



Full length article

# Approximation of rough functions

M.F. Barnsley<sup>a,\*</sup>, B. Harding<sup>a</sup>, A. Vince<sup>b</sup>, P. Viswanathan<sup>a,1</sup>

<sup>a</sup> Australian National University, Canberra, ACT 2601, Australia

<sup>b</sup> Department of Mathematics, University of Florida, Gainesville, FL 32611-8105, USA

Received 1 May 2015; received in revised form 25 January 2016; accepted 25 April 2016

Available online 18 May 2016

Communicated by Martin Buhmann

---

## Abstract

For given  $p \in [1, \infty]$  and  $g \in L^p(\mathbb{R})$ , we establish the existence and uniqueness of solutions  $f \in L^p(\mathbb{R})$ , to the equation

$$f(x) - af(bx) = g(x),$$

where  $a \in \mathbb{R}$ ,  $b \in \mathbb{R} \setminus \{0\}$ , and  $|a| \neq |b|^{1/p}$ . Solutions include well-known nowhere differentiable functions such as those of Bolzano, Weierstrass, Hardy, and many others. Connections and consequences in the theory of fractal interpolation, approximation theory, and Fourier analysis are established.

© 2016 The Author(s). Published by Elsevier Inc. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

MSC: 26A15; 26A18; 26A27; 42A38; 39B12

Keywords: Functional equations; Fractal interpolation; Iterated function system; Fractal geometry; Fourier series

---

## 1. Introduction

The subject of this paper, in broad terms, is fractal analysis. More specifically, it concerns a constellation of ideas centered around the single unifying functional equation (1). In practice,

---

\* Corresponding author.

E-mail addresses: [michael.barnsley@anu.edu.au](mailto:michael.barnsley@anu.edu.au) (M.F. Barnsley), [brendan.harding@anu.edu.au](mailto:brendan.harding@anu.edu.au) (B. Harding), [avince@ufl.edu](mailto:avince@ufl.edu) (A. Vince), [amritaviswa@gmail.com](mailto:amritaviswa@gmail.com) (P. Viswanathan).

<sup>1</sup> Current address: Department of Mathematics, Indian Institute of Technology New Delhi, New Delhi, 110016, India.

the given function  $g(x)$  may be smooth and the solution  $f(x)$  is often rough, possessing fractal features. Classical notions from interpolation and approximation theory are extrapolated, via this equation, to the fractal realm, the basic goal being the utilization of fractal functions to analyze real world rough data.

For given  $p \in [1, \infty]$  and  $g : \mathbb{R} \rightarrow \mathbb{R}$  with  $g \in L^p(\mathbb{R})$ , we establish the existence and uniqueness of solutions  $f \in L^p(\mathbb{R})$ , to the equation

$$f(x) - af(bx) = g(x), \quad (1)$$

where  $a \in \mathbb{R}$ ,  $b \in \mathbb{R} \setminus \{0\}$ , and  $|a| \neq |b|^{1/p}$ . By uniqueness we mean that any solution is equal to  $f$  almost everywhere in  $\mathbb{R}$ . When  $a$ ,  $b$  and  $g$  are chosen appropriately, solutions include the classical nowhere differentiable functions of Bolzano, Weierstrass, Hardy, Takagi, and others; see the reviews [2,13]. For example, the continuous, nowhere differentiable function presented by Weierstrass in 1872 to the Berlin Academy, defined by

$$f(x) = \sum_{k=0}^{\infty} a^k \cos(\pi b^k x), \quad (2)$$

where  $0 < a < 1$ ,  $b$  is an integer, and  $ab \geq 1 + \frac{3}{2}\pi$  (see [12]), is a solution to the functional equation (1) when  $g(x) = \cos(\pi x)$ . The graph of  $f$  was studied as a fractal curve in the plane by Besicovitch and Ursell [5]. An elementary and readable account of the history of nowhere differentiable functions is [23]; it includes the construction by Bolzano (1830) of one of the earliest examples of such a function. Analytic solutions to the functional equation (1) for various values of  $a$  and  $b$ , when  $g$  is analytic, have been studied by Fatou in connection with Julia sets [9,22]. If  $g(x) = e^{\lambda x}$ , then  $f(x) = \sum_{k=0}^{\infty} a^k e^{b^k \lambda x}$  is a solution to Eq. (1) and is a special case of the Dirichlet series studied by Iserles and Wang [14] in the context of solutions to ordinary differential equations.

If  $|ab| > 1$ ,  $b > 1$  is an integer, and  $g$  has certain properties, see [2,13], then the graph of  $f$ , restricted to  $[0, 1]$ , has box-counting (Minkowski) dimension

$$D = 2 + \frac{\ln |a|}{\ln b}.$$

In particular, if  $g(x) = \cos(\pi x)$ , then by a recent result of Bárány, Romanowska, and Barański [1] the Hausdorff dimension of the graph of  $f$  is  $D$ , for a large set of values of  $|a| < 1$ .

Notation that is used in this paper is set in Section 2. In Section 3 we establish existence and uniqueness of solutions to Eq. (1) in various function spaces (see Theorem 1, Corollaries 1 and 4, Proposition 1). Although the emphasis has been on the pathology of the solution to the functional equation (1), it is shown that, if  $g$  is continuous, then the solution  $f$  is continuous (see Corollaries 2 and 3).

A widely used method for constructing fractal sets, in say  $\mathbb{R}^2$ , is as the attractor of an iterated function system (IFS). Indeed, starting in the mid 1980s, IFS fractal attractors  $A$  were systematically constructed so that  $A$  is the graph of a function  $f : J \rightarrow \mathbb{R}$ , where  $J$  is a closed bounded interval on the real line [3]. Moreover  $f$  can be made to interpolate the data  $(x_0, y_0), (x_1, y_1), \dots, (x_N, y_N)$ , where  $x_0 < x_1 < \dots < x_N$  and  $J = [x_0, x_N]$ . The basic idea is to consider an IFS on  $\mathbb{R}^2$  of the form  $F = (\mathbb{R}^2; w_1, w_2, \dots, w_N)$  where  $w_n(x, y) = (L_n(x), F_n(x, y))$ ;  $L_n$  is a linear function that maps the interval  $J$  to the interval  $[x_{n-1}, x_n]$ ; and  $w_n$  takes  $(x_0, y_0)$  to  $(x_{n-1}, y_{n-1})$  and  $(x_N, y_N)$  to  $(x_n, y_n)$ . Under appropriate conditions on the functions  $L_n$  and  $F_n$  (see Section 4 for details), there exists a unique closed

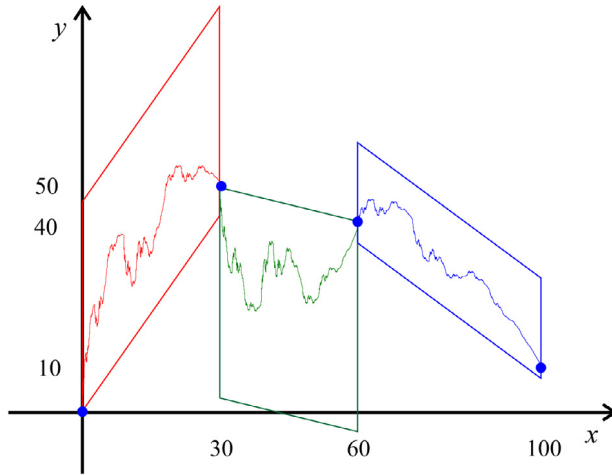


Fig. 1. This illustrates how the graph of a fractal interpolation function is made of affinely transformed copies of itself, and how it interpolates the data. The four large dots represent the data and the parallelograms are affine transforms of a rectangle that contains the graph and has sides parallel to the axes.

bounded nonempty set  $A \subset \mathbb{R}^2$  that obeys the self-referential equation  $A = \cup_{n=1}^N w_n(A)$ , which says  $A$  is made of transformed copies of itself, for example as illustrated in Fig. 1. This set  $A$  is called the attractor of the IFS, and has the property that it is the graph of a continuous function, defined on  $J$ , that interpolates the data.

The book [18] is a reference on such fractal interpolation functions constructed via an IFS. One of the appeals of the theory is that it is possible to control the box-counting dimension and smoothness of the graph of the interpolant. The solutions to the functional equation (1) include, not only the classical nowhere differentiable functions, but also fractal interpolation functions. This is the subject of Section 4, in particular Theorems 2 and 3. One impetus for the research reported here is the work on fractal interpolation by Massopust [18], Navascués [21,19,20], and Chand and his students [6].

In Section 5, Eq. (1) and the theory surrounding it are leveraged to obtain orthogonal expansions – that we call Weierstrass Fourier series – and corresponding approximants, for various functions, both smooth and rough, using approximants with specified Minkowski and even Hausdorff dimension.

Some ideas in the present work are anticipated, at least in flavor, in Deliu and Wingren [7] and Kigami and his collaborators [16,25]. But, as far as we know, our main observations, namely Theorem 1 and its corollaries, Theorems 2 and 3, and Theorem 4, are new.

