Legendrean and G2 Contact Structures

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Contents

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To Yeh Yeh

Abstract

We investigate two parabolic contact geometries: Legendrean contact structures and G2 contact structures. The methods used are mostly independent of the general theory of parabolic geometry.

Building on the work of Eastwood and Nurowski we present a new method of generating G2 contact structures from five-dimensional Legendrean contact structures. This construction requires some input data, a choice of sections, and we calculate the minimal partial torsion, the obstruction to flatness of the G2 contact geometry, in terms of this input data. We show that in fact every G2 contact structure arises (locally) via this construction.

Separately, inspired by the prolongation of the conformal-to-Einstein equation, we construct the standard tractor bundle for five-dimensional integrable Legendrean contact structures via prolongation of an invariantly defined partial differential equation. We compute the partial curvature of the invariant partial connection on this bundle and show, by constructing an explicit isomorphism, that the geometry is locally isomorphic to the homogeneous model if and only if the partial curvature vanishes. We outline a similar prolongation procedure for G2 contact structures.

There is a detailed review of relevant facts about contact manifolds, including the construction of the Rumin complex. We work out the required theory of partial connections on contact manifolds. We explain how to write many of the natural differential operators on a contact manifold, for example, the Rumin complex, in terms of a suitably adapted partial connection.

Introduction

History

In differential geometry the most famous tensor invariant of them all is the Riemannian curvature tensor. On a Riemannian manifold (M, g) for which this vanishes, one can construct a local diffeomorphism on a neighbourhood of each point to $\mathbb{R}^n$ such that the geometric structure of interest, the Riemannian metric, pulls back to the Euclidean metric on $\mathbb{R}^n$. On the other hand, the non-vanishing of the Riemannian curvature tensor is the obstruction to the existence of such a local diffeomorphism.

The desire to generalise this idea to other geometries has been a driving motivation in differential geometry. In particular it shows up in the development of Cartan geometries. These are families (G, H) of spaces "modelled" on the homogeneous space G/H where G is a Lie group and H is a closed subgroup. Abstractly they are defined to be principal-H bundles $\mathcal{P} \to M$ with some extra structure called a Cartan connection, though often these geometries have descriptions in terms of more elementary geometric data. The curvature of the Cartan connection is precisely the obstruction to the existence of a local isomorphism to the canonical model G/H, and therefore this invariant generalises the Riemannian curvature tensor.

Long before the terms "principal bundle" or "Cartan geometry" were invented, Élie Cartan, in his 1910 article [Car10], colloquially known as the "five variables" paper, investigated smoothly varying, generic configurations of planes in 5-dimensional space. In the article he wrote down a complete set of invariant quantities, in the sense that the configuration was locally isomorphic to the canonical model if and only if these quantities vanished. The canonical model had been noticed by Cartan [Car93] and Engel [Eng93], independently, yet simultaneously some years earlier, as a space with the surprising symmetry algebra, the exceptional Lie algebra $\mathfrak{g}_2$. In the language of Cartan geometries introduced above, modern geometers, for example [LNS17], would view the 1910 article as having solved the problem of finding the local invariants of Cartan geometries of type

(G_2, P_1) where G_2 is defined as the stabiliser, as in [Bry87],

$$G_2 := \mathrm{stab}(e^{123} - e^{145} - e^{167} - e^{246} + e^{257} + e^{347} + e^{356}) \qquad (0.0.1)$$

where we write e^{ijk} for $e^i \wedge e^j \wedge e^k$ where e^i is the ith standard basis vector for $(\mathbb{R}^7)^*$, and P_1 is the closed subgroup defined as the stabiliser of a line that is null with respect to the standard signature $(3, 4)$ metric on $\mathbb{R}^7$ (which it turns out is automatically preserved by G_2, that is $G_2 \hookrightarrow SO(3, 4)$). At this point it is worth mentioning that there are three Lie groups commonly called G_2, each with the (complexified) Lie algebra $\mathfrak{g}_2$. For the purposes of this thesis, G_2 is the "split-form" of G_2 defined above.

P_1 is a subgroup of a special type called a parabolic subgroup. Cartan geometries of type (G, P), where P is a parabolic subgroup are called parabolic geometries and turn out to be particularly amenable to study. The reason for this is that their study is very much a generalisation of the study of flag manifolds, which are configuration spaces of nested subspaces in a vector space. This allows a vast array of tools from representation theory to be applied to the setting. Many seemingly unrelated structures being studied in their own right, for example projective, conformal, and CR structures, were discovered to fall under the umbrella of parabolic geometry. This led to a confluence of ideas that led to the general theory being worked out and presented in [ČS09], the main reference in the field.

There is another parabolic subgroup P_2 of G_2 such that G_2/P_2 is a 5-dimensional space. The canonical model was again found by Cartan and Engel, in the same publications as cited above. The group G_2 turns out to act transitively on 2-planes null with respect to the signature $(3, 4)$ metric on $\mathbb{R}^7$, and P_2 is defined as the stabiliser of a plane that is null with respect to this metric as well as null with respect to the 3-form 0.0.1 above, and so the canonical model can be thought of as the space of such planes. The tangent bundle of G_2/P_2 has a canonical codimension-1 subbundle that is a special type of non-integrable subbundle called a contact distribution. Accordingly, parabolic geometries of type (G_2, P_2) are called G_2 contact structures. There is scant literature that explicitly deals with G_2 contact geometries beyond the canonical model. The literature that does is recent, see [ČS09] for a definition, and [LNS17], [EN20a], [EN20b].

A more well known parabolic geometry coming equipped with a contact distribution is a Legendrean contact structure. Their description in terms of familiar geometric data are contact manifolds for which the contact distribution H splits as $H = P \oplus V$ such that P, V are of equal rank and the canonical skew-form, defined up to scale, vanishes on each of the subbundles. From the perspective of parabolic geometries, much is explained in [ČS09] and a classification of such structures in 5 dimensions, with the subbundles P and V, both integrable, and satisfying a type of transitivity condition, is given in [DMT19]. Also of interest will be the construction in [Tak94].

References and further reading for the historical information in this section are [Agr08], [BM06], and [Bry00] for Cartan, Engel and G_2, and [Čap] for parabolic geometry.

Content and structure of the thesis

This thesis is an exploration of two seemingly unrelated parabolic geometries, Legendrean and G_2 contact structures. The main results of interest are as follows: In Chapter 6 we give a simplification of the construction in [EN20b] to get a method of generating G_2 contact structures starting from a Legendrean contact structure. The only extra input data this construction requires basically amounts to a choice of appropriate non-vanishing sections from each of the subbundles P and V. Then, in Section 6.2 we calculate a formula for the fundamental invariant of a G_2 contact structure, the minimal partial torsion, written in terms of this input data. Next, via a simple argument we show Theorem 6.3.20, that locally, all G_2 contact structures in fact arise as this construction.

In the setting of Legendrean contact structures, we show in Chapter 7 that, in the 5-dimensional integrable case, via prolongation of an invariant partial differential equation, one can obtain the standard tractor bundle, a type of canonical vector bundle on a parabolic geometry coming equipped with an invariant connection. We use this construction to explicitly calculate curvature invariants of the contact Legendrean structure. As is understood by the general theory of parabolic geometry, this connection's curvature is the obstruction to the existence of an isomorphism to the flat model. In Section 7.6 we explicitly construct this isomorphism in a relatively elementary fashion, in the sense it can be understood without knowledge of the general theory.

Concluding with Chapter 8, we prolong another invariant differential equation, this time on a G_2 contact structure, and outline why the resulting rank-7 bundle and invariant connection is the standard tractor bundle with canonical connection corresponding to a G_2 contact structure. Specifically, we obtain an invariantly defined signature-$(3, 4)$ metric and 3-form on this bundle, and the bundle is filtered in such a way that there is an invariantly defined rank-2 subbundle that inserts trivially into these two objects.

First however, Chapter 1 and Chapter 2 spend time setting up calculus on contact manifolds. We do this with varying levels of generality so there may be some new formulae of interest beyond the applications in later chapters. In particular, in Chapter 1 we introduce the Rumin complex from [Rum90], an important canonical differential complex on contact manifolds. Chapter 2 works out the necessary theory for partial connections on vector bundles, generalisations of connections on vector bundles in the usual sense, but as we will see, better adapted to the study of contact manifolds. Importantly, we show Theorem 2.2.25, that a partial connection on a vector bundle over a contact manifold of dimension $2n+1 \geq 5$ with vanishing partial curvature, an analogue of curvature, promotes canonically to a flat connection [1]. Throughout Chapter 2 we calculate many differential operators in terms of partial connections that respect the contact structure and for which the partial torsion, an analogue of torsion, vanishes.

Chapter 3 introduces G_2 contact geometries and a calculus on them reminiscent of the "spin calculus" from [PR84], [PR86]. The crucial construction is of the preferred partial connection for a G_2 contact structure in the presence of a contact form. Chapter 4 recounts the construction of Legendrean contact structures on the projectivised cotangent bundle of a projective manifold from [Tak94] and building on that, the construction of G_2 contact geometries in [EN20b]. Chapter 5 examines Legendrean contact structures in more detail. Once again, the crucial construction is of the adapted partial connection for this structure, given a contact form.

While parabolic geometry plays a prominent role in motivating the work in this thesis, the general theory itself is applied very little. Instead, we take a more bottom-up approach, starting with descriptions of parabolic geometries in terms of more familiar geometric data, and only later recover the usual machinery of parabolic geometries, for example tractor bundles.

Future directions

Legendrean and G_2 contact structures are just two of a plethora of examples of parabolic contact structures for which definitions in terms of elementary geometric data are given in [ČS09]. For instance, there are contact parabolic geometries corresponding to various real forms of the other exceptional Lie groups: F_4, E_6, E_7, E_8. These exceptional contact structures do not appear elsewhere in the literature as far as we know. In both the Legendrean and G_2 examples we employed the idea of setting up a calculus in terms of a preferred partial connection associated with a choice of contact form. Doing explicit computations for these higher-dimensional geometries may be difficult though, as the reduction of the structure group of the contact distribution H to $GL(2,\mathbb{R})$ in the case of a G_2 contact structure was invaluable. In the case of the other contact parabolics, the structure group of H is reduced to a Lie group with a more complicated representation theory. In practice this makes decomposing tensor bundles into irreducible components hard. In principle however, we can see a use for some of the results in Chapter 2 in this setting. In particular, from [ČS09] it is known that the other parabolic contact structures

[1] I could not find a reference for this very useful fact even though I suspect it is well understood. I asked on StackExchange to no avail. See: `math.stackexchange.com/questions/3881334/does-a-flat-partial-connection-promote-to-a-flat-connection`

corresponding to the exceptional Lie groups, just like in the G_2 case, have precisely one obstruction to local flatness, a torsion. In Chapter 2 we write the Rumin complex in terms of a contact partial torsion free partial connection. So in the flat case these formulae should allow the Rumin complex to be written down in terms of a partial connection adapted to the exceptional contact structure. One could then couple these operators with flat tractor connections to obtain canonical complexes of invariant differential operators on these flat parabolic contact geometries. One might hope to obtain explicit expressions for the Bernstein-Gelfand-Gelfand sequences (see [ČSS01], [CD01]) in the flat case this way. Furthermore, many of the formulae in Chapter 2 could conceivably be carefully adapted to incorporate non-vanishing torsion.

Preliminaries and Notation

The aim is that this thesis will be accessible to someone with working knowledge of abstract algebra at the undergraduate level, smooth manifolds, differential forms and vector bundles. We will also need some basic facts about Lie groups, Lie algebras. When they appear, more advanced notions will either be introduced in the text, or the reader will be directed to standard references.

0.1 Notation for differential operators

As appears in the literature, see for example [BEGN19], we will write a linear differential operator between vector bundles E, F like $D : E \to F$. We could also have written $D : \Gamma(E) \to \Gamma(F)$ but omitting the "sections" symbol Γ emphasises that these operators are well-defined for sections defined on an arbitrarily small neighbourhood of any point, and this choice also tidies notation. In the same vein, we will sometimes write $s \in E$ for a section, local or otherwise, of E. Whether $s \in E$ refers to a point or a section will either be unimportant (for example, sometimes when dealing with linear maps between vector bundles) or clear from the context (for example if we apply a differential operator to s).

0.2 Abstract index notation

One of the crucial tools that this thesis will bring to bear is Penrose's abstract index notation. We will give a working overview here and point the reader to [PR84] and [PR86] for a precise presentation of the formalism.

Abstract index notation is a tool that combines the computational efficiency of concrete index notation with the basis independent nature of coordinate-free notation. In concrete index notation, when working with a vector bundle E over a manifold M (all manifolds are taken to be smooth), we may write σ^i for the ith component function in the expansion of the section $\sigma \in E$ with respect to a particular local trivialisation $\{e_i\}$. That is, summing over repeated indices

$$\sigma = \sigma^i e_i, \tag{0.2.1}$$

so that, for example, if some section of the endomorphism bundle has components ${A^i}_j$ with respect to the induced trivialisation for $\mathrm{End}(E)$, the image of σ under this endomorphism has components ${A^i}_j\sigma^j$.

Superficially, abstract index notation looks similar to concrete index notation. However, instead of a symbol adorned with an index representing a component function, one writes σ^a for the section. The index should be considered just to be a label, with the number of indices and their placement indicating the type of tensor for sections of the bundle, its dual and tensor powers thereof. For instance, a section of E^* may be written like ω_a while a section of $\otimes^2 E$ may be written like T^{ab}.

While the index itself could be an arbitrary symbol, it is desirable to adorn sections of different bundles with indices from a different alphabet. For example if we have different vector bundle F we may choose to write its sections like ξ^α, using lower-case indices drawn from the Greek alphabet. We reserve lower-case Latin indices for sections of E so then $X^{ab\beta}$ should be recognised as a section of the tensor product $\otimes^2 E \otimes F$.

Abstract index notation is helpful for cleanly writing down operations involving tensor powers. For example the natural pairing $E \otimes E^* \to \mathbb{R}$ is written

$$(\sigma^a, \omega_b) \mapsto \sigma^a \omega_a. \tag{0.2.2}$$

For another example, given $\mu_{abcd} \in \otimes^4 E^*$, the interior product $\otimes^4 E^* \otimes E \to \otimes^3 E^*$ given by pairing E with the third factor is given by

$$(\mu_{abcd}, X^e) \mapsto X^c \mu_{abcd}. \tag{0.2.3}$$

The identity endomorphism, a section of $E^* \otimes E$, is denoted

$$\delta_a{}^b \quad \text{or} \quad \delta^b{}_a. \tag{0.2.4}$$

0.3 Tensor symmetries

Let V be a vector space. Picking a basis identifies $V \cong \mathbb{R}^n$ and thus we get a representation of $GL(n, \mathbb{R})$ on V. We also get an induced representation on $\otimes^n V$ defined on simple tensors by

$$g(v_1 \otimes v_2 ... \otimes v_k) = gv_1 \otimes gv_2 \otimes ... \otimes gv_k, \ v_i \in V, \tag{0.3.1}$$

and extending by linearity.

Lemma 0.3.2. *Given $\otimes^n V$ for some vector space V of dimension k, for each permutation $\sigma \in S_n$, the braiding map*

$$e_{a_1} \otimes e_{a_2} \otimes ... e_{a_n} \mapsto e_{a_{\sigma(1)}} \otimes e_{a_{\sigma(2)}} \otimes ... \otimes e_{a_{\sigma(n)}} \tag{0.3.3}$$

defined with reference to the induced basis for $\otimes^k V$, where $\{e_i\}_{i=1}^k$ is a basis for V, then extending by linearity, is independent of the choice of basis for V and is a homomorphism of $GL(n, \mathbb{R})$ representations.

Proof. Let $\{f_i\}_{i=1}^k$ be some other basis with $f_i = A_i^j e_j$ for $\{A_i^j\}$ an invertible $k \times k$ matrix and let $\rho = \sigma^{-1} \in S_n$. We have

$$\begin{aligned} f_{i_{\sigma(1)}} \otimes ... \otimes f_{i_{\sigma(n)}} &= A^{j_1}_{i_{\sigma(1)}} ... A^{j_n}_{i_{\sigma(n)}} e_{j_1} \otimes ... \otimes e_{j_n} \\ &= A^{j_{\rho(1)}}_{i_1} ... A^{j_{\rho(n)}}_{i_n} e_{j_1} \otimes ... \otimes e_{j_n} \\ &= A^{j_1}_{i_1} ... A^{j_n}_{i_n} e_{j_{\sigma(1)}} \otimes ... \otimes e_{j_{\sigma(n)}}, \end{aligned} \tag{0.3.4}$$

which shows that the map is independent of chosen basis. It also immediately follows that the map commutes with the group action, since $\{A_i^j\}$ is an arbitrary invertible matrix. □

The above lemma means we have a well-defined notion of tensors which are symmetric under transposition of any two indices, a subrepresentation which will be denoted by $\odot^n V$. The subrepresentation consisting of tensors which are anti-symmetric will be denoted $\Lambda^n V$.

Furthermore, this lemma also means we get a well-defined notion of these braiding maps for vector bundles. For this we need the following fact:

Lemma 0.3.5. *Let U, V be representations of a Lie group G and $\phi : U \to V$ be a homomorphism of G-representations. Let E be a vector bundle with structure group G and write $F(E)$ for the G-frame bundle. Then there are vector bundle homomorphisms between associated bundles*

$$F(E) \times_G U \to F(E) \times_G V. \tag{0.3.6}$$

Proof. That ϕ commutes with the G-action means the vector bundle homomorphism $[(f, u)] \mapsto [(f, \phi(u))]$ is well-defined. □

See [Lee09] for the background on associated bundles and frame bundles.

Let E be a rank n-vector bundle. We have an isomorphism

$$\otimes^k E \cong F(E) \times_G \otimes^k \mathbb{R}^n. \tag{0.3.7}$$

So the point is, the braiding maps on $\otimes^k \mathbb{R}^n$ give us braiding maps on $\otimes^k E$. We can define these braiding maps with respect to any trivialisation, since the permutation commutes with the transition functions of the bundle.

For example, the isomorphism $E \otimes E \xrightarrow{\sim} E \otimes E$ defined by

$$e_i \otimes e_j \mapsto e_j \otimes e_i, \tag{0.3.8}$$

in any trivialisation $\{e_i\}_{i=1}^n$ will be denoted

$$T^{ab} \mapsto T^{ba}. \tag{0.3.9}$$

We will continually use symmetrisation and antisymmetrisation operators. Define

$$T_{(a_1 \ldots a_n)} := \sum_{i=1}^n \frac{1}{n!} T_{a_{\sigma(1)} \ldots a_{\sigma(n)}} \tag{0.3.10}$$

where we are taking our abstract indices from the set of symbols $\{a_1, \ldots a_n\}$. Next,

$$T_{[a_1 \ldots a_n]} := \sum_{i=1}^n \frac{\mathrm{sgn}(\sigma)}{n!} T_{a_{\sigma(1)} \ldots a_{\sigma(n)}} \tag{0.3.11}$$

where $\mathrm{sgn}(\sigma)$ is the parity of the permutation.

Then by

$$T_{(ab|c|d)} \tag{0.3.12}$$

we mean to skip over symmetrising the index between the vertical bars, and so on.

The subbundle of $\otimes^n E$ consisting of tensors which are symmetric under transposition of any two indices will be denoted by $\odot^n E$ while the subbundle which consists of those tensors which are antisymmetric under any transposition will be denoted $\Lambda^n E$.

We have isomorphisms

$$\odot^k E \xrightarrow{\sim} F(E) \times_G \odot^k \mathbb{R}^n \tag{0.3.13}$$

$$\Lambda^k E \xrightarrow{\sim} F(E) \times_G \Lambda^k \mathbb{R}^n. \tag{0.3.14}$$

0.4 Conventions for differential forms

Given a manifold M we will usually denote its tangent bundle by TM and its cotangent bundle by Λ^1. Vector fields will be written with their indices raised like X^a and accordingly 1-forms with their indices lowered like ω_a. The bundle of k-forms is denoted Λ^k. Of particular importance is the wedge product. Given a k-form ω and a l-form μ, which we will alternately denote $\omega_{a_1 \ldots a_k}$ and $\mu_{b_1 \ldots b_l}$ respectively, the wedge product is

$$\omega \wedge \mu = \omega_{[a_1 \ldots a_k} \mu_{b_1 \ldots b_l]}. \tag{0.4.1}$$

In particular we are adopting the normalisation convention of [KN63] rather than [Spi65]. A consequence of this is that, the exterior derivative $d : \Lambda^k \to \Lambda^{k+1}$ is normalised according to the former.

Chapter 1

Calculus on contact manifolds

In this chapter we recall the basic local structure of contact geometry and provide techniques to perform calculus on contact manifolds.

While Darboux's theorem shows there are no local invariants in contact geometry, it will be useful to define, and fix notation for the local machinery. Most importantly, we will define a complex of differential operators present on every contact manifold, the Rumin complex. This complex was first given by Rumin in [Rum90], and is constructed from and retains the useful features of the de Rham complex, for example, local exactness and self-adjointness.

We will also define the Reeb field, which is a distinguished vector field transverse to the contact distribution, induced by a choice of contact form.

Lastly, we briefly discuss how contact manifolds fit into a broader family of filtered manifolds with prescribed symbol algebras.

