is really the pullback $i^*\eta$ of η to Y by the inclusion $i : Y \hookrightarrow X$, and trust this will cause no confusion.

We recall also the de Rham isomorphism between de Rham cohomology and singular cohomology with real coefficients. This again takes the form of a non-degenerate pairing $H_p(X, \mathbb{R}) \times H^p_{dR}(X) \to \mathbb{R}$:

$$([\sigma], [\omega]) \mapsto \int_\sigma \omega.$$

This definition actually requires the introduction of *smooth* singular homology, which is naturally isomorphic to the standard continuous singular homology; see Chapter 18 of [Lee13] for details. What's more, a cohomology class $[\omega]$ is in the image of the canonical map $H^p(X, \mathbb{Z}) \to H^p_{dR}(X)$ if and only if the integral of ω over each singular p-cycle σ is an integer. In this case, we say the form ω has *integer periods*, and call the cohomology class $[\omega]$ *integral*. Note that if $H^p(X, \mathbb{Z})$ has torsion then there will be distinct classes in $H^p(X, \mathbb{Z})$ which map to $[\omega]$.

We are mainly interested in the first cohomology group $H^1(X, \mathbb{Z})$. By the de Rham theorem, a class $[\alpha]$ in $H^1_{dR}(X)$ is integral if and only if the integral of α over each smooth closed loop is an integer. By the universal coefficient theorem $H^1(X, \mathbb{Z})$ has no torsion and the canonical map $H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{R})$ is injective. It follows that every cohomology class $\gamma \in H^1(X, \mathbb{Z})$ can be represented by a smooth closed 1-form α with integer periods, and this representation is unique up to addition of exact 1-forms.

1.2 Technical preparations

Let X be a compact oriented connected manifold. Suppose $\gamma \in H^1(X, \mathbb{Z})$ is a primitive cohomology class, meaning that γ is non-zero and not an integer multiple of any other cohomology class (aside from $-\gamma$, of course). We will use the manifold X and the class γ to create a non-compact manifold which will be used as the 'end' of an end-periodic manifold.

Lemma 1.2.1. *Let $[d\theta]$ denote the standard generator of $H^1(S^1, \mathbb{Z})$. For any $\gamma \in H^1(X, \mathbb{Z})$ there exists a smooth map $f : X \to S^1$ such that $f^*[d\theta] = \gamma$. This map is unique up to homotopy.*

Remark 1.2.2. For such a smooth map $f : X \to S^1$, the derivative df takes values in $TS^1 \cong S^1 \times \mathbb{R}$. We will thus abuse notation and write df for the global 1-form on X defined by the composition $TX \to TS^1 \xrightarrow{\sim} S^1 \times \mathbb{R} \to \mathbb{R}$. Using this notation, we have $f^*[d\theta] = [df]$ in de Rham cohomology.

Proof of Lemma 1.2.1. First, represent γ by a smooth closed one-form $\alpha \in \Omega^1(X)$. Choose some initial point $x_0 \in X$. If $x \in X$, let σ be a smooth path from x_0 to x. Integrating α over the path σ, we obtain a real number $\int_\sigma \alpha$. Since α has integer periods, this number is independent modulo integers of the choice of path. We therefore get a well-defined map $f : X \to \mathbb{R}/\mathbb{Z} \cong S^1$. It is easy to check that f is locally smooth, and therefore smooth. The fundamental theorem of calculus gives $df = \alpha$, hence $f^*[d\theta] = [df] = [\alpha] = \gamma$.

Suppose $g : X \to S^1$ is another smooth map such that $g^*[d\theta] = \gamma$. Then $[df] = [dg]$ so that $f = g + p \circ h$ for some smooth function $h : X \to \mathbb{R}$, where $p : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ is the projection. Then $F(t, x) = g(x) + p(t \cdot h(x))$ is a homotopy from f to g. □

It follows that there is a canonical isomorphism $H^1(X, \mathbb{Z}) \cong [X, S^1]$, where $[X, S^1]$ denotes the set of homotopy classes of maps $X \to S^1$. The circle S^1 is in fact the classifying space for the integers. Hence we have established a bijection between isomorphism classes of principal $\mathbb{Z}$-bundles over the manifold X and the first integer cohomology $H^1(X, \mathbb{Z})$. Note that a principal $\mathbb{Z}$-bundle is in particular a covering space. We will see later how to explicitly construct the corresponding principal $\mathbb{Z}$-bundle.

Lemma 1.2.3. *Let $\gamma \in H^1(X, \mathbb{Z})$ be non-zero. There exists a compact oriented submanifold $Y \subset X$ that is Poincaré dual to γ.*

Proof. By Sard's theorem, almost every point in S^1 is a regular value of f (a point in the complement of the critical values). If none of these points are in the image of f, then f is not surjective. If f is not surjective, it is clearly nullhomotopic, which contradicts $f^*[d\theta] = \gamma \neq 0$. Hence there is a regular value y of f which is also in the image of f. Define $Y = f^{-1}(y)$. Then Y is a compact submanifold of X by the inverse function theorem. The orientation of Y is chosen such that df followed by the orientation of Y is the orientation of X.

Consider $X \backslash Y$. Since X and Y are compact, this is an open manifold with finitely many connected components $X_1, \dots, X_k$. For $i = 1, \dots, k$, let W_i be the compact manifold with boundary obtained by compactifying X_i using the connected components of Y as the boundary. Denote by Y_i^+ (resp. Y_i^-) the positively (resp. negatively) oriented boundary component of W_i. Over each W_i, the form df has a primitive f_i such that $f_i|_{Y_i^-} = 0$ and $f_i|_{Y_i^+} = 1$. If $\omega \in \Omega^{n-1}(X)$ is closed, we apply Stokes' theorem to get

$$\int_X df \wedge \omega = \sum_{i=1}^k \int_{W_i} df \wedge \omega = \sum_{i=1}^k \int_{W_i} d(f_i\,\omega) = \sum_{i=1}^k \int_{Y_i^+} \omega = \int_Y \omega.$$

Since $[df] = \gamma$, Y is Poincaré dual to γ. □

Lemma 1.2.4 ([MP77])**.** *If $\gamma \in H^1(X, \mathbb{Z})$ is primitive, the Poincaré dual submanifold Y may be chosen to be connected.*

It is essential that γ be primitive. If $\gamma \neq 0$ is not primitive, we may write $\gamma = k\eta$ for some primitive class η, and then the Poincaré dual submanifold to γ can be chosen to have k connected components. A further reduction in the number of connected components is not possible.

Lemma 1.2.5. *If $\gamma \in H^1(X, \mathbb{Z})$ is primitive and Y is a connected Poincaré dual submanifold to γ, the open manifold $X \setminus Y$ is connected.*

Proof. Clearly, since Y is connected, either $X \setminus Y$ is connected or has exactly two connected components. Suppose it has two connected components X^+ and X^-. Then for any closed $(n-1)$-form ω on X,

$$\int_X df \wedge \omega = \int_Y \omega = \int_{X^+} d\omega = 0.$$

This contradicts the class γ being non-zero. □

1.3 Definition of an end-periodic manifold

Let X be a compact oriented connected manifold, and $\gamma \in H^1(X, \mathbb{Z})$ a primitive cohomology class. Choose a connected Poincaré dual submanifold $Y \subset X$ to γ. Compactify $X \setminus Y$ to get a manifold W with boundary $Y \amalg -Y$. Glue infinitely many copies of W end-to-end along Y to obtain a manifold $\widetilde{X}_{\geqslant 0} = \bigcup_{k=0}^{\infty} W_k$. We choose to leave the negatively oriented component $-Y$ of W_0 exposed. The manifold $\widetilde{X}_{\geqslant 0}$ is then a non-compact manifold with boundary $-Y$. Supposing that Y (with its positive orientation) is the boundary of some compact oriented manifold Z, we may glue Z to $\widetilde{X}_{\geqslant 0}$ along Y, to obtain an oriented manifold $Z_\infty = Z \cup_Y \widetilde{X}_{\geqslant 0}$ without boundary.

Definition 1.3.1. We call any manifold Z_∞ obtained as in the above paragraph an *end-periodic manifold* (with one end). When convenient, we say the end of Z_∞ is *modelled* on the pair (X, γ).

Remark 1.3.2. Our boundary orientation convention is as follows: If Z is an oriented manifold with boundary $\partial Z = Y$, the boundary orientation of Y is such that the outward unit normal of Z followed by the orientation of Y is the orientation of Z.

Recall the function $f : X \to S^1$ classifying the cohomology class γ. By rotating the circle we may assume that $Y = f^{-1}(1)$. Pulling back the $\mathbb{Z}$-cover $\mathbb{R} \to S^1$ by f, we get a $\mathbb{Z}$-cover $\widetilde{X} \to X$. The commutative pullback diagram is

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{f}} & \mathbb{R} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & S^1. \end{array}$$

The manifold $\widetilde{X}_{\geqslant 0}$ is precisely the preimage of $\mathbb{R}_{\geqslant 0}$ by the map $\widetilde{f}$, justifying our choice of notation. We can therefore think of the end $\widetilde{X}_{\geqslant 0}$ of an end-periodic manifold as 'half' of a $\mathbb{Z}$-cover $\widetilde{X} \to X$.

Examples 1.3.3.

- Let $X = S^1 \times Y$, and $\gamma = [d\theta]$. Then $\widetilde{X} = \mathbb{R} \times Y$ and $\widetilde{X}_{\geqslant 0} = \mathbb{R}_{\geqslant 0} \times Y$. Such an end is called *cylindrical.* If an end-periodic manifold has only ends of this form, we call it a *manifold with cylindrical ends.*
- Any $\mathbb{Z}$-cover $\widetilde{X} \to X$ is an end-periodic manifold with two ends. To see this, consider X with its cohomology class γ. We proceed as above to obtain $\widetilde{X}_{\geqslant 0}$. If we *reverse the sign of* γ, this has the effect of changing the orientation of the Poincaré dual submanifold Y (since the orientation of X is fixed). By our boundary orientation conventions, we instead glue the copies of W end to end, leaving the copy of Y with its original orientation exposed. The resulting manifold is denoted $\widetilde{X}_{\leqslant 0}$, which is appropriate since it is the preimage of $\mathbb{R}_{\leqslant 0}$ by the original map $\widetilde{f} : \widetilde{X} \to \mathbb{R}$. The two manifolds $\widetilde{X}_{\leqslant 0}$ and $\widetilde{X}_{\geqslant 0}$ glue along Y (or, if you like, an extra copy of W) to form the covering $\widetilde{X}$.

- Consider $\widetilde{X}_{\geqslant 0}$. The boundary of this non-compact manifold is Y. We can therefore attach a cylinder $\mathbb{R}_{\leqslant 0} \times Y$ to the manifold and obtain an end-periodic manifold with two ends—one cylindrical. This manifold will play an important role later on in showing the relationship between the end-periodic index theorem and the Atiyah-Patodi-Singer index theorem.

Throughout, we have assumed that our manifolds X and $Y \subset X$ are connected. It is very often convenient to take X disconnected, and require the restriction of γ to each connected component of X be primitive. This essentially allows an end-periodic manifold modelled on a single pair (X, γ) to have *multiple* ends. End-periodic manifolds with multiple ends will play an important role later on, so we must allow for this eventuality.

1.4 End-periodic objects

Let Z_∞ be an end-periodic manifold. We want to consider objects over the end which are 'compatible' with the periodic end. That is to say, they should repeat themselves periodically off to infinity in unison with the end. The cover $\widetilde{X}$ has a fundamental covering translation, mapping each segment W_i diffeomorphically onto W_{i+1}; we write this map $T_i : W_i \to W_{i+1}$. For $i \geqslant 0$, T_i can be considered as a map of subsets of Z_∞.

Definition 1.4.1. An object $\mathcal{E}$ on Z_∞ (e.g. vector bundle, differential operator, etc.) is called *end-periodic* if $T_i^*(\mathcal{E}|_{W_{i+1}}) \cong \mathcal{E}|_{W_i}$ for all $i \geqslant 0$.

In other words, $\mathcal{E}$ is invariant under the fundamental translation.

We can equivalently describe end-periodic objects in terms of pullbacks. The covering map $\widetilde{X} \to X$ restricts to a map $p : \widetilde{X}_{\geqslant 0} \to X$. An object $\mathcal{E}$ on Z_∞ is end-periodic if and only if there is a corresponding object $\mathcal{E}_X$ on X such that

$$\mathcal{E}|_{\widetilde{X}_{\geqslant 0}} \cong p^*(\mathcal{E}_X).$$

In words, the restriction of $\mathcal{E}$ to the periodic end is isomorphic to the pullback of the same kind of object on X.

For example, the tangent bundle to an end-periodic manifold is clearly end-periodic. Canonical operators on the manifold, such as the exterior derivative, are clearly end-periodic. We can define an end-periodic Riemannian metric on Z_∞ by pulling back one from X to the end, and then extending it smoothly over the compact piece Z. It is clear that end-periodic objects are in abundance.

1.5 Notes

1.5.1 Ends

The term *end* has a precise meaning in topology.

Definition 1.5.1. Let X be a Hausdorff topological space and $K_1 \subset K_2 \subset \cdots$ be an exhaustion of X by compact subsets. That is, the union of the K_i's is X, and $K_i \subset K_{i+1}^{\circ}$ for each i. An *end* of X is a decreasing sequence of connected open subsets $U_1 \supset U_2 \supset \cdots$ such that each U_i is a connected component of $X - K_i$.

This captures the idea of a non-compact piece of X which goes off to infinity. Clearly $\mathbb{R}$ has two ends and $\mathbb{R}^n$ has one end for all $n \geqslant 2$. As it stands, the definition depends on a choice of exhaustion, but we can remedy this by taking ends up to a suitable equivalence. It is evident that the 'ends' of an end-periodic manifold really are ends in the sense of this definition.

1.5.2 Taubes' work

Taubes was the first to study end-periodic manifolds in detail [Tau87]. His main result was a non-existence theorem for certain end-periodic manifolds, which implied that $\mathbb{R}^4$ admits uncountably many non-diffeomorphic smooth structures. The proof goes along these lines: let $\mathbf{R}$ be an exotic $\mathbb{R}^4$ (existence due to earlier work of Donaldson [Don83] Freedman [Fre82]), and let $\psi : \mathbf{R} \to \mathbb{R}^4$ be a homeomorphism. For $t > 0$ define $\mathbf{B}_t = \psi^{-1}(B(0,t))$, the preimage of the ball of radius t centred at 0 in $\mathbb{R}^4$. There exists some $r > 0$ such that $\mathbf{B}_r$ is not diffeomorphic to $\mathbb{R}^4$, otherwise $\mathbf{R}$ would not be exotic. Suppose that $s > r$, and $\mathbf{B}_r$ is diffeomorphic to $\mathbf{B}_s$. One can then consider the annulus $\mathbf{B}_s \setminus \mathbf{B}_r$. Gluing the annulus end-to-end infinitely many times, one obtains the end of an end-periodic manifold, which can be closed off using $\mathbf{B}_r$. Taubes' main theorem is that such an end-periodic manifold cannot exist. It follows that $\mathbf{B}_s$ is not diffeomorphic to $\mathbf{B}_r$ for all $s > r$, and so there is an uncountable family of mutually non-diffeomorphic $\mathbb{R}^4$'s. In Chapter 4 of this thesis, we will see some of the analysis that went into the proof of Taubes' result.

Chapter 2

Positive scalar curvature

In this chapter, we move on from topology and begin our discussion of geometry. An abstract smooth manifold M does not have any inherent 'shape'; in order to measure geometric notions such as distances and curvature one can equip a manifold with a *Riemannian metric*[1]—a smoothly varying assignment of inner products g_x to each tangent space T_xM of the manifold. The pair (M, g) is then called a *Riemannian manifold*. The *scalar curvature* of (M, g) is a smooth function $\kappa : M \to \mathbb{R}$ which gives a rough measure of how the space curves at each point. For two-dimensional Riemannian manifolds, the scalar curvature is twice the Gaussian curvature, and this is the only curvature invariant that one can ascribe to a surface.

Given a smooth manifold M, we might ask whether it is possible to find a metric on it whose scalar curvature is everywhere positive. This is in general not possible. For example, the only compact orientable surface which can be given a metric of positive scalar curvature is the two-sphere S^2. This is a simple consequence of the Gauss-Bonnet theorem, and shows that one typically cannot expect metrics of positive scalar curvature to exist.

The simple question of whether or not a manifold admits a metric of positive scalar curvature has led to remarkable research, employing a vast array of techniques borrowed from different areas of mathematics. For a general introduction to the field of positive scalar curvature, the surveys of Rosenberg [Ros07], and Rosenberg and Stolz [RS94, RS01] are excellent sources.

We will be primarily concerned with the applications of index theory to positive scalar curvature. An early result in this area is the vanishing theorem of Lichnerowicz [Lic63]: a compact spin manifold M with positive

[1]There are other ways of equipping a manifold with some geometric structure, for instance in Finsler geometry one equips each tangent space with a smoothly varying norm. We will only consider Riemannian geometry in this thesis.

scalar curvature must have vanishing A-hat genus. Note this does not provide any information for manifolds of dimension not a multiple of 4, since these automatically have vanishing A-hat genus. Lichnerowicz's result was later improved upon by Hitchin [Hit74], who showed that a refined version $\widehat{a}(M)$ of the A-hat genus, taking values in the real K-theory group $KO^{-n}(\mathrm{pt})$, must vanish if M has positive scalar curvature. Bott periodicity [Bot59] classifies these groups as follows:

$n \mod 8$	$KO^{-n}(\mathrm{pt})$
0	$\mathbb{Z}$
1	$\mathbb{Z}_2$
2	$\mathbb{Z}_2$
3	0
4	$\mathbb{Z}$
5	0
6	0
7	0

The $\widehat{a}$ invariant agrees with the A-hat genus in dimensions $8k$ and is half the A-hat genus in dimensions $8k+4$, but can also take non-zero values in dimensions $8k+1$ and $8k+2$. It turns out the $\widehat{a}$-invariant is enough to classify all simply connected manifolds in dimensions $\geqslant 5$ which admit a metric of positive scalar curvature: those which do not have a spin structure admit a metric of positive scalar curvature [GL80, Corollary C], and those which do have a spin structure admit a metric of positive scalar curvature if and only if $\widehat{a}(M)$ vanishes [Sto92]. The presence of a fundamental group complicates things immensely, and no complete classification is available—see the Gromov-Lawson [GL83] and Gromov-Lawson-Rosenberg [Ros91] conjectures, and also Rosenberg's remarks on them [Ros07, pages 11-13].

A key result on positive scalar curvature is the surgery theorem, proven independently and via different methods by Gromov and Lawson [GL80], and Schoen and Yau [SY79b]. It allows one to construct many new examples of manifolds with positive scalar curvature; if M is a manifold of positive scalar curvature and M' is obtained from M by a surgery of codimension at least 3, then M' has positive scalar curvature also. A very nice exposition of Gromov and Lawson's proof is given in [RS01].

If a manifold M does have a metric of positive scalar curvature, one might ask how many 'distinct' metrics of positive scalar curvature it has. Given a metric of positive scalar curvature g on M, we can obtain another metric of positive scalar curvature $\varphi^* g$ by pulling back g via a diffeomorphism $\varphi : M \to M$. This is of course cheating—the new metric is isometric to the original metric. Perhaps we might therefore consider distinct metrics to be elements of $\mathcal{M}^+(M) = \mathcal{R}^+(M)/\mathrm{Diff}(M)$, where $\mathcal{R}^+(M)$ is the space of positive scalar curvature metrics on M. The space $\mathcal{M}^+(M)$ is called the *moduli space of positive scalar curvature metrics on M* (modulo diffeomorphisms).

This is somehow not quite the right notion of distinctness however, as the space $\mathcal{R}^+(M)$ is open in the space of all Riemannian metrics and we can therefore always slightly deform a metric of positive scalar curvature to obtain a new metric of positive scalar curvature. Since this method of obtaining new metrics is somewhat underhand, we could instead call two metrics distinct if they cannot be connected by a path in the moduli space $\mathcal{M}^+(M)$. This turns out to be the right definition, but only after one establishes a path lifting property for the moduli space using Ebin's slice theorem [Ebi70]; see Section 2.3 of this chapter. It is for this reason that we are interested in counting the number of path components of the moduli space for a given manifold. Note the corresponding problem for arbitrary Riemannian metrics is trivial—the space of Riemannian metrics on M is path connected, in fact contractible. We will see that the corresponding problem for positive scalar curvature is highly non-trivial. A nice introduction to spaces of positive scalar curvature metrics, containing many recent results, is the book [TW15].

An early (dis)connectedness result for the space $\mathcal{R}^+(M)$ of positive scalar curvature metrics was deduced by Hitchin in [Hit74]; he proved that if a compact spin manifold of dimension $8k$ or $8k+1$ admits a metric of positive scalar curvature, then the space $\mathcal{R}^+(M)$ is not connected. More relevant to this thesis is the application of the Atiyah-Patodi-Singer index theorem [APS75a] to counting path components of the moduli space $\mathcal{M}^+(M)$, discovered by Botvinnik and Gilkey [BG95]. Their results yield manifolds whose moduli spaces of positive scalar curvature metrics have infinitely many path components! The methods only work for odd-dimensional manifolds, however the more recent work of Mrowka, Ruberman and Saveliev [MRS16] allows the Bottvinnik-Gilkey method to be applied to even-dimensional manifolds.

The purpose of this chapter is to exposit the important facts about positive scalar curvature which will be used later in the thesis; we will only provide proofs or references for those results which will be relevant later on. Section 2.1 is an introduction to scalar curvature, giving the definition and a few simple examples and results. In Section 2.2 are some more technical results on positive scalar curvature that will be needed later in the thesis, in particular the surgery theory for positive scalar curvature of Gromov and Lawson, and Schoen and Yau. We also give a proof that a smooth path in $\mathcal{R}^+(M)$ gives rise to metrics of positive scalar curvature on $[0,1]\times M$; a standard proof is in [GL80], however the methods used here give rise to sharper estimates, as mentioned in [RS01]. Section 2.3 is devoted to the moduli space of positive scalar curvature metrics, in particular the proof that any continuous path in the moduli space $\mathcal{M}^+(M)$ can be represented up to homotopy by a smooth path in the space of positive scalar curvature metrics $\mathcal{R}^+(M)$. The notes at the end of the chapter provide a sample of various results and problems in the theory of scalar curvature.

2.1 Sectional, Ricci, and scalar curvatures

> The curvature tensor of a Riemannian manifold is a little monster of (multi)linear algebra whose full geometric meaning remains obscure.
>
> *M. Gromov* [Gro94].

Throughout, let (M, g) be a Riemannian manifold. We will assume the basic definitions and facts of Riemannian geometry, and use without further comment the Einstein summation convention in what follows.

Definition 2.1.1. The *Ricci curvature* of M is the covariant 2-tensor

$$\mathrm{Ric} = R_{ij}\, dx^i \otimes dx^j$$

obtained by contracting the full Riemann curvature tensor:

$$R_{ij} = R_{ki}{}^{k}{}_{j}.$$

The Ricci curvature is symmetric; $\mathrm{Ric}(u, v) = \mathrm{Ric}(v, u)$ for all $u, v \in T_xM$. Let a unit vector $v \in T_xM$ span a line. We define the *Ricci curvature of the line spanned by* v to be $\mathrm{Ric}(v, v)$. This does not depend on whether we choose v or $-v$ to represent the line.

Definition 2.1.2. The *scalar curvature* of M is the smooth function $\kappa : M \to \mathbb{R}$ given by contracting the Ricci curvature:

$$\kappa = g^{ij} R_{ij} = R^i_i.$$

Here κ is used for the scalar curvature instead of R, which we reserve to denote the full Riemann curvature tensor. As the above quote by Gromov illustrates, the full Riemann curvature tensor of a Riemannian manifold is a highly complicated object. While the Ricci and scalar curvatures do not provide all the information about curvature, their study is far more tractable than that of the 'little monster'. It is for this reason that we will only consider scalar curvature in what follows.

The scalar curvature at a point $x \in M$ gives a rough measure of how M curves at that point. Denote by $S^{n-1}(x)$ the unit sphere in the inner product space (T_xM, g_x), with the induced metric from g_x. The scalar curvature is equivalently defined as

$$\kappa(x) = \frac{1}{\mathrm{Vol}(S^{n-1}(x))} \int_{S^{n-1}(x)} \mathrm{Ric}(v, v)\, d\mathrm{vol}_{S^{n-1}(x)};$$

see page 107 of [dC92]. Hence the scalar curvature at x is the average of the Ricci curvatures of lines through the origin in T_xM.

One can also define the scalar curvature by comparing volumes of small geodesic balls centred at x to volumes of the standard Euclidean balls. In fact,

$$\frac{\mathrm{Vol}(B_{(M,g)}(x, \epsilon))}{\mathrm{Vol}(B_{\mathbb{R}^n}(0, \epsilon))} = 1 - \frac{\kappa(x)}{6(n+2)}\epsilon^2 + O(\epsilon^4), \tag{2.1}$$

where $n = \dim M$ and $\mathbb{R}^n$ is given the standard Euclidean metric; see [Gra73] for a proof. It follows that if $\kappa(x) < 0$, small geodesic balls centred at x will have *larger* volumes than their Euclidean counterparts. If $\kappa(x) > 0$ then small geodesic balls at x will have *smaller* volumes than the Euclidean balls. If $\kappa(x) = 0$ then the volumes of small balls at x closely approximate the volumes of the corresponding Euclidean balls, to fourth order in fact.

There is an explicit local formula for κ in terms of the metric coefficients and the Christoffel symbols of the Levi-Civita connection:

$$\kappa = g^{ij}\left(\partial_k \Gamma^k_{ij} - \partial_j \Gamma^k_{ik} + \Gamma^\ell_{ij}\Gamma^k_{k\ell} - \Gamma^\ell_{ik}\Gamma^k_{j\ell}\right). \tag{2.2}$$

To prove this, one simply computes $R(\partial_i, \partial_j)\partial_k = \nabla_{\partial_i}\nabla_{\partial_j}\partial_k - \nabla_{\partial_j}\nabla_{\partial_i}\partial_k$ and takes traces.

Definition 2.1.3. Let $u, v \in T_xM$ be linearly independent. The *sectional curvature* of (u, v) is defined as

$$K(u, v) = \frac{\langle R(u, v)v, u\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2}.$$

Suppose u and v span a plane $V \subset T_xM$. If u', v' also span V then $K(u, v) = K(u', v')$. Hence we can define the sectional curvature of the plane V as $\mathrm{Sec}(V) = K(u, v)$ for some choice of basis $\{u, v\}$ of V. It follows there is a well-defined map $\mathrm{Sec} : \mathrm{Gr}_2(T_xM) \to \mathbb{R}$ on the Grassmannian of 2-planes in T_xM. Allowing x to vary, we have a smooth map $\mathrm{Sec} : \mathrm{Gr}_2(TM) \to \mathbb{R}$, where $\mathrm{Gr}_2(TM)$ is the smooth fibre bundle on M whose fibre at x is $\mathrm{Gr}_2(T_xM)$.

While the full Riemann curvature tensor cannot be recovered from the Ricci and scalar curvatures, it can in fact be recovered from the sectional curvatures. It should therefore be possible to write down the scalar curvature in terms of the sectional curvatures.

Proposition 2.1.4. *If e_i is an orthonormal frame for T_xM then*

$$\kappa(x) = \sum_{i \neq j} K(e_i, e_j).$$

Proof. Since the frame is orthonormal, $g_{ij} = \delta_{ij}$, and we can freely raise and lower indices, e.g. $R_{ij}{}^{k}{}_{\ell} = g^{km} R_{ijm\ell} = \delta^{km} R_{ijm\ell} = R_{ijk\ell}$. Hence

$$\kappa = g^{ij} R_{ij} = g^{ij} R_{ki}{}^{k}{}_{j} = \sum_{i} R_{ki}{}^{k}{}_{i} = \sum_{i,k} R_{kiki} = \sum_{i \neq j} R_{ijij} = \sum_{i \neq j} K(e_i, e_j).$$

□

We also have the following, which is useful for calculating sectional curvature in certain cases (e.g. the n-sphere).

Proposition 2.1.5. *Let $V \subset T_xM$ be a plane, and let $W \subset M$ be the diffeomorphic image of $V \cap B$ under the exponential map, where $B \subset T_xM$ is a normal neighbourhood of the origin. Then W is a 2-dimensional (locally closed) submanifold in M and inherits an induced metric from M. The sectional curvature of V is equal to the Gaussian curvature of W at x.*

Proof. Choose an orthonormal basis $\{e_1, e_2\}$ for V, and complete it to an orthonormal basis $e_1, \ldots, e_n$ for T_xM. These define normal coordinates in a neighbourhood of x. It is easy to see that the exponential map for W at x is the exponential map for M at x restricted to the subspace V. One way of recovering the curvature is through an expansion of the metric coefficients in normal coordinates:

$$g_{ij} = \delta_{ij} + \frac{1}{3} \sum_{p,q} R_{jpqi} x^p x^q + \mathrm{O}(|x|^4);$$

see page 21 of [Roe98]. In this way we see the Gaussian curvature of W at x is R_{1212}, which is the sectional curvature of V. □

We now use these facts to give a computation of the scalar curvature of the n-sphere S^n.

Example 2.1.6 (Scalar curvature of the n-sphere)**.** Consider the tangent space to the north pole $(0, \ldots, 0, 1) \in S^n$. This has an orthonormal basis consisting of the vectors $e_1, \ldots, e_n$ in $\mathbb{R}^{n+1}$. The geodesics of S^n are great circles, and so the image of $\operatorname{span}\{e_i, e_j\}$ under the exponential map is the unit 2-sphere in $\operatorname{span}\{e_i, e_j, e_{n+1}\} \subset \mathbb{R}^{n+1}$. This has Gaussian curvature equal to 1. Thus the sectional curvatures are all 1, and so the scalar curvature is $\sum_{i \neq j} 1 = n(n-1)$. This is at the north pole. Since the group $\mathrm{O}(n+1)$ acts transitively on S^n by isometries, the scalar curvature is the same at every point.

We also state the following examples without proof.

Examples 2.1.7.

- Euclidean space $(\mathbb{R}^n, \langle\, , \rangle)$ has $\kappa = 0$.
- Any one-dimensional Riemannian manifold, being locally isometric to $\mathbb{R}$, has zero scalar curvature.
- The n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ with the induced metric from Euclidean space has scalar curvature $\kappa = 0$.
- The upper half space $\mathbb{H}^n = \{x \in \mathbb{R}^n : x_n > 0\}$ with the metric $g(\partial_i, \partial_j) = \frac{1}{x_n^2}\langle \partial_i, \partial_j \rangle$ has constant negative scalar curvature $\kappa = -n(n-1)$; see Petersen [Pet16], page 121.[2]
- If (M, g) is a surface, the scalar curvature is twice the Gaussian curvature K; $\kappa = 2K$. To see this, note that the definition of Gaussian curvature (with respect to an orthonormal frame) is $K = R_{1212}$, whereas the scalar curvature is the sum of sectional curvatures $R_{1212} + R_{2121} = 2R_{1212} = 2K$.
- A lens space, being a quotient of a sphere S^{2n-1} by a free action of $\mathbb{Z}_m$ by isometries, inherits a metric of constant positive scalar curvature. This includes every $\mathbb{RP}^n$ with $n > 1$.
- The Fubini-Study metric on $\mathbb{CP}^n$ has constant positive scalar curvature. In fact, the Ricci curvature satisfies $\operatorname{Ric}(v, v) = 2n + 2$ for all v with length 1, so the scalar curvature $\kappa = 2n + 2$ everywhere; see [Pet16, pages 148-150].
- Each Riemann surface of genus $g \geqslant 2$ admits a constant metric of negative scalar curvature, while the torus $(g = 1)$ has a metric of zero scalar curvature. This is a consequence of the uniformization theorem

[2] In Petersen's notation, $\mathbb{H}^n$ is isometric to S^n_k with $k = -1$.

for Riemann surfaces; each surface with $g \geqslant 2$ is the quotient of $\mathbb{H}^2$ by a discrete group of isometries. See Lemma 2.3.5, Theorem 2.3.4, and Theorem 4.4.1 of [Jos06],

- If a Lie group G has a bi-invariant Riemannian metric (that is, all left and right actions by elements of G are isometries) then for orthonormal left-invariant vector fields X and Y, one has $\sec(X, Y) = \frac{1}{4}\|[X, Y]\|^2 \geqslant 0$; [dC92, page 103]. It follows that if the Lie algebra of G is non-abelian, then G has a metric of positive scalar curvature. In particular, since any compact Lie group admits a bi-invariant metric [dC92, page 46], any compact Lie group with non-abelian Lie algebra has a metric of positive scalar curvature.

- Let G be a compact Lie group and H a closed subgroup. Then the homogeneous space G/H has a metric of positive scalar curvature; this is a simple corollary of Theorem 3.5 in Chapter X of [KN69], see also [Hit74, page 8].

Proposition 2.1.8. *Let κ_1 be the scalar curvature of (M_1, g_1), κ_2 the scalar curvature of (M_2, g_2). Then the scalar curvature of $(M, g) := (M_1 \times M_2, g_1 \oplus g_2)$ is $\kappa(x_1, x_2) = \kappa_1(x_1) + \kappa_2(x_2)$.*

Sketch proof. The curvature of the product is the direct sum of the curvatures of the factors; $R_M = R_{M_1} \oplus R_{M_2}$. Taking the first trace to get the Ricci tensor respects the direct sum decomposition, so $\mathrm{Ric}_M = \mathrm{Ric}_{M_1} \oplus \mathrm{Ric}_{M_2}$. Taking the second trace, we get $\kappa = \kappa_1 + \kappa_2$. □

2.2 Positive scalar curvature

Definition 2.2.1. We say (M, g) has *positive scalar curvature* if $\kappa(x) > 0$ for all $x \in M$.

It is convenient to adopt the abbreviation 'PSC' to mean positive scalar curvature.

For example, the n-sphere with its standard metric has PSC for all $n > 1$. A longstanding problem in Riemannian geometry is to classify compact manifolds which admit a metric of PSC. We begin with the following classical obstruction for surfaces of genus $g \geqslant 1$.

Theorem 2.2.2 (Gauss-Bonnet)**.** *Let Σ_g be a compact oriented surface of genus g. Then with respect to any metric on Σ_g,*

$$\frac{1}{4\pi} \int_{\Sigma_g} \kappa \, d\mathrm{vol} = \chi(\Sigma_g),$$

where $\chi(\Sigma_g) = 2 - 2g$ *is the Euler characteristic of* Σ_g*.*

Now, since $\chi(\Sigma_g) = 2 - 2g \leqslant 0$ for $g \geqslant 1$, it follows that a g-holed torus Σ_g with $g \geqslant 1$ cannot have positive scalar curvature. So by the classification of compact oriented surfaces, the only such surface admitting a metric of positive scalar curvature is the two-sphere S^2. We say that non-zero genus is an *obstruction* to positive scalar curvature for compact oriented surfaces. It is intriguing that a global topological feature such as the genus can influence a local feature such as the curvature.

Remark 2.2.3. We will only consider orientable manifolds in this thesis, however the PSC classification for non-orientable compact surfaces is so near that it is difficult to resist. If Σ is a non-orientable compact surface then Σ has an orientable double cover $\widetilde{\Sigma}$, diffeomorphic to a g-holed torus Σ_g. If Σ admits a metric of positive scalar curvature then so must $\widetilde{\Sigma}$. It follows that $\widetilde{\Sigma}$ is the two-sphere. The only non-orientable compact surface having the two-sphere as its orientation double cover is $\mathbb{RP}^2$, hence the only compact surfaces (orientable or not) which admit metrics of positive scalar curvature are S^2 and $\mathbb{RP}^2$.

The Gauss-Bonnet theorem solves the PSC classification problem in the two-dimensional case, but the higher dimensional cases are still unknown and there are many sophisticated and varied techniques for finding obstructions. In [Ros07], Rosenberg proposes that all such techniques stem from three basic principles: the Lichnerowicz formula in spin geometry [Lic63], minimal surface techniques in dimensions $\leqslant 7$ [SY79b], and Seiberg-Witten theory in dimension 4 [SW94a, SW94b, Wit94]. The aim of this thesis is to study the applications of index theory to positive scalar curvature, which are based on the Lichnerowicz formula approach.

The original such application is the following theorem of Lichnerowicz:

Theorem 2.2.4 ([Lic63])**.** *Let M be a compact spin manifold of positive scalar curvature. Then*

$$\widehat{A}(M) = 0,$$

where $\widehat{A}(M)$ is the A-hat genus of M, defined in terms of the pontryagin classes.

The proof uses the index theorem of Atiyah and Singer [AS63, AS68a, AS68b] for spin Dirac operators. This remarkable theorem provides deep links between geometry, topology and analysis, and lends itself to applications in many different areas of mathematics and theoretical physics. Accounts of spin manifolds, Dirac operators, the Atiyah-Singer index theorem, and the proof of the Lichnerowicz obstruction, are all given in Appendix B. The Lichnerowicz obstruction and an example are found specifically in Appendix B.7.

Existence results

Proposition 2.2.5. *Let M_1, M_2 be compact manifolds. If either M_1 or M_2 admits a metric of positive scalar curvature, then the product $M_1 \times M_2$ also admits a metric of positive scalar curvature.*

Proof. Let g_1 be a metric of PSC on M_1 and g_2 an arbitrary metric on M_2. If we rescale g_1 and define $g_1' = \frac{1}{c} g_1$, then formula (2.2) tells us that the scalar curvature changes by $\kappa_1' = c \cdot \kappa_1$ (the Christoffel symbols do not change under constant rescalings, and the inverse metric coefficients g^{ij} change by the inverse of the scaling factor). Using compactness, we take c large enough so that $c \cdot \min_{x_1 \in M_1} \kappa_1(x_1) > \max_{x_2 \in M_2} |\kappa_2(x_2)|$. In view of Proposition 2.1.8, the resulting metric has PSC. □

We will describe another method for obtaining metrics of PSC, proved independently by Gromov-Lawson [GL80] and Schoen-Yau [SY79b]. First, we must review what 'surgery' on a manifold is. In what follows, D^k will denote the closed unit ball in $\mathbb{R}^k$, and B^k the open unit ball in $\mathbb{R}^k$.

Definition 2.2.6. Let M be a manifold of dimension n, and consider an embedded k-sphere $S^k \subset M$, where $k < n$. Assume the normal bundle of the sphere is trivial (S^k with a given trivialisation of the normal bundle is called a *framed sphere* in the literature). Then there is a normal neighbourhood of the sphere, diffeomorphic to $S^k \times D^{n-k}$. This neighbourhood has boundary $S^k \times S^{n-k-1}$, which is also the boundary of $D^{k+1} \times S^{n-k-1}$. We can therefore delete the interior of the normal neighbourhood and attach $D^{k+1} \times S^{n-k-1}$ to the boundary. This process is called a *surgery of dimension k*.

Remark 2.2.7. Of course this is only a topological description of surgery; the resulting space is a manifold with a standard smooth structure. To get the smooth structure, one really considers small open neighbourhoods of the boundaries involved, and glues these together via diffeomorphisms.

Example 2.2.8. The connected sum of manifolds $M_1 \# M_2$ is an example of surgery. To see this, consider an embedded 0-sphere (the disjoint union of two points) with one point in M_1 and the other in M_2. Then $S^0 \times D^n$ is a closed ball about each point. Removing these balls and attaching $D^1 \times S^{n-1}$ to the boundary, we get the connected sum of M_1 and M_2. This is a surgery of codimension n.

Theorem 2.2.9 (PSC surgery theorem, [GL80], [SY79b]). *Let (M, g) be a Riemannian manifold of positive scalar curvature. If one performs a surgery on M of codimension $\geqslant 3$, then the resulting manifold admits a metric of positive scalar curvature.*

Corollary 2.2.10. *If M_1 and M_2 are compact orientable manifolds of dimension $\geqslant 3$ which admit metrics of positive scalar curvature, then their connected sum $M_1 \# M_2$ also admits a metric of positive scalar curvature.*

Note that the result of the corollary is actually true in dimension 2 also, since in that case $M_1 = M_2 = S^2$ and $M_2 \# M_2 = S^2$ which has PSC.

The following will be needed in studying path components of the moduli space of PSC metrics.

Proposition 2.2.11 ([GL80], Lemma 3). *Let M be compact, and let h_t be a smooth path $[0,1] \to \mathcal{R}^+(M)$ in the space of positive scalar curvature metrics on M. Then for all sufficiently large $c > 0$, the metric $g = c\,dt^2 + h_t$ on $[0,1] \times M$ has positive scalar curvature.*

The intuitive idea is that of Proposition 2.1.8—if we scale the metric up in one factor, the curvature contributions of that factor will diminish. By compactness, we can scale it up enough so that the positive scalar curvature of the other factor dominates.

Proof. This is a rather tedious exercise in keeping track of how Christoffel symbols change. Denote the Christoffel symbols on $[0,1] \times M$ by Γ^k_{ij}. We cover M by finitely many coordinate balls that are precompact and extend to slightly larger coordinate balls, and take product coordinates $(t, x^1, \ldots, x^n)$ on $[0,1] \times M$, so that the 1 in Γ^k_{1j} corresponds to the t-variable. The symbol γ^k_{ij} is reserved for the Christoffel symbols of $\{t\} \times M$ for some fixed t. Now, the metric g on $[0,1] \times M$ locally has the form

$$[g_{ij}] = \begin{pmatrix} c & 0 \\ 0 & [h_{ij}] \end{pmatrix}.$$

The inverse metric coefficients of g therefore take the form

$$[g^{ij}] = \begin{pmatrix} 1/c & 0 \\ 0 & [h^{ij}] \end{pmatrix}.$$

The formula for the Christoffel symbols is

$$\Gamma^k_{ij} = \frac{1}{2}\, g^{k\ell}(\partial_j g_{\ell i} - \partial_\ell g_{ij} + \partial_i g_{\ell j}).$$

Comparing this formula to the above forms of $[g_{ij}]$ and $[g^{ij}]$ we have the following estimates.

i,j,k	Γ_{ij}^k
$i,j,k\neq 1$	$\Gamma_{ij}^k=\gamma_{ij}^k$
$i,j\neq 1,\ k=1$	$\Gamma_{ij}^1\sim 1/c$
$j=k=1$	$\Gamma_{i1}^1=0$
$i=1,\ j,k\neq 1$	$\Gamma_{1j}^k\sim 1$
$i=j=1$	$\Gamma_{11}^k=0$

As an example, for $i,j\neq 1$ and $k=1$, we have $g^{k\ell}\neq 0$ only for $\ell=k=1$. Then

$$\Gamma_{ij}^1=\frac{1}{2c}(\partial_j g_{1i}-\partial_1 g_{ij}+\partial_i g_{1j})\sim 1/c.$$

The estimate $\sim 1/c$ holds over the whole coordinate neighbourhood $[0,1]\times B$, since we have chosen B to be precompact and to extend to a slightly larger coordinate ball B'. Since we have covered $[0,1]\times M$ by finitely many such neighbourhoods, the estimate $\sim 1/c$ holds uniformly over the whole manifold. Note the factors of $1/c$ are constant, so the same estimates hold for any derivatives of the Christoffel symbols.

Using these estimates we study how the scalar curvature changes. We have

$$\kappa=g^{ij}(\partial_k\Gamma_{ij}^k-\partial_j\Gamma_{ik}^k+\Gamma_{ij}^\ell\Gamma_{k\ell}^k-\Gamma_{ik}^\ell\Gamma_{j\ell}^k).$$

For $i=j=1$,

$$\begin{aligned}&g^{11}(\partial_k\Gamma_{11}^k-\partial_1\Gamma_{1k}^k+\Gamma_{11}^\ell\Gamma_{k\ell}^\ell-\Gamma_{1k}^\ell\Gamma_{1\ell}^k)\\ \sim&\frac{1}{c}(0-1+0-1)\\ \to&0.\end{aligned}$$

For $i,j\neq 1$, we split into the three cases below:

$k,\ell\neq 1$:

$$\begin{aligned}&g^{ij}(\partial_k\Gamma_{ij}^k-\partial_j\Gamma_{ik}^k+\Gamma_{ij}^\ell\Gamma_{k\ell}^k-\Gamma_{ik}^\ell\Gamma_{j\ell}^k)\\ =&h^{ij}(\partial_k\gamma_{ij}^k-\partial_j\gamma_{ik}^k+\gamma_{ij}^\ell\gamma_{k\ell}^k-\gamma_{ik}^\ell\gamma_{j\ell}^k)\\ =&\kappa_{\{t\}\times M}.\end{aligned}$$

$k=1$:

$$\begin{aligned}&g^{ij}(\partial_1\Gamma_{ij}^1-\partial_j\Gamma_{i1}^1+\Gamma_{ij}^\ell\Gamma_{1\ell}^1-\Gamma_{i1}^\ell\Gamma_{j\ell}^1)\\ \sim&1(1/c-0+0+1/c)\\ \to&0.\end{aligned}$$

$k \neq 1, \ell = 1$:

$$\begin{aligned} & g^{ij}(\Gamma^1_{ij}\Gamma^1_{k1} - \Gamma^1_{ik}\Gamma^k_{j1}) \\ \sim & \, 1(1/c + 1/c) \\ \to & \, 0. \end{aligned}$$

Hence we see that as $c \to \infty$, all terms except for $\kappa_{\{t\}\times M}$ go to zero and, since there are only finitely many precompact coordinate neighbourhoods involved, this term eventually dominates and the resulting metric has positive scalar curvature. □

Remark 2.2.12. There is a proof in [GL80] which uses more sophisticated techniques than our brute force method of checking term by term. However, their methods only yield estimates of order $1/\sqrt{c}$, whereas these give the stronger estimate of order $1/c$. The possibility of a better estimate is mentioned by Rosenberg and Stolz [RS01] in the proof of Proposition 3.3.

2.3 The moduli space of PSC metrics

The aim of this section is to establish the path lifting property for the map $\mathcal{R}^+(M) \to \mathcal{R}^+(M)/\mathrm{Diff}(M) = \mathcal{M}^+(M)$, which will be used to study path components of the moduli space $\mathcal{M}^+(M)$. The main ingredient is the Ebin slice theorem [Ebi70]. The rough structure of the proof is borrowed from [Wie16].

2.3.1 The space of PSC metrics as a Fréchet manifold

Let $E = \mathrm{Symm}^2(T^*M)$ be the second symmetric power of the cotangent bundle of M. We topologise the space of sections $\Gamma(E)$ as in Appendix A.2, with the topology of uniform convergence of all derivatives on compact subsets. The set of Riemannian metrics on M is a subset of the space of sections $\Gamma(E)$, specifically the subset of sections which are pointwise positive definite.

Proposition 2.3.1. *The set of Riemannian metrics on a compact manifold M, considered as a subset of the space of sections of $E = \mathrm{Symm}^2(T^*M)$, is an open subset.*

It follows that the set of Riemannian metrics on a compact manifold is naturally a Fréchet manifold, being an open subset of a Fréchet space.

Proof. Let $g \in \Gamma(E)$ be a Riemannian metric. Then g is pointwise positive definite. Choose local coordinates $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$. These give rise to a trivialisation $\psi : E|_U \to U \times \mathrm{Symm}(n)$, where $\mathrm{Symm}(n) \subset \mathbb{R}^{n^2}$ denotes the vector subspace of $n \times n$ symmetric matrices. Positive definiteness of g can be checked in local coordinates through Sylvester's criterion; g is positive definite at $x \in U$ if and only if all principal minors of $[g_{ij}(x)]$ are positive. The principal minors appearing in Sylvester's criterion are all continuous functions $\mathrm{Symm}(n) \to \mathbb{R}$. It follows that on each compact subset $K \subset \varphi_\alpha(U_\alpha)$, we can vary the entries of the matrix $[g_{ij}]$ by some small $\epsilon > 0$, and the principal minors will remain positive. The data $(U, \varphi, \psi, K, 0)$ and ϵ (see Appendix A.2 for notations) define a neighbourhood of g in $\Gamma(E)$, although it gives us no control outside the compact subset K. Since M is compact we can cover M by finitely many such compact sets K_i, with corresponding $\epsilon_i > 0$. These will define finitely many neighbourhoods of g, and taking their intersection gives a neighbourhood of g containing only Riemannian metrics. □

Remark 2.3.2. If M is not compact then we will have no control over sections of E outside compact subsets of M. Hence the set of Riemannian metrics will not be open in the above topology.

Proposition 2.3.3. *Let M be compact. The set of positive scalar curvature metrics $\mathcal{R}^+(M)$, considered as a subset of $\mathcal{R}(M)$, is an open subset.*

It follows that the space of positive scalar curvature metrics is a Fréchet manifold.