## 2. Notation

For  $p \in [1, \infty)$ ,  $L^p(X)$  denotes the Banach space of functions  $f : X \rightarrow \mathbb{R}$  such that

$$\int_X |f(x)|^p dx < \infty,$$

where the integration is with respect to Lebesgue measure on  $X$ . In this paper,  $X$  will be  $\mathbb{R}$ , a closed interval of  $\mathbb{R}$ , or an interval of the form  $[c, \infty)$ ,  $(-\infty, c]$  or  $(-\infty, c] \cup [c', \infty)$ . The space  $L^\infty(X)$  denotes the Banach space of functions  $f : X \rightarrow \mathbb{R}$  such that the essential supremum of

$|f|$  is bounded. For all  $p \in [0, \infty]$ , the norm of  $f \in L^p(X)$  is denoted  $\|f\|_p$ , where

$$\|f\|_p = \left( \int_X |f(x)|^p dx \right)^{1/p} \quad \text{when } p \in [1, \infty),$$

$$\|f\|_\infty = \inf\{M \in [0, \infty) : |f(x)| \leq M \text{ for almost all } x \in X\}.$$

The norm of a bounded linear operator  $H : L^p(X) \rightarrow L^p(X)$  is defined by

$$\|H\|_p = \max\{\|Hf\|_p : \|f\|_p = 1\}.$$

The space of bounded uniformly continuous real valued functions with the supremum norm is denoted  $C_B(X)$ . Further let

$$C_B^k(X) = \{f : f^{(j)} \in C_B(X), j = 0, 1, \dots, k\}.$$

For a bounded continuous function  $f$  and  $\alpha \in (0, 1]$ , let

$$[f]_\alpha = \sup_{x, y \in X, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

For  $k \in \mathbb{N} \cup \{0\}$ , the Hölder space

$$C_B^{k, \alpha}(\mathbb{R}) := \{f \in C_B(\mathbb{R}) : f^{(j)} \in C_B(\mathbb{R}), j = 0, 1, \dots, k, \|f\|_{C^{k, \alpha}} < \infty\}$$

where

$$\|f\|_{C^{k, \alpha}} := \sum_{j=0}^k \|f^{(j)}\|_\infty + [f^{(k)}]_\alpha$$

is a Banach space.

Let  $k \geq 1$  be an integer and  $f \in L^1_{\text{loc}}(\mathbb{R})$ , the space of all locally integrable functions. A function  $g \in L^1_{\text{loc}}(\mathbb{R})$  is a weak-derivative of  $f$  of order  $k$  if

$$\int_{\mathbb{R}} g(x)\phi(x)dx = (-1)^k \int_{\mathbb{R}} f(x)\phi^{(k)}(x)dx$$

for all  $\phi \in C_c^\infty(\mathbb{R})$ , where  $C_c^\infty(\mathbb{R})$  is the space of continuous functions with compact support, having continuous derivatives of every order.

For  $1 \leq p \leq \infty$  and  $k \in \mathbb{N} \cup \{0\}$ , let  $W^{k, p}(\mathbb{R})$  denote the usual Sobolev space. That is,

$$f \in W^{k, p}(\mathbb{R}) \iff f^{(j)} \in L^p(\mathbb{R}), \quad j = 0, 1, \dots, k,$$

where  $f^{(j)}$  denotes the  $j$ th weak or distributional derivative of  $f$ . The space  $W^{k, p}(\mathbb{R})$  endowed with the norm

$$\|f\|_{W^{k, p}} := \left\| f^{(k)} \right\|_p + \|f\|_p$$

is a Banach space.

Consider the difference operator

$$\Delta_h f(x) = f(x - h) - f(x)$$

and define the modulus of continuity by

$$\omega_p^2(f, t) = \sup_{|h| \leq t} \|\Delta_h^2 f\|_p.$$

For  $n \in \mathbb{N} \cup \{0\}$ ,  $s = n + \alpha$ ,  $0 < \alpha \leq 1$  and  $1 \leq p, q \leq \infty$ , the Besov space  $B_{p,q}^s(\mathbb{R})$  consists of all functions  $f$  such that

$$f \in W^{n,p}(\mathbb{R}), \quad \int_0^\infty \left| \frac{w_p^2(f^{(n)}, t)}{t^\alpha} \right|^q \frac{dt}{t} < \infty.$$

The functional

$$\|f\|_{B_{p,q}^s} := \left( \|f\|_{W^{n,p}}^q + \int_0^\infty \left| \frac{w_p^2(f^{(n)}, t)}{t^\alpha} \right|^q \frac{dt}{t} \right)^{\frac{1}{q}}$$

is a norm which turns  $B_{p,q}^s(\mathbb{R})$  into a Banach space.

### 3. Solutions of the functional equation

The functional equation (1) can be expressed as

$$M_{a,b} f = g, \tag{3}$$

where the linear operator  $M_{a,b}$  is defined as follows.

**Definition 1.** For all  $p \in [1, \infty]$ ,  $a, b \in \mathbb{R}$ ,  $b \neq 0$ , the linear operators  $T_b : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$  and  $M_{a,b} : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$  are given by

$$\begin{aligned} (T_b f)(x) &= f(bx) \\ M_{a,b} f &= (I - aT_b) f \end{aligned}$$

for all  $x \in \mathbb{R}$  and all  $f \in L^p(\mathbb{R})$ . By convention, if  $p = \infty$  and  $b \neq 0$ , then  $|b|^{\frac{1}{p}} = 1$ .

It is easy to check that Eq. (3) is equivalent to

$$M_{\frac{1}{a}, \frac{1}{b}} f = \bar{g}, \tag{4}$$

where  $\bar{g} = -\frac{1}{a} T_{\frac{1}{b}} g$ . This fact is used in the proof of the following theorem.

**Theorem 1.** For all  $p \in [1, \infty]$ ,  $a, b \in \mathbb{R}$ ,  $b \neq 0$ , and  $|a| \neq |b|^{\frac{1}{p}}$ , the linear operators  $T = T_b$  and  $M = M_{a,b}$  are homeomorphisms from  $L^p(\mathbb{R})$  to itself. In particular,

1.

$$T_b^{-1} = T_{\frac{1}{b}}$$

2.

$$\|T_b\|_p = |b|^{-\frac{1}{p}}$$

3.

$$\left| 1 - \frac{|a|}{|b|^{\frac{1}{p}}} \right| \|f\|_p \leq \|M_{a,b} f\|_p \leq \left( 1 + \frac{|a|}{|b|^{\frac{1}{p}}} \right) \|f\|_p$$

4.

$$M_{a,b}^{-1} = \begin{cases} \sum_{n=0}^{\infty} a^n T_b^n & \text{if } |a| < |b|^{\frac{1}{p}}, \\ -\sum_{n=1}^{\infty} \left(\frac{1}{a}\right)^n T_{\frac{1}{b}}^n & \text{if } |a| > |b|^{\frac{1}{p}}. \end{cases}$$

**Proof.** It is readily verified that  $T$  is invertible with inverse  $T_b^{-1} = T_{\frac{1}{b}}$  and that the formula (2) for the  $p$ -norm of  $T_b$  holds. Consequently

$$\|T_b^{-1}\|_p = |b|^{\frac{1}{p}}.$$

Inequality (3) follows from (2) and the triangle inequality.

Assume that  $|a| < |b|^{\frac{1}{p}}$ . To show that  $M$  is injective in this case, assume that  $Mf = 0$ , i.e.,  $\|f - aT_b f\|_p = 0$ . Then

$$\begin{aligned} 0 &= \|f - aT_b f\|_p \geq \|f\|_p - |a| \|T_b\|_p \|f\|_p = (1 - |a| \|T_b\|_p) \|f\|_p \\ &= \left(1 - |a| |b|^{-\frac{1}{p}}\right) \|f\|_p \geq 0, \end{aligned}$$

which implies that  $\|f\|_p = 0$ . To show that  $M$  is surjective and that a solution to  $Mf = g$  in  $L^p(\mathbb{R})$  is

$$f = \sum_{n=0}^{\infty} a^n T_b^n g$$

first note that the series is absolutely and uniformly convergent in  $L^p(\mathbb{R})$ . This is because the partial sums are Cauchy sequences. Now, using the continuity of  $M : L^p(\mathbb{R}) \rightarrow L^p(\mathbb{R})$  and equality (2) in the statement of the theorem, we have

$$M \left( \sum_{n=0}^{\infty} a^n T_b^n g \right) = \lim_{k \rightarrow \infty} \left( \sum_{n=0}^k a^n M T_b^n g \right) = \lim_{k \rightarrow \infty} (I - a^{k+1} T_b^{k+1}) g = g.$$

Now assume that  $|a| > |b|^{\frac{1}{p}}$ . By the paragraph above  $M_{\frac{1}{a}, \frac{1}{b}}$  is injective. But it is easily checked, using statement (2) in the theorem, that  $M_{a,b} f = 0$  if and only if  $M_{\frac{1}{a}, \frac{1}{b}} f = 0$ . Therefore  $M_{a,b}$  is injective. To show that  $M := M_{a,b}$  is surjective and that a solution to  $Mf = g$  in  $L^p(\mathbb{R})$  is

$$f = -\sum_{n=1}^{\infty} \left(\frac{1}{a}\right)^n T_{\frac{1}{b}}^n g, \tag{5}$$

note that, by the paragraph above,

$$M_{\frac{1}{a}, \frac{1}{b}}^{-1} \left( -\frac{1}{a} T_{\frac{1}{b}} g \right) = \sum_{n=0}^{\infty} \left(\frac{1}{a}\right)^n T_{\frac{1}{b}}^n \left( -\frac{1}{a} T_{\frac{1}{b}} g \right) = -\sum_{n=1}^{\infty} \left(\frac{1}{a}\right)^n T_{\frac{1}{b}}^n g.$$

Referring to Eq. (4), this verifies Eq. (5).  $\square$

The next corollary on existence and uniqueness of solutions to Eq. (1) follows at once from Theorem 1.