1.1 Local contact geometry

A symplectic manifold is a manifold M with a closed 2-form $\omega \in \Lambda^2$ such that ω defines a non-degenerate bilinear form on T_xM $\forall x \in M$. The elementary fact, that for an $n \times n$ matrix A, $\det(-A) = (-1)^n \det(A)$, implies that non-degenerate, skew bilinear forms, and hence symplectic manifolds, can exist only in even dimension.

The best one can hope for on an odd-dimensional manifold is a non-degenerate, skew bilinear form on some smooth subbundle $H \subset TM$ of the tangent bundle which is of even rank. The case where this subbundle is integrable, that is, corresponds to the tangent spaces of a foliation of integral submanifolds, should reduce to symplectic case. Then the interesting case is when H is non-integrable.

One characterisation of integrability for a smooth subbundle H is it being involutive, which is to say $[X, Y] \in H$ $\forall X, Y \in \Gamma(H)$. Locally at least, any corank-1 vector subbundle of TM is the kernel of a 1-form $\alpha \in \Lambda^1$. There is a well known formula for the exterior derivative $d : \Lambda^1 \to \Lambda^2$

$$d\omega(X,Y) = \frac{1}{2}X(\omega(Y)) - \frac{1}{2}Y(\omega(X)) - \frac{1}{2}\omega([X,Y]). \tag{1.1.1}$$

Letting $H = \ker\alpha$ we have $d\alpha(X,Y) = -\frac{1}{2}\alpha([X,Y])$ for $X, Y \in H$. So H is integrable in a neighbourhood if and only if $d\alpha|_H$ is vanishing in that neighbourhood.

With the above considerations in mind, if one were to try to define an odd-dimensional analogue of symplectic geometry, one might be led to the following definition:

Definition 1.1.2 (Contact manifold)**.** *A contact manifold (M, H) is a manifold M of dimension $2n+1 \geq 5$ equipped with a codimension-1 subbundle H of TM, such that there exists a (globally defined) 1-form $\alpha \in \Lambda^1$ such that $H = \ker\alpha$ and such that $d\alpha|_H$ a non-degenerate bilinear form on H_x $\forall x \in M$. H is called the contact distribution and is non-integrable. Such an α is non-vanishing and is called a contact form for (M, H).*

Since the focus will mostly be on local questions, unlike other authors we do not give the usual, more general definition of a contact structure, and instead just consider contact manifolds, for which their exists a globally defined contact form. The point is that given a contact structure in the wider sense, it is always possible to restrict to a sufficiently small neighbourhood around any point with a well-defined contact form. We also restrict our consideration to contact manifolds of dimension greater than 5 to avoid the somewhat degenerate behaviour in 3 dimensions. We may still mention the 3-dimensional case in passing. The other possible difference is that the above definition does not distinguish a particular contact form, rather, we emphasise the subbundle.

Note the condition on α is stronger than the non-integrability of its kernel, hence the condition is sometimes called the maximal non-integrability of H. This condition has a sometimes useful restatement, often taken as the base definition of a contact form:

Lemma 1.1.3. *An n-multilinear alternating map ω on a n-dimensional vector space V is non-zero if and only if $\omega(v_1, ...v_n) \neq 0$ for a basis $\{v_i\}_{i=1}^n$.*

Proof. From the linearity and the alternating property, given $\{u_i\}_{i=1}^n$, arbitrary vectors in V, it is clear $\omega(u_1, ..., u_n)$ can be expanded as a sum in $\omega(v_1, ...v_n)$. So ω is non-zero if and only if $\omega(v_1, ...v_n) \neq 0$. □

Proposition 1.1.4. *Let $\alpha \in \Lambda^1$ with $H := \ker\alpha$. Then $d\alpha|_H$ is non-degenerate if and only if the $2n+1$-form $\alpha \wedge d\alpha \wedge ... \wedge d\alpha$ is non-vanishing on M.*

Proof. If $d\alpha$ is non-degenerate on H_x we can pick a basis $\{U_1, V_1, ..., U_n, V_n\}$ for H_x such that $d\alpha|_H(U_i, V_j) = \delta_{ij}$. Extend to a basis $\{T, U_1, V_1, ..., U_n, V_n\}$ for T_xM with $\alpha(T) = 1$. Then a simple calculation gives $\alpha \wedge d\alpha \wedge ... \wedge d\alpha(T, U_1, V_1, ..., U_n, V_n) = (2^n \cdot n!)/(2n+1)! \neq 0$.

Conversely, let $\alpha \wedge d\alpha \wedge ... \wedge d\alpha$ be non-vanishing. Take an arbitrary vector $V \in H_x \setminus \{0\}$. Extend V to a basis $\{V, U_1, ..., U_{2n-1}\}$ for H_x and then a basis $\{T, V, U_1, ..., U_{2n-1}\}$ for T_xM.

$$\alpha \wedge d\alpha \wedge ... \wedge d\alpha(T, V, U_1, ..., U_{2n-1}) = \alpha(T) d\alpha \wedge ... \wedge d\alpha(V, U_1, ..., U_{2n-1}) \quad (1.1.5)$$

is non-zero. When expanding the right-hand side we get a sum in $d\alpha(V, U_i)$, so $d\alpha(V, U_i)$ must be non-zero for some i. Thus $d\alpha(V, \cdot)$ is non-zero on H_x for any $V \in H_x \setminus \{0\}$. That is, it is a non-degenerate bilinear form on H_x. □

The most important, and as we will see, in a local sense, only example of a contact manifold is the following.

Example 1.1.6 (Contact geometry of $\mathbb{R}^{2n+1}$)**.** *Give $\mathbb{R}^{2n+1}$ with $n \geq 2$ the global coordinates $(t, p_1, ..., p_n, q^1, ..., q^n)$ and define*

$$\alpha = dt - p_i dq^i. \quad (1.1.7)$$

Then $d\alpha = -dp_i \wedge dq^i$ and $\alpha \wedge d\alpha \wedge ... \wedge d\alpha = (-1)^n \cdot n!\ dt \wedge dp_1 \wedge dq^1 \wedge ... \wedge dp_n \wedge dq^n$ is non-vanishing and hence $(M, \ker \alpha)$ is a contact manifold.

We will come back to this example often.

The next theorem is the most important in contact geometry.

Theorem 1.1.8 (Darboux)**.** *Let (M, H) be a contact manifold with contact form α. For each $x \in M$, there exists a coordinate chart $(U, (t, p_1, ..., p_n, q^1, ..., q^n))$ with $x \in U$ such that α takes the form 1.1.7.*

Proof. Omitted. See [Ber00] for a detailed proof. It is originally due to Moser. □

The existence of such standard coordinates is a marked contrast to Riemannian geometry, for example, where the obstruction to being locally isometric to $\mathbb{R}^n$ with the Euclidean metric is the Riemannian curvature, a local invariant. All contact manifolds are locally isomorphic to the standard example and hence there are no local invariants in contact geometry. Nonetheless it will be useful to start by studying the local model in preparation for adding more structure. In particular, it will turn out that some of the invariants of more complicated structures can be interpreted as obstructions (torsions) to writing the local contact machinery in a way compatible with the added structure.

Since a contact form α of a contact manifold is non-vanishing, we can use it to trivialise a line subbundle of Λ^1 denoted L. By linearity, any contact form must be a section of L. In the sequel we denote by Λ^k_H the set of k-forms restricted as linear maps to the distribution H. Restriction to H gives surjective vector bundle homomorphisms $\Lambda^k \to \Lambda^k_H$.

Definition 1.1.9 (Levi map)**.** *On a contact manifold (M, H) define the Levi map $\mathcal{L} : L \to \Lambda^2_H$ as the composition $L \to \Lambda^1 \xrightarrow{d} \Lambda^2 \to \Lambda^2_H$.*

A priori the Levi map looks like a first order differential operator, but it is in fact a vector bundle homomorphism. Given f a smooth function, $d(f\alpha) = df \wedge \alpha + f d\alpha$ so $\mathcal{L}(f\alpha) = d(f\alpha)|_H = f d\alpha|_H$.

Note that the Levi map has image consisting of non-degenerate 2-forms, since $d\alpha|_H$ is non-degenerate. So non-vanishing sections of L are contact forms for (M, H). We think of $\mathcal{L}$ as an identification of some line-subbundle of Λ^2_H with L and decompose $\Lambda^2_H \cong \Lambda^2_{H\perp} \oplus L$ where $\Lambda^2_{H\perp} := \Lambda^2_H / L$.

We will represent sections of Λ^1_H and its associated bundles using abstract index notation. Unless otherwise stated, symbols adorned with lower case Latin indices represent sections of these bundles. For example, ϕ_a should be interpreted as a section of Λ^1_H and we write J_{ab} for $d\alpha|_H$. By non-degeneracy there is a unique $J^{ab} \in \Lambda^2 H$ such that $J_{ab} J^{cb} = \delta_a{}^c$. We will also use J^{ab} and J_{ab} to raise and lower indices respectively. That is $\phi^a := J^{ab}\phi_b$.

Identifying L as a subbundle of Λ^2_H, in this notation, we have the projection $\Lambda^2_H \to L$ given by

$$\omega_{ab} \mapsto \frac{1}{2n} J_{ab} \omega_c{}^c. \tag{1.1.10}$$

Then this implies there is a canonical identification of $\Lambda^2_{H\perp}$ with elements of Λ^2_H trace-free with respect to J^{ab}, with the projection $\Lambda^2_H \to \Lambda^2_{H\perp}$ given by

$$\omega_{ab} \mapsto (\omega_{ab})_\perp := \omega_{ab} - \frac{1}{2n} J_{ab} \omega_c{}^c. \tag{1.1.11}$$

These projections are independent of the chosen contact form α since if we replace α by $\hat{\alpha} = f\alpha$ for non-vanishing f we have $\hat{J}_{ab} = f J_{ab}$ and $\hat{J}^{ab} = \frac{1}{f} J^{ab}$.

Remark 1.1.12. *Note this canonical identification uses the maximal non-integrability condition, that is, J_{ab} is non-degenerate. So this algebra does not work for arbitrary non-integrable codimension-1 distributions. Since some of the constructions in the first few chapters still work in this setting, we will sometimes choose not to employ the above identification even though it would sometimes tidy arguments.*

We will continually need:

Definition 1.1.13 (Induced Levi maps)**.** *The Levi map $\mathcal{L} : L \otimes \Lambda^k_H \to \Lambda^{k+2}_H$ is defined as the composition of the induced homomorphism followed by the wedge product, that is:* $L \otimes \Lambda^k_H \xrightarrow{\mathcal{L}\otimes \mathrm{Id}} \Lambda^2_H \otimes \Lambda^k_H \to \Lambda^{k+2}_H$.

In indices these induced homomorphisms are just $J_{ab}\omega_{c_1\ldots c_k} \mapsto J_{[ab}\omega_{c_1\ldots c_k]}$.

A well known fact about symplectic forms due to Lefshetz will come in handy. The straightforward proof here is from [BGG03][1].

Lemma 1.1.14 (Lefshetz)**.** *The induced Levi map $\mathcal{L} : L\otimes\Lambda^{k-1}_H \to \Lambda^{k+1}_H$ is injective for $k \leq n$ and surjective for $k \geq n$*

Proof. If we pick a contact form α we get an isomorphism $\tau : \Lambda^p_H \to L \otimes \Lambda^p_H$ for $p = 0, ..., 2n$ simply given by $\omega \mapsto \alpha \otimes \omega$. Define $l = \mathcal{L} \circ \tau$. We then get a map $l^m : \Lambda^{n-m}_H \to \Lambda^{n+m}_H$ for $m = 1, ..., n$ which is just

$$\omega \mapsto \underbrace{d\alpha|_H \wedge ... \wedge d\alpha|_H}_{m \text{ times}} \wedge \omega \tag{1.1.15}$$

We show that l^m isomorphism, which clearly implies the lemma. Firstly note that $\operatorname{rank} \Lambda^{n-m}_H = \operatorname{rank} \Lambda^{n+m}_H$ so we only need to show injectivity. The result is true for $m = n$ by the non-degeneracy of $d\alpha|_H$. Assume $l : \Lambda^{n-p}_H \to \Lambda^{n+p}_H$ is injective. Let $\omega \in \ker l : \Lambda^{n-p+1}_H \to \Lambda^{n+p-1}_H$, then

$$\underbrace{d\alpha|_H \wedge ... \wedge d\alpha|_H}_{p-1 \text{ times}} \wedge \omega = 0. \tag{1.1.16}$$

We have

$$\underbrace{d\alpha|_H \wedge ... \wedge d\alpha|_H}_{p \text{ times}} \wedge \omega = 0 \tag{1.1.17}$$

and plugging in some $X \in H$ yields

$$\begin{aligned} &\frac{2p}{n+p+1} d\alpha(X, \cdot)|_H \wedge \underbrace{d\alpha|_H \wedge ... \wedge d\alpha|_H}_{p-1 \text{ times}} \wedge \omega \\ +&\frac{n-p+1}{n+p+1} \underbrace{d\alpha|_H \wedge ... \wedge d\alpha|_H}_{p \text{ times}} \wedge \omega(X, \cdot, ..., \cdot) = 0 \end{aligned} \tag{1.1.18}$$

but the first term vanishes by the assumption on ω so

$$\underbrace{d\alpha|_H \wedge ... \wedge d\alpha|_H}_{p \text{ times}} \wedge \omega(X, \cdot, ..., \cdot) = 0. \tag{1.1.19}$$

[1] Bryant attributes it to Eugenio Calabi. See the interesting discussion at `mathoverflow.net/questions/316505/the-lefschetz-operator`.

Invoking the inductive hypothesis

$$\omega(X, \cdot, ..., \cdot) = 0 \tag{1.1.20}$$

and since X was arbitrary, $\omega = 0$. So the result holds for $m = n, ..., 1$. □

Define $\Lambda^k_{H\perp} := \Lambda^k_H / \mathcal{L}(L \otimes \Lambda^{k-2}_H)$ for $k \geq 2$. Just like for 2-forms there is an identification of $\Lambda^k_{H\perp}$ with forms in Λ^k_H which are trace-free with respect to J^{ab}. Proving this turns out to be somewhat cumbersome without invoking the representation theory of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$, which we will do here. This is the approach taken by Tseng and Yau [TY12].

Lemma 1.1.21 (Trace-free k-forms)**.** *For $2 \leq k \leq n$ there is a canonical isomorphism $\Lambda^k_{H\perp} \cong \{\omega_{a_1...a_k} \in \Lambda^k_H \mid \omega_{a_1...a_{k-2}b}{}^b = 0\}$.*

Proof. One can check that for $0 \leq k \leq 2n$

$$\begin{aligned}&(k+1)(k+2) J_{[a_1 a_2} \omega_{a_3 ... a_{k+2}]} J^{a_{k+1} a_{k+2}} \\ &= 4(n-k)\omega_{a_1...a_k} + k(k-1) J_{[a_1 a_2} \omega_{a_3 ... a_k] b_1}{}^{b_2}.\end{aligned} \tag{1.1.22}$$

In particular if we define $\Lambda_H := \Lambda^0_H \oplus \Lambda^1_H \oplus ... \oplus \Lambda^{2n}_H$ and define $r : \Lambda_H \to \Lambda_H$ by

$$r(\omega_{a_1...a_k}) := \frac{1}{4} k(k-1) \omega_{a_1 ... a_{k-1} b}{}^b \tag{1.1.23}$$

then $l \circ r - r \circ l = h$ where l is as in 1.1.14 so in indices is

$$\omega_{a_1...a_k} \mapsto J_{[a_1 a_2} \omega_{a_3 ... a_k + 2]} \tag{1.1.24}$$

and $h : \Lambda_H \to \Lambda_H$ is defined by

$$h(\omega_{a_1...a_k}) := (n-k)\omega_{a_1...a_k}. \tag{1.1.25}$$

Furthermore $h \circ r - r \circ h = -2r$ and $h \circ l - l \circ h = 2l$. So we recognise $\{h, l, r\}$ as a representation of the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ on each fibre of Λ_H. See [Hum72] for a detailed account of representations of this Lie algebra. Now $\omega_{a_1...a_k}$ being trace free is the same as being in the kernel of r, which is to say that is a highest weight vector in a root string of length $n - k + 1$ and this occurs if and only if we have $\Lambda^{2n-k+2}_H \ni l^{n-k+1} \omega_{a_1...a_k} = 0$.

Now given an arbitrary $\omega \in \Lambda^k_H$, where $2 \leq k \leq n$ we know that $l^{n-k+2} : \Lambda^{k-2}_H \to \Lambda^{2n-k+2}_H$ is an isomorphism and so there is a unique $\mu \in \Lambda^{k-2}_H$ such that

$$l^{n-k+1}(\omega) = l^{n-k+2}(\mu). \tag{1.1.26}$$

Accordingly, the map $\Lambda^k_H \to \{\omega_{a_1...a_k} \in \Lambda^k_H \mid \omega_{a_1...a_{k-2}b}{}^b = 0\}$ defined by

$$\omega \mapsto \omega - l(\mu) \tag{1.1.27}$$

has kernel $\mathcal{L}(L \otimes \Lambda^{k-2}_H)$ by uniqueness of μ which therefore gives an identification $\Lambda^k_{H\perp} \cong \{\omega_{a_1...a_k} \in \Lambda^k_H \mid \omega_{a_1...a_{k-2}b}{}^b = 0\}$. □

Although we will not give formulae for the general case here, as the algebra is clearly messy, the above lemma means we can write down canonical projections $\Lambda^k_H \to \mathcal{L}(L \otimes \Lambda^{k-2}_H)$ by iteratively taking traces with J^{ab}. For example, for contact manifolds with $2n+1 > 5$ we have projections:

$$\begin{aligned}
&\Lambda^3_H \to \mathcal{L}(L \otimes \Lambda^1_H) : \\
&\qquad\qquad \omega_{abc} \mapsto \frac{3}{2n-2} J_{[ab}\omega_{c]d}{}^{d} && (1.1.28) \\
&\Lambda^4_H \to \mathcal{L}(L \otimes \Lambda^2_H) : \\
&\qquad\qquad \omega_{abcd} \mapsto \frac{6}{2n-4} J_{[ab}\omega_{cd]e}{}^{e} - \frac{6}{(4n-4)(2n-4)} J_{[ab}J_{cd]}\omega_{c}{}^{c}{}_{d}{}^{d}. && (1.1.29)
\end{aligned}$$

The kernels consist precisely of the trace-free forms.

Proposition 1.1.30. *The short exact sequence*

$$0 \to L \to \Lambda^1 \to \Lambda^1_H \to 0 \qquad (1.1.31)$$

of vector bundles induces short exact sequences

$$0 \to L \otimes \Lambda^{k-1}_H \underset{i_k}{\to} \Lambda^k \underset{q_k}{\to} \Lambda^k_H \to 0 \qquad (1.1.32)$$

for each $k \in \mathbb{N}$.

Proof. We proceed by induction. For $k > 1$ define $i_k : L \otimes \Lambda^{k-1}_H \hookrightarrow \Lambda^k$ by $\alpha \otimes \omega \mapsto \alpha \wedge \tilde{\omega}$ where $\tilde{\omega} \in \Lambda^{k-1}$ satisfies $\tilde{\omega}|_H = \omega$. This is well defined since any two choices for $\tilde{\omega}$ will differ by something in the kernel of $\Lambda^{k-1} \to \Lambda^{k-1}_H$, and hence, by the inductive hypothesis, the difference takes the form $\alpha \wedge \rho$ for $\rho \in \Lambda^{k-2}$, which vanishes when wedged with α. If $\alpha \otimes \omega$ is non-zero pick linearly independent $\{u_1, ..., u_{k-1}\}$ in the contact distribution such that $\omega(u_1, ..., u_{k-1}) \neq 0$ and given a non-zero $\alpha \in L$ pick v such that $\alpha(v) \neq 0$ then $\alpha \wedge \tilde{\omega}(v, u_1, ..., u_{k-1}) = \frac{1}{k}\alpha(v)\omega(u_1, ..., u_{k-1}) \neq 0$ which shows injectivity of i_k.

Define $q_k : \Lambda^k \to \Lambda^k_H$ to be the restriction, which is obviously a surjection. Clearly $q_k \circ i_k = 0$. Considerations on the ranks of the bundles show that in fact $\operatorname{im} i_k = \ker q_k$, so 1.1.32 is a short exact sequence. □

1.2 The Rumin complex

We now construct the Rumin complex, an analogue of the de Rham complex, present on any contact manifold. In some sense, the Rumin complex is more efficient, since it resolves the locally constant functions while only taking derivatives in contact directions, but we will find it retains many of the useful properties of the de Rham complex.

The complex is due to Michel Rumin in [Rum90] and the treatment here basically follows that article, with some superficial changes as to how we identify the bundles involved.

Definition 1.2.1 (Zeroth Rumin operator)**.** *The Rumin operator* $d_\perp : \Lambda^0 \to \Lambda^1_H$ *is defined as the composition* $\Lambda^0 \xrightarrow{d} \Lambda^1 \to \Lambda^1_H$.