Proof. Let g be a metric of positive scalar curvature on M, $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ a local coordinate system, and $\psi : E|_U \to U \times \mathrm{Symm}(n)$ the corresponding local trivialisation of E. The scalar curvature is a continuous function of the g_{ij} and their first and second partials. Hence over a compact subset K, we can choose $\epsilon > 0$ such that if the g_{ij} and their first and second partials are varied by no more than ϵ, the resulting matrix will be a Riemannian metric with positive scalar curvature. The data $(U, \varphi, \psi, K, 2)$ and $\epsilon > 0$ define a neighbourhood of g such that all sections in this neighbourhood are Riemannian over K and have positive scalar curvature over K. As before we can cover M by finitely many such compact subsets K_i with corresponding $\epsilon_i > 0$ in order to define a neighbourhood of g consisting only of Riemannian metrics of positive scalar curvature. □

2.3.2 The moduli space of PSC metrics

Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms of a compact manifold M. This is an infinite dimensional Fréchet Lie group, with Lie algebra the space of vector fields $\Gamma(TM)$ with the usual Lie bracket; see [Mil84], for example. The group $\mathrm{Diff}(M)$ acts on the space of Riemannian metrics on the right via pullback; if $\varphi \in \mathrm{Diff}(M)$ and $g \in \mathcal{R}(M)$ then the pullback metric $\varphi^* g$ is defined by

$$(\varphi^* g)_x(X_x, Y_x) := g_{\varphi(x)}(d\varphi|_x(X_x), d\varphi|_x(Y_x)).$$

The resulting map $\varphi : (M, \varphi^* g) \to (M, g)$ is an isometry of Riemannian manifolds. Since pulling back metrics preserves all curvature invariants, the action takes metrics of PSC to metrics of PSC. The action therefore restricts to a well-defined action of $\mathrm{Diff}(M)$ on the space of PSC metrics $\mathcal{R}^+(M)$.

Definition 2.3.4. Let M be a compact manifold which admits a metric of PSC. The *moduli space of PSC metrics* on M is the set $\mathcal{M}^+(M) = \mathcal{R}^+(M)/\mathrm{Diff}(M)$ equipped with the quotient topology from $\mathcal{R}^+(M)$.

Remark 2.3.5. We cannot expect the action of $\mathrm{Diff}(M)$ on $\mathcal{R}^+(M)$ to be free; for example the n-sphere S^n with its standard metric of PSC has $\mathrm{O}(n+1)$ as its isotropy subgroup. We will see that the quotient $\mathcal{M}^+(M)$ still has a workable topology, however.

We recall from the introduction that, once we have established a path lifting property for the map $\mathcal{R}^+(M) \to \mathcal{M}^+(M)$, we will think of two metrics of PSC as distinct if their classes in $\mathcal{M}^+(M)$ lie within distinct path components. There are some interesting results on this in low dimensions. Recall the only compact surfaces admitting metrics of positive scalar curvature are S^2 and $\mathbb{RP}^2$.

Theorem 2.3.6 ([RS01], Theorem 3.4)**.** *The moduli spaces of PSC metrics for S^2 and $\mathbb{RP}^2$ are contractible.*

There is also the more recent, and much more difficult theorem of Marques [Mar12]:

Theorem 2.3.7 ([Mar12])**.** *If M is a compact orientable 3-manifold admitting a metric of positive scalar curvature, then the moduli space of PSC metrics on M is path connected.*

We will see that the behaviour of the moduli space in higher dimensions differs violently from these low-dimensional cases.

Paths in the moduli space

In this section we will show that every continuous path in the moduli space $\mathcal{M}^+(M)$ can be represented up to homotopy by a smooth path in the space $\mathcal{R}^+(M)$.

Lemma 2.3.8. *Let $\gamma : [a, b] \to \mathcal{R}$ be a continuous path in a Fréchet manifold $\mathcal{R}$. Then γ is homotopic to a smooth path.*

Hence, if we can lift a continuous path in $\mathcal{M}^+(M)$ to a continuous path in $\mathcal{R}^+(M)$, then we can homotopy the lift to a smooth path, and the original path in $\mathcal{M}^+(M)$ will be homotopic to the projection of this smooth path.

Proof. For each $t \in [0, 1]$ there is a neighbourhood U_t of $\gamma(t)$ that is diffeomorphic to a convex open subset of a Fréchet space via $\varphi_t : U_t \to \varphi_t(U_t)$. We can cover $[0, 1]$ by finitely many neighbourhoods $\gamma^{-1}(U_{t_1}), \ldots, \gamma^{-1}(U_{t_m})$, and choose finitely many points $0 = a_0 < a_1 < \cdots < a_n = 1$ such that $[a_i, a_{i+1}] \subset \gamma^{-1}(U_{t_j})$ for some j between 1 and m (depending on i). Since $\varphi_{t_j}(U_{t_j})$ is convex, we can homotopy the path $\varphi_{t_j} \circ \gamma|_{[a_i,a_{i+1}]}$ to a straight line path, which corresponds to a smooth path $[a_i, a_{i+1}] \to \mathcal{R}$. Then γ is homotopic to the composition of these smooth paths. However, the composition of these smooth paths will not be smooth at the a_i. To fix this, we can homotopy the speed at which we traverse γ; choose a smooth function $\chi : [0, 1] \to [0, 1]$ such that $\chi^{-1}(a_i)$ is a neighbourhood of a_i. Then the composition $\gamma(\chi(t))$ will be smooth, and homotopic to γ via $F(s, t) = \gamma(st + (1 - s)\chi(t))$. □

The following path lifting property will also be needed.

Lemma 2.3.9 ([Bre72], Theorem 6.2 of Chapter II)**.** *Let G be a compact Lie group acting continuously on a Hausdorff topological space X, and let $\pi : X \to X/G$ be the projection. If $\gamma : [0, 1] \to X/G$ is a continuous path then there is a continuous lift $\widetilde{\gamma} : [0, 1] \to X$ of γ, that is to say, a continuous path satisfying $\pi \circ \widetilde{\gamma} = \gamma$.*

As is, we cannot apply this result to the moduli space of PSC metrics, since it is a quotient by the action of a non-compact group. However, we will see from the Ebin slice theorem that the moduli space locally looks like a quotient by a compact Lie group, and from this we can deduce a path lifting property.

Let $\mathcal{R}(M)$ denote the space of Riemannian metrics on M. If $g \in \mathcal{R}(M)$, we write $\mathrm{Iso}(g)$ for the set of isometries $(M, g) \to (M, g)$. This is a Lie subgroup of the diffeomorphism group $\mathrm{Diff}(M)$ of M, and it is compact if M is compact; see [KN63, Chapter VI, Theorem 3.4]. There is a canonical right action of $\mathrm{Diff}(M)$ on $\mathcal{R}(M)$ via pullback: $g \cdot \varphi = \varphi^* g$, where $g \in \mathcal{R}(M)$ and $\varphi \in \mathrm{Diff}(M)$. We let $A(g, \varphi) = \varphi^* g$ denote this action.

Theorem 2.3.10 (Ebin slice theorem [Ebi70], Theorem 7.4)**.** *For each $g \in \mathcal{R}(M)$, there exists a* slice *for the action A at g, that is, a contractible subset $S \subset \mathcal{R}(M)$ containing g with the following properties:*

1. *If $\varphi \in \mathrm{Iso}(g)$ then $A(S, \varphi) = S$ (the isometry group of g restricts to an action on S).*

2. *If $\varphi \in \mathrm{Diff}(M)$ and $A(S, \varphi) \cap S \neq \emptyset$, then $\varphi \in \mathrm{Iso}(g)$.*

3. *There is a neighbourhood U of the identity coset in $\mathrm{Diff}(M)/\mathrm{Iso}(g)$ and a section $\chi : U \to \mathrm{Diff}(M)$ such that $S \times U$ is homeomorphic to a neighbourhood of g via the map $(s, u) \mapsto A(s, \chi(u))$.*

Let $g \in \mathcal{R}(M)$ and let N be a neighbourhood of g homeomorphic to $S \times U$ as in the above theorem, where S is a slice for the action at g. Let us see what the quotient $N/\mathrm{Diff}(M)$ looks like. Let u_0 be the identity coset in $\mathrm{Diff}(M)/\mathrm{Iso}(g)$. Then any (s, u) is identified with (s, u_0) under the action of $\mathrm{Diff}(M)$, since $\chi(u)^*\chi(u_0)^*s = \chi(u)^*s$ (by part 3.). The subsequent identifications within $S \times \{u_0\}$ are simply due to the action of $\mathrm{Iso}(g)$ (by part 2.), which is a compact Lie group. Hence we have a homeomorphism $N/\mathrm{Diff}(M) \cong S/\mathrm{Iso}(g)$, and we see that $\mathcal{R}(M)/\mathrm{Diff}(M)$ is locally homeomorphic to a quotient by the action of a compact Lie group. Since $\mathcal{R}^+(M) \subset \mathcal{R}(M)$ is open, the same holds for the moduli space $\mathcal{M}^+(M)$.

Theorem 2.3.11. *If $\gamma : [0, 1] \to \mathcal{M}^+(M)$ is a continuous path, then γ has a continuous lift $\widetilde{\gamma} : [0, 1] \to \mathcal{R}^+(M)$.*

Proof. Given a path $[0, 1] \to \mathcal{M}^+(M)$, we can cover the unit interval by finitely many subintervals $[a_i, a_{i+1}]$ such that $\gamma([a_i, a_{i+1}])$ lies within a neighbourhood homeomorphic to $S_i/\mathrm{Iso}(g_i)$ for some g_i. The path lifting property for compact Lie groups (Lemma 2.3.9) then allows us to lift $\gamma|_{[a_i,a_{i+1}]}$ to a path which lies within the slice $S_i \subset \mathcal{R}^+(M)$. The chosen lifts may not agree on the overlaps of the subintervals, however this easily remedied by noting that any two lifts of $\gamma(a_i)$ are related by a diffeomorphism, and we can translate successive pieces of the lifts by these diffeomorphisms so as to make the total lift continuous. □

2.3.3 Conclusion

We wish to study path components of the moduli space, as the number of path components can be thought of as the number of 'distinct' metrics of PSC on M. Given a path $\gamma : [0, 1] \to \mathcal{M}^+(M)$, we can lift γ to a path $\widetilde{\gamma} : [0, 1] \to \mathcal{R}^+(M)$, by Theorem 2.3.11. This can then be homotopied to a smooth path $\eta : [0,1] \to \mathcal{R}^+(M)$ by Lemma 2.3.8 and Proposition 2.3.3. The composition $\pi \circ \eta : [0,1] \to \mathcal{M}^+(M)$ will be homotopic to the original path γ, so any path in the moduli space can (up to homotopy) be thought of as a smooth path in $\mathcal{R}^+(M)$. Finally, if we have a smooth path $\eta : [0,1] \to \mathcal{R}^+(M)$, then we can equip $[0,1] \times M$ with a metric $\eta_t + c\,dt^2$ which has PSC, by Proposition 2.2.11. This remarkably allows us to represent paths within the moduli space as Riemannian manifolds (with boundary), and moreover these manifolds have PSC!

We will see later, using the above method of converting paths in $\mathcal{M}^+(M)$ into manifolds with PSC, that there exist manifolds with PSC whose moduli spaces have infinitely many path components, and so admit infinitely many 'non-distinct' metrics of positive scalar curvature.

2.4 Notes

2.4.1 Negative scalar curvature and the trichotomy theorem

Until this point, the reader might wonder whether there is any reason to be interested in negative scalar curvature. Unfortunately the existence problem for negative scalar curvature is, in a very precise sense, boring:

Theorem 2.4.1 ([KW75a], [KW75b])**.** *Let M be a compact manifold of dimension $\geqslant 3$. If $f : M \to \mathbb{R}$ is a strictly negative function on M then there exists a metric on M whose scalar curvature is f.*

The result is a corollary of the more informative *trichotomy theorem* of Kazdan and Warner:

Theorem 2.4.2 (Trichotomy theorem [KW75a], [KW75b])**.** *If M is a compact smooth manifold of dimension $\geqslant 3$. Then M satisfies precisely one of the following three conditions:*

1. *M has a metric of positive scalar curvature. In this case, any smooth function on M is the scalar curvature of some metric.*

2. *M has a metric of zero scalar curvature, and any metric on M with $\kappa \geqslant 0$ must be identically 0. In this case, a smooth function $f : M \to \mathbb{R}$ is the scalar curvature of some metric if and only if $f(x) < 0$ for some $x \in M$ or f is identically 0. Any metric with zero scalar curvature is in fact Ricci flat.*

3. *M has no metric of non-negative scalar curvature at all, and a smooth function $f : M \to \mathbb{R}$ is the scalar curvature of some metric if and only if $f(x) < 0$ for some $x \in M$.*

Moreover, the following theorem of Lohkamp [Loh92] tells us that the corresponding moduli space problem for negative scalar curvature is also boring.

Theorem 2.4.3 ([Loh92])**.** *Let M be a compact manifold of dimension $\geqslant 3$. Then the space of negative scalar curvature metrics $\mathcal{R}^-(M)$ is contractible.*

Hence the quotient $\mathcal{M}^-(M) = \mathcal{R}^-(M)/\mathrm{Diff}(M)$ is also contractible.

2.4.2 Scalar curvature in physics

The scalar curvature of a manifold plays an important role in physics, for instance in general relativity where it appears in the Einstein field equations:

$$R_{ij} - \frac{\kappa}{2} g_{ij} + \Lambda g_{ij} = \frac{8\pi G}{c^4} T_{ij},$$

where T_{ij} is the stress-energy tensor. These equations (in the case $\Lambda = 0$ and $T_{ij} = 0$) arise as the variation of the action functional

$$g \mapsto \int_M \kappa_g \, d\mathrm{vol}_g;$$

see [Hil15]. Of course, general relativity deals primarily with Lorentzian metrics, whereas we consider only Riemannian metrics.

A celebrated result in general relativity is the positive energy theorem, also known as the positive mass conjecture. The theorem was first proven in a special case by Schoen and Yau in [SY79a] and then in full generality in [SY81]. A different proof was subsequently given by Witten in [Wit81]. In the context of the positive energy theorem, the scalar curvature involved is that of a Riemannian manifold, namely a space-like hypersurface of space-time.

2.4.3 Yamabe problem

The Yamabe problem is the following: Given a compact Riemannian manifold (M, g), is there a metric in the conformal class of g with constant scalar curvature? Or in other words, is there a smooth function $f : M \to \mathbb{R}$ such that $e^f g$ has constant scalar curvature? The answer is indeed yes, and was proven in 1984 by Schoen [Sch84], following earlier work of Yamabe [Yam60], Trudinger [Tru68], and Aubin [Aub76]. The proof in the 3-dimensional case is one application of the positive energy theorem.

2.4.4 Higher homotopy groups of the moduli space

Recently, Botvinnik *et al.* [BHSW10] have shown that for any integer $N \geqslant 0$, there is a manifold M with PSC metric g whose moduli space $\mathcal{M}^+(M)$ has a nontrivial homotopy group $\pi_n(M^+(M), [g])$ for some $n \geqslant N$.

Theorem 2.4.4 (Theorem 1.12, [BHSW10])**.** *For any $N > 0$ there is a compact smooth manifold M admitting a metric of positive scalar curvature g such that for any $0 < q \leqslant N$ the homotopy group $\pi_{4q}(\mathcal{M}^+(M), [g])$ is non-trivial.*

2.4.5 Concordance vs. isotopy

Another open problem in positive scalar curvature is the *concordance vs. isotopy problem.* Let M be a compact manifold, and let g_0 and g_1 be Riemannian metrics on M having positive scalar curvature. The metrics g_0 and g_1 are *isotopic* if there is a path in $\mathcal{R}^+(M)$ connecting g_0 to g_1. That is, g_0 and g_1 lie within the same path component of $\mathcal{R}^+(M)$. The metrics g_0 and g_1 are *concordant* if there is a metric of positive scalar curvature on $[0,1] \times M$ which restricts to $dt^2 + g_0$ on a neighbourhood of $\{0\} \times M$ and to $dt^2 + g_1$ on a neighbourhood of $\{1\} \times M$. Note that Proposition 2.2.11 gives that isotopy implies concordance, after one reparamatrises the path h_t so that it is constant near 0 and 1.

In 4 dimensions, concordance does not imply isotopy. In fact, Ruberman [Rub01] deduced the following using the techniques of Seiberg-Witten theory.

Theorem 2.4.5 ([Rub01])**.** *There is a simply connected compact 4-dimensional manifold which has an infinite family of concordant positive scalar curvature metrics, none of which are isotopic.*

In a recent paper on the arXiv [Bot16], Botvinnik proposes a proof of concordance implies isotopy for simply connected manifolds of dimension $\geqslant 5$.

Chapter 3

The Atiyah-Patodi-Singer index theorem

In this chapter we review the Atiyah-Patodi-Singer (APS) index theorem [APS75a, APS75b, APS76], a few important aspects of its proof, and some applications to positive scalar curvature. Unlike the Atiyah-Singer index theorem for closed manifolds, the APS index theorem applies to Riemannian manifolds Z with boundary $\partial Z = Y$, under the assumption that the metric is a product near the boundary. It still computes the index of a Dirac operator on such a manifold, although there are a couple of key differences: (i) the index is taken with respect to a certain global boundary condition, and (ii) there is an *eta invariant* which appears as a correction term accounting for the presence of the boundary.

The eta invariant is defined in terms of the spectrum of the Dirac operator on the boundary, so the 'invariant' part of its name refers to its spectral invariance. The Dirac operator on the boundary has a discrete spectrum of real eigenvalues, and the eta invariant can be thought of formally as the number of positive eigenvalues minus the number of negative eigenvalues in the spectrum. Of course, these numbers are infinite so this must be interpreted in a regularised sense.

The original proof of the APS index theorem in [APS75a] used heat kernel analysis and is modelled on the proof of the local index theorem for manifolds without boundary. There are considerable differences however, notably one must work with the non-compact manifold $\mathbb{R}_{\leqslant 0} \times Y$, where $Y = \partial Z$ is the boundary, and keep track of the boundary condition throughout the analysis. The proof proceeds by analysing the heat kernel on $\mathbb{R}_{\leqslant 0} \times Y$ with the global boundary condition, and then patching together information from this non-compact manifold with information from the closed double of the original manifold Z. There is also an alternative proof due to Melrose [Mel93], a central element of which is a certain regularised trace called the b-trace.

It should be noted that the index of a Dirac operator on an odd-dimensional manifold vanishes, and so does not provide useful information such as obstructions to positive scalar curvature. It seems that the saving grace of odd-dimensional manifolds is that they can be boundaries of even-dimensional manifolds—a fact which can be systematically exploited using the APS index theorem.

The eta invariant, being defined in terms of the spectrum of a Dirac operator, is rather sensitive. For example, if one varies (non-conformally) the metric of a Riemannian manifold then most likely one will produce changes in the eta invariant. However, in order to produce a more robust invariant, one can twist the Dirac operator by two unitary representations of the fundamental group, and take the difference of the resulting eta invariants. The invariant so obtained is called the *rho invariant*, and it turns out to have many more invariance properties than the eta invariant. For example, the mod $\mathbb{Z}$ reduction of the rho invariant of the signature operator is independent of the choice of Riemannian metric, and so is an (orientation preserving) diffeomorphism invariant [APS75b].

The invariance properties of rho invariants can be elegantly captured by way of homology theories such as geometric K-homology [HR10] and spin bordism [BG95]. There is also a spin bordism group tailored to positive scalar curvature metrics, and the rho invariant turns out to be a genuine $\mathbb{R}$-valued PSC spin bordism invariant. This was exploited by Botvinnik and Gilkey [BG95] in order to show that the moduli space of PSC metrics for certain manifolds has a countably infinite number of path components.

The literature on eta invariants and positive scalar curvature is extremely rich and diverse, see for example the works of Benameur and Mathai [BM13, BM15], Botvinnik and Gilkey [BG95], Higson and Roe [HR10], Keswani [Kes00], Mathai [Mat92], Piazza and Schick [PS07a, PS07b, PS14], and Weinberger [Wei88].

The chapter begins in Section 3.1 with a review of the definition and fundamental properties of the eta invariant. In Section 3.2 we give a brief overview of the APS index theorem and its proof, noting the formulation of the theorem in terms of manifolds with cylindrical ends. Even though Melrose's proof of the APS theorem is more relevant to this thesis, we choose to summarise the original proof instead. The reason for this is that there is already a perfectly clear summary of Melrose's proof in the introduction to his book [Mel93]. It is also interesting to contrast the two approaches. Section 3.3 is on rho invariants, and we give a description of how Botvinnik and Gilkey [BG95] used them to detect path components of the moduli space of PSC metrics.

For the basic definitions and facts concerning elliptic operators, Sobolev spaces, Fredholm theory and the like, see Chapter 1 of [Gil95] or Chapter 3 of [LM89]. For our conventions on Clifford algebras and Dirac operators, see Appendix B.

3.1 The eta invariant

Let A be a Dirac operator on a compact odd-dimensional Riemannian manifold Y without boundary. Then the spectrum $\mathrm{Spec}(A)$ of A is a discrete subset of $\mathbb{R}$, consisting of eigenvalues with finite dimensional eigenspaces [Gil95, Lemma 1.6.3]. The *eta function* of A is the function of the complex variable s given by

$$\eta_A(s) = \sum_{\lambda \in \mathrm{Spec}(A)\setminus\{0\}} \mathrm{sign}(\lambda)|\lambda|^{-s},$$

where the sum over each eigenvalue is repeated according to its multiplicity.

Proposition 3.1.1. *The eta function $\eta_A(s)$ converges absolutely and uniformly for* $\mathrm{Re}(s)$ *sufficiently positive.*

Hence the resulting function is holomorphic on some domain $\mathrm{Re}(s) > C > 0$.

Proof. This is a straightforward consequence of estimates for the growth of eigenvalues of a Dirac operator. If we arrange the non-zero eigenvalues (with multiplicities) in order of increasing absolute value $0 < |\lambda_1| \leqslant |\lambda_2| \leqslant \cdots$, then there are constants $C, \epsilon > 0$ such that

$$|\lambda_n| \geqslant Cn^{\epsilon}$$

for all n; see [Gil95, Lemma 1.6.3]. Applying this estimate, we have

$$\begin{aligned} |\eta_A(s)| &\leqslant \sum_{n=1}^{\infty} |\lambda_n|^{-\mathrm{Re}(s)} \\ &\leqslant \sum_{n=1}^{\infty} \frac{C^{-\mathrm{Re}(s)}}{n^{\epsilon \mathrm{Re}(s)}}. \end{aligned}$$

Choosing $\mathrm{Re}(s) \geqslant c > 1/\epsilon$ we have absolute and uniform convergence of the sum. □

Theorem 3.1.2. *The eta function $\eta_A(s)$ has a meromorphic continuation to the complex plane. Furthermore, the continuation has no pole at the origin, and its well-defined value at zero is written $\eta_A(0)$.*

The proof of this theorem emerges naturally during the proof of the APS index theorem, which we will discuss in the next section.

Definition 3.1.3. The value $\eta_A(0)$ of the meromorphic continuation of $\eta_A(s)$ at zero is called the *eta invariant* of A.

Remark 3.1.4. Disregarding convergence and setting $s = 0$ in the formula for the eta invariant, we get

$$\eta_A(0) = \sum_{\lambda \neq 0} \operatorname{sign}|\lambda|.$$

Hence the eta invariant has an interpretation as a regularised difference between the amount of positive and negative eigenvalues.

A common formula for the eta invariant is:

Theorem 3.1.5 ([Gil89], page 20 and [Mel93], Section 8.13)**.**

$$\eta_A(0) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \operatorname{Tr}(Ae^{-tA^2})\, dt.$$

Here the operator e^{-tA^2} is defined using the spectral theorem for self-adjoint operators—any Dirac operator on a complete Riemannian manifold has self-adjoint closure as an unbounded operator on $L^2(S)$ [LM89, Theorem 5.7].

We will call the quantity

$$\eta(A) := (\eta_A(0) + h)/2$$

the *full eta invariant*, where $h = \dim \ker(A)$. The reason for this is that $\eta(A)$ is the full correction term which appears in the Atiyah-Patodi-Singer index theorem, so it is often more useful to have a name for it.

We will end by looking at some basic properties of the eta invariant, and some explicit examples of eta invariants. First, the eta invariant is a measure of the asymmetry of the spectrum of A about the origin. So one should expect that if the spectrum of A is symmetric about the origin (i.e. $\operatorname{Spec}(A) = \operatorname{Spec}(-A)$, including multiplicities) then $\eta_A(0) = 0$. This is easy to see because the eta function will be zero for s sufficiently large, so by the identity theorem from complex analysis its analytic continuation is zero at the origin.

Example 3.1.6. The simplest example of an eta invariant is of the Dirac operator on the circle, $\frac{1}{i}\frac{\partial}{\partial\theta}$. The spectrum of this operator is the integers $\mathbb{Z}$, and each eigenvalue has multiplicity 1. Therefore the spectrum is symmetric about zero, and hence the eta invariant is zero.

Example 3.1.7. Let (M, g) be an oriented Riemannian manifold and let $A = \pm(*d - d*)$ be the signature operator on even forms. Suppose that M has an orientation reversing isometry. Since the isometry preserves the metric but reverses orientation, the Hodge star is replaced by its negative under the isometry. Since A depends linearly on the Hodge star, this implies that the signature operator is isomorphic to its negative: $A \cong -A$. Hence its spectrum is symmetric about the origin and so its eta invariant vanishes. For example, the eta invariant of the signature operator on any sphere with its standard metric is zero.

It is quite difficult to obtain non-trivial calculations of eta invariants; see for example [Goe12] and [Don78].

3.2 The Atiyah-Patodi-Singer index theorem

Let Z be a compact oriented even-dimensional Riemannian manifold with boundary $\partial Z = Y$. We assume that the metric of Z is isometric to the product metric on $(0, 1] \times Y$ in a neighbourhood of the boundary.[1] Let $E \to Z$ be a $\mathbb{Z}_2$-graded Dirac bundle with Dirac operator $D = D^+ \oplus D^-$ and $S \to Y$ an ungraded Dirac bundle with Dirac operator A. We assume that on $(0, 1] \times Y$ there is an isomorphism $\psi : E \cong p^*S \oplus p^*S$, where $p : (0, 1] \times Y \to Y$ is the projection, such that the following hold:

1. If $p^*S \oplus p^*S$ is given the pullback hermitian metric from S, then ψ preserves hermitian metrics.
2. Under the isomorphism ψ, Clifford multiplication takes the form
$$c_Z(v) = -\begin{pmatrix} 0 & c_Y(v) \\ c_Y(v) & 0 \end{pmatrix} \qquad c(\partial_t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
for all $v \in TY$.
3. The Clifford connections are preserved by ψ, where the Clifford connection on $p^*S \to (0, 1] \times Y$ is given by $\nabla_{\partial_t} s = \partial s/\partial t$ and $\nabla_v s$ is given by the connection on S when $v \in TY$.

[1] It is not essential that the product have length 1 in the t-variable—we could equally well assume an isometry with $(0, \epsilon] \times Y$ for some $\epsilon > 0$. However, we prefer to avoid the extra notation.

Under these assumptions, it is easy to see that the corresponding Dirac operator takes the form

$$D = \begin{pmatrix} 0 & -\partial_t - A \\ \partial_t - A & 0 \end{pmatrix}.$$

We will sometimes abuse notation by writing $D^+ = \partial_t - A$, identifying negative and positive spinors via the isomorphism $c(\partial_t)$.

Definition 3.2.1. If all data Z, Y, D, A are as above, we call the collection $(Z, D\,; Y, A)$ a *boundary Dirac problem.*

Remark 3.2.2. All naturally occurring Dirac operators (spin Dirac, signature operator, etc.) take this form when the metric is a product near the boundary.

The Dirac operator A on the boundary Y has a discrete spectrum of real eigenvalues. Denote by $P_{\geqslant 0} : L^2(Y, S) \to L^2(Y, S)$ the orthogonal projection onto the closure of the span of the eigensections of A with non-negative eigenvalues. We then consider the boundary problem

$$\begin{cases} D^+ s = 0, \\ P_{\geqslant 0}(s|_Y) = 0, \end{cases}$$

for $s \in \Gamma(Z, E^+)$, whose space of solutions is denoted $\ker(D^+, P_{\geqslant 0})$. In the boundary condition, we have made the identification $E^+|_Y \cong S$. The corresponding adjoint problem is

$$\begin{cases} D^- s = 0, \\ P^\perp_{\geqslant 0}(s|_Y) = 0, \end{cases}$$

with space of solutions written $\ker(D^-, P^\perp_{\geqslant 0})$. Here $P^\perp_{\geqslant 0} = \mathrm{Id} - P_{\geqslant 0}$ is the orthogonal projection onto the closure of the span of the eigensections of A with negative eigenvalues. Whenever both of these solution spaces are finite dimensional, we say $(D^+, P_{\geqslant 0})$ is *Fredholm*, and define

$$\mathrm{Ind}(D^+, P_{\geqslant 0}) = \dim\ker(D^+, P_{\geqslant 0}) - \dim\ker(D^-, P^\perp_{\geqslant 0}).$$

Theorem 3.2.3 (Atiyah-Patodi-Singer)**.** *Let $(Z, D\,; Y, A)$ be a boundary Dirac problem on a compact manifold Z with boundary Y, where the Riemannian metric is a product near the boundary. Then $(D^+, P_{\geqslant 0})$ is Fredholm, and*

$$\mathrm{Ind}(D^+, P_{\geqslant 0}) = \int_Z \mathbf{I}(D^+) - \eta(A),$$

where $\mathbf{I}(D^+)$ is the index form of D^+, and $\eta(A) = (\eta_A(0) + h)/2$ is the full eta invariant of A.

We will only sketch the proof, as there are a plethora of details to be worked out in full rigour. The original proof is in [APS75a], but see also Chapter 22 of the book [BBW93] for another exposition. An alternative proof involving Melrose's b-trace is the subject of Melrose's book [Mel93].

Proof outline. The proof begins with the analysis of the operator $D_c^+ = \partial_t - A$ on the half-cylinder $\mathbb{R}_{\leqslant 0} \times Y$.[2] First one constructs a suitable parametrix Q for $(D_c^+, P_{\geqslant 0})$ by the standard Fourier series separation of variables method. Using the parametrix, it can be proven that D_c^+ and $D_c^- = (D_c^+)^*$ with the relevant boundary conditions have closed extensions $\mathcal{D}_c^+$ and $\mathcal{D}_c^-$ as unbounded operators on L^2, and these closures are adjoint to each other. For example, the L^2-closure $\mathcal{D}_c^+$ of $(D_c^+, P_{\geqslant 0})$ has domain the intersection of $H^1(\mathbb{R}_{\leqslant 0} \times Y, S)$ with the kernel of the composition

$$H^1(\mathbb{R}_{\leqslant 0} \times Y, S) \xrightarrow{r} L^2(Y, S) \xrightarrow{P_{\geqslant 0}} L^2(Y, S),$$

where r is restriction to the boundary.[3]

One then obtains unbounded self-adjoint operators $\mathcal{D}_c^- \mathcal{D}_c^+$ and $\mathcal{D}_c^+ \mathcal{D}_c^-$ on $L^2(\mathbb{R}_{\leqslant 0} \times Y, S)$ satisfying the boundary conditions. The corresponding heat operators $e^{-t\mathcal{D}_c^- \mathcal{D}_c^+}$ and $e^{-t\mathcal{D}_c^+ \mathcal{D}_c^-}$ defined via the spectral theorem have heat kernels which can be written down very explicitly in terms of the eigensections of A—see equations (2.16) and (2.17) of [APS75a]. Integrating the supertrace of the heat kernel (that is, the trace of the heat kernel for $e^{-t\mathcal{D}_c^- \mathcal{D}_c^+} - e^{-t\mathcal{D}_c^+ \mathcal{D}_c^-}$) over the diagonal of $(\mathbb{R}_{\leqslant 0} \times Y)^2$ to get a function $K(t)$ for $t \in (0, \infty)$, the eta invariant then emerges from the calculation

$$\int_0^\infty (K(t) + h/2)\, t^{s-1}\, dt = -\frac{\Gamma(s+1/2)}{2s\sqrt{\pi}} \eta_A(2s), \tag{3.1}$$

where Γ is the gamma function from complex analysis, and $\mathrm{Re}(s)$ is sufficiently positive to ensure the integral converges.[4] This formula later yields a meromorphic extension of $\eta_A(s)$ to the complex plane, once an asymptotic expansion for $K(t)$ near 0 is established.

Next, a parametrix for the boundary value problem $(D^+, P_{\geqslant 0})$ on Z is constructed, using the index theorist's favourite trick of patching together parametrices. The parametrix Q_1 for $(D_c^+, P_{\geqslant 0})$ on $\mathbb{R}_{\leqslant 0} \times Y$ is used near the

[2] c for cylinder.

[3] Restriction to the boundary extends to a well-defined bounded linear map $H^s(\mathbb{R}_{\leqslant 0} \times Y, S) \to H^{s-1/2}(Y, S)$ for any $s \in \mathbb{R}$, see Theorem 11.4 of [BBW93].

[4] Convergence on the infinite end of the integral is assured due to exponential decay of $K(t) + h/2$ as $t \to \infty$, but a large positive $\mathrm{Re}(s)$ is needed to ensure convergence at the $t = 0$ end, as $|K(t)| \leqslant Ct^{-n/2}$ for small t.

boundary of Z, and the parametrix Q_2 for D^+ on the double of Z is used on the remainder of the manifold; the total parametrix is given by

$$R := \phi_1 Q_1 \psi_1 + \phi_2 Q_2 \psi_2,$$

where ϕ_1, ψ_1 are 1 near the boundary and 0 away from the boundary, ϕ_2 and ψ_2 are 0 near the boundary and 1 away from the boundary, and $\phi_i \equiv 1$ on $\mathrm{supp}(\psi_i)$. As for the non-boundary case, the existence of a parametrix implies $(D^+, P_{\geqslant 0})$ on Z is Fredholm. Just like for the operators on $\mathbb{R}_{\leqslant 0} \times Y$, the operators $(D^+, P_{\geqslant 0})$ and $(D^-, P_{\geqslant 0}^{\perp})$ on Z have closures $\mathcal{D}^+$ and $\mathcal{D}^-$, and these closures are adjoint to each other. Each has domain a subspace of $H^1(Z, E^{\pm})$ consisting of elements satisfying the relevant boundary condition. We let $\mathcal{D}$ be the direct sum of these operators.

In a similar manner as for the parametrix R, the heat kernels of the operators $(\mathcal{D}_c^+, P_{\geqslant 0})$ on $\mathbb{R}_{\leqslant 0} \times Y$ and D^+ on the double of Z may be glued together to form, not a heat kernel, but an *approximate* heat kernel (Definition C.2.4) for $(D^+, P_{\geqslant 0})$ on the manifold Z. Given an approximate heat kernel, the standard convolution procedure (Appendix C, page 150) allows for the construction of the true heat kernel. But of course, the asymptotic expansion for the approximate heat kernel works equally well for the actual heat kernel. The asymptotic expansion is seen to be a sum of two terms, one coming from the boundary and the other from the interior of the manifold. Analysis of the asymptotics shows that the boundary contribution can be replaced by $K(t)$, and the interior contribution by the usual asymptotic expansion $t^{-n/2} \sum_{k=-0}^{\infty} t^k u_k$ of the heat kernel of D, so that

$$\mathrm{Tr}_s(e^{-t\mathcal{D}^2}) \sim K(t) + t^{-n/2} \sum_{k=0}^{\infty} t^k \int_Z \mathrm{tr}_s(u_k(x,x))\, d\mathrm{vol}_x. \tag{3.2}$$

Now, the operators $\mathcal{D}^-\mathcal{D}^+$ and $\mathcal{D}^+\mathcal{D}^-$ have the same non-zero eigenvalues, since $\mathcal{D}^-\mathcal{D}^+\phi = \lambda\phi$ implies $\mathcal{D}^+\mathcal{D}^-(\mathcal{D}^+\phi) = \lambda\mathcal{D}^+\phi$. Hence the contributions from the non-zero eigenvalues in the super-trace vanish, and all we're left with is the dimensions of the kernels. Since $\ker \mathcal{D}^-\mathcal{D}^+ = \ker \mathcal{D}^+$ and $\ker \mathcal{D}^+\mathcal{D}^- = \ker \mathcal{D}^-$, the analogue of the Mckean-Singer formula [MS67] holds for these operators:

$$\mathrm{Tr}_s(e^{-t\mathcal{D}^2}) = \mathrm{Ind}(\mathcal{D}^+) = \mathrm{Ind}(D^+, P_{\geqslant 0}).$$

Combining this with the expansion (3.2), we get an asymptotic expansion for $K(t)$. Combining this with the integral formula (3.1) for the eta invariant yields the formula

$$\eta_A(2s) = -\frac{2s\sqrt{\pi}}{\Gamma(s+1/2)}\left(\frac{h/2+\operatorname{Ind}(\mathcal{D}^+)}{s}\right.$$
$$\left. - \sum_{k=0}^{N} \frac{1}{k+n/2+s}\int_Z \operatorname{tr}_s(u_k(x,x))\,dx + \theta_N(s)\right),$$

where we have truncated the asymptotic expansion for $K(t)$ at $k = N$ and $\theta_N(s)$ comes from the difference between $K(t)$ and the N-th partial sum of its expansion. The function $\theta_N(s)$ is in fact holomorphic for $\operatorname{Re}(s) \geqslant -(N+1)/2$, as analysis of the integral in equation (3.1) shows. Thus the above formula gives the meromorphic continuation of $\eta(2s)$ to the complex plane, and the value $\eta(0)$ is evaluated as

$$\begin{aligned}\eta_A(0) &= 2\int_Z \operatorname{tr}_s(u_{n/2}(x,x))\,d\mathrm{vol}_x - (h + 2\operatorname{Ind}(\mathcal{D}^+)) \\ &= 2\int_Z \mathbf{I}(D^+) - (h + 2\operatorname{Ind}(D^+, P_{\geqslant 0})).\end{aligned} \qquad \square$$

We end by noting that the APS index theorem can be equally well considered as an index theorem for non-compact manifolds with cylindrical ends. To do this, one attaches the manifold $\mathbb{R}_{\geqslant 0} \times Y$ to Z to obtain a manifold Z_∞ with a cylindrical end, and all structures extend canonically over Z_∞. An *extended L^2 section* of $E^\pm \to Z_\infty$ is a section which for $t \geqslant 0$ takes the form

$$s(t,x) = g(t,x) + f_\infty(x)$$

over the cylindrical end, where $g \in L^2(Z_\infty, E^\pm)$ and $f_\infty \in \ker A$. Atiyah-Patodi-Singer [APS75a, Proposition 3.11] show that $\ker(D^+, P_{\geqslant 0})$ is isomorphic to the space of L^2 solutions to $D^+ s = 0$ on Z_∞. In fact, if $s \in \ker(D^+, P_{\geqslant 0})$ then taking an orthonormal basis $\{\phi_\lambda\}$ for L^2 consisting of eigensections for A, and writing

$$s(t,x) = \sum_\lambda a_\lambda(t)\phi_\lambda(x)$$

all Fourier coefficients $a_\lambda(t)$ of s satisfy $\partial_t a_\lambda - A a_\lambda = \partial_t a_\lambda - \lambda a_\lambda = 0$, so take the form $a_\lambda(0)e^{\lambda t}$. Since s satisfies the boundary condition, only the negative eigenvalues remain, and so exponential decay along the cylinder is ensured. Conversely, any L^2 solution can clearly only have non-zero a_λ for $\lambda < 0$. Similar statements hold for the adjoint boundary problem $(D^-, P_{\geqslant 0}^\perp)$, although '$L^2$ solution' must be replaced with 'extended L^2 solution', because the boundary condition allows a_λ to be non-zero when $\lambda = 0$.

Remark 3.2.4. If A is invertible, then the index of $D^+(Z_\infty)$ is the genuine index of a Freholm operator. This is because, for the adjoint, extended L^2 solutions are precisely L^2 solutions.

3.3 Rho invariants and an application to PSC

It was mentioned in the introduction that the eta invariant is sensitive, and difficult to compute. We will now introduce rho invariants, which satisfy many more invariance properties, and have been applied to the study of positive scalar curvature.

Let Y be an odd-dimensional manifold and $S \to Y$ a Dirac bundle with Dirac operator A. Let π be a discrete group and $P \to Y$ a principal π-bundle. Given unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, we can form the associated bundles $E_i = P \times_{\sigma_i} \mathbb{C}^n$ for $i = 1, 2$. These vector bundles come equipped with flat hermitian connections, since the structure group π of P is discrete and the representations σ_1, σ_2 are unitary. We can therefore twist the Dirac operator A by the bundles E_1 and E_2 to get twisted Dirac operators A_1 and A_2.[5] Since the bundles E_1 and E_2 are flat unitary bundles, A_1 and A_2 are locally isomorphic to the direct sum of N copies of A (Proposition B.4.7).

Definition 3.3.1. The *rho invariant* associated to A, σ_1, σ_2 and P is the difference

$$\rho(\sigma_1, \sigma_2\,; A, P) = \eta(A_1) - \eta(A_2)$$

of the full eta invariants of the twisted operators A_1 and A_2.

Rho invariants have been studied and used by many authors in different contexts—see [BM13, BM15], Botvinnik and Gilkey [BG95], Higson and Roe [HR10], Keswani [Kes00], Mathai [Mat92], Piazza and Schick [PS07a, PS07b, PS14], and Weinberger [Wei88]. In the remainder of this chapter, we will review the application by Botvinnik and Gilkey [BG95] to counting path components of the moduli space of PSC metrics.

The following is a key ingredient in applications of eta invariants to positive scalar curvature.

Proposition 3.3.2. *Let D^+ be the spin Dirac operator on a compact even-dimensional Riemannian spin manifold Z with boundary Y, satisfying the usual hypotheses of product structures near the boundary. If the metric on Z has positive scalar curvature $\kappa > 0$, then*

$$\operatorname{Ind}(D^+, P_{\geqslant 0}) = 0.$$

[5] See Appendix B.4.3 for information on twisted Dirac operators.

The following proof is from [BG95, Lemma 2]

Proof. The proof uses the Lichnerowicz formula (Proposition B.4.6) for the spin Dirac operator. Let $s \in \ker(D^+, P_{\geqslant 0})$. Then

$$\begin{aligned} 0 &= \langle D^- D^+ s, s\rangle \\ &= \langle \nabla^* \nabla s, s\rangle + \frac{1}{4}\int_Z \kappa(s,s)\, d\mathrm{vol}_Z \\ &= \|\nabla s\|^2 - \int_Y (\nabla_{\partial_t} s, s)\, d\mathrm{vol}_Y + \frac{1}{4}\int_Z \kappa(s,s)\, d\mathrm{vol}_Z, \end{aligned}$$

where ∂_t is the outward unit normal at the boundary. The first term is non-negative, and the third is strictly positive whenever $s \neq 0$. We must analyse the second term—the boundary contribution from the integration by parts. Since the structures near the boundary are product, the term $\nabla_{\partial_t} s$ is merely $\partial_t s$ under the identification $S^+ \oplus S^- \cong p^* S_Y \oplus p^* S_Y$. Since $D^+ s = 0$, and $D^+ \cong \partial_t - A$ near the boundary, we have $\nabla_{\partial_t} s = As$. The second term is therefore

$$-\int_Y (As, s)\, d\mathrm{vol}_Y.$$

Since $P_{\geqslant 0} s = 0$ we have $\langle As, s\rangle_{L^2(Y)} \leqslant 0$, and hence the second term is non-negative. Thus all terms are non-negative and the third is strictly positive if $s \neq 0$, hence $s = 0$.

A similar argument shows that the kernel of $(D^-, P^{\perp}_{\geqslant 0})$ vanishes, and hence the result on the index follows. □

The same holds when D^+ is twisted by a flat unitary bundle since, by Proposition B.4.7, the Lichnerowicz formula still holds in this instance.

We are now in a position to state some results on the number of path components in the moduli space of PSC metrics, which we recall we think of as the number of 'distinct' PSC metrics on a manifold. The results in this section are due to Botvinnik and Gilkey [BG95].

If two (equivalence classes of) PSC metrics are connected by a path in the moduli space of PSC metrics $\mathcal{M}^+(M)$, then we can lift this path to the space $\mathcal{R}^+(M)$ of metrics of PSC. Deforming this to a smooth path and then reparametrising so that the path is constant near the start and endpoints, we obtain a manifold $[0,1] \times M$ with a metric $c\, dt^2 + g_t$ of PSC that is a product near the boundary. If M is spin and we choose a definite spin structure σ on M, then $[0,1] \times M$ comes equipped with a canonical product spin structure. Applying the APS index theorem to this manifold and using Proposition 3.3.2, we get the following:

Proposition 3.3.3. *Let M be a compact odd-dimensional spin manifold with fixed spin structure, and let g_0, g_1 be metrics of positive scalar curvature on M. If g_0 and g_1 lie in the same path component of $\mathcal{M}^+(M)$, then for any principal π-bundle P and unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the rho invariants are equal:*

$$\rho(\sigma_1, \sigma_2; A_{g_0}, P) = \rho(\sigma_1, \sigma_2; A_{g_1}, P),$$

where A_{g_0} and A_{g_1} are the spin Dirac operators on M corresponding to the two metrics g_0 and g_1.

Proof. Let $[0,1] \times M$ be the Riemannian spin manifold described in the above paragraph, and let D^+ be its spin Dirac operator. Applying the APS index theorem to the twisted operators D_1^+ and D_2^+, we get

$$0 = \int_{[0,1]\times M} \mathbf{I}(D_i^+) - \eta(A_i^+) \tag{3.3}$$

for $i = 1, 2$, where we have applied Proposition 3.3.2. Since we are twisting by flat bundles, the twisted operators have the same index forms:

$$\mathbf{I}(D_1^+) = \mathbf{I}(D_2^+) = N\,\mathbf{I}(D^+),$$

by Proposition B.5.4. We can therefore subtract the two equations (3.3) for $i = 1, 2$, and the integral terms vanish, leaving the equality of the rho invariants.[6] □

Corollary 3.3.4. *If M is a compact odd-dimensional spin manifold admitting a metric of PSC, and if g_0 and g_1 are metrics of PSC which have distinct rho invariants (with respect to some principal π-bundle and unitary representations), then g_0 and g_1 lie in distinct path components of $\mathcal{M}^+(M)$.*

Hence the rho invariants can detect distinct path components of the moduli space of PSC metrics! This is assuming that one can: (a) construct different metrics of PSC on M and (b) calculate the corresponding rho invariants and show that they're different. This was indeed done by Botvinnik and Gilkey [BG95]. The precise statement of their result requires the definition of a certain representation theoretic quantity $r_m(\pi)$, which gives a lower bound for the rank of the PSC spin bordism groups. When π is non-trivial and finite, and $m = 4k - 1$ with $k > 1$, $r_m(\pi) > 0$.

[6] This also uses the obvious fact that the eta invariant on a disjoint union $M_1 \amalg M_2$ is just the sum of the eta invariants on M_1 and M_2.

Theorem 3.3.5 (Theorem 0.2. [BG95]). *Let M be a compact connected spin manifold of dimension $4k-1$ with $k>1$. Suppose that $\pi_1(M)$ is non-trivial and finite, and that M admits a metric g of positive scalar curvature. Then $\pi_0(\mathcal{M}^+(M))$ is countably infinite.*

The main elements of the proof are: (a) explicit calculations of rho invariants for certain manifolds from [Don78] and [Gil89], and (b) the PSC surgery theory of Miyazaki [Miy84, Theorem 1.1.] and Rosenberg [Ros86], which allows one to 'push' a PSC metric across a cobordism satisfying some hypotheses.

3.4 Notes

3.4.1 Geometric K-homology and spin bordism

The invariance properties of rho invariants are more naturally expressed using geometry K-homology groups [HR10], and spin bordism groups [BG95]. For example, if $B\pi$ is the classifying space of a discrete group π, then the rho invariant descends to a well-defined map

$$\rho(\sigma_1, \sigma_2) : K_1(B\pi) \to \mathbb{R}/\mathbb{Z},$$

see [HR10]. This captures independence of the rho invariant of the signature operator on the choice of metric, among other invariance properties such as mod $\mathbb{Z}$ bordism invariance.

There are also certain PSC spin bordism groups, described in [BG95], on which the rho invariants give a well-defined real-valued invariant. This is due to Proposition 3.3.2, and the proof is essentially the same as for Proposition 3.3.3.

These relations between geometric K-homology, spin bordism theories and rho invariants will be discussed further in the final chapter, where we will carry out the analogous theory for end-periodic manifolds.

3.4.2 Kreck-Stolz s-invariant

Kreck and Stolz [KS93] obtain similar disconnectedness results to Botvinnik and Gilkey on the moduli space of PSC metrics. Their work relies on the construction of a certain *s-invariant*, which is defined for manifolds whose pontryagin numbers all vanish (e.g. an n-sphere). They obtain the following:

Theorem 3.4.1 ([KS93]). *Let M be a connected spin manifold of dimension $4k-1$ with $k>1$. Assume that $H^1(M,\mathbb{Z}_2)=0$ and that all the pontryagin numbers of M vanish. Then $\pi_0(\mathcal{M}^+(M))$ is at least countably infinite.*

3.4.3 Manifolds with holonomy in G_2

Eta invariants have also recently found applications in G_2 geometry. In the unpublished paper [CGN15], the authors study the moduli space of Riemannian metrics having holonomy which lies in the exceptional group G_2 (G_2 metrics, for short). There is an analytic invariant (Definition 1.4 of [CGN15]), defined as a difference of eta invariants, which can be used to detect distinct path components of the moduli space of G_2 metrics. They then use this invariant to construct an example of a manifold whose space of G_2 metrics is disconnected. This seems highly analogous to the work of Botvinnik and Gilkey, although, unlike the Botvinnik-Gilkey case, an infinitude of path components cannot be determined due to the lack of a suitable surgery theory for G_2 metrics (the author thanks D. Crowley for explaining this point to him).