**Corollary 1.** Assume that  $a, b \in \mathbb{R}$ ,  $b \neq 0$ , and  $|a| \neq |b|^{\frac{1}{p}}$ . For any  $g \in L^p(\mathbb{R})$ ,  $p \in [1, \infty]$ , there is a unique solution  $f \in L^p(\mathbb{R})$  to the equation

$$f(x) - af(bx) = g(x),$$

and the solution is given by the following series that are absolutely and uniformly convergent in  $L^p(\mathbb{R})$ :

$$f(x) = \begin{cases} \sum_{n=0}^{\infty} a^n g(b^n x) & \text{if } |a| < |b|^{\frac{1}{p}} \\ -\sum_{n=1}^{\infty} \left(\frac{1}{a}\right)^n g\left(\frac{x}{b^n}\right) & \text{if } |a| > |b|^{\frac{1}{p}}. \end{cases} \tag{6}$$

**Remark 1.** Recall that the adjoint of a bounded linear operator  $A : X \rightarrow Y$  is the operator  $A^* : Y^* \rightarrow X^*$  defined by  $(A^*\mu)(x) = \mu(Ax)$  for all  $\mu \in Y^*$  and  $x \in X$ , where  $X^*$  denotes the dual space of  $X$ . For  $1 \leq p < \infty$  there is a canonical isomorphism between  $L^p(\mathbb{R})^*$  and  $L^q(\mathbb{R})$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ . For each linear functional  $\mu \in L^p(\mathbb{R})^*$ , this isomorphism associates a unique representative  $g \in L^q(\mathbb{R})$  such that  $\mu(f) = \int_{\mathbb{R}} f(x)g(x) dx$  for all  $f \in L^p(\mathbb{R})$ . It is routine to show that

$$T_b^* = \frac{1}{b} T_{\frac{1}{b}}$$

in the sense that, if the representative of  $\mu \in L^p(\mathbb{R})^*$  in the space  $L^q(\mathbb{R})$  is  $g$ , then the representative of  $T_b^*\mu$  is  $b^{-1}T_{b^{-1}}g$ . Similarly

$$M_{a,b}^* = M_{\frac{a}{b}, \frac{1}{b}}$$

**Remark 2.** If  $b = 0$ , then Eq. (1) has solution

$$f(x) = g(x) + \frac{a}{1-a} g(0).$$

So, for all  $a \neq 1$ , there is a well-defined solution  $f(x)$  for all  $x \in \mathbb{R}$ , for each specified value of  $g(0)$ . Since, as a element of  $L^p(\mathbb{R})$ , the function  $g$  is defined only up to a set of measure 0, the value  $g(0)$  has little meaning. Thus it does not make sense to consider Eq. (1) in  $L^p(\mathbb{R})$  when  $b = 0$ . However, the problem of finding  $f$  for a given  $g$  is well-posed in spaces such as  $C_B(\mathbb{R})$ , even when  $b = 0$ .

In view of Remark 2, except where otherwise stated, it is assumed throughout this paper that  $b \neq 0$  and  $|a| \neq |b|^{\frac{1}{p}}$ . The results in Theorem 1 and its Corollary 1 hold for various spaces related to the  $L^p$ -spaces. Corollaries 2, 3, and 4 concern these related spaces.

**Corollary 2.** Assume that  $a, b \in \mathbb{R}$ ,  $b \neq 0$ , and  $|a| \neq 1$ . For any  $g \in C_B(\mathbb{R})$ , there is a unique solution  $f \in C_B(\mathbb{R})$  to the equation

$$f(x) - af(bx) = g(x),$$

and the solution is given by the following series that are absolutely and uniformly convergent:

$$f(x) = \begin{cases} \sum_{n=0}^{\infty} a^n g(b^n x) & \text{if } |a| < 1 \\ -\sum_{n=1}^{\infty} \left(\frac{1}{a}\right)^n g\left(\frac{x}{b^n}\right) & \text{if } |a| > 1. \end{cases}$$

**Proof.** If  $|a| < 1$ , then  $\left| \sum_{n=M}^{\infty} a^n g(b^n x) \right| < \frac{|a|^M}{1-|a|} \|g\|_{\infty}$ . Therefore

$$\lim_{M \rightarrow \infty} \sup_{x \in \mathbb{R}} \left| \sum_{n=M}^{\infty} a^n g(b^n x) \right| = 0,$$

which implies  $\sum_{n=0}^{\infty} a^n g(b^n x)$  is absolutely and uniformly convergent. Since  $g \in C_B(\mathbb{R})$ , it follows that the infinite sum is a continuous function. That the series is a solution of the functional equation can be verified at once by substitution, see also [Corollary 1](#). A similar argument applies in the case  $|a| > 1$ .  $\square$

The following relationships between the continuity of  $f$  and the continuity of  $g$  follow as in [Corollary 2](#).

**Corollary 3.** For the equation  $M_{a,b} f = g$  in  $L^{\infty}(\mathbb{R})$ , if  $|a| < 1$ , then the following hold.

1. If  $b > 0$ , then  $f \in C_B([0, \infty))$  if and only if  $g \in C_B([0, \infty))$ .
2. If  $b \geq 1$ , then  $f \in C_B([1, \infty))$  if and only if  $g \in C_B([1, \infty))$ .
3. If  $0 < b \leq 1$ , then  $f \in C_B([0, 1])$  if and only if  $g \in C_B([0, 1])$ .

A similar set of statements hold when  $C_B(X)$  is replaced by  $C'_B(X)$ , the set of functions in  $C_B(X)$  with countably many discontinuities.

**Remark 3.** If, in [Corollary 2](#),  $g \in L^{\infty}(\mathbb{R})$  is assumed piecewise continuous with countably many points of discontinuity, rather than continuous, then it follows by a similar argument that the solution  $f \in L^{\infty}(\mathbb{R})$  to  $Mf = g$  is piecewise continuous with at most countably many points of discontinuity.

**Remark 4.** Examples related to fractal interpolation (see [Example 1](#)) show that  $f = M^{-1}g$  may be continuous on  $[0, 1]$  even if  $g$  possesses discontinuities.

Unlike continuity, it is well-known from basic real analysis that  $f = M_{a,b}^{-1}g$  may fail to be differentiable even if  $g$  is differentiable. Vice versa, when  $|a| > 1$  and  $g$  is continuous,  $f$  may be more differentiable than  $g$ . Thus, in a general sense, for  $|a| < 1$ , the mapping  $M_{a,b}^{-1}$  is a “roughing” operation, and for  $|a| > 1$ , it is a “smoothing” operation.

The following estimate is worth mentioning.

**Proposition 1.** Consider the equation  $M_{a,b} f = g$  for  $f \in C_B([0, \infty))$ ,  $|a| < 1$  and  $b > 0$ . Then the uniform distance between  $f$  and  $g$  satisfies

$$\|g - f\|_{\infty} = \|g - M_{a,b}^{-1}g\|_{\infty} \leq \frac{|a|}{1 - |a|} \|g\|_{\infty}.$$



Consequently

$$\|I - M_{a,b}^{-1}\|_\infty \leq \frac{|a|}{1 - |a|}.$$

**Proof.** Note that

$$\begin{aligned} |g(x) - M_{a,b}^{-1}g(x)| &= \left|g(x) - \sum_{n=0}^\infty a^n g(b^n x)\right| \\ &\leq \sum_{n=1}^\infty |a|^n \|g\|_\infty \\ &= \frac{|a|}{1 - |a|} \|g\|_\infty. \end{aligned}$$

Therefore  $\|g - f\|_\infty \leq \frac{|a|}{1 - |a|} \|g\|_\infty$ , proving the assertion.  $\square$

The term automorphism in the next corollary refers to a linear map that is a homeomorphism of a space to itself. In particular, statement (6) in the corollary is used in Section 5.

**Corollary 4.** *If  $M = M_{a,b}$  is the operator of Definition 1, with  $|a| \neq |b|^{\frac{1}{p}}$ , then  $M$  is an automorphism when considered as a mapping on*

1.  $L^p([0, \infty))$  or  $L^p((-\infty, 0])$  if  $b > 0$ ;
2.  $L^p([1, \infty))$  or  $L^p((-\infty, -1])$  if  $b > 1$ ;
3.  $L^p((-\infty, -1] \cup [1, \infty))$  if  $|b| > 1$ ;
4.  $L^p([0, 1])$  or  $L^p([-1, 0])$  if  $0 < b < 1$ ;
5.  $L^p([-1, 1])$  if  $0 < |b| \leq 1$ ;
6.  $L^\infty([0, \infty)) \cap \mathcal{P}$  if  $b \in \mathbb{N}$ ,  $|a| < 1$ , where  $\mathcal{P}$  is the set of functions  $f : [0, \infty) \rightarrow \mathbb{R}$  such that  $f(x) = f(x + 1)$  for all  $x \in (0, \infty)$ . This is with the understanding that, for  $g \in L^\infty([0, \infty)) \cap \mathcal{P}$ , a representative of  $M^{-1}g$  can be chosen to lie in  $\mathcal{P}$ .