Proposition 1.2.2. *The kernel of the zeroth Rumin operator is the locally constant functions.*

Proof. The key idea is that the contact distribution is bracket generating. That is, $[H, H] = TM$. We do not need the maximal non-integrability. If $d_\perp f(X) = X(f) = 0\ \forall X \in \Gamma(H)$ then taking Lie brackets we have $[X, Y](f) = X(Y(f)) - Y(X(f)) = 0\ \forall X, Y \in \Gamma(H)$. From the standard identity $d\alpha(X, Y) = \frac{1}{2}X(\alpha(Y)) - \frac{1}{2}Y(\alpha(X)) - \frac{1}{2}\alpha([X, Y])$ we get $d\alpha(X, Y) = -\alpha([X, Y])\ \forall \alpha \in L$ and $X, Y \in \Gamma(H)$. Since $d\alpha$ is non-zero on H_x for each $x \in M$ we can find $X, Y \in \Gamma(H)$ such that $d\alpha(X|_x, Y|_x) = -\alpha([X, Y]|_x) \neq 0$ which is to say $[X, Y]|_x \notin H$. Since $[X, Y]|_x(f) = 0$ and H is of codimension one we have shown any function f in the kernel of the zeroth Rumin operator is annihilated by T_xM for each $x \in M$ and hence is locally constant. The converse is clear.

To extend this into a complex of differential operators we inspect the diagram. Choosing a splitting $\Lambda^1_H \hookrightarrow \Lambda^1$ and then composing with the exterior derivative we get an obvious differential operator $\Lambda^1_H \to \Lambda^2_H$. To construct a canonical differential operator, we need to quotient by the image of the Levi form to get $d_\perp : \Lambda^1_H \to \Lambda^2_{H\perp}$.

$$\begin{tikzcd}
& 0 & 0 & \\
& \Lambda^1_H \arrow[u] & \Lambda^2_H \arrow[u] & \\
\Lambda^0 \arrow[r, "d"] & \Lambda^1 \arrow[u] \arrow[r, "d"] & \Lambda^2 \arrow[u] \arrow[r, "d"] & \dots \\
& L \arrow[u] \arrow[uur, bend right, "\mathcal{L}"'] & L \otimes \Lambda^1_H \arrow[u] & \\
& 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

In the sequel, given a form ω in Λ^k_H, the same symbol with a tilde, $\tilde{\omega}$, will denote a choice of some k-form in Λ^k satisfying $\tilde{\omega}|_H = \omega$. We will also usually denote sections of Λ^k by a symbol with a tilde in order to avoid confusion between the bundles Λ^k_H and Λ^k, where confusion may arise. Omission of the tilde then means the restriction.

Definition 1.2.3 (First Rumin operator). *The first Rumin operator is defined as* $d_\perp : \Lambda^1_H \to \Lambda^2_{H\perp}$ *given by*

$$d_\perp \omega := d\tilde{\omega}|_H + L. \tag{1.2.4}$$

Proposition 1.2.5. *The first Rumin operator is well defined.*

Proof. The difference between two choices for $\tilde{\omega}$ in Λ^1 must be a contact form, and by design, the definition quotients out by the image of such forms in Λ^2_H. $\square$

Proposition 1.2.6. $d_\perp \circ d_\perp = 0$.

Proof. Given $f \in C^\infty(M)$, we may as well take $\widetilde{d_\perp f} = df$ and then use the fact $d \circ d = 0$. $\square$

The first Rumin operator defined above is the model for the next operators in the Rumin complex, and how the sequence relates to the de Rham complex.

$$\begin{tikzcd}
 & 0 & 0 & \\
 & \Lambda_H^{k-1} \arrow[u] & \Lambda_H^{k} \arrow[u] & \\
\cdots \arrow[r, "d"] & \Lambda^{k-1} \arrow[u] \arrow[r, "d"] & \Lambda^{k} \arrow[u] \arrow[r, "d"] & \cdots \\
 & L \otimes \Lambda_H^{k-2} \arrow[u] \arrow[uur, hook, bend right, "\mathcal{L}"'] & L \otimes \Lambda_H^{k-1} \arrow[u] & \\
 & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Definition 1.2.7 (kth Rumin operator for $2 \le k < n$). *On a* $2n+1$ *dimensional contact manifold* (M, H), *for* $2 \le k < n$ *define the* k*th Rumin operator* $d_\perp : \Lambda^k_{H\perp} \to \Lambda^{k+1}_{H\perp}$ *by*

$$d_\perp\big(\omega + \mathcal{L}(L \otimes \Lambda_H^{k-2})\big) := d\tilde{\omega}|_H + \mathcal{L}(L \otimes \Lambda_H^{k-1}) \tag{1.2.8}$$

To check that this is well defined, fixing a representative $\omega \in \Lambda^k_H$, the image in Λ^{k+1}_H of the difference of two choices for $\tilde{\omega}$ lies in the image of $\mathcal{L} : L \otimes \Lambda^{k-1}_H \to \Lambda^{k+1}_H$, by construction. Then, note that two representatives for $\omega + \mathcal{L}(L \otimes \Lambda^{k-2}_H)$ differ by $d\alpha|_H \wedge \rho$ for some $\rho \in \Lambda^{k-2}_H$ and this also lands in $\mathcal{L}(L \otimes \Lambda^{k-1}_H)$.

We may denote the kth Rumin operator by $d_\perp^{(k)}$ where necessary, to distinguish between the operators in the complex.

Proposition 1.2.9. *$d_\perp^{(k+1)} \circ d_\perp^{(k)} = 0$ for $1 \leq k < n-1$.*

Proof. Given $d\tilde{\omega}|_H + \mathcal{L}(L \otimes \Lambda^{k-1}_H) \in \Lambda^{k+1}_{H\perp}$ take $\widetilde{d\tilde{\omega}|_H} = d\tilde{\omega}$ and then use $d \circ d = 0$.

While in principle the operators defined this way continue to be well defined for all $k \in \mathbb{N}$, for $k \geq n$, the Levi map $\mathcal{L} : L \otimes \Lambda^{k-1}_H \to \Lambda^{k+1}_H$ is surjective and hence these differential operators vanish. So using the above construction we can only replace the first n operators of the de Rham sequence, consisting of $2n+1$ non-trivial differential operators on a manifold of dimension $2n+1$. As we will now see, it turns out it is similarly easy to construct adapted operators from the final n operators of the de Rham sequence.

Given a vector bundle E we will write without comment, any section of $L \otimes E$ as $\alpha \otimes e$ where e is a section of E and α is a choice of contact form, so is non-vanishing.

Define $\Lambda^k_{H\times} := \{\rho \in \Lambda^k_H \mid d\alpha|_H \wedge \rho = 0\}$. Obviously $\Lambda^k_{H\times} = \Lambda^k_H$ for $k \geq 2n-1$, but non-degeneracy of $d\alpha|_H$ also implies that $\Lambda^k_{H\times}$ vanishes for $k < n$. In addition we have $\Lambda^n_{H\times} = \Lambda^n_{H\perp}$ if we identify the latter bundle with trace-free forms.

The key observation is that if $\alpha \otimes \rho \in L \otimes \Lambda^{k-1}_{H\times}$ then $d(\alpha \wedge \tilde{\rho})$ lies in the kernel of the restriction map and hence lies in the image of the canonical inclusion i_{k+1}. Write i_k^{-1} for the inverse of i_k on its image.

Definition 1.2.10 (kth Rumin operator for $n < k \leq 2n$)**.** *On a $2n+1$ dimensional contact manifold (M, H), for $n < k \leq 2n$, define the kth Rumin operator $d_\perp : L \otimes \Lambda^{k-1}_{H\times} \to L \otimes \Lambda^k_{H\times}$ as the composition $i_{k+1}^{-1} \circ d \circ i_k$.*

It's not immediately clear that the operator factors through $L \otimes \Lambda^k_{H\times}$ as opposed to merely landing in $L \otimes \Lambda^k_H$. To see this, first note that if $d_\perp(\alpha \otimes \rho) = \alpha \otimes \omega \in L \otimes \Lambda^k_H$ then

$$\alpha \wedge \tilde{\omega} = d\alpha \wedge \tilde{\rho} - \alpha \wedge d\tilde{\rho}. \tag{1.2.11}$$

Applying the exterior derivative shows that $d\alpha|_H \wedge \omega = 0$ and hence $\alpha \otimes \omega$ is a section of $\Lambda^{k-1}_{H\times}$.

Proposition 1.2.12. *$d_\perp^{(k+1)} \circ d_\perp^{(k)} : L \otimes \Lambda^{k-1}_{H\times} \to L \otimes \Lambda^{k+1}_{H\times} = 0$ for $n < k \leq 2n-1$*

Proof. We calculate $d_\perp^{(k+1)} \circ d_\perp^{(k)} = i_{k+2} \circ d \circ i_{k+1}^{-1} \circ i_{k+1} \circ d \circ i_k = 0$. □

As it stands we have two complexes of differential operators.

$$\dots \longrightarrow \Lambda^{n-1}_{H\perp} \longrightarrow \Lambda^{n}_{H\perp} \dashrightarrow L\otimes\Lambda^{n}_{\times} \longrightarrow L\otimes\Lambda^{n+1}_{\times} \longrightarrow \dots$$

We now show there is an appropriate differential operator $\Lambda^n_{H\perp} \to L\otimes\Lambda^n_{H\times}$ that will complete a differential complex analogous to the de Rham complex.

The key point of the following construction is the use of the Levi homomorphism $\mathcal{L} : L\otimes\Lambda^{n-1}_H \to \Lambda^{n+1}_H$ which by 1.1.14 is an isomorphism.

The obvious thing to try is to map into Λ^n by taking some right inverse to the projection $\Lambda^n \to \Lambda^n_{H\perp}$ then apply the exterior derivative $d : \Lambda^n \to \Lambda^{n+1}$. We want to identify the result with a section of $L\otimes\Lambda^n_H$ but this isn't possible since the image of this composition does not in general lie in the kernel of the projection $\Lambda^{n+1} \to \Lambda^{n+1}_H$. That is, given $\omega \in \Lambda^n_{H\perp}$, $d\tilde{\omega}|_H$ need not vanish. Note however, that because $\mathcal{L} = q_{n+1}\circ d\circ i_n$ by definition, the section $d\tilde{\omega} - (d\circ i_n\circ\mathcal{L}^{-1})(d\tilde{\omega}|_H)$ lies in the kernel of the projection $q_{n+1} : \Lambda^n \to \Lambda^{n+1}_H$ and can therefore be identified as a section of $L\otimes\Lambda^n_H$, by virtue of exactness.

Definition 1.2.13 (nth Rumin operator)**.** *On a contact manifold (M,H), define the nth operator $d_\perp : \Lambda^n_{H\perp} \to L\otimes\Lambda^n_{H\times}$*

$$d_\perp\big(\omega + \mathcal{L}(L\otimes\Lambda^{n-2}_H)\big) = i^{-1}_{n+1}\big(d\tilde{\omega} - (d\circ i_n\circ\mathcal{L}^{-1})(d\tilde{\omega}|_H)\big). \tag{1.2.14}$$

There are a few things to check.

Proposition 1.2.15. *The nth Rumin operator is well defined.*

Proof. Let $\tilde{\omega},\hat{\omega} \in \Lambda^n$ be two representatives of $\omega + \mathcal{L}(L\otimes\Lambda^{n-2}_H) \in \Lambda^n_{H\perp}$ then $(\tilde{\omega}-\hat{\omega})|_H$ lies in $\mathcal{L}(L\otimes\Lambda^{n-2}_H)$. So $\tilde{\omega}-\hat{\omega} = d\alpha\wedge\tilde{\mu} + \alpha\wedge\tilde{\nu}$ for some $\mu\in\Lambda^{n-2}_H$ and $\nu\in\Lambda^{n-1}_H$. Applying the exterior derivative gives $d(d\alpha\wedge\tilde{\mu}+\alpha\wedge\tilde{\nu}) = d\alpha\wedge d\tilde{\mu} + d\alpha\wedge\tilde{\nu} - \alpha\wedge d\tilde{\nu}$. Now $\mathcal{L}^{-1}\big((d\alpha\wedge d\tilde{\mu}+d\alpha\wedge\tilde{\nu}-\alpha\wedge d\tilde{\nu})|_H\big) = \alpha\otimes(d\mu+\nu)$ and hence

$$\begin{aligned}
&\tilde{\omega}-\hat{\omega}-(d\circ i_n\circ\mathcal{L}^{-1})(d\tilde{\omega}|_H - d\hat{\omega}|_H)\\
&= d\alpha\wedge d\tilde{\mu} + d\alpha\wedge\tilde{\nu} - \alpha\wedge d\tilde{\nu} - (d\circ i_n)(\alpha\otimes(d\mu+\nu))\\
&= d\alpha\wedge d\tilde{\mu} + d\alpha\wedge\tilde{\nu} - \alpha\wedge d\tilde{\nu} - (d\alpha\wedge d\tilde{\mu} + d\alpha\wedge\tilde{\nu} - \alpha\wedge d\tilde{\nu})\\
&= 0
\end{aligned} \tag{1.2.16}$$

which shows independence from the choice of representative. □

Proposition 1.2.17. $i^{-1}_{n+1}\big(d\omega - (d\circ i_n\circ\mathcal{L}^{-1})(d\omega|_H)\big)$ *is a section of* $L\otimes\Lambda^n_{H\times}$

Proof. $d\big(d\tilde{\omega} - (d\circ q_n\circ\mathcal{L}^{-1})(d\tilde{\omega}|_H)\big) = 0$ so if $d\tilde{\omega} - (d\circ i_n\circ\mathcal{L}^{-1})(d\tilde{\omega}|_H) = \alpha\wedge\nu$ then $d\alpha\wedge\tilde{\nu} - \alpha\wedge d\tilde{\nu} = 0$ and hence $d\alpha|_H\wedge\nu = 0$. □

Proposition 1.2.18. *The nth Rumin operator completes a differential complex.*

Proof. This follows from the definitions and $d \circ d = 0$. □

So we have constructed the Rumin complex for a contact manifold of dimension $2n + 1 \geq 5$. Following the above, it is easy enough to work out a complex in the case $n = 1$ although this case is degenerate in the sense the operator on Λ^1_H is immediately second order, since the bundle $\Lambda^2_{H\perp}$ vanishes.

Lastly for this section, there is a coordinate free expression for the first Rumin operator $d_\perp : \Lambda^1_H \to \Lambda^2_{H\perp}$ derived from the corresponding expression for the exterior derivative $d : \Lambda^1 \to \Lambda^2$, which we will occasionally find useful.

Proposition 1.2.19. *Let $\rho \in \Lambda^1_H$, then, identifying $\Lambda^2_{H\perp}$ with trace-free forms,*

$$d_\perp \rho(X, Y) = \frac{1}{2}X(\rho(Y)) - \frac{1}{2}Y(\rho(X)) - \frac{1}{2}(\tilde{\rho} - \frac{1}{n}\operatorname{Tr}(d\tilde{\rho}|_H)\alpha)([X, Y]) \tag{1.2.20}$$

where $\tilde{\rho} \in \Lambda^1$ is some lift of $\rho \in \Lambda^1_H$ and X, Y are sections of H. Here $\operatorname{Tr}(d\tilde{\rho}|_H)\alpha$ refers taking the trace with respect to the form $H \wedge H \to \mathbb{R}$ distinguished by α.

Proof. Let X, Y be sections of H then

$$d_\perp \rho(X, Y) = d\tilde{\rho}(X, Y) - \frac{1}{2n}d\alpha(X, Y)\operatorname{Tr}(d\tilde{\rho}|_H). \tag{1.2.21}$$

Now we use the usual coordinate free formula for the exterior derivative of a 1-form.

$$\begin{aligned} d_\perp \rho(X, Y) =& \frac{1}{2}X(\rho(Y)) - \frac{1}{2}Y(\rho(X)) - \frac{1}{2}\tilde{\rho}([X, Y]) \\ &- \frac{1}{2n}\operatorname{Tr}(d\tilde{\rho}|_H)d\alpha(X, Y) \end{aligned} \tag{1.2.22}$$

Then $d\alpha(X, Y) = -\alpha([X, Y])$ gives the result. □

1.3 Cohomology groups of the Rumin complex

That the nth Rumin operator is a second order differential operator is really the only price one pays for greater efficiency compared to the de Rham complex. As we will now see, it turns out that this complex computes the de Rham cohomology groups of M.

Proposition 1.3.1. *The cohomology groups of the Rumin complex are isomorphic to the de Rham cohomology groups.*

Proof. This can be shown using a spectral sequence, converging quickly at the E_1-level, as is done in [BEGN19].

It is a very good exercise however, in navigating the Rumin complex to construct explicit isomorphisms

$$\frac{\ker d_\perp^{(k)}}{\operatorname{im} d_\perp^{(k-1)}} \xrightarrow{\sim} \frac{\ker d^{(k)}}{\operatorname{im} d^{(k-1)}} \cong H^k_{dR}(M,\mathbb{R}) \tag{1.3.2}$$

for $k = 0, ..., 2n$. Here we take $\operatorname{im} d_\perp^{(-1)} := \{0\}$.

Firstly, the statement that the kernel of the first Rumin operator is the sheaf of locally constant functions makes it clear that the isomorphism is true for $k = 0$. That is, the zeroth cohomology is just $\mathbb{R}^m$ where m is the number of connected components of M.

Then, the isomorphism is given by

$$[\omega] \mapsto [\tilde{\omega} - (i_k \circ \mathcal{L}^{-1})(d\tilde{\omega}|_H)] \tag{1.3.3}$$

for $k = 1$, and

$$[\omega + \mathcal{L}(L \otimes \Lambda_H^{k-2})] \mapsto [\tilde{\omega} - (i_k \circ \mathcal{L}^{-1})(d\tilde{\omega}|_H)] \tag{1.3.4}$$

for $k = 2, ..., n$. $\mathcal{L}^{-1}$ makes sense in each of the above definitions since $d\tilde{\omega}|_H \in \mathcal{L}(L \otimes \Lambda_H^{k-1})$. Next

$$[\alpha \otimes \omega] \mapsto [\alpha \wedge \tilde{\omega}] \tag{1.3.5}$$

gives the isomorphism for $k = n+1, ..., 2n$. Note that this is just the canonical inclusion pushed down to cohomology.

That these maps are well defined on cohomology, and are isomorphisms follows from diagram chases which are in Appendix A. □

1.4 The Reeb field

We now define the Reeb field, which will be an important object. This is a preferred vector field transverse to the contact distribution that is defined in the presence of a contact form.

Definition 1.4.1 (Reeb field)**.** *On a contact manifold (M, H), given a choice of contact form α, the Reeb field T is the vector field such that $\alpha(T) = 1$ and $d\alpha(T, \cdot) = 0$.*

Proposition 1.4.2. *Given a choice of contact form α, the Reeb field exists and is unique.*

Proof. Let T, T' be locally defined vector fields satisfying $\alpha(T') = \alpha(T) = 1$ and $d\alpha(T, \cdot) = d\alpha(T', \cdot) = 0$ on their common domain. The first condition implies $T' - T \in H$, but use the non-degeneracy of $d\alpha|_H$ to see that the second condition implies $T = T'$ on their common domain. Now cover the manifold with Darboux coordinate patches $\{U_i\}_{i \in I}$ and define $T = \frac{\partial}{\partial t}$ on U_i, which locally satisfies the required conditions. The above argument shows both that T is a well-defined vector field and is the unique vector field satisfying $\alpha(T) = 1$ and $d\alpha(T, \cdot) = 0$. □

Proposition 1.4.3. *The Reeb field gives distinguished splittings of the short exact sequences $0 \to L \otimes \Lambda_H^{k-1} \to \Lambda^k \to \Lambda_H^k \to 0$ for $k \in \mathbb{N}$.*

Proof. The most important case is $k = 1$ where we define $\Lambda^1 \xrightarrow{\sim} L \oplus \Lambda_H^1$ by

$$\omega \mapsto (\omega(T)\alpha, \omega|_H). \tag{1.4.4}$$

$(\omega|_x(T)\alpha, \omega|_{H_x}) = (0, 0)$ implies $\omega|_x = 0$ since T, H generate $T_x M$ so this is an isomorphism. Moreover this defines a splitting since the inclusion $L \hookrightarrow \Lambda^1$ composes with the isomorphism to get the natural inclusion $L \hookrightarrow L \oplus \Lambda_H^1$ and the projection $L \oplus \Lambda_H^1 \to \Lambda_H^1$ is precisely the inverse of 1.4.4 followed by the projection $\Lambda^1 \to \Lambda_H^1$.

More generally, splittings of $0 \to L \otimes \Lambda_H^{k-1} \to \Lambda^k \to \Lambda_H^k \to 0$ for any $k \in \mathbb{N}$ are are given by the isomorphism $\Lambda^k \xrightarrow{\sim} (L \otimes \Lambda_H^{k-1}) \oplus \Lambda_H^k$ defined by

$$\omega \mapsto (\alpha \otimes \omega(T, \cdot, ..., \cdot), \omega|_H) \tag{1.4.5}$$

and the same argument works. □

In particular a contact form gives a distinguished lift $\Lambda_H^k \hookrightarrow \Lambda^k$ which takes $\rho \in \Lambda_H^k$ to the unique $\tilde{\rho} \in \Lambda^k$ such that $\tilde{\rho}|_H = \rho$ and $\tilde{\rho}(T, \cdot, ..., \cdot) = 0$.