Chapter 4

Analysis on end-periodic manifolds and Taubes' theorem

This chapter reviews the Fredholm theory of end-periodic operators on end-periodic manifolds, whose study was initiated by Taubes [Tau87]. It is well known that on a non-compact manifold, elliptic operators are no longer Fredholm in general. The culprit is Rellich compactness, which does not hold for non-compact manifolds—the inclusions $H^s(M) \hookrightarrow H^t(M)$ for $s > t$ are not compact if M is not a compact manifold. To remedy this, one can introduce *weighted* Sobolev spaces. The weighted Sobolev space $H^k_\delta(M)$, where $\delta \in \mathbb{R}$ is a weight, consists of sections in $H^k_{\text{loc}}(M)$ which satisfy a certain growth condition at infinity determined by δ. The embeddings $H^s_\delta(M) \hookrightarrow H^t_\delta(M)$ are still no longer compact, but by varying the weight δ it is possible to find spaces on which the elliptic operator is Fredholm. The index depends on the weight however, so due caution must be paid when extracting information from these indices.

Instead of beginning with the end-periodic case, we work in the less general case of cylindrical ends carried out in [AN63] and [LM85]. There are a few simplifications which can be made by restricting to these manifolds, and it serves to motivate the end-periodic case quite well. On a cylindrical manifold $\mathbb{R} \times M$, one has a Fourier transform in the $\mathbb{R}$-variable which facilitates the study of differential equations. If a manifold Z_∞ has a cylindrical end $\mathbb{R}_{\geqslant 0} \times M$, then one can use an excision principle to transfer information from the cylinder $\mathbb{R} \times M$ to Z_∞. The resulting analysis yields Fredholm theory for operators of the form $\partial_t - A$ over the end, where A is a Dirac operator on M. Note that in this instance, the Fourier transform converts ∂_t into multiplication by the dual variable $i\lambda$, which suggests that the invertibility problem for $\partial_t - A$ should in some way correspond to the eigenvalue problem for A.

In the end-periodic setting, there is no $\mathbb{R}$-variable to integrate over in the Fourier transform, which complicates things. However, if $\widetilde{X} \to X$ is the $\mathbb{Z}$-cover from which the periodic end is built, we do have a fundamental covering translation $T : \widetilde{X} \to \widetilde{X}$ which gives us a discrete identification of each segment of the end. This allows one to define a discrete 'Fourier-Laplace transform' on the cover. Again, the results obtained on $\widetilde{X}$ can be transferred to an end-periodic manifold with end $\widetilde{X}_{\geqslant 0}$ via an excision argument.

The arguments for the cylindrical setting are due to Agmon and Nirenberg [AN63], and Lockhart and McOwen [LM85]. These papers deal with considerably more general situations than we wish to consider. The end-periodic case treated by Taubes [Tau87] is also more general than what we need—he deals with elliptic complexes whereas we consider only Dirac operators.

Section 4.1 deals with manifolds having only cylindrical ends. Conditions are established under which Dirac operators that are translation invariant over the end are Fredholm. The translation invariant Dirac operator D^+ over $\mathbb{R} \times M$ is isomorphic to $\partial_t - A$, where A is a Dirac operator on M, and we see that the Fredholm properties of D^+ are closely tied to the spectrum of A. Section 4.2 deals with the end-periodic case, beginning with the introduction of the Fourier-Laplace transform and its basic properties in Section 4.2.1. Section 4.2.2 is the work of Taubes on the Fredholm theory of end-periodic operators. Fredholmness of an end-periodic Dirac operator is tied to a 'spectral set' of a family $\{D^+_\lambda(X)\}$ formed via conjugation with the Fourier-Laplace transform. Taubes' result is that if the symbol of $D^+(X)$ satisfies a certain requirement, then this spectral set turns out to be discrete. Finally in Section 4.2.3 we review the result of [MRS11] which details how the index depends on the choice of weight δ. The use of weighted Sobolev spaces is a common theme throughout the chapter.

4.1 Manifolds with cylindrical ends

In this section we consider the cylindrical case. Let Z_∞ be an oriented even-dimensional manifold with cylindrical ends; Z_∞ is the union of a compact piece Z with boundary $\partial Z = M$ and the cylinder $\mathbb{R}_{\geqslant 0} \times M$. We take a Riemannian metric on Z_∞ that is a product over the cylinder: $dt^2 + g_M$. Let $p : \mathbb{R}_{\geqslant 0} \times M \to M$ be the projection. We assume there is a $\mathbb{Z}_2$-graded Dirac bundle $S^+ \oplus S^- \to Z_\infty$ which takes the form $S^+ \oplus S^- \cong p^*S \oplus p^*S$ over the cylinder, where $S \to M$ is an ungraded Dirac bundle. We assume the metric and connection on $S^+ \oplus S^-$ are pulled back from S over the end, and that

Clifford multiplication takes the form

$$c(\partial_t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad c(v) = -\begin{pmatrix} 0 & c_M(v) \\ c_M(v) & 0 \end{pmatrix}$$

for all $v \in TM$. Under these assumptions, the Dirac operator on Z_∞ is isomorphic to

$$D = \begin{pmatrix} 0 & -\partial_t - A \\ \partial_t - A & 0 \end{pmatrix}$$

over the cylindrical end, where A is the Dirac operator on $S \to M$. In particular $D^+ = c(\partial_t)(\partial_t - A)$ over the cylinder. The $c(\partial_t)$ is merely there to swap the positive and negative spinors; to perform analysis we may identify D^+ with $\partial_t - A$ over the cylinder.

4.1.1 Weighted Sobolev spaces

Instead of the usual Sobolev spaces H^k, we will now consider modified Sobolev spaces which are particularly useful for analysis on cylindrical manifolds. We assume all data on the cylindrical-end manifold Z_∞ as above. For k a non-negative integer, the *k-th Sobolev norm* is defined as

$$\|s\|_{H^k(Z_\infty, S^\pm)} = \left(\sum_{i=0}^{k} \int_{Z_\infty} |\nabla^{(i)} s|^2 \, d\mathrm{vol} \right)^{1/2},$$

where $s \in \Gamma_c(Z_\infty, S^\pm)$. We can modify this by adding a *weight* at infinity; let u be a smooth function on Z_∞ with $u(t,x) = t$ on $\mathbb{R}_{\geqslant 1} \times M = \{(t,x) : t \geqslant 1, x \in M\}$. For $\delta \in \mathbb{R}$ we define the *k-th Sobolev norm with weight δ* to be

$$\|s\|_{H^k_\delta(Z_\infty, S^\pm)} = \|e^{\delta u} s\|_{H^k(Z_\infty, S^\pm)}. \tag{4.1}$$

In shorthand, $H^k_\delta = e^{-\delta u} H^k$. This definition is clearly (up to equivalence of norms) independent of the choice of such function u. The norm is also equivalent to

$$\left(\sum_{i=0}^{k} \int_{Z_\infty} |e^{\delta u} \nabla^{(i)} s|^2 \, d\mathrm{vol} \right)^{1/2}.$$

As we shall see, these weights arise naturally when one allows complex parameters in the Fourier transform. We define the *k-th weighted Sobolev space with weight δ*, $H^k_\delta(Z_\infty, S^\pm)$, to be the completion of $\Gamma_c(Z_\infty, S^\pm)$ in the norm (4.1). We will often write H^k_δ or $\|\cdot\|_{H^k_\delta}$ to avoid overly cumbersome notation, provided there is no ambiguity.

Proposition 4.1.1. *For any $\delta \in \mathbb{R}$, the Dirac operator D^+ extends to a bounded linear operator $H^k_\delta(Z_\infty, S^+) \to H^{k-1}_\delta(Z_\infty, S^-)$.*

Proof. There are obvious isomorphisms $H^k(Z_\infty, S^\pm) \xrightarrow{\sim} H^k_\delta(Z_\infty, S^\pm)$ given by $s \mapsto e^{-\delta u}s$. The following diagram commutes:

$$\begin{array}{ccc} H^k(Z_\infty, S^+) & \xrightarrow{D^+ - \delta c(\nabla u)} & H^{k-1}(Z_\infty, S^-) \\ {\scriptstyle e^{\delta u}}\uparrow & & \downarrow{\scriptstyle e^{-\delta u}} \\ H^k_\delta(Z_\infty, S^+) & \xrightarrow[D^+]{} & H^{k-1}_\delta(Z_\infty, S^-). \end{array}$$

Since the operator on top is bounded (it is translation invariant over the cylindrical end) and the vertical arrows are isomorphisms, the result follows. □

Note that as we vary the weight δ, the space H^k_δ on which D^+ is defined changes. The above commutative diagram permits us to make sense of the idea that these operators are varying continuously with δ. We can then draw conclusions such as:

Corollary 4.1.2. *If D^+ with domain H^k_δ is Fredholm (resp. invertible) for some $\delta \in \mathbb{R}$, then D^+ with domain H^k_μ is Fredholm (resp. invertible) for all μ in some open neighbourhood of δ.*

We can also prove Garding's inequality for the weighted spaces:

Theorem 4.1.3 (Weighted Garding inequality)**.** *For any k and δ there is a constant $C > 0$ such that*

$$\|s\|_{H^k_\delta} \leqslant C\left(\|D^+s\|_{H^{k-1}_\delta} + \|s\|_{H^{k-1}_\delta}\right).$$

Proof. The operator $D^+ + \delta c(\nabla u)$ is uniformly elliptic,[1] so satisfies Garding's inequality on the unweighted Sobolev spaces [Shu92, Appendix 1, Lemma 1.4]. We have

$$\begin{aligned} \|s\|_{H^k_\delta} &= \|e^{\delta u}s\|_{H^k} \\ &\leqslant C\left(\|(D^+ - \delta c(\nabla u))(e^{\delta u}s)\|_{H^{k-1}} + \|e^{\delta u}s\|_{H^{k-1}}\right) \\ &= C\left(\|e^{\delta u}D^+s\|_{H^k} + \|e^{\delta u}s\|_{H^{k-1}}\right) \\ &= C\left(\|D^+s\|_{H^{k-1}_\delta} + \|s\|_{H^{k-1}_\delta}\right). \end{aligned}$$ □

[1] A uniform differential operator P on a manifold M of bounded geometry is *uniformly elliptic* if its principal symbol admits a uniform inverse over the unit sphere bundle in T^*M; see [Shu92, Appendix 1] for details. Since the operator $D^+ + \delta c(\nabla u)$ is translation invariant outside of a compact subset, it is clearly uniformly elliptic.

Corollary 4.1.4. *For any k and ℓ,*

$$\ker(D^+ : H^k_\delta \to H^{k-1}_\delta) = \ker(D^+ : H^\ell_\delta \to H^{\ell-1}_\delta).$$

By equality, we mean once we have canonically included one weighted Sobolev space into the other, for example if $k \geqslant \ell$ then $H^k_\delta \hookrightarrow H^\ell_\delta$.

Proof. Suppose without loss of generality that $k > \ell$. If $s \in \ker(D^+ : H^\ell_\delta \to H^{\ell-1}_\delta)$ then

$$\|s\|_{H^k_\delta} \leqslant C(\|s\|_{H^{k-1}_\delta} + \|D^+ s\|_{H^{k-1}_\delta}) = C\|s\|_{H^{k-1}_\delta}.$$

Repeated application of this process gives $\|s\|_{H^k_\delta} \leqslant C'\|s\|_{H^\ell_\delta} < \infty$. □

Remark 4.1.5. Given a manifold with cylindrical end $\mathbb{R}_{\geqslant 0} \times M$, we will have need of considering the total cylinder $\mathbb{R} \times M$. When working on this manifold, the weighted Sobolev spaces are defined using the weight function $e^{\delta t}$, and **not** by considering $\mathbb{R} \times M$ as a manifold with two cylindrical ends $\mathbb{R}_{\leqslant 0} \times M$ and $\mathbb{R}_{\geqslant 0} \times M$ (in this case the weight function would be $e^{\delta u}$ where $u = -t$ for $t \leqslant -1$ and $u = t$ for $t \geqslant 1$). It is clear that Proposition 4.1.1 and Theorem 4.1.3 continue to hold for the cylindrical manifold $\mathbb{R} \times M$.

We end with some brief remarks on duality and adjoints. For k a negative integer, we define

$$H^k_\delta := (H^{-k}_{-\delta})^*.$$

Note the sign reversal in the weight. The adjoint of $D^+ : H^k_\delta \to H^{k-1}_\delta$ is $(D^+)^* : H^{-k+1}_{-\delta} \to H^{-k}_{-\delta}$, and through the commutative diagram below Proposition 4.1.1 it is easy to see that this is the bounded extension of D^- to these spaces. All the above results, including the weighted Garding inequality, hold for H^k_δ with $k \leqslant 0$.

4.1.2 Fredholm theory

In order to deduce information on the Dirac operator on Z_∞ it will be necessary to consider the Dirac operator on the whole cylinder $\mathbb{R} \times M$.

We will need the following.

Lemma 4.1.6. *Suppose $f \in L^2(\mathbb{R})\backslash\{0\}$. For $n \in \mathbb{Z}$ define*

$$f_n(t) = f(t-n).$$

Then $\{f_n\}_{n\in\mathbb{Z}}$ has infinite dimensional span in $L^2(\mathbb{R})$.

Proof. It suffices to consider the case $\|f\| = 1$. Suppose the span is finite dimensional, and choose a basis $e_1, \ldots, e_n$ for the span. Given $\epsilon > 0$, there is some compact subset $K \subset \mathbb{R}$ outside of which all the e_i have L^2-norm less than ϵ. Choose m so large that f_m has norm greater than $1 - \epsilon$ outside of K, and write f_m uniquely as $f_m = \sum_{i=1}^n \alpha^i e_i$. Then

$$1 - \epsilon \leqslant \|f_m\|_{L^2(\mathbb{R}\setminus K)} \leqslant \sum |\alpha^i| \, \|e_i\|_{L^2(\mathbb{R}\setminus K)} \leqslant \epsilon \sum |\alpha^i|.$$

Hence $\sum |\alpha^i| \geqslant 1/\epsilon - 1$. Sending $\epsilon \to 0$ we see that there are f_m whose $\sum |\alpha^i|$ can be arbitrarily large. But the span of the f_i is finite dimensional, so $\sum |\alpha^i|$ is an equivalent norm for the span. Since $\|f_m\|_{L^2(\mathbb{R})} = 1$ for all m, we have a sequence in a finite dimensional vector space that is bounded with respect to one norm, but unbounded with respect to another; contradiction. □

Proposition 4.1.7. *The following are equivalent:*

1. $D^+ : H^k_\delta(Z_\infty, S^+) \to H^{k-1}_\delta(Z_\infty, S^-)$ *is Fredholm.*
2. $D^+ : H^k_\delta(\mathbb{R} \times M, S^+) \to H^{k-1}_\delta(\mathbb{R} \times M, S^-)$ *is Fredholm.*
3. $D^+ : H^k_\delta(\mathbb{R} \times M, S^+) \to H^{k-1}_\delta(\mathbb{R} \times M, S^-)$ *is invertible.*

Proof. For the lower two operators, we can work with the isomorphic operator $D^+ - \delta c(\partial_t) : H^k(\mathbb{R} \times M, S^+) \to H^{k-1}(\mathbb{R} \times M, S^-)$.

- 3. $\Rightarrow$ 2.) Any invertible operator is Fredholm.
- 2. $\Rightarrow$ 3.) Let $D^+ - \delta c(\partial_t) : H^k(\mathbb{R} \times M, S^+) \to H^{k-1}(\mathbb{R} \times M, S^-)$ be Fredholm, and suppose there is a nontrivial element s in its kernel. Define $s_m(t, x) = s(t - m, x)$ for each $m \in \mathbb{Z}$. Since $D^+ - \delta c(\partial_t)$ is translation invariant, each s_m also lies in the kernel of $D^+ - \delta c(\partial_t)$. Lemma 4.1.6 clearly generalises to the case at hand, implying span$\{s_m : m \in \mathbb{Z}\}$ is an infinite dimensional subspace of the kernel; contradiction. A similar argument applies to the translation invariant adjoint $D^- + \delta c(\partial_t)$, showing it too has trivial kernel. Since $D^+ - \delta c(\partial_t) : H^k \to H^{k-1}$ is Fredholm with trivial kernel and cokernel, it is invertible.
- 1. $\Rightarrow$ 2.) Suppose $D^+ : H^k_\delta(\mathbb{R} \times M, S^+) \to H^{k-1}_\delta(\mathbb{R} \times M, S^-)$ has nontrivial kernel. Let s be a non-zero element of the kernel. Define $s_m(t, x) = e^{-\delta m} s(t - m, x)$ for $m \in \mathbb{Z}$. Let ϕ be a smooth function on Z_∞ equal to 1 on $\mathbb{R}_{\geqslant 2} \times M$ and equal to 0 on $Z \cup ([0, 1] \times M)$. Then ϕs_m can be considered as a section of S^+ over Z_∞. Over $\mathbb{R}_{\geqslant 0} \times M$ we have
$$D^+(\phi s_m) = c(d\phi)s_m + \phi D^+ s_m = c(d\phi)s_m$$

and on Z we have $D^+(\phi s_m) = 0$, hence $D^+(\phi s_m) = c(d\phi)s_m$ on all of Z_∞. Note that $\phi s_m \in H^k_\delta(Z_\infty, S^+)$ and $c(d\phi)s_m \to 0$ in $H^k_\delta(Z_\infty, S^+)$ as $m \to \infty$.[2]

Now, suppose $D^+ : H^k_\delta(Z_\infty, S^+) \to H^{k-1}_\delta(Z_\infty, S^-)$ is Fredholm and let Q be an inverse modulo compact operators: $QD^+ = \mathrm{Id} + K$, where K is compact. Then

$$\phi s_m = Q(c(d\phi)s_m) - K(\phi s_m).$$

Since $c(d\phi)s_m \to 0$ as $m \to \infty$, and since K is compact and ϕs_m is bounded in $H^k_\delta(Z_\infty, S^+)$,[3] the right hand side has a convergent subsequence in $H^k_\delta(Z_\infty, S^+)$. However, the left hand side clearly has no convergent subsequence (cf. Lemma 4.1.6); contradiction.

We can apply a similar argument to the adjoint $D^- : H^{-k+1}_{-\delta}(\mathbb{R} \times M, S^-) \to H^{-k}_{-\delta}(\mathbb{R} \times M, S^+)$, to show that it too has trivial kernel. Hence D^+ on the cylinder is Fredholm.

- 2. ⇒ 1.) The following argument is from pages 420–421 of [LM85]. Let:

  i) P be an inverse modulo compact operators for D^+ on the double of $Z \cup ([0,3] \times M)$,
  ii) Q be an inverse modulo compact operators for $D^+ : H^k_\delta(\mathbb{R} \times M, S^+) \to H^{k-1}_\delta(\mathbb{R} \times M, S^-)$,
  iii) $\varphi_1 : Z_\infty \to \mathbb{R}$ be smooth with $\varphi_1 = 1$ on $Z \cup ([0,1] \times M)$ and $\varphi_1 = 0$ on $[2,\infty) \times M$,
  iv) $\varphi_2 = 1 - \varphi_1$,
  v) $\psi_1 : Z_\infty \to \mathbb{R}$ be smooth with support in $Z \cup ([0,3] \times M)$ and $\psi_1 = 1$ on $\mathrm{supp}(\varphi_1)$,
  vi) $\psi_2 : Z_\infty \to \mathbb{R}$ with support in $[0,\infty) \times M$ and $\psi_2 = 1$ on $\mathrm{supp}(\varphi_2)$.

  Using this, define

  $$R = \psi_1 P \varphi_1 + \psi_2 Q \varphi_2,$$

  which is a bounded operator $H^{k-1}_\delta(Z_\infty, S^-) \to H^k_\delta(Z_\infty, S^+)$. Note φ_2 has support on the cylinder, and we are identifying $\varphi_2 s$ with a section of S^- over the cylinder so that we can apply Q. Multiplying by ψ_2 afterwards, we then have a section $\psi_2 Q \varphi_2 s$ of S^+ over Z_∞. A short computation gives that R is a two-sided inverse for $D^+ : H^k_\delta(Z_\infty, S^+) \to H^{k-1}_\delta(Z_\infty, S^-)$ modulo compact operators. □

[2] The $e^{-\delta m}$ factor in s_m is needed to ensure this convergence to 0.

[3] The norms $\|\phi s_m\|_{H^k_\delta} \to \|s\|_{H^k_\delta}$ as $m \to \infty$, again due to the factor of $e^{-\delta m}$ in s_m.

Consider the operator $\partial_t - A$ over the infinite cylinder $\mathbb{R} \times M$. Recall the *Fourier transform* in the t-variable is defined on compactly supported smooth sections of S over $\mathbb{R} \times M$ as

$$\widehat{s}(\mu, x) = \int_{-\infty}^{\infty} e^{-i\mu t} s(t, x)\, dt.$$

We also have the Fourier inversion formula:

$$s(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mu t}\, \widehat{s}(\mu, x)\, d\mu,$$

which holds for compactly supported smooth sections. The Fourier transform and Fourier inverse extend to isometries of $L^2(\mathbb{R} \times M, S)$. In particular, for $x \in M$ fixed we have

$$\|\widehat{s}(\mu, x)\|_{L^2(\mathbb{R}_\mu, S_x)} = \|s(t, x)\|_{L^2(\mathbb{R}_t, S_x)}$$

by the fact that the standard Fourier transform is an isometry. Integrating the square of both sides over M gives that the Fourier transform is an isometry of $L^2(\mathbb{R} \times M, S)$. The following is the standard statement that the Fourier transform converts differentiation to multiplication.

Proposition 4.1.8. *For $s \in \Gamma_c(\mathbb{R} \times M, S)$,*

$$(i\mu + A)\widehat{s}\,(\mu, x) = \mathcal{F}\left[(\partial_t + A)s\right](\mu, x)$$

Note that if we allow the variable in the Fourier transform to be *complex*, say $\lambda = \mu + i\delta$ with $\mu, \delta \in \mathbb{R}$, then we get

$$\widehat{s}(\lambda, x) = \int_{-\infty}^{\infty} e^{-i\mu t} e^{\delta t} s(t, x)\, dt.$$

The $e^{\delta t}$ term in the integral is the familiar weight factor for the weighted Sobolev spaces.

Definition 4.1.9. For $\delta \in \mathbb{R}$, define the *weighted Fourier transform* $\mathcal{F}_\delta$ via the commutative diagram

$$\begin{array}{ccc} L^2(\mathbb{R}_t \times M, S) & \xrightarrow{\ \mathcal{F}\ } & L^2(\mathbb{R}_\mu \times M, S) \\ {\scriptstyle e^{\delta t}} \uparrow & \nearrow_{\mathcal{F}_\delta} & \\ L^2_\delta(\mathbb{R}_t \times M, S) & & \end{array}$$

Explicitly,

$$\mathcal{F}_\delta(s)(\mu, x) = \int_{-\infty}^{\infty} e^{-i(\mu+i\delta)t} s(t,x)\, dt.$$

Since the top and vertical arrows in the diagram are isometries, so is $\mathcal{F}_\delta$. The weighted Fourier inversion formula is

$$\mathcal{F}_\delta^{-1}(s)(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\mu+i\delta)t}\, \mathcal{F}_\delta(s)(\mu, x)\, d\mu.$$

As usual, these formulas apply only on a dense subspace (e.g. compactly supported smooth sections) but extend to isometries. Note that if we fix an $x \in M$ then the maps $\mathcal{F}_\delta : L^2_\delta(\mathbb{R}, S_x) \to L^2(\mathbb{R}, S_x)$ are also defined and are isometries.

For $\lambda \in \mathbb{C}$, consider the operator $i\lambda - A$ defined on M. This is invertible if and only if $i\lambda \notin \mathrm{Spec}(A)$. Now A has a discrete spectrum of real eigenvalues, so the set of $\lambda \in \mathbb{C}$ for which $i\lambda - A$ is not invertible is a discrete subset of the imaginary axis in $\mathbb{C}$. Writing $\lambda = \mu + i\delta$, we have that $i\lambda \in \mathrm{Spec}(A)$ if and only if $\mu = 0$ and $-\delta \in \mathrm{Spec}(A)$.

Proposition 4.1.10. *The operator $\partial_t - A : H^k_\delta(\mathbb{R} \times M) \to H^{k-1}_\delta(\mathbb{R} \times M)$ is invertible if and only if the operators $i\lambda - A : H^k(M, S) \to H^{k-1}(M, S)$ are invertible for all λ with $\mathrm{Im}(\lambda) = \delta$. That is, if and only if $-\delta \notin \mathrm{Spec}(A)$.*

Proof. ($\Leftarrow$) The first part of the proof is from [LM85, pages 419–420]. Suppose $i\lambda - A$ is invertible for all $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) = \delta$. Define

$$T_\delta(s)(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\mu+i\delta)t}\, \left[i(\mu + i\delta) - A \right]^{-1} \mathcal{F}_\delta(s)(\mu, x)\, d\mu.$$

Rewritten in terms of the standard Fourier transform and the complex variable $\lambda = \mu + i\delta$, this is

$$(T_\delta s)(t,x) = \frac{1}{2\pi} \int_{\mathrm{Im}(\lambda)=\delta} e^{i\lambda t} (i\lambda - A)^{-1} \widehat{s}_\lambda(x)\, d\lambda,$$

where $\widehat{s}_\lambda(x) = \mathcal{F}_\delta(s)(\mu, x)$. It is straightforward to check that T_δ is an inverse for $\partial_t - A$ on compactly supported smooths sections using Fourier inversion. To check that $T_\delta : H^{k-1}_\delta(\mathbb{R} \times M, S) \to H^k_\delta(\mathbb{R} \times M, S)$ is *bounded*, we first use the weighted Garding inequality to reduce to showing that $T_\delta : H^0_\delta(\mathbb{R} \times M) \to$

$H^0_\delta(\mathbb{R}\times M)$ is bounded. If $\sim$ denotes equivalence of norms then

$$\begin{aligned}\|s\|_{H^k_\delta} &\sim \|(\partial_t - A)s\|_{H^{k-1}_\delta} + \|s\|_{H^{k-1}_\delta}\\ &\sim \|(\partial_t - A)s\|_{H^{k-1}_\delta} + \|(\partial_t - A)s\|_{H^{k-2}_\delta} + \|s\|_{H^{k-2}_\delta}\\ &\vdots\\ &\sim \sum_{i=0}^{k-1} \|(\partial_t - A)s\|_{H^i_\delta} + \|s\|_{H^0_\delta}.\end{aligned}$$

To show T_δ is bounded, we want $\|T_\delta s\|_{H^k_\delta} \leqslant C\|s\|_{H^{k-1}_\delta}$ for all s smooth with compact support. Using the above equivalent norm, we want

$$\sum_{i=0}^{k-1} \|s\|_{H^i_\delta} + \|T_\delta s\|_{H^0_\delta} \leqslant C\|s\|_{H^{k-1}_\delta}.$$

It is therefore enough to show $\|T_\delta s\|_{H^0_\delta} \leqslant C\|s\|_{H^0_\delta}$ for s smooth with compact support. Now,

$$\begin{aligned}\|T_\delta s\|^2_{H^0_\delta(\mathbb{R}\times M)} &= \int_M \int_{-\infty}^{\infty} \left| e^{\delta t} \int_{\operatorname{Im}(\lambda)=\delta} e^{i\lambda t}(i\lambda - A)^{-1}\widehat{s}_\lambda(x)\, d\lambda\right|^2 dt\, dx\\ &= \int_M \int_{-\infty}^{\infty} \left| \int_{\operatorname{Im}(\lambda)=\delta} e^{i\mu t}(i\lambda - A)^{-1}\widehat{s}_\lambda(x)\, d\mu\right|^2 dt\, dx,\end{aligned}$$

where $\mu = \operatorname{Re}(\lambda)$. Since Fourier inversion is an isometry of L^2 spaces, this is equal to

$$\int_M \int_{\operatorname{Im}(\lambda)=\delta} \left|(i\lambda - A)^{-1}\widehat{s}_\lambda(x)\right|^2 d\lambda\, dx,$$

see also [Kon67, page 216]. Since the integrand is non-negative we can change the order of integration (Tonelli's theorem) to get

$$\int_{\operatorname{Im}(\lambda)=\delta} \|(i\lambda - A)^{-1}\widehat{s}_\lambda\|^2_{H^0(M)}\, d\lambda.$$

Now $\|(i\lambda - A)^{-1}\|_{\mathcal{B}(H^0(M))} \sim 1/|\lambda|$ for large λ, hence this is less than or equal to

$$C\int_{\operatorname{Im}(\lambda)=\delta} \|\widehat{s}_\lambda\|^2_{H^0(M)}\, d\lambda = C\|\mathcal{F}_\delta(s)\|^2_{H^0(\mathbb{R}\times M)} = C\|s\|^2_{H^0_\delta(\mathbb{R}\times M)},$$

and so T_δ is bounded.

($\Rightarrow$) This part of the proof is from [Tau87, Lemma 4.3]. Suppose $i\lambda - A$ is not invertible for some λ with $\operatorname{Im}(\lambda) = \delta$. Then $\lambda = i\delta$ since $\operatorname{Spec}(A)$ is real, so $-\delta - A$ is not invertible. Since $-\delta - A$ is self-adjoint, Fredholm, and not invertible, it must have non-trivial kernel. Let s be a non-zero element of the kernel. For $n \in \mathbb{N}$ let $\beta_n : \mathbb{R} \to [0,1]$ be a smooth even function with $\beta_n(t) = 1$ if $|t| \leqslant n$, $\beta_n(t) = 0$ if $|t| \geqslant n+1$, and $\beta_{n+1}(t+1) = \beta_n(t)$ for $t \in [n, n+1]$. Define

$$s_n(t,x) = e^{-\delta t}\beta_n(t)s(x)/n^{1/2}.$$

Then there are constants $C_1, C_2 > 0$ such that

$$C_1 \leqslant \|s_n\|_{H^k_\delta(\mathbb{R}\times M,S)} \leqslant C_2$$

for all n, as one can check by calculating with the equivalent norm from Garding's inequality. Then

$$(\partial_t - A)s_n = e^{-\delta t}\beta_n'(t)s(x)/n^{1/2} \to 0$$

as $n \to \infty$, using $As = -\delta s$. It follows that $\partial_t - A$ is not invertible. □

We collect all of the above in the following, which is the main theorem of this section.

Theorem 4.1.11. *The following are equivalent:*

1. $D^+ : H^k_\delta(Z_\infty, S^+) \to H^{k-1}_\delta(Z_\infty, S^-)$ *is Fredholm.*

2. $D^+ : H^k_\delta(\mathbb{R}\times M, S^+) \to H^{k-1}_\delta(\mathbb{R}\times M, S^-)$ *is Fredholm.*

3. $D^+ : H^k_\delta(\mathbb{R}\times M, S^+) \to H^{k-1}_\delta(\mathbb{R}\times M, S^-)$ *is invertible.*

4. $i\lambda - A : H^k(M,S) \to H^{k-1}(M,S)$ *is invertible for all* λ *with* $\mathrm{Im}(\lambda) = \delta$.

5. $-\delta \notin \mathrm{Spec}(A)$.

Most importantly, 1. and 5. are equivalent. This condenses the Fredholmness of $D^+(Z_\infty)$ on the weighted Sobolev spaces to a more understandable criterion. Setting $\delta = 0$, we get:

Corollary 4.1.12. $D^+ : H^k(Z_\infty, S^+) \to H^{k-1}(Z_\infty, S^-)$ *is Fredholm if and only if* A *is invertible.*

4.1.3 Change of index

The set $\mathbb{R} \setminus -\mathrm{Spec}(A)$ is an open subset of $\mathbb{R}$. For δ and δ' in the same connected component of $\mathbb{R} \setminus -\mathrm{Spec}(A)$, we have $\mathrm{Ind}(D^+ : H^k_\delta \to H^{k-1}_\delta) = \mathrm{Ind}(D^+ : H^k_{\delta'} \to H^{k-1}_{\delta'})$. This follows from the fact that $D^+ : H^k_\mu \to H^{k-1}_\mu$ is isomorphic to the operator $D^+ - \mu c(\nabla u) : H^k \to H^{k-1}$ via the commutative diagram in the proof of Proposition 4.1.1. The operators $D^+ - \delta c(\nabla u)$ and $D^+ - \delta' c(\nabla u)$ are then homotopic through Fredholm operators by the obvious straight path, hence their indices are the same.

All that's left is to understand how the index of $D^+ : H^k_\delta(Z_\infty, S^+) \to H^{k-1}_\delta(Z_\infty, S^-)$ changes when $-\delta$ passes over an eigenvalue of A. There is an obvious candidate for the difference between the indices on either side of the eigenvalue, namely the dimension of the corresponding eigenspace for the eigenvalue. We will see this later, as a corollary of the more general case for end-periodic manifolds carried out in [MRS11].

4.2 Manifolds with periodic ends

Let Z_∞ be an end-periodic manifold modelled on (X, γ), where $\gamma \in H^1(X, \mathbb{Z})$ is a primitive class. Choose a smooth function $h : X \to S^1$ with $h^*[d\theta] = \gamma$. Then dh is a closed 1-form on X that is not exact. It is however exact over the $\mathbb{Z}$-cover $\widetilde{X} \to X$. Let f be a primitive for dh over $\widetilde{X}$. Restrict f to $\widetilde{X}_{\geqslant 0}$, and assume that $f = 0$ on the boundary. Extend f to a smooth function (still denoted f) on Z_∞. If $T : \widetilde{X} \to \widetilde{X}$ is the fundamental covering translation, we will write $x + 1 = Tx$, and then $f(x+1) = f(x) + 1$ for all $x \in \widetilde{X}$.

We assume an end-periodic Riemannian metric on Z_∞, and let $S = S^+ \oplus S^- \to Z_\infty$ be an end-periodic Dirac bundle with end-periodic Dirac operator D. We will denote the bundle over X, on which S is modelled, by the same letter: $S \to X$.

4.2.1 Fredholm properties

For $\delta \in \mathbb{R}$, define the norm

$$\|s\|_{H^k_\delta(Z_\infty, S)} = \|e^{\delta f} s\|_{H^k(Z_\infty, S)}$$

for $s \in \Gamma_c(Z_\infty, S)$. The completion of $\Gamma_c(Z_\infty, S)$ in this norm is denoted $H^k_\delta(Z_\infty, S)$, and is called the *k-th weighted Sobolev space with weight δ*. We similarly define the weighted Sobolev spaces $H^k_\delta(\widetilde{X}, S)$ on $\widetilde{X}$, using the function $f : \widetilde{X} \to \mathbb{R}$.

As for the cylindrical case,

Proposition 4.2.1. *For any k and δ, $D^+(Z_\infty)$ extends to a bounded operator $H^k_\delta(Z_\infty, S^+) \to H^{k-1}_\delta(Z_\infty, S^-)$.*

Proof. The following diagram commutes:

$$\begin{array}{ccc} H^k(Z_\infty, S^+) & \xrightarrow{D^+ - \delta c(df)} & H^{k-1}(Z_\infty, S^-) \\ {\scriptstyle e^{\delta f}}\uparrow & & \downarrow{\scriptstyle e^{-\delta f}} \\ H^k_\delta(Z_\infty, S^+) & \xrightarrow[D^+]{} & H^{k-1}_\delta(Z_\infty, S^-), \end{array}$$

where the vertical arrows are isomorphisms and the top arrow is bounded since $c(df)$ is periodic. □

Again, the weighted Garding inequality holds for D^+ on Z_∞ or on $\widetilde{X}$:

$$\|s\|_{H^k_\delta} \leqslant C\|D^+ s\|_{H^{k-1}_\delta} + \|s\|_{H^{k-1}_\delta}.$$

This implies that for any k and ℓ, the kernel of $D^+ : H^k_\delta(Z_\infty, S^+) \to H^{k-1}_\delta(Z_\infty, S^-)$ agrees with the kernel of $D^+ : H^\ell_\delta(Z_\infty, S^+) \to H^{\ell-1}_\delta(Z_\infty, S^-)$. The weighted Sobolev spaces for $k < 0$ are defined via duality:

$$H^k_\delta(Z_\infty, S) = H^{-k}_{-\delta}(Z_\infty, S)^*.$$

It is an observation of Taubes [Tau87] that Proposition 4.1.7 continues to hold in the end-periodic case.

Proposition 4.2.2. *The following are equivalent:*

1. $D^+ : H^k_\delta(Z_\infty, S^+) \to H^{k-1}_\delta(Z_\infty, S^-)$ *is Fredholm.*
2. $D^+ : H^k_\delta(\widetilde{X}, S^+) \to H^{k-1}_\delta(\widetilde{X}, S^-)$ *is Fredholm.*
3. $D^+ : H^k_\delta(\widetilde{X}, S^+) \to H^{k-1}_\delta(\widetilde{X}, S^-)$ *is invertible.*

Proof. A change of notation and terminology in the proof of Proposition 4.1.7.[4] □

On the cylindrical manifold $\mathbb{R} \times M$ we had a Fourier transform in the $\mathbb{R}$-variable which allowed us to prove Fredholm properties. The corresponding tool in the end-periodic case is the *Fourier-Laplace transform*. Let us motivate this transform by recalling Fourier series for L^2 functions on the circle.

[4] Note that even though a cylinder has a continuous family of translations $t \mapsto t + a$ for $a \in \mathbb{R}$, the proof of Proposition 4.1.7 only ever used integer translations $t \mapsto t + n$ for $n \in \mathbb{Z}$, which the $\mathbb{Z}$-cover $\widetilde{X} \to X$ certainly has.

There is a well known correspondence $L^2(S^1) \cong L^2(\mathbb{Z})$ given by Fourier series:

$$f \mapsto (\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots),$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\,d\theta$. The function f can then be written as the L^2-limit

$$f = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

On the cover $p : \widetilde{X} \to X$, consider the fibre $p^{-1}(x)$ of a point $x \in X$. The fundamental covering translation makes $p^{-1}(x)$ into an 'affine group' modelled on $\mathbb{Z}$, so given an L^2 function $s : p^{-1}(x) \to S_x$ we would like a corresponding 'Fourier transform' $S^1 \to S_x$. We would further like these Fourier transforms to vary continuously in x, but the set $p^{-1}(x)$ has no canonical zero element, and we clearly cannot define such an element to be continuous in x.[5] To fix this, let $\widetilde{x} \in p^{-1}(x)$ be any element of the fibre and define

$$\widehat{s}(\theta, x) = e^{-i\theta f(\widetilde{x})} \sum_{n=-\infty}^{\infty} e^{-in\theta} s(\widetilde{x} + n).$$

The factor of $e^{-i\theta f(\widetilde{x})}$ at the front of the summation corrects for the choice of $\widetilde{x}$. Unfortunately we lose 2π-periodicity in θ, but only in a very mild way.

Definition 4.2.3 ([Tau87])**.** The *Fourier-Laplace transform* of a compactly supported smooth section $s \in \Gamma_c(\widetilde{X}, S)$ is

$$\widehat{s}(\theta, x) = e^{-i\theta f(x)} \sum_{n=-\infty}^{\infty} e^{-i\theta n} s(x + n). \tag{4.2}$$

We will also use the notation $\mathcal{F}(s)(\theta, x) = \widehat{s}_\theta(x) = \widehat{s}(\theta, x)$.

Remarks 4.2.4.

- For fixed x, the sum is finite since s has compact support.
- For any θ and x,

$$\begin{aligned}\widehat{s}(\theta, x+1) &= e^{-i\theta f(x+1)} \sum_{n=-\infty}^{\infty} e^{-i\theta n} s(x + n + 1) \\ &= e^{-i\theta f(x) - i\theta} \sum_{n=-\infty}^{\infty} e^{-i\theta n + i\theta} s(x + n) \\ &= \widehat{s}(\theta, x).\end{aligned}$$

[5] That is, there is no continuous section $X \to \widetilde{X}$ of $p : \widetilde{X} \to X$.

Hence $\widehat{s}_\theta$ is a periodic section of S over $\widetilde{X}$ and so descends to a well-defined section of S over X. When convenient we will write $\widehat{s}(\theta, p(x))$ in place of $\widehat{s}(\theta, x)$.

- Since
$$\widehat{s}(\theta + 2\pi, x) = e^{-2\pi i f(x)}\,\widehat{s}(\theta, x),$$
we no longer have 2π-periodicity in θ. However, $e^{2\pi i f}$ is well-defined on X and defines an automorphism of $L^2(X, S)$ since $|e^{-2\pi i f(x)}| = 1$, so this hardly matters.

- Taubes [Tau87] and Mrowka, Ruberman and Saveliev [MRS16] use a different form of the transform, namely
$$\widehat{u}_z(x) = z^{f(x)} \sum_{n=-\infty}^{\infty} z^n s(x+n)$$
for a chosen branch of $\ln(z)$. The transform (4.2) is clearly equivalent to this by the holomorphic transformation $\theta = i \ln(z)$.

Lemma 4.2.5. *For any $x \in X$ and $s \in \Gamma_c(\widetilde{X}, S)$,*
$$\|\widehat{s}(\theta, x)\|_{L^2([0,2\pi],S_x)} = \left\|\left(s|_{p^{-1}(x)}\right)\right\|_{L^2(p^{-1}(x),S_x)}.$$

Proof. Let $\widetilde{x} \in p^{-1}(x)$. Then
$$\left\|\left(s|_{p^{-1}(x)}\right)\right\|^2_{L^2(p^{-1}(x),S_x)} = \sum_{n=-\infty}^{\infty} |s(\widetilde{x}+n)|^2.$$

By the Plancherel formula for Fourier series on the circle, this is equal to
$$\int_0^{2\pi} \left| \sum_{n=-\infty}^{\infty} e^{-in\theta} s(\widetilde{x}+n)\right|^2 d\theta$$
$$= \int_0^{2\pi} \left| e^{-i\theta f(\widetilde{x})} \sum_{n=-\infty}^{\infty} e^{-in\theta} s(\widetilde{x}+n)\right|^2 d\theta,$$

since $|e^{-i\theta f(\widetilde{x})}| = 1$. This equals $\|\widehat{s}(\theta, x)\|^2_{L^2([0,2\pi],S_x)}$. □

Integrating the square of the above equality over $x \in X$ we get:

Proposition 4.2.6. *For $s \in \Gamma_c(\widetilde{X}, S)$,*
$$\|\widehat{s}(\theta, x)\|_{L^2([0,2\pi]\times X,S)} = \|s\|_{L^2(\widetilde{X},S)}.$$

Let $s \in \Gamma([0,2\pi] \times X, S)$. The *Fourier-Laplace inverse* $\mathcal{F}^{-1}(s) \in \Gamma(\widetilde{X}, S)$ of s is defined as

$$\mathcal{F}^{-1}(s)(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta f(x)} s(\theta, p(x))\, d\theta.$$

Proposition 4.2.7 (Fourier-Laplace inversion formula)**.** *Let $s \in \Gamma_c(\widetilde{X}, S)$. Then*

$$s(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta f(x)}\, \widehat{s}(\theta, p(x))\, d\theta.$$

Proof. The right hand side is

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta f(x)} e^{-i\theta f(x)} \sum_{n=-\infty}^{\infty} e^{-i\theta n} s(x+n)\, d\theta \\ =&\frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} e^{-in\theta}\, d\theta\, s(x+n) \\ =&s(x). \end{aligned}$$

□

Similarly by Fourier inversion on the circle, we have $\mathcal{F}\mathcal{F}^{-1}s = s$ for any $s \in \Gamma([0,2\pi] \times X, S)$. Hence Fourier-Laplace and Fourier-Laplace inversion are inverse to each other on compactly supported smooth sections. The next lemma and proposition imply that they extend to isometries of L^2 spaces which invert each other.

Lemma 4.2.8. *For any $x \in X$,*

$$\|\mathcal{F}^{-1}(s)\|_{L^2(p^{-1}(x), S_x)} = \|s\|_{L^2([0,2\pi], S_x)}$$

Proof. The standard Fourier inverse on the circle is an isometry. □

Integrating the norm squared of the above over X,

Proposition 4.2.9. *For any $s \in \Gamma([0,2\pi] \times X, S)$,*

$$\|\mathcal{F}^{-1}(s)\|_{L^2(\widetilde{X}, S)} = \|s\|_{L^2([0,2\pi]\times X, S)}.$$

We will now bring weights into the mix. As for the cylindrical case, the weights correspond to complex parameters in the Fourier-Laplace transform. Replacing θ with $\lambda = \theta + i\delta$,

$$\widehat{s}(\lambda, x) = e^{\delta f(x)} e^{-i\theta f(x)} \sum_{n=-\infty}^{\infty} e^{-i\theta n} e^{\delta n} s(x+n) \tag{4.3}$$

$$= e^{-i\theta f(x)} \sum_{n=-\infty}^{\infty} e^{-i\theta n} e^{\delta f(x+n)} s(x+n). \tag{4.4}$$

Definition 4.2.10. For $\delta \in \mathbb{R}$, define the *weighted Fourier-Laplace transform* $\mathcal{F}_\delta$ as the composition

$$\begin{array}{ccc} L^2(\widetilde{X}, S) & \xrightarrow{\mathcal{F}} & L^2([0,2\pi] \times X, S) \\ {\scriptstyle e^{\delta f}}\uparrow & \nearrow_{\mathcal{F}_\delta} & \\ L^2_\delta(\widetilde{X}, S) & & \end{array}$$

Again, the weighted Fourier-Laplace transform is an isometry, as it is a composition of two isometries. The weighted Fourier-Laplace inversion formula is

$$s(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{i\lambda f(x)} \mathcal{F}_\delta(s)(\theta, x)\, d\theta,$$

where $\lambda = \theta + i\delta$.

Proposition 4.2.11. *Denote $D^+_\lambda(X) = D^+(X) + i\lambda\, c(df)$. Then for any $\lambda \in \mathbb{C}$ the following diagram commutes:*

$$\begin{array}{ccc} \Gamma_c(\widetilde{X}, S^+) & \xrightarrow{D^+(\widetilde{X})} & \Gamma_c(\widetilde{X}, S^-) \\ {\scriptstyle \mathcal{F}_\lambda}\downarrow & & \downarrow{\scriptstyle \mathcal{F}_\lambda} \\ \Gamma(X, S^+) & \xrightarrow[D^+_\lambda(X)]{} & \Gamma(X, S^-), \end{array}$$

where $\mathcal{F}_\lambda(s) := \widehat{s}_\lambda$.

Proof. We will identify sections over X with periodic sections over $\widetilde{X}$. For any λ and x,

$$\begin{aligned} & D^+(X)\widehat{s}(\lambda, x) \\ &= D^+(\widetilde{X}) \left(e^{-i\lambda f(x)} \sum_{n=-\infty}^{\infty} e^{-i\lambda n} s(x+n) \right) \\ &= -\, i\lambda\, c(df)\, e^{-i\lambda f(x)} \sum_{n=-\infty}^{\infty} e^{-i\lambda n} s(x+n) + e^{-i\lambda f(x)} \sum_{n=-\infty}^{\infty} e^{-i\lambda n} D^+(\widetilde{X}) s(x+n) \\ &= -\, i\lambda\, c(df)\, \widehat{s}(\lambda, x) + \mathcal{F}(D^+(\widetilde{X})s)(\lambda, x). \end{aligned}$$

□

Proposition 4.2.12. *For any $\lambda \in \mathbb{C}$, the operators $D^+_\lambda(X)$ and $D^+_{\lambda+2\pi}(X)$ are isomorphic.*

Proof. Conjugating $D^+_\lambda(X)$ with the isomorphism $e^{-2\pi i f(x)}$,[6] we get

$$e^{-2\pi i f(x)} D^+_\lambda(X) e^{2\pi i f(x)} = D^+_\lambda(X) + 2\pi i c(df) = D^+_{\lambda+2\pi}(X).$$

□

[6] This function is well-defined on X.

The following is the analogue of Proposition 4.1.10 for end-periodic manifolds.