**Proof.** (1) The space  $L^p(\mathbb{R})$  is the direct sum of two subspaces  $L_+$  and  $L_-$ , the first consisting of functions which vanish over the negative reals and the second consisting of functions which vanish over the positive reals. Since each of these two subspaces is mapped into itself by  $M$  and since  $M$  is bijective on  $L^p(\mathbb{R})$ , it follows that  $M$  restricted to  $L_+$  and  $M$  restricted to  $L_-$  are both bijective. The proofs of (2)–(6) are similar, some using Corollary 1.  $\square$

For appropriate values of  $a$  and  $b$ , the operator  $M_{a,b}$  also defines an automorphism in some standard spaces of smooth functions that occur frequently in various fields of analysis such as approximation theory, numerical analysis, functional analysis, harmonic analysis, and in particular in connection with partial differential equations. The proof is similar to that of Theorem 1, and hence is omitted.

**Proposition 2.** *For the operator  $M_{a,b}$  specified in Definition 1 the following properties hold.*

1. If  $|a| < \min\{|b|^{\frac{1}{p}}, |b|^{\frac{1}{p}-k}\}$  or  $|a| > \max\{|b|^{\frac{1}{p}}, |b|^{\frac{1}{p}-k}\}$ , then  $M_{a,b}$  is an automorphism on Sobolev space  $W^{k,p}(\mathbb{R})$ .
2. If  $|a| < \min\{1, |b|^{-1}, |b|^{-2}, \dots, |b|^{-k}, |b|^{-\alpha}\}$  or  $|a| > \max\{1, |b|^{-1}, |b|^{-2}, \dots, |b|^{-k}, |b|^{-\alpha}\}$ , then  $M_{a,b}$  is an automorphism on Hölder space  $C_B^{k,\alpha}(\mathbb{R})$ .
3. If  $|a| < \min\{|b|^{\frac{1}{p}}, |b|^{\frac{1}{p}-n}, |b|^{\frac{1}{p}-n-\alpha}\}$  or  $|a| > \max\{|b|^{\frac{1}{p}}, |b|^{\frac{1}{p}-n}, |b|^{\frac{1}{p}-n-\alpha}\}$ , then  $M_{a,b}$  is an automorphism on Besov space  $B_{p,q}^s(\mathbb{R})$ .

In all the above cases  $M_{a,b}^{-1} = \sum_{n=0}^{\infty} a^n T_b^n$  for the first set of admissible values of parameters  $a, b$  and  $M_{a,b}^{-1} = -\sum_{n=1}^{\infty} \left(\frac{1}{a}\right)^n T_{\frac{1}{b}}^n$  for the second set of admissible values of parameters  $a, b$ .

**Remark 5.** A straightforward but useful consequence of the fact that  $M_{a,b}^{-1}$  is an automorphism on various spaces is the following. It is well known that Schauder bases are preserved under an isomorphism. Consequently, if  $\{f_n\}_{n=1}^{\infty}$  is a Schauder basis for  $X$ , where  $X$  is one of the spaces  $L^p(\mathbb{R})$ ,  $W^{k,p}(\mathbb{R})$ ,  $C_B^{k,\alpha}(\mathbb{R})$  or  $B_{p,q}^s(\mathbb{R})$ , then  $\{M_{a,b}^{-1}f_n\}_{n=1}^{\infty}$  is a Schauder basis consisting of rough analogues of the functions  $\{f_n\}_{n=1}^{\infty}$ . In particular, if  $\{f_n\}_{n=1}^{\infty}$  is an orthonormal basis for the Hilbert space  $L^2(\mathbb{R})$  or  $W^{k,2}(\mathbb{R})$ , then  $\{M_{a,b}^{-1}f_n\}_{n=1}^{\infty}$  is a Riesz basis for  $L^2(\mathbb{R})$  or  $W^{k,2}(\mathbb{R})$ . Some orthonormal bases consisting of rough functions obtained via  $M_{a,b}^{-1}$  are discussed in detail in Section 5.

### 4. Fractal Interpolation

To illustrate how standard fractal interpolation theory fits into the functional equation framework, consider a given set of data points  $\{(x_n, y_n)\}_{n=0}^N \subset \mathbb{R}^2$ ,  $N > 1$ , with  $0 = x_0 < x_1 < x_2 \cdots < x_N = 1$ . At minimum what one seeks is a function  $f : [0, 1] \rightarrow \mathbb{R}$ , such that

1.  $f$  interpolates the data, i.e.,  $f(x_n) = y_n$ ,  $n = 0, 1, \dots, N$ ;
2. there is an IFS  $F = (\mathbb{R}^2; w_1, w_2, \dots, w_N)$  whose attractor is the graph of the function  $f$  on the interval  $[0, 1]$ ;
3. parameters of the IFS can be varied to control continuity and differentiability of  $f$  and the Minkowski dimension of the graph of  $f$ .

The IFS maps  $w_n$ ,  $n = 1, 2, \dots, N$ , that are studied extensively in fractal interpolation theory [3] are of the form

$$w_n(x, y) = (L_n(x), F_n(x, y)), \tag{7}$$

where

$$L_n(x) = a_n x + b_n, \quad F_n(x, y) = \alpha_n y + g_n(x), \tag{8}$$

$|\alpha_n| < 1$ ;  $g_n : [0, 1] \rightarrow \mathbb{R}$  is continuous; and

$$\begin{aligned} L_n(x_0) &= x_{n-1}, & F_n(x_0, y_0) &= y_{n-1}, \\ L_n(x_N) &= x_n, & F_n(x_N, y_N) &= y_n, \end{aligned} \tag{9}$$

for all  $n = 1, 2, \dots, N$ . In this case there is a unique attractor of  $F$ , and it is the graph of a continuous function  $f$  that interpolates the data [3]. The parameters  $\alpha_n$  and  $g_n$  can be varied to control continuity and differentiability of  $f$  and the Minkowski dimension of the graph of  $f$ .

We specialize to the uniform partition of  $[0, 1]$  and a constant scaling factor, i.e.,

$$L_n(x) = \frac{x + n - 1}{N}, \quad \alpha_n = a, \quad |a| < 1, \tag{10}$$

for all  $n = 1, 2, \dots, N$ .

The next two theorems make precise the close relationship between fractal interpolation functions and solutions to the ‘‘Weierstrass-type’’ functional equation.

**Theorem 2.** Given data points  $\{(x_n, y_n)\}_{n=0}^N \subset \mathbb{R}^2$ ,  $N > 1$ , let  $F$  be the IFS defined by Eqs. (7)–(10), and let  $f$  be the function on  $[0, 1]$  whose graph is the attractor of  $F$  and that interpolates the data.

Then  $f$  is the unique solution to the functional equation  $f(x) - af(Nx) = g(x)$  considered in the space  $L^\infty([0, \infty)) \cap \mathcal{P}$  of Corollary 4, where

$$g(x) = \begin{cases} g_n(L_n^{-1}(x)) & \text{if } x \in [x_{n-1}, x_n), n = 1, 2, \dots, N, \\ g_N(1) & \text{if } x = 1, \\ g(x - 1) & \text{if } x \in (1, \infty). \end{cases}$$

**Proof.** It follows immediately from the fact that the graph of  $f(x)$  is the attractor of the IFS with functions as in Eq. (7) that

$$\begin{aligned} \{(x, f(x)) : x \in [0, 1]\} &= \bigcup_{n=1}^N \{(L_n(x)), af(x) + g_n(x) : x \in [0, 1]\} \\ &= \bigcup_{n=1}^N \left\{ (x, af(L_n^{-1}(x))) + g_n(L_n^{-1}(x)) : x \in \left[ \frac{n-1}{N}, \frac{n}{N} \right] \right\}. \end{aligned}$$

This implies, for  $x \in [(n-1)/N, n/N]$ ,  $n = 1, 2, \dots, N$  and in the space  $L^\infty([0, \infty)) \cap \mathcal{P}$ , that

$$f(x) = af(Nx - (n - 1)) + g_n(L_n^{-1}(x)) = af(Nx) + g(x). \quad \square$$

**Theorem 3.** Let  $f$  be the unique solution to the functional equation  $f(x) - af(Nx) = g(x)$  considered in the space  $L^\infty([0, \infty)) \cap \mathcal{P}$ , where  $g \in L^\infty([0, \infty)) \cap \mathcal{P}$  has the following properties

1.  $g$  is continuous on the intervals  $[x_0, x_1], [x_1, x_2], \dots, [x_{N-1}, x_N]$ ,
2. the limit from the right  $g(\frac{n}{N}+)$  exists for  $n = 1, \dots, N - 1$ .

Then  $f$  interpolates the data  $\{(x_n, y_n), n = 0, 1, 2, \dots, N\}$ , where  $x_n = n/N$  and

$$\begin{aligned} y_0 &= g(0)/(1 - a) \\ y_N &= g(1)/(1 - a) \\ y_n &= g(x_n) + \frac{a}{1 - a} g(1), \quad n = 1, 2, \dots, N - 1. \end{aligned}$$

Moreover, the closure of the graph of  $f$  restricted to the domain  $[0, 1]$  is the unique attractor of the IFS  $W = ([0, 1] \times \mathbb{R}; w_1, w_2, \dots, w_N)$ , where  $w_n(x, y) = (L_n(x), ay + g_n(x))$ ,  $n = 1, 2, \dots, N$ , and

$$\begin{aligned} L_n(x) &= (x + n - 1)/N \\ g_n(x) &= \begin{cases} g(L_n(x)) & \text{if } 0 < x < 1 \\ g\left(\frac{n-1}{N}+\right) & \text{if } x = 0 \\ g\left(\frac{n}{N}\right) & \text{if } x = 1. \end{cases} \end{aligned}$$

If, in addition to properties (1–2) of the function  $g$ , we have

$$(3) \quad g\left(\frac{n}{N}+\right) - g\left(\frac{n}{N}\right) = \frac{a}{1 - a} (g(1) - g(0))$$

for  $n = 1, 2, \dots, N - 1$ , then  $f$  is continuous on  $[0, 1]$ , and the graph of  $f$  restricted to the domain  $[0, 1]$  is the unique attractor of the IFS  $W$ .