Proposition 1.4.6. *Let T be the Reeb field corresponding to a contact form α. Then given a change of contact form $\hat{\alpha} = \Omega\alpha$ for some positive function Ω, the new Reeb field $\hat{T}$ is given by*

$$\hat{T} = \frac{1}{\Omega}\big(T + \frac{1}{2}F\big) \tag{1.4.7}$$

where $F \in H$ is the unique vector field in the contact direction satisfying

$$J_{ba}F^b = d_\perp \ln \Omega. \tag{1.4.8}$$

Proof. Clearly $\hat{\alpha}(\hat{T}) = 0$ and then calculate

$$d\hat{\alpha}(\hat{T}, \cdot) = \frac{1}{2}d\alpha(F, \cdot) + \frac{1}{2\Omega}d\Omega(T + \frac{1}{2}F)\alpha - \frac{1}{2\Omega}d\Omega. \tag{1.4.9}$$

Now this expression vanishes when restricted to H thanks to the definition of F. On the other hand plugging in $T + \frac{1}{2}F$

$$\begin{aligned} &\frac{1}{2}d\alpha(F, T + \frac{1}{2}F) + \frac{1}{2\Omega}d\Omega(T + \frac{1}{2}F)\alpha(T + \frac{1}{2}F) - \frac{1}{2\Omega}d\Omega(T + \frac{1}{2}F) \\ =&\frac{1}{2\Omega}d\Omega(T + \frac{1}{2}F) - \frac{1}{2\Omega}d\Omega(T + \frac{1}{2}F) = 0 \end{aligned} \tag{1.4.10}$$

but H and $T + \frac{1}{2}F$ span each tangent space and so $d\hat{\alpha}(\hat{T}, \cdot)$ vanishes. □

1.5 Integration and the Rumin complex

We now briefly discuss integration. In particular we will recover the formal self adjointness of the De Rham complex for the Rumin complex. See [Lee09] for the background to integration on manifolds.

The first thing to note is that there is a canonical isomorphism $L \otimes \Lambda^{2n}_H \xrightarrow{\sim} \Lambda^{2n+1}$ given by $\alpha \otimes \omega \mapsto \alpha \wedge \tilde{\omega}$. This isomorphism means it makes sense to integrate sections of $L \otimes \Lambda^{2n}_H$ (with compact support).

Definition 1.5.1 (Integrating sections of $L \otimes \Lambda^{2n}_H$)**.** *For a section $\alpha \otimes \omega$ of $L \otimes \Lambda^{2n}_H$ with compact support define*

$$\int_M \alpha \otimes \omega = \int_M \alpha \wedge \tilde{\omega} \tag{1.5.2}$$

Recall Stokes' theorem for forms with compact support:

Theorem 1.5.3 (Stokes')**.** *Let M be a manifold (without boundary) of dimension $2n + 1$. Let ω be a $2n$-form with compact support, then*

$$\int_M d\omega = 0. \tag{1.5.4}$$

One gets integration by parts from the way d acts as an anti-derivation on forms. For $\omega \in \Lambda^k$ and $\mu \in \Lambda^{2n-k}$ with compact support

$$0 = \int_M d(\omega \wedge \mu) = \int_M d\omega \wedge \mu + (-1)^k \int_M \omega \wedge d\mu. \tag{1.5.5}$$

So if one defines pairings on forms of compact support $\Lambda^k \times \Lambda^{2n+1-k} \to \mathbb{R}$ by

$$(\omega, \mu) := \int_M \omega \wedge \mu, \tag{1.5.6}$$

then we have the "self-adjoint" or "integration by parts" property

$$(d\rho, \mu) = (-1)^{k+1}(\rho, d\mu). \tag{1.5.7}$$

for $\rho \in \Lambda^{k-1}$ and $\mu \in \Lambda^{2n+1-k}$. Moreover, $d : \Lambda^{2n+1-k} \to \Lambda^{2n+2-k}$ is the unique differential operator satisfying 1.5.7 for all sections for $\rho \in \Lambda^{k-1}$ and $\mu \in \Lambda^{2n+1-k}$. with compact support. This follows from the non-degeneracy of the pairing, which is the simple fact that if $\mu \in \Lambda^{2n+1-k}$ with compact support satisfies

$$\int_M \omega \wedge \mu = 0 \tag{1.5.8}$$

for all $\omega \in \Lambda^k$ with compact support, then $\mu = 0$.

Definition 1.5.9. *For $0 \leq k \leq n$, define pairings $\Lambda^k_{H\perp} \times (L \otimes \Lambda^{2n-k}_{H\times}) \to \mathbb{R}$ on sections of compact support by*

$$([\omega], \alpha \otimes \mu) = \int_M \alpha \wedge \tilde{\omega} \wedge \tilde{\mu} \tag{1.5.10}$$

To see that this is well defined first note that

$$(\omega, \alpha \otimes \mu) \mapsto \int_M \alpha \wedge \tilde{\omega} \wedge \tilde{\mu} \tag{1.5.11}$$

c is well defined. Next, two representatives in Λ^k_H of $[\omega] \in \Lambda^k_{H\perp}$ differ by $d\alpha|_H \wedge \rho$ but

$$\alpha \wedge d\alpha \wedge \tilde{\rho} \wedge \tilde{\mu} = 0 \tag{1.5.12}$$

by virtue of $\mu \in \Lambda^{2n-k}_{H\times}$.

Proposition 1.5.13. *The Rumin complex is self-adjoint with respect to the pairing above in the sense that for $\omega \in \Lambda^k_H$ and $\alpha \otimes \mu \in L \otimes \Lambda^{2n-1-k}_H$*

$$(d_\perp[\omega], \alpha \otimes \mu) = (-1)^{k+1}([\omega], d_\perp(\alpha \otimes \mu)) \tag{1.5.14}$$

for $0 \leq k < n$ and

$$([\omega], d^{(n)}_\perp[\mu]) = (-1)^n([\mu], d^{(n)}_\perp[\omega]). \tag{1.5.15}$$

Proof. For $0 \leq k < n$,

$$(d_\perp[\omega], \mu) = \int_M \alpha \wedge d\tilde{\omega} \wedge \tilde{\mu} \tag{1.5.16}$$

where we may as well take $\tilde{\omega}(T) = 0$ and $\tilde{\mu}(T) = 0$, where T is the Reeb field associated with α. By Stokes' theorem

$$(d_\perp[\omega], \mu) = \int_M \alpha \wedge d\tilde{\omega} \wedge \tilde{\mu} = \int_M \tilde{\omega} \wedge d(\alpha \wedge \tilde{\mu}). \tag{1.5.17}$$

Now $\Lambda^{2n+1} \ni \tilde{\omega} \wedge d\alpha \wedge \tilde{\mu} = 0$ since it vanishes when plugging in T. Thus

$$\int_M \tilde{\omega} \wedge d(\alpha \wedge \tilde{\mu}) = -\int_M \tilde{\omega} \wedge \alpha \wedge d\tilde{\mu} = (-1)^{k+1} \int_M \alpha \wedge \tilde{\omega} \wedge d\tilde{\mu}. \tag{1.5.18}$$

which is just $(-1)^{k+1}([\omega], d_\perp \mu)$.

For $k = n$, using the definition of the nth Rumin operator

$$([\omega], d_\perp^{(n)}[\mu]) = (-1)^n \int_M \tilde{\omega} \wedge (d\tilde{\mu} - d(\alpha \wedge \tilde{\nu})) \tag{1.5.19}$$

where $\nu \in \Lambda_H^{n-1}$ with $d\alpha \wedge \nu = d\tilde{\mu}|_H$. Defining $\rho \in \Lambda_H^{n-1}$ by $d\alpha|_H \wedge \rho = d\omega|_H$ the above is

$$(-1)^n \Big(\int_M (\tilde{\omega} - \alpha \wedge \tilde{\rho}) \wedge (d\tilde{\mu} - d(\alpha \wedge \tilde{\nu})) + \alpha \wedge \tilde{\rho} \wedge (d\tilde{\mu} - d(\alpha \wedge \tilde{\nu})) \tag{1.5.20}$$

but the trailing term vanishes since $d\tilde{\mu} - d(\alpha \wedge \tilde{\nu}) \in L \otimes \Lambda_H^n$ by construction of the nth Rumin operator, so is killed when wedged with α. Then Stokes' theorem allows us to rewrite the above as

$$\begin{aligned} & -\int_M d(\tilde{\omega} - \alpha \wedge \tilde{\rho}) \wedge (\tilde{\mu} - \alpha \wedge \tilde{\nu}) \\ = & -\int_M d(\tilde{\omega} - \alpha \wedge \tilde{\rho}) \wedge \tilde{\mu} \\ = & \int_M \tilde{\mu} \wedge d(\tilde{\omega} - \alpha \wedge \tilde{\rho}) \\ = & \ (-1)^n([\mu], d_\perp^{(n)}[\omega]). \end{aligned} \tag{1.5.21}$$

□

The above property could also be rephrased by introducing a "symplectic star" operator analogous to the Hodge star of Riemannian geometry and defining pairings on k-forms this way. This approach is taken by [TY12]. The presentation above remains agnostic to the choice of contact form.

1.6 Contact manifolds as filtered manifolds

There is another definition of a contact manifold which identifies a contact manifold as a special case of a filtered manifold with prescribed symbol algebra.

Definition 1.6.1 (k-filtered manifold)**.** *A k-filtered manifold is a smooth manifold M with tangent bundle TM, such that the tangent bundle has a filtration*

$$TM = T^{-k} \supset T^{-k+1} \supset ... \supset T^{-1}M \tag{1.6.2}$$

satisfying $[X,Y] \in T^{i+j}M$ for sections $X \in T^iM$ and $Y \in T^jM$.

A consequence of the above definition is that the Lie bracket is well defined on $\mathrm{gr}(TM) := TM/T^{-k+1}M \oplus T^{-k+1}M/T^{-k+2}M \oplus ... \oplus T^{-1}M$ and furthermore preserves the grading. That is, $[X,Y] \in T^{i+j}M/T^{i+j+1}M$ for sections $X \in T^iM/T^{i+1}M$ and $Y \in T^jM/T^{j+1}M$. Even better, this map is tensorial, which follows from the usual identity $[X, fY] = X(f)Y + f[X,Y]$.

Definition 1.6.3 (Levi bracket)**.** *Given a filtered manifold M, the tensorial pairing on $\mathrm{gr}(TM)$ induced by the Lie bracket is called the (generalised) Levi bracket.*

Definition 1.6.4 (Symbol Lie algebra)**.** *Given a filtered manifold M and $x \in M$, the vector space $\mathrm{gr}(T_xM)$ equipped with the generalised Levi bracket is a Lie algebra, called the symbol algebra of the filtered manifold at x.*

Definition 1.6.5 (Heisenberg algebra)**.** *The (real) Heisenberg algebra of dimension $2n+1$ is the Lie algebra with basis $X_1, ...X_n, Y_1, ...Y_n, Z$ and commutation relations*

$$[X_i, Y_j] = \delta_{ij} Z, \ [X_i, Z] = 0, \ [Y_i, Z] = 0, \ \forall i, j = 1, ..., n \tag{1.6.6}$$

For example when $n = 1$ the Heisenberg algebra is easily realised as the set of matrices

$$\left\{ \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \middle| \ x, y, z \in \mathbb{R} \right\} \tag{1.6.7}$$

equipped with the matrix commutator.

If M is a contact manifold, it is filtered with $TM =: T^{-2}M \supset T^{-1}M := H$. The associated graded bundle is $\mathrm{gr}(TM) := TM/H \oplus H$. Choosing a trivialisation for TM/H means the Levi bracket gives a non-degenerate skew form $H_x \times H_x \to \mathbb{R}$ on each fibre. From the fact there is a unique non-degenerate skew form on a $2n$-dimensional vector space up to change of basis, it follows that the symbol algebra corresponding to a contact manifold of dimension $2n + 1$ is isomorphic to the Heisenberg algebra $\mathfrak{h}_{2n+1}$.

On the other hand, a k-filtered manifold with the symbol algebras $\mathrm{gr}(T_xM)$ being isomorphic to $\mathfrak{h}_{2n+1}$ for each $x \in M$ implies that the manifold 2-filtered by a rank-$2n$ subbundle. If we also insist that $T^{-2}M/T^{-1}M$ is orientable then choosing a non-vanishing section T induces a globally defined one form $\alpha \in \Lambda^1$ by $X \mapsto \lambda \in C^\infty(M)$ such that $\lambda T = X \bmod T^{-1}M$. Then α annihilates $H := T^{-1}M$ and $d\alpha|_H$ is non-degenerate on each fibre.

All this means we have definition of a contact manifold equivalent to 1.1.2:

Definition 1.6.8 (Contact manifold II). *A contact manifold is a filtered manifold of dimension $2n + 1 \geq 5$ with $T^{-2}M/T^{-1}M$ orientable and $\mathrm{gr}(T_xM)$ isomorphic to $\mathfrak{h}_{2n+1}$ for each $x \in M$.*

The above definition leads to an obvious analogue of a contact manifold.

Definition 1.6.9 (Quaternions). *Given the set $\mathbb{R}^4$ write a for $(a, 0, 0, 0) \in \mathbb{R}^4$ and write $i, j, k \in \mathbb{R}^4$ for the second, third and fourth standard basis vectors respectively. Then the quaternions, denoted $\mathbb{H}$, is the associative algebra defined as the set $\mathbb{R}^4$ equipped with the $\mathbb{R}$-bilinear map $\mathbb{H} \times \mathbb{H} \to \mathbb{H}$ given by the relations $1 \cdot q = q \ \forall q \in \mathbb{H}$, $i^2 = j^2 = k^2 = -1, ij = k$.*

We call quaternions in the span of $\{i, j, k\}$ imaginary quaternions and denote the set of these $\mathrm{im}(\mathbb{H})$ which is a 3-dimensional subalgebra. We have a conjugation operation given by $q = a + bi + cj + dk \mapsto a - bi - cj - dk =: \bar{q}$ and $\bar{q}q$ is real. Define $\mathbb{H}^n$ to be the set of n-tuples of quaternions which, is a right $\mathbb{H}$-module in the obvious way. There is a canonical $\mathbb{H}$-linear form $\langle \cdot, \cdot \rangle : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ defined exactly analogously to the canonical Hermitian form on $\mathbb{C}^n$ and this is Hermitian with respect to the notion of conjugation defined above.

Definition 1.6.10 (Quaternionic Heisenberg algebra). *The quaternionic Heisenberg algebra of real dimension $4n + 3$ is the real Lie algebra defined as $\mathbb{H}^n \oplus \mathrm{im}(\mathbb{H})$ with Lie bracket*

$$[(x, q), (y, p)] = \mathrm{im}\langle x, y\rangle \tag{1.6.11}$$

where $\langle \cdot, \cdot \rangle : \mathbb{H}^n \times \mathbb{H}^n \to \mathbb{H}$ is the canonical Hermitian form.

Definition 1.6.12 (Quaternionic contact structure). *A quaternionic contact structure on a $4n+3$-dimensional manifold with $n \geq 1$ is a filtration such that the symbol algebras $\mathrm{gr}(T_xM)$ are isomorphic to the quaternionic Heisenberg algebra for each $x \in M$.*

Such structures have been studied by Biquard [Biq00] and Duchemin [Duc06], among others. It is an obvious direction for future investigation to see if one can reconstruct the analogue of the Rumin complex for these manifolds using a diagram chase similar to, but presumably more elaborate than the one given here.

Chapter 2

Partial connections

In this chapter we investigate the notion of a partial connection on a vector bundle. These are analogues of connections, except the directional derivative is defined only in directions lying in some subbundle $H \subset TM$. We begin in as much generality as possible. In the first section we provide definitions of useful tensor invariants, in particular an analogue of curvature, the partial curvature, and, in the case of a partial connection on Λ^1_H, an analogue of torsion, the partial torsion.

Next, we specialise to the case of a partial connection on a vector bundle over a contact manifold where we take H to be the contact distribution. In the contact setting, there is a canonical extension to a full connection of any partial connection, and this extension satisfies some desirable properties. We explain this construction and show that vanishing of the partial curvature implies vanishing of the curvature of the promoted connection. We also prove symmetries, analogous to the algebraic Bianchi identities, of the partial curvature of partial connections which respect the contact structure and have vanishing partial torsion.

Just like how the de Rham complex can be written in terms of a torsion-free affine connection, we will write the Rumin complex in terms of an affine partial connection with vanishing partial torsion. In order to do this, we end up needing to calculate many more differential operators in terms of abstract indices, including an explicit formula for the canonical extension described above. Lastly, we attempt to use a partial connection to write out the coupled Rumin sequence, analogous to the coupled de Rham sequence (whose operators are sometimes called exterior covariant derivatives), and show that there is a curvature obstruction to doing so. We also give index formulae for differential identities for partial curvature derived from the usual differential Bianchi identity.

2.1 General partial connections

First we recall one of the many definitions of a connection on a vector bundle.

Definition 2.1.1 (Connection). *Given a smooth manifold M and a smooth vector bundle $\pi : E \to M$ a (linear) connection $\nabla : E \to \Lambda^1 \otimes E$ is a smooth map of local sections which satisfies the Leibniz rule*

$$\nabla fs = df \otimes s + f\nabla s \tag{2.1.2}$$

for $s \in E$ a local section and f a smooth function.

In the case $E = \Lambda^1$ then we say ∇ is an affine connection. (Usually an affine connection is defined to be a connection on the tangent bundle, but we will see these notions are equivalent.)

Let $(x_1, ...x_n)$ be local coordinates for M and let $\{e_i\}_{i=1}^k$ be a local trivialisation for the vector bundle E contained in the coordinate neighbourhood. In the trivialising neighbourhood, we write, for any local section s of E, $s = s^i e_i$ for some smooth functions s^i. In particular

$$\nabla s = ds^i \otimes e_i + s^i \nabla e_i, \tag{2.1.3}$$

and if we write $\nabla e_i = \Gamma^k{}_{ij} dx^j \otimes e_k$ then the functions $\Gamma^k{}_{ij}$ are called the connection coefficients, or Christoffel symbols (in most texts, the latter name is reserved for the Riemannian case).

It's clear from the above that the value at a point of a connection applied to a local section depends only on the first derivatives of the component functions at that point. In other words, a connection is a first order differential operator.

A choice of local trivialisation then gives rise to a preferred connection over that trivialisation by simply decreeing that the Christoffel symbols vanish. That is, one defines

$$ds = ds^i \otimes e_i. \tag{2.1.4}$$

From the Leibniz rule it is clear that the difference of two connections must be a tensorial quantity and hence given two connections $\hat{\nabla}_a$ and ∇_a on E (and using abstract indices) we may write

$$\hat{\nabla}_a s^\mu = \nabla_a s^\mu + \Gamma_a{}^\mu{}_\nu s^\nu \tag{2.1.5}$$

for some tensor $\Gamma_a{}^\mu{}_\nu \in \Lambda^1 \otimes \mathrm{End}(E)$. Indeed, the right-hand side is a connection for arbitrary $\Gamma_a{}^\mu{}_\nu$.

Now given a connection on the bundle E we get an induced connection on the dual bundle E^* by decreeing that for sections $s \in E$ and $\omega \in E^*$ we get the Leibniz rule

$$d(\omega s) = (\nabla\omega)s + \omega(\nabla s) \tag{2.1.6}$$

where juxtaposition means to take the natural pairing between sections of the bundles.

Next, given connections on E, F one gets a connection on $E \otimes F$ by enforcing the Leibniz rule. Decree on decomposable sections that

$$\nabla(e \otimes f) = \nabla e \otimes f + e \otimes \nabla f \tag{2.1.7}$$

then extend to arbitrary sections by linearity. This construction respects the associativity of the tensor product and furthermore one can see that we now get an induced connection on $\mathrm{End}(E) \cong E \otimes E^*$.

There is an alternate construction, see [KMS93], for induced connections that involves showing that a connection on E induces something called a principal connection on $F(E)$, the frame bundle of E, which then induces connections on all associated bundles of $F(E)$, which include the dual and tensor powers of E. The two constructions agree.

Finally, given a connection $\nabla : E \to \Lambda^1 \otimes E$ one gets a well-defined first order differential operator $d^\nabla : \Lambda^k \otimes E \to \Lambda^{k+1} \otimes E$ given by

$$d^\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^k \omega \wedge \nabla s \tag{2.1.8}$$

on simple tensors and extending by $\mathbb{R}$-linearity. Then it turns out the composition $d^\nabla \circ \nabla : E \otimes \Lambda^2 \to E$ is a tensorial map which is called the curvature κ of the connection. If we think of κ as a section of $\Lambda^2 \otimes \mathrm{End}(E)$ we have

$$d^\nabla \kappa = 0. \tag{2.1.9}$$

This fact is called the differential Bianchi identity.

For now let $H \subset TM$ be an arbitrary subbundle (of constant rank). For a smooth function f define $d_\perp f := df|_H$ and write $\Lambda^k_H := \Lambda^k H^*$.

Definition 2.1.10 (Partial Connection)**.** *Given a smooth manifold M, distribution $H \subseteq TM$ and a smooth vector bundle $\pi : E \to M$ a partial connection (with respect to H) is a smooth map $\nabla : E \to \Lambda^1_H \otimes E$ of local sections which satisfies*

$$\nabla f s = d_\perp f \otimes s + f \nabla s \tag{2.1.11}$$

Given coordinates $(x^1, ..., x^n)$, a local trivialisation $\{e_i\}_{i=1}^r$ for E over that coordinate patch and a section $s = s^i e_i$, any partial connection ∇ satisfies (in concrete index notation)

$$\nabla s = d_\perp s^i \otimes e_i + s^i \Gamma^k{}_{ij} d_\perp x^j \otimes e_k \tag{2.1.12}$$

where the $\Gamma^k{}_{ij}$ are smooth functions defined in the trivialising neighbourhood.