Theorem 4.2.13 (Lemma 4.3. [Tau87]). *The operator $D^+ : H^k_\delta(\widetilde{X}, S^+) \to H^{k-1}_\delta(\widetilde{X}, S^-)$ is invertible if and only if the operators $D^+_\lambda(X)$ are invertible for all $\lambda \in \mathbb{C}$ with $\mathrm{Im}(\lambda) = \delta$.*

Proof. ($\Leftarrow$) We will go over this implication in some detail, so that the reader is assured the Fourier-Laplace transform works as expected. Suppose that all the $D^+_\lambda(X)$ are invertible for $\mathrm{Im}(\lambda) = \delta$. For $s \in \Gamma_c(\widetilde{X}, S^-)$, define

$$R(s)(x) = \frac{1}{2\pi} \int_{\mathrm{Im}(\lambda)=\delta} e^{i\lambda f(x)} D^+_\lambda(X)^{-1}(\widehat{s}_\lambda)(x)\, d\lambda,$$

where the integral is from $\mathrm{Re}(\lambda) = 0$ to $\mathrm{Re}(\lambda) = 2\pi$. By the Fourier-Laplace inversion formula we have $D^+(\widetilde{X})R(s) = s$ for s compactly supported. Also, by the F-L inversion formula and Proposition 4.2.11, we have $RD^+(\widetilde{X})(s) = s$ for s with compact support. It remains to show that R extends to a bounded operator $H^{k-1}_\delta(\widetilde{X}, S^-) \to H^k_\delta(\widetilde{X}, S^+)$. As for the proof of Proposition 4.1.10, the weighted Garding inequality reduces this to proving that the operator $R : H^0_\delta(\widetilde{X}, S^-) \to H^0_\delta(\widetilde{X}, S^+)$ is bounded. Computing norms,

$$\begin{aligned}
\|Rs\|^2_{H^0_\delta(\widetilde{X},S^+)} &= \int_{\widetilde{X}} \left| e^{\delta f(x)} \frac{1}{2\pi} \int_{\mathrm{Im}(\lambda)=\delta} e^{i\lambda f(x)} D^+_\lambda(X)^{-1}(\widehat{s}_\lambda)(x)\, d\lambda \right|^2 dx \\
&= \int_X \int_{\mathrm{Im}(\lambda)=\delta} \left| D^+_\lambda(X)^{-1}(\widehat{s}_\lambda)(x) \right|^2 d\lambda\, dx \\
&= \int_{\mathrm{Im}(\lambda)=\delta} \|D^+_\lambda(X)^{-1}(\widehat{s}_\lambda)\|^2_{H^0(X,S^-)}\, d\lambda \\
&\leqslant C \int_{\mathrm{Im}(\lambda)=\delta} \|\widehat{s}_\lambda\|^2_{H^0(X,S^-)}\, d\lambda \\
&= C\|\mathcal{F}_\delta(s)\|^2_{H^0([0,2\pi]\times X,S^-)} \\
&= C\|s\|^2_{H^0_\delta(\widetilde{X},S^-)}.
\end{aligned}$$

Hence R is a bounded inverse for $D^+(\widetilde{X})$. Note that on the fourth line we use that $\|D^+_\lambda(X)^{-1}\|_{\mathcal{B}(H^0)}$ is uniformly bounded, since λ runs over a compact set.

($\Rightarrow$) This is essentially the same as the proof of the forward implication in Proposition 4.1.10. Note however that even though $D_\lambda(X)$ is not self-adjoint, the argument in that proof applies equally well to its adjoint $D^-_{\overline{\lambda}}(X)$, so that either $D^+(\widetilde{X}) : H^k_\delta(\widetilde{X}, S^+) \to H^{k-1}_\delta(\widetilde{X}, S^-)$ or its adjoint is not invertible if $D^+_\lambda(X)$ is not invertible for some λ with $\mathrm{Im}(\lambda) = \delta$.

□

Combining all of the above,

Theorem 4.2.14. *The following are equivalent:*

1. $D^+ : H^k_\delta(Z_\infty, S^+) \to H^{k-1}_\delta(Z_\infty, S^-)$ *is Fredholm.*
2. $D^+ : H^k_\delta(\widetilde{X}, S^+) \to H^{k-1}_\delta(\widetilde{X}, S^-)$ *is Fredholm.*
3. $D^+ : H^k_\delta(\widetilde{X}, S^+) \to H^{k-1}_\delta(\widetilde{X}, S^-)$ *is invertible.*
4. $D^+_\lambda(X) : H^k(X, S^+) \to H^{k-1}(X, S^-)$ *is invertible for all* λ *with* $\mathrm{Im}(\lambda) = \delta$.

Of course in the fourth condition we only really need $0 \leqslant \mathrm{Re}(\lambda) < 2\pi$ by Proposition 4.2.12.

Corollary 4.2.15. $D^+ : H^k(Z_\infty, S^+) \to H^{k-1}(Z_\infty, S^-)$ *is Fredholm if and only if* $D^+_\lambda(X) : H^k(X, S^+) \to H^{k-1}(X, S^-)$ *is invertible for all* $\lambda \in \mathbb{R}$.

Corollary 4.2.16. *The signature operator on* Z_∞ *is never Fredholm.*

Proof. The signature operator on X has a non-trivial element in its kernel, namely $1 \oplus \mathrm{vol}_X$. □

Unlike the cylindrical case, there is no fifth equivalent condition relating Fredholmness of $D^+(Z_\infty)$ to the spectrum of a differential operator on X. The next section is dedicated to proving Taubes' theorem, which gives a sufficient condition for the set of λ for which $D^+_\lambda(X)$ is not invertible to be a discrete subset of $\mathbb{C}$.

4.2.2 Taubes' theorem

Definition 4.2.17. The *spectral set* of the family $\{D^+_\lambda(X)\}_{\lambda\in\mathbb{C}}$ is the set of $\lambda \in \mathbb{C}$ for which $D^+_\lambda(X)$ is not invertible.

Observation 4.2.18. *If the spectral set of* $\{D^+_\lambda(X)\}$ *is not all of* $\mathbb{C}$*, then the index of* $D^+(X)$ *is zero.*

Proof. If $D^+_\lambda(X)$ is invertible for some λ, it is in particular Fredholm with zero index. Since $D^+_\lambda(X) = D^+(X) + i\lambda\, c(df)$, we can homotopy away the second (lower order) term without changing the index. Hence $D^+(X)$ has zero index. □

Corollary 4.2.19. *In order for* $D^+(Z_\infty)$ *to be Fredholm on some weighted Sobolev space, we require that* $\mathrm{Ind}(D^+(X)) = 0$.

The end-periodic index theorem will assume that $\{D^+_\lambda(X)\}$ has discrete spectral set, which in particular implies that $\mathrm{Ind}(D^+(X)) = 0$.

Lemma 4.2.20 (Lemma 4.4. [Tau87])**.** *Suppose* $c(df) : \ker(D^+(X)) \to \mathrm{coker}(D^+(X))$ *is an isomorphism. Then there is* $\epsilon > 0$ *such that* $D^+_\lambda(X)$ *is invertible for all* $\lambda \in \mathbb{C}$ *with* $0 < |\lambda| < \epsilon$.

Proof. Suppose to the contrary that there are arbitrarily small λ such that $D^+_\lambda(X)$ is not invertible. For now assume that there is a sequence $\lambda_n \to 0$ such that $\ker D^+_{\lambda_n}(X)$ is nontrivial for each n. We choose $\psi_n \in \ker D^+_{\lambda_n}(X)$ with $\|\psi_n\|_{L^2} = 1$. Then for any n,

$$D^+(X)\psi_n + i\lambda_n c(df)\psi_n = 0. \tag{4.5}$$

Let P be a parametrix for $D^+(X)$; recall $P : \Gamma(S^-) \to \Gamma(S^+)$ is a pseudo-differential operator of order -1 such that $PD^+(X) = \mathrm{Id} + S$, where S is a smoothing operator. Applying P to equation (4.5) we get

$$\psi_n = -S\psi_n - i\lambda_n P(c(df)\psi_n).$$

Denote by A_n the operator on the right hand side; $A_n := -S - i\lambda_n Pc(df)$. Then A_n is a pseudo-differential operator of order -1. We have

$$\|\psi_n\|_{H^1} = \|A_n\psi_n\|_{H^1} \leqslant \|A_n\|\,\|\psi_n\|_{H^0} = \|A_n\|,$$

where A_n is considered as a bounded linear map $H^0(S^+) \to H^1(S^+)$. Since $\lambda_n \to 0$, the norms of the A_n are uniformly bounded, and hence $\|\psi_n\|_{H^1}$ is bounded. By Rellich compactness there is a subsequence ψ_{n_j} which converges in $L^2(S^+)$. We now apply the same argument as above to the subsequence ψ_{n_j}, considering A_{n_j} as a bounded linear map $H^1(S^+) \to H^2(S^+)$. This produces a subsequence $\psi_{n_{j_k}}$ which converges in $H^1(S^+)$. Continuing in this vein, we take the diagonal sequence and get a subsequence of the ψ_n which converges in every H^k, and hence in the C^∞ topology, to some $\psi \in \Gamma(S^+)$. Since each ψ_n has L^2-norm equal to 1, it is clear that $\psi \neq 0$. We now replace ψ_n with the subsequence. Taking the limit as $n \to \infty$ in equation (4.5) (in the C^∞ topology) then gives $D^+(X)\psi = 0$, so $\psi \in \ker(D^+(X)) \setminus \{0\}$.

Now, take $\phi \in \ker(D^-(X))$. Then

$$\begin{aligned}
\langle \phi, c(df)\psi\rangle_{L^2} &= \langle \phi, c(df)(\psi - \psi_n)\rangle_{L^2} + \langle \phi, c(df)\psi_n\rangle_{L^2} \\
&= \langle \phi, c(df)(\psi - \psi_n)\rangle_{L^2} + \frac{1}{i\overline{\lambda_n}}\langle \phi, D^+(X)\psi_n\rangle_{L^2} \\
&= \langle \phi, c(df)(\psi - \psi_n)\rangle_{L^2} + \frac{1}{i\overline{\lambda_n}}\langle D^-(X)\phi, \psi_n\rangle_{L^2} \\
&= \langle \phi, c(df)(\psi - \psi_n)\rangle_{L^2} \\
&\to 0.
\end{aligned}$$

Hence $c(df)\psi$ is orthogonal to the kernel of $D^-(X)$, and so is in the image of $D^+(X)$. Thus we have exhibited a non-zero element ψ of $\ker(D^+(X))$ whose image under $c(df) : \ker(D^+(X)) \to \operatorname{coker}(D^+(X))$ is zero, implying this map is not an isomorphism.

It may instead be the case that there is a sequence of non-zero $\lambda_n \to 0$ such that $D^+_{\lambda_n}(X)$ is not onto. If we take a corresponding sequence ψ_n with $\|\psi_n\|_{L^2} = 1$ and ψ_n orthogonal to the image of $D^+_{\lambda_n}(X)$, then ψ_n is in the kernel of $D^+_{\lambda_n}(X)^* = D^-_{\overline{\lambda_n}}(X)$, and the argument is entirely the same as before. □

Lemma 4.2.21 (Lemma 4.5. [Tau87])**.** *Suppose that $D_{\lambda_0} : H^k(S^+) \to H^{k-1}(S^-)$ is invertible for some $\lambda_0 \in \mathbb{C}$. Then there is a discrete subset Γ of $\mathbb{C}$ with no accumulation points such that $D^+_\lambda(X)$ is invertible for all $\lambda \in \mathbb{C}\backslash\Gamma$.*

The following proof, from [MRS11, Theorem 4.6], uses the compact resolvent theorem from functional analysis.

Proof. Consider

$$Q_\lambda = D^+_\lambda(X) D^+_{\lambda_0}(X)^{-1} = (T + \lambda K)(T + \lambda_0 K)^{-1},$$

where we write $T = D^+(X) : H^1(X,S) \to H^0(X,S)$ and $K = ic(df) : H^1(X,S) \to H^0(X,S)$. Note K is compact by Rellich compactness. Now

$$Q_\lambda = \mathrm{Id} + (\lambda - \lambda_0)K(T + \lambda_0 K)^{-1}$$

which is a compact perturbation of the identity. For $\lambda \neq \lambda_0$, we see that Q_λ is invertible if and only if $(\lambda_0 - \lambda)^{-1} \notin \operatorname{Spec}(K(T + \lambda_0 K)^{-1})$. Thus, set of λ for which Q_λ is invertible corresponds to the inverse-translate of the spectrum of a compact operator, which is discrete. □

Combining the above two lemmas, we get the following:

Theorem 4.2.22 ([Tau87])**.** *If $c(df) : \ker D^+(X) \to \operatorname{coker} D^+(X)$ is an isomorphism, then the spectral set of the family $\{D^+_\lambda(X)\}$ is discrete.*

Remark 4.2.23. If we are in the cylindrical setting, then $f = t$ and $c(df) = c(\partial_t) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ which is an isomorphism. It follows that the spectral set of the family $\{i\lambda - A\}$ is discrete. We already knew this though, since the spectral set is basically the spectrum of A.

Corollary 4.2.24. *If $D^+(X)$ is invertible, then the spectral set of the family $\{D^+_\lambda(X)\}$ is discrete.*

Proof. Both $\ker(D^+(X))$ and $\operatorname{coker}(D^+(X))$ are trivial. □

4.2.3 The index change formula

We will assume in this section that the spectral set of the family $\{D^+_\lambda(X)\}$ is discrete. This given, we can say that $D^+(Z_\infty) : H^k_\delta \to H^{k-1}_\delta$ is Fredholm for all but a discrete set of $\delta \in \mathbb{R}$. In the cylindrical setting where $D^+ \cong \partial_t - A$, this set of δ's is just the spectrum of $-A$. We want to understand how the index changes when we move over a point in this set. This section summarises the results in [MRS11] on how the index changes with the weight. Since their exposition is clear and self-contained, there is little reason to repeat the proofs. We will therefore give a brief summary of the results, and consider what these mean in the case of cylindrical ends.

For $\lambda_0 \in \mathbb{C}$ an element of the spectral set of $\{D^+_\lambda(X)\}$, consider the following system of equations:

$$\begin{cases} D^+_{\lambda_0}(X)b_{-m} = 0 \\ D^+_{\lambda_0}(X)b_{-m+1} + ic(df)b_{-m} = 0 \\ D^+_{\lambda_0}(X)b_{-m+2} + ic(df)b_{-m+1} = 0 \\ \vdots \\ D^+_{\lambda_0}(X)b_{-1} + ic(df)b_{-2} = 0, \end{cases}$$

where m is the degree of the pole of $D^+_\lambda(X)^{-1}$ at λ_0. Define $d(\lambda_0)$ to be the dimension of the space of solutions to this system. If $\lambda \in \mathbb{C}$ is not in the spectral set, we define $d(\lambda) = 0$. The significance of this number is given by the following:

Theorem 4.2.25 (Section 6.4, [MRS11]). *Let Z_∞ be an end-periodic manifold, and suppose $D^+ : H^k_{\delta_i}(Z_\infty, S^+) \to H^{k-1}_{\delta_i}(Z_\infty, S^-)$ is Fredholm for $i = 1, 2$, where $\delta_1 < \delta_2$. Then*

$$\mathrm{Ind}_{\delta_2} D^+(Z_\infty) - \mathrm{Ind}_{\delta_1} D^+(Z_\infty) = \sum_\lambda d(\lambda),$$

where the sum is over $\lambda \in \mathbb{C}$ with $\delta_1 < \mathrm{Im}(\lambda) < \delta_2$ and $0 \leqslant \mathrm{Re}(\lambda) < 2\pi$.

Note the sum is finite, since we have assumed the spectral set of $\{D^+_\lambda(X)\}$ is discrete.

Let us ponder what this says in the cylindrical case $X = S^1 \times Y$. Suppose $\lambda \in \mathbb{C}$ is such that $D^+_\lambda(X) \cong \partial_t - A + i\lambda$ is not invertible. We will argue that $\mathrm{Re}(\lambda) = k2\pi$ for some $k \in \mathbb{Z}$. Suppose $\mathrm{Re}(\lambda) \neq k2\pi$. Let s be an element of $\ker(\partial_t - A + i\lambda)$, and decompose s as

$$s = \sum_{\ell,\alpha} a_{\ell,\alpha} e^{2\pi i \ell t} \phi_\alpha,$$

where $\{\phi_\alpha\}$ is an orthonormal basis for $L^2(S)$ consisting of eigensections of A with eigenvalues $\alpha \in \mathbb{R}$. Then $a_{\ell,\alpha}(2\pi i\ell - \alpha + i\lambda) = 0$ for all ℓ, α, and by assumption on λ this means $a_{\ell,\alpha} = 0$ for all ℓ, α. Hence $\partial_t - A + i\lambda$ has trivial kernel. Since it also has index zero, it is invertible. This proves that if $\partial_t - A + i\lambda$ is not invertible then λ is an integer multiple of 2π. Since $D^+_\lambda(X) \cong D^+_{\lambda+2\pi}(X)$, we may assume $\mathrm{Re}(\lambda) = 0$ in what follows.

So, suppose that $\partial_t - A + i\lambda$ is not invertible. Then $\lambda = i\delta$ and $-\delta \in \mathrm{Spec}(A)$ by a combination of Theorem 4.1.11 and Theorem 4.2.14. On the manifold $S^1 \times Y$ we have $D^+_\lambda(X) = c(\partial_t)(\partial_t - A - \delta)$, and the system of equations becomes

$$\begin{cases} (\partial_t - A - \delta)b_{-m} = 0 \\ (\partial_t - A - \delta)b_{-m+1} + ib_{-m} = 0 \\ (\partial_t - A - \delta)b_{-m+2} + ib_{-m+1} = 0 \\ \vdots \\ (\partial_t - A - \delta)b_{-1} + ib_{-2} = 0. \end{cases}$$

Suppose we have a solution $(b_{-m}, \ldots, b_{-1})$ of this system. Decompose b_{-m} as $b_{-m} = \sum_{\ell,\alpha} a_{\ell,\alpha} e^{2\pi i\ell t}\phi_\alpha$. Then the first equation gives $a_{\ell,\lambda}(2\pi i\ell - \alpha - \delta) = 0$ for all ℓ, α, and hence $b_{\ell,\alpha} = 0$ for $\ell \neq 0$. Writing $b_{-m} = \sum_\alpha a_\alpha \phi_\alpha$, we get $(A + \delta)b_{-m} = 0$.

Now, write $b_{-m+1} = \sum_{\ell,\alpha} c_{\ell,\alpha} e^{2\pi i\ell t}\phi_\alpha$. The second equation similarly gives $c_{\ell,\alpha} = 0$ for $\ell \neq 0$, and hence we have $(A+\delta)b_{-m} = ib_{-m}$. Now, $b_{-m} \in \ker(A+\delta) = \mathrm{Im}((A+\delta)^*)^\perp = \mathrm{Im}(A+\delta)^\perp$. Hence the equation $(A+\delta)b_{-m+1} = ib_{-m}$ cannot be solved unless $b_{-m} = 0$. Applying the same reasoning to the second and third equations similarly gives $b_{-m+1} = 0$. Repeating the argument we get $b_{-m} = b_{-m+1} = \cdots = b_{-2} = 0$. Then, in the last equation, b_{-1} can be anything in the kernel of $A + \delta$. It follows that the dimension of the solution space is simply the dimension of the eigenspace for the eigenvalue $-\delta$, as claimed.

Proposition 4.2.26. *Let Z_∞ be a cylindrical manifold and suppose $D^+ : H^k_\delta(Z_\infty, S) \to H^{k-1}_\delta(Z_\infty, S)$ is not Fredholm. Then $-\delta$ is an eigenvalue of A. Let $\epsilon > 0$ be such that $D^+ : H^k_\mu(Z_\infty, S) \to H^{k-1}_\mu(Z_\infty, S)$ is Fredholm for all $\mu \in (\delta - \epsilon, \delta + \epsilon) \setminus \{\delta\}$, and choose $\delta - \epsilon < \delta_1 < \delta < \delta_2 < \delta + \epsilon$. Then*

$$\mathrm{Ind}_{\delta_2} D^+(Z_\infty) - \mathrm{Ind}_{\delta_1} D^+(Z_\infty) = \dim\ker(A + \delta).$$

Chapter 5

The end-periodic index theorem

In this final chapter before the preprint, we discuss the index theorem for end-periodic manifolds, proved by Mrowka, Ruberman and Saveliev [MRS16]. This index theorem applies to certain end-periodic Dirac operators on an end-periodic manifold Z_∞ with end modelled on (X, γ), and takes a similar form to the Atiyah-Patodi-Singer index theorem. Namely, the index of the Dirac operator $D^+(Z_\infty)$ (possibly interpreted in some 'extended' sense due to Z_∞ not being compact) is equal to the integral of the Dirac operator's index form $\mathbf{I}(D^+)$ over the compact piece Z, plus a correction term $\eta^{\mathrm{ep}}(D^+(X))$ coming from the periodic end—the *end-periodic eta invariant*. We saw in Chapter 3 that the Atiyah-Patodi-Singer index theorem for manifolds with boundary had an equivalent formulation in terms of manifolds with cylindrical ends. When one considers a cylindrical manifold as a manifold with periodic ends, the end-periodic index theorem reduces to the APS index theorem.

Unfortunately the end-periodic index theorem does not apply to any end-periodic Dirac operator on an end-periodic manifold; in order to apply the theorem the spectral set of the family $\{D^+_\lambda(X)\}$, introduced in the previous chapter, must be assumed discrete. We have seen that this is always true for cylindrical Dirac operators, but it is not true in general (certainly if $\mathrm{Ind}(D^+(X)) \neq 0$ then the spectral set of the family is all of $\mathbb{C}$). If the Dirac operator $D^+(Z_\infty)$ happens to be Fredholm, then the index in the MRS index theorem is the usual Fredholm index. If $D^+(Z_\infty)$ is not Fredholm but the family $D^+_\lambda(X)$ still has discrete spectral set, then the index is defined through use of weighted Sobolev spaces. In the cylindrical case, this corresponds to the situation where the Dirac operator A on the boundary is not invertible, and one considers 'extended L^2 solutions' on Z_∞.

The proof of the theorem is via heat kernel analysis, as for the proof of the local index theorem and the APS index theorem. There are some additional complications due to: (a) the manifold Z_∞ being non-compact, and (b) there being no explicit formula for the heat kernel on $\widetilde{X}$. The proof is therefore modelled on Melrose's proof [Mel93] of the APS index theorem, which does not use a formula for the heat kernel, but employs a certain 'b-trace', see Definition (In.19) of [Mel93]. The analogue in the end-periodic setting is the 'regularised trace' $\mathrm{Tr}^\flat$. Roughly, this trace exploits the fact that far away from the compact piece Z, the heat kernel for $\widetilde{X}$ very closely approximates the heat kernel for Z_∞, if times and distances are small.

Once the regularised trace is introduced, the proof proceeds by calculating the regularised supertrace of the heat kernel for $D(Z_\infty)$ and taking limits as $t \to 0$ and $t \to \infty$. As usual the $t \to 0$ limit gives the integral of the index form over Z, and the $t \to \infty$ limit gives the index. However, due to the regularised trace not vanishing on commutators, the supertrace is not constant in t, so these are not equal. The end-periodic eta invariant emerges from the integral of the t-derivative of the supertrace from $t = 0$ to $t = \infty$.

In Section 5.1, we state the end-periodic index theorem in both the Fredholm and non-Fredholm cases, and describe how these theorems reduce to the usual APS theorem when the end-periodic manifold has cylindrical ends. Section 5.2 then moves on to some aspects of the proof, namely a formula relating the heat kernel on X to the heat kernel on $\widetilde{X}$, and also the regularised trace. Section 5.3 then gives an outline of the proof of the theorem, in the Fredholm case. We end in Section 5.4 by describing the applications to positive scalar curvature from [MRS16].

5.1 Statement of the theorem

Throughout this chapter, Z_∞ will be an end-periodic manifold with periodic end modelled on the pair (X, γ). We assume an end-periodic Riemannian metric on Z_∞, an end-periodic Dirac bundle $S \to Z_\infty$ with end-periodic $\mathbb{Z}_2$-grading, and an end-periodic Dirac operator D on S. As usual, Y is a connected submanifold of X Poincaré dual to the class γ, and the function $h : X \to S^1$ satisfies $h^*[d\theta] = \gamma$ and $h^{-1}(1) = Y$. Its lift to the cover is denoted by $f : \widetilde{X} \to \mathbb{R}$, and $f : Z_\infty \to \mathbb{R}$ is any smooth extension of $f|_{\widetilde{X}_{\geqslant 0}}$ to Z_∞.

We will first state the MRS index theorem in the case where the Dirac operator $D^+(Z_\infty)$ is Fredholm, and then in the case where it is not Fredholm but the family $\{D^+_\lambda(X)\}$ has discrete spectral set. In both instances, we will explain how the theorem reduces to the APS index theorem in the case of manifolds with cylindrical ends.

5.1.1 Fredholm case

Suppose that $D^+(Z_\infty)$ is Fredholm. Then by Corollary 4.2.19, the index of $D^+(X)$ is zero. Hence by Atiyah-Singer (and Poincaré duality) the index form $\mathbf{I}(D^+(X))$ is exact, so there is an $(n-1)$-form ω on X with $d\omega = \mathbf{I}(D^+(X))$.

Theorem 5.1.1 (Theorem A, [MRS16])**.** *Assume that the operator $D^+(Z_\infty) : H^k(Z_\infty, S^+) \to H^{k-1}(Z_\infty, S^-)$ is Fredholm, and let ω be a form such that $d\omega = \mathbf{I}(D^+(X))$. Then*

$$\operatorname{Ind} D^+(Z_\infty) = \int_Z \mathbf{I}(D^+(Z)) - \int_Y \omega + \int_X df \wedge \omega - \frac{1}{2}\eta^{\mathrm{ep}}(D^+(X)),$$

where

$$\eta^{\mathrm{ep}}(D^+(X)) = \frac{1}{\pi}\int_0^\infty \int_0^{2\pi} \operatorname{Tr}(c(df)D_\lambda^+ \exp(-tD_\lambda^- D_\lambda^+))\, d\lambda\, dt.$$

The quantity $\eta^{\mathrm{ep}}(D^+(X))$ defined in the theorem is called the *end-periodic eta invariant*. Thus, the theorem can be interpreted as the statement that the integral defining η^{ep} converges, and that it is given by the above formula.

The operators $\exp(-tD_\lambda^- D_\lambda^+)$ in the theorem are defined using the spectral theorem for unbounded self-adjoint operators on Hilbert spaces. Recall also that $D^+(Z_\infty) : H^1(Z_\infty) \to H^0(Z_\infty)$ being Fredholm is equivalent to $D_\lambda^+(X)^{-1}$ having no poles for $\operatorname{Im}(\lambda) = 0$ (or equivalently $\operatorname{Im}(\lambda) = 0$ and $0 \leqslant \lambda < 2\pi$), so the operators $D_\lambda^+(X)$ in the eta integrand are all invertible.

Let us consider what the theorem says for the case of manifolds with cylindrical ends. First, $X = S^1 \times Y$, and we may choose $h : X \to S^1$ to be projection onto the circle factor. Then $\widetilde{X} = \mathbb{R} \times Y$ and $f(t, x) = t$. Then the integral terms involving ω vanish, leaving

$$\operatorname{Ind}(D^+(Z_\infty)) = \int_Z \mathbf{I}(D^+(Z)) - \frac{1}{2}\eta^{\mathrm{ep}}(D^+(X)).$$

Next, note the Dirac operator $D^+(Z_\infty)$ takes the form $c(\partial_t)(\partial_t - A)$ over the cylindrical end, where A is the Dirac operator on Y. Recall that Corollary 4.1.12 says that $D^+(Z_\infty)$ being Fredholm corresponds exactly to A being invertible. If A is invertible, we noted in Remark 3.2.4 that the index occurring in the APS index theorem is just the index of the Fredholm operator $D^+(Z_\infty)$. Equating the index in the APS theorem and the index in the MRS theorem, we get

$$\int_Z \mathbf{I}(D^+(Z)) - \frac{1}{2}\eta_A(0) = \int_Z \mathbf{I}(D^+(Z)) - \frac{1}{2}\eta^{\mathrm{ep}}(D^+(X)),$$

and hence the end-periodic eta invariant reduces to the standard eta invariant.

There is a more elementary way to observe that the eta invariants agree, of course. In Section 6.3 of [MRS16], the L^2-basis $\psi_{n,\lambda} = e^{in\theta}\phi_\lambda$ of eigensections for $\partial_t - A$ is used to compute directly that

$$\eta(D^+(X)) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} \mathrm{Tr}(Ae^{-tA^2})\, dt,$$

which is a standard formula for the eta invariant of A; see page 20 of [Gil89] and Section 8.13 of [Mel93]. In any case, it follows that the MRS index theorem reduces to the APS index theorem in the Fredholm case.

5.1.2 Non-Fredholm case

Suppose now that the spectral set of $\{D^+_\lambda(X)\}$ is discrete. It may then have poles which lie on the real axis, in which case $D^+(Z_\infty)$ is not Fredholm in the usual L^2 sense. In the cylindrical case, this corresponds to the situation $\ker(A) \neq 0$. We know that even if $\ker(A) \neq 0$, a version of the APS theorem still holds, but the formula must be corrected in a couple of ways. First, the extended L^2 spaces must be used to calculate the index of $D^+(Z_\infty)$ rather than the usual L^2 spaces, and second, the dimension h of the kernel of A appears in the formula alongside the eta invariant. We must therefore seek replacements for these corrections in the end-periodic setting. In addition, a replacement for the eta invariant is needed, since we generally wish to avoid the poles of the family $\{D^+_\lambda(X)\}$. The rough idea is to consider some sufficiently small $\delta > 0$ for which $D^+_\lambda(X)^{-1}$ has no poles with $\mathrm{Im}(\lambda) = \delta$, and then prove the index theorem for the operator $D^+(Z_\infty) : H^k_\delta \to H^{k-1}_\delta$.

The eta invariant

Let us start with the eta invariant. The formula in the non-Fredholm case cannot be used since the spectral set of $\{D^+_\lambda(X)\}$ has elements along the real axis. To correct for this, suppose that $\epsilon > 0$ is sufficiently small so that $D^+_\lambda(X)^{-1}$ for $-\epsilon < \mathrm{Im}(\lambda) < \epsilon$ has poles only on the real axis. For $0 < |\delta| < \epsilon$, define

$$\eta^{\mathrm{ep}}_\delta(D^+(X)) = \frac{1}{\pi} \int_0^\infty \int_{\mathrm{Im}(\lambda)=\delta} \mathrm{Tr}\left(c(df) D^+_\lambda \exp(-t(D^+_\lambda)^* D^+_\lambda)\right) d\lambda\, dt,$$

where the λ-integral runs from $\mathrm{Re}(\lambda) = 0$ to $\mathrm{Re}(\lambda) = 2\pi$. The integral does not actually converge at the $t = 0$ end, so $\eta^{\mathrm{ep}}_\delta(D^+(X))$ is really taken to be the constant term in the asymptotic expansion of

$$\frac{1}{\pi}\int_a^\infty \int_{\mathrm{Im}(\lambda)=\delta} \operatorname{Tr}\left(c(df)D_\lambda^+ \exp(-t(D_\lambda^+)^* D_\lambda^+)\right)\, d\lambda\, dt,$$

in $a > 0$; see [Gil89, Theorem 1.2.7], [Mül94, Lemma 1.17], and [Mel93, Sections 8.13-8.14] for the details of such expansions.

Note that since λ is not real, we do not have $D_\lambda^+(X)^* = D_\lambda^-(X)$, so we must use the adjoint in the integral instead. This adjoint is of course given by $D_\lambda^+(X)^* = D_{\bar{\lambda}}^-(X)$. The *end-periodic eta invariant* is defined as

$$\eta^{\mathrm{ep}}(D^+(X)) = \lim_{\delta\to 0^+} \frac{\eta_\delta^{\mathrm{ep}}(D^+(X)) + \eta_{-\delta}^{\mathrm{ep}}(D^+(X))}{2}.$$

The index

Next we discuss how the index should be modified. Recall that if $\dim\ker(A) > 0$, then the formulation of the APS theorem for manifolds with cylindrical ends involves 'extended L^2 solutions'. By definition, such a solution is asymptotically of the form

$$s = g + a_\infty,$$

where $g \in L^2(Z_\infty, S^+)$ and $a_\infty \in \ker(A)$. It is easy to see that such a solution is precisely one which lies in all the weighted Sobolev spaces $H_\delta^k(Z_\infty, S^+)$ for all sufficiently small $\delta < 0$. First, over $\mathbb{R}_{\geqslant 0} \times Y$ decompose $s = \sum_{\lambda\in\mathrm{Spec}(A)} a_\lambda(t)\phi_\lambda(x)$, where $\{\phi_\lambda\}$ is an orthonormal L^2 basis consisting of eigensections of A. Computing $(\partial_t - A)s = 0$ we get $a_\lambda'(t) = \lambda a_\lambda(t)$, hence $a_\lambda = C_\lambda e^{\lambda t}$, where C_λ is some constant. We require $\|e^{\delta t}s\|_{L^2} < \infty$, from which it is seen that $C_\lambda = 0$ for $\lambda > -\delta$. However, $C_\lambda \neq 0$ for $\lambda = 0$ is still allowed, since $C_\lambda e^{\delta t}$ decays rapidly. If we choose δ close enough to 0 so that A has no eigenvalues between $-\delta$ and 0, then s is precisely an extended L^2 solution.

Now, since extended L^2 solutions are only allowed for the adjoint $D^-(Z_\infty)$ in the APS index theorem, and since taking duals changes the sign of the weight, we really want to consider $\delta > 0$. The APS index is therefore the index of $D^+(Z_\infty)$ acting on weighted Sobolev spaces with weight $\delta > 0$ sufficiently small. For the non-Fredholm end-periodic case, the index in the theorem is replaced by the index of

$$D^+(Z_\infty) : H_\delta^k(Z_\infty, S^+) \to H_\delta^{k-1}(Z_\infty, S^-),$$

where $\delta > 0$ is sufficiently small so that the spectral set of $\{D_\lambda^+(X)\}$ has no elements with imaginary part in $(0, \delta]$. Note the index is independent of the choice of such δ by the remarks in Section 4.1.3, which apply equally well to the end-periodic case. We denote the index by $\mathrm{Ind}_+(D^+(Z_\infty))$.

The term h

Finally we consider the correction h, which is equal to $\dim\ker(A)$ in the APS index theorem. We have already encountered a suitable replacement for h in Section 4.2.3. Recall that if λ is a pole of $D^+_\lambda(X)^{-1}$ with degree m, then the integer $d(\lambda)$ is defined as the dimension of the space of solutions to the system:

$$\begin{cases} D^+_\lambda(X)b_{-m} = 0 \\ D^+_\lambda(X)b_{-m+1} + ic(df)b_{-m} = 0 \\ D^+_\lambda(X)b_{-m+2} + ic(df)b_{-m+1} = 0 \\ \vdots \\ D^+_\lambda(X)b_{-1} + ic(df)b_{-2} = 0. \end{cases}$$

We observed that for the cylindrical case, this dimension for $\lambda = 0$ is precisely the dimension of the kernel of A. The number which will play the role of $\dim\ker(A)$ in the end-periodic index theorem is

$$h = \sum_{0 \leqslant \lambda < 2\pi} d(\lambda).$$

This number h is then the difference between the indices of $D^+(Z_\infty)$ acting on weighted Sobolev spaces with small positive and negative weights.

Theorem 5.1.2. *Assume that the family $\{D^+_\lambda(X)\}$ has discrete spectral set. Since* $\mathrm{Ind}(D^+(X)) = 0$*, we can choose $\omega \in \Omega^{n-1}(X)$ such that $d\omega = \mathbf{I}(D^+(X))$. Then*

$$\mathrm{Ind}_+(D^+(Z_\infty)) = \int_Z \mathbf{I}(D^+(Z)) - \int_Y \omega + \int_X df \wedge \omega - \frac{\eta^{\mathrm{ep}}(D^+(X)) + h}{2}.$$

From the above discussion of the new terms, if we apply this theorem and the APS theorem to a manifold with cylindrical ends then the indices are equal, the Z-integrals are equal, the terms involving ω vanish, and the h-terms are equal. Hence, we indirectly observe that the MRS and APS eta invariants are also equal.

5.2 Ingredients of the proof

5.2.1 A heat kernel formula

For a summary of basic facts concerning heat kernels, including existence and asymptotic expansion for Dirac operators associated to Dirac bundles with bounded geometry, see Appendix C . Since end-periodic manifolds with end-periodic Riemannian metrics and end-periodic Dirac bundles clearly have bounded geometry, all of those facts hold for the situation under consideration.

The proof of the index theorem proceeds by analysing the heat kernels of operators of the form $D^m e^{-tD^2}$ on Z_∞. Given the heat kernel $K_t(x,y)$ for e^{-tD^2}, the heat kernel for $D^m e^{-tD^2}$ is simply

$$D_x^m K_t(x,y),$$

where D_x is the Dirac operator in the x-variable.

In the original proof of the APS index theorem, there is an explicit formula for the heat kernel of e^{-tD^2} on the half cylinder $\mathbb{R}_{\geqslant 0} \times Y$, with the boundary condition. In the end-periodic setting there is no such formula. However, the heat kernel on $\widetilde{X}$ can be related to the heat kernel on X via the following proposition:

Proposition 5.2.1 (Proposition 2.5, [MRS16])**.** *Let $\widetilde{K}$ be the heat kernel for an operator of the form $D^m e^{-tD^2}$ on $\widetilde{X}$, and let K be the kernel of the corresponding operator on X. Then*

$$\widetilde{K}(t;x,y) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\theta f(x)} K_\theta(t;p(x),p(y)) e^{-i\theta f(y)}\, d\theta,$$

where K_θ is the kernel for $D_\theta(X)^m e^{-tD_\theta(X)^2}$.

Sketch proof. We will check the defining properties of the heat kernel (Definition C.2.1), and the theorem will follow from uniqueness (Theorem C.2.3). Foremost among these are

$$(\partial_t + D_x^2)\widetilde{K}(t;x,y) = 0$$

and

$$\lim_{t\to 0}\int_{\widetilde{X}} \widetilde{K}(t;x,y)s(y)\,dy \to D_x^m s(x), \tag{5.1}$$

where s is any compactly supported smooth section. First,

$$\begin{aligned}
&\partial_t\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta f(x)} K_\theta(t;p(x),p(y)) e^{-i\theta f(y)}\, d\theta\right)\\
=&\frac{-1}{2\pi}\int_0^{2\pi} e^{i\theta f(x)} D_{\theta,x}^+(X)^2 K_\theta(t;p(x),p(y)) e^{-i\theta f(y)}\, d\theta\\
=&\frac{-1}{2\pi}\int_0^{2\pi} D_x^+(X)^2\left(e^{i\theta f(x)} K_\theta(t;p(x),p(y)) e^{-i\theta f(y)}\right) d\theta\\
=&-D_x^+(X)^2\left(\frac{1}{2\pi}\int_0^{2\pi} e^{i\theta f(x)} K_\theta(t;p(x),p(y)) e^{-i\theta f(y)}\, d\theta\right).
\end{aligned}$$

The differentiation under the integral is valid, since the family K_θ of heat kernels is smooth in θ [BGV04, Theorem 2.48].

Next, note that both sides of (5.1) are linear under addition of sections. By using a partition of unity, it therefore suffices to consider the case where s has compact support in some region small enough to be identified with a subset of the manifold X. Then

$$\begin{aligned}
&\int_{\widetilde{X}} \frac{1}{2\pi} \int_0^{2\pi} e^{i\theta f(x)} K_\theta(t; p(x), p(y)) e^{-i\theta f(y)}\, d\theta\, s(y)\, dy \\
=&\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta f(x)} \int_{\widetilde{X}} K_\theta(t; p(x), p(y)) e^{-i\theta f(y)} s(y)\, dy\, d\theta \\
\xrightarrow{(t\to 0)}&\frac{1}{2\pi} \int_0^{2\pi} e^{i\theta f(x)} D^+_{\theta,x}(X)^m (e^{-i\theta f(x)} s(x))\, d\theta \\
=&\frac{1}{2\pi} \int_0^{2\pi} D^+_x(X)^m \left(e^{i\theta f(x)} e^{-i\theta f(x)} s(x)\right)\, d\theta \\
=&D^+_x(X)^m s(x),
\end{aligned}$$

where we have assumed we may pass the limit as $t \to 0$ under the integral. Note that when we apply the limit property of the heat kernel K_θ, we are identifying $e^{-i\theta f(y)} s(y)$ with a section over X, using the assumption on the support of s.

Since the manifold $\widetilde{X}$ is not compact, we also need estimates on the heat kernel. For any $T > 0$, we require a constant $C > 0$ such that

$$\left| \frac{\partial^i}{\partial t^i} \nabla^j_x \nabla^k_y \widetilde{K}(t; x, y) \right| \leqslant C t^{-n/2-i-j-k} e^{-d(x,y)^2/4t} \tag{5.2}$$

for all $t \in (0, T]$ and $0 \leqslant i, j, k \leqslant 1$, where C depends only on T. Such estimates are automatic for heat kernels on compact manifolds, but must be taken as part of the definition for heat kernels on non-compact manifolds. The estimates will follow from the corresponding heat kernel estimates for the K_θ on the compact manifold X. That these estimates hold locally uniformly in θ can be observed during the heat kernel construction proof—recall the construction proceeds by building an approximate heat kernel H_ℓ, defining $R_\ell = (\partial_t + D_x^2) H_\ell$, and then

$$K = H_\ell - \left(\sum_{N=1}^{\infty} (-1)^{N+1} R_\ell^{*N} \right) * H_\ell,$$

see Appendix C.2 for details. The heat kernel estimates then come from estimates on R_ℓ and the above convolution formula, as in [BGM71, page 212] or [Don79, page 489]. Berline, Getzler and Vergne [BGV04, page 99] remark that for a smooth family of Dirac operators depending on a parameter θ, these estimates for the R_ℓ hold locally uniformly in θ, and hence so do the estimates for the heat kernels. Since θ ranges over the compact space $[0, 2\pi]$, the estimates hold uniformly in θ. This allows one to differentiate under the integral and apply the estimates for the heat kernel on a compact manifold.

In a similar manner one can justify the passage of the limit as $t \to 0$ under the integral above. This involves similarly noting that the limits as $t \to 0$ of $\int_X K_\theta(t; x, y)s(y)\, dy$ are uniform in θ, as can be observed during the heat kernel construction proof. □

Corollary 5.2.2. *The kernel $\widetilde{K}$ is translation invariant, in the sense that*

$$\widetilde{K}_t(x, y) = \widetilde{K}_t(x + n, y + n)$$

for any $n \in \mathbb{Z}$.

5.2.2 The regularised trace

On a compact manifold, in order to prove the index theorem via heat kernel methods one must integrate the supertrace of the heat kernel. The integral is well-defined since the kernel is smooth and the manifold is compact, but the method generally fails for non-compact manifolds. To account for this failure on end-periodic manifolds, MRS introduce a 'regularised trace', which is essentially the b-trace of Melrose [Mel93, Equation (In.19)].[1]

We denote $Z_{\leqslant N} = Z \cup W_0 \cup \cdots \cup W_{N-1} = f^{-1}(-\infty, N] \subset Z_\infty$.

Theorem 5.2.3 ([MRS16])**.** *Let K be the smoothing kernel of an operator of the form $D^m e^{-tD^2}$ on Z_∞, and $\widetilde{K}$ the kernel for the corresponding operator on the cover $\widetilde{X}$. Then the limit*

$$\mathrm{Tr}_s^\flat(D^m e^{-tD^2}) = \lim_{N\to\infty} \left(\int_{Z_{\leqslant N}} \mathrm{tr}_s(K_t(x, x))\, dx - N \int_{W_0} \mathrm{tr}_s(\widetilde{K}_t(x, x))\, dx \right)$$

exists, and is called the regularised supertrace *of $D^m e^{-tD^2}$.*

The idea is intuitively very simple. As we move further and further away from the compact piece Z, the end-periodic manifold looks locally more and more like the cover $\widetilde{X}$. So for finite time t, the local heat transfer on Z_∞ far away from Z is basically indistinguishable from the local heat transfer on $\widetilde{X}$. Since it is this 'far away' heat transfer which causes the integral of the

[1] To see this, make the transformation $t = \ln(x)$ in Melrose's definition.

supertrace not to converge, we can correct for it by subtracting off the heat transfer on $\widetilde{X}$. Of course, a heat kernel estimate is required to make this intuition rigorous, and this is by no means easy to establish.

Lemma 5.2.4 (Corollary 10.8, Remark 10.10 [MRS16]). *Let K and $\widetilde{K}$ be as in Theorem 5.2.3. Then for any $T > 0$ there are $\gamma, C > 0$ such that*

$$|K_t(x,x) - \widetilde{K}_t(x,x)| \leqslant Ce^{-\gamma d(x,W_0)^2/t}$$

for all $t \in (0,T]$ and $x \in Z_{\geqslant 1}$.

So as x moves further away from Z, the difference between the heat kernels on the diagonal decreases at an exponential rate. This makes proving the convergence of the regularised trace rather easy.

Proof of theorem 5.2.3. Ignoring the integral over Z, we have the difference

$$\int_{W_0\cup\cdots\cup W_{N-1}} \mathrm{tr}_s(K_t(x,x))\,dx - \int_{W_0\cup\cdots\cup W_{N-1}} \mathrm{tr}_s(\widetilde{K}_t(x,x))\,dx,$$

by translation invariance of the kernel $\widetilde{K}$. This is equal to

$$\sum_{k=0}^{N-1} \int_{W_k} \mathrm{tr}_s(K_t(x,x) - \widetilde{K}_t(x,x))\,dx, \tag{5.3}$$

the absolute value of which is bounded by[2]

$$C\sum_{k=0}^{N-1} \int_{W_k} |K_t(x,x) - \widetilde{K}_t(x,x)|\,dx.$$

Using the heat kernel estimate, this is bounded by

$$C'\sum_{k=0}^{N-1} \mathrm{Vol}(X)e^{-\gamma k^2/t}.$$

Hence the sum converges absolutely, and the limit as $N \to \infty$ of (5.3) exists. Note that the convergence is uniform in $t \in (0,T]$, for any $T > 0$. □

[2]Here end-periodicity is used to uniformly bound $|\mathrm{tr}(s)|$ by $|s|$, where $s \in \Gamma(S \otimes S^*)$.

5.3 Proof outline

We will only account for the proof of the Fredholm case; for the modifications needed in the non-Fredholm case see Section 7 of [MRS16].

Consider the regularised supertrace

$$\mathrm{Tr}_s^\flat(e^{-tD^2}) = \mathrm{Tr}^\flat(e^{-tD^-D^+}) - \mathrm{Tr}^\flat(e^{-tD^+D^-}),$$

of operators on Z_∞. For closed manifolds, the supertrace of the heat operator tends to the index of the operator as $t \to \infty$, and the integral of the index form as $t \to 0$. The analogous results continue to hold for the regularised supertrace.

Proposition 5.3.1 (Propositions 5.2 and 5.3, [MRS16])**.**

$$\lim_{t\to 0} \mathrm{Tr}_s^\flat(e^{-tD^-D^+}) = \int_Z \mathbf{I}(D^+(Z)),$$

and

$$\lim_{t\to \infty} \mathrm{Tr}_s^\flat(e^{-tD^-D^+}) = \mathrm{Ind}(D^+(Z_\infty)).$$

Sketch proof. We will only look at the limit as $t \to 0$. Consider

$$\int_{Z_{\leqslant N}} \mathrm{tr}_s(K_t(x,x))\,dx - N\int_{W_0} \mathrm{tr}_s(\widetilde{K}_t(x,x))\,dx.$$

As $t \to 0$ this approaches

$$\int_{Z_{\leqslant N}} \mathbf{I}(D^+(Z_\infty)) - N\int_{W_0} \mathbf{I}(D^+(\widetilde{X})),$$

as in the usual proof of the local index theorem. Since the index form is locally determined by D, and $D^+(Z_\infty)$ is isomorphic to $D^+(\widetilde{X})$ over the periodic end, this is simply $\int_Z \mathbf{I}(D^+(Z))$. It remains to justify swapping the limit $t \to 0$ with the limit $N \to \infty$ in the definition of the regularised trace. This follows from the comment at the end of the proof of Theorem 5.2.3, that the convergence of the regularised trace is uniform in $t \in (0, T]$ for any $T > 0$. □

For closed manifolds, the supertrace of the heat operator is constant, so interpolating between $t = 0$ and $t = \infty$ gives the index theorem. In the current situation,

$$\frac{d}{dt}\mathrm{Tr}_s^\flat(e^{-tD^2}) = -\mathrm{Tr}^\flat[D^-, D^+e^{-tD^-D^+}].$$

The regularised trace does not vanish on commutators, and integrating the above from $t = 0$ to $t = \infty$ and applying Proposition 5.3.1 gives

$$\operatorname{Ind}(D^+(Z_\infty)) - \int_Z \mathbf{I}(D^+(Z)) = -\int_0^\infty \operatorname{Tr}^\flat[D^-, D^+ e^{-tD^-D^+}].$$

It remains to show that the term on the right hand side reduces to the correction terms in the index theorem. This is a huge analytic effort, involving numerous heat kernel estimates. We will give a brief outline of the main steps in proving the reduction to the eta invariant.

An intermediate goal is to show that

$$\begin{aligned} -\operatorname{Tr}^\flat[D^-, D^+ e^{-tD^-D^+}] &= \frac{1}{2\pi}\frac{d}{dt}\int_0^{2\pi}\int_{W_0} f \cdot \operatorname{tr}_s\left(K_\theta(t;x,x)\right)\,dx\,d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \operatorname{Tr}\left(c(df)\,D_\theta^+ \exp(-tD_\theta^- D_\theta^+)\right)\,d\theta. \qquad (5.4)\end{aligned}$$

The second term on the right hand side is the familiar eta integrand, and integrating this from $t = 0$ to $t = \infty$ gives the end-periodic eta invariant. Integrating the first term on the right hand side gives the $t \to \infty$ limit of the integral minus the $t \to 0$ limit of the integral. The $t \to \infty$ limit vanishes since the kernels converge uniformly to the projection onto the kernels of $D_\theta^\pm$, which are assumed trivial. The $t \to 0$ limit reduces to $\int_Y \omega - \int_X df \wedge \omega$.