**Proof.** Concerning the interpolation of the data, assume that  $|a| < 1$ . Statement (1) of Corollary 4 guarantees a unique solution given by  $f(x) = \sum_{k=0}^{\infty} a^k g(N^k x)$ . Substituting  $x = 0$  into the functional equation, we obtain  $f(0) - af(0) = g(0)$  which implies  $f(0) = \frac{g(0)}{1-a} = y_0$ . Substituting  $x = 1$  in the series expansion yields

$$f(1) = \sum_{k=0}^{\infty} a^k g(N^k) = \sum_{k=0}^{\infty} a^k g(1) = y_N.$$

With  $1 \leq n \leq N - 1$ , substituting  $x = x_n$  and using properties of  $g$ , we have

$$\begin{aligned} f(x_n) &= \sum_{k=0}^{\infty} a^k g(N^k x_n) = \sum_{k=0}^{\infty} a^k g\left(N^k \frac{n}{N}\right) = g\left(\frac{n}{N}\right) + \sum_{k=1}^{\infty} a^k g(N^{k-1}n) \\ &= g\left(\frac{n}{N}\right) + \sum_{k=1}^{\infty} a^k g(1) = g(x_n) + ay_N = y_n. \end{aligned}$$

Concerning the statement about the closure of the graph of  $f$ , we consider the following set-valued map associated with the IFS  $W$ . With a slight abuse of notation, we shall denote the associated map also by  $W$  and let  $W : 2^{[0,1] \times \mathbb{R}} \rightarrow 2^{[0,1] \times \mathbb{R}}$  defined by

$$W(B) = \bigcup_{i=1}^N w_i(B).$$

Let  $G := \{(x, f(x)) : x \in [0, 1]\}$ . It is well known, see for example [4, Theorem 3.2], that under the stated conditions the IFS  $W$  possesses a unique attractor. The attractor is the unique compact set  $A \subset [0, 1] \times \mathbb{R}$  such that  $W(A) = A$ . It suffices to show that  $W(\overline{G}) = \overline{G}$ . Note that  $f$  is periodic with period 1.

To show that  $W(\overline{G}) = \overline{G}$ , we first show that

$$W(\widehat{G}) \subseteq G, \tag{11}$$

where  $\widehat{G} = G \setminus \{(0, f(0)), (1, f(1))\}$ . For any  $n = 1, 2, \dots, N$ , let  $(x', y') \in w_n(\widehat{G})$ . Then there is an  $(x, y)$  such that  $x \in (0, 1]$ ,  $y = f(x)$ ,  $x' = L_n(x) = (n - 1 + x)/N$ , and

$$\begin{aligned} y' &= ay + g_n(x) = a f(x) + g(L_n(x)) = a f(Nx' - n + 1) + g(L_n(L_n^{-1}(x'))) \\ &= a f(Nx') + g(x'). \end{aligned}$$

This implies that  $y' = f(x')$ , so that  $(x', y') \in G$ .

We next show that

$$\widetilde{G} \subseteq W(G), \tag{12}$$

where  $\widetilde{G} = G \setminus \{(n/N, f(n/N)), n = 0, 1, 2, \dots, N\}$ . Assume that  $(x, y) \in \widetilde{G}$  and, without loss of generality, that  $x \in ((n - 1)/N, n/N)$ . Let  $x' = L_n^{-1}(x)$ ,  $y' = f(x')$ . Then

$$y = f(x) = a f(Nx) + g(x) = a f(x' + N - 1) + g(L_n(x')) = a f(x') + g_n(x').$$

Therefore  $(x, y) = w_n(x', y') \in W(G)$ .

Note that the map  $w_n : [0, 1] \times \mathbb{R} \rightarrow ((n - 1)/n, n/N] \times \mathbb{R}$  is a homeomorphism. From Eqs. (11) and (12), respectively,

$$\begin{aligned} W(\overline{G}) &= W(\widehat{G}) = \overline{W(\widehat{G})} \subseteq \overline{G} \\ \overline{G} &= \widetilde{G} \subseteq \overline{W(G)} = W(\overline{G}). \end{aligned}$$

With the additional assumption (3) we have,  $F_1(x_0, y_0) = y_0$  and for  $n = 2, 3, \dots, N$

$$\begin{aligned} F_n(x_0, y_0) &= ay_0 + g_n(0) \\ &= ay_0 + g\left(\frac{n-1}{N}\right) \\ &= ay_0 + g\left(\frac{n-1}{N}\right) + \frac{a}{1-a}(g(1) - g(0)) \\ &= y_{n-1}. \end{aligned}$$

Similarly  $F_n(x_N, y_N) = y_n$ . Therefore the functions  $F_n$  satisfy Eq. (9), in which case the attractor of the IFS is the graph of a continuous function.  $\square$

The present formalism allows both continuous and discontinuous interpolants, as illustrated in Example 1, in contrast to continuous interpolants in the traditional theory of fractal interpolation functions. Furthermore, the fractal interpolation functions obtained herein can be evaluated pointwise to desired precision, by summing absolutely and uniformly convergent series. We note that discontinuous fractal functions are also mentioned in [20].

**Example 1.** It follows from Theorems 2 and 3 that the attractor  $A \subset [0, 1] \times [-1, 1]$  of the contractive IFS

$$W = \{\mathbb{R}^2; w_1(x, y) = (x/2, ay), w_2(x, y) = (x/2 + 1/2, (1 - a) + ay)\},$$

where  $-1 < a < 1$ , is the closure of the graph, restricted to the domain  $[0, 1]$ , of the unique function  $f$  in the space  $L^\infty([0, \infty)) \cap \mathcal{P}$  that is the solution to the equation

$$f(x) - af(2x) = g(x),$$

where

$$g(x) = \begin{cases} 0 & \text{for } x \in [0, 1/2] \\ 1 - a & \text{for } x \in (1/2, 1] \\ g(x - n) & \text{for } x \in (n, n + 1], n \in \mathbb{N}. \end{cases}$$

Moreover, the function  $f$  interpolates the data  $\{(0, 0), (0.5, a), (1, 1)\}$ . The function  $g : [0, \infty) \rightarrow \mathbb{R}$  is not continuous on  $[0, 1]$ . The function  $f : [0, 1] \rightarrow \mathbb{R}$ , that can be represented by the uniformly and absolutely convergent series

$$f(x) = \sum_{k=0}^{\infty} a^k g(2^k x),$$

is not continuous for  $a \neq 1/2$ . That the function  $f$  is discontinuous on  $[0, 1]$  for  $a \neq 1/2$  can be verified, for instance, by showing that  $f(1/2^+) = f(1/2)$  and  $f(0^+) = f(0)$  cannot be satisfied simultaneously. When  $a = 1/2$ , the function  $f$  is continuous; in fact  $f(x) = x$ .

For a formulation more closely related to continuous fractal interpolation functions, as illustrated in the next paragraph, let  $f_0, g_0 \in L^\infty([0, \infty)) \cap \mathcal{P}$  be such that  $f_0(x)$  is continuous for  $x \in [0, 1]$  (from the right at  $x = 0$  and from the left at  $x = 1$ ) with

$$\begin{aligned} f_0(0) &= y_0 \\ f_0(1) &= y_N \\ f_0(x) &= f_0(x - 1) \quad \text{for } x \in (1, \infty), \end{aligned}$$

and  $g_0 : [0, \infty) \rightarrow \mathbb{R}$  is continuous and such that

$$\begin{aligned} g_0(0) &= g_0(1) = 0 \\ g_0(x) &= g_0(x + 1) \quad \text{for all } x \in [0, \infty) \\ g_0(x_n) &= y_n - f_0(x_n) \quad \text{for } n = 1, 2, \dots, N - 1. \end{aligned}$$

Then it is readily confirmed that

$$g(x) := g_0(x) + f_0(x) - af_0(Nx) \tag{13}$$

satisfies the conditions (1), (2), (3) of **Theorem 3**. In particular, the solution to the functional equation  $f(x) - af(Nx) = g(x)$  is continuous on  $[0, 1]$  and passes through the data. That is, the solution  $f(x)$  is a continuous fractal interpolation function on the interval  $[0, 1]$ . Note, however, that typically  $g(x)$  is not continuous for  $x \in (0, 1)$  even though  $g_0(x)$  is continuous for all  $x \in [0, \infty)$  and  $f_0(x)$  is continuous for  $x \in (0, 1)$ .