Just as in the case of a full connection, for any two partial connections with respect to H on E we may write

$$\hat{\nabla}_a s^\mu = \nabla_a s^\mu + \Gamma_a{}^\mu{}_\nu s^\nu \tag{2.1.13}$$

for some tensor $\Gamma_a{}^\mu{}_\nu \in \Lambda^1_H \otimes \mathrm{End}(E)$ and the right-hand side is a partial connection for arbitrary $\Gamma_a{}^\mu{}_\nu$.

One gets induced partial connections on the dual and tensor powers by enforcing the Leibniz rule analogously to usual connections but with the obvious modification that the induced partial connection on E^* is determined by

$$d(\omega s)|_H = (\nabla \omega)s + \omega(\nabla s). \tag{2.1.14}$$

We will call a partial connection $\nabla : \Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ an affine partial connection. For a usual affine connection it is possible to define the torsion as the vector bundle homomorphism $\Lambda^1 \to \Lambda^2$ defined as $\omega_b \mapsto (d\omega)_{ab} - \nabla_{[a}\omega_{b]}$. In the case of a 2-filtered manifold, $H \subset TM$ we have a rank-k subbundle $V \hookrightarrow \Lambda^1$, (which we call L in the case of a contact manifold) of 1-forms which annihilate H. The problem in defining the torsion of an affine partial connection is that there is in general no natural differential operator $\Lambda^1_H \to \Lambda^2_H$ with the same symbol as $\wedge \circ \nabla$ that we can use to construct a tensorial quantity. However, we have the (generalised) Levi map $\mathcal{L} : V \to \Lambda^2_H$ defined in exactly the same way as in the contact case. That is,

$$V \ni \alpha \mapsto d\alpha|_H. \tag{2.1.15}$$

To avoid major complications, we will only consider 2-filtered manifolds such that the Levi map $\mathcal{L} : V \to \Lambda^2_H$ is of constant rank. We get a canonical differential operator $d_\perp : \Lambda^1_H \to \Lambda^2_H/\mathcal{L}(V) =: \Lambda^2_{H\perp}$ defined exactly analogously to, and generalising the Rumin operator. Furthermore, $d_\perp \circ d_\perp : \mathbb{R} \to \Lambda^2_{H\perp}$ clearly vanishes, just as in the contact case, although local exactness doesn't follow in general. The result of applying the partial connection, skew-symmetrising, and then projection onto the quotient, is a differential operator with the same symbol as $d_\perp$ and hence the difference between these first order differential operators defines a vector bundle homomorphism $\Lambda^1_H \to \Lambda^2_{H\perp}$.

Remark 2.1.16. *The assumption that $\mathcal{L}$ is of constant rank is called sometimes regularity and is true, for example, for contact structures, quaternionic contact structures, or when V admits local trivialisation by closed 1-forms, in which case the rank of $\mathcal{L}$ is zero. In fact it is true for any filtered manifold with isomorphic symbol algebras at each point (such structures were the subject of a programme of study by Tanaka, and following him, Morimoto [Mor93]). This is easily seen if we note the assumption is equivalent to $[H, H]$ being of constant rank.*

Definition 2.1.17 (Partial torsion)**.** *Let M be a 2-filtered manifold with subbundle $H \subset M$. The partial torsion of a partial connection ∇ on Λ^1_H is the vector bundle homomorphism $\Lambda^1_H \to \Lambda^2_{H\perp}$ defined as $d_\perp - \wedge_\perp \circ \nabla$, where $\wedge_\perp$ is the projection $\Lambda^1_H \otimes \Lambda^1_H \to \Lambda^2_{H\perp}$.*

For a general partial connection we also wish to recover some notion of curvature. For this define a differential operator $d^\nabla_\perp : \Lambda^1_H \otimes E \to \Lambda^2_{H\perp} \otimes E$, a 'twisted' version of the Rumin operator, analogous to the exterior covariant derivative, by

$$d^\nabla_\perp(\omega \otimes s) = d_\perp\omega \otimes s - \wedge_\perp(\omega \otimes \nabla s), \tag{2.1.18}$$

on simple tensors, extending by linearity, where by $\wedge_\perp$ we really mean the induced homomorphism $\Lambda^1_H \otimes \Lambda^1_H \otimes E \to \Lambda^2_{H\perp} \otimes E$ with the identity in the E factor. Alternatively, given $e \in \Lambda^1_H \otimes E$, this operator can be defined as follows: pick a full connection $\tilde{\nabla}$ extending ∇ and then define $d^\nabla_\perp e \in \Lambda^2_{H\perp} \otimes E$ as the image of $d^{\tilde{\nabla}}\tilde{e} \in \Lambda^2 \otimes E$ under the usual projection $\Lambda^2 \otimes E \to \Lambda^2_{H\perp} \otimes E$ where $\tilde{e} \in \Lambda^1 \otimes E$, is some lift of e.

The Leibniz rule for multiplication by smooth functions can be checked in the case of simple tensors, but then holds in general by linearity.

$$d^\nabla_\perp(f\omega \otimes s) = d_\perp f \wedge \omega \otimes s + f d_\perp\omega \otimes s - f \wedge_\perp (\omega \otimes \nabla s) \tag{2.1.19}$$

$$= d_\perp f \wedge \omega \otimes s + f d^\nabla_\perp(\omega \otimes s). \tag{2.1.20}$$

Thus the composition $d^\nabla_\perp \circ \nabla$ is tensorial.

$$(d^\nabla_\perp \circ \nabla)(fs) = d^\nabla_\perp(d_\perp f \otimes s + f\nabla s) = f(d^\nabla_\perp \circ \nabla)(s). \tag{2.1.21}$$

Definition 2.1.22 (Partial curvature)**.** *Let M be a 2-filtered manifold with subbundle $H \subset M$. The partial curvature of a partial connection ∇ on a vector bundle $E \to M$ is the vector bundle homomorphism $\kappa : E \to \Lambda^2_{H\perp} \otimes E$ defined by $\kappa := d^\nabla_\perp \circ \nabla$.*

2.2 Partial connections on contact manifolds

We now specialise to the case of contact manifolds. In the sequel, let M be a contact manifold with contact distribution $H \subset M$. As mentioned previously we have a natural identification of $\Lambda^2_{H\perp}$ as those elements of Λ^2_H which are trace-free with respect to a non-vanishing section J^{ab} of $\mathcal{L}(L)$ with projection 1.1.11.

We will give equivalent definitions of the partial torsion and partial curvature in terms of indices.

Definition 2.2.1 (Partial torsion)**.** *Given a partial connection $\nabla : \Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ the partial torsion is the vector bundle homomorphism $\Lambda^1_H \to \Lambda^2_{H\perp}$ defined by*

$$\tau_{ab}{}^c\omega_c = (d_\perp\omega)_{ab} - (\nabla_{[a}\omega_{b]})_\perp. \tag{2.2.2}$$

Note that by using the freedom as per 2.1.13, we can always find a partial torsion free connection on Λ^1_H. Then the Rumin operator $d_\perp : \Lambda^1_H \to \Lambda^2_{H\perp}$ is just

$$\omega_a \mapsto (\nabla_{[a}\omega_{b]})_\perp = \nabla_{[a}\omega_{b]} - \frac{1}{2n}J_{ab}\nabla_c\omega^c. \tag{2.2.3}$$

Next, given any partial connection $\nabla_a : \Lambda^1_H \to \Lambda^1_H \otimes E$, we want to calculate the twisted Rumin operator $d^\nabla_\perp : \Lambda^1_H \otimes E \to \Lambda^2_{H\perp} \otimes E$. To do this we need to pick an auxiliary partial torsion free partial connection (which we will also denote ∇_a) on Λ^1_H. Then on simple tensors we can translate the definition 2.1.18 into indices:

$$\omega_a s^\mu \mapsto \nabla_{[a}\omega_{b]}s^\mu - \frac{1}{2n}J_{ab}\nabla_c\omega^c s^\mu - \omega_{[a}\nabla_{b]}s^\mu + \frac{1}{2n}J_{ab}\omega_c\nabla^c s^\mu. \tag{2.2.4}$$

So in general it is

$$e_a{}^\mu \mapsto \nabla_{[a}e_{b]}{}^\mu - \frac{1}{2n}J_{ab}\nabla_c e^{c\mu}. \tag{2.2.5}$$

In particular

$$(d^\nabla_\perp\nabla s)_{ab}{}^\mu = (\nabla_{[a}\nabla_{b]})_\perp s^\mu. \tag{2.2.6}$$

In light of this we restate the definition of partial curvature.

Definition 2.2.7 (Partial curvature)**.** *Given a partial connection $\nabla : E \to \Lambda^1_H \otimes E$ for a vector bundle $E \to M$ define the partial curvature $R_{ab}{}^\mu{}_\nu \in \Lambda^2_{H\perp} \otimes \mathrm{End}(E)$ by*

$$R_{ab}{}^\mu{}_\nu s^\nu := (\nabla_{[a}\nabla_{b]})_\perp s^\mu - \tau_{ab}{}^c\nabla_c, \phi^\mu \tag{2.2.8}$$

where, in order for the above expression to make sense, we need to choose some auxiliary partial connection $\Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ with partial torsion $\tau_{ab}{}^c$.

We need to check this is independent of the auxiliary partial connection. Given another partial connection $\hat{\nabla}_a$ on Λ^1_H with partial torsion $\hat{\tau}_{ab}{}^c$, and labelling the induced connection $\Lambda^1_H \otimes E \to \Lambda^1_H \otimes \Lambda^1_H \otimes E$ by $\hat{\nabla}_a$ we have

$$(\hat{\nabla}_{[a}\nabla_{b]})_\perp\phi_\nu - \hat{\tau}_{ab}{}^c\nabla_c = (\nabla_{[a}\nabla_{b]})_\perp\phi_\nu + \Gamma_{[ab]}{}^c\nabla_c\mu_n - \hat{\tau}_{ab}{}^c\nabla_c \tag{2.2.9}$$

where

$$\hat{\nabla}_a \omega_b = \nabla_a \omega_b + \Gamma_{ab}{}^c \omega_c, \tag{2.2.10}$$

but then $\Gamma_{[ab]}{}^c$ is just the difference in the partial torsions of the two partial connections on Λ^1_H and so the above definition is independent of this choice. Picking a partial connection with vanishing partial torsion and using 2.2.6 this definition is then evidently equivalent to 2.1.22. The reason we do not assume the auxiliary partial connection in 2.2.7 has vanishing partial torsion is that in some cases we may be obliged to use a partial connection on Λ^1_H with non-vanishing partial torsion.

Example 2.2.11. *In a Darboux coordinate patch* (t, p_i, q^i) *we have the canonical partial connection* $\nabla : \Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ *defined by*

$$\nabla(\omega^i d_\perp p_i + \omega_i d_\perp q^i) = d_\perp \omega^i \otimes d_\perp p_i + d_\perp \omega_i \otimes d_\perp q^i. \tag{2.2.12}$$

We claim this has vanishing partial torsion and partial curvature.

Proof. The Rumin operator satisfies

$$d_\perp(\omega^i d_\perp p_i + \omega_i d_\perp q^i) = \wedge_\perp(d_\perp \omega^i \otimes d_\perp p_i + d_\perp \omega_i \otimes d_\perp q^i) \tag{2.2.13}$$

so this connection has vanishing partial torsion. Then from the definition of the 'twisted' Rumin operator 2.1.18

$$d^\nabla_\perp \circ \nabla(\omega^i d_\perp p_i + \omega_i d_\perp q^i) = d^\nabla_\perp(d_\perp \omega^i \otimes d_\perp p_i + d_\perp \omega_i \otimes d_\perp q^i) \tag{2.2.14}$$

$$= - \wedge_\perp(d_\perp \omega^i \otimes \nabla d_\perp p_i + d_\perp \omega_i \otimes \nabla d_\perp q^i) = 0 \tag{2.2.15}$$

so the connection has vanishing partial curvature. □

Given a partial connection, an obvious question is whether there is a canonical representative connection in the equivalence class of connections that restrict to the given partial connection. It turns out that in the contact case the question can be answered in the affirmative, as explained below, the proof from Eastwood and Gover [EG11] Proposition 3.5. Since we will be dealing with both connections and partial connections together, we will reserve the symbol ∇ for a partial connection for the rest of the section, and we will use $\tilde{\nabla}$ will denote a full connection.

Theorem 2.2.16 ([EG11])**.** *Let* $\nabla : E \to \Lambda^1_H \otimes E$ *be a partial connection with respect to a contact distribution* $H \hookrightarrow M$*. Then there exists a unique connection* $\tilde{\nabla} : E \to \Lambda^1 \otimes E$ *such that* ∇ *is the composition* $E \to \Lambda^1 \otimes E \to \Lambda^1_H \otimes E$ *and the composition*

$$E \xrightarrow{\tilde{\kappa}} \Lambda^2 \otimes E \to \mathcal{L}(L) \otimes E \tag{2.2.17}$$

vanishes, where $\tilde{\kappa}$ *is the curvature of* $\tilde{\nabla}$*, and* $\Lambda^2 \otimes E \to \mathcal{L}(L) \otimes E$ *is the canonical projection (the trace).*

Proof. Firstly, there is a $\tilde{\nabla}$ with ∇ being the composition $E \xrightarrow{\tilde{\nabla}} \Lambda^1 \otimes E \to \Lambda^1_H \otimes E$. To see this, pick a connection $\tilde{\nabla}$ on E. Recall that the freedom for picking a new connection is

$$s \mapsto \tilde{\nabla}s + \Gamma(s) \tag{2.2.18}$$

for an arbitrary $\Gamma \in \Lambda^1 \otimes \mathrm{End}(E)$. Write $\Gamma|_H$ for the image of Γ under the projection to $\Lambda^1_H \otimes \mathrm{End}(E)$. The difference between ∇ and the restriction of $\tilde{\nabla}$ to H is a homomorphism $E \to \Lambda^1_H \otimes E$ and we can simply set $\Gamma|_H$ to this homomorphism to ensure that the new connection extends ∇. The remaining freedom lies in the transverse subbundle $L \otimes \mathrm{End}(E) \hookrightarrow \Lambda^1 \otimes \mathrm{End}(E)$, so we need to inspect how such a change effects the curvature. We calculate the curvature as the composition $\tilde{\kappa} := d^{\tilde{\nabla}} \circ \tilde{\nabla} : E \to \Lambda^2 \otimes E$. Given a change of connection by $\Gamma \in \Lambda^1 \otimes \mathrm{End}(E)$, the new exterior covariant derivative $\Lambda^1 \otimes E \to \Lambda^2 \otimes E$ is

$$d^{\tilde{\nabla}} - \mathrm{Id} \wedge \Gamma \tag{2.2.19}$$

where Id is taken to be the identity of 1-forms. Accordingly, the curvature of the new connection is

$$\begin{aligned} s \mapsto\ & (d^{\tilde{\nabla}} - \mathrm{Id} \wedge \Gamma)(\tilde{\nabla}s + \Gamma(s)) \\ &= \tilde{\kappa}(s) - \Gamma(\tilde{\nabla}s) + (d^{\tilde{\nabla}}\Gamma)(s) + \Gamma(\tilde{\nabla}s) - (\Gamma \wedge \Gamma)(s) \\ &= \tilde{\kappa}(s) + (d^{\tilde{\nabla}}\Gamma)(s) - (\Gamma \wedge \Gamma)(s). \end{aligned} \tag{2.2.20}$$

Now considering Γ as a homomorphism $E \to \Lambda^1 \otimes E$, if we assume the range is in $L \otimes E$ then it follows that $\Gamma \wedge \Gamma = 0$ and thus the curvature κ of the new connection is just

$$s \mapsto \tilde{\kappa}(s) + (d^{\tilde{\nabla}}\Gamma)(s). \tag{2.2.21}$$

Now since $\Gamma \in L \otimes \mathrm{End}(E)$ we can write $\Gamma = \alpha \otimes A$ for some $\alpha \in L$ and some $A \in \mathrm{End}(E)$. Then $d^{\tilde{\nabla}}\Gamma = d\alpha \otimes A - \alpha \wedge \tilde{\nabla}A$ and $(d^{\tilde{\nabla}}\Gamma)|_H = d\alpha|_H \otimes A$. In particular the part of the new curvature in $\Lambda^2_H \otimes \mathrm{End}(E)$ is

$$s \mapsto \tilde{\kappa}|_H(s) + d\alpha|_H \otimes A(s). \tag{2.2.22}$$

Now $\mathcal{L}(L) \otimes \mathrm{End}(E) \hookrightarrow \Lambda^2_H \otimes \mathrm{End}(E)$ is spanned by elements of the form $d\alpha|_H \otimes A$, and so there is a unique $\Gamma = \alpha \otimes A \in L \otimes \mathrm{End}(E)$ such that the component of the curvature of $\tilde{\nabla} + \Gamma$ in $\mathcal{L}(L) \otimes \mathrm{End}(E)$ vanishes, which completes the proof. $\square$

Note that the columns of the following diagram are exact, where we are taking the maps to be induced by the short exact sequences 1.1.32, taking the identity in the E factor.

$$\begin{tikzcd}
 & \Lambda^1_H \otimes E & \Lambda^2_H \otimes E & \Lambda^3_H \otimes E & \\
E \arrow[r, "\tilde{\nabla}"] \arrow[ur] & \Lambda^1 \otimes E \arrow[u] \arrow[r, "d^{\tilde{\nabla}}"] & \Lambda^2 \otimes E \arrow[u] \arrow[r, "d^{\tilde{\nabla}}"] & \Lambda^3 \otimes E \arrow[u] \arrow[r] & \dots \\
 & L \otimes E \arrow[u] & L \otimes \Lambda^1_H \otimes E \arrow[u] & L \otimes \Lambda^2_H \otimes E \arrow[u] &
\end{tikzcd}$$

Also note that for any connection $\tilde{\nabla}$ extending ∇, the partial curvature $\kappa = d^{\nabla}_{\perp} \circ \nabla$ is the composition of $d^{\tilde{\nabla}} \circ \tilde{\nabla}$ followed by the projection $\Lambda^2 \otimes E \to \Lambda^2_{H\perp} \otimes E$. Since $\Lambda^2_H = \Lambda^2_{H\perp} \oplus \mathcal{L}$, the unique extension is characterised by the fact $\tilde{\kappa}|_H = \kappa$.

Lemma 2.2.23. *Let $\tilde{\nabla}$ be the canonical representative of a partial connection ∇ as constructed above. Then for a section $s \in E$ we have $\nabla s = 0$ if and only if $\tilde{\nabla} s = 0$.*

Proof. If $\tilde{\nabla} s = 0$ then clearly $\nabla s = 0$. By exactness of $L \otimes E \to \Lambda^1 \otimes E \to \Lambda^1_H \otimes E$, if $\nabla s = 0$, then $\tilde{\nabla} s$ is in the image of $L \otimes E \to \Lambda^1 \otimes E$ so we can write $\tilde{\nabla} s = \alpha \otimes e$ for some non-vanishing $\alpha \in L$ and $e \in E$. Now

$$\tilde{\kappa}(s) = d^{\tilde{\nabla}}(\alpha \otimes e) = d\alpha \otimes e - \alpha \wedge \tilde{\nabla} e \tag{2.2.24}$$

but the component in $\mathcal{L}(L) \otimes E$ must vanish by the theorem 2.2.16 which is to say $d\alpha|_H \otimes e = 0$ which implies $e = 0$. □

The next obvious question to ask is if the curvature of the lifted connection $\tilde{\nabla}$ vanishes if the partial curvature of ∇ vanishes. This turns out to be true by a similar argument. First, note that since the partial curvature is the composition $E \xrightarrow{\tilde{\kappa}} \Lambda^2 \otimes E \to \Lambda^2_{H\perp} \otimes E$, the theorem 2.2.16 ensures that we can eliminate the component of the curvature in $\Lambda^2_H \otimes \mathrm{End}(E)$, given vanishing partial curvature.