The hardest part of the proof of the index theorem (aside from proving the ubiquitous heat kernel estimates) is establishing formula (5.4). It is enough to prove this formula for the commutator $[P, Q]$, where $P = D^- e^{-sD^+D^-}$ and $Q = D^+ e^{-tD^-D^+}$ on Z_∞, and take the limit $s \to 0$. The first step is to use Proposition 5.2.1 to write

$$\operatorname{Tr}^\flat([P,Q]) = \lim_{N\to\infty}\left(\int_{Z_{\leqslant N}} \operatorname{tr}(K_{[P,Q]}(x,x))\,dx - \frac{N}{2\pi}\int_0^{2\pi} \operatorname{Tr}([P,Q]_\theta)\,d\theta\right).$$

Since $[P,Q]_\theta = [P_\theta, Q_\theta]$, where for example $P_\theta = D_\theta^- e^{-tD_\theta^+ D_\theta^-}$, the term on the right is zero since the standard trace vanishes on commutators. This leaves

$$\operatorname{Tr}^\flat([P,Q]) = \lim_{N\to\infty}\int_{Z_{\leqslant N}} \operatorname{tr}(K_{[P,Q]}(x,x))\,dx. \qquad (5.5)$$

Now, as for compact manifolds one has

$$K_{PQ}(x,x) = \int_{Z_\infty} K_P(x,y) K_Q(y,x)\,dy$$

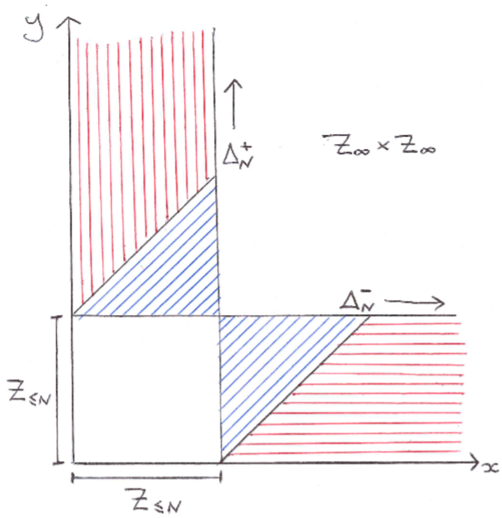

Figure 5.1: A sketch of the regions of integration.

and

$$K_{QP}(x,x) = \int_{Z_\infty} K_Q(x,y)K_P(y,x)\,dy.$$

Using these, and the fact that the trace tr vanishes on commutators, equation (5.5) reduces to

$$\int_{Z_{\leqslant N}} \mathrm{tr}(K_{[P,Q]}(x,x))\,dx = \iint_{\Delta_N^+} \mathrm{tr}(K_P(x,y)K_Q(y,x))\,dx\,dy - \iint_{\Delta_N^-} \mathrm{tr}(K_Q(x,y)K_P(y,x))\,dx\,dy, \quad (5.6)$$

where $\Delta_N^+ = Z_{\leqslant N} \times (Z_\infty \backslash Z_{\leqslant N})$, and $\Delta_N^- = (Z_\infty \backslash Z_{\leqslant N}) \times Z_{\leqslant N}$. These regions are sketched in Figure 5.1, where they have been further divided into diagonally and horizontally/vertically shaded regions—these are respectively blue and red in the colour version of this document. Now, here is the intuition for the situation:

1. The red regions (horizontally/vertically shaded) move further and further away from the diagonal as $N \to \infty$. Since the heat kernel decays exponentially as (x,y) moves away from the diagonal (see the estimate (5.2)), the trace contributions from these regions decay exponentially in the limit, and can therefore be ignored in equation (5.6).

2. The blue regions (diagonally shaded) stay near the diagonal, but move

further and further from Z in both the x and y variables as $N \to \infty$.[3]

3. Near the diagonal and far away from Z, the heat kernel on $\widetilde{X}$ very closely approximates the heat kernel on Z_∞. This and the above two points allow one to *replace* the traces in equation (5.6) with traces of heat kernels on $\widetilde{X}$. One eventually arrives at

$$\mathrm{Tr}^\flat([P,Q]) = -\sum_{m=-\infty}^{\infty} m \iint_{W_0 \times W_0} \mathrm{tr}(\widetilde{K}_P(x+m,y)\widetilde{K}_Q(y,x+m)\,dx\,dy,$$

where the kernels are for the corresponding operators on $\widetilde{X}$ [MRS16, equation (20)].

Once the regularised trace of the commutator has been expressed in terms of the kernels of operators on $\widetilde{X}$, Proposition 5.2.1 can then be applied to express the trace in terms of the kernels operators P_θ and Q_θ on X. Some more calculating eventually yields equation (5.4).

5.4 Applications to positive scalar curvature

In order to study applications to positive scalar curvature, an index vanishing result for end-periodic spin manifolds with positive scalar curvature is required. The results in Section 2 of [GL83] imply the following.

Theorem 5.4.1 (Section 2, [GL83])**.** *Let Z_∞ be an end-periodic spin manifold with an end-periodic metric of positive scalar curvature and end-periodic spin Dirac operator $D^+(Z_\infty)$. Then $D^+(Z_\infty)$ is Fredholm and*

$$\mathrm{Ind}(D^+(Z_\infty)) = 0.$$

The proof uses the Lichnerowicz formula in the same way as for the compact case (Theorem B.7.1), although one must worry about whether everything stays in L^2. In particular, the integration by parts must be carefully justified.

Recall that to study path components of the moduli space $\mathcal{M}^+(M)$ in the odd-dimensional case, we showed that the manifold $[0,1] \times M$ can be equipped with a metric $c\,dt^2 + g_t$ of positive scalar curvature, for any path g_t in the space $\mathcal{R}^+(M)$ of positive scalar curvature metrics. This gave us a

[3]Strictly speaking, the parts of the blue region closest to the 'origin' of the diagram have x or y coordinates near Z, but in the limit $N \to \infty$ these do not contribute for the exact same reason as the red region.

geometric realisation of paths in the moduli space, to which we could then apply the APS index theorem and thereby obtain results on rho invariants. The following result is the analogous construction in the end-periodic case.

Theorem 5.4.2 (Theorem 9.1, [MRS16]). *Let g_t, $t \in [0,1]$, be a path in the space $\mathcal{R}^+(X)$ of positive scalar curvature metrics on X. Then the manifold $\widetilde{X}$ can be given a metric of positive scalar curvature g such that:*

1. *$g = p^* g_0$ on $\widetilde{X}_{\leqslant 0}$, and*
2. *$g = p^* g_1$ on $\widetilde{X}_{\geqslant N}$ for some $N \in \mathbb{N}$,*

where $p : \widetilde{X} \to X$ is the projection.

Thus, if we consider $\widetilde{X}$ as an end-periodic manifold with two ends, then g is an end-periodic metric of positive scalar curvature on $\widetilde{X}$.

Proof. Let $Y \subset X$ be Poincaré dual to γ, and choose a neighbourhood U of Y diffeomorphic to $[-1,1] \times Y$. Let $\beta : [-1,1] \to [0,1]$ be a smooth function with $\beta(u) = 0$ near -1 and $\beta(u) = 1$ near 1.

We claim that for any g_{t_0}, there is a neighbourhood of t_0 in $[0,1]$ such that the metric

$$(1 - \beta(u))g_{t_0} + \beta(u)g_t$$

has positive scalar curvature on U for all t in the neighbourhood. To see this, note that:

(a) The map $t \mapsto (1 - \beta(u))g_{t_0} + \beta(u)g_t$ is continuous $[0,1] \to \mathcal{R}(U)$, and

(b) Since U is compact, $\mathcal{R}^+(U) \subset \mathcal{R}(U)$ is an open subset.

By fact (b) $g_{t_0}|_U$ has a neighbourhood in $\mathcal{R}(U)$ consisting of positive scalar curvature metrics. By fact (a), the metric $(1 - \beta(u))g_{t_0} + \beta(u)g_t$ will be in this chosen neighbourhood of g_{t_0} for t sufficiently near t_0. This establishes the claim.

By compactness of the unit interval $[0,1]$, we can find finitely many $0 = t_0 < t_1 < \cdots < t_N = 1$ such that the metric $(1 - \beta(u))g_{t_i} + \beta(u)g_{t_{i+1}}$ has positive scalar curvature on U for each i. These metrics agree with the g_{t_i} near the boundary of U, so we can therefore construct a metric on $\widetilde{X}$ which interpolates between g_0 and g_1 using $g_{t_0}, \ldots, g_{t_N}$ on each component $W_0, \ldots, W_N$, gluing consecutive metrics over $[-1,1] \times Y$ as above. □

Let π be a discrete group, $P \to Z_\infty$ an end-periodic principal π-bundle, and $\sigma_1, \sigma_2 : \pi \to U(N)$ unitary representations of π. We twist the operator $D^+(Z_\infty)$ by σ_1, σ_2 and the bundle P to obtain two end-periodic twisted Dirac operators $D_1^+(Z_\infty)$ and $D_2^+(Z_\infty)$. Assuming the families associated to these operators have discrete spectral sets, we define the *end-periodic rho invariant* associated to σ_1, σ_2, P and D to be

$$\rho^{\text{ep}}(\sigma_1, \sigma_2; D, P) = \frac{\eta^{\text{ep}}(D_1^+(X)) + h_1}{2} - \frac{\eta^{\text{ep}}(D_2^+(X)) + h_2}{2}.$$

Now, suppose we have two metric g_0 and g_1 of positive scalar curvature on X that are homotopic through PSC metrics via g_t. By the above theorem we get an end-periodic manifold $\widetilde{X}$ with two ends, having positive scalar curvature, such that one end is modelled on (X, g_0) and the other on (X, g_1). Suppose also that we have an end-periodic principal π-bundle $P \to \widetilde{X}$ and two unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, from which we form the twisted Dirac operators $D_1^+(\widetilde{X})$ and $D_2^+(\widetilde{X})$. Applying the MRS index theorem to these operators, we get

$$0 = \int_Z \mathbf{I}(D_i^+(X)) - \int_Y \omega_i + \int_X df \wedge \omega_i - \frac{\eta^{\text{ep}}(D_i^+(X)) + h_i}{2}$$

for $i = 1, 2$, by Theorem 5.4.1 which applies equally well to spin Dirac operators twisted by flat bundles. Now the index forms of $D_1^+(\widetilde{X})$ and $D_2^+(\widetilde{X})$ agree, since they are both twists of $D^+(\widetilde{X})$ by flat unitary bundles of the same dimension. We may therefore assume $\omega_1 = \omega_2$. Subtracting the two equations then yields:

Theorem 5.4.3 ([MRS16], Theorem 9.1)**.** *Let X be a spin manifold with fixed spin structure, and g_1, g_2 be metrics of positive scalar curvature on X that are homotopic through metrics of positive scalar curvature. Let $\widetilde{X}$ be the end-periodic manifold of PSC described above, modelled on $(X, g_0) \amalg (X, g_1)$. Then for any end-periodic principal π-bundle $P \to \widetilde{X}$ and unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$,*

$$\rho^{\text{ep}}(\sigma_1, \sigma_2; D^+(X, g_0), P) = \rho^{\text{ep}}(\sigma_1, \sigma_2; D^+(X, g_1), P).$$

As for the odd-dimensional case, we now have a systematic approach for detecting distinct path components of the moduli space of PSC metrics for even-dimensional manifolds: if two metrics of PSC have differing rho invariants, then they lie in distinct path components of the moduli space. The question then remains, how exactly can one actually calculate the end-periodic rho invariants?

Theorem 5.4.4 ([MRS16], Proposition 8.5). *Let $D^+(Z_\infty)$, σ_1, σ_2 and P be as above. Assume the families associated to the operators $D_1^+(Z_\infty)$ and $D_2^+(Z_\infty)$ have discrete spectral sets. Denote by A the Dirac operator on the manifold Y. Then*

1. $$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2; D^+(X), P) = \rho(\sigma_1, \sigma_2; A, P|_Y) \mod \mathbb{Z}$$

2. *If X is spin, both g and g_Y have positive scalar curvature, and the metric on X is a product $dt^2 + g_Y$ in a product neighbourhood of Y, then*
$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2; D^+(X), P) = \rho(\sigma_1, \sigma_2; A, P|_Y),$$
and the equality is of real numbers.

Proof. Apply the MRS index theorem to the twisted operators on the end-periodic manifold $(Y \times \mathbb{R}_{\leqslant 0}) \cup W_0 \cup \widetilde{X}_{\geqslant 0}$ with two ends. This gives us equations

$$\begin{aligned} \operatorname{Ind}(D_i^+(Z_\infty)) = \int_{W_0} \mathbf{I}(D_i^+(X)) - \int_Y \omega_i + \int_X df \wedge \omega_i \\ - \frac{\eta^{\mathrm{ep}}(D_i^+(X \amalg -S^1 \times Y)) + h_i}{2}, \end{aligned}$$

for $i = 1, 2$. The index forms of the twisted operators are the same, so subtracting these equations gives that the rho invariant is an integer, proving the first point.

For the second, note that since the metric is a product metric in a neighbourhood of Y, it extends to $(Y \times \mathbb{R}_{\leqslant 0}) \cup W_0 \cup \widetilde{X}_{\geqslant 0}$. Applying the MRS index theorem along with Theorem 5.4.1 gives equality of the rho invariants. □

Mrowka, Ruberman and Saveliev then apply all of the above theory to deduce the following result:

Theorem 5.4.5 (Theorem 9.2 [MRS16]). *Let Y compact connected spin manifold of dimension $4k - 1$ for $k > 1$, with non-trivial finite fundamental group $\pi_1(Y)$. Let M be a compact connected spin manifold of dimension $4k$. If both Y and M admit metrics of positive scalar curvature then the moduli space $\mathcal{M}^+((S^1 \times Y)\#M)$ has infinitely many path components.*

Proof. The manifold $X = (S^1 \times Y)\#M$ has the salient feature that any metric of PSC on Y extends to a metric of PSC on X. This is a consequence of Proposition 2.1.8 and the PSC surgery theory of Gromov-Lawson [GL80] and Schoen-Yau [SY79b], which in particular implies that the connected sum of manifolds with PSC metrics also admits a PSC metric. The work of Botvinnik-Gilkey [BG95], summarised in Theorem 3.3.5, implies that Y admits uncountably many metrics of PSC with differing rho invariants. Theorem 5.4.4, proved by [MRS16], implies that the rho invariants of X agree with those on Y. Finally Theorem 5.4.3, also proved by [MRS16], gives that each of the PSC metrics on X lies in a different component of the moduli space $\mathcal{M}^+(X)$. □

Thus, there is a certain correspondence between the MRS and the APS situations, which allows us to transfer results for odd-dimensional manifolds to even-dimensional manifolds. The aim of the paper in the next chapter is to determine just how deep this correspondence goes.

Chapter 6

End-periodic K-homology and spin bordism

What follows is the preprint [HM17] with my supervisor V. Mathai, and constitutes the new work accomplished in this thesis. In it, we construct new variants of geometric K-homology and spin bordism that are suited to manifolds with periodic ends. The invariance theorems for rho invariants mentioned in Chapter 3 extend to the new setting, through use of the end-periodic index theorem. The theory provides a conceptual framework for the methods of [MRS16], which allow one to transfer results on PSC for odd-dimensional manifolds to even-dimensional manifolds.

The paper was written before much of the preceding thesis. Consequently there is some inevitable overlap, and some unfortunate differences in notation. Since the paper is self-contained, these should not cause too much grief for the reader. One notable difference is the use of Taubes' convention for the Fourier-Laplace transform, rather than what is in Chapter 4.

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Certification:	This paper reports on original research I conducted during the period of my Higher Degree by Research candidature and is not subject to any obligations or contractual agreements with a third party that would constrain its inclusion in this thesis. I am the primary author of this paper.		
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Co-Author Contributions

By signing the Statement of Authorship, each author certifies that:

i. the candidate's stated contribution to the publication is accurate (as detailed above);

ii. permission is granted for the candidate in include the publication in the thesis; and

iii. the sum of all co-author contributions is equal to 100% less the candidate's stated contribution.

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Contribution to the Paper			
Signature		Date	

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POSITIVE SCALAR CURVATURE METRICS VIA END-PERIODIC MANIFOLDS

MICHAEL HALLAM AND VARGHESE MATHAI

Abstract. We obtain two types of results on positive scalar curvature metrics for compact spin manifolds that are even dimensional. The first type of result are obstructions to the existence of positive scalar curvature metrics on such manifolds, expressed in terms of end-periodic eta invariants that were defined by Mrowka-Ruberman-Saveliev [28]. These results are the even dimensional analogs of the results by Higson-Roe [20]. The second type of result studies the number of path components of the space of positive scalar curvature metrics modulo diffeomorphism for compact spin manifolds that are even dimensional, whenever this space is non-empty. These extend and refine certain results in Botvinnik-Gilkey [12] and also [28]. End-periodic analogs of K-homology and bordism theory are defined and are utilised to prove many of our results.

Contents

2010 *Mathematics Subject Classification.* Primary 58J28; Secondary 19K33, 19K56, 53C21.
Key words and phrases. positive scalar curvature metrics, maximal Baum-Connes conjecture, end-periodic manifolds, end-periodic eta invariant, vanishing theorems, end-periodic K-homology, end-periodic bordism.

1. Introduction

Eta invariants were originally introduced by Atiyah, Patodi and Singer [2, 3, 4] as a correction term appearing in an index theorem for manifolds with odd-dimensional boundary. The eta invariant itself is a rather sensitive object, being defined in terms of the spectrum of a Dirac operator. However, when one considers the *relative* eta invariant (or *rho invariant*), defined by twisting the Dirac operator by a pair of flat vector bundles and subtracting the resulting eta invariants, many marvellous invariance properties emerge. For example, Atiyah, Patodi and Singer showed that the mod $\mathbb{Z}$ reduction of the relative eta invariant of the signature operator is in fact independent of the choice of Riemannian metric on the manifold. Key to the approach is their index theorem for even dimensional manifolds with global boundary conditions, which they show is equivalent to studying manifolds with cylindrical ends and imposing (weighted) L^2 decay conditions.

The links between eta invariants and metrics of positive scalar curvature metrics have been studied using different approaches by Mathai [24, 25], Keswani [22] and Weinberger [36]. A conceptual proof of the approach by Keswani, was achieved in the paper by Higson-Roe [20] using K-homology; see also the recent papers by Deeley-Goffeng [13], Benameur-Mathai [6, 7, 8] and Piazza-Schick [32, 31].

Our goal in this paper to use the results of Mrowka-Ruberman-Saveliev [28] instead of those by Atiyah-Patodi-Singer [2]. Manifolds with cylindrical ends studied in [2] are special cases of end-periodic manifolds studied in [28]. More precisely, let Z be a compact manifold with boundary Y and suppose that Y is a connected submanifold of a compact oriented manifold X that is Poincaré dual to a primitive cohomology class $\gamma \in H^1(X, \mathbb{Z})$. Let W be the fundamental segment obtained by cutting X open along Y (Figure 1).

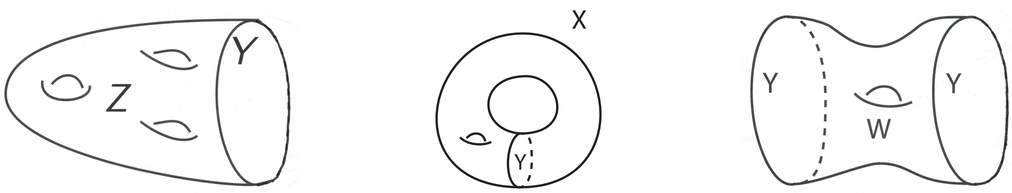

Figure 1. Pieces of an end-periodic manifold

If W_k are isometric copies of W, then we can attach $X_1 = \bigcup_{k\geq 0} W_k$ to the boundary component Y of Z, forming the *end-periodic manifold* Z_∞ (Figure 2). Often in the paper, we also deal with manifolds with more than one periodic end.

The motivations for considering such manifolds are from gauge theory; it was Taubes [35] who originally developed the analysis of end-periodic elliptic operators on end-periodic manifolds, and successfully calculated the index of the end-periodic anti-self dual operator in Yang-Mills theory.

We adapt the results by Higson-Roe [20], using end-periodic K-homology, to obtain obstructions to the existence of positive scalar curvature metrics in terms of end-periodic eta invariants (see section 3) that were defined by Mrowka-Ruberman-Saveliev [28] for even dimensional manifolds, using the b-trace approach of Melrose [26]. This is established in section 6. Roughly speaking, end-periodic K-homology is an analog of geometric K-homology, where

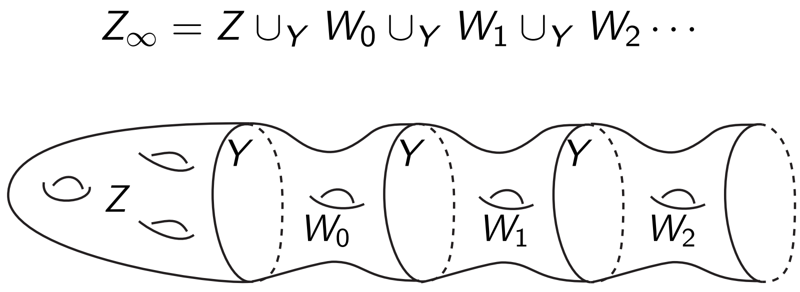

FIGURE 2. End-periodic manifold

the representatives have in addition, a choice of degree 1 cohomology class determining the codimension 1 submanifold. It is defined and studied in Section 2.

We also adapt the results by Botvinnik-Gilkey [12], using end-periodic bordism, to obtain results on the number of components of the moduli space of Riemannian metrics of positive scalar curvature metrics in terms of end-periodic eta invariants, generalising certain results Mrowka-Ruberman-Saveliev [28] for even dimensional manifolds. End-periodic bordism is defined and studied in Section 4.

In Section 5 we define the end-periodic analogues of the structure groups of Higson and Roe, and study the end-periodic rho invariant on these groups.

Section 6 contains the applications to positive scalar curvature, using the established end-periodic K-theory and end-periodic spin bordism of the previous sections.

In Section 7 we give a proof of the vanishing of the end-periodic rho invariant of the twisted Dirac operator with coefficients in a flat Hermitian vector bundle on a compact even dimensional Riemannian spin manifold X of positive scalar curvature using the representation variety of $\pi_1(X)$.

It seems to be a general theme that for any geometrically defined homology theory, there is an analogous theory tailored to the setting of end-periodic manifolds, and that this end-periodic theory is isomorphic to the original geometric theory in a natural way. These isomorphisms are built on the foundation of Poincaré duality.

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2. END-PERIODIC K-HOMOLOGY

2.1. **Review of K-homology.** We begin by reviewing the definition of K-homology of Baum and Douglas [5], using the (M, S, f)-formulation introduced by Keswani [22], and used by Higson and Roe [20].

Definition 2.1. A *K-cycle* for a discrete group π is a triple (M, S, f), where M is a compact oriented odd-dimensional Riemannian manifold, S is a smooth Hermitian bundle over M with Clifford multiplication $c : TM \to \mathrm{End}(S)$, and $f : M \to B\pi$ is a continuous map to the classifying space of π.

Such a bundle S with the above data is called a *Dirac bundle*. We remark that M may be disconnected, and that its connected components are permitted to have different odd dimensions.

Definition 2.2. Two K-cycles (M, S, f) and (M', S', f') for $B\pi$ are said to be *isomorphic* if there is an orientation preserving diffeomorphism $\varphi : M \to M'$ covered by an isometric bundle isomorphism $\psi : S \to S'$ such that

$$\psi \circ c_M(v) = c_{M'}(\varphi_* v) \circ \psi$$

for all $v \in TM$, and such that $f' \circ \varphi = f$.

A *Dirac operator* for the cycle (M, S, f) is any first order linear partial differential operator D acting on smooth sections of S whose principal symbol is the Clifford multiplication. That is to say, for any smooth function $\phi : M \to \mathbb{R}$ one has

$$[D, \phi] = c(\operatorname{grad} \phi) : \Gamma(S) \to \Gamma(S).$$

The K-homology group $K_1(B\pi)$ will consist of geometric K-cycles for π modulo an equivalence relation, which we will now describe.

Definition 2.3. A K-cycle (M, S, f) is a *boundary* if there exists a compact oriented even-dimensional manifold W with boundary $\partial W = M$ such that:

(a) W is isometric to the Riemannian product $(0, 1] \times M$ near the boundary.

(b) There is a $\mathbb{Z}_2$-graded Dirac bundle over W that is isomorphic to $S \oplus S$ in the collar with Clifford multiplication given by

$$c_W(v) = \begin{pmatrix} 0 & c_M(v) \\ c_M(v) & 0 \end{pmatrix}, \qquad c_W(\partial_t) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

for $v \in TM$.

Remark 2.4. Our orientation convention for boundaries is the following: If W is an oriented manifold with boundary ∂W then the orientation on W at the boundary is given by the outward unit normal followed by the orientation of ∂W. The isometry in part (a) is required to be orientation preserving.

We define the *negative* of a K-cycle (M, S, f) to be $(-M, -S, f)$, where $-M$ is M with its orientation reversed, and $-S$ is S with the negative Clifford multiplication $c_{-S} = -c_S$. Two K-cycles (M, S, f) and (M', S', f') are *bordant* if the disjoint union $(M, S, f) \amalg (-M', -S', f')$ is a boundary, and we write $(M, S, f) \sim (M', S', f')$. This is the first of the relations defining K-homology; there are two more to define:

(1) *Direct sum/disjoint union*:

$$(M, S_1, f) \amalg (M, S_2, f) \sim (M, S_1 \oplus S_2, f).$$

(2) *Bundle modification*: Let (M, S, f) be a K-cycle. If P is a principal $SO(2k)$-bundle over M, we define

$$\hat{M} = P \times_\rho S^{2k}.$$

Here ρ denotes the action of $SO(2k)$ on S^{2k} given by the standard embedding of $SO(2k)$ into $SO(2k+1)$. The metric on $\hat{M}$ is any metric agreeing with that of M on horizontal tangent vectors and with that of S^{2k} on vertical tangent vectors. The map $\hat{f} : \hat{M} \to B\pi$ is the obvious one. Over S^{2k} is an $SO(2k)$-equivariant vector bundle $C\ell_\theta(S^{2k}) \subset C\ell(TS^{2k})$, defined as the $+1$ eigenspace of the *right* action by the oriented volume element θ on the Clifford bundle $C\ell(TS^{2k})$. The $SO(2k)$-equivariance of this bundle implies that it lifts to a well defined bundle over $\hat{M}$. We thus define the bundle

$$\hat{S} = S \otimes C\ell_\theta(S^{2k})$$

over $\hat{M}$. Clifford multiplication on $\hat{S}$ is given by

$$c(v) = \begin{cases} c_M(v) \otimes \epsilon & \text{if } v \text{ is horizontal,} \\ I \otimes c_{S^{2k}}(v) & \text{if } v \text{ is vertical,} \end{cases}$$

where ϵ is the grading element of the Clifford bundle over S^{2k}. The K-cycle $(\hat{M}, \hat{S}, \hat{f})$ is called an *elementary bundle modification* of (M, S, f), and we write $(M, S, f) \sim (\hat{M}, \hat{S}, \hat{f})$.

Remark 2.5. If D is a given Dirac operator for the cycle (M, S, f), then there is a preferred choice of Dirac operator for an elementary bundle modification $(\hat{M}, \hat{S}, \hat{f})$ of (M, S, f). If D_θ denotes the $SO(2k)$-equivariant Dirac operator acting on $C\ell(S^{2k})$, then the Dirac operator on $S \otimes C\ell_\theta(S^{2k})$ is

$$\hat{D} = D \otimes \epsilon + I \otimes D_\theta$$

where ϵ is the grading element of $C\ell_\theta(S^{2k})$.

Definition 2.6. The *K-homology group* $K_1(B\pi)$ is the abelian group of K-cycles modulo the equivalence relation generated by isomorphism of cycles, bordism, direct sum/disjoint union, and bundle modification. The addition of equivalence classes of K-cycles is given by disjoint union

$$(M, S, f) \amalg (M', S', f') = (M \amalg M', S \amalg S', f \amalg f').$$

One must of course check that this operation descends to a well-defined binary operation on K-homology which satisfies the group axioms. The details are straightforward.

Remark 2.7. There is another group $K_0(B\pi)$ defined in terms of even-dimensional cycles, which is well suited to the original Atiyah-Singer index theorem. We will not need it here.

2.2. **Definition of End-Periodic K-homology.** With the above definition of K-homology reviewed, we now adapt the definition to the setting of manifolds with periodic ends.

Definition 2.8. An *end-periodic K-cycle*, or simply a *K^{ep}-cycle* for a discrete group π is a quadruple (X, S, γ, f), where X is a compact oriented even-dimensional Riemannian manifold, $S = S^+ \oplus S^-$ is a $\mathbb{Z}_2$-graded Dirac bundle over X, $\gamma \in H^1(X, \mathbb{Z})$ is a cohomology class whose restriction to each connected component of X is primitive, and f is a continuous map $f : X \to B\pi$.

The $\mathbb{Z}_2$-graded structure of S includes a Clifford multiplication by tangent vectors to X which swaps the positive and negative sub-bundles. Again, the manifold X is allowed to be disconnected, with the connected components possibly having different even dimensions. Note that the definition of a K^{ep}-cycle imposes topological restrictions on X, namely each connected component of X must have non-trivial first cohomology in order for the class γ to be primitive on each component.

Definition 2.9. Two K^{ep}-cycles (X, S, γ, f) and (X', S', γ', f') are *isomorphic* if there exists an orientation preserving diffeomorphism $\varphi : X \to X'$ which is covered by a $\mathbb{Z}_2$-graded isometric bundle isomorphism $\psi : S \to S'$ such that

$$\psi \circ c_X(v) = c_{X'}(\varphi_* v) \circ \psi$$

for all $v \in TX$. The diffeomorphism φ must additionally satisfy $\varphi^*(\gamma') = \gamma$, and $f' \circ \varphi = f$.

We now define what it means for a K^{ep}-cycle (X, S, γ, f) to be a boundary. First, let $Y \subset X$ be a connected codimension-1 submanifold that is Poincaré dual to γ. The orientation of Y is such that for all closed forms α of codimension 1 (over each component of X),

$$\int_Y \iota^*(\alpha) = \int_X \gamma \wedge \alpha,$$

where $\iota : Y \to X$ is the inclusion and we abuse notation by writing γ for what is really a closed 1-form representing the cohomology class γ. In other words, the orientation of Y is such that the signs of the above two integrals always agree. Now, cut X open along Y to obtain a compact manifold W with boundary $\partial W = Y \amalg -Y$, with our boundary orientation conventions as in Remark 2.4. Glue infinitely many isometric copies W_k of W end to end along Y to obtain the complete oriented Riemannian manifold $X_1 = \bigcup_{k \geq 0} W_k$ with boundary $\partial X_1 = -Y$. Pull back the Dirac bundle S on X to get a $\mathbb{Z}_2$-graded Dirac bundle on X_1, also denoted S, and pull back the map f to get a map $f : X_1 \to B\pi$.

Definition 2.10. The K^{ep}-cycle (X, S, γ, g) is a *boundary* if there exists a compact oriented Riemannian manifold Z with boundary $\partial Z = Y$, which can be attached to X_1 along Y to form a complete oriented Riemannian manifold $Z_\infty = Z \cup_Y X_1$, such that the bundle S extends to a $\mathbb{Z}_2$-graded Dirac bundle on Z_∞ and the map f extends to a continuous map $f : Z_\infty \to B\pi$.

Remark 2.11. Being a boundary is clearly independent of the choice of Y; if Y' is another choice of submanifold Poincaré dual to γ we simply embed Y' somewhere in the periodic end of Z_∞, and take Z' to be the compact piece in Z_∞ bounded by Y'.

Definition 2.12. The manifold Z_∞ from Definition 2.10 is called an *end-periodic* manifold. It is convenient to say the end is *modelled* on (X, γ), or sometimes just X if γ is understood. Any object on Z_∞ whose restriction to the periodic end X_1 is the pullback of an object from X is called *end-periodic*. For example, the bundle S, the map f, and the metric on Z_∞ in the previous definition are all end-periodic.

Remark 2.13. We allow end-periodic manifolds to have multiple ends. This situation arises when the manifold X, on which the end of Z_∞ is modelled, is disconnected.

The *negative* of a K^{ep}-cycle (X, S, γ, f) is simply $(X, S, -\gamma, f)$. This is so that the disjoint union of a K^{ep}-cycle with its negative is a boundary—it is clear that the $\mathbb{Z}$-cover $\tilde{X}$ of X

corresponding to γ is an end-periodic manifold with end modelled on $(X \amalg X, \gamma \amalg -\gamma)$. The definitions of bordism and direct sum/disjoint union are exactly the same as before, with the class γ left unchanged. In the case of bundle modification, the class $\hat{\gamma}$ on $\hat{X} = X \times_\rho S^{2k}$ is the pullback of γ by the projection $p : \hat{X} \to X$, and we endow the tensor product bundle $S \otimes C\ell_\theta(S^{2k})$ with the standard tensor product grading of $\mathbb{Z}_2$-graded modules. There is also one more relation we define which relates the orientation on X to the one-form γ:

$$(X, S, -\gamma, f) \sim (-X, \Pi(S), \gamma, f)$$

where $-X$ is X with the reversed orientation and $\Pi(S)$ is S with its $\mathbb{Z}_2$-grading reversed. We call this relation *orientation/sign*, as it links the orientation on X to the sign of γ. The need for this relation will become apparent in (2) of the proof of Lemma 2.16.

Definition 2.14. The *end-periodic K-homology group*, $K_1^{\mathrm{ep}}(B\pi)$, is the abelian group consisting of K^{ep}-cycles up to the equivalence relation generated by isomorphism of K^{ep}-cycles, bordism, direct sum/disjoint union, bundle modification, and orientation/sign. Addition is given by disjoint union of cycles

$$(X, S, \gamma, f) \amalg (X', S', \gamma', f') = (X \amalg X', S \amalg S', \gamma \amalg \gamma', f \amalg f').$$

Remark 2.15. As for K-homology we could also define the group $K_0^{\mathrm{ep}}(B\pi)$ using odd-dimensional K^{ep}-cycles, although we will not pursue this here.

2.3. **The isomorphism.** We will now show that there is a natural isomorphism $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$.

First we describe the map $K_1(B\pi) \to K_1^{\mathrm{ep}}(B\pi)$. Let (M, S, f) be a K-cycle for $B\pi$. Define $X = S^1 \times M$ an even dimensional manifold with the product orientation and Riemannian metric, the Dirac bundle $S \oplus S \to X$ with Clifford multiplication as in (b) of Definition 2.3, $\gamma = d\theta \in H^1(X, \mathbb{Z})$ the standard generator of the first cohomology of S^1, and $f : X \to B\pi$ the extension of $f : M \to B\pi$. We map the equivalence class of (M, S, f) in $K_1(B\pi)$ to the equivalence class of $(S^1 \times M, S \oplus S, d\theta, f)$ in $K_1^{\mathrm{ep}}(\pi)$.

Lemma 2.16. *The map sending a cycle (M, S, f) to the end-periodic cycle $(S^1 \times M, S \oplus S, d\theta, f)$ descends to a well-defined map of K-homologies.*

Proof. It must be checked that each of the relations defining $K_0(B\pi)$ are preserved by this map.

(1) **Boundaries:** Let (M, S, f) be a boundary. Then we have a compact manifold W with boundary $\partial W = M$ satisfying conditions (a) and (b) in Definition 2.3. To show that $(S^1 \times M, S \oplus S, d\theta, f)$ is a boundary, we attach W to the half-cover $X_1 = \mathbb{R}_{\geq 0} \times M$ to obtain a Riemannian manifold Z_∞. Over X_1 is the bundle $S \oplus S$, and over W is a bundle isomorphic to $S \oplus S$. We use the isomorphism to glue the bundles together and define $S \oplus S$ over Z_∞. The assumptions on the Clifford multiplication imply that it extends over this bundle. Since the map f on M extends to W, the map f on $S^1 \times M$ extends to Z_∞.

(2) **Negatives:** The negative of (M, S, f) is $(-M, -S, f)$, which maps to $(-S^1 \times M, -S \oplus -S, d\theta, f)$. The negative of $(-S^1 \times M, -S \oplus -S, d\theta, f)$ is

$$(-S^1 \times M, -S \oplus -S, -d\theta, f) \sim (S^1 \times M, \Pi(-S \oplus -S), d\theta, f)$$

by the orientation/sign relation. The only difference between this cycle and $(X, S \oplus S, d\theta, f)$ is that the Clifford multiplication is negative; Clifford multiplication by vectors tangent to M has become negative and reversing the $\mathbb{Z}_2$-grading has caused ∂_θ to act negatively. This cycle is isomorphic to

$$(S^1 \times M, S \oplus S, d\theta, f)$$

via the identity map $\varphi : M \to M$ and the isometric bundle isomorphism $\psi : -S \oplus -S \to S \oplus S$, $\psi(s \oplus t) = c(\omega)(s \oplus t)$, where ω is the oriented volume element of $S^1 \times M$. Hence negatives are preserved by the mapping.

(3) **Disjoint union:** Obvious.

(4) **Bordism:** Since negatives map to negatives, boundaries map to boundaries, and disjoint union is preserved, it follows that bordism is also preserved.

(5) **Direct sum/disjoint union:** Also obvious.

(6) **Bundle modification:** Let $(\hat{M}, \hat{S}, \hat{f})$ be an elementary bundle modification for (M, S, f) associated to the principal $SO(2k)$-bundle $P \to M$. We pullback P to a bundle over $X = S^1 \times M$, and use it to construct our bundle modification $(\hat{X}, (S \oplus S)\hat{}, d\theta, f)$ of $(S^1 \times M, S \oplus S, d\theta, f)$. It is clear that $\hat{X} = S^1 \times \hat{M}$. Now $\hat{S} = S \otimes C\ell_\theta(S^{2k})$, so

$$\hat{S} \oplus \hat{S} \cong (S \oplus S) \otimes C\ell_\theta(S^{2k}) = (S \oplus S)\hat{}.$$

It is straightforward yet tedious to verify that Clifford multiplication is preserved by this isomorphism. So the K^{ep}-cycle obtained via bundle modification then mapping, is isomorphic to the K^{ep}-cycle obtained by mapping then bundle modification. □

Now for the inverse map. Let (X, S, γ, f) be an end-periodic cycle. Choose a submanifold $Y \subset X$ Poincaré dual to γ, oriented as in the paragraph after Definition 2.9. We map the cycle (X, S, γ, f) to (Y, S^+, f), where S^+ and f are restricted to Y. If ω is an oriented volume form for Y then we let ∂_t be the unit normal to Y such that $\partial_t \wedge \omega$ is the orientation on X. The Clifford multiplication on S^+ is then defined to be

$$c_Y(v) = c_X(\partial_t)c_X(v)$$

for $v \in TY$. Note that this agrees with the conventions of (b) in Definition 2.3. One easily verifies that this indeed defines a Clifford multiplication on S^+.

Lemma 2.17. *The map sending an end-periodic cycle (X, S, γ, f) to the cycle (Y, S^+, f) described above, descends to a well-defined map of K-homologies.*

Proof. We must not only check that the relations defining end-periodic K-homology are preserved, but that the class in K-homology obtained is independent of the choice of Y.

(1) **Boundaries:** Let (X, S, γ, f) be a boundary. Then there is a compact oriented manifold Z with boundary $\partial Z = Y$ over which the $\mathbb{Z}_2$-graded Dirac bundle S and map f extend. We modify the metric near the boundary of Z to make it a product. It follows that the cycle (Y, S^+, f) is a boundary.

(2) **Choice of Y:** Suppose Y_1 and Y_2 are submanifolds of X that are Poincaré dual to γ. The class γ determines a $\mathbb{Z}$-cover $\tilde{X}$ of X, and Y_1, Y_2 may be considered as submanifolds

of this cover. Since both Y_1 and Y_2 are compact, they can be embedded in $\tilde{X}$ so that they are disjoint. We delete the open subset of $\tilde{X}$ lying outside of Y_1 and Y_2, and leave only the points in $\tilde{X}$ between and including Y_1 and Y_2. We call the remaining manifold

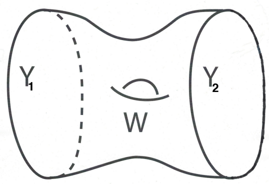

FIGURE 3. Piece of an end-periodic manifold

W; it is a compact manifold with boundary $\partial W = Y_1 \amalg -Y_2$. We pull back the bundle S and the map f to W, and modify the metric near the boundary so that it is a product. The result is that $Y_1 \amalg -Y_2$ is a boundary.

(3) **Negatives:** Reversing the sign of γ changes the orientation of Y. Clifford multiplication on Y also becomes negative, since changing the orientation on Y reverses the unit normal to Y. Hence negatives of cycles map to negatives.

(4) **Disjoint union:** Obvious.

(5) **Bordism:** Since boundaries map to boundaries, negatives map to negatives, and disjoint union is preserved, it follows that bordism is also preserved.

(6) **Direct sum/disjoint union:** Obvious.

(7) **Orientation/sign:** From (3) in this proof, the K-cycle obtained from $(X, S, -\gamma, f)$ is the negative of the cycle (Y, S^+, f). Now consider the K-cycle obtained from $(-X, \Pi(S), \gamma, f)$. Reversing the orientation on X will also reverse it on Y. Instead of S^+, we now take S^- with Clifford multiplication

$$c_{S^-}(v) = c(-\partial_t)c(v) = -c(\partial_t)c(v)$$

where $v \in TY$ and $-\partial_t$ is the unit normal to $-Y$. We now show $(-Y, S^+, f)$ and $(-Y, S^-, f)$ are isomorphic. Let ω be the oriented volume element of $+Y$ (or $-Y$, it does not matter) and define a map $\psi : S^+ \to S^-$ by $\psi(s) = c(\omega)s$. Then

$$\psi \circ c_{S^+}(v) = c_{S^-}(v) \circ \psi$$

and the cycles are therefore isomorphic.

(8) **Bundle modification:** Let $(\hat{X}, \hat{S}, \hat{\gamma}, \hat{f})$ be an elementary bundle modification for (X, S, γ, f), associated to the principal $SO(2k)$-bundle $P \to X$. We restrict this principal bundle to Y and consider the corresponding bundle modification $(\hat{Y}, \hat{S^+}, \hat{f})$ for (Y, S^+, f). It is clear that $\hat{Y} \subset \hat{X}$ is Poincaré dual to $\hat{\gamma}$. The bundle

$$\hat{S} = S \otimes C\ell_\theta(S^{2k})$$

has even part

$$\hat{S}^+ = (S^+ \otimes C\ell_\theta^+(S^{2k})) \oplus (S^- \otimes C\ell_\theta^-(S^{2k})),$$

while over $\hat{Y}$ we have the bundle

$$\hat{S^+} = S^+ \otimes C\ell_\theta(S^{2k}).$$

Identifying S^+ with S^- via the isomorphism $c(\partial_t)$, we see that $\hat{S}^+ \cong \hat{S^+}$. It is routine to check that the Clifford multiplications are preserved under this isomorphism. □

Theorem 2.18. *The above maps between K-homologies define an isomorphism of groups $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$.*

Proof. We must check that the above maps on K-homologies are inverse to each other. If we begin with a cycle (M, S, f), this maps to $(S^1 \times M, S \oplus S, d\theta, f)$. Mapping this again, we get (M, S, f) back, so this direction is easy. Now suppose we begin with a cycle (X, S, γ, f). This maps to (Y, S^+, f) which then maps to $(S^1 \times Y, S^+ \oplus S^+, d\theta, f)$. We will show this cycle is bordant to the original cycle (X, S, γ, f). Consider the half cover X_1 of X obtained using $-\gamma$. Near the boundary, this is diffeomorphic to a product $(-\delta, 0] \times Y$. The half cover of $S^1 \times Y$ obtained from $d\theta$ is $\mathbb{R}_{\geq 0} \times Y$. The two half covers clearly glue together to produce and end-periodic manifold with two ends. The Dirac bundles and maps to $B\pi$ extend over

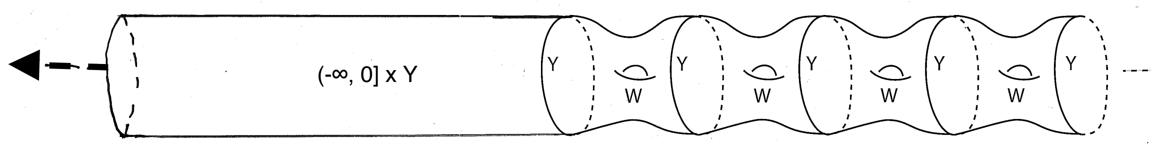

FIGURE 4. End-periodic manifold with two ends

this manifold, and hence the two cycles are bordant. □

3. Relative eta/rho invariants

In this section, we use the end-periodic eta invariant of MRS to define homomorphisms from the end-periodic K-homology group $K_1^{\mathrm{ep}}(B\pi)$ to $\mathbb{R}/\mathbb{Z}$. Any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$ will determine such a homomorphism, and we see that this homomorphism agrees with that constructed in Higson-Roe [20] under the natural isomorphism $K_1(B\pi) \cong K_1^{\mathrm{ep}}(B\pi)$.

3.1. **Rho invariant for K-homology.** Let (M, S, f) be a K-cycle. Any Dirac operator for this cycle is a self-adjoint elliptic first order operator on S, and so has a discrete spectrum of real eigenvalues. The *eta function* of this operator is defined to be the sum over the non-zero eigenvalues of D

$$\eta(s) = \sum_{\lambda \neq 0} \mathrm{sign}(\lambda)|\lambda|^{-s},$$

which converges absolutely for $\mathrm{Re}(s)$ sufficiently large. It is a theorem of Atiyah, Patodi and Singer (APS) that this function admits a meromorphic continuation to the complex plane, and that this continuation takes a finite value $\eta(0)$ at the origin. The *eta invariant* of the chosen Dirac operator D is by definition

$$\eta(D) = \frac{\eta(0) - h}{2} \tag{1}$$

where $h = \dim \ker(D)$ is the multiplicity of the zero eigenvalue.

The eta invariant plays a central role in the Atiyah-Patodi-Singer index theorem, appearing as a correction term for the boundary. Suppose W is an even dimensional manifold with boundary $\partial W = M$, equipped with a Dirac bundle satisfying the conditions of Definition 2.3. Further, suppose we have a Dirac operator $D(W)$ on W so that

$$D(W) = \begin{pmatrix} 0 & -\partial_t + D \\ \partial_t + D & 0 \end{pmatrix} \tag{2}$$

in a product neighbourhood of the boundary, where D is the Dirac operator on M. In this instance we say that $D(W)$ *bounds* D. Then the APS index theorem [2] states

$$\operatorname{Ind}_{\mathrm{APS}} D^+(W) = \int_W \mathbf{I}(D^+(W)) - \eta(D). \tag{3}$$

The left-hand side is the index of $D^+(W)$ with respect to a certain global boundary condition – the projection onto the non-negative eigenspace of D must vanish. The integrand $\mathbf{I}(D^+(W))$ is the constant term in the asymptotic expansion of the supertrace of the heat operator for $D^+(W)$, called the *index form* of the Dirac operator.

Remark 3.1. In equation (3), the eta invariant is as in (1), where the sign of the term $h = \dim \ker D$ is negative. This is contingent on the orientation of M being consistent with the boundary orientation inherited from W. If the orientations are not compatible, then the sign of h is reversed in equation (3).

The map f in the cycle (M, S, f) determines a principle π-bundle over M. Given a representation $\sigma_1 : \pi \to U(N)$, we can then form a flat vector bundle $E_1 \to M$ and twist the Dirac operator D on S to obtain a Dirac operator D_1 acting on sections of $S \otimes E_1$. Given a second representation $\sigma_2 : \pi \to U(N)$ we form another operator D_2 on $S \otimes E_2$ in the same way.

Definition 3.2. The *relative eta invariant*, or *rho invariant* associated to the two unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the K-cycle (M, S, f) for $B\pi$, and the choice of Dirac operator D for the K-cycle, is defined to be

$$\rho\,(\sigma_1, \sigma_2\,; M, S, f) = \eta(D_1) - \eta(D_2).$$

The eta invariant of an operator depends sensitively on the operator itself, whereas the relative eta invariant is much more robust. The following is a restatement of Theorem 6.1 from Higson-Roe [20], and is the reason for our omission of D in the above notation for the rho invariant.

Theorem 3.3. *The* mod $\mathbb{Z}$ *reduction of the rho invariant* $\rho(\sigma_1, \sigma_2\,; M, S, f)$ *for representations* $\sigma_1, \sigma_2 : \pi \to U(N)$*, depends only on the equivalence class of* (M, S, f) *in* $K_1(B\pi)$*, and on* σ_1, σ_2*. There is therefore a well-defined group homomorphism*

$$\rho\,(\sigma_1, \sigma_2) : K_1(B\pi) \to \mathbb{R}/\mathbb{Z}.$$

The most complicated part of the proof is showing invariance under bundle modification. We will not repeat the full proof, however we will show invariance under bordism since the argument serves to motivate the end-periodic case.