In this setting, the free parameters, namely the “base function”  $f_0$ , the function  $g_0$ , and the vertical scaling parameter  $a$ , may be chosen to obtain diverse fractal interpolation systems, for instance, Hermite and spline fractal interpolation functions [3,6,21]. They can also be chosen to control the Minkowski dimension and other properties of the graph of the approximant  $f$ . For example, it is reported in [2] that both the Minkowski dimension and the packing dimension of the graph of  $f$  are given by  $D = \max\{2 + \frac{\ln|a|}{\ln N}, 1\}$ , for various classes of function  $g_0$ . Consistent formulas for the Minkowski dimensions related to graphs of a fractal interpolation function are established in [8,10,11].

In Refs. [3,19] it is observed that the notion of fractal interpolation can be used to associate an entire family of fractal functions  $\{h^\alpha : \alpha \in (-1, 1)^N\}$  with a prescribed continuous function  $h$  on a compact interval. To this end, one may consider Eq. (8) with  $g_n(x) = h(L_n(x)) - \alpha_n q(x)$ , where  $q : [0, 1] \rightarrow \mathbb{R}$  is a continuous function such that  $q \not\equiv h$  and  $q$  interpolates  $h$  at the extremes of the interval  $[0, 1]$ . Each function  $h^\alpha$  in this family is referred to as  $\alpha$ -fractal function or “fractal perturbation” corresponding to  $h$ . In our present setting, the function  $f$  is the fractal perturbation corresponding to  $g_0 + f_0$  with base function  $f_0$  and constant scale vector  $\alpha$  whose components are  $a$ . Therefore, the  $\alpha$ -fractal function and the approximation classes obtained through the corresponding fractal operator (see, for instance, [19,24]) can also be discussed using the present formalism.

### 5. Weierstrass Fourier approximation

This section deals with a framework for a “fractal” Fourier analysis. A natural complete orthonormal basis set of fractal functions is provided that serves as a rough analog of the standard sine–cosine Fourier basis. These fractal counterparts are obtained as solutions  $f$  to the functional equation (1), with  $g \in \{\sin 2k\pi x, \cos 2k\pi x\}_{k=1}^\infty \cup \{1\}$ .

**Proposition 3.** *Let  $f(x)$  be the solution to  $f(x) - af(bx) = g(x)$  in  $L^2(\mathbb{R})$  where  $|a| < |b|^{1/2}$ . If  $\{g_k\}_{k=1}^\infty$  is an orthonormal basis for  $L^2(\mathbb{R})$ , then*

$$\langle f_k, f_l \rangle = c + \sum_{n=1}^\infty \frac{a^{2n}}{b^n} \sum_{m=1}^n \frac{b^m}{a^m} \langle g_k, (T_{b^m} + T_{b^m}^*)g_l \rangle,$$

where  $c = (1 - a^2/b)^{-1}$ .

**Proof.** Define  $T_{a,b} = aT_b$ . We have  $T_{a,b}^* = T_{\frac{a}{b}, \frac{1}{b}}$ , and also  $T_{a,b}T_{c,d} = T_{ac,bd} = T_{c,d}T_{a,b}$ . On taking the product, term-by-term, of two absolutely and uniformly convergent series of linear operators, we obtain

$$\begin{aligned} ((I - T_{a,b})^*(I - T_{a,b}))^{-1} &= (I - T_{a,b})^{-1}(I - T_{a,b}^*)^{-1} \\ &= (I - T_{a,b})^{-1} \left( I - T_{\frac{a}{b}, \frac{1}{b}} \right)^{-1} \\ &= \left( \sum_{n=0}^{\infty} a^n T_b^n \right) \left( \sum_{m=0}^{\infty} \left( \frac{a}{b} \right)^m T_{\frac{1}{b}}^m \right) \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^{n+m}}{b^m} T_b^n T_{\frac{1}{b}}^m = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^{n+m}}{b^m} T_{b^{n-m}}. \end{aligned}$$

Now let  $\{g_k\}_{k=1}^{\infty}$  be an orthonormal basis for  $L^2(\mathbb{R})$ . Let  $f_k = M_{a,b}^{-1}(g_k) = (I - T_{a,b})^{-1}g_k$ . Since, by [Theorem 1](#),  $(I - T_{a,b})^{-1}$  is a linear homeomorphism on  $L^2(\mathbb{R})$ , the set of functions  $\{f_k\}_{k=1}^{\infty}$  is a Riesz basis for  $L^2(\mathbb{R})$ . Then

$$\begin{aligned} \langle f_k, f_l \rangle &= \langle g_k, ((I - T_{a,b})^*(I - T_{a,b}))^{-1}g_l \rangle \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{a^{n+m}}{b^m} \langle g_k, T_{b^{n-m}}g_l \rangle \\ &= \sum_{n=0}^{\infty} \left( \frac{a^2}{b} \right)^n + \sum_{\substack{n,m=0 \\ m < n}}^{\infty} \frac{a^{n+m}}{b^m} \langle g_k, T_{b^{n-m}}g_l \rangle + \sum_{\substack{n,m=0 \\ m > n}}^{\infty} \frac{a^{n+m}}{b^m} \langle g_k, T_{b^{n-m}}g_l \rangle \\ &= c + \sum_{\substack{n,m=0 \\ m < n}}^{\infty} \frac{a^{n+m}}{b^m} \langle g_k, T_{b^{n-m}}g_l \rangle + \sum_{\substack{n,m=0 \\ m < n}}^{\infty} \frac{a^{n+m}}{b^n} \langle g_k, T_{b^{m-n}}g_l \rangle \\ &= c + \sum_{\substack{n,m=0 \\ m < n}}^{\infty} \frac{a^{n+m}}{b^m} \langle g_k, T_{b^{n-m}}g_l \rangle + \sum_{\substack{n,m=0 \\ m < n}}^{\infty} \frac{a^{n+m}}{b^m} \langle g_k, T_{b^{n-m}}^*g_l \rangle \\ &= c + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{a^{n+m}}{b^m} \langle g_k, (T_{b^{n-m}} + T_{b^{n-m}}^*)g_l \rangle \\ &= c + \sum_{n=1}^{\infty} \sum_{m=1}^n \frac{a^{2n-m}}{b^{n-m}} \langle g_k, (T_{b^m} + T_{b^m}^*)g_l \rangle \\ &= c + \sum_{n=1}^{\infty} \frac{a^{2n}}{b^n} \sum_{m=1}^n \frac{b^m}{a^m} \langle g_k, (T_{b^m} + T_{b^m}^*)g_l \rangle \end{aligned}$$

where  $c = (1 - a^2/b)^{-1}$ .  $\square$

A similar looking but different expression can be obtained in the case  $|a| > |b|^{1/2}$ . Clearly, such series are amenable to computation, as we illustrate in the next section. For another example, the  $g_k$  in [Proposition 3](#) could be  $(\sqrt{\pi}2^k k!)^{-1/2} H_k(x) \exp(-x^2/2)$ , where the  $H_k$  are Hermite polynomials [15].

5.1. Weierstrass Fourier basis

Working in  $L^2([0, 1])$ , the inner product is  $\langle f, h \rangle := \int_0^1 f(x)h(x)dx$ . The set of functions  $\{\sqrt{2} \cos k2\pi x\}_{k=1}^\infty \cup \{\sqrt{2} \sin k2\pi x\}_{k=1}^\infty \cup \{1\}$  is a complete orthonormal basis for  $L^2([0, 1])$ . Consider these as functions on  $\mathbb{R}$ , periodic of period 1. Let

$$\begin{aligned} c_k(x) &= \sqrt{2} \cos k2\pi x, \\ s_k(x) &= \sqrt{2} \sin k2\pi x, \\ e(x) &= 1, \end{aligned}$$

for all  $k \in \mathbb{N}$  and  $x \in \mathbb{R}$ . Inner products are given by

$$\begin{aligned} \langle s_k, s_l \rangle &= \langle c_k, c_l \rangle = \delta_{k,l}, \\ \langle s_k, c_l \rangle &= \langle e, c_k \rangle = \langle e, s_k \rangle = 0, \langle e, e \rangle = 1, \end{aligned}$$

for all  $k, l \in \mathbb{N}$ .

Let  $b = 2$ ,  $|a| < 1$ , and  $M = M_{a,b}$ . In view of statement (6) of Corollary 4, and the fact that the restriction to  $[0, 1]$  of functions in  $L^\infty([0, \infty)) \cap \mathcal{P}$  can be endowed with the  $L^2$  norm, a new normalized basis for  $L^2([0, 1])$  is  $\{\widehat{e}, \widehat{c}_k, \widehat{s}_k : k \in \mathbb{N}\}$ , where

$$\begin{aligned} \widehat{e} &= (1 - a) M^{-1}(e) = e \\ \widehat{c}_k &= \sqrt{1 - a^2} M^{-1}(c_k) \\ \widehat{s}_k &= \sqrt{1 - a^2} M^{-1}(s_k). \end{aligned}$$