Theorem 2.2.25. *Let $\nabla : E \to \Lambda^1_H \otimes E$ be a partial connection with respect to a contact distribution $H \hookrightarrow M$ on a contact manifold (M, H). Suppose that the partial curvature of ∇ vanishes. Then there is a unique flat connection $\tilde{\nabla}$ such that ∇ is the composition $E \to \Lambda^1 \otimes E \to \Lambda^1_H \otimes E$.*

Proof. Let $\tilde{\kappa} : E \to \Lambda^2 \otimes E$ be the curvature of the unique $\tilde{\nabla}$ from 2.2.16 extending ∇, such that the composition $E \to \Lambda^2 \otimes E \to \Lambda^2_H \otimes E$ vanishes. Let r be the rank of E. By exactness of $L \otimes \Lambda^1_H \otimes E \to \Lambda^2 \otimes E \to \Lambda^2_H \otimes E$, we can write (over a suitable trivialising neighbourhood)

$$\tilde{\kappa}(s) = \alpha \wedge \tilde{\rho}^i \otimes e_i \tag{2.2.26}$$

for $s \in E$ where $\{e_i\}_{i=1}^r$ is some local trivialisation of E, α is a chosen contact form and $\tilde{\rho}^i \in \Lambda^1$ are lifts of some $\rho^i \in \Lambda^1_H$. We are summing over repeated indices. Next we apply the exterior covariant derivative $d^{\tilde{\nabla}} : \Lambda^2 \otimes E \to \Lambda^3 \otimes E$. The left-hand side is $d^{\tilde{\nabla}}(\tilde{\kappa}(s)) = (d^{\tilde{\nabla}}\tilde{\kappa})(s) + \tilde{\kappa}(\tilde{\nabla}(s))$ but the first term vanishes due to the Bianchi identity. Hence

$$\tilde{\kappa}(\tilde{\nabla} s) = d\alpha \wedge \tilde{\rho}^i \otimes e_i - \alpha \wedge d\tilde{\rho}^i \otimes e_i + \alpha \wedge \tilde{\rho}^i \wedge \tilde{\nabla} e_i \tag{2.2.27}$$

Note that the composition $\Lambda^1 \otimes E \xrightarrow{\tilde{\kappa}} \Lambda^3 \otimes E \to \Lambda^3_H \otimes E$ vanishes since the induced map $\tilde{\kappa} : \Lambda^1 \otimes E \to \Lambda^3 \otimes E$ is just $\mathrm{Id} \wedge \tilde{\kappa}$ and $\tilde{\kappa}|_H$ vanishes. Accordingly, projecting $\Lambda^3 \otimes E \to \Lambda^3_H \otimes E$, the left-hand side of 2.2.27 vanishes, and furthermore the right-hand side simplifies since $\alpha|_H = 0$. This means (by the linear independence of $\{e_i\}_{i=1}^r$), that

$$d\alpha|_H \wedge \rho^i = 0 \quad \forall\ i = 1, .., r \tag{2.2.28}$$

but non-degeneracy of $d\alpha|_H$ assures us that $\rho^i = 0$, since $2n + 1 \geq 5$. So $\tilde{\kappa}(s) = 0$ and since s was arbitrary, the curvature vanishes as claimed. □

Remark 2.2.29. *Note the failure of 2.2.25 in the case of a 3-dimensional contact manifold despite 2.2.16 and 2.2.23 still following. This is another case of 3-dimensional contact geometry being a degenerate case. In this case the partial curvature is always 0 since $\mathcal{L}(L) = \Lambda^2_H$ and accordingly any partial connection on a 3-dimensional contact manifold promotes to a full connection that is flat in the contact directions, that is, with curvature contained in $L \otimes \Lambda^1_H \otimes \mathrm{End}(E)$.*

Remark 2.2.30. *It is interesting to consider how much can be recovered if H is not a contact distribution. If H is merely non-integrable codimension-1 distribution, and accordingly, $d\alpha|_H$ is non-vanishing then 2.2.16 and 2.2.23 still follow. However 2.2.25 relies on the non-degeneracy of $d\alpha|_H$ which is often called the maximal non-integrability condition. In the extreme case of an integrable distribution, $\mathcal{L}$ is the zero map, and so 2.2.16 does not follow and a partial connection on E with respect to such a distribution gives rise to full connections on the bundle E pulled back over integral submanifolds.*

An important result for full connections is the following. The result follows from interpreting a linear connection as a splitting of TE into horizontal and vertical subbundles as explained later in section 4.1, and showing that the horizontal bundle is integrable precisely if the curvature vanishes. See [Lee09] Theorem 12.25.

Theorem 2.2.31 (Frobenius' theorem for flat connections)**.** *Let $\tilde{\nabla} : E \to \Lambda^1 \otimes E$ be a flat connection on a rank-r vector bundle $E \to M$. Then for any simply connected open set $U \subset M$ there exists r everywhere linearly independent sections $\{e_i\}_{i=1}^r$ of E such that $\tilde{\nabla} e_i = 0 \ \forall i = 1, ..., r$.*

Putting this together with 2.2.25, we get an important corollary.

Corollary 2.2.32. *Let $\nabla : E \to \Lambda^1_H \otimes E$ be a partial connection on a vector bundle $E \to M$ with respect to a contact distribution $H \hookrightarrow M$ on a contact manifold (M, H) of dimension $2n + 1$. Suppose that the partial curvature of ∇ vanishes, then for any simply connected open set $U \subset M$ there exists r everywhere linearly independent sections $\{e_i\}_{i=1}^r$ of E such that $\nabla e_i = 0 \ \forall i = 1, ..., r$.*

2.3 Bianchi symmetry of partial connections

Let M be a smooth manifold with torsion-free affine connection $\nabla_a : \Lambda^1 \to \Lambda^1 \otimes \Lambda^1$, and let f be a smooth function. It is simple to check that

$$\begin{aligned} &\nabla_a \nabla_{[b} \nabla_{c]} f + \nabla_c \nabla_{[a} \nabla_{b]} f + \nabla_b \nabla_{[c} \nabla_{a]} f \\ &= \nabla_{[a} \nabla_{b]} \nabla_c f + \nabla_{[c} \nabla_{a]} \nabla_b f + \nabla_{[b} \nabla_{c]} \nabla_a f. \end{aligned} \tag{2.3.1}$$

Then, writing $R_{abc}{}^d$ for the curvature tensor and noting that the first line vanishes due to vanishing torsion we have

$$\begin{aligned} &R_{[ab]c}{}^d \nabla_d f + R_{[ca]b}{}^d \nabla_d f + R_{[bc]a}{}^d \nabla_d f = 0 \\ \implies &R_{[abc]}{}^d \nabla_d f = 0. \end{aligned} \tag{2.3.2}$$

By picking an appropriate f, $\nabla_d f$ can take any value we wish at a point on M and so we conclude $R_{[abc]}{}^d = 0$. This is called the Bianchi symmetry, or sometimes the algebraic Bianchi identity.

Unfortunately, letting $\nabla_a : \Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ be a partial connection with respect to a contact distribution H with vanishing partial torsion, no such slick proof of a similar symmetry for the partial connection seems to be available. It turns out however, that if we restrict our consideration to partial connections which are somehow compatible with the contact structure, something analogous can be recovered.

Definition 2.3.3 (Contact partial connection)**.** *Let $H \subset TM$ be a contact distribution and $\alpha \in L$. Then we say a partial connection $\nabla : \Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ is a contact partial connection (associated with α) if $\nabla d\alpha|_H = 0$.*

Recall that the Levi-Civita connection on a Riemannian manifold (M, g) is a torsion-free connection with $\nabla g = 0$. Just like the Riemann curvature tensor, the partial curvature of a contact partial connection with vanishing partial torsion has an obvious symmetry of its endomorphism indices.

Proposition 2.3.4. *Let ∇_a be a contact partial connection with vanishing partial torsion and partial curvature $R_{abc}{}^d$ then $R_{abcd} = R_{ab(cd)}$.*

Proof. Write J_{ab} for the non-degenerate skew-form $d\alpha|_H$, which we will use to raise and lower indices. Since ∇_a is contact

$$\begin{aligned} &\nabla_{[a}\nabla_{b]}J_{cd} - \frac{1}{2n}J_{ab}\nabla_e\nabla^e J_{cd} = 0. \\ \implies &R_{abc}{}^e J_{ed} + R_{abd}{}^e J_{ce} = 0 \\ \implies &R_{ab[cd]} = 0 \end{aligned} \tag{2.3.5}$$

Unlike the Levi-Civita connection a contact partial connection with vanishing torsion is not unique. We will exploit this freedom to prove an analogue of the Bianchi symmetry.

Proposition 2.3.6 (Bianchi symmetry of contact connections). *Let $\nabla_a : \Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ be a contact partial connection on a contact manifold of dimension $2n+1$, with vanishing partial torsion, then*

$$R_{[abc]}{}^d = J_{[ab}\Xi_{c]}{}^d \tag{2.3.7}$$

for some $\Xi_{cd} \in \odot^2\Lambda^1_H$ and hence

$$R_{d[ab]}{}^d = 0. \tag{2.3.8}$$

Proof. Suppose that $\hat{\nabla}_a$ and ∇_a are partial connections on Λ^1_H. Then

$$\hat{\nabla}_a\phi_b = \nabla_a\phi_b + \Gamma_{ab}{}^c\phi_c \tag{2.3.9}$$

for some $\Gamma_{ab}{}^c \in \Lambda^1_H \otimes \mathrm{End}(\Lambda^1_H)$. Now suppose that $\hat{\nabla}_a$ and ∇_a both satisfy the hypotheses of the proposition. The skew form J_{ab} associated with the choice of contact form must be parallel for both connections. Accordingly,

$$\Gamma_{ab}{}^d J_{dc} + \Gamma_{ac}{}^d J_{bd} = 0 \implies \Gamma_{abc} - \Gamma_{acb} = 0. \tag{2.3.10}$$

Then the torsion free condition gives

$$\Gamma_{[ab]c} - \frac{1}{2n}J_{ab}\Gamma_d{}^d{}_c = 0. \tag{2.3.11}$$

Next, antisymmetrising gives

$$J_{[ab}\Gamma_{|d|}{}^{d}{}_{c]} = 0, \tag{2.3.12}$$

but then, since $2n+1 \geq 5$

$$\Gamma_{d}{}^{d}{}_{c} = 0. \tag{2.3.13}$$

Putting it all together we see that Γ_{abc} is symmetric on its first two and last two indices and so $\Gamma_{abc} = \Gamma_{(abc)}$ is the difference between two connections satisfying the hypotheses. Now we investigate the change in partial curvature. One gets

$$\hat{\nabla}_a\hat{\nabla}_b\phi_c = \nabla_a\nabla_b\phi_c + \nabla_a(\Gamma_{bc}{}^{d}\phi_d) + 2\Gamma_{a(b|}{}^{d}\nabla_d\phi_{|c)} + 2\Gamma_{a(b|}{}^{d}\Gamma_{d|c)}{}^{e}\phi_e. \tag{2.3.14}$$

We antisymmetrise to get

$$\begin{aligned}\hat{\nabla}_{[a}\hat{\nabla}_{b]}\phi_c &= \nabla_{[a}\nabla_{b]}\phi_c + \nabla_{[a}(\Gamma_{b]c}{}^{d}\phi_d) + \Gamma_{c[a}{}^{d}\nabla_{b]}\phi_d + \Gamma_{c[a}{}^{d}\Gamma_{b]d}{}^{e}\phi_e \\ &= \nabla_{[a}\nabla_{b]}\phi_c + \nabla_{[a}\Gamma_{b]c}{}^{d}\phi_d + \Gamma_{c[a}{}^{d}\Gamma_{b]d}{}^{e}\phi_e.\end{aligned} \tag{2.3.15}$$

Taking the trace free part gives the partial curvature.

$$\begin{aligned}\hat{R}_{abcd} = R_{abcd} + \nabla_{[a}\Gamma_{b]cd} - \frac{1}{2n}J_{ab}\nabla_e\Gamma_{cd}{}^{e} \\ + \Gamma_{c[a}{}^{e}\Gamma_{b]ed} - \frac{1}{2n}J_{ab}\Gamma_{cf}{}^{e}\Gamma_{ed}{}^{f}.\end{aligned} \tag{2.3.16}$$

Antisymmetrising gives

$$\hat{R}_{[abc]d} = R_{[abc]d} - \frac{1}{2n}J_{[ab|}\nabla_e\Gamma_{|c]d}{}^{e} - \frac{1}{2n}J_{[ab}\Gamma_{c]f}{}^{e}\Gamma_{ed}{}^{f}. \tag{2.3.17}$$

Now if $R_{[abc]}{}^{d} = J_{[ab}\Xi_{c]}{}^{d}$ for some $\Xi_{cd} \in \odot^2\Lambda^1_H$ for one connection satisfying the hypotheses, the above implies that the curvature takes such a form for all such connections, and this argument can be made locally. In particular, over any Darboux coordinate neighbourhood, we can take the canonical flat connection, which satisfies the hypotheses, and has curvature which takes the form $J_{[ab}\Xi_{c]}{}^{d}$ (of course, with $\Xi_{cd} = 0$).

In the process we have also shown

Proposition 2.3.18. *Let $\nabla_a : \Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ and $\hat{\nabla}_a : \Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ be contact partial connections on a contact manifold of dimension $2n+1$, each with vanishing partial torsion, then*

$$\hat{\nabla}_a\phi_b = \nabla_a\phi_b + \Gamma_{ab}{}^{c}\phi_c \tag{2.3.19}$$

for all sections ϕ_b of Λ^1_H for some $\Gamma_{abc} = \Gamma_{(abc)}$.

Remark 2.3.20. *That the space of contact partial connections with vanishing partial torsion is an affine space diffeomorphic to the space of symmetric tensors* $\odot^3\Lambda^1_H$ *is exactly analogous to a well known fact from symplectic geometry regarding the space of symplectic connections, which are torsion free connections on a symplectic manifold for which the symplectic form is parallel. See for instance [BCG+06].*

2.4 The Rumin complex in terms of a partial connection

On a smooth manifold M, the de Rham complex can be written in terms of a torsion-free affine connection ∇_{a_0}. The exterior derivative $d : \Lambda^k \to \Lambda^{k+1}$ is

$$\omega_{a_1\ldots a_k} \mapsto \nabla_{[a_0}\omega_{a_1\ldots a_k]}. \tag{2.4.1}$$

We would like to have similar formulae for the operators of the Rumin complex. The motivation is that a contact manifold endowed with additional structure may have a distinguished, or class of distinguished partial connections with which one would want to perform differential calculus. These formulae will then allow one to write down the machinery intrinsic to the contact manifold in terms of these partial connections.

Firstly however, we need to write down some differential operators that arise in the presence of a chosen contact form. These are of course, not intrinsic to a contact manifold, but will serve as stepping stones along the way to our goal.

Given a choice of contact form $\alpha \in L$ we get an obvious differential operator $\Lambda^1_H \to \Lambda^2_H$ given by

$$\omega \mapsto d\tilde{\omega}|_H \tag{2.4.2}$$

where the lift $\tilde{\omega}$ is uniquely determined by $\tilde{\omega}(T) = 0$ and $\tilde{\omega}|_H = \omega$, where T is the Reeb field associated with α. We have, for any contact partial connection with vanishing partial torsion, locally

$$\nabla_a\omega_b = D_a\omega_b + \Gamma_{ab}{}^c\omega_c, \tag{2.4.3}$$

where D_a is the canonical flat connection 2.2.11 associated with a Darboux coordinate patch, for some tensor $\Gamma_{(abc)} = \Gamma_{abc}$. In particular

$$\nabla_{[a}\omega_{b]} = D_{[a}\omega_{b]} \tag{2.4.4}$$

but

$$D_{[a}\omega_{b]} = (d\tilde{\omega}|_H)_{ab} \tag{2.4.5}$$

as can be easily verified: For $\omega = \omega^i d_\perp p_i + \omega_j d_\perp q^j$ we have $\tilde{\omega} = \omega^i dp_i + \omega_i dq^i$. Then

$$\begin{aligned} D\omega &= d_\perp \omega^i \otimes d_\perp p_i + d_\perp \omega_i \otimes d_\perp q^i \\ \implies (\wedge \circ D)(\omega) &= d_\perp \omega^i \wedge d_\perp p_i + d_\perp \omega_i \wedge d_\perp q^i = d\tilde{\omega}|_H. \end{aligned} \tag{2.4.6}$$

What about $d\tilde{\omega}$ where $\omega \in \Lambda^k_H$ for $k > 1$? We can use the same strategy. It is easy to verify that locally

$$(d\tilde{\omega}|_H)_{a_1 \dots a_{k+1}} = D_{[a_1} \omega_{a_2 \dots a_{k+1}]} \tag{2.4.7}$$

and

$$\nabla_{a_1} \omega_{a_2 \dots a_{k+1}} = D_{a_1} \omega_{a_2 \dots a_{k+1}} + k \Gamma_{a_1 [a_2|}{}^c \omega_{c | a_3 \dots a_{k+1}]} \tag{2.4.8}$$

and so we have shown:

Proposition 2.4.9. *A contact partial connection ∇_a on a contact manifold of dimension $2n+1$ with vanishing partial torsion satisfies*

$$\nabla_{[a_1} \omega_{a_2 \dots a_{k+1}]} = (d\tilde{\omega}|_H)_{a_1 \dots a_{k+1}} \tag{2.4.10}$$

for $k \geq 0$ and where $\tilde{\omega}$ is the lift of ω uniquely determined by $\tilde{\omega}(T) = 0$ and $\tilde{\omega}|_H = \omega$.

Next, it will turn out to be important to examine the differential operator $\Lambda^k_H \to \mathbb{R}$ given by:

$$f \mapsto \nabla_c \nabla^c f, \tag{2.4.11}$$

where again ∇_a is a contact partial connection with vanishing partial torsion. A simple calculation shows

$$\nabla_c \nabla^c fg = f \nabla_c \nabla^c g + f \nabla_c \nabla^c g. \tag{2.4.12}$$

That is, this operator is actually a first order differential operator. Now 2.3.18 immediately implies that $\nabla_c \nabla^c f = D_c D^c f$, so we can simply compute this in Darboux coordinates.

$$Df = \frac{\partial f}{\partial p_j} d_\perp p_j + \Big(\frac{\partial f}{\partial q^j} + p_j \frac{\partial f}{\partial t} \Big) d_\perp q^j \tag{2.4.13}$$

$$\begin{aligned} (D \circ D) f = &\frac{\partial^2 f}{\partial p_i \partial p_j} d_\perp p_i \otimes d_\perp p_j + \Big(\frac{\partial^2 f}{\partial p_i \partial q^j} + p_j \frac{\partial^2 f}{\partial p_i \partial t} \Big) d_\perp p_i \otimes p_\perp q^j \\ &+ \frac{\partial f}{\partial t} d_\perp p_i \otimes d_\perp q^i + \Big(\frac{\partial^2 f}{\partial q^i \partial p_j} + p_i \frac{\partial^2 f}{\partial t \partial p_j} \Big) d_\perp q^i \otimes d_\perp p_j \\ &+ \Big(\frac{\partial^2 f}{\partial q_i \partial q^j} + p_j \frac{\partial^2 f}{\partial q_i \partial t} \Big) d_\perp q_i \otimes d_\perp q^j \end{aligned} \tag{2.4.14}$$

$$(\wedge \circ D \circ D) f = \frac{\partial f}{\partial t} d_\perp p_i \wedge d_\perp q^i + p_j \frac{\partial^2 f}{\partial q_i \partial t} d_\perp q_i \wedge d_\perp q^j. \tag{2.4.15}$$

Recalling that $d\alpha|_H = d_\perp q^i \wedge d_\perp p_i$ we then have the following:

Proposition 2.4.16. *A contact partial connection ∇_a on a contact manifold of dimension $2n+1$ with vanishing partial torsion satisfies*

$$\nabla_c \nabla^c f = -2nT(f). \tag{2.4.17}$$

Another formula we need before embarking is the following:

Proposition 2.4.18. *A contact partial connection ∇_a on a contact manifold of dimension $2n+1$ with vanishing partial torsion satisfies*

$$[T, X]^a = -\frac{1}{2n}\nabla_c \nabla^c X^a + \frac{1}{n-1} R^a{}_{cb}{}^c X^b. \tag{2.4.19}$$

Proof. First examine the case for the canonical connection in a Darboux neighbourhood where we have $T = \frac{\partial}{\partial t}$. For $X \in H$ write (locally) $X = X^i \frac{\partial}{\partial p_i} + X_i(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t})$ so that

$$\begin{aligned}
[T, X] &= \frac{\partial X^i}{\partial t}\frac{\partial}{\partial p_i} + \frac{\partial X_i}{\partial t}\Big(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\Big) + X^i \Big[\frac{\partial}{\partial t}, \frac{\partial}{\partial p_i}\Big] + X^i \Big[\frac{\partial}{\partial t}, \frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\Big] \\
&= \frac{\partial X^i}{\partial t}\frac{\partial}{\partial p_i} + \frac{\partial X_i}{\partial t}\Big(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\Big) \\
&= -\frac{1}{2n}(D_c D^c X^i)\frac{\partial}{\partial p_i} - \frac{1}{2n}(D_c D^c X_i)\Big(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\Big) \\
&= -\frac{1}{2n} D_c D^c X.
\end{aligned} \tag{2.4.20}$$

So there is the temptation the write

$$[T, X]^a \stackrel{?}{=} -\frac{1}{2n}\nabla_c \nabla^c X^a \tag{2.4.21}$$

but the left-hand side is evidently independent of the choice of torsion free partial connection satisfying $\nabla_a J_{bc} = 0$, whereas the right-hand side is not. We can easily fix this by adding an appropriate curvature correction. Firstly, given a change of connection by

$$\hat{\nabla}_c X^a = \nabla_c X^a - \Gamma_{cb}{}^a X^b, \tag{2.4.22}$$

we have

$$\hat{\nabla}_c \hat{\nabla}^c X^a = \nabla_c \nabla^c X^a - \nabla_c \Gamma^c{}_b{}^a X^b + \Gamma_{cd}{}^a \Gamma^c{}_b{}^d X^b. \tag{2.4.23}$$

On the other hand contracting 2.3.16 gives

$$\begin{aligned}\hat{R}_{adc}{}^{d} =& R_{adc}{}^{d} - \frac{1}{2}\nabla_{d}\Gamma_{ac}{}^{d} + \frac{1}{2n}\nabla_{e}\Gamma_{ca}{}^{e} \\ & -\frac{1}{2}\Gamma_{cd}{}^{e}\Gamma_{ae}{}^{d} + \frac{1}{2n}\Gamma_{cf}{}^{e}\Gamma_{ea}{}^{f}.\end{aligned} \tag{2.4.24}$$

Thus, both sides of

$$[T, X]^{a} \stackrel{?}{=} -\frac{1}{2n}\nabla_{c}\nabla^{c}X^{a} + \frac{1}{n-1}R^{a}{}_{cb}{}^{c}X^{b} \tag{2.4.25}$$

are independent of the choice of contact torsion free partial connection. So since the equality holds for the canonical flat partial connection given by a Darboux coordinate neighbourhood as demonstrated by 2.4.20 it is true for any other contact partial torsion free connection. □

Later, when constructing the coupled Rumin complex we will need a useful alternate characterisation of the canonical extension.