Proof. Let (M, S, f) be a boundary—we will show that the rho invariant $\rho(\sigma_1, \sigma_2; M, S, f)$ vanishes modulo $\mathbb{Z}$. Let W be as in Definition 2.3 and let $D(W)$ be a Dirac operator on W which bounds the Dirac operator D on M. Since the map f to $B\pi$ extends to W, we find twisted Dirac operators $D_1(W)$ and $D_2(W)$ on W bounding the twisted operators D_1 and D_2 on M. Applying the APS index theorem separately to these operators gives

$$\operatorname{Ind}_{\mathrm{APS}} D_i^+(W) = \int_W \mathbf{I}(D_i^+(W)) - \eta(D_i) \tag{4}$$

for $i = 1, 2$. Since $D_1(W)$ and $D_2(W)$ are both twists of the same Dirac operator $D(W)$ by flat bundles of dimension N, we have

$$\mathbf{I}(D_1^+(W)) = \mathbf{I}(D_2^+(W)) = N \cdot \mathbf{I}(D^+(W)).$$

Subtracting the two equations (4) from each other therefore yields

$$\rho(\sigma_1, \sigma_2; M, S, f) = \eta(D_1) - \eta(D_2) = \operatorname{Ind}_{\mathrm{APS}} D_2^+(W) - \operatorname{Ind}_{\mathrm{APS}} D_1^+(W)$$

which is an integer.

Now, consider the negative cycle $(-M, -S, f)$ for (M, S, f). If D is a Dirac operator for (M, S, f), then $-D$ is a Dirac operator for $(-M, -S, f)$. From the definition of the eta invariant (1) and from Remark 3.1, we see that $\eta(-D) = -\eta(D)$. Finally, the eta invariant is clearly additive under disjoint unions of cycles. It follows that if two cycles are bordant, then their eta invariants agree modulo integers. □

Higson and Roe [20] used this map on K-homology to obtain obstructions to positive scalar curvature for odd-dimensional manifolds. Our isomorphism of K-homologies will allow us to transfer their results to the even dimensional case.

3.2. **Index theorem for end-periodic manifolds** [28]**.** In [28], Mrowka, Ruberman and Saveliev prove an index theorem for end-periodic Dirac operators on end-periodic manifolds, which generalises the Atiyah-Patodi-Singer index theorem. Rather than the eta invariant appearing as a correction term for the end, a new invariant called the *end-periodic eta invariant* appears, and this new invariant agrees with the eta invariant of Atiyah-Patodi-Singer in the case of a cylindrical end. In this section, we review the end-periodic index theorem of MRS, and give the necessary definitions and theorems required to define the end-periodic rho invariants. There is nothing new here, so the reader who is already familiar with the MRS index theorem may safely skip to Section 3.3

Let (X, S, γ, f) be a K^{ep}-cycle, and let $D(X)$ be a Dirac operator for the cycle. Let $\tilde{X}$ be the $\mathbb{Z}$-cover associated to γ, and let $F : \tilde{X} \to \mathbb{R}$ be the map which covers the classifying map $X \to S^1$ for the $\mathbb{Z}$-cover $\tilde{X}$. Then F satisfies $F(x+1) = F(x) + 1$, where $x+1$ denotes the image of $x \in \tilde{X}$ under the fundamental covering translation. It follows that dF descends to a well-defined one-form on X, also denoted dF. Fixing a branch of the complex logarithm, define a family of operators

$$D_z(X) = D(X) - \ln(z)\, c(dF)$$

on X, where $c(dF)$ is Clifford multiplication by dF, and $z \in \mathbb{C}^*$. These are in fact the operators obtained by conjugating the Dirac operator on $\tilde{X}$ with the *Fourier-Laplace transform*—see Section 2.2 of [28] for more details. The *spectral set* of this family of operators is defined to be the set of z for which $D_z(X)$ is not invertible. The spectral sets of the families $D_z^{\pm}(X)$ are defined similarly.

Henceforth, we will take Z_∞ to be an end-periodic manifold with end modelled on (X, γ). All objects on Z_∞ will be taken to be end-periodic, unless stated otherwise. Now, the Fredholm properties of the end-periodic operator $D^+(Z_\infty)$ are linked to the spectral set of the family $D_z^+(X)$. In fact, it follows from Lemma 4.3 of Taubes [35], that $D^+(Z_\infty)$ is Fredholm if and only if the spectral set of the family $D_z^+(X)$ is disjoint from the unit circle $S^1 \subset \mathbb{C}$. Thus, a necessary (but not sufficient) condition for $D^+(Z_\infty)$ to be Fredholm is that $\operatorname{Ind} D^+(X) = 0$.

Definition 3.4 ([28])**.** Suppose that the spectral set of the family $D_z^+(X)$ is disjoint from the unit circle $S^1 \subset \mathbb{C}$. The *end-periodic eta invariant* for the Dirac operator $D^+(X)$ is then defined as

$$\eta^{\text{ep}}(D^+(X)) = \frac{1}{\pi i} \int_0^\infty \oint_{|z|=1} \operatorname{Tr}(c(dF) \cdot D_z^+ \exp(-tD_z^- D_z^+)) \, \frac{dz}{z} \, dt,$$

where the Dirac operators in the integral are on X, and the contour integral over the unit circle is taken in the anti-clockwise direction.

Remark 3.5. There is an equivalent definition of the eta invariant in terms of the von Neumann trace—see Proposition 6.2 of [28], also [1] for information on the von Neumann trace.

Suppose $X = S^1 \times Y$, where Y is a compact oriented odd dimensional manifold, and X is endowed with the product Riemannian metric. Assume the Dirac operator $D(X)$ on X takes the form of that in the RHS of equation (2), with D being the Dirac operator on Y. Then it is shown in section 6.3 [28] that for $dF = d\theta$,

$$\eta^{\text{ep}}(D^+(X)) = \eta(D).$$

We now state the end-periodic index theorem of Mrowka, Ruberman and Saveliev, in the case when the end-periodic operator $D^+(Z_\infty)$ is Fredholm. Recall that for $D^+(Z_\infty)$ to be Fredholm, it is necessary that $\operatorname{Ind} D^+(X) = 0$. The Atiyah-Singer index theorem then implies that the index form $\mathbf{I}(D^+(X))$ is exact, so one can find a form ω on X satisfying $d\omega = \mathbf{I}(D^+(X))$.

Theorem 3.6 (MRS Index Theorem, Theorem A, [28])**.** *Suppose that the end-periodic operator $D^+(Z_\infty)$ is Fredholm, and choose a form ω on X such that $d\omega = \mathbf{I}(D^+(X))$. Then*

$$\operatorname{Ind} D^+(Z_\infty) = \int_Z \mathbf{I}(D^+(Z)) \; - \int_Y \omega + \int_X dF \wedge \omega \; - \frac{1}{2}\, \eta^{\text{ep}}(X). \tag{5}$$

Remarks 3.7. The form ω is called the *transgression class* – see Gilkey [14], page 306 for more details. In the case that the metric is a product near Y, one can choose F so that the two integrals involving the transgression class cancel, leaving a formula similar to the original APS formula. The theorem reduces to the APS index theorem [2] when Z_∞ only has cylindrical ends.

When $D^+(Z_\infty)$ is *not* Fredholm, Mrowka, Ruberman and Saveliev are still able to prove an index theorem under the assumptions that the spectrum of the family $D_z^+(X)$ is discrete, which in particular implies $\mathrm{Ind} D^+(X) = 0$. This is analogous to the case in the APS index theorem when the Dirac operator D on the boundary has a non-zero kernel, and the correction $h = \dim\ker D$ appears in the formula.

The key is to introduce the *weighted Sobolev spaces* on Z_∞ as follows. First recall that the Sobolev space $L^2_k(Z_\infty, S)$ for an integer $k \geq 0$, is defined as the completion of $C_0^\infty(Z_\infty, S)$ in the norm

$$\|u\|^2_{L^2_k(Z_\infty,S)} = \sum_{j\leq k} \int_{Z_\infty} |\nabla^j u|^2$$

for a fixed choice of end-periodic metric and compatible end-periodic Clifford connection on Z_∞. Now, restrict the upstairs covering map $F : \tilde{X} \to \mathbb{R}$ to the half-cover $X_1 = \bigcup_{k\geq 0} W_k$, and choose an extension of this map to Z_∞, which we continue to denote F. Given a weight $\delta \in \mathbb{R}$ and an integer $k \geq 0$, we say that $u \in L^2_{k,\delta}(Z_\infty, S)$ if $e^{\delta F} u \in L^2_k(Z_\infty, S)$. Define the $L^2_{k,\delta}$-norm by

$$\|u\|_{L^2_{k,\delta}(Z_\infty,S)} = \| e^{\delta F} u\|_{L^2_k(Z_\infty,S)}.$$

It is easy to check that up to equivalence of norms, this is independent of the choice of extension of F to Z_∞, since the region over which we are choosing an extension is compact. The spaces $L^2_{k,\delta}(Z_\infty, S)$ are all complete in this norm, and the operator $D^+(Z_\infty)$ extends to a bounded operator

$$D^+(Z_\infty) : L^2_{k+1,\delta}(Z_\infty, S^+) \to L^2_{k,\delta}(Z_\infty, S^-) \tag{6}$$

for every k and δ. The following theorem of Taubes [35] classifies Fredholmness of the operator (6) in terms of the family $D_z^+(X) = D^+(X) - \ln(z)\, c(dF)$.

Lemma 3.8 (Lemma 4.3 [35]). *The operator $D^+(Z_\infty) : L^2_{k+1,\delta}(Z_\infty, S^+) \to L^2_{k,\delta}(Z_\infty, S^-)$ is Fredholm if and only if the operators $D_z^+(X)$ are invertible for all z on the circle $|z| = e^\delta$.*

The usual L^2-case corresponds to the weighting $\delta = 0$, and hence we see by setting $z = 1$:

Corollary 3.9. *A necessary condition for the operator $D^+(Z_\infty)$ to be Fredholm is that $\mathrm{Ind}\, D^+(X) = 0$.*

The following result on the spectral set of the family is also due to Taubes, which suffices for our purposes.

Theorem 3.10 (Theorem 3.1, [35]). *Suppose that $\mathrm{Ind}\, D^+(X) = 0$ and that the map $c(dF)$: $\ker D^+(X) \to \ker D^-(X)$ is injective. Then the spectral set of the family $D_z^+(X)$ is a discrete subset of $\mathbb{C}^*$, and the operator $D^+(Z_\infty)$ is a Fredholm operator.*

It follows that the operator $D^+(Z_\infty)$ acting on the Sobolev spaces of weight δ is Fredholm for all but a closed discrete set of $\delta \in \mathbb{R}$.

Remark 3.11. There are two important instances where the hypothesis of Theorem 3.10 is satisfied:

(1) When $X = S^1 \times M$ with the product metric, and the Dirac operator on X taking the form of equation (2). In this case $dF = d\theta$, and $c(d\theta)$ is as in part (b) of Definition 2.3. This example shows that every class in $K^{\mathrm{ep}}(B\pi)$ has a representative with discrete spectral set.

(2) When X is spin with positive scalar curvature and $D^+(X)$ is the spin Dirac operator on X (or more generally, $D^+(X)$ twisted by a flat bundle). In this case Lichnerowicz' vanishing theorem implies that $\ker D^+(X)$ and $\ker D^-(X)$ are trivial. In the applications to positive scalar curvature, we will always assume X to be spin, so that this assumption is satisfied.

Theorem C, [28] extends Theorem 3.6 to the non-Fredholm case that applies to operators such as the signature operator and is analogous to the extended L^2 case considered in [2].

We allow for the case where the family has poles lying on the unit circle, in which case the operator $D^+(X)$ is not Fredholm. By discreteness of the spectral set, the family $D_z^+(X)$ has no poles for z sufficiently close to (but not lying on) the unit circle, and hence there is $\epsilon > 0$ such that for all $0 < \delta < \epsilon$ the operators $D_z^+(Z_\infty)$ acting on the δ-weighted Sobolev spaces are all Fredholm (see Lemma 3.8). The index does not change under small variations of δ in this region, and we denote it by $\operatorname{Ind}_{\mathrm{MRS}} D^+(Z_\infty)$. This is the regularised form of the index which appears in the full MRS index theorem.

There are two more quantities to define which appear in the full MRS index theorem. First of all, the end-periodic eta invariant in Definition 3.4 is no longer well defined if the family $D_z^+(X)$ has poles on the unit circle. Letting $\epsilon > 0$ be sufficiently small so that there are no poles in $e^{-\epsilon} < |z| < e^{\epsilon}$ except for those with $|z| = 1$, define

$$\eta_\epsilon^{\mathrm{ep}}(D^+(X)) = \frac{1}{\pi i} \int_0^\infty \oint_{|z|=e^\epsilon} \operatorname{Tr}\left(df \cdot D_z^+ \exp(-t(D_z^+)^* D_z^+)\right) \frac{dz}{z}\, dt, \tag{7}$$

where the integral is taken to be the constant term of its asymptotic expansion in powers of t. Define

$$\eta_\pm^{\mathrm{ep}}(D^+(X)) \;=\; \lim_{\epsilon\to 0\pm} \eta_\epsilon^{\mathrm{ep}}(D^+(X)),$$

and

$$\eta^{\mathrm{ep}}(D^+(X)) \;=\; \frac{1}{2}\,[\eta_+^{\mathrm{ep}}(D^+(X)) + \eta_-^{\mathrm{ep}}(D^+(X))]. \tag{8}$$

It is this incarnation of the eta invariant which will appear in the MRS index theorem. Since $(D_z^+)^* = D_z^-$ for $|z| = 1$ this definition of $\eta^{\mathrm{ep}}(X)$ agrees with Definition 3.4 when there are no poles on the unit circle.

The last term to define is the analogue of $h = \dim\ker D$ appearing in the APS index theorem. The family $D_z^+(X)^{-1}$ is meromorphic, so if $z \in S^1$ is a pole then it has some finite order m. Define $d(z)$, as in Section 6.3 of [29], to be the dimension of the vector space solutions $(\varphi_1, \ldots, \varphi_m)$ to the system of equations

$$\begin{cases} D_z^+(X)\varphi_1 = c(dF)\varphi_2 \\ \vdots \\ D_z^+(X)\varphi_{m-1} = c(dF)\varphi_m \\ D_z^+(X)\varphi_m = 0. \end{cases}$$

For z not in the spectral set of the family $D_z^+(X)$, we have $d(z) = 0$. The term h in the MRS index theorem is defined as the finite sum of integers

$$h = \sum_{|z|=1} d(z).$$

Remark 3.12. The integers $d(z)$ give a formula for the change in index when one varies the weight δ; if $\operatorname{Ind}_\delta D^+(Z_\infty)$ denotes the index of $D^+(Z_\infty)$ acting on the δ-weighted Sobolev spaces, then one has for $\delta < \delta'$ that

$$\operatorname{Ind}_\delta D^+(Z_\infty) - \operatorname{Ind}_{\delta'} D^+(Z_\infty) \;=\; \sum_{e^\delta < |z| < e^{\delta'}} d(z).$$

Theorem 3.13 (MRS Index Theorem, Theorem C, [28])**.** *Suppose the spectral set of $D_z^+(X)$ is a discrete subset of $\mathbb{C}^*$, and let ω be a form on X such that $d\omega = \mathbf{I}(D^+(X))$. Then*

$$\operatorname{Ind}_{\text{MRS}} D^+(Z_\infty) \;=\; \int_Z \mathbf{I}(D^+(Z)) - \int_Y \omega + \int_X dF \wedge \omega \;-\; \frac{h + \eta^{\text{ep}}\,(D^+(X))}{2}.$$

3.3. **End-periodic $\mathbb{R}/\mathbb{Z}$-index theorem.** Let $\sigma_1, \sigma_2 : \pi \to U(N)$ be unitary representations of the discrete group π. Using the end-periodic eta invariant of MRS, we will define an end-periodic rho invariant $\rho^{\text{ep}}(\sigma_1, \sigma_2)$ analogous to the rho invariant in the APS case. This will determine a map from end-periodic K-homology to $\mathbb{R}/\mathbb{Z}$, however we must be more careful about how we define the rho invariant due to the MRS index theorem not being applicable to all operators.

Definition 3.14. Let (X, S, γ, f) be a K^{ep}-cycle. Assume we can choose a covering function $F : \tilde{X} \to \mathbb{R}$ so that the spectral sets of the families of the twisted operators $D_1^+(X)$ and $D_2^+(X)$ are discrete. Then we define the *end-periodic rho invariant* to be

$$\rho^{\text{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f) = \frac{1}{2}[h_1 + \eta^{\text{ep}}(D_1^+(X)) - h_2 - \eta^{\text{ep}}(D_2^+(X))].$$

By Lemma 8.2 of [28], this definition is independent of the choice of such function F, if it exists.

Theorem 3.15. *Whenever it is defined, the mod $\mathbb{Z}$ reduction of the end-periodic rho invariant $\rho^{\text{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f)$ associated to $\sigma_1, \sigma_2 : \pi \to U(N)$ depends only on the representations σ_1, σ_2 and the equivalence class of (X, S, γ, f) in $K_1^{\text{ep}}(B\pi)$. Moreover, every equivalence class has a representative with a well-defined rho invariant. Hence there is a well-defined group homomorphism*

$$\rho^{\text{ep}}(\sigma_1, \sigma_2) : K_1^{\text{ep}}(B\pi) \to \mathbb{R}/\mathbb{Z}.$$

Furthermore, the following diagram commutes:

$$\begin{array}{ccc} K_1^{\text{ep}}(B\pi) & \overset{\sim}{\longleftrightarrow} & K_1(B\pi) \\ {\scriptstyle \rho^{\text{ep}}(\sigma_1,\sigma_2)} \searrow & & \swarrow {\scriptstyle \rho(\sigma_1,\sigma_2)} \\ & \mathbb{R}/\mathbb{Z} & \end{array}$$

Hence, even if the spectral set of $D^+(X)$ is not discrete, we can still define its $\mathbb{R}/\mathbb{Z}$ end-periodic rho invariant in a perfectly reasonable and consistent manner. This allows us to define the $\mathbb{R}/\mathbb{Z}$ invariant, for instance, in the case where $\operatorname{Ind} D^+(X) \neq 0$. For the applications to positive scalar curvature, the end-periodic rho invariant is well-defined and given by the usual formula (8), since in Remark 3.11 we have noted that the spectral sets of its twisted operators are discrete.

Proof. That every equivalence class in K^{ep}-homology has a representative with discrete spectral set follows from the proof of Theorem 3.3—the cycle (X, S, γ, f) is bordant to the cycle $(S^1 \times Y, S^+ \oplus S^+, d\theta, f)$, which has discrete spectral set by part (1) of Remark 3.11.

As we shall see, it is only necessary to prove invariance of ρ^{ep} under bordism, and then Theorem 3.3 will imply invariance under the other relations defining K^{ep}-homology. First suppose that (X, S, γ, f) is a boundary with Dirac operator $D^+(X)$ such that the families associated to the twisted operators $D_1^+(X)$ and $D_2^+(X)$ have discrete spectral sets. We apply the MRS index theorem to each operator separately to get

$$\operatorname{Ind}_{\text{MRS}} D_i^+(Z_\infty) \;=\; \int_Z \mathbf{I}(D_i^+(Z)) - \int_Y \omega_i \;+ \int_X dF \wedge \omega_i \;-\; \frac{h_i + \eta^{\text{ep}}\,(D_i^+(X))}{2}$$

for $i = 1, 2$. Now, since we are twisting by flat vector bundles, both the index form and the transgression classes for the twisted operators are constant multiples of the index form and transgression class of the original operator. Hence when we subtract the two equations, the terms involving these vanish and we are left with

$$\rho^{\text{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f) = \operatorname{Ind}_{\text{MRS}} D_2^+(Z_\infty) - \operatorname{Ind}_{\text{MRS}} D_1^+(Z_\infty)$$

which is an integer. The end-periodic rho invariant behaves additively under disjoint unions of cycles and changes sign when the negative of a cycle is taken. This proves bordism invariance mod $\mathbb{Z}$.

Now the K^{ep}-cycle (X, S, γ, f) with discrete spectral sets is bordant to $(S^1 \times Y, S^+ \oplus S^+, d\theta, f)$, where Y is Poincaré dual to γ. By Section 6.3 of MRS [28], the end-periodic rho invariant of $(S^1 \times Y, S^+ \oplus S^+, d\theta, f)$ is equal to the rho invariant of the K-cycle (Y, S^+, f). Hence

$$\rho^{\text{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f) = \rho\,(\sigma_1, \sigma_2\,; Y, S^+, f) \mod \mathbb{Z}.$$

The isomorphism $K_1(B\pi) \cong K_1^{\text{ep}}(B\pi)$ then immediately implies the theorem. □

4. End-Periodic Bordism Groups

In this section, we recall the definition of the spin bordism groups, and introduce the analogous bordism groups in the end-periodic setting. As for K-homology, there are natural isomorphisms between the spin bordism groups and the end-periodic spin bordism groups. We also consider the PSC spin bordism groups described in Botvinnik-Gilkey [12], and define the corresponding end-periodic PSC spin bordism groups. Throughout, we take $m \geq 5$ to be a positive odd integer.

4.1. **Spin bordism and end-periodic spin bordism.** We recall the definition of the spin bordism group $\Omega_m^{\text{spin}}(B\pi)$ for a discrete group π.

Definition 4.1. An Ω_m^{spin}*-cycle* for $B\pi$ is a triple (M, σ, f), where M is a compact oriented Riemannian spin manifold of dimension m, σ is a choice of spin structure on M, and $f : M \to B\pi$ is a continuous map.

The *negative* of an Ω_m^{spin}-cycle (M, σ, f) is $(-M, \sigma, f)$, where $-M$ is M with the reversed orientation. An Ω_m^{spin}-cycle (M, σ, f) is a *boundary* if there exists a compact oriented Riemannian manifold W with boundary $\partial W = M$, a spin structure on W whose restriction to the boundary is the spin structure σ, and a continuous map $W \to B\pi$ extending the map f. Two Ω_m^{spin}-cycles (M, σ, f) and (M', σ', f') are *bordant* if $(M, \sigma, f) \amalg (M', \sigma', f')$ is a boundary.

Definition 4.2. The m-dimensional spin bordism group $\Omega_m^{\mathrm{spin}}(B\pi)$ for $B\pi$, consists of Ω_m^{spin}-cycles for $B\pi$ modulo the equivalence relation of bordism. It is an abelian group with addition given by disjoint union of cycles.

The end-periodic spin bordism group $\Omega_m^{\mathrm{ep,spin}}(B\pi)$, is defined in an analogous way to the end-periodic K-homology group.

Definition 4.3. An $\Omega_m^{\mathrm{ep,spin}}$-cycle for $B\pi$ is a quadruple (X, σ, γ, f) where X is a compact oriented Riemannian spin manifold of dimension $m+1$, σ is a spin structure on X, γ is a cohomology class in $H^1(X, \mathbb{Z})$ that is primitive on each component of X, and $f : X \to B\pi$ is a continuous map.

The definition of a boundary is essentially the same as for end-periodic K-homology.

Definition 4.4. An $\Omega_m^{\mathrm{ep,spin}}$-cycle (X, σ, γ, f) is a *boundary* if there exists an end-periodic oriented Riemannian spin manifold Z_∞ with end modelled on (X, γ), such that the pulled back spin structure σ on the periodic end extends to Z_∞, as does the pulled back map f to $B\pi$.

The *negative* of a cycle (X, σ, γ, f) is $(X, \sigma, -\gamma, f)$. As before, we introduce the additional relation of *orientation/sign*:

$$(X, \sigma, -\gamma, f) \sim (-X, \sigma, \gamma, f).$$

Two $\Omega_m^{\mathrm{ep,spin}}$-cycles (X, γ, σ, f) and $(X', \gamma', \sigma', f')$ are *bordant* if $(X, \sigma, \gamma, f) \amalg (X, \sigma, -\gamma, f)$ is a boundary.

Definition 4.5. The m-dimensional *end-periodic spin bordism group* $\Omega_m^{\mathrm{ep,spin}}(B\pi)$ consists of $\Omega_m^{\mathrm{ep,spin}}$-cycles modulo the equivalence relation generated by bordism and orientation/sign, with addition given by disjoint union.

Analogous to the K-homology groups from Section 2, there is a canonical isomorphism between the spin bordism and end-periodic spin bordism groups which we will now describe.

The map $\Omega_m^{\mathrm{spin}}(B\pi) \to \Omega_m^{\mathrm{ep,spin}}(B\pi)$ takes a $\Omega_m^{\mathrm{spin}}(B\pi)$-cycle (M, σ, f) to $(S^1 \times M, 1 \times \sigma, d\theta, f)$, where $S^1 \times M$ has the product orientation and Riemannian metric, $1 \times \sigma$ is the product spin structure of the trivial spin structure 1 on S^1 with the spin structure σ on M, $d\theta$ is the standard generator of the first cohomology of S^1, and f is the obvious extension of $f : M \to B\pi$ to $S^1 \times M$.

Proposition 4.6. *The map which sends an $\Omega_m^{\mathrm{spin}}(B\pi)$-cycle (M, σ, f) to the $\Omega_m^{\mathrm{ep,spin}}(B\pi)$-cycle $(S^1 \times M, 1 \times \sigma, d\theta, f)$ is well-defined on spin bordism groups.*

Proof. If (M, σ, f) and (M', σ', f') are bordant, with W bounding their disjoint union, then $\mathbb{R}_{\geq 0} \times M$ and $\mathbb{R}_{\leq 0} \times M'$ can be joined using W to form and end-periodic manifold Z_∞ with multiple ends. All structures extend to Z_∞ by assumption, hence the two $\Omega_m^{\mathrm{ep,spin}}(B\pi)$-cycles $(S^1 \times M, 1 \times \sigma, d\theta, f)$ and $(-S^1 \times M, 1 \times \sigma', -d\theta, f')$ are bordant. Using the orientation/sign relation, we see that $(S^1 \times M, 1 \times \sigma, d\theta, f)$ and $(S^1 \times M', 1 \times \sigma', d\theta, f')$ are equivalent. ☐

Now for the map $\Omega_m^{\mathrm{ep,spin}}(B\pi) \to \Omega_m^{\mathrm{spin}}(B\pi)$. Let (X, σ, γ, f) be an $\Omega_m^{\mathrm{ep,spin}}$-cycle for $B\pi$, and Y be a submanifold of X Poincaré dual to γ. We equip Y with the induced spin structure and orientation from γ. Explicitly, the orientation of Y is as in the paragraph after Definition 2.9, and the restricted spin structure is obtained first by cutting X open along Y to get a manifold W with boundary $\partial W = Y \amalg -Y$, and then taking the boundary spin structure on the positively oriented component Y of ∂W. This yields an Ω_m^{spin}-cycle (Y, σ, f), where σ and f are restricted to Y.

Proposition 4.7. *The map taking an $\Omega_m^{\mathrm{ep,spin}}(B\pi)$-cycle (X, σ, γ, f) to the $\Omega_m^{\mathrm{spin}}(B\pi)$-cycle (Y, σ, f) described above is well-defined on bordism groups.*

Proof. Independence of the choice of Y is proved as for the K-homology case, only with spin structures instead of Dirac bundles. It is clear that the orientation/sign relation is respected, since both $(X, \sigma, -\gamma, f)$ and $(-X, \sigma, \gamma, f)$ get sent to $(-Y, \sigma, f)$. If (X, σ, γ, f) and $(X', \sigma', \gamma', f')$ are bordant, then there is a compact manifold Z with boundary $\partial Z = Y \amalg -Y'$ such that the spin structures and maps extend over Z. But this shows that (Y, σ, f) and (Y', σ', f') are bordant. ☐

Theorem 4.8. *The above maps of bordism groups are inverse to each other, and so define a natural isomorphism of abelian groups $\Omega_m^{\mathrm{spin}}(B\pi) \cong \Omega_m^{\mathrm{ep,spin}}(B\pi)$.*

Proof. A cycle (M, σ, f) gets mapped to $(S^1 \times M, 1 \times \sigma, d\theta, f)$, which gets returned to $(M, 1 \times \sigma, f)$, where the latter two entries are restricted to M. It is straightforward to check that the product spin structure $1 \times \sigma$ restricted to M yields the original spin structure σ. Therefore we obtain our original cycle (M, σ, f) after mapping it to and from end-periodic bordism.

Now let (X, σ, γ, f) be an end-periodic cycle, with submanifold Y Poincaré dual to γ. This maps to a cycle (Y, σ, f), where the latter two structures are restricted from X, and this maps back to $(S^1 \times Y, 1 \times \sigma, d\theta, f)$. The same argument as in the proof of Theorem 2.9 shows that this is bordant to (X, σ, γ, f). ☐

4.2. **PSC spin bordism and end-periodic PSC spin bordism.** In [12], Botvinnik and Gilkey use a variant of spin cobodism tailored to the setting of manifolds with positive scalar curvature, which we now recall.

Definition 4.9. A $\Omega_m^{\mathrm{spin},+}$-cycle is a quadruple (M, g, σ, f), where M is a compact oriented Riemannian spin manifold of dimension m with a metric g of positive scalar curvature, σ is a spin structure on M, and $f : M \to B\pi$ is a continuous map.

The negative of (M, g, σ, f) is $(-M, g, \sigma, f)$, as before. A cycle (M, g, σ, f) is called a *boundary* if there is a compact oriented Riemannian spin manifold W with boundary $\partial W = M$ so that the spin structure σ and map f extend to W. It is also required that W has a metric of positive scalar curvature that is a product metric in a neighbourhood of the boundary. Two cycles are *bordant* if the disjoint union of one with the negative of the other is a boundary.

Definition 4.10. The *PSC spin bordism group* $\Omega_m^{\mathrm{spin},+}(B\pi)$ for $B\pi$ consists of $\Omega_m^{\mathrm{spin},+}$-cycles modulo bordism, with addition given by disjoint union.

We now define the end-periodic PSC spin bordism group $\Omega_m^{\mathrm{ep,spin},+}(B\pi)$ for $B\pi$.

Definition 4.11. An $\Omega_m^{\mathrm{ep,spin},+}$*-cycle* is a quintuple $(X, g, \sigma, \gamma, f)$, where X is a compact oriented Riemannian spin manifold of dimension $m+1$ with a metric g of positive scalar curvature, σ is a choice of spin structure on X, γ is a cohomology class in $H^1(X, \mathbb{Z})$ whose restriction to each component of X is primitive, and $f : X \to B\pi$ is a continuous map. We further require that there is a submanifold Y of X that is Poincaré dual to γ, such that the induced metric on Y has positive scalar curvature, and the metric on X is a product metric $dt^2 + g_Y$ in a neighbourhood of Y.

Let $(X, g, \sigma, \gamma, f)$ be an $\Omega_m^{\mathrm{ep,spin,+}}$-cycle and take $Y \subset X$ to be a submanifold with PSC that is Poincaré dual to γ. As before we form $X_1 = \bigcup_{k \geq 0} W_k$, where the W_k are isometric copies of X cut open along Y. For $(X, g, \sigma, \gamma, f)$ to be a *boundary* means that there is a compact oriented Riemannian spin manifold Z of positive scalar, whose metric is a product near the boundary, which can be attached to X_1 along Y to form a complete oriented Riemannian spin manifold of PSC $Z_\infty = Z \cup_Y X_1$, such that the pulled back spin structure σ and map f on X_1 extend over Z.

The *negative* of $(X, g, \sigma, \gamma, f)$ is $(X, g, \sigma, -\gamma, f)$, and we have the *orientation/sign* relation

$$(X, g, \sigma, -\gamma, f) \sim (-X, g, \sigma, \gamma, f).$$

Two $\Omega_m^{\mathrm{ep,spin,+}}$-cycles are *bordant* if the disjoint union of one with the negative of the other is a boundary.

Definition 4.12. The m-dimensional *end-periodic PSC spin bordism group* $\Omega_m^{\mathrm{ep,spin,+}}(B\pi)$ for $B\pi$ consists of $\Omega_m^{\mathrm{ep,spin,+}}$-cycles modulo bordism and orientation/sign, with addition given by disjoint union.

Theorem 4.13. *There is a canonical isomorphism* $\Omega_m^{\mathrm{spin,+}}(B\pi) \cong \Omega_m^{\mathrm{ep,spin,+}}(B\pi)$.

The maps are exactly as for the spin bordism theories, only we must take the submanifold Y with positive scalar curvature and a product metric in a tubular neighbourhood when mapping from $\Omega_m^{\mathrm{ep,spin,+}}(B\pi)$ to $\Omega_m^{\mathrm{spin,+}}(B\pi)$.

Proof. As before. □

4.3. **Rho invariants.** Given a triple (M, σ, f) and two unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, we define the rho invariant $\rho\,(\sigma_1, \sigma_2\,; M, \sigma, f)$ as before, using the spin Dirac operator for the cycle (M, S, f). We also define the end-periodic rho invariant for cycles (X, σ, γ, f) in an entirely analogous manner, using the end-periodic eta invariant of MRS instead. Of course, we must again be careful with the definition, allowing only the rho invariant for cycles whose twisted operators have discrete spectral sets to be defined in terms of the true end-periodic eta invariants—all others are defined by taking bordant cycles with discrete spectra.

Theorem 4.14. *The rho invariant extends to a well-defined homomorphism*

$$\rho\,(\sigma_1, \sigma_2) : \Omega_m^{\mathrm{spin}}(B\pi) \to \mathbb{R}/\mathbb{Z},$$

as does the end-periodic rho invariant

$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2) : \Omega_m^{\mathrm{ep,spin}}(B\pi) \to \mathbb{R}/\mathbb{Z}.$$

Furthermore, the following diagram commutes:

$$\begin{array}{ccc} \Omega_m^{\mathrm{ep,spin}}(B\pi) & \xleftrightarrow{\;\sim\;} & \Omega_m^{\mathrm{spin}}(B\pi) \\ & {\scriptstyle \rho^{\mathrm{ep}}(\sigma_1,\sigma_2)} \searrow \quad \swarrow {\scriptstyle \rho\,(\sigma_1,\sigma_2)} & \\ & \mathbb{R}/\mathbb{Z} & \end{array}$$

Proof. Apply the APS and MRS index theorems respectively, and use the isomorphism of Theorem 4.8. □

Now for the positive scalar curvature case.

Theorem 4.15. *The rho invariant extends to a well-defined homomorphism*

$$\rho\,(\sigma_1, \sigma_2) : \Omega_m^{\mathrm{spin},+}(B\pi) \to \mathbb{R},$$

as does the end-periodic rho invariant

$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2) : \Omega_m^{\mathrm{ep,spin},+}(B\pi) \to \mathbb{R}.$$

Furthermore, the following diagram commutes:

$$\begin{array}{ccc} \Omega_m^{\mathrm{ep,spin},+}(B\pi) & \xleftrightarrow{\ \sim\ } & \Omega_m^{\mathrm{spin},+}(B\pi) \\ & {\scriptstyle \rho^{\mathrm{ep}}(\sigma_1,\sigma_2)} \searrow \quad \swarrow {\scriptstyle \rho\,(\sigma_1,\sigma_2)} & \\ & \mathbb{R} & \end{array}$$

Remark 4.16. The end-periodic rho invariant appearing in the theorem is given on all representatives of equivalence classes as the genuine difference of the twisted eta invariants as in formula (8), due to Remark 3.11.

For the proof, we will need the following (cf. [28], Proposition 8.5 (ii)).

Lemma 4.17. *If $(X, g, \sigma, \gamma, f)$ is an $\Omega_m^{\mathrm{ep,spin},+}$-cycle and (Y, g, σ, f) is the $\Omega_m^{\mathrm{spin},+}$-cycle it maps to, then*

$$\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; X, g, \sigma, \gamma, f) = \rho(\sigma_1, \sigma_2\,; Y, g, \sigma, f)$$

Proof. We join $\mathbb{R}_{\geq 0} \times Y$ to $X_1 = \cup_{k \geq 0} W_k$ together as in Figure 4 to form an end-periodic spin manifold Z_∞ with two ends. Lemma 8.1 of [28] (which uses the results of Gromov-Lawson [18]) gives that the spin Dirac operator $D^+(Z_\infty)$ is Fredholm and has zero index. The same holds for its twisted counterparts. Applying the MRS index theorem to the two twisted spin Dirac operators $D_1^+(Z_\infty)$ and $D_2^+(Z_\infty)$, and subtracting the equations as per usual then yields the result. □

Proof of Theorem 4.15. See Theorem 1.1 of Botvinnik-Gilkey [12] for the proof that the map $\rho\,(\sigma_1, \sigma_2) : \Omega_m^{\mathrm{spin},+}(B\pi) \to \mathbb{R}$ is well-defined. Lemma 4.17 and the isomorphism of Theorem 4.13 then immediately imply the result. □

5. End-periodic structure group

Let $\sigma_1, \sigma_2 : \pi \to U(N)$ be unitary representations of the discrete group π. Recall the definition of the structure group $S_1(\sigma_1, \sigma_2)$ of Higson-Roe, starting from Definition 8.7 of [20].

Definition 5.1. An *odd* (σ_1, σ_2)*-cycle* is a quintuple (M, S, f, D, n) where (M, S, f) is an odd K-cycle for $B\pi$, D is a Dirac operator for (M, S, f), and $n \in \mathbb{Z}$.

A (σ_1, σ_2)-cycle (M, S, f, D, n) is a *boundary* if the K-cycle (M, S, f) is a bounded by a manifold W (as in Definition 2.3) and there are Dirac operators $D_1(W)$ and $D_2(W)$ on W which bound the twisted Dirac operators D_1 and D_2 on M, such that

$$\mathrm{Ind}_{\mathrm{APS}}\, D_1^+ - \mathrm{Ind}_{\mathrm{APS}}\, D_2^+ = n.$$

Since we are no longer looking at rho invariants modulo integers or at spin Dirac operators, we will denote by $\rho(\sigma_1, \sigma_2\,; D, f)$ the rho invariant of definition 3.2, indicating its possible dependence on the Dirac operator D.

Lemma 5.2 ([20] Lemma 8.10)**.** *If a (σ_1, σ_2)-cycle (M, S, f, D, n) is a boundary, then* $\rho(\sigma_1, \sigma_2\,; D, f) + n = 0$.

Definition 5.3. The *relative eta invariant*, or *rho invariant* of the (σ_1, σ_2)-cycle (M, S, f, D, n) is $\rho(\sigma_1, \sigma_2\,; D, f) + n$.

The *disjoint union* of (σ_1, σ_2)-cycles is defined as,

$$(M, S, f, D, n) \amalg (M', S', f', D', n') = (M \amalg M', S \amalg S', f \amalg f', D \amalg D', n + n').$$

The *negative* of a (σ_1, σ_2)-cycle (M, S, f, D, n), is defined as,

$$-(M, S, f, D, n) = (M, -S, f, -D, h_1 - h_2 - n),$$

where $h_1 = \dim\ker(D_1)$ and $h_2 = \dim\ker(D_2)$. Two (σ_1, σ_2)-cycles are *bordant* if the disjoint union of one cycle with the negative of the other is a boundary.

The two remaining relations to define are:

- *Direct sum/disjoint union*:
$$(M, S \oplus S', f, D \oplus D', n) \sim (M \amalg M, S \amalg S', f \amalg f, D \amalg D', n).$$
- *Bundle Modification*: If $(\hat{M}, \hat{S}, \hat{f})$ is an elementary bundle modification of (M, S, f) with the Dirac operator $\hat{D}$ from 2.5, then $(M, S, f, D, n) \sim (\hat{M}, \hat{S}, \hat{f}, \hat{D}, n)$.

Definition 5.4. The structure group $S(\sigma_1, \sigma_2)$, is the set of equivalence classes of (σ_1, σ_2)-cycles under the equivalence relation generated by bordism, direct sum/disjoint union, and bundle modification. It is an abelian group with addition is given by disjoint union.

In [20] Proposition 8.14, it is proved that the relative eta invariant of a (σ_1, σ_2)-cycle depends only on the class that the cycle determines in $S(\sigma_1, \sigma_2)$. Hence there is a well-defined group homomorphism $\rho : S(\sigma_1, \sigma_2) \to \mathbb{R}$, defined by

$$\rho(M, S, f, D, n) = \rho(\sigma_1, \sigma_2\,; D, f) + n.$$

5.1. **End-periodic structure group.** We define in a parallel manner the end-periodic structure group $S_1^{\mathrm{ep}}(\sigma_1, \sigma_2)$.

Definition 5.5. An *odd* $(\sigma_1, \sigma_2)^{\mathrm{ep}}$*-cycle* is a sextuple (X, S, γ, f, D, n) where (X, S, γ, f) is a K^{ep}-cycle for $B\pi$, D is a Dirac operator for (X, S, γ, f), and $n \in \mathbb{Z}$. We additionally assume that the spectral set of the family $D_z^+(X)$ is discrete.

A $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycle (X, S, f, γ, D, n) is a *boundary* if the K^{ep}-cycle (X, S, γ, f) is a boundary (Definition 2.10), and moreover there is a Dirac operator $D(Z_\infty)$ on the manifold Z_∞ extending the Dirac operator D on $X_1 = \bigcup_{k\geq 0} W_k$ such that the difference of the MRS indices

$$\mathrm{Ind}_{\mathrm{MRS}}(D_1^+(Z_\infty)) - \mathrm{Ind}_{\mathrm{MRS}}(D_2^+(Z_\infty)) = n.$$

Here the $D_i^+(Z_\infty)$ are the twists of $D^+(Z_\infty)$ by the flat vector bundles determined by the extension of f to Z_∞ and by σ_1, σ_2. We can show the analog of Lemma 5.2

Lemma 5.6. *If a* $(\sigma_1, \sigma_2)^{\mathrm{ep}}$*-cycle* (X, S, γ, f, D, n) *is a boundary, then* $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; D, f, \gamma) + n = 0$.

We call the quantity $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; D, f, \gamma)+n$ the *end-periodic rho invariant* of the $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycle (X, S, γ, f, D, n).

The *disjoint union* of $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycles is defined as

$$(X, S, f, \gamma, D, n) \amalg (X', S', \gamma', f', D', n') = (X \amalg X', S \amalg S', \gamma \amalg \gamma', f \amalg f', D \amalg D', n + n').$$

The *negative* of a $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycle (X, S, γ, f, D, n), is

$$-(X, S, \gamma, f, D, n) = (X, S, -\gamma, f, D, h_1 - h_2 - n),$$

where h_1, h_2 are the integers occurring in the MRS index theorem associated to σ_1, σ_2. Two $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycles are *bordant* if the disjoint union of one with the negative of the other is a boundary. We also have:

- *Direct sum/disjoint union*:
$$(X, S \oplus S', \gamma + \gamma' f, D \oplus D', n) \sim (X \amalg M, S \amalg S', \gamma \amalg \gamma', f \amalg f, D \amalg D', n).$$
- *Bundle Modification*: If $(\hat{X}, \hat{S}, \hat{\gamma}, \hat{f})$ is an elementary bundle modification of (X, S, γ, f) and $\hat{D}$ is the Dirac operator of Remark 2.5, then $(X, S, \gamma, f, D, n) \sim (\hat{X}, \hat{S}, \hat{\gamma}, \hat{f}, \hat{D}, n)$.
- *Orientation/sign*:
$$(X, S, -\gamma, f, D, n) \sim (-X, \Pi(S), \gamma, f, D, n).$$

Definition 5.7. The end-periodic structure group, denoted by $S_1^{\mathrm{ep}}(\sigma_1, \sigma_2)$, is the set of equivalence classes of $(\sigma_1, \sigma_2)^{\mathrm{ep}}$-cycles under the equivalence relation generated by bordism, direct sum/disjoint union, bundle modification, and orientation/sign. It is is an abelian group with unit and addition is given by disjoint union.

Define the group homomorphism $\rho^{\mathrm{ep}} : S_1^{\mathrm{ep}}(\sigma_1, \sigma_2) \to \mathbb{R}$ by the formula,

$$\rho^{\mathrm{ep}}(X, S, \gamma, f, D, n) = \rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; D, f, \gamma) + n.$$

Then the following theorem is the analog of Theorem 3.15 is is proved in a similar way.

Theorem 5.8. *The end-periodic rho invariant* $\rho^{\mathrm{ep}}(X, S, \gamma, f, \sigma_1, \sigma_2) + n$ *associated to the* $(\sigma_1, \sigma_2)^{\mathrm{ep}}$*-cycle* (M, S, γ, f, D, n) *depends only on the equivalence class of* (M, S, γ, f, D, n) *in* $S_1^{\mathrm{ep}}(\sigma_1, \sigma_2)$*. Hence there is a well-defined group homomorphism*

$$\rho^{\mathrm{ep}} : S_1^{\mathrm{ep}}(\sigma_1, \sigma_2) \to \mathbb{R}.$$

Furthermore, the following diagram commutes:

$$\begin{array}{ccc} S_1^{\mathrm{ep}}(\sigma_1, \sigma_2) & \overset{\sim}{\longleftrightarrow} & S_1(\sigma_1, \sigma_2) \\ & {\scriptstyle \rho^{\mathrm{ep}}} \searrow \quad \swarrow {\scriptstyle \rho} & \\ & \mathbb{R} & \end{array}$$

Here the maps $S_1^{\mathrm{ep}}(\sigma_1, \sigma_2) \leftrightarrow S_1(\sigma_1, \sigma_2)$ are the analog of the maps in K-homologies given earlier.

Also, Higson-Roe establish a commuting diagram of short exact sequences, cf. [20] the paragraph below Definition 8.6,

$$
(9)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & S_1(\sigma_1,\sigma_2) & \longrightarrow & K_1(B\pi) & \longrightarrow & 0\\
 & & \Big\downarrow{=} & & \Big\downarrow{\rho} & & \Big\downarrow{\rho(\sigma_1,\sigma_2)} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} & \longrightarrow & 0.
\end{array}
$$

By Theorems 5.8 and 3.15, we deduce that there is a commuting diagram of short exact sequences,

$$
(10)\qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & S_1^{\mathrm{ep}}(\sigma_1,\sigma_2) & \longrightarrow & K_1^{\mathrm{ep}}(B\pi) & \longrightarrow & 0\\
 & & \Big\downarrow{=} & & \Big\downarrow{\rho^{\mathrm{ep}}} & & \Big\downarrow{\rho^{\mathrm{ep}}(\sigma_1,\sigma_2)} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & \mathbb{R}/\mathbb{Z} & \longrightarrow & 0.
\end{array}
$$

This tells us when the $\mathbb{R}/\mathbb{Z}$-index theorem can be refined to an $\mathbb{R}$-index theorem.

6. Applications to positive scalar curvature

Using the above isomorphisms of K-homologies and cobordism theories, we can immediately transfer results on positive scalar curvature from the odd-dimensional case to the even-dimensional case in which a primitive 1-form is given.

6.1. **Odd-dimensional results in the literature.**

First we will state the odd-dimensional results that we will be generalising to the even-dimensional case using our isomorphisms. The first ones are obstructions to positive scalar curvature.

Theorem 6.1 (Weinberger [36], Higson-Roe Theorem 6.9 [20])**.** *Let (M, S, f) be an odd K-cycle for $B\pi$, where M is an odd dimensional spin manifold with a Riemannian metric of positive scalar curvature, and S is the bundle of spinors on M. Then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated rho invariant $\rho(\sigma_1, \sigma_2\,; M, S, f)$ is a rational number.*

Theorem 6.2 (Higson-Roe Remark 6.10 [20])**.** *Let (M, S, f) be an odd K-cycle for $B\pi$, where M is an odd dimensional spin manifold with a Riemannian metric of positive scalar curvature, and S is the bundle of spinors on M. If the maximal Baum-Connes map for π is injective, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated rho invariant $\rho(\sigma_1, \sigma_2\,; M, S, f)$ is an integer.*

Remarks 6.3. The maximal Baum-Connes map for π is injective whenever for instance π is a torsion-free linear discrete group, [19].

Theorem 6.4 (Higson-Roe Theorem 1.1 [20], Keswani [23])**.** *Let (M, S, f) be an odd K-cycle for $B\pi$, where M is an odd dimensional spin manifold with a Riemannian metric of positive scalar curvature, and S is the bundle of spinors on M. If the maximal Baum-Connes conjecture holds for π, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated rho invariant $\rho(\sigma_1, \sigma_2\,; M, S, f)$ is zero.*

Remarks 6.5. The maximal Baum-Connes conjecture holds for π whenever π is K-amenable.

We now turn to a result on the number of path components of the moduli space of PSC metrics modulo diffeomorphism, $\mathfrak{M}^+(M)$. Denote for a group π, the representation ring $R(\pi)$ consisting of formal differences of finite dimensional unitary representations, and let $R_0(\pi)$ be those formal differences with virtual dimension zero (an element of $R_0(\pi)$ can be thought of as an ordered pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$). Following Botvinnik and Gilkey [12], introduce the subgroups

$$R_0^{\pm}(\pi) = \{\alpha \in R_0(\pi) : \mathrm{tr}(\alpha(\lambda)) = \pm\, \mathrm{tr}(\alpha(\lambda^{-1})) \text{ for all } \lambda \in \pi\}$$

and define

$$r_m(\pi) = \begin{cases} \mathrm{rank}_{\mathbb{Z}}\, R_0^+(\pi) & \text{if} \quad m = 3 \mod 4, \\ \mathrm{rank}_{\mathbb{Z}}\, R_0^-(\pi) & \text{if} \quad m = 1 \mod 4. \end{cases}$$

The following is a result of Botvinnik and Gilkey on the number of path components of the moduli space of PSC metrics modulo diffeomorphism.