For  $1 \leq k \leq l$ , the inner products are

$$\begin{aligned} \langle \widehat{c}_k, \widehat{c}_l \rangle &= 2(1 - a^2) \sum_{n,m=0}^\infty a^{n+m} \int_0^1 (\cos k\pi 2^{n+1}x)(\cos l\pi 2^{m+1}x)dx \\ &= (1 - a^2) \sum_{n,m=0}^\infty a^{n+m} \delta_{2^n k, 2^m l} = (1 - a^2) \sum_{\substack{n,m=0 \\ n \geq m}}^\infty a^{n+m} \delta_{2^n k, 2^m l} \\ &= (1 - a^2) \sum_{\substack{n,m=0 \\ n \geq m}}^\infty a^{n+m} \delta_{2^{n-m} k, l} = (1 - a^2) \sum_{i,m \geq 0} a^{2m+i} \delta_{2^i k, l} \\ &= \begin{cases} a^j & \text{if } l = 2^j k, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Similar expressions are obtained for  $\{\widehat{s}_k\}_{k=1}^\infty$ . In summary, for all  $k, l \in \mathbb{N}$ ,

$$\begin{aligned} \langle \widehat{c}_k, \widehat{e} \rangle &= \langle \widehat{s}_k, \widehat{e} \rangle = \langle \widehat{c}_k, \widehat{s}_l \rangle = 0 \quad \text{and} \quad \langle \widehat{e}, \widehat{e} \rangle = 1, \\ \langle \widehat{c}_k, \widehat{c}_l \rangle &= \langle \widehat{s}_k, \widehat{s}_l \rangle = \begin{cases} a^j & \text{if } k = 2^j l \text{ or } l = 2^j k \text{ for some } j \in \mathbb{N} \cup \{0\}, \\ 0 & \text{if } k \neq 2^j l \text{ and } l \neq 2^j k \text{ for all } j \in \mathbb{N} \cup \{0\}. \end{cases} \end{aligned} \tag{14}$$

The Gram matrix of inner products of these basis functions, as displayed below, is relatively sparse. See the work of Per-Olof Löwdin on overlap matrices in quantum mechanics, for



example [17].

$$(\widehat{s}_k, \widehat{s}_l)_{k,l=1}^\infty = (\widehat{c}_k, \widehat{c}_l)_{k,l=1}^\infty = \begin{pmatrix} 1 & a^1 & 0 & a^2 & 0 & 0 & 0 & a^3 & \cdot \\ a^1 & 1 & 0 & a^1 & 0 & 0 & 0 & a^2 & \cdot \\ 0 & 0 & 1 & 0 & 0 & a^1 & 0 & 0 & \cdot \\ a^2 & a^1 & 0 & 1 & 0 & 0 & 0 & a^1 & \cdot \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdot \\ 0 & 0 & a^1 & 0 & 0 & 1 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdot \\ a^3 & a^2 & 0 & a^1 & 0 & 0 & 0 & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

Note that, for  $m = 0, 1, 2, 3$ ,

$$\det ((\widehat{c}_k, \widehat{c}_l)_{k,l=1}^{2^m}) = (1 - a^2)^{2^m},$$

which suggests that this formula holds for all  $m \in \mathbb{N} \cup \{0\}$ .

The graph of each function  $\widehat{c}_k, \widehat{s}_k$  has Minkowski (and in “many cases” Hausdorff) dimension  $D = 2 + (\ln a) / \ln 2$  when  $a > 0.5$ ; see [1,13]. It is straightforward to apply the Gram–Schmidt algorithm to obtain the complete orthonormal basis of Weierstrass nowhere differentiable functions given in the following theorem.

**Theorem 4.** *The set of functions  $\{1, \widetilde{c}_k, \widetilde{s}_k : k \in \mathbb{N}\}$ , where*

$$\widetilde{c}_i = \begin{cases} \widehat{c}_i & \text{if } i \text{ is odd} \\ \frac{\widehat{c}_i - a \widehat{c}_{i/2}}{\sqrt{1 - a^2}} = \sqrt{1 - a^2} \widehat{c}_i - a c_{i/2} & \text{if } i \text{ is even} \end{cases}$$

$$\widetilde{s}_i = \begin{cases} \widehat{s}_i & \text{if } i \text{ is odd} \\ \frac{\widehat{s}_i - a \widehat{s}_{i/2}}{\sqrt{1 - a^2}} = \sqrt{1 - a^2} \widehat{s}_i - a s_{i/2} & \text{if } i \text{ is even,} \end{cases}$$

is a complete orthonormal basis for  $L^2([0, 1])$ .

**Proof.** Using the relations in Eq. (14) it follows readily that, if  $k$  is odd and  $l$  is even, then  $\langle \widetilde{c}_k, \widetilde{c}_l \rangle = 0$  unless  $l = k2^j$  for some positive integer  $j$ , in which case,

$$\sqrt{1 - a^2} \langle \widetilde{c}_k, \widetilde{c}_l \rangle = \langle \widehat{c}_k, \widehat{c}_l \rangle - a \langle \widehat{c}_k, \widehat{c}_{l/2} \rangle = a^j - a a^{j-1} = 0.$$

For  $k < l$ , both even, it again readily follows that  $\langle \widetilde{c}_k, \widetilde{c}_l \rangle = 0$  unless  $l = k2^j$  for some positive integer  $j$ , in which case,

$$(1 - a^2) \langle \widetilde{c}_k, \widetilde{c}_l \rangle = \langle \widehat{c}_k, \widehat{c}_l \rangle + a^2 \langle \widehat{c}_{k/2}, \widehat{c}_{l/2} \rangle - a \langle \widehat{c}_k, \widehat{c}_{l/2} \rangle - a \langle \widehat{c}_{k/2}, \widehat{c}_l \rangle$$

$$= a^j + a^{j+2} - a a^{j-1} - a a^{j+1} = 0.$$

For  $k = l$ , both even,

$$(1 - a^2) \langle \widetilde{c}_k, \widetilde{c}_k \rangle = \langle \widehat{c}_k, \widehat{c}_k \rangle + a^2 \langle \widehat{c}_{k/2}, \widehat{c}_{k/2} \rangle - 2a \langle \widehat{c}_k, \widehat{c}_{k/2} \rangle = 1 + a^2 - 2a a = 1 - a^2.$$

To show the equality of the two expressions in the even cases, express  $\widehat{c}_i$  (or  $\widehat{s}_i$ ) as a sum of the  $c_i$ 's (or  $s_i$ 's) using Eq. (6) and simplify.  $\square$

A given function  $h \in L^2([0, 1])$  has a Fourier expansion in terms of the complete orthonormal basis  $\{1, s_k, c_k : k \in \mathbb{N}\}$ . If  $h$  is, in addition, bounded and extended periodically, it has an expansion, that we refer to as a **Weierstrass Fourier series**, in terms of the complete orthonormal basis  $\{1, \tilde{s}_k, \tilde{c}_k : k \in \mathbb{N}\}$  of fractal functions.

**Theorem 5.** *If  $h \in L^2([0, 1])$  has Fourier expansion*

$$h(x) = \alpha_0 + \sum_{n=1}^{\infty} [\alpha_n c_n(x) + \beta_n s_n(x)],$$

*then on the interval  $[0, 1]$  it also has Weierstrass Fourier expansion*

$$h(x) = \tilde{\alpha}_0 + \sum_{n=1}^{\infty} [\tilde{\alpha}_n \tilde{c}_n(x) + \tilde{\beta}_n \tilde{s}_n(x)],$$

where  $\tilde{\alpha}_0 = \alpha_0$  and

$$\tilde{\alpha}_n = \begin{cases} \sqrt{1 - a^2} \sum_{m=0}^{\infty} a^m \alpha_{n2^m} & \text{if } n \text{ is odd} \\ -a \alpha_{n/2} + (1 - a^2) \sum_{m=0}^{\infty} a^m \alpha_{n2^m} & \text{if } n \text{ is even,} \end{cases}$$

$$\tilde{\beta}_n = \begin{cases} \sqrt{1 - a^2} \sum_{m=0}^{\infty} a^m \beta_{n2^m} & \text{if } n \text{ is odd} \\ -a \beta_{n/2} + (1 - a^2) \sum_{m=0}^{\infty} a^m \beta_{n2^m} & \text{if } n \text{ is even.} \end{cases}$$

**Proof.** To compute  $\tilde{\alpha}_n = \langle h, \tilde{c}_n \rangle$ , express  $\tilde{c}_n$  and  $\tilde{s}_n$  in terms of the  $\hat{c}_n$  and  $c_n$  using [Theorem 4](#), then just express in terms of the  $c_n$  using [Eq. \(6\)](#). The orthogonality relations for the respective sine and cosine functions yield the formulas in the statement of the theorem, similarly for the computation of  $\tilde{\beta}_n = \langle h, \tilde{s}_n \rangle$ .  $\square$

**Remark 6.** If  $a = 0$ , then  $\tilde{c}_n = c_n$  and  $\tilde{s}_n = s_n$ , for all  $n$ , and the Weierstrass Fourier series reduces to the classical Fourier series.

**Example 2.** [Figs. 2–4](#) illustrate both classical Fourier and Weierstrass Fourier approximations of the function  $h(x) = x - 0.5$  over the interval  $[0, 1]$ .

**Example 3.** Other examples, using a discretized version of the theory and both theoretical and experimental data, are reported in [\[26\]](#). In one example, a discretized version of [Example 2](#) with  $a = 0.5$ , the  $L^2$  errors, obtained by subsampling both the approximants and  $h(x)$  at 512 equally spaced points, were compared: it was found that the Weierstrass Fourier series performed slightly better than the classical Fourier series, for all partial sums of length  $l$  for  $l = 1, 2, \dots, 510$ . In some other examples, the performance was worse, as measured by the  $L^2$  error.