Proposition 2.4.26. *Let $\nabla_a : E \to \Lambda^1_H \otimes E$ (using abstract index notation) be a partial connection with respect to a contact distribution $H \hookrightarrow M$ where (M, H) is a contact manifold of dimension $2n+1$. Let $\tilde{\nabla} : E \to \Lambda^1 \otimes E$ be the canonical extension as in 2.2.16. Then for a section $e \in E$ we have*

$$\nabla_{c}\nabla^{c}e = -2n\tilde{\nabla}_{T}e \tag{2.4.27}$$

where T is the Reeb field corresponding to some $\alpha \in L$, and the auxiliary torsion free affine partial connection (which we need in order for the left-hand side makes sense) satisfies $\nabla_a J_{bc} = 0$, where $J_{bc} = d\alpha|_H$.

Proof. As usual pick coordinates such that $\alpha = dt - p_i dq^i$. Let $e \in E$. Then the point of 2.2.16 is that we know the full curvature is trace free with respect to the non-degenerate skew-form. To abuse notation, $J^{ab} = -4\big(\frac{\partial}{\partial p_i} \wedge (\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial t})\big)$, since this gives the identity when paired with $J_{ab} = -d_{\perp}p_i \wedge d_{\perp}q^i$. So

$$\tilde{\kappa}\Big(\frac{\partial}{\partial p_i} \wedge (\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial t}), e\Big) = 0 \tag{2.4.28}$$

where we are interpreting the curvature $\tilde{k}$ of $\tilde{\nabla}$ as a map $\tilde{\kappa} : H \wedge H \times E \to \mathbb{R}$ and summing $i = 1, ..., n$. Interpreting the curvature this way, and using that

$$\Big[\frac{\partial}{\partial p_i}, \frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial t}\Big] = \frac{\partial}{\partial t} = T \tag{2.4.29}$$

as well as the definition of $\tilde{\kappa}$ as

$$\tilde{\kappa}(X \wedge Y, e) = \frac{1}{2}\nabla_X \nabla_Y e - \frac{1}{2}\nabla_Y \nabla_X e - \frac{1}{2}\nabla_{[X,Y]} e, \tag{2.4.30}$$

we have

$$\nabla_{\frac{\partial}{\partial p_i}} \nabla_{\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}} e - \nabla_{\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial p_i}} e = n\tilde{\nabla}_T e. \tag{2.4.31}$$

and abusing notation as before

$$\nabla_a (J^{cd} \nabla_b e) = -4\nabla\Big(\frac{\partial}{\partial p_i} \wedge (\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}) \otimes \nabla e\Big). \tag{2.4.32}$$

Then, taking the natural pairing, we compute

$$\begin{aligned} \nabla_a \nabla^c e = & -2\nabla\Big(\frac{\partial}{\partial p_i} \otimes \nabla_{\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}} e - \Big(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\Big) \otimes \nabla_{\frac{\partial}{\partial p_i}} e\Big) \\ = & -2\Big\{\frac{\partial}{\partial p_i} \otimes \nabla\Big(\nabla_{\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}} e\Big) - \Big(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\Big) \otimes \ \nabla\Big(\nabla_{\frac{\partial}{\partial p_i}} e\Big) \\ & + \nabla\frac{\partial}{\partial p_i} \otimes \nabla_{\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}} e - \nabla\Big(\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}\Big) \otimes \nabla_{\frac{\partial}{\partial p_i}} e\Big\}. \end{aligned} \tag{2.4.33}$$

This time take the natural pairing $\Lambda^1_H \otimes H \otimes E \to E$ to get

$$\nabla_c \nabla^c e = -2\big\{\nabla_{\frac{\partial}{\partial p_i}} \nabla_{\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}} e - \nabla_{\frac{\partial}{\partial q^i} + p_i \frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial p_i}} e\big\}. \tag{2.4.34}$$

We have used that $-2\nabla\frac{\partial}{\partial p_i}$ vanishes under the pairing $\Lambda^1_H \otimes H \to \mathbb{R}$. To see this note that by "lowering an index", the image under the pairing is proportional to the trace of $-2\nabla d_\perp q_i$ but this trace vanishes since the trace vanishes for the canonical connection D in the Darboux coordinate neighbourhood, and ∇ and D are each contact and partial torsion free, so differ by a symmetric tensor. Similarly for $-2\nabla\Big(\frac{\partial}{\partial q^i} + p_i\frac{\partial}{\partial t}\Big)$. Finally 2.4.34 put together with 2.4.31 gives the result. $\square$

We are now ready to compute the Rumin complex with respect to a contact torsion free partial connection. We will continue to take $\tilde{\omega}$ to satisfy $\tilde{\omega}(T, \cdot, ..., \cdot) = 0$ for $\omega \in \Lambda^k_H$.

We know the first Rumin operator $\mathbb{R} \to \Lambda^1_H$ is

$$f \mapsto \nabla_a f \tag{2.4.35}$$

and because of the vanishing partial torsion, the Rumin operator $\Lambda^1_H \to \Lambda^2_{H\perp}$ is

$$\nabla_{[a}\omega_{b]} - \frac{1}{2n} J_{ab} \nabla_c \omega^c. \tag{2.4.36}$$

If we are in the $n > 2$ case we have more operators $\Lambda^k_{H\perp} \to \Lambda^{k+1}_{H\perp}$, until $k+1 = n$. By definition, each is just the composition $\Lambda^1_H \to \Lambda^{k+1}_H \to \Lambda^{k+1}_{H\perp}$ where the first map is as in 2.4.9 and the second is the projection which removes the trace. For instance when $n = 3$ (so we are on a 7-dimensional contact manifold) the operator $\Lambda^2_{H\perp} \to \Lambda^3_{H\perp}$ is

$$\omega_{ab} = \nabla_{[a}\omega_{bc]} - \frac{1}{12}J_{[ab}\nabla_{c]}\omega_d{}^d - \frac{1}{6}J_{[ab}\nabla^d\omega_{c]d}. \tag{2.4.37}$$

The only complication here is finding the projection $\Lambda^k_H \to \Lambda^k_{H\perp}$ onto totally trace-free tensors. As explained in Chapter 1, while this is possible, notationally this will quickly get out of hand, so we will not give a general formula.

The nth Rumin operator is more interesting. In index free notation, and for the moment identifying $L \otimes \Lambda^n_{H\times} \hookrightarrow \Lambda^{n+1}$ we have $d_\perp : \Lambda^n_{H\perp} \to L \otimes \Lambda^n_{H\times}$ given by

$$\omega \mapsto d\tilde{\omega} - d(\alpha \wedge \tilde{\rho}) \tag{2.4.38}$$

Where ρ is defined by $d\alpha|_H \wedge \rho = d\tilde{\omega}|_H$. Since we are identifying $L \otimes \Lambda^n_{H\times} \hookrightarrow \Lambda^{n+1}$ we can write

$$\alpha \wedge \tilde{\mu} = d\tilde{\omega} - d(\alpha \wedge \tilde{\rho}) = d\tilde{\omega} - d\alpha \wedge \tilde{\rho} + \alpha \wedge d\tilde{\rho} \tag{2.4.39}$$

for some μ which we would like to solve for. Inserting the Reeb field T gives

$$\frac{1}{n+1}\tilde{\mu} = d\tilde{\omega}(T, \cdot, ..., \cdot) + \frac{1}{n+1}d\tilde{\rho} - \frac{n}{n+1}\alpha \wedge d\tilde{\rho}(T, \cdot, ..., \cdot). \tag{2.4.40}$$

Now we restrict to H to get

$$\mu = (n+1)d\tilde{\omega}(T, \cdot, ..., \cdot)|_H + d\tilde{\rho}|_H. \tag{2.4.41}$$

Remember that

$$d\alpha|_H \wedge \rho = d\tilde{\omega}|_H. \tag{2.4.42}$$

In abstract indices this is just the statement

$$J_{[ab}\rho_{c]} = \nabla_{[a}\omega_{bc]}. \tag{2.4.43}$$

We continue only in the $n = 2$ case to avoid the notation getting out of hand, but it should clear how to adapt these calculations for dimension greater than 5. Contracting with J^{bc} gives

$$\rho_a = \frac{1}{2}\nabla_a\omega_d{}^d + \nabla_d\omega^d{}_a = \nabla_d\omega^d{}_a, \tag{2.4.44}$$

where we have used the fact ω_{ab} is trace free. Thus

$$(d\tilde{\rho}|_H)_{ab} = \nabla_{[a}\nabla^d\omega_{b]d}. \tag{2.4.45}$$

Now we have to deal with $d\tilde{\omega}(T,\cdot,\cdot)$. We use the index free formula for the exterior derivative. Let $X, Y \in H$ then

$$\begin{aligned} d\tilde{\omega}(T,X,Y) =& \frac{1}{3}T(\omega(X,Y)) - \frac{1}{3}X(\tilde{\omega}(T,Y)) + \frac{1}{3}Y(\tilde{\omega}(T,X)) \\ &- \frac{1}{3}\tilde{\omega}([T,X],Y) + \frac{1}{3}\tilde{\omega}([T,Y],X) - \frac{1}{3}\tilde{\omega}([X,Y],T) \qquad (2.4.46) \\ =& \frac{1}{3}T(\omega(X,Y)) - \frac{1}{3}\omega([T,X],Y) + \frac{1}{3}\omega([T,Y],X) \qquad (2.4.47) \end{aligned}$$

which is, in indices, using 2.4.18

$$\begin{aligned} &-\frac{1}{12}\nabla_c\nabla^c(\omega_{ab}X^aY^b) + \omega_{ab}\frac{1}{12}(\nabla_c\nabla^c X^a)Y^b - \omega_{ab}\frac{1}{12}(\nabla_c\nabla^c Y^a)X^b \\ &-\frac{1}{3}\omega_{ab}R^a{}_{cd}{}^c X^dY^b + \frac{1}{3}\omega_{ab}R^a{}_{cd}{}^c X^bY^d \qquad (2.4.48) \\ =& -\frac{1}{12}(\nabla_c\nabla^c\omega_{ab})X^aY^b - \frac{1}{3}\omega_{ab}R^a{}_{cd}{}^c X^dY^b + \frac{1}{3}\omega_{ab}R^a{}_{cd}{}^c X^bY^d. \qquad (2.4.49) \end{aligned}$$

So we see

$$d\tilde{\omega}(T,\cdot,\cdot)|_H = -\frac{1}{12}\nabla_c\nabla^c\omega_{ab} + \frac{2}{3}\omega_{d[a}R_{b]c}{}^{dc} \tag{2.4.50}$$

and obtain a new formula for the nth Rumin operator $d_\perp : \Lambda^2_{H\perp} \to L \otimes \Lambda^2_{H\perp}$ on a contact manifold with $n = 2$. It is

$$\omega_{ab} \mapsto J_{ab}(\nabla_{[c}\nabla^e\omega_{d]e} - \frac{1}{4}\nabla_e\nabla^e\omega_{cd} + 2\omega_{f[c}R_{d]e}{}^{fe}). \tag{2.4.51}$$

Lastly, we write down the Rumin operators $L \otimes \Lambda^{k-2}_{H\times} \to L \otimes \Lambda^{k-1}_{H\times}$ for $k > n+1$. These are defined by including $L \otimes \Lambda^{k-2}_{H\times} \hookrightarrow \Lambda^{k-1}$ and then applying the exterior derivative. We write

$$\alpha \otimes \omega \mapsto \alpha \wedge \tilde{\omega} \mapsto d\alpha \wedge \tilde{\omega} - \alpha \wedge d\tilde{\omega} = \alpha \wedge \tilde{\mu} \tag{2.4.52}$$

and want to solve for $\mu \in \Lambda^{k-1}_{H\times}$. Plug in the Reeb field to the above equation to get

$$-\frac{1}{k}d\tilde{\omega} + \frac{k-1}{k}\alpha \wedge d\tilde{\omega}(T,\cdot,...,\cdot) = \frac{1}{k}\tilde{\mu}. \tag{2.4.53}$$

Hence $\mu = -d\tilde{\omega}|_H$. So in terms of the partial connection the Rumin operator is

$$J_{ab}\omega_{c_1\ldots c_{k-2}} \mapsto -J_{ab}\nabla_{[c_1}\omega_{c_2\ldots c_{k-1}]}. \tag{2.4.54}$$

If $J_{[ab}\omega_{c_1\ldots c_{k-2}]} = 0$ then so does $J_{[ab}\nabla_{c_1}\omega_{c_2\ldots c_{k-1}]}$ since ∇_{c_1} is a contact connection. So the range of the above operator lies in $L \otimes \Lambda^{k-1}_{H\times}$ as it should.

We summarise our results for the $n = 2$ case:

Rumin operator:	Formula:
$\mathbb{R} \to \Lambda^1_H$	$f \mapsto \nabla_a f$
$\Lambda^1_H \to \Lambda^2_{H\perp}$	$\omega_a \mapsto \nabla_{[a}\omega_{b]} - \frac{1}{4}J_{ab}\nabla_c\omega^c$
$\Lambda^2_{H\perp} \to L \otimes \Lambda^2_{H\perp}$	$\omega_{ab} \mapsto J_{ab}(\nabla_{[e}\nabla^c\omega_{d]e} - \frac{1}{4}\nabla_e\nabla^e\omega_{cd} + 2\omega_{f[c}R_{d]e}{}^{fe})$
$L \otimes \Lambda^2_{H\perp} \to L \otimes \Lambda^3_H$	$J_{ab}\omega_{cd} \mapsto -J_{ab}\nabla_{[c}\omega_{de]}$
$L \otimes \Lambda^3_H \to L \otimes \Lambda^4_H$	$J_{ab}\omega_{cde} \mapsto -J_{ab}\nabla_{[c}\omega_{def]}$

2.5 The coupled Rumin sequence

Recall 2.1.9 for a full connection $\tilde{\nabla}$ with curvature $\tilde{\kappa}$ we have the Bianchi identity:

$$d^{\tilde{\nabla}}\tilde{\kappa} = 0. \tag{2.5.1}$$

We will write down what this means for partial curvature. First we will attempt to construct the coupled analogue of the Rumin complex. For a vector bundle E we previously defined $d^{\nabla}_{\perp} : \Lambda^1_H \otimes E \to \Lambda^2_{H\perp} \otimes E$ and the pattern works for $d^{\nabla}_{\perp} : \Lambda^k_{H\perp} \otimes E \to \Lambda^{k+1}_{H\perp} \otimes E$ for $0 \leq k < n$.

Definition 2.5.2 (kth coupled Rumin operator for $0 \leq k < n$)**.** *On a $2n+1$ dimensional contact manifold (M, H), for $0 \leq k < n$, with vector bundle E equipped with a partial connection $\nabla : E \to \Lambda^1_H \otimes E$, define the kth Rumin coupled operator $d_{\perp} : \Lambda^k_{H\perp} \otimes E \to \Lambda^{k+1}_{H\perp} \otimes E$ by*

$$d^{\nabla}_{\perp}(\omega \otimes s) = d_{\perp}\omega \otimes s + (-1)^k \wedge_{\perp} (\omega \otimes \nabla s) \tag{2.5.3}$$

on simple tensors, extending by linearity.

Equivalently, these can be defined analogously to the Rumin operators. Given $e \in \Lambda^k_{H\perp} \otimes E$, take a full connection $\tilde{\nabla}$ extending ∇ and then define $d^{\nabla}_{\perp}e$ as the projection to $\Lambda^{k+1}_{H\perp} \otimes E$ of $d^{\tilde{\nabla}}\tilde{e} \in \Lambda^{k+1} \otimes E$ where $\tilde{e}$ is in the preimage of e under the projection $\Lambda^k \otimes E \to \Lambda^k_{H\perp} \otimes E$.

If we use ∇_a to indicate both the partial connection on E and a auxiliary affine partial connection with vanishing partial torsion then in abstract indices the operator is simply

$$e_{a_1 \ldots a_k}{}^{\mu} \mapsto (\nabla_{[a_1} e_{a_2 \ldots a_{k+1}]}{}^{\mu})_{\perp}. \tag{2.5.4}$$

It is similarly easy to define the coupled versions of the operators of second half of the Rumin complex.

Definition 2.5.5 (kth coupled Rumin operator for $n < k \leq 2n$)**.** *On a $2n+1$ dimensional contact manifold (M, H), for $n < k \leq 2n$, with vector bundle E equipped with a partial connection $\nabla : E \to \Lambda^1_H \otimes E$, define the kth coupled Rumin operator $d_\perp : L \otimes \Lambda^{k-1}_{H\times} \otimes E \to L \otimes \Lambda^k_{H\times} \otimes E$ by*

$$d^\nabla_\perp(\omega \otimes s) = d_\perp \omega \otimes s + (-1)^k \omega \wedge \nabla s \tag{2.5.6}$$

on simple tensors, extending by linearity.

As with the first half of the sequence, there is a useful equivalent definition of this operator. Given $e \in L \otimes \Lambda^{k-1}_{H\times} \otimes E$, take a full connection $\tilde{\nabla}$ extending ∇ and then define $d^\nabla_\perp e$ as $i^{-1}_{k+1}(d^{\tilde{\nabla}} i_k e) \in L \otimes \Lambda^k_H \otimes E$ where i_k is the canonical inclusion. In indices the operator is

$$J_{a_1 a_2} e_{a_3 \ldots a_{k+1}}{}^\mu \mapsto -J_{a_1 a_2} \nabla_{[a_3} e_{a_4 \ldots a_{k+2}]}{}^\mu. \tag{2.5.7}$$

Based on the above pattern we may hope to define the nth coupled Rumin operator $\Lambda^n_{H\perp} \otimes E \to L \otimes \Lambda^n_{H\perp} \otimes E$ like

$$d^\nabla_\perp(\omega \otimes s) \stackrel{?}{=} d_\perp \omega \otimes s + (-1)^n \omega \wedge \nabla s \tag{2.5.8}$$

but the problem is that the second term does not in general lie in $L \otimes \Lambda^k_H \otimes E$ let alone $L \otimes \Lambda^k_{H\perp} \otimes E$. There are a few complications in construction such an operator, which was noticed by Calderbank in the PhD thesis [Cal96].

We will proceed by attempting to define the operator by mimicking the definition of the nth Rumin operator. That is, we try

$$d^\nabla_\perp(e) = i^{-1}_{n+1}\big(d^{\tilde{\nabla}} \tilde{e} - (d^{\tilde{\nabla}} \circ i_n \circ \mathcal{L}^{-1})(d^{\tilde{\nabla}} \tilde{e}|_H)\big). \tag{2.5.9}$$

Where $\tilde{e} \in \Lambda^n \otimes E$ is any lift of $e \in \Lambda^n_{H\perp} \otimes E$ and $\tilde{\nabla}$ is any proper connection extending ∇. First note that the expression inside the argument of i^{-1}_{n+1} really lies in the image of $L \otimes \Lambda^n_H \to \Lambda^{n+1}_H$ since the expression vanishes when restricted to H. Let $e = \omega \otimes s$ for $s \in E$. Two lifts of $\omega \otimes s$ differ by $\alpha \wedge \tilde{\mu} \otimes s$ for some $\mu \in \Lambda^{n-1}_H$. So 2.5.9 is independent of this choice if

$$d(\alpha \wedge \tilde{\mu}) \otimes s + (-1)^n \alpha \wedge \tilde{\mu} \wedge \tilde{\nabla} s - (d^{\tilde{\nabla}} \circ i_n \circ \mathcal{L}^{-1})(d\alpha|_H \wedge \mu \otimes s) = 0 \tag{2.5.10}$$

but this follows from $\mathcal{L}^{-1}(d\alpha|_H \wedge \mu \otimes s) = \alpha \otimes \mu \otimes s$. So 2.5.9 is well-defined given a fixed connection $\tilde{\nabla}$ extending ∇.

Now we consider whether or not 2.5.9 is independent of the choice of extension of this extension. Consider instead choosing the extension $\tilde{\nabla} + \Gamma$ with arbitrary $\Gamma \in L \otimes \mathrm{End}(E)$, which is the freedom available, as explained in 2.2.16.