Theorem 6.6 (Botvinnik-Gilkey Theorem 0.3 [12])**.** *Let M be a compact connected spin manifold of odd dimension $m \geq 5$ admitting a metric of positive scalar curvature. Suppose that $\pi = \pi_1(M)$ is finite and nontrivial, and that $r_m(\pi) > 0$. Then the moduli space of PSC metrics modulo diffeomorphism $\mathfrak{M}^+(M)$ has infinitely many path components.*

Their proof involves finding a countably indexed family of metrics g_i of positive scalar curvature on M so that $\rho(M, g_i) \neq \rho(M, g_j)$ for $i \neq j$. If these metrics were homotopic through PSC metrics, then they would lie in the same PSC bordism class and hence have equal rho invariants. We will extend this result to the even-dimensional case under the additional hypothesis of 'psc-adaptability'; see Definition 6.11.

6.2. **Our even dimensional results.** In the following theorems, we assume that Y is a submanifold of X that is Poincaré dual to a primitive class $\gamma \in H^1(X, \mathbb{Z})$ such that the scalar curvature of Y in the induced metric is positive. By a theorem of [34], if $\dim(X) = n \leq 7$, then every homology class in $H_{n-1}(X, \mathbb{Z})$ has a representative that is a smooth, orientable minimal hypersurface. It follows that if X is spin with positive scalar curvature, then Poincaré dual to a primitive class $\gamma \in H^1(X, \mathbb{Z})$ can be chosen to be a smooth, spin minimal hypersurface Y, and it follows that the scalar curvature of Y in the induced metric is positive. So our assumption in the Theorems below are automatically true when $\dim(X) = n \leq 7$.

The following is our even dimensional analog of Theorem 6.1.

Theorem 6.7. *Let (X, S, γ, f) be an odd K^{ep}-cycle for $B\pi$, where X is an even dimensional spin manifold with a Riemannian metric of positive scalar curvature, S is the bundle of spinors on X and γ a primitive class in $H^1(X, \mathbb{Z})$ such that there is a Poincaré dual submanifold Y whose scalar curvature in the induced metric is positive. Then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated end-periodic rho invariant $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f)$ is a rational number.*

Proof. The odd K^{ep}-cycle for $B\pi$, (X, S, γ, f) determines an odd K-cycle for $B\pi$, (Y, S^+, f) where Y is a Poincaré dual submanifold for γ having positive scalar curvature, where Y has an induced spin structure. By Theorem 6.1, $\rho(\sigma_1, \sigma_2\,; Y, S^+, f) \in \mathbb{Q}$. By Theorem 3.15 it follows that $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f) \in \mathbb{Q}$ as claimed. ☐

Next is our even dimensional analog of Theorem 6.2, and is argued as above.

Theorem 6.8. *Let (X, S, γ, f) be an odd K^{ep}-cycle for $B\pi$, where X is an even dimensional spin manifold with a Riemannian metric of positive scalar curvature, S is the bundle of spinors on X and γ a primitive class in $H^1(X, \mathbb{Z})$ such that there is a Poincaré dual submanifold Y whose scalar curvature in the induced metric is positive. If the maximal Baum-Connes map for π is injective, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated end-periodic rho invariant $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f)$ is an integer.*

Proof. As for Theorem 6.7. □

Here is the even dimensional analog of Theorem 6.4.

Theorem 6.9. *Let (X, S, γ, f) be an odd K^{ep}-cycle for $B\pi$, where X is an even dimensional spin manifold with a Riemannian metric of positive scalar curvature, S is the bundle of spinors on X and γ a primitive class in $H^1(X, \mathbb{Z})$ such that there is a Poincaré dual submanifold Y whose scalar curvature in the induced metric is positive. If the maximal Baum-Connes conjecture holds for π, then for any pair of unitary representations $\sigma_1, \sigma_2 : \pi \to U(N)$, the associated end-periodic rho invariant $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f)$ is zero.*

Proof. The odd K^{ep}-cycle for $B\pi$, (X, S, γ, f) determines an odd K-cycle (Y, S^+, f) for $B\pi$, where Y is a Poincaré dual submanifold for γ having positive scalar curvature, and is endowed with the induced spin structure. By Theorem 6.4, $\rho(\sigma_1, \sigma_2\,; Y, S^+, f) = 0$. By 4.17 it follows that $\rho^{\mathrm{ep}}(\sigma_1, \sigma_2\,; X, S, \gamma, f) = 0$. □

Example 6.10. Although ρ-invariants are difficult to compute, nevertheless thanks to many authors, there is now a decent set of computations that are available. We can use these to compute end-periodic rho invariants, which we will show in a simple example. Consider $Y = S^1$ with the trivial spin structure. Then unitary characters σ_1, σ_2 of the fundamental group of S^1 can be identified with real numbers, and a computation (cf. page 82, [14]) says that the rho invariant of the spin Dirac operator is, $\rho(S^1, \sigma_1, \sigma_2) = \sigma_1 - \sigma_2 \mod \mathbb{Z}$. In particular, $\rho(S^1, \sigma_1, \sigma_2)$ can take on any real value $\mod \mathbb{Z}$. Let W be a spin cobordism from S^1 to S^1, and Σ be the compact spin Riemann surface (whose genus is ≥ 1) obtained as a result of gluing the two boundary components of W. Then S^1 is a codimension one submanifold of Σ that represents a generator a of $\pi_1(\Sigma)$. We can extend the characters σ_1, σ_2 of $a\mathbb{Z}$ to all of $\pi_1(\Sigma)$ by declaring them to be trivial on the other generators. Then by Theorem 3.15, it follows that $\rho^{\mathrm{ep}}(\Sigma, \gamma, \sigma_1, \sigma_2) = \sigma_1 - \sigma_2 \mod \mathbb{Z}$, can take on any real value $\mod \mathbb{Z}$, where γ is the degree one cohomology class on Σ which is Poincaré dual to S^1. We conclude by Theorem 6.7 that the Riemann surface Σ does not admit a PSC metric. This of course can also be proved by the Gauss-Bonnet theorem and is well known.

The construction generalises easily to any odd dimensional spin manifold Y with non-zero rho invariant $\rho(Y, \sigma_1, \sigma_2) \neq 0 \mod \mathbb{Z}$. We conclude by Theorem 3.15 that the resulting even dimensional spin manifold X constructed from a spin cobordism from Y to itself, has non-zero end-periodic rho invariant $\rho^{\mathrm{ep}}(X, \gamma, \sigma_1, \sigma_2) \neq 0 \mod \mathbb{Z}$ where γ is the degree one cohomology class on X which is Poincaré dual to the submanifold Y. In particular, such an X does not admit a PSC metric. Examples of Y include odd-dimensional lens spaces $L(p; \vec{q})$, where it is shown in Theorem 2.5, part (c) [15], that for any spin structure on $L(p; \vec{q})$, there is a representation σ of $\pi_1(L(p; \vec{q}))$ such that $\rho(L(p; \vec{q}), \mathrm{Id}, \sigma) \neq 0 \in \mathbb{Q}/\mathbb{Z}$. Explicitly, for 3 dimensional lens spaces $L(p, q)$, consider the representation $\sigma : \pi_1(L(p; \vec{q})) \longrightarrow U(1)$ taking the generator $t \in \pi_1(L(p; q))$ to the unit complex number $\exp(2\pi\sqrt{-1}/p)$. Then

$\rho(L(p;q), \mathrm{Id}, \sigma) = -\left(\frac{d}{2p}\right)(p+1) \neq 0 \in \mathbb{Q}/\mathbb{Z}$ where d is a certain integer relatively prime to $48p$. Then $\rho^{\mathrm{ep}}(X, \gamma, \mathrm{Id}, \sigma) \neq 0 \in \mathbb{Q}/\mathbb{Z}$. These results confirm Theorem 6.7 in these examples.

6.3. **Size of the space of components of positive scalar curvature metrics.** Hitchin [21] proved the first results on the size of the space of components of the space of Riemannian metrics of positive scalar curvature metrics on a compact spin manifold, when non-empty. This sparked much interest in the topic and results by Botvinnik-Gilkey, Piazza-Schick and many others.

We now extend Theorem 6.6 to the even dimensional case. We would like to say something like 'Given an even-dimensional manifold X with PSC having a submanifold Y of PSC Poincaré dual to a primitive one-form γ, if $\mathfrak{M}^+(Y)$ has infinitely many path components then so does $\mathfrak{M}^+(X)$.' The argument would involve using a countable family of PSC metrics on Y with distinct rho invariants to find a countable such family on X. There are complications however, since given an arbitrary PSC metric on Y, there is not necessarily a PSC metric on X whose restriction to Y is the given metric. Because we are already assuming that there is at least one PSC metric on X which restricts to a metric of PSC on Y, there are no obstructions from topology preventing this from being the case.

Definition 6.11. Let X be a compact even dimensional manifold, and $\gamma \in H^1(X, \mathbb{Z})$ a primitive cohomology class with accompanying Poincaré dual submanifold Y. Suppose that there is at least one PSC metric on X which restricts to a PSC metric on Y. We say that X is *psc-adaptable with respect to* Y if for *every* PSC metric g_Y on Y, there is a PSC metric g_X on X whose restriction to Y is g_Y.

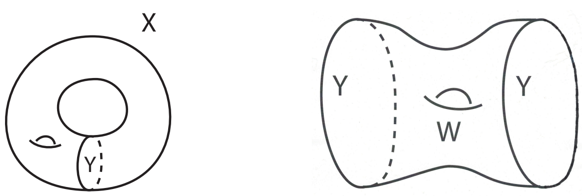

FIGURE 5. explaining psc-adaptable

Some notes and comments on the notion of psc-adaptability. Let X and Y be as in the above definition, and take an arbitrary PSC metric g_Y on Y. Cutting X open along Y, we obtain a self cobordism W of Y; see Figure 5. Under suitable assumptions on the topology of X and Y, a construction of Miyazaki [27] and Rosenberg [33] (using the theory of Gromov-Lawson [17] and Schoen-Yau [34]) enables one to *push* the psc metric on Y across the bordism (pictured on the right in the figure) to get a PSC metric on W restricting to metrics of PSC on each boundary component. One might then try to glue the manifold back together to obtain a PSC metric on X which restricts to the given metric g_Y on Y. The problem is that one doesn't know whether the new psc metric on Y is isotopic to the original (this would be true if the general *concordance = isotopy conjecture* were true, cf. Botvinnik [9, 10, 11]). Hence the concept of psc-adaptability which hypothesizes that this is true. It is the case when the bordism is *symmetric* for instance. That is, starting with a bordism W' from Y to Y', we get a bordism from Y to itself by thinking of W' as a bordism from Y' to Y and gluing to the original bordism, see Figure 6.

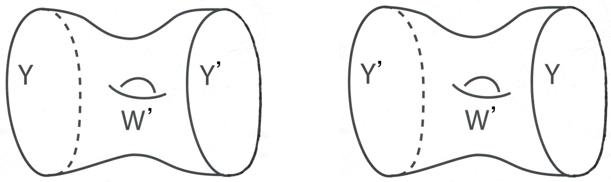

FIGURE 6. explaining psc-adaptable

Then one can use the Miyazaki-Rosenberg construction starting with the PSC metric Y to get another another PSC metric on Y' halfway through, and then reverse the Miyazaki-Rosenberg construction from the PSC metric on the halfway Y' to get a PSC metric on Y on the other end. In this case, we end up with the original PSC metric on Y. Since the metrics agree on either end, the bordisms can be glued together.

Mrowka, Ruberman and Saveliev also note a class of psc-adaptable manifolds – those of the form $(S^1 \times Y)\#M$ where Y and M are manifolds of positive scalar curvature, see [28] Theorem 9.2. The end-periodic bordism groups provide a more natural framework for their proof of the following:

Theorem 6.12 (Theorem 9.2, [28])**.** *Let X be a compact even-dimensional spin manifold of dimension ≥ 6 admitting a metric of positive scalar curvature. Suppose there is a submanifold $Y \subset X$ of PSC that is Poincaré dual to a primitive cohomology class $\gamma \in H^1(X, \mathbb{Z})$, such that $\pi = \pi_1(Y)$ is finite and non-trivial. Further assume that the classifying map $f : Y \to B\pi$ of the universal cover extends to X, and that X is psc-adaptable with respect to Y. If $r_m(\pi_1(Y)) > 0$, then $\pi_0(\mathfrak{M}^+(X))$ is infinite, where $\mathfrak{M}^+(X)$ denotes the quotient of the space of positive scalar curvature metrics by the diffeomorphism group.*

Proof. In the terminology of Section 4, we have an $\Omega_m^{\mathrm{ep,spin},+}(B\pi)$-cycle $(X, g, \sigma, \gamma, f)$, with associated $\Omega_m^{\mathrm{spin},+}(B\pi)$-cycle (Y, g, σ, f). Botvinnik and Gilkey [12] construct a representation $\alpha : \pi \to U(N)$ of π and a countable family of metrics g_i on Y with

$$\rho(\alpha, 1\,; Y, g_i, \sigma, f) \neq \rho(\alpha, 1\,; Y, g_j, \sigma, f)$$

for $i \neq j$, where $1 : \pi \to U(N)$ is the trivial representation. Our assumption of psc-adaptability and Theorem 4.15 imply there is an countable family of metrics g_i on X with

$$\rho^{\mathrm{ep}}(\alpha, 1\,; X, g_i, \sigma, \gamma, f) \neq \rho(\alpha, 1\,; X, g_j, \sigma, \gamma, f)$$

for $i \neq j$. But Theorem 9.1 of MRS [28] says that homotopic metrics of PSC on X should have the same rho invariants. □

7. Vanishing of end-periodic rho using the representation variety

In this section we give a proof of the vanishing of the end-periodic rho invariant of the twisted Dirac operator with coefficients in a flat Hermitian vector bundle on a compact even dimensional Riemannian spin manifold X of positive scalar curvature using the representation variety of $\pi_1(X)$ instead.

Let $\iota \colon Y \hookrightarrow X$ be a codimension one submanifold of X which is Poincaré dual to a generator $\gamma \in H^1(X, \mathbb{Z})$.

Let $\mathfrak{R} = \mathrm{Hom}(\pi, U(N))$ denote the representation variety of $\pi = \pi_1(Y)$, and $\tilde{\mathfrak{R}}$ denote the representation variety of $\pi_1(X)$. We now construct a generalization of the Poincaré vector bundle $\mathcal{P}$ over $B\pi \times \mathfrak{R}$. Let $E\pi \to B\pi$ be a principal π-bundle over the space $B\pi$ with contractible total space $E\pi$. Let $h\colon Y \to B\pi$ be a continuous map classifying the universal π-covering of Y. We construct a tautological rank N Hermitian vector bundle $\mathcal{P}$ over $B\pi\times\mathfrak{R}$ as follows: consider the action of π on $E\pi \times \mathfrak{R} \times \mathbb{C}^N$ given by

$$\begin{aligned} E\pi \times \mathfrak{R} \times \mathbb{C}^N \times \pi &\longrightarrow E\pi \times \mathfrak{R} \times \mathbb{C}^N \\ ((q,\sigma,v),\tau) &\longrightarrow (q\tau,\sigma,\sigma(\tau^{-1})v). \end{aligned}$$

Define the universal rank N Hermitian vector bundle $\mathcal{P}$ over $B\pi \times \mathfrak{R}$ to be the quotient $(E\pi\times\mathfrak{R}\times\mathbb{C}^N)/\pi$. Then $\mathcal{P}$ has the property that the restriction $\mathcal{P}\big|_{B\pi\times\sigma}$ is the flat Hermitian vector bundle over $B\pi$ defined by σ. Let I denote the closed unit interval $[0,1]$ and $\beta : I \to \mathfrak{R}$ be a smooth path in $\mathfrak{R}$ joining the unitary representation α to the trivial representation. Define $E = (f\times\beta)^*\mathcal{P} \to X\times I$ to be the Hermitian vector bundle over $X\times I$. By the Kunneth Theorem in cohomology, we have $\operatorname{ch}(F) = \sum_i x_i\xi_i$, where $\operatorname{ch}(F)$ is the Chern character of F, for some $x_i \in H^*(B\pi,\mathbb{R})$ and $\xi_i \in H^*(\mathfrak{R},\mathbb{R})$, by the Kunneth theorem. It follows that if $y_i = f^*(x_i)$ and $\mu_i = \beta^*(\xi_i)$,then $\operatorname{ch}(E) = \sum_i y_i\mu_i$. Note that the pullback connection makes E into a Hermitian vector bundle over $Y \times I$.

Theorem 7.1 (PSC and vanishing of end-periodic rho)**.** *Let (X,g) be a compact spin manifold of even dimension, and let $\iota\colon Y \hookrightarrow X$ be a codimension one submanifold of X which is Poincaré dual to a primitive class $\gamma \in H^1(X,\mathbb{Z})$. Suppose that*

(1) g is a Riemannian metric of positive scalar curvature;

(2) the restriction $g\big|_Y$ is also a metric of positive scalar curvature.

Let π denote the fundamental group of Y and $\alpha\colon \pi \to U(N)$ a unitary representation that can be connected by a smooth path $\beta : I \to \mathfrak{R}$ to the trivial representation in the representation space $\mathfrak{R}$, and the induced unitary representation $\tilde\alpha\colon \tilde\pi \to U(N)$, where $\tilde\pi = \pi_1(X)$. Then $\rho^{\mathrm{ep}}(X,S,\gamma,g;\tilde\alpha,1) = 0$, where the flat hermitian bundle $E_{\tilde\alpha}$ is determined by $\tilde\alpha$.

Proof. Observe that the unitary connection induced on E has curvature which is a multiple of dt, so that $\operatorname{ch}(E) = N + c_1(E)$, where $c_1(E)$ is the first Chern class of E and t is the variable on the interval I. It follows that $\operatorname{ch}(E) = N + y\mu$ where $y \in H^1(Y,\mathbb{R})$ and $\mu \in H^1(I,\mathbb{R})$, as $c_1(E)$ can be represented by the trace of the curvature of a unitary connection on E. Since $c_1(E) = (f\times\beta)^*c_1(F)$, we see that $y = f^*(x)$ and $\mu = \beta^*(\xi)$ for some $x \in H^1(B\pi,\mathbb{R})$ and $\xi \in H^1(\mathfrak{R},\mathbb{R})$. Let E_t denote the flat hermitian bundle over Y determined by the representation $\gamma(t) : \pi \to U(N)$. Consider the integrand $\int_{Y\times I} \widehat{A}(Y \times I)\operatorname{ch}(E)$. Since $\widehat{A}(Y \times I) = \widehat{A}(Y)$, where $\widehat{A}(Y)$ is the A-hat characteristic class of Y. From the discussion above

$$\int_{Y\times I} \widehat{A}(Y)\operatorname{ch}(E) = \int_Y \widehat{A}(Y)y \int_I \mu.$$

Since (Y,g) is a spin Riemannian manifold of positive scalar curvature, it follows from the work of Gromov-Lawson [16] that $\displaystyle\int_Y \widehat{A}(Y)f^*(x) = 0$ for all $x \in H^1(B\pi,\mathbb{R})$.

Therefore we conclude that $\displaystyle\int_{Y\times I} \widehat{A}(Y)\operatorname{ch}(E) = 0.$

Consider the manifold $Y \times I$. It can be made into an end-periodic manifold with two ends as follows. Let W be the fundamental segment obtained by cutting X open along Y, and W_k

be isometric copies of W. Then we can attach $X_1 = \cup_{k\geq 0} W_k$ to one boundary component of $Y \times I$ and $X_0 = \cup_{k<0} W_k$ to the other boundary component . Call the resulting end-periodic manifold Z_∞ (see the Figure 7). It is clear that Z_∞ is diffeomorphic to $\tilde{X}$, the cyclic Galois cover of X corresponding to γ. Let $f_0 = -f$ and $f_1 = f$ for a choice of real-valued function f on Z_∞ such that $\gamma = [df]$.

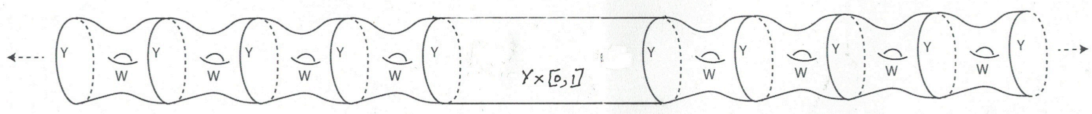

FIGURE 7. End-periodic manifold with 2 ends

The flat hermitian bundle $E_{\tilde{\alpha}}$ over X induces a flat hermitian bundle $p^*(E_{\tilde{\alpha}})$ over $\tilde{X}$, where $p\colon \tilde{X} \to X$ is the projection. The restriction of $p^*(E_{\tilde{\alpha}})$ to the subset X_1 is denoted by E_1. Let E_0 denote the trivial bundle over X_0. We use the smooth path γ to define the bundle E over $Y \times I$ which has the property that the restriction of $\tilde{E}$ to the boundary components agree with E_0 and E_1, thereby defining a global vector bundle $\tilde{E}$ over Z_∞.

We can apply Theorem C in [28] to see that

$$\begin{aligned}\operatorname{index}(D^+_{\tilde{E}}(Z_\infty)) = &\int_{Y\times I} \widehat{A}(Y\times I)\operatorname{ch}(E) - \int_Y \omega + \int_X df\wedge\omega - \frac{1}{2}(h_1 + \eta_{ep}(X, E_{\tilde{\alpha}}, \gamma, g) \\ &+ \int_Y \omega - \int_X df\wedge\omega - \frac{1}{2}(h_0 - \eta_{ep}(X, E_{id}, \gamma, g)\end{aligned}$$

Since g and $g\big|_Y$ are metrics of positive scalar curvature by hypothesis, it follows that $\operatorname{index}(D^+_E(Z_\infty)) = 0$ by Lemma 8.1 in [28] and that $\displaystyle\int_{Y\times I} \widehat{A}(Y\times I)\operatorname{ch}(E) = 0$ by the earlier argument. Therefore $\rho^{ep}(X, S, \gamma, g; \tilde{\alpha}, 1) = 0$ as claimed.

□

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Appendix A

Topology on the space of smooth sections

Here we give a quick overview of the definitions and basic facts on the topology on the space $\Gamma(E)$ of smooth sections of vector bundle E. The Fréchet topology on $\Gamma(E)$ is needed to study the moduli space of positive scalar curvature metrics, the metrics being certain sections of the second symmetric power of the cotangent bundle.

A.1 Fréchet spaces

In this section we take our vector spaces to be complex, although there is no difference in taking real vector spaces. Let E be a topological vector space (not necessarily Hausdorff). We call E a *Fréchet* space if:

1. The topology of E is locally convex (each point has arbitrarily small convex open neighbourhoods),
2. There exists a translation invariant metric inducing the topology of E, and
3. E is complete.

Although it is possible to equip E with a translation invariant metric, we do not really care what the metric is, not unless E happens to be a Hilbert or Banach space. The third condition can be phrased entirely in terms of the topological vector space structure of E, although it turns out to be equivalent that any translation invariant metric inducing the topology be complete.

Although the above definition is a nice abstract characterisation of Fréchet spaces, it is often convenient to work with a more practical definition. Recall a *seminorm* on a vector space is a map $p : E \to [0, \infty)$ such that

- $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in E$, and
- $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{C}$ and $x \in E$.

Thus, a seminorm is like a norm, although we don't require $p(x) = 0$ to imply that $x = 0$. Given a family of seminorms $\{p_i\}_{i \in I}$ on E, we can define a topology on E as follows. First, for $i \in I$, $x \in E$ and $\epsilon > 0$ define

$$B_i(x, \epsilon) = \{y \in E : p_i(x - y) < \epsilon\}.$$

The topology on E is then defined by taking the following as a subbasis for the topology:

$$\{B_i(x, \epsilon) : i \in I, x \in E, \epsilon > 0\}.$$

Theorem A.1.1. *A topological vector space E is Fréchet if and only if its topology is induced by a countable family of seminorms $\{p_n\}_{n \in \mathbb{N}}$ such that*

1. *The family $\{p_n\}_{n \in \mathbb{N}}$* separates points*, meaning that $p_n(x) = 0$ for all $n \in \mathbb{N}$ implies $x = 0$.*
2. *The topology induced by the family $\{p_n\}$ is complete.*

The first condition is there to ensure the resulting topology is Hausdorff. The countability assumption on the family of seminorms implies the resulting topology is induced by a translation invariant metric. It is a basic principle that all important properties of Fréchet spaces can be phrased in terms of the seminorms. For instance, $x_n \to x$ in E if and only if $p_N(x_n - x) \to 0$ for all $N \in \mathbb{N}$.

Example A.1.2. Let $U \subset \mathbb{R}^n$ be a non-empty open subset. We equip the vector space $C^\infty(U, \mathbb{C})$ with the following family of seminorms: for $K \subset U$ compact and $N \in \mathbb{N}$, define

$$p_{K,N}(f) = \sup_{|\alpha| \leqslant N} \sup_{x \in K} |\partial^\alpha f(x)|.$$

We take the topology on $C^\infty(U, \mathbb{C})$ to be induced by the seminorms $p_{K,N}$ as K ranges over all compact subsets of U and N ranges over all the natural numbers. By taking a countable exhaustion of U by compact subsets, it follows that the topology is induced by a countable family of seminorms. It is a consequence of the Arzela-Ascoli theorem that the resulting topology is complete—see [Rud91, Chapter 1].

A.2 Topology on $\Gamma(E)$

Now, let M be a smooth manifold (not necessarily compact) and $E \to M$ a smooth vector bundle. We will equip the vector space of smooth sections $\Gamma(E)$ with a Fréchet topology. Again we work with complex vector bundles, but everything goes through for real vector bundles in the same way.

The seminorms on $\Gamma(E)$ will be indexed by the set $\mathcal{I}$ consisting of all quintuples $Q = (U, \varphi, \psi, K, N)$, where:

- $U \subset M$ is an open subset,
- $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ is a coordinate chart,
- $\psi : E|_U \to U \times \mathbb{C}^k$ is a trivialisation of E,
- $K \subset \varphi(U)$ is compact, and
- $N \in \mathbb{N}$.

Given a section $s \in \Gamma(E)$ and a quintuple $Q \in \mathcal{I}$, we define the map $f_s : \varphi(U) \to \mathbb{C}^k$ to be the composition

$$\varphi(U) \xrightarrow{\varphi^{-1}} U \xrightarrow{s} E|_U \xrightarrow{\psi} U \times \mathbb{C}^k \xrightarrow{\mathrm{pr}_2} \mathbb{C}^k$$

where $\mathrm{pr}_2 : U \times \mathbb{C}^k \to \mathbb{C}^k$ is the projection. We then associate to the quintuple $Q = (U, \varphi, \psi, K, N)$ the seminorm

$$p_Q(s) = \sup_{|\alpha| \leqslant N} \sup_{x \in K} |\partial^\alpha f_s(x)|. \tag{A.1}$$

The topology on $\Gamma(E)$ is then taken to be induced by the collection of seminorms $\{p_Q\}_{Q \in \mathcal{I}}$.

Proposition A.2.1. *With the topology induced from the family of seminorms* $\{p_Q\}_{Q \in \mathcal{I}}$*, where* p_Q *is defined by* (A.1)*, the space of sections* $\Gamma(E)$ *is a Fréchet space.*

Sketch proof. It is clear that if $p_Q(s) = 0$ for all Q, then $s = 0$. Thus, the topology induced by these seminorms is Hausdorff.

We must show that the topology can be induced by a countable family of seminorms. For this, let $\{U_\alpha\}$ be an cover of M such that each U_α is a coordinate domain, and $E|_{U_\alpha}$ is trivial for each α. Since M is second countable, every open cover has a finite subcover, so we can take $\{U_\alpha\}$ to be countable. For each α, we take an exhaustion $K_{\alpha,1} \subset K_{\alpha,2} \subset \cdots$ of $\varphi_\alpha(U_\alpha)$ by compact subsets. It is routine to check that the collection of seminorms

$p_{Q_{\alpha,i}}$ with $Q_{\alpha,i} = (U_\alpha, \varphi_\alpha, \psi_\alpha, K_{\alpha,i}, N)$ induces the same topology on $\Gamma(E)$, and this collection is countable.

Finally, it remains to check completeness. The argument is essentially the same as for $C^\infty(U)$. One uses the Arzela-Ascoli theorem locally, using the trivialisations and coordinate charts to transfer information from the vector bundle to Euclidean space. □

Remark A.2.2. Setting $E = M \times \mathbb{C}$, we recover the Fréchet space of smooth functions $C^\infty(M, \mathbb{C})$.

Let V be a Fréchet space and let $\gamma : (a, b) \to V$ be a curve in V. We say that γ is *differentiable at* $t \in (a, b)$ if the limit

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t + h) - \gamma(t)}{h}$$

exists. If γ is differentiable at t, then it is continuous at t. If γ is differentiable at every $t \in (a, b)$, we say γ is *differentiable*, and the function $\gamma' : (a, b) \to V$ is called the *derivative of* γ. If the derivative of γ is continuous, we say γ is *continuously differentiable*. We define, when they exist, the iterated derivatives

$$\gamma^{(k)}(t) = (\gamma^{(k-1)})'(t),$$

with $\gamma^{(0)}(t) = \gamma(t)$. We call γ *smooth* if all iterated derivatives $\gamma^{(k)}$ exist (they are then automatically continuous). A map $\gamma : [a, b] \to V$ is called *smooth* if it extends to a smooth map on some slightly larger open interval containing $[a, b]$.

If $\gamma \in \Gamma([0, 1] \times M, E)$ is a smooth section of E pulled back to $[0, 1] \times M$, define the *transpose* of γ to be

$$\overline{\gamma}(t)(x) = \gamma(t, x).$$

The following is proved by routine, yet tedious verifications.

Theorem A.2.3. *Let M be a smooth compact manifold and $E \to M$ a smooth vector bundle. There is a natural bijection*

$$\Gamma([a, b] \times M, E) \cong C^\infty([a, b], \Gamma(E))$$

given by $\gamma \mapsto \overline{\gamma}$.

Alternatively this can be deduced from more general and sophisticated results, as may be found in [Mic80] and [KM97].

Appendix B

Dirac operators and index theory

In this appendix, we first cover the fundamental algebraic concepts needed to define Dirac operators, including Clifford algebras and Spin groups. Following this, we introduce Dirac operators on manifolds, give important geometric examples, and state basic formulas for Dirac operators that are used in the thesis. We end by stating the local index theorem, and the various incarnations of the Atiyah-Singer index theorem for Dirac operators.

B.1 Algebraic foundations

The references for this section are Section 1 of [ABS64] and Chapter 1 of [LM89].

B.1.1 Clifford algebras

Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$ with quadratic form q. The Clifford algebra $\mathrm{C}\ell(V,q)$ is defined as the quotient of the tensor algebra $\bigoplus_{i=0}^{\infty} V^{\otimes i}$ by the two-sided ideal generated by all elements of the form $v \otimes v + q(v) \cdot 1$. The most important case for us is that where $V = \mathbb{R}^n$ and q is given by the standard inner product, $q(v) = \langle v, v \rangle$. The algebra is then denoted $\mathrm{C}\ell_n$, and is the universal $\mathbb{R}$-algebra generated by $1, e_1, \ldots, e_n$ subject to the relations

$$\begin{cases} e_i^2 = -1 & i = 1, \ldots, n \\ e_i e_j + e_j e_i = 0 & i \neq j. \end{cases}$$

Then $\mathrm{C}\ell_n$ is an $\mathbb{R}$-algebra of dimension 2^n with basis elements 1 and $e_{i_1} \cdots e_{i_k}$, $i_1 < \cdots < i_k$ for $k = 1, \ldots, n$. The complex Clifford algebras are defined as

$\mathbb{C}\ell_n = \mathrm{C}\ell_n \otimes \mathbb{C}$. There is a canonical vector space inclusion $V \overset{\iota}{\hookrightarrow} \mathrm{C}\ell(V, q)$. Clifford algebras have the following universal property:

Proposition B.1.1 ([LM89] Proposition I.1.1)**.** *Let (V, q) be a vector space with quadratic form, and let A be an algebra with 1 over the same field. If $f : V \to A$ satisfies $f(v)^2 = -q(v) \cdot 1$ for all $v \in V$, then there exists a unique homomorphism of algebras $\widetilde{f} : \mathrm{C}\ell(V, q) \to A$ such that $\widetilde{f} \circ \iota = f$.*

Define $\alpha : V \to \mathrm{C}\ell(V, q)$ by $\alpha(v) = -v$. Then $\alpha(v)^2 = -q(v)$, so α extends to a well-defined map $\alpha : \mathrm{C}\ell(V, q) \to \mathrm{C}\ell(V, q)$. The map α is an involution, and the Clifford algebra splits into the ± 1 eigenspaces of α. We define

$$\mathrm{C}\ell^0(V, q) = \{\varphi : \alpha(\varphi) = \varphi\}, \qquad \mathrm{C}\ell^1(V, q) = \{\varphi : \alpha(\varphi) = -\varphi\}.$$

These definitions give $\mathrm{C}\ell(V, q)$ the structure of a $\mathbb{Z}_2$-graded algebra; we call elements of $\mathrm{C}\ell^0(V, q)$ *even* and elements of $\mathrm{C}\ell^1(V, q)$ *odd*.

There is a well known classification of Clifford algebras in terms of matrix algebras that is 8-periodic in the real case and 2-periodic in the complex case; see pages 28-29 of [LM89]. This classification immediately reveals the representation theory of Clifford algebras, for example $\mathbb{C}\ell_n$ has two inequivalent irreducible representations when n is odd, and a unique irreducible representation when n is even [LM89, page 32]. The irreducible representations have dimension $2^{[n/2]}$, where $[n/2]$ is the greatest integer less than or equal to $n/2$.

We will be interested only in complex representations of Clifford algebras. We call a complex vector space S a (complex) *Clifford module* if it is equipped with a homomorphism of algebras

$$c : \mathrm{C}\ell_n \to \mathrm{End}_{\mathbb{C}}(S),$$

called *Clifford multiplication*. We require that Clifford multiplication by φ commutes with multiplication by $i \in \mathbb{C}$, so that c extends to a $\mathbb{C}$-linear map from the complex Clifford algebra $\mathbb{C}\ell_n$ to $\mathrm{End}_{\mathbb{C}}(S)$. Note that by the universal property, a Clifford module is equivalent to a linear map $c : \mathbb{R}^n \to \mathrm{End}_{\mathbb{C}}(S)$ such that $c(v)^2 = -\|v\|^2 \mathrm{Id}_S$ for all $v \in \mathbb{R}^n$, and $c(v)$ commutes with i.

A *$\mathbb{Z}_2$-graded Clifford module* is a Clifford module S equipped with a direct sum decomposition $S = S^+ \oplus S^-$ such that Clifford multiplication by elements in $\mathrm{C}\ell^0_n$ takes $S^\pm$ to $S^\pm$, and multiplication by elements in $\mathrm{C}\ell^1_n$ takes $S^\pm$ to $S^\mp$. Elements of a Clifford module are called *spinors*; spinors in S^+ or S^- are respectively called positive or negative spinors.

Example B.1.2. In the even-dimensional case, the irreducible complex representation of $\mathbb{C}\ell_{2n}$ can be expressed as follows. For $v \in \mathbb{C}^n$ and $\xi \in \Lambda^*_{\mathbb{C}}\mathbb{C}^n$, define

$$c(v)\xi = v \wedge \xi - v\llcorner\xi.$$

Here $\llcorner$ is the interior multiplication map

$$v\llcorner(v_0 \wedge \cdots \wedge v_p) = \sum_{i=0}^{p} (-1)^i \langle v_p, v\rangle v_0 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p.$$

The hat on v_i signifies that it does not appear in the product. Note our hermitian inner products are taken to be conjugate linear in the *second* variable, so that the expression $v\llcorner\xi$ is complex linear in ξ. A straightforward but tedious calculation shows that $c(v)^2\xi = -\|v\|^2\xi$. Identifying $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ we thus have a complex representation of $\mathrm{C}\ell_{2n}$. A dimension count shows that it must be the unique irreducible one.

There is more to this example than meets the eye; there is a canonical filtration of the Clifford algebra $F_0 \subset F_1 \subset \cdots \subset F_n = \mathrm{C}\ell_n$, where F_k is the image of $\mathrm{span}\{v_1 \otimes \cdots \otimes v_k : v_i \in \mathbb{R}^n\}$ under the projection from the tensor algebra to $\mathrm{C}\ell_n$. The associated graded algebra of this filtration is in fact the exterior algebra $\Lambda^*(\mathbb{R}^n)$. It follows there is a canonical isomorphism $\mathrm{C}\ell_n \cong \Lambda^*(\mathbb{R}^n)$, but only of *vector spaces*—this is certainly not an isomorphism of algebras. The Clifford multiplication described above is simply left Clifford multiplication on $\mathrm{C}\ell_n$ under this isomorphism.

The complex Clifford algebras have canonical volume elements, depending on the orientation of the vector space. For $\mathbb{R}^n$, we take an oriented orthonormal basis $e_1, \ldots, e_n$ and define the *oriented volume element*

$$\omega = i^{[(n+1)/2]} e_1 \cdots e_n.$$

This definition is independent of the choice of oriented orthonormal basis. The volume element satisfies $\omega^2 = 1$, and commutes (resp. anti-commutes) with elements of $\mathbb{R}^n$ if n is odd (resp. even). In the even-dimensional case, any Clifford module S splits as the ± 1 eigenspaces $S^{\pm}$ of the involution defined by multiplication by ω:

$$S = S^+ \oplus S^-.$$

It is easy to see that this gives S the structure of a $\mathbb{Z}_2$-graded Clifford module, although it is essential that n be even so that elements in $\mathbb{R}^n$ anti-commute with ω.

If S is a Clifford module then there exists a hermitian metric $\langle\,,\rangle$ such that Clifford multiplication by unit vectors is unitary [LM89, Proposition I.5.16]. Moreover, if $S = S^+ \oplus S^-$ is $\mathbb{Z}_2$-graded, then the metric can be chosen so that $S^+ \perp S^-$.

B.1.2 Spin groups

Sitting inside the real Clifford algebras are the all-important spin groups,

$$\mathrm{Spin}(n) = \{v_1 \cdots v_{2k} : \|v_j\| = 1\} \subset \mathrm{C}\ell_n^0 \subset \mathrm{C}\ell_n.$$

The group $\mathrm{Spin}(n)$ is a Lie group of dimension $\frac{n(n-1)}{2}$, and is in fact the universal covering group of $\mathrm{SO}(n)$ (unless $n = 2$, in which case it is the nontrivial double cover of $\mathrm{SO}(2)$). The covering homomorphism is the *adjoint homomorphism* defined as

$$\mathrm{Ad}_\varphi(x) = \varphi x \varphi^{-1}$$

for $\varphi \in \mathrm{Spin}(n)$ and $x \in \mathbb{R}^n$. To see that $\mathrm{Ad}_\varphi \in \mathrm{SO}(n)$, one first proves that for $\|v\| = 1$, $-\mathrm{Ad}_v$ is reflection in the hyperplane orthogonal to v. The result for general $\varphi \in \mathrm{Spin}(n)$ is then clear, since the composition of an even number of reflections belongs to $\mathrm{SO}(n)$. The kernel of the adjoint homomorphism is $\mathbb{Z}_2 = \{\pm 1\}$, and there is an exact sequence

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 0$$

so that $\mathrm{Spin}(n)$ is a double cover of $\mathrm{SO}(n)$. It is in fact the non-trivial cover, since 1 can be connected to -1 in $\mathrm{Spin}(n)$ via the path

$$t \mapsto (\cos(t)e_1 + \sin(t)e_2)(\cos(t)e_1 - \sin(t)e_2); \qquad t \in [0, \pi/2].$$

If $n \geqslant 3$ then $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$, and $\mathrm{Ad} : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the universal cover.

Since $\mathrm{Spin}(n)$ sits inside the Clifford algebra $\mathrm{C}\ell_n$, we can restrict irreducible representations of $\mathrm{C}\ell_n$ to the spin group. Any representation obtained in this way is called a *spin representation* of $\mathrm{Spin}(n)$. When one restricts an irreducible representation to the spin group, the representation will generally split up and become reducible. This is indeed the case in even dimensions, where any Clifford module S has a decomposition into positive and negative spinors $S^+ \oplus S^-$, given by the eigenspaces of multiplication by the oriented volume element ω. The spin group, consisting of even-graded Clifford elements, will take S^+ to itself and S^- to itself. Hence a spin representation S of $\mathrm{Spin}(n)$ (n even) splits into invariant subspaces $S^\pm$.

In low dimensions, there are a few standard isomorphisms that one should keep in mind [LM89, page 50]:

1. $\mathrm{Spin}(2) \cong \mathrm{U}(1)$
2. $\mathrm{Spin}(3) \cong \mathrm{SU}(2)$
3. $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$.

B.2 Spin manifolds

Let M be a smooth, oriented Riemannian manifold. Then we can form the principal $\mathrm{SO}(n)$-bundle P_{SO} of oriented orthonormal tangent frames. We say that M is *spin* if there exists a principal $\mathrm{Spin}(n)$-bundle P_{Spin} and a smooth fibre preserving map

$$\xi : P_{\mathrm{Spin}} \to P_{\mathrm{SO}}$$

that is a double cover and is equivariant, in the sense that

$$\xi(pg) = \xi(p)\mathrm{Ad}(g)$$

for all $p \in P_{\mathrm{Spin}}$ and $g \in \mathrm{Spin}(n)$. We call such a cover of P_{SO} a choice of *spin structure* for M.

Example B.2.1. Suppose that M is a manifold whose tangent bundle has a trivialisation $TM \cong M \times \mathbb{R}^n$ (then M is called *parallelizable*). Such a trivialisation induces an isomorphism $P_{\mathrm{SO}}(TM) \cong M \times \mathrm{SO}(n)$, and there is an obvious equivariant double cover $M \times \mathrm{Spin}(n) \to M \times \mathrm{SO}(n)$ given by the adjoint representation. It follows that any parallelizable manifold has a spin structure. As a corollary, all Lie groups are spin manifolds.

Definition B.2.2. Two spin structures $\xi : P_{\mathrm{Spin}} \to P_{\mathrm{SO}}$ and $\xi' : P'_{\mathrm{Spin}} \to P_{\mathrm{SO}}$ are *isomorphic* if there is a principal $\mathrm{Spin}(n)$-bundle isomorphism $\varphi : P_{\mathrm{Spin}} \to P'_{\mathrm{Spin}}$ such that $\xi' \circ \varphi = \xi$.

We will often abuse terminology, and allow 'spin structure' to mean either a spin structure as above, or an isomorphism class of spin structures.

From here we will assume basic knowledge of characteristic classes, in particular the Stiefel-Whitney and Chern classes of vector bundles, as can be found in [MS74]. If M is orientable, a spin structure on M exists if and only if the second Stiefel-Whitney class $w_2(M) \in H^2(M, \mathbb{Z}_2)$ vanishes [LM89, Theorem II.1.7].

Example B.2.3. Consider the n-sphere S^n as a submanifold of $\mathbb{R}^{n+1}$. The normal bundle N satisfies $TS^n \oplus N \simeq S^n \times \mathbb{R}^{n+1}$. The Whitney summation formula [MS74, page 38] then implies $w_i(TS^n) = 0$ for $i \geqslant 1$, so every n-sphere is spin.

For $n \geqslant 3$ we can see that S^n is spin in another way. The tangent bundle TS^n is determined up to isomorphism by a map $S^{n-1} \to \mathrm{SO}(n)$ (the transition function for the tangent bundle over the equator). Since S^{n-1} is simply connected for $n \geqslant 3$, this map lifts to the universal cover $\mathrm{Spin}(n)$, and so allows us to define a spin structure on S^n.

Proposition B.2.4 ([MS74], page 171)**.** *If X is a complex manifold, then*

$$w_2(X) = c_1(X) \mod 2.$$

Example B.2.5. Consider complex projective space $\mathbb{CP}^n$. There is a short exact sequence of vector bundles

$$0 \to \mathbb{CP}^n \times \mathbb{C} \to (H^*)^{\oplus(n+1)} \to T\mathbb{CP}^n \to 0,$$

where H is the tautological bundle over $\mathbb{CP}^n$ [LM89, page 47]. Hence $c(T\mathbb{CP}^n) \cong c(H^*)^{n+1}$. Now $c = c_1(H^*)$ is the canonical generator of $H^2(\mathbb{CP}^n, \mathbb{Z})$, and we have $c(H^*) = 1 + c$. Hence $c(H^*)^{n+1} = (1+c)^{n+1}$, and therefore

$$c_1(\mathbb{CP}^n) := c_1(T\mathbb{CP}^n) = (n+1)c.$$

Hence the mod 2 reduction of $c_1(\mathbb{CP}^n)$ is zero for n odd, and non-zero for n even. We conclude, via Proposition B.2.4, that all the odd complex projective spaces are spin, and all the even complex projective space are not spin.

Example B.2.6. Recall *lens space* $L(p; q_1, \ldots, q_m)$ is the quotient of $S^{2m-1} \subset \mathbb{C}^m$ by the action of $\mathbb{Z}_p$ given on the generator $e^{2\pi i/p}$ by

$$(z_1, \ldots, z_m) \mapsto (e^{2\pi i q_1/p} z_1, \ldots, e^{2\pi i q_m/p} z_m).$$

Here $p \geqslant 2$ and $q_1, \ldots, q_m$ are non-negative integers $< p$. According to [Fra87], one has the following classification:

1. If p is odd, $L(p; q_1, \ldots, q_m)$ has a unique spin structure.
2. If p is even,
   (a) For m odd, $L(p; q_1, \ldots, q_m)$ admits no spin structure.
   (b) For m even, $L(p; q_1, \ldots, q_m)$ has two inequivalent spin structures.

For example, $\mathbb{RP}^n$ is spin if $n = 3\,(4)$.

The following two propositions are special cases of the 'two-out-of-three' principle for spin structures [LM89, Proposition II.1.15].

Proposition B.2.7. *The boundary ∂M of a spin manifold M is spin, and inherits a canonical spin structure from any given spin structure on M.*

Recall from Example B.2.3, there were two different ways in which the sphere was seen to be a spin manifold. The above result gives us a third, for S^n is the boundary of the unit disk D^{n+1}, which is spin since it has trivial tangent bundle.

Proposition B.2.8. *Let M and N be spin manifolds. Then $M \times N$ is spin and inherits a canonical spin structure from any given spin structures on M and N.*

B.3 Theory of Dirac operators

Let (M, g) be a smooth oriented Riemannian manifold. Each tangent space T_xM is then a vector space with inner product, and we can apply the Clifford functor $\mathrm{C}\ell$ to get a smooth vector bundle $\mathrm{C}\ell(TM) \to M$ called the *Clifford bundle*. Each fiber $\mathrm{C}\ell(T_xM, g_x)$ of the Clifford bundle is a Clifford algebra, and the multiplication map $\mathrm{C}\ell(TM) \otimes \mathrm{C}\ell(TM) \to \mathrm{C}\ell(TM)$ is smooth. The tangent bundle is canonically embedded as a subbundle: $TM \subset \mathrm{C}\ell(TM)$. There is a canonical isomorphism of vector bundles $\mathrm{C}\ell(TM) \cong \Lambda^*(TM)$ given by the associated graded algebra map. As before, the isomorphism is not an algebra isomorphism on the fibres, only a vector space isomorphism.

Suppose that $S \to M$ is a complex vector bundle equipped with a smooth bundle homomorphism

$$c : TM \to \mathrm{End}_{\mathbb{C}}(S)$$

such that $c(v)$ commutes with multiplication by i and satisfies

$$c(v)^2 s = -\|v\|^2 s$$

for all $v \in T_xM$ and $s \in S_x$. We call c a *Clifford multiplication*. The universal property of $\mathrm{C}\ell$ implies that c extends uniquely to a smooth map $c : \mathrm{C}\ell(TM) \to \mathrm{End}_{\mathbb{C}}(S)$ that is an algebra homomorphism on each fibre. The fiber S_x is then a Clifford module for $\mathrm{C}\ell(T_xM)$ in the sense of Section B.1.1, and we call S a *bundle of spinors*. Given a Clifford multiplication, there exists a Hermitian metric on S such that Clifford multiplication by tangent vectors is skew adjoint:

$$\langle c(v)s_1, s_2\rangle = \langle s_1, -c(v)s_2\rangle,$$

and we always equip S with such a metric—see Lemma 2.2 of [BBW93]. This condition of skew-adjointness is equivalent to Clifford multiplication by unit tangent vectors being unitary:

$$\langle c(u)s_1, c(u)s_2\rangle = \langle s_1, s_2\rangle$$

whenever $\|u\| = 1$.