**Remark 7 (Error Analysis).** In various spaces, such as  $L^2([0, 1])$ , the finite Fourier sum  $\sum C_n c_n(x) + \sum S_n s_n(x) + Ee(x)$  is close to  $g(x)$  if and only if the corresponding Weierstrass Fourier sum  $\sum C_n \hat{c}_n(x) + \sum S_n \hat{s}_n(x) + E\hat{e}(x)$  is close to  $M_{a,b}^{-1}g(x)$ . While the errors remain

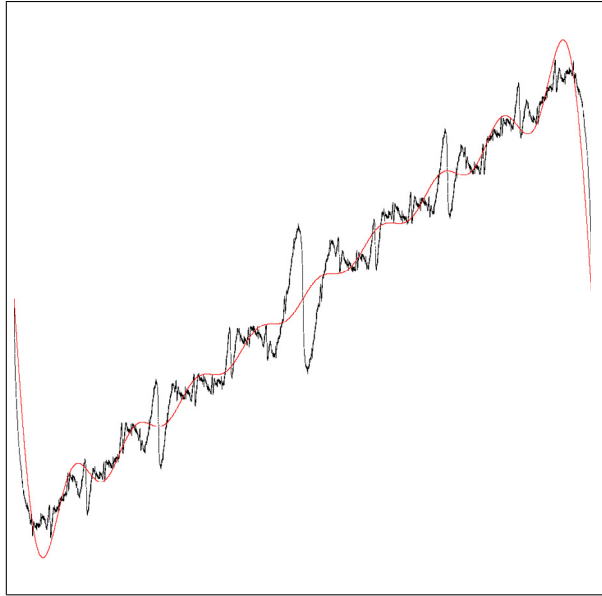


Fig. 2. The sum of the first ten terms of the Fourier (red) and the Weierstrass Fourier ( $a = 0.6$ ) series (black) approximations of the function  $h(x) = x - 0.5$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

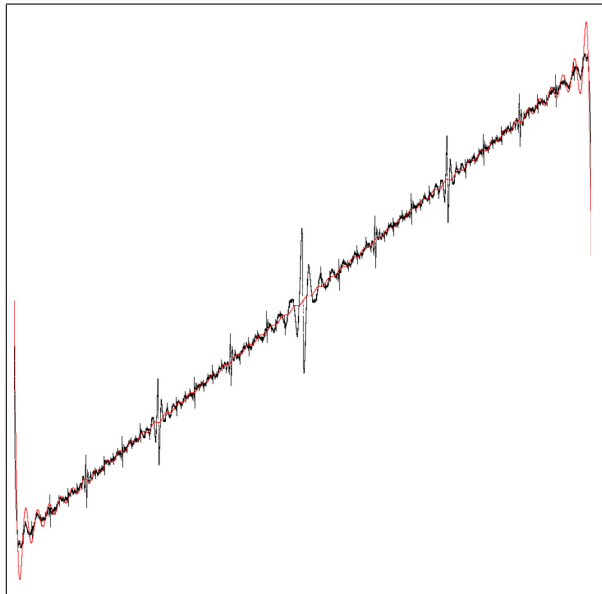


Fig. 3. The sum of the first fifty terms of the Fourier (red) and the Weierstrass Fourier ( $a = 0.6$ ) series (black) approximations of the function  $h(x) = x - 0.5$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

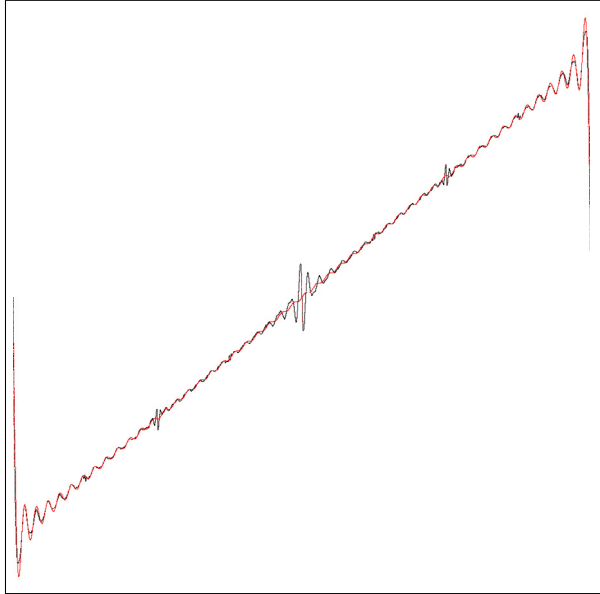


Fig. 4. The sum of the first fifty terms of the Fourier (red) and the Weierstrass Fourier ( $a = 0.3$ ) series (black) approximations of the function  $h(x) = x - 0.5$ . (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

under control, the smoothness of functions, as measured by their differentiability and box-counting dimensions, can be altered. There is a huge literature, and a good understanding, of error issues for classical Fourier analysis. A future direction of research is to derive the Weierstrass Fourier analogues based on the orthonormal basis  $\{\tilde{c}_n, \tilde{s}_n, \tilde{e}\}$  instead of the classical basis  $\{c_n, s_n, e\}$ . This may provide a systematic approach, founded in classical approximation theory, for including deterministic roughness in approximation and interpolation procedures.

## Acknowledgments

We acknowledge support for this work by Australian Research Council grant DP13 0101738. We thank Louisa Barnsley for her help with this paper. This work was partially supported by a grant from the Simons Foundation (322515 to Andrew Vince).

## References

- [1] K. Barański, B. Bárány, J. Romanowska, On the dimension of the graph of the classical Weierstrass function, *Adv. Math.* 265 (2014) 32–59.
- [2] K. Barański, Dimension of the graphs of the Weierstrass-type functions, in: C. Bandt, K. Falconer, M. Zähle (Eds.), *Fractal Geom. Stochastics V*, Birkhauser, 2015, Chapter 5, pages 77–92.
- [3] M.F. Barnsley, Fractal functions and interpolation, *Constr. Approx.* 2 (1986) 303–329.
- [4] M.F. Barnsley, A. Vince, Fractal continuation, *Constr. Approx.* 38 (2013) 311–337.
- [5] A.S. Besicovitch, H.D. Ursell, Sets of fractional dimensions (V): On dimensional numbers of some continuous curves, *J. Lond. Math. Soc.* 12 (1937) 18–25.
- [6] A.K.B. Chand, P. Viswanathan, A constructive approach to cubic Hermite fractal interpolation function and its constrained aspects, *BIT* 53 (2013) 841–865.
- [7] A. Deliu, P. Wingren, The Takagi operator, Bernoulli sequences, smoothness conditions, and fractal curves, *Proc. Amer. Math. Soc.* 121 (1994) 871–881.

- [8] L. Dalla, V. Drakopoulos, M. Prodromou, On the box dimension for a class of nonaffine fractal interpolation functions, *Anal. Theory Appl.* 19 (2003) 220–233.
- [9] P. Fatou, Sur les équations fonctionnelles, *Bull. Math. France XLVIII* (1919) 261.
- [10] J.S. Geronimo, D.P. Hardin, An exact formula for the measure dimensions associated with a class of piecewise linear maps, *Constr. Approx.* 5 (1989) 89–98.
- [11] D.P. Hardin, P. Massopust, The capacity for a class of fractal functions, *Comm. Math. Phys.* 105 (1986) 455–460.
- [12] G.H. Hardy, Weierstrass's non-differentiable function, *Trans. Amer. Math. Soc.* 17 (1916) 301–325.
- [13] B.R. Hunt, The Hausdorff dimension of graphs of Weierstrass functions, *Proc. Amer. Math. Soc.* 126 (1998) 791–800.
- [14] A. Iserles, B. Wang, Dirichlet series for dynamical systems of first-order ordinary differential equations, *Discrete Contin. Dyn. Syst. B* 19 (2014) 281–298.
- [15] W. Johnston, The weighted Hermite polynomials form a basis for  $L^2(\mathbb{R})$ , *Amer. Math. Monthly* 121 (2014) 249–253.
- [16] J. Kigami, Some functional equations which generate both crinkly broken lines and fractal curves, *J. Math. Kyoto Univ.* 27 (1987) 141–149.
- [17] Per-Olof Löwdine, *Linear Algebra for Quantum Theory*, Wiley-Interscience, 1998.
- [18] P. Massopust, *Interpolation and Approximation with Splines and Fractals*, Oxford University Press, New York, 2010.
- [19] M.A. Navascués, Fractal polynomial interpolation, *Z. Anal. Anwend.* 24 (2) (2005) 401–418.
- [20] M.A. Navascués, Fractal functions of discontinuous approximation, *J. Basic Appl. Sci.* 10 (2014) 173–176.
- [21] M.A. Navascués, M.V. Sebastián, Generalization of Hermite functions by fractal interpolation, *J. Approx. Theory* 131 (1) (2004) 19–29.
- [22] A.H. Read, The solutions to a functional equation, *Proc. Roy. Soc. Edinburgh Ser. A* 63 (1952) 336–345.
- [23] J. Thim, Continuous nowhere differentiable functions (Masters Thesis), Luleå, 2003.
- [24] P. Viswanathan, A.K.B. Chand, Fractal rational functions and their approximation properties, *J. Approx. Theory* 185 (2014) 31–50.
- [25] M. Yamaguti, M. Hata, J. Kigami, *Mathematics of fractals*, *Transl. Math. Monographs* 167 (1997) 1–78.
- [26] S. Zhang, B. Harding, Discrete Weierstrass Fourier transform and experiments, *ArXiv:1502.07734v4 [math.NA]* 6 Jan 2016.