We are interested in connections on S that are compatible with the Clifford module structure. A connection ∇ on S is called a *Clifford connection* if it is Hermitian and satisfies

$$\nabla_X(c(v)s) = c(\nabla^{\mathrm{LC}}_X v)s + c(v)\nabla_X s$$

for all $X, v \in \Gamma(TM)$ and $s \in \Gamma(S)$, where ∇^{LC} is the Levi-Civita connection on TM. Note we can think of Clifford multiplication as an $\mathrm{End}(S)$-valued 1-form $c \in \Gamma(T^*M \otimes S \otimes S^*)$, and the above condition is equivalent to $\nabla c = 0$ where ∇ is the induced tensor product connection.

Proposition B.3.1 ([BBW93], Lemma 2.4)**.** *Let S be a Hermitian bundle with Clifford multiplication $c : TM \to \mathrm{End}(S)$. Then there exists a Clifford connection for S.*

Definition B.3.2. Let S be a bundle of Clifford modules, with compatible Hermitian metric h and Clifford connection ∇. We call the collection (S, c, h, ∇) a *Dirac bundle* over M.

We will often just say that S is a Dirac bundle on M, understanding the extra data c, h and ∇ to be present. A Dirac bundle provides sufficient data to define a Dirac operator. Recall that on a Riemannian manifold (M, g), the tangent bundle is canonically identified with the cotangent bundle via the map $v \mapsto g(-, v)$.

Definition B.3.3. The *Dirac operator* on a Dirac bundle S is the following composition of maps:

$$D : \Gamma(S) \xrightarrow{\nabla} \Gamma(T^*M \otimes S) \xrightarrow{\sim} \Gamma(TM \otimes S) \xrightarrow{c} \Gamma(S),$$

where the first map is ∇, the second induced by the canonical isomorphism $T^*M \cong TM$ coming from the Riemannian metric, and the third is Clifford multiplication.

The Dirac operator is a first order linear partial differential operator $D : \Gamma(S) \to \Gamma(S)$. We will abuse notation and write $c(\alpha)$ for a one-form α, when we really mean Clifford multiplication by the tangent vector corresonding to α under the isomorphism $TM \cong T^*M$ induced by the metric. We have, in local coordinates x^i,

$$Ds = c\left(dx^i \otimes \nabla_{\partial_i} s\right) = c(dx^i)\nabla_{\partial_i} s.$$

If instead we have a local orthonormal frame e_i for TM, we get

$$Ds = c(e^i)\nabla_{e_i} s = \sum_{i=1}^{n} c(e_i)\nabla_{e_i} s.$$

The same formula will not hold for an arbitrary frame, since the metric will generally not identify the dual frame e^i with the original frame e_i. In particular, one cannot take the e_i to be a coordinate frame ∂_i.

Proposition B.3.4 ([LM89], Lemma II.5.1)**.** *The principal symbol of a Dirac operator is Clifford multiplication, i.e.*

$$\sigma(D)(x, \xi) = ic(\xi) : S_x \to S_x$$

for $x \in M$ and $\xi \in T_xM$.

Remark B.3.5. This can be equivalently expressed by saying that $[D, f]s = c(\mathrm{grad} f)s$ for any smooth function f and $s \in \Gamma(S)$.

For $\xi \neq 0$, we have $(ic(\xi))^2 = \|\xi\|^2\mathrm{Id}$ which is invertible, hence $ic(\xi)$ is also invertible.

Corollary B.3.6. *D is an elliptic operator.*

The operator D^2, being the square of an elliptic operator, is also elliptic, with principal symbol $\sigma(D^2)(x, \xi) = \|\xi\|^2\mathrm{Id}$. Hence D^2 is a second order operator whose highest order part is the Laplacian. However, we cannot escape D^2 having lower order terms.

Definition B.3.7. Let e_i be a local oriented orthonormal frame for TM, and let K be the curvature of the Clifford connection on S. Then we call

$$\underline{K} = \sum_{i<j} c(e_i)c(e_j)K(e_i, e_j)$$

the *Clifford contracted curvature* of S. It is a smooth endomorphism of S which is independent of the choice of such frame e_i.

We can write $\underline{K}$ equivalently as

$$\sum_{i<j} c(e_i)c(e_j)K(e_i, e_j) = \frac{1}{2}c(e^i)c(e^j)K(e_i, e_j).$$

From this last representation it is clear that we are taking traces in some way, hence the name ‘contracted curvature’. The significance of $\underline{K}$ is that it appears in the following *Bochner-Lichnerowicz-Weitzenbock (BLW)* formula.

Proposition B.3.8 ([Roe98], page 44)**.** *For any Dirac bundle S with Dirac operator D,*

$$D^2 = \nabla^*\nabla + \underline{K},$$

where ∇^* is the formal adjoint of the Clifford connection and $\underline{K}$ is the Clifford contracted curvature of S.

Proposition B.3.9 ([Roe98], Proposition 3.11)**.** *With respect to the pointwise inner product on S, one has the formula*

$$(Ds_1, s_2) - (s_1, Ds_2) = d^*(s_1, c(-)s_2).$$

Recall that the L^2-*inner product* on sections of S is given on compactly supported smooth sections by

$$\langle s_1, s_2 \rangle = \int_M (s_1, s_2)\, d\mathrm{vol}.$$

The Hilbert space completion of $\Gamma_c(S)$ in this inner product is denoted $L^2(S)$. By integrating the above formula over M and applying the divergence theorem from Riemannian geometry, we get:

Corollary B.3.10 ([Roe98], Proposition 3.11)**.** *With respect to the L^2-inner product on sections of S, one has that for any $s_1 \in \Gamma_c(S)$ and $s_2 \in \Gamma(S)$,*

$$\langle Ds_1, s_2 \rangle = \langle s_1, Ds_2 \rangle.$$

This result is often stated as "D is *formally self-adjoint*". Here we are viewing D as an unbounded operator on $L^2(S)$, and it is self-adjoint on some dense subset, but it may not have self-adjoint closure.

$\mathbb{Z}_2$-gradings

We will also have an interest in $\mathbb{Z}_2$-gradings of Dirac bundles. A Dirac bundle S is $\mathbb{Z}_2$-*graded* if there is a direct sum decomposition $S = S^+ \oplus S^-$ such that

- S^+ and S^- are orthogonal,
- The connection ∇ on S preserves the subbundles S^+ and S^-,
- If $v \in T_xM$ then Clifford multiplication by v maps S^+ to S^- and S^- to S^+. That is, $c(v) : S^+_x \to S^-_x$ and $c(v) : S^-_x \to S^+_x$.

In this situation it is easy to see that the Dirac operator for S splits accordingly into positive and negative components:

$$D = \begin{pmatrix} 0 & D^- \\ D^+ & 0 \end{pmatrix}.$$

Any Dirac bundle over an even-dimensional manifold is $\mathbb{Z}_2$-graded, and a decomposition $S^+ \oplus S^-$ is given from the ± 1 eigenspaces of multiplication by the oriented volume element $\omega \in \Gamma(\mathbb{C}\ell(TM))$. That ∇ preserves the eigenspaces of ω follows from Lemma B.4.1 in the next section.

B.4 Examples of Dirac operators

B.4.1 The Dirac operator on $\mathrm{C}\ell(TM)$

Let $S = \mathrm{C}\ell(TM)$. There is a canonical Clifford connection on S extending the Levi-Civita connection. It satisfies

$$\nabla_X(v_1 v_2) = (\nabla_X v_1)v_2 + v_1(\nabla_X v_2).$$

It follows that $\mathrm{C}\ell(TM)$ is a (real) Dirac bundle, and complexifying we get a complex Dirac bundle $\mathbb{C}\ell(TM)$.

Lemma B.4.1 ([BBW93], Lemma 7.1)**.** *Let ω be the oriented volume section of $\mathrm{C}\ell(TM)$. Then $\nabla\omega = 0$.*

The Clifford multiplication on S is simply left Clifford multiplication within $\mathrm{C}\ell(TM)$. We therefore have an associated Dirac operator D.

Proposition B.4.2 ([LM89], Theorem II.5.12)**.** *Under the canonical isomorphism of vector bundles $\mathrm{C}\ell(TM) \cong \Lambda^*(TM) \cong \Lambda^*(T^*M)$, the Dirac operator satisfies*

$$D \cong d + d^*,$$

where d is the exterior derivative on differential forms and d^* is its formal adjoint.

The Euler characteristic operator

Recall there is a splitting of $\mathrm{C}\ell(TM)$ into even and odd elements, $\mathrm{C}\ell(TM) = \mathrm{C}\ell^0(TM) \oplus \mathrm{C}\ell^1(TM)$. Under the isomorphism $\mathrm{C}\ell(TM) \cong \Lambda^*(T^*M)$, this corresponds to the splitting $\Lambda^*(T^*M) = \Lambda^{\mathrm{ev}}(T^*M) \oplus \Lambda^{\mathrm{odd}}(T^*M)$ into even and odd differential forms. The corresponding operator $d+d^*$ takes the form

$$\begin{pmatrix} 0 & (d+d^*)^{\mathrm{odd}} \\ (d+d^*)^{\mathrm{ev}} & 0 \end{pmatrix} : \Omega^{\mathrm{ev}}(M) \oplus \Omega^{\mathrm{odd}}(M) \to \Omega^{\mathrm{ev}}(M) \oplus \Omega^{\mathrm{odd}}(M)$$

and is called the *Euler characteristic operator*.

The signature operator

If we instead consider the *complex* Clifford bundle $\mathbb{C}\ell(TM)$, then a second splitting is available. Recall the oriented volume element ω acting by Clifford multiplication is an involution of $\mathbb{C}\ell(TM)$, and $\mathbb{C}\ell(TM)$ splits into the ± 1 eigenspaces of left multiplication by the oriented volume element ω:

$$\mathbb{C}\ell(TM) = \mathbb{C}\ell_+(TM) \oplus \mathbb{C}\ell_-(TM).$$

Under the isomorphism $\mathbb{C}\ell(TM) \cong \Lambda^*(T^*_{\mathbb{C}}M)$, Clifford multiplication by ω corresponds to the Hodge star operator $*$. The eigenspaces of ω therefore correspond to the self-dual and anti-self-dual differential forms:

$$\Lambda^*(T^*_{\mathbb{C}}M) \cong \Lambda_+(T^*_{\mathbb{C}}M) \oplus \Lambda_-(T^*_{\mathbb{C}}M).$$

The corresponding Dirac operator $d + d^*$ takes the form

$$\begin{pmatrix} 0 & (d+d^*)^- \\ (d+d^*)^+ & 0 \end{pmatrix} : \Omega^{\mathbb{C}}_+(M) \to \Omega^{\mathbb{C}}_-(M) \to \Omega^{\mathbb{C}}_+(M) \oplus \Omega^{\mathbb{C}}_-(M)$$

and is called the (complex) *signature operator*.

Remark B.4.3. If $n = 4k$ then $i^{[(n+1)/2]} = \pm 1$ is real and the splitting goes through for the real case as well; in this case the real Dirac operator corresponds to the real signature operator acting on real differential forms.

B.4.2 The spin Dirac operator

If M is spin then given a choice of spin structure P_{Spin} for M we can form a bundle of spinors S over M via the associated bundle construction:

$$S = P_{\mathrm{Spin}} \times_\rho \Delta^{\mathbb{C}}_n$$

where $\Delta^{\mathbb{C}}_n$ is a complex spin representation (recall a spin representation is an irreducible representation of $\mathrm{C}\ell_n$ restricted to $\mathrm{Spin}(n) \subset \mathrm{C}\ell_n$; there is a unique complex spin representation in each dimension [LM89, Proposition I.5.15]). To get the Clifford multiplication, one first observes that $TM \cong P_{\mathrm{Spin}} \times_{\mathrm{Ad}} \mathbb{R}^n$. Then Clifford multiplication is a bilinear map $TM \otimes S \to S$, or rather

$$(P_{\mathrm{Spin}} \times_{\mathrm{Ad}} \mathbb{R}^n) \otimes (P_{\mathrm{Spin}} \times_\rho \Delta^{\mathbb{C}}_n) \to P_{\mathrm{Spin}} \times_\rho \Delta^{\mathbb{C}}_n.$$

The map is given by

$$([p, v], [p, \phi]) \mapsto [p, c(v)\phi].$$

An inspection of this map in local trivialisations shows that it is a smooth bundle morphism.

To construct the Dirac operator on S, one needs a connection. There is in fact a canonical one. Since M is Riemannian, we have a preferred choice of connection on TM—the Levi-Civita connection. This connection induces an equivalent connection on the principal $\mathrm{SO}(n)$-bundle P_{SO} of oriented orthonormal frames. The connection on P_{SO} can then be lifted to P_{Spin} via the double cover $P_{\mathrm{Spin}} \to P_{\mathrm{SO}}$. Finally, the connection on P_{Spin} canonically induces a connection on the associated bundle $S = P_{\mathrm{Spin}} \times_\rho \Delta^{\mathbb{C}}_n$. We call this the *spin connection* on S.

Proposition B.4.4 ([LM89], Proposition II.4.11)**.** *The spin connection is compatible with the Clifford multiplication.*

Definition B.4.5. Let M be a spin manifold with a given spin structure P_{Spin}. We call the Dirac operator D for the bundle of spinors $S = P_{\mathrm{Spin}} \times_\rho \Delta_n^{\mathbb{C}}$ the *spin Dirac operator* on M.

For spin Dirac operators, the BLW formula (Proposition B.3.8) has an extremely useful refinement.

Proposition B.4.6 (Lichnerowicz formula, [Lic63])**.** *Let M be a Riemannian spin manifold with spin Dirac operator D. Then*

$$D^2 = \nabla^*\nabla + \frac{1}{4}\kappa,$$

where κ is the scalar curvature of M.

For a proof, see [LM89, page 161]. It is this formula which allows the methods of index theory to be applied to the study of positive scalar curvature, as we will see later in this appendix.

B.4.3 Twisted Dirac operators

A standard move one can perform is to 'twist' a Dirac operator by a vector bundle. Let D be the Dirac operator for a Dirac bundle S, and let E be a Hermitian vector bundle with connection ∇. We will define a Dirac operator D_E acting on sections of $S \otimes E$. First, equip $S \otimes E$ with the usual tensor product connection:

$$\nabla_X(s \otimes e) = \nabla_X s \otimes e + s \otimes \nabla_X e.$$

Next, extend the Clifford multiplication c to act on $S \otimes E$ by

$$c(v)(s \otimes e) = c(v)s \otimes e.$$

The *twisted Dirac operator* D_E on $S \otimes E$ is then defined as the Dirac operator associated to this data. If $S = S^+ \oplus S^-$ is $\mathbb{Z}_2$-graded then $S \otimes E$ inherits a $\mathbb{Z}_2$-grading $(S^+ \otimes E) \oplus (S^- \otimes E)$, and the twisted Dirac operator splits accordingly into positive and negative parts $D_E^\pm$.

We will be particularly interested in twisting Dirac operators by flat vector bundles; recall a connection on a vector bundle E is called *flat* if its curvature form K is identically zero. If E is flat then for each $x \in M$ there is a local frame $e_1, \ldots, e_N$ for E on a neighbourhood of x such that $\nabla e_i = 0$ for each i. This can be proven either by parallel transporting a given frame from E_x (on a sufficiently small neighbourhood of x, the parallel translates will not depend on the chosen paths), or by considering E as the associated bundle of a flat principal bundle (a flat principal G-bundle is locally isomorphic to $U \times G$ with its trivial Maurer-Cartan connection). Hence if E is flat, it is locally unitarily isomorphic to $U \times \mathbb{C}^N$ with its trivial metric and connection.

So, let E be a flat Hermitian bundle and let D_E be a Dirac operator twisted by E. Choosing a local frame of covariantly constant sections $e_1, \ldots, e_N$ for E, we may write an arbitrary section of $S \otimes E$ uniquely as $\sum_i s_i \otimes e_i$, where the s_i are sections of S. The twisted Dirac operator then takes the form

$$D_E \left(\sum_{i=1}^{N} s_i \otimes e_i \right) = \sum_{i=1}^{N} Ds_i \otimes e_i.$$

From this formula, it is clear that the twisted Dirac operator D_E is locally isomorphic to the direct sum $D \oplus \cdots \oplus D$ of N copies of D. We record this for reference:

Proposition B.4.7. *Let D_E be a twisted Dirac operator, where E is a flat vector bundle of rank N. Then D_E is locally isomorphic to the direct sum of N copies of D.*

Twisting Dirac operators by flat vector bundles is an important operation in constructing rho invariants, which are central to the results of this thesis. It also plays an important role in the K-homology proof of the Atiyah-Singer index theorem—see [BD82] or [BvE16].

B.4.4 Bott Generator

We now describe a certain Dirac operator on the sphere S^{2k} that plays an important part; it allows one to implement Bott periodicity into the definition of geometric K-homology. Let $\theta = e_1 \cdots e_{2k}$ be the oriented volume element for $\mathrm{C}\ell(TS^{2k})$. We define $\mathrm{C}\ell_\theta(S^{2k}) \subset \mathrm{C}\ell(TS^{2k})$ to be the subbundle consisting of $+1$ eigenvectors for the *right* Clifford multiplication by θ. Since we defined $\mathrm{C}\ell_\theta$ as a right eigenspace, we still have a well-defined left Clifford multiplication on $\mathrm{C}\ell_\theta(S^{2k})$. The canonical connection on $\mathrm{C}\ell(TS^{2k})$ restricts to a well-defined connection on $\mathrm{C}\ell_\theta(S^{2k})$; this is due to the volume element θ being parallel (Lemma B.4.1). Hence there is a Dirac operator D_θ on our bundle $\mathrm{C}\ell_\theta(S^{2k})$.

We will now show that this whole set-up is $\mathrm{SO}(2k)$-equivariant. Recall that if G is a Lie group and M is a G-manifold (a manifold with a smooth action $G \times M \to M$), a *G-equivariant vector bundle* on M is a vector bundle $E \to M$ equipped with a G-action on E such that

1. The projection $\pi : E \to M$ is G-equivariant, meaning
$$\pi(ge) = g\pi(e)$$
for all $e \in E$ and $g \in G$, and
2. Left action by g gives a linear map $E_x \to E_{gx}$ for any $g \in G$ and $x \in M$ (the map $E_x \to E_{gx}$ is then automatically a linear isomorphism).

Now, $\mathrm{SO}(2k)$ acts on S^{2k} via the diagonal inclusion $\mathrm{SO}(2k) \hookrightarrow \mathrm{SO}(2k+1)$. A G-action on M canonically induces a G-action on TM, making TM a G-equivariant bundle on M. Hence we have an $\mathrm{SO}(2k)$-equivariant bundle $TS^{2k} \to S^{2k}$. The action of $\mathrm{SO}(2k)$ on TS^{2k} canonically extends to an $\mathrm{SO}(2k)$-action on $\mathrm{C}\ell(TS^{2k})$ via
$$g(v_1 \cdots v_\ell) = (gv_1) \cdots (gv_\ell)$$
for $v_i \in TS^{2k}$, which may be checked from the universal property of Clifford algebras. Since $g \in \mathrm{SO}(2k)$ takes an oriented orthonormal basis for TS^{2k} to an oriented orthonormal basis, $g\theta = \theta$, and hence the $\mathrm{SO}(2k)$ action preserves the $+1$ eigenspace of right action by θ. We therefore have an $\mathrm{SO}(2k)$-equivariant bundle $\mathrm{C}\ell_\theta(S^{2k}) \to S^{2k}$.

If $E \to M$ is a G-equivariant vector bundle, there is a canonical G-action on the sections $\Gamma(E)$, given by
$$(gs)(x) = g\, s(g^{-1}x)$$
for $g \in G$ and $s \in \Gamma(E)$. A differential operator $P : \Gamma(E) \to \Gamma(E)$ is called *G-equivariant* if
$$P(gs) = gPs$$
for all $g \in G$ and $s \in \Gamma(E)$.

Proposition B.4.8. *The Dirac operator D_θ on $\mathrm{C}\ell_\theta(S^{2k})$ is $\mathrm{SO}(2k)$-equivariant.*

Proof. Since $\mathrm{SO}(2k)$ acts by isometries on S^{2k}, the Levi-Civita connection and its extension to $\mathrm{C}\ell_\theta(S^{2k})$ are equivarient. Hence
$$\begin{aligned} D_\theta(gs) &= \sum_i c(e_i)\nabla_{e_i}(gs) \\ &= \sum_i c(e_i)(g\nabla_{g^{-1}e_i}s) \\ &= g\left(\sum_i c(g^{-1}e_i)\nabla_{g^{-1}e_i}s\right) \\ &= g(D_\theta s). \end{aligned}$$

The last equality follows because $g \in \mathrm{SO}(2k)$ implies that $\{g^{-1}e_i\}$ is an orthonormal frame. $\square$

Given a G-equivariant operator P, the kernel of P is canonically a representation space for G, since $s \in \ker P$ implies $P(gs) = g(Ps) = 0$ for any $g \in G$.

Theorem B.4.9. $\mathrm{Ind}(D_\theta^+) = 1$. *In fact,* $\dim\ker D_\theta^+ = 1$ *and* $\dim\ker D_\theta^- = 0$.

Proof. Recall that D_θ is isomorphic to the signature operator $d+d^*$, restricted to an appropriate subspace of forms. The kernel of the full signature operator is identified via Hodge theory with the de Rham cohomology of the sphere:

$$H^*(S^{2k}, \mathbb{R}) = H^0_{dR}(S^{2k}) \oplus H^{2k}_{dR}(S^{2k}) \cong \mathbb{R} \oplus \mathbb{R}.$$

We are interested in the subspace which is invariant under the right action by the oriented volume element. It is not difficult to see that this is generated by $1 \oplus \mathrm{vol}_{S^{2k}}$, so is 1-dimensional. This element is also invariant under left action by the oriented volume element, so is a positive spinor. It follows immediately that $\ker D_\theta^+ \cong \mathrm{span}\{1 \oplus \mathrm{vol}_{S^{2k}}\}$ and $\ker D_\theta^- = 0$. $\square$

Note that $1 \oplus \mathrm{vol}_{S^{2k}}$ is a fixed point for the action of $\mathrm{SO}(2k)$, and so the kernel of D_θ^+ is the 1-dimensional trivial representation of $\mathrm{SO}(2k)$.

B.5 The local index theorem for Dirac operators

Given a Dirac operator on a manifold M, we might ask if there exist any non-trivial solutions to the equation $Ds = 0$. Index theory provides powerful tools which help to answer such existence questions. For simplicity we will avoid references to Fredholm operators and Banach spaces, although this is the natural viewpoint from which to study index theory.

Theorem B.5.1 ([Gil95], Lemma 1.4.5)**.** *Let* $P : \Gamma(E) \to \Gamma(F)$ *be an elliptic differential operator on a compact manifold* M*. Then* $\ker P$ *is finite dimensional.*

Since the formal adjoint P^* of P is elliptic whenever P is elliptic, this implies that $\ker P^*$ is finite dimensional also.

Definition B.5.2. Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator on a compact manifold M. We define the *index* of P to be the integer

$$\mathrm{Ind}(P) = \dim\ker P - \dim\ker P^*.$$

Since a Dirac operator D on a compact manifold is elliptic, it has a well-defined index. Of course, this index is zero since D is self-adjoint. However, if S is $\mathbb{Z}_2$-graded then the positive part D^+ of D is also elliptic and has a finite index. The index of D^+ is generally quite interesting. The proof of the following may be found in [BGV04] and [Roe98]

Theorem B.5.3 (Local index theorem, [Pat71], [Gil73])**.** *Let D be the Dirac operator on a compact even-dimensional manifold M. Then*

$$\operatorname{Ind}(D^+) = \int_M \mathbf{I}(D^+)$$

where $\mathbf{I}(D^+)$ is the index form *of D^+.*

The index form $\mathbf{I}(D^+)$ is a top degree differential form on M that emerges from the *asymptotic expansion* of the heat kernel of the Dirac operator—see Appendix C for more information. The index form satisfies the following properties:

- $\mathbf{I}(D^+|_U) = \mathbf{I}(D^+)|_U$.
- Let $U_1 \subset M_1$ and $U_2 \subset M_2$ be open subsets of Riemannian manifolds M_1 and M_2 respectively, and suppose there is an isometry $\varphi : U_1 \to U_2$. Let $S_1 \to M_1$ and $S_2 \to M_2$ be Dirac bundles, and suppose there is an isomorphism $\psi : S_1|_{U_1} \xrightarrow{\sim} S_1|_{U_2}$ of Dirac bundles compatible with φ, that is, an isomorphism of vector bundles which preserves the Hermitian metrics, Clifford multiplications, and Clifford connections (the later two under the identification $U_1 \cong U_2$ via the isometry φ). Then $\mathbf{I}(D_1^+)|_{U_1} = \varphi^*(\mathbf{I}(D_2^+)|_{U_2})$.
- $\mathbf{I}(D \oplus D') = \mathbf{I}(D) + \mathbf{I}(D')$.

These properties can be easily observed from the proof of the local index theorem—see [Roe98, Chapter 7] or [BGV04, Theorem 2.26] While the index form has an abstract characterisation in terms of heat kernels, for the standard examples of Dirac operators it reduces to polynomials in characteristic classes of bundles, as we shall see in the next section.

Proposition B.5.4. *Let D_E be the twist of a Dirac operator D by a flat bundle E of rank N. Then*

$$\operatorname{Ind}(D_E^+) = N \operatorname{Ind}(D^+).$$

Proof. By Proposition B.4.7, D_E is locally isomorphic to a direct sum of N copies of D. Since the index form is a local invariant and is additive,

$$\mathbf{I}(D_E) = N\,\mathbf{I}(D)$$

and the result follows immediately from the local index theorem. □

B.6 The Atiyah-Singer index theorem

When we replace a general Dirac operator by a specific one in the index theorem, such as the spin Dirac operator or the signature operator, the index form $\mathbf{I}(D^+)$ takes a particularly nice form. Instead of being an abstract term in an asymptotic expansion, the index form is realised as an explicit polynomial in the pontryagin (or chern) classes of the bundles involved. The resulting formulae, originally proven without the use of heat kernels, are all subsumed by the general *Atiyah-Singer index theorem* [AS63, AS68a, AS68b]. We will not prove that the index form reduces to these characteristic classes, but instead refer to [ABP73] for the proofs.

Chern-Gauss-Bonnet theorem

Theorem B.6.1 (Chern-Gauss-Bonnet theorem)**.** *Let* $(d+d^*)^{\mathrm{ev}} : \Omega^{\mathrm{ev}}(M) \to \Omega^{\mathrm{odd}}(M)$ *be the Euler-Characteristic operator of a compact oriented even-dimensional Riemannian manifold* M*. Then*

$$\mathrm{Ind}((d+d^*)^{\mathrm{ev}}) = \chi(M) = \int_M \mathrm{Pf}(M),$$

where $\mathrm{Pf}(M)$ *is the Pfaffian of* M*.*

Example B.6.2. In dimension 2, we have $\mathrm{Pf}(M) = \frac{1}{4\pi}\kappa\, d\mathrm{vol}$, where κ is the scalar curvature of M. Hence this generalises the Gauss-Bonnet theorem for surfaces to arbitrary dimensions.

Hirzebruch signature theorem

Let M be an oriented $2k$-dimensional manifold. Then there is a canonical quadratic form on the middle cohomology

$$H^k(M,\mathbb{R}) \times H^k(M,\mathbb{R}) \to \mathbb{R}$$

given by the composition of the cup product and the Poincaré duality isomorphism: $H^k(M,\mathbb{R}) \times H^k(M,\mathbb{R}) \xrightarrow{\smile} H^{2k}(M,\mathbb{R}) \xrightarrow{P.D.} \mathbb{R}$. In terms of de Rham cohomology, the quadratic form is

$$([\omega],[\eta]) \mapsto \int_M \omega \wedge \eta.$$

The *signature* $\sigma(M)$ of M is defined as the signature of this quadratic form.

Theorem B.6.3 (Hirzebruch signature theorem). *Let $(d+d^*)^+ : \Omega^*_+(M) \to \Omega^*_-(M)$ be the signature operator of a compact oriented even-dimensional Riemannian manifold M. Then*

$$\operatorname{Ind}((d+d^*)^+) = \sigma(M) = \int_M L(M),$$

where $\sigma(M)$ is the signature of M and $L(M)$ is the Hirzebruch L-polynomial in the pontryagin classes of M.

Atiyah-Singer index theorem for spin Dirac

Theorem B.6.4 (Atiyah-Singer index theorem for spin Dirac). *Let D be the spin Dirac operator of a compact even-dimensional Riemannian spin manifold M. Then*

$$\operatorname{Ind}(D^+) = \int_M \widehat{A}(M),$$

where $\widehat{A}(M)$ is the A-hat polynomial in the pontryagin classes of M.

We will see an application of this theorem to positive scalar curvature in the next section. For the meantime, we will compute the index of the spin Dirac operator on spheres:

Example B.6.5. The sphere is stably trivial; $TS^n \oplus N \cong S^n \times \mathbb{R}^{n+1}$. It follows that $p(TS^n) = p(TS^n)p(N) = p(\mathbb{R}^{n+1}) = 1$. Hence the A-hat genus of S^n, being a polynomial in the pontryagin classes, has top degree part equal to zero. Therefore $\int_{S^n} \widehat{A}(M) = 0$.

Hirzebruch-Riemann-Roch theorem

Let F be a holomorphic vector bundle of rank N on a complex manifold X. Denote by $H^i(X, F)$ the i-th cohomology group of the sheaf of holomorphic sections of F. The *holomorphic Euler characteristic* of F is defined as

$$\chi(X, F) = \sum_{i=0}^{N} \dim_{\mathbb{C}} H^i(X, F).$$

Being a complex vector bundle on a complex manifold, there is a natural operator $\bar{\partial}_F : \Omega^{p,q}(X, F) \to \Omega^{p,q+1}(X, F)$, defined locally by the formula $\bar{\partial}_F(\alpha^i \otimes s_i) = \bar{\partial}(\alpha^i) \otimes s_i$ for any local holomorphic frame $s_1, \ldots, s_N$ of F. Equipping F with a hermitian metric, we get an adjoint operator $\bar{\partial}_F^* : \Omega^{p,q+1}(X, F) \to \Omega^{p,q}(X, F)$. Letting $\bar{\partial}_F + \bar{\partial}_F^*$ act on the total space

$\Omega^{0,*}(X,F)$ of F-valued differential forms of type $(0,q)$, there is a decomposition of $\bar\partial_F + \bar\partial_F^*$ into

$$\begin{pmatrix} 0 & (\bar\partial_F + \bar\partial_F^*)^{\text{odd}} \\ (\bar\partial_F + \bar\partial_F^*)^{\text{ev}} & 0 \end{pmatrix} : \Omega^{0,\text{ev}}(X,F) \oplus \Omega^{0,\text{odd}}(X,F) \\ \to \Omega^{0,\text{ev}}(X,F) \oplus \Omega^{0,\text{odd}}(X,F).$$

Theorem B.6.6 (Hirzebruch-Riemann-Roch theorem)**.** *Let F be a holomorphic vector bundle on a compact complex manifold X, and equip F with a hermitian metric. Let $\bar\partial_F + \bar\partial_F^*$ be the canonically induced operator on $\Omega^{0,*}(X,F)$, defined above. Then*

$$\operatorname{Ind}((\bar\partial_F + \bar\partial_F^*)^{\text{ev}}) = \chi(X,F) = \int_X \operatorname{Td}(X)\operatorname{ch}(F),$$

where $\operatorname{Td}(X)$ *is the Todd class of X and $ch(F)$ is the Chern character of F.*

B.7 An application of the index theorem to PSC

Theorem B.7.1 (Lichnerowicz, [Lic63])**.** *Let M be a compact even-dimensional Riemannian spin manifold with positive scalar curvature. Then the topological invariant $\int_M \widehat{A}(M)$ vanishes.*

Proof. Suppose $D\psi = 0$. Then by the Lichnerowicz formula,

$$\begin{aligned} 0 &= \langle D^2\psi, \psi\rangle \\ &= \langle \nabla^*\nabla\psi, \psi\rangle + \frac{1}{4}\langle \kappa\psi, \psi\rangle \\ &= \langle \nabla\psi, \nabla\psi\rangle + \frac{1}{4}\int_M \kappa(\psi,\psi)\, d\text{vol}. \end{aligned}$$

Since $\kappa > 0$, this implies that $\psi = 0$. Decomposing $\psi = \psi^+ \oplus \psi^-$, we see that the kernels of both D^+ and D^- are trivial. Applying Atiyah-Singer (Theorem B.6.4), $\int_M \widehat{A}(M)$ vanishes. □

The theorem gives an obstruction to positive scalar curvature—any compact even-dimensional spin manifold M with $\widehat{A}(M) \neq 0$ cannot carry a metric of positive scalar curvature. The simplest example of such a manifold is a K3 sufrace; it is a compact 4-dimensional spin manifold with $\widehat{A}(X) = 2$. By definition, a K3 surface is a compact complex manifold X of dimension 2 (real dimension 4) with first Betti number $b_1(X) = 0$ and holomorphically trivial canonical bundle $K_X := \Lambda^2_{\mathbb{C}} T^*X \cong X \times \mathbb{C}$. An example of such a surface is the non-singular quartic in $\mathbb{CP}^3$ defined as the zero set of the homogeneous polynomial $z_0^4 + z_1^4 + z_2^4 + z_3^4$.

Example B.7.2 (Obstruction to PSC for a K3 surface)**.** Let X be a K3 surface. For a complex vector bundle E, we have $c_1(E) = c_1(\det E)$. Hence, since X has trivial canonical bundle, $c_1(X) = 0$. We apply the Hirzebruch-Riemann-Roch theorem to the trivial canonical bundle in order to compute $c_2(X)$. One has

$$\chi(X, \mathcal{O}) = \int_X \mathrm{Td}(X)\mathrm{ch}(\mathcal{O}) = \int_X \mathrm{Td}(X) = \frac{c_1(X)^2 + c_2(X)}{12}.$$

Hence $c_2(X) = 12\,\chi(X, \mathcal{O})$ and we are reduced to computing the holomorphic Euler characteristic of $\mathcal{O}$. One always has $h^0(X, \mathcal{O}) = 1$. By Serre duality,

$$H^2(X, \mathcal{O}) \cong H^0(X, \mathcal{O}^* \otimes K_X)^* \cong H^0(X, \mathcal{O})$$

and so $h^2(X, \mathcal{O}) = 1$. Finally, X is a Kähler manifold with $b_1(X) = 0$, so

$$0 = H^1(X, \mathbb{C}) \cong H^1(X, \mathcal{O}) \oplus \overline{H^1(X, \mathcal{O})}$$

by the Hodge theory—see [GH94, page 116]—and therefore $h^1(X, \mathcal{O}) = 0$. It follows that $\chi(X, \mathcal{O}) = h^0(X, \mathcal{O}) - h^1(X, \mathcal{O}) + h^2(X, \mathcal{O}) = 2$, and finally $c_2(X) = 24$.

To compute the A-hat genus of X we finish with

$$\widehat{A}(X) = -\frac{1}{24} p_1(X) = -\frac{c_1(X)^2 - 2c_2(X)}{24} = 2.$$

Since X is a complex manifold, its second Stiefel-Whitney class is the mod-2 reduction of its first Chern class. Hence $w_2(X) = 0$ and X is spin. We conclude that X cannot admit a metric of positive scalar curvature.

The following example demonstrates that the spin assumption may not be omitted in the Lichnerowicz obstruction.

Example B.7.3. The manifold $\mathbb{CP}^2$ is not spin. We have

$$\widehat{A}(\mathbb{CP}^2) = -\frac{1}{24} p_1 = -\frac{c_1^2 - 2c_2}{24} = -\frac{3^2 - 6}{24} c^2 = -\frac{1}{8} c^2,$$

and so $\int_{\mathbb{CP}^2} \widehat{A}(\mathbb{CP}^2) = \frac{-1}{8}$ which is non-zero. The manifold $\mathbb{CP}^2$ admits a metric of positive scalar curvature, hence the spin assumption cannot be omitted. Note this illustrates another point, that if a manifold is not spin then its $\widehat{A}$-genus may not be an integer.

Appendix C

Heat kernels

This appendix contains some of the basic theory for heat kernels of Dirac operators on manifolds of bounded geometry. On a compact manifold, the heat kernel for $\partial_t + D^2$ can be constructed using an orthonormal L^2-basis of eigensections for D. In the non-compact case, there is no such basis, and the existence proof is more difficult. The proof here only applies to certain nice manifolds—those of 'bounded geometry'. We begin in the first section by defining such manifolds and bundles. The next section sketches the proof of existence and asymptotic expansion for heat kernels on manifolds of bounded geometry. The chapter ends with a few heat kernel estimates.

The proof of the existence and asymptotic expansion of the heat kernel for Dirac operators is adapted from [BGM71, Chapter 3.E], and also [Don79] for the case of bounded geometry. These authors only consider the heat kernel for the scalar Laplacian; the necessary modifications needed to treat the case of Dirac operators are adapted from [Roe98, Chapter 7]. Another exposition of the proof of the local index theorem given in [BGV04].

C.1 Manifolds of bounded geometry

Performing analysis on an arbitrary non-compact manifolds is rather difficult, and there is very little one can say in general. For our purposes, it will be enough to restrict attention to a class of Riemannian manifolds and vector bundles which are 'tame' in a certain sense.

Definition C.1.1. Let M be a Riemannian manifold. We say that M has *bounded geometry* if the following conditions are satisfied:

(B1) The Riemann curvature tensor R and all of its covariant derivatives are bounded above in norm. Explicitly, for any $k \geqslant 0$ there exists a

constant C_k such that
$$|\nabla^k R(x)| \leqslant C_k$$
for all $x \in M$, where the norm is taken in the induced tensor product metric on $(T_x^* M)^{\otimes(k+3)} \otimes T_x M$.

(B2) The injectivity radius of M is bounded below by a positive constant. That is to say, there exists some $\epsilon > 0$ such that for each $x \in M$ the exponential map
$$\exp|_x : T_x M \to M$$
is a diffeomorphism when restricted to the open ball with centre 0 and radius ϵ in $T_x M$.

The condition (B2) implies that geodesics in M can be extended indefinitely, so that M is complete as a metric space by the Hopf-Rinow theorem. A Dirac bundle S has *bounded geometry* if the curvature of the Clifford connection and all of its covariant derivatives are bounded above in norm, as in (B1). If both M and S have bounded geometry, then we say the pair (M, S) has bounded geometry.

Example C.1.2. If M is compact, then (M, S) clearly has bounded geometry. End-periodic manifolds with end-periodic Dirac bundles also have bounded geometry, and are the most important examples for us. The punctured plane $\mathbb{R}^2 \backslash \{0\}$ does not have bounded geometry, since the injectivity radius approaches 0 near the origin and (B2) fails.

Basic facts on manifolds with bounded geometry may be found in [Shu92], and [Roe88a, Roe88b]. The latter two are particularly useful for heat kernels on manifolds of bounded geometry.

C.2 Heat kernels on manifolds of bounded geometry

Given a Dirac bundle S, we denote by $S \boxtimes S^*$ the *exterior product* of S with S^*—it is the vector bundle over $M \times M$ defined as $\pi_L^*(S) \otimes \pi_R^*(S^*)$, where π_L and π_R are the projections from $M \times M$ to the left and right factors respectively. The fibre of $S \boxtimes S^*$ over the point $(x, y) \in M \times M$ is then $S_x \otimes S_y^*$, which is naturally isomorphic to $\mathrm{Hom}(S_y, S_x)$. We will also denote by $S \boxtimes S^*$ the pullback of this bundle to the manifold $\mathbb{R}_{>0} \times M \times M$ or $\mathbb{R}_{\geqslant 0} \times M \times M$ (by the obvious projection onto $M \times M$), taking care to clearly indicate when we are doing so.

Definition C.2.1 (Heat kernel)**.** Let M be a Riemannian manifold, and let S be a Dirac bundle on M with associated Dirac operator D. A *heat kernel* for D^2 is a section $K_t(x,y)$ of the bundle $S \boxtimes S^* \to \mathbb{R}_{>0} \times M \times M$ which satisfies the following conditions:

(K1) $K_t(x,y)$ is C^1 in $t > 0$ and C^2 in $(x,y) \in M \times M$.

(K2) For any fixed $y \in M$, $K_t(x,y)$ satisfies the heat equation:

$$\frac{\partial}{\partial t} K_t(x,y) + D_x^2 K_t(x,y) = 0,$$

where D_x denotes the Dirac operator acting on the x-variable.

(K3) For any $s \in \Gamma_c(M,S)$,

$$\int_M K_t(x,y)s(y)\,dy \to s(x)$$

as $t \to 0$, where dy is the Riemannian volume measure on M.

(K4) For any $T > 0$, there is a constant $C > 0$ such that

$$\left| \frac{\partial^i}{\partial t^i} \nabla_x^j \nabla_y^k K(t;x,y) \right| \leqslant C t^{-n/2-i-j-k} e^{-d(x,y)^2/4t}$$

for all $t \in (0,T]$ and $0 \leqslant i,j,k \leqslant 1$, where C depends only on T. Here ∇_x is the Clifford connection on S and ∇_y is the dual connection on S^*. The norm on the left hand side is taken in the induced tensor product metric on $(T_x^* M)^{\otimes j} \otimes (T_y^* M)^{\otimes k} \otimes S_x \otimes S_y^*$.

The condition (K3) stipulates that for fixed x, the sections $K_t(x,y)$ of the bundle $S_x \otimes S^* \to M$ converge to the Dirac delta distribution at x as $t \to 0$. We sometimes call K the *heat kernel for the Dirac operator*, or simply the *heat kernel* when D is understood. An explicit calculation of the heat kernel is beyond hope in most cases, however we will see that the heat kernel has an *asymptotic expansion* which allows us to recover the important geometric information held in the heat kernel.

Definition C.2.2 (Asymptotic expansion)**.** Let E be a Fréchet space, and $f : \mathbb{R}_{>0} \to E$ a function. Let $\{p_n\}$ be a countable family of seminorms defining the topology of E. An infinite sum $\sum_{k=0}^{\infty} a_k\, t^k$ with $a_k \in E$ is called an *asymptotic expansion* for f if for every positive integer m and seminorm p_n, there exists $N(m,n)$ such that if $M \geqslant N(m,n)$ then

$$p_n\left(f(t) - \sum_{k=0}^{M} a_k\, t^k\right) = O(t^m),$$

for small positive t.

The definition is of course independent of the chosen family of seminorms. It is easy to show that if such an asymptotic expansion exists then it is unique. While the definition appears complicated, the concept is quite simple:

For any positive integer m and any seminorm p_n, all but finitely many of the partial sums approximate f to within an order of t^m, with respect to p_n.

A Taylor series for a function is a simple example of an asymptotic expansion. In general, one cannot expect the infinite sum $\sum_{k=0}^{\infty} a_k\, t^k$ to converge, and we should only really think of it as a sequence of partial sums rather than an infinite sum.

In the case when M is compact, there is a very simple formula for the heat kernel. One takes an orthonormal basis for $L^2(M,S)$ consisting of smooth eigensections ϕ_λ for D^2 and defines

$$K_t(x,y) = \sum_{\lambda} e^{-t\lambda} \phi_\lambda(x) \otimes \phi_\lambda(y)^*.$$

Unfortunately we do not have the luxury of this spectral decomposition in the non-compact case, and we must proceed by other means.

Theorem C.2.3 (Existence and uniqueness of heat kernels)**.** *Let (M,S) have bounded geometry. Then*

(1) *There exists a heat kernel $K_t(x,y)$ for $\partial_t + D^2$.*

(2) *The heat kernel is unique.*

(3) *$K_t(x,y)$ is smooth.*

(4) *The estimate in* (K4) *is satisfied* **for all** *i, j, k.*

(5) *There exists an asymptotic expansion of the heat kernel along the diagonal of the form*

$$K_t(x,x) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} u_k(x,x) t^k$$

where the u_k *are smooth sections of* $S \boxtimes S^* \to M \times M$*, and* $u_0(x,x) = \mathrm{Id}_{S_x}$*. The expansion is valid in the Fréchet space of smooth sections of* $\mathrm{End}(S)$*. Furthermore, the* u_k *are determined purely by the local coefficients of the metric and the Dirac operator.*

The proof is long and involved, so we will only sketch it, referring to [BGM71, Chapter 3.E], [Roe98, Chapter 7],and [Don79] for the details. Although the following construction is much more complicated than in the compact case, the method has the advantage of revealing additional information about the heat kernel. In particular, the estimate in (K4) and the asymptotic expansion of the heat kernel emerge quite naturally.

Proof outline. A key notion in the proof is the following:

Definition C.2.4 (Définition E.III.2, [BGM71])**.** An *approximate heat kernel* for the heat operator $\frac{\partial}{\partial t} + D^2$ is a section H of $S \boxtimes S^*$ over $\mathbb{R}_{>0} \times M \times M$ satisfying:

(A1) H is smooth.

(A2) $\left(\frac{\partial}{\partial t} + D_x^2\right) H$ extends to a continuous section of $S \boxtimes S^*$ over $\mathbb{R}_{\geqslant 0} \times M \times M$.

(A3) $H(t,x,-) \to \delta_x$ as $t \to 0$.

The hardest part of the proof is the existence—properties (2) through (5) can be observed without too much difficulty during the course of the proof. The existence proof can be roughly divided into two steps:

1. Construct an approximate heat kernel by solving a recursive system of ordinary differential equations.
2. Show that any approximate heat kernel gives rise to a true heat kernel via a 'convolution trick'.

We begin by describing the construction of the approximate heat kernel. It is convenient to change notation and write $K_t(x,y) = K(t;x,y)$. Since M has bounded geometry, there exists $\epsilon > 0$ such that if $d(x,y) < \epsilon$ then y lies in a normal neighbourhood of x. Define the open subset

$$U_\epsilon = \{(x,y) \in M \times M : d(x,y) < \epsilon\}$$

of $M \times M$, and write

$$h(t;x,y) = (4\pi t)^{-n/2} e^{-d(x,y)^2/4t}.$$

The function h mimics the scalar Laplacian on Euclidean space, and is smooth when restricted to U_ϵ. We then seek a formal power series solution to the heat equation of the form

$$S_\infty(t;x,y) = h(t;x,y)\sum_{k=0}^{\infty} u_k(x,y)t^k,$$

where the $u_k \in \Gamma(U_\epsilon, S^* \boxtimes S)$ are certain smooth sections. Applying the heat operator and setting the resulting formal sum to 0, we get a recursive system of ordinary differential equations

$$\nabla_{\partial_r}(r^k g^{1/4} u_k) = \begin{cases} 0 & k = 0, \\ -r^{k-1}\, g^{1/4}\, D^2 u_{k-1} & k \geqslant 1 \end{cases} \tag{C.1}$$

which uniquely determines the u_k. Here all differential operators act on the x-variable, and r is the radial coordinate in the x-direction. The formal series does not converge however, so we must instead consider some ℓ-th partial sum S_ℓ. We take $\ell > n/2$ and extend S_ℓ from U_ϵ to all of $M \times M$ by defining

$$H_\ell(t,x,y) = \eta(x,y)S_\ell(t,x,y),$$

where η is a smooth function on $M \times M$ satisfying $\eta = 1$ on $U_{\epsilon/4}$ and $\eta = 0$ outside of $U_{\epsilon/2}$. The function H_ℓ is not a heat kernel, but is indeed an approximate heat kernel.[1]

Now for the 'convolution trick' that turns an approximate heat kernel into a heat kernel. For continuous sections A and B of $S \boxtimes S^* \to \mathbb{R}_{>0} \times M \times M$, define their *convolution*

$$(A * B)(t,x,y) = \int_0^t \int_M A(s,x,z)B(t-s,z,y)\, dz\, ds,$$

whenever the integral exists. This convolution is associative, and we write $A^{*N} = A * \cdots * A$ for the N-fold convolution of A with itself. Now take

$$R_\ell = \left(\frac{\partial}{\partial t} + D_x^2\right) H_\ell$$

and define

$$Q_\ell = \sum_{N=1}^{\infty} (-1)^{N+1} R_\ell^{*N}.$$

[1] This is a crucial step in the proof, as it is where the bounded geometry is needed. If M did not have bounded geometry, we might instead define some continuous function $\epsilon : M \to \mathbb{R}$ so that each $x \in M$ has a normal neighbourhood of radius $\epsilon(x)$, and then try and use ϵ to define η and extend S_ℓ smoothly to M. However, in doing this we cannot prove that H_ℓ is an approximate heat kernel.

After showing this expression is well defined and is a C^m section of $S \boxtimes S^* \to \mathbb{R}_{\geqslant 0} \times M \times M$ for $m < \ell - n/2$, we finally define

$$K = H_\ell - Q_\ell * H_l$$

which is the heat kernel we've been looking for. Showing that K satisfies axioms (K1) to (K4) of a heat kernel completes the existence proof.

Uniqueness is fairly straightforward—see [Don79, page 490]. Smoothness comes from uniqueness and the fact that any S_ℓ, where ℓ is sufficiently large, can be used to construct a heat kernel—remember for $\ell > m + n/2$, the heat kernel is C^m. The estimates emerge naturally from the convolution formula for the heat kernel. The formal series

$$h(t, x, y) \sum_{k=0}^{\infty} u_k(x, y) t^k$$

gives an asymptotic expansion of the heat kernel when restricted to the diagonal $x = y$. □

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