Writing $e = \omega \otimes s$ the formula 2.5.9 is independent of the chosen extension if and only if

$$(-1)^n \tilde{\omega} \wedge \Gamma(s) - ((d^{\tilde{\nabla}+\Gamma} - d^{\tilde{\nabla}}) \circ i_n \circ \mathcal{L}^{-1})(d\tilde{\omega}|_H \otimes s - \omega \wedge \nabla s) = 0 \qquad (2.5.11)$$

for all $\Gamma \in L \otimes \mathrm{End}(E)$. We can write $d\tilde{\omega}|_H \otimes s - \omega \wedge \nabla s = d\alpha|_H \wedge f$ for some $f \in \Lambda_H^{n-1} \otimes E$ so that $\mathcal{L}^{-1}(d\tilde{\omega}|_H \otimes s - \omega \wedge \nabla s) = \alpha \otimes f$. Then we calculate

$$(d^{\tilde{\nabla}})(\alpha \wedge \tilde{f}) - (d^{\tilde{\nabla}+\Gamma})(\alpha \wedge \tilde{f}) = (-1)^{n+1} \alpha \wedge (\mathrm{Id} \wedge \Gamma)(\tilde{f}) = 0, \qquad (2.5.12)$$

where the last equality follows because $\mathrm{Id} \wedge \Gamma$ has range in L, hence vanishes when wedged with α. So 2.5.9 is independent of the chosen extension if and only if

$$(-1)^n \tilde{\omega} \wedge \Gamma(s) = 0 \qquad (2.5.13)$$

for all $\Gamma \in L \otimes \mathrm{End}(E)$. This is manifestly not true, so the expression 2.5.9 depends on what extension of ∇ we pick.

We do however, have a canonical extension, so we get a well-defined differential operator if we insist we take the canonical extension of ∇. The trade-off is one has less flexibility in calculating this operator compared to the other coupled Rumin operators.

As defined, the codomain of the operator is not quite as we hope. That is, we cannot employ the same proof that showed the codomain of the nth Rumin operator was $L \otimes \Lambda^n_{H\times} \otimes E$, which used the fact $d \circ d = 0$. Nonetheless we define:

Definition 2.5.14 (nth coupled Rumin operator)**.** *On a $2n+1$ dimensional contact manifold (M, H) with vector bundle E equipped with a partial connection $\nabla : E \to \Lambda^1_H \otimes E$, define the nth Rumin coupled operator $d_\perp : \Lambda^n_{H\perp} \otimes E \to L \otimes \Lambda^n_H \otimes E$ by*

$$d_\perp^\nabla(e) = i_{n+1}^{-1}\big(d^{\tilde{\nabla}}\tilde{e} - (d^{\tilde{\nabla}} \circ i_n \circ \mathcal{L}^{-1})(d^{\tilde{\nabla}}\tilde{e}|_H)\big) \qquad (2.5.15)$$

where $\tilde{e} \in \Lambda^n \otimes E$ is any lift of $e \in \Lambda^n_{H\perp} \otimes E$ and $\tilde{\nabla}$ is the canonical extension of ∇ from 2.2.16

To calculate the operator in abstract indices (in the $n = 2$ case) we imitate the derivation for the uncoupled Rumin operator. We will need to take an auxiliary contact partial torsion free partial connection on Λ^1_H which we will also denote ∇_a. Denote its curvature by $R_{abc}{}^d$.

For $\omega \otimes s \in \Lambda^2_{H\perp} \otimes E$, suppose $d_\perp^\nabla(\omega \otimes s) = \alpha \otimes g$ so that

$$d^{\tilde{\nabla}}(\tilde{\omega} \otimes s) - (d^{\tilde{\nabla}} \circ i_n \circ \mathcal{L}^{-1})(d^{\tilde{\nabla}}(\tilde{\omega} \otimes s)|_H) = \alpha \wedge \tilde{g} \qquad (2.5.16)$$

and we would like to solve for g. We can write $d\tilde{\omega}|_H \otimes s + \omega \wedge \nabla s = d\alpha|_H \wedge f$ for some $f \in \Lambda^1_H \otimes E$ so that $(d^{\tilde{\nabla}} \circ i_n \circ \mathcal{L}^{-1})(d^{\tilde{\nabla}}(\tilde{\omega} \otimes s)|_H) = d^{\tilde{\nabla}}(\alpha \wedge \tilde{f})$. Making this substitution then inserting the Reeb field into 2.5.16 before restricting to H we can solve for g as

$$g = 3d\tilde{\omega}(T, \cdot, \cdot)|_H \otimes s + \omega \otimes \tilde{\nabla}_T s + d^{\tilde{\nabla}} \tilde{f}|_H. \tag{2.5.17}$$

Where, as usual we have taken $\tilde{f} \in \Lambda^1 \otimes E$ to be the lift of f satisfying $\tilde{f}(T) = 0$. The first term can be calculated exactly analogously to 2.4.50. It is

$$-\frac{1}{4}(\nabla_c \nabla^c \omega_{ab}) s^\mu + 2\omega_{f[a} R_{b]e}{}^{fe} s^\mu + \omega_{ab}. \tag{2.5.18}$$

The second term is

$$-\frac{1}{4}\omega_{ab} \nabla_c \nabla^c s^\mu, \tag{2.5.19}$$

where we've used the characterisation of the canonical extension 2.4.26. Then since f is defined by

$$J_{[ab} f_{c]}{}^\mu = \nabla_{[a} \omega_{bc]} s^\mu + \omega_{[ab} \nabla_{c]} s^\mu, \tag{2.5.20}$$

we have $f_c{}^\mu = \nabla_d(\omega^d{}_a s^\mu)$ and so

$$d^{\tilde{\nabla}} f|_H = \nabla_{[a|} \nabla_d (\omega^d{}_{|b]} s^\mu). \tag{2.5.21}$$

All in all we see the coupled middle Rumin operator in the $n = 2$ case is

$$e_{ab}{}^\mu \mapsto J_{ab}(\nabla_{[c} \nabla^e e_{d]e}{}^\mu - \frac{1}{4}\nabla_e \nabla^e e_{cd}{}^\mu + 2e_{f[c}{}^\mu R_{d]e}{}^{fe}) \tag{2.5.22}$$

in terms of an auxiliary contact partial torsion free connection ∇_a.

Proposition 2.5.23. *Let (M, H) be a contact manifold of dimension $2n+1$, then the nth coupled Rumin operator $d^\nabla_\perp : \Lambda^n_{H\perp} \otimes E \to L \otimes \Lambda^n_H \otimes E$ factors through $L \otimes \Lambda^n_{H\perp} \otimes E$ if and only if $\nabla : E \to \Lambda^1_H \otimes E$ has vanishing partial curvature.*

Proof. As before write $d^\nabla_\perp(\omega \otimes s) = \alpha \otimes g$ so that

$$d^{\tilde{\nabla}}(\tilde{\omega} \otimes s) - (d^{\tilde{\nabla}} \circ i_n \circ \mathcal{L}^{-1})(d^{\tilde{\nabla}}(\tilde{\omega} \otimes s)|_H) = \alpha \wedge \tilde{g}. \tag{2.5.24}$$

Again take f such that $\alpha \otimes f = \mathcal{L}^{-1}(d^{\tilde{\nabla}}(\tilde{\omega} \otimes s)|_H)$ so $(d^{\tilde{\nabla}} \circ i_n \circ \mathcal{L}^{-1})(d^{\tilde{\nabla}}(\tilde{\omega} \otimes s)|_H) = d^{\tilde{\nabla}}(\alpha \wedge \tilde{f})$. Make the substitution and apply $d^{\tilde{\nabla}}$ to both sides of 2.5.24 to get

$$\tilde{\omega} \wedge \tilde{\kappa}(s) - \alpha \wedge \tilde{\kappa}(\tilde{f}) = d\alpha \wedge \tilde{g} - \alpha \wedge d^{\tilde{\nabla}} g. \tag{2.5.25}$$

Restricting to H, and recalling that $\tilde{\kappa}|_H = \kappa$ for the canonical extension, gives

$$\omega \wedge \kappa(s) = d\alpha|_H \wedge g. \tag{2.5.26}$$

Now, the operator factors through $L \otimes \Lambda^n_{H\perp} \otimes E = L \otimes \Lambda^n_{H\times} \otimes E$ if and only if the right-hand side always vanishes, but this occurs if and only if $\kappa = 0$ since ω, is arbitrary. □

We can also obtain this result in the $n = 2$ case from 2.5.22. Contracting with J^{cd} yields (after some computation)

$$J_{ab}\kappa_{cd}{}^{\mu}{}_{\nu}e^{cd\nu} \tag{2.5.27}$$

which vanishes for all $e_{cd}{}^{\nu} \in \Lambda^2_{H\perp} \otimes E$ if and only if $\kappa_{ab}{}^{\mu}{}_{\nu} = 0$.

So we conclude the coupled Rumin operators fit together into a complex if and only if the partial curvature of the coupled connection vanishes.

We now write down the consequences of the differential Bianchi identity for partial curvature.

Proposition 2.5.28 (Bianchi identity for partial curvature)**.** *Let (M, H) be a contact manifold and ∇ be a partial connection with partial curvature κ, then $d^{\nabla}_{\perp}\kappa = 0$.*

Proof. Take the canonical extension $\tilde{\nabla}$, and by construction its curvature $\tilde{\kappa}$ satisfies $\tilde{\kappa}|_H = \kappa$. Let $2n+1 > 5$. We have $\kappa \in \Lambda^2_{H\perp} \otimes E$. Then $d^{\nabla}_{\perp}\kappa$ is just the projection of $d^{\tilde{\nabla}}\tilde{\kappa} = 0$ to $\Lambda^3_{H\perp} \otimes E$. So we have $d^{\nabla}_{\perp}\kappa = 0$. In the case $n = 2$, this follows immediately from the formula 2.5.15 and $d^{\tilde{\nabla}}\tilde{\kappa} = 0$. □

In the $2n+1 > 5$ case, taking an auxiliary affine contact partial connection ∇_a with vanishing partial torsion we have

$$\begin{aligned} &\nabla_{[a}\kappa_{bc]}{}^{\mu}{}_{\nu} - \frac{1}{2n-2}J_{[ab}\nabla_{c]}\kappa_{d}{}^{d\mu}{}_{\nu} - \frac{1}{n-1}J_{[ab}\nabla^{d}\kappa_{c]d}{}^{\mu}{}_{\nu} = 0 \\ \implies &\nabla_{[a}\kappa_{bc]}{}^{\mu}{}_{\nu} - \frac{1}{n-1}J_{[ab}\nabla^{d}\kappa_{c]d}{}^{\mu}{}_{\nu} = 0. \end{aligned} \tag{2.5.29}$$

Note that this holds as a trivial index identity in the $2n+1 = 5$ case. The reason being $\Lambda^3_{H\perp} = \{0\}$.

Also in the $2n+1 = 5$ case using our formula 2.5.22 and the above proposition 2.5.28 produces

$$\nabla_{[a}\nabla^{c}\kappa_{b]c}{}^{\mu}{}_{\nu} - \frac{1}{4}\nabla_{e}\nabla^{e}\kappa_{ab}{}^{\mu}{}_{\nu} + 2\kappa_{f[a|}{}^{\mu}{}_{\nu}R_{b]e}{}^{fe} = 0 \tag{2.5.30}$$

where $R_{abc}{}^{d}$ is the partial curvature of the auxiliary contact partial torsion free partial connection $\Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$. There is no reason to suspect this contains any non-trivial differential information. That is, this can likely alternatively be derived by differentiating a known trivial index identity. Nonetheless it happens to be a useful thing to write down for later use.

Chapter 3

G_2 contact geometry

In this chapter we introduce G_2 contact geometries. After a review of some facts about the representation theory of $GL(2, \mathbb{R})$, we outline how the definition of a G_2 contact geometry in terms of an auxiliary rank-2 “spin” vector bundle is related to the canonical model. Starting from this definition we set up a calculus by showing the existence of a canonical partial connection in the presence of a contact form. We derive useful formulae regarding the partial curvature of this partial connection. Lastly, we show there is an equivalent definition of a G_2 contact geometry in terms of a field of twisted cubic curves in the contact distribution.

3.1 $GL(2, \mathbb{R})$ and rank-2 vector bundles

In this section we review some facts about the representation theory of $GL(2, \mathbb{R})$ that will be of use, particularly in regard to decompositions of tensor powers of rank-2 vector bundles.

Let $\mathbb{R}^n$ be the standard (matrix) representation of $GL(n, \mathbb{R})$. Recall we get a representation on the n^k-dimensional vector space $\otimes^k \mathbb{R}^n$ by defining

$$g(v_1 \otimes v_2 ... \otimes v_k) = gv_1 \otimes gv_2 \otimes ... \otimes gv_k, \ v_i \in \mathbb{R}^n \tag{3.1.1}$$

one simple tensors then extending by linearity. $\odot^k \mathbb{R}^n$ and $\Lambda^k \mathbb{R}^n$ are subrepresentations. We are interested in $n = 2$. In this case $\odot^k \mathbb{R}^2$ is of dimension $k + 1$ while $\Lambda^2 \mathbb{R}^2$ is a 1-dimensional representation given by $g(e_1 \wedge e_2) \mapsto \det(g) e_1 \wedge e_2$. $\Lambda^k \mathbb{R}^2$ vanishes for $k > 2$.

Proposition 3.1.2. *$\odot^k \mathbb{R}^2$ is an irreducible representation of $GL(2, \mathbb{R})$.*

Proof. For fun, we present a novel proof without any representation theory.

We have a basis $\{e_1^k, e_1^{k-1}e_2, ..., e_2^k\}$. Let $V \leq \odot^k \mathbb{R}^2$ be a non-zero subrepresentation. We will start by showing that $e_2^k \in V$. Suppose that

$$v = a_{k-n}e_1^n e_2^{k-n} + a_{k-n+1}e_1^{n-1}e_2^{k-n+1} + ... + a_k e_2^k \in V \tag{3.1.3}$$

where $a_{k-n} \neq 0$ and $0 < n \leq k$. Note that if we cannot find such a v, then $e_2^k \in V$. Otherwise we claim that $e_1^n e_2^{k-n} \in V$.

To see this consider the action of the matrix

$$\begin{bmatrix} r^{(k-n)/n} & 0 \\ 0 & 1/r \end{bmatrix} \tag{3.1.4}$$

for $r > 0$, on v. We get

$$v \mapsto a_{k-n}e_1^n e_2^{k-n} + r^{-k/n}a_{k-n+1}e_1^{n-1}e_2^{k-n+1} + ... + r^{-k}a_k e_2^k. \tag{3.1.5}$$

Since V is a subrepresentation, these images must lie in V. Since V is closed (in the topology induced, for example, by the Euclidean norm), the limit as $r \to \infty$ must also lie in V, hence $a_{k-n}e_1^n e_2^{k-n} \in V$. In other words (block matrix notation)

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ x \\ * \\ \vdots \\ * \end{bmatrix} \in V, x \neq 0, \implies \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in V. \tag{3.1.6}$$

Now suppose that $e_1^n e_2^{k-n} \in V$, where V is as above. Then we claim $e_2^k \in V$. Consider the action of the matrix

$$\begin{bmatrix} 1/r & 0 \\ 1 & 1 \end{bmatrix} \tag{3.1.7}$$

on $e_1^n e_2^{k-n}$ we get

$$e_1^n e_2^{k-n} \mapsto (r^{-1}e_1 + e_2)^n e_2^{k-n} = r^{-n}e_1^n e_2^{k-n} + ... + nr^{-1}e_1 e_2^{k-1} + e_2^k. \tag{3.1.8}$$

Of course, the full expansion involves some binomial coefficients, but we are not concerned with this since it is clear the limit as $r \to \infty$ is e_2^k. So we have shown a subrepresentation of $\odot^k \mathbb{R}^2$ contains e_2^k. Now we will use this to generate a spanning set.

First note that

$$\begin{bmatrix} 1 & 1 \\ 0 & x \end{bmatrix} \tag{3.1.9}$$

for $x \neq 0$ maps $e_2^k \mapsto (e_1 + xe_2)^k$. So $(e_1 + xe_2)^k \in V$ for $x \neq 0$. Assume $u \in \odot^k \mathbb{R}^2$ is orthogonal to V with respect to the Euclidean inner product taken with respect to the basis $\{e_1^k, e_1^{k-1}e_2, ..., e_n^k\}$. We can write $u = \sum_{n=0}^k c_n e_1^{k-n} e_2^n$ for $c_n \in \mathbb{R}$.

$$\langle \sum_{n=0}^k c_n e_1^{k-n} e_2^n | (e_1 + xe_2)^k \rangle = 0 \ \forall x \neq 0 \tag{3.1.10}$$

$$\implies \sum_{n=0}^k \binom{k}{n} c_n x^n = 0 \ \forall x \neq 0 \tag{3.1.11}$$

The above expression is then a polynomial in x vanishing on $\mathbb{R} \setminus \{0\}$. Thus $c_n = 0$ for each $n = 0, ..., k$ and we have shown the orthogonal complement to V is $\{0\}$ and hence $V = \odot^k \mathbb{R}^2$, which completes the proof. □

Lemma 3.1.12 (Clebsch-Gordan)**.** *Let $k, l \in \mathbb{N}$ with $k \geq l$. There is a canonical direct sum decomposition into irreducible representations*

$$\begin{aligned} &\odot^k \mathbb{R}^2 \otimes \odot^l \mathbb{R}^2 \cong \\ &\odot^{k+l} \mathbb{R}^2 \oplus (\odot^{k+l-2}\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^2) \oplus ... \oplus (\odot^{k-l}\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^2 \otimes ... \otimes \Lambda^2\mathbb{R}^2). \end{aligned} \tag{3.1.13}$$

Proof. We use abstract index notation. For each $n = 0, ..., l$ define a map into the nth summand, $\odot^k\mathbb{R}^2 \otimes \odot^l\mathbb{R}^2 \to (\odot^{k+l-2n}\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^2 \otimes ... \otimes \Lambda^2\mathbb{R}^2)$ by

$$\phi_{a_1...a_l}\psi_{b_1...b_k} \mapsto \phi_{(a_1...a_{l-n}|c_1...c_n}\psi_{|b_1...b_{k-n})}{}^{c_1...c_n}\varepsilon_{a_{l-n+1}b_{k-n+1}}...\varepsilon_{a_l b_k}. \tag{3.1.14}$$

Firstly, we can think of this map as a linear combination of braiding maps and hence it is a homomorphism of $GL(2,\mathbb{R})$ representations.

Take $\rho := \rho_{a_1...a_{k+l-2n}}\varepsilon_{a_{l-n+1}b_{k-n+1}}...\varepsilon_{a_l b_k} \in (\odot^{k+l-2n}\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^2 \otimes ... \otimes \Lambda^2\mathbb{R}^2)$, which represents an arbitrary element of the nth summand. We claim that this is in the image of the above homomorphism. Even without doing a tedious calculation to determine the constant of proportionality (usually called the Clebsch-Gordan coefficient in angular momentum calculations in quantum mechanics) it is clear that the image of

$$\rho_{(a_1...a_{l-n}|b_1...b_{k-n}}\varepsilon_{|a_{l-n+1}|b_{k-n+1}} \cdots \varepsilon_{|a_l)b_k} \in \odot^l\mathbb{R}^2 \otimes \odot^k\mathbb{R}^2 \tag{3.1.15}$$

in $(\odot^{k+l-2n}\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^2 \otimes ... \otimes \Lambda^2\mathbb{R}^2)$ is non-zero and proportional to ρ as desired. Therefore the map is surjective. Finally it is a matter of using the rank theorem to see that the isomorphism 3.1.21 is given by the using 3.1.14 to project onto each summand. □

If we set $k = l$ and suppose that $\tau_{a_1 \ldots a_k \; a_{k+1} \ldots a_{2k}} \in \odot^k \mathbb{R}^2 \otimes \odot^k \mathbb{R}^2$ satisfies

$$\tau_{a_1 \ldots a_k \; a_{k+1} \ldots a_{2k}} = \tau_{a_{k+1} \ldots a_{2k} \; a_1 \ldots a_k} \tag{3.1.16}$$

the projection 3.1.14 vanishes for odd n since then

$$\begin{aligned} &\tau_{(a_1 \ldots a_{k-n}|c_1 \ldots c_n|b_. \ldots b_{k-n})}{}^{c_1 \ldots c_n} = \tau_{(b_1 \ldots b_{k-n}|}{}^{c_1 \ldots c_n}{}_{|a_1 \ldots a_{k-n})c_1 \ldots c_n} \\ = &-\tau_{(b_1 \ldots b_{k-n}|c_1 \ldots c_n|a_1 \ldots a_{k-n})}{}^{c_1 \ldots c_n} = -\tau_{(a_1 \ldots a_{k-n}|c_1 \ldots c_n|b_1 \ldots b_{k-n})}{}^{c_1 \ldots c_n} \end{aligned} \tag{3.1.17}$$

Similarly if $\tau_{a_1 \ldots a_k \; a_{k+1} \ldots a_{2k}}$ is antisymmetric on exchange of its first and last k indices, then 3.1.14 vanishes for even n. So we have:

Lemma 3.1.18. *Let $k \in \mathbb{N}$. Then there is a canonical direct sum decomposition into irreducible representations*

$$\begin{aligned} \Lambda^2(\odot^k \mathbb{R}^2) &\cong (\odot^{2k-2}\mathbb{R}^2 \otimes \Lambda^2 \mathbb{R}^2) \oplus (\odot^{2k-6}\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^2) \oplus \ldots \\ \odot^2(\odot^k \mathbb{R}^2) &\cong \odot^{2k}\mathbb{R}^2 \oplus (\odot^{2k-4}\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^2 \otimes \Lambda^2\mathbb{R}^2) \oplus \ldots \end{aligned} \tag{3.1.19}$$

where $\odot^0 \mathbb{R} := \mathbb{R}$ and $\odot^l \mathbb{R} := \{0\}$ for $l < 0$.

It is important to note that for odd and even k respectively, $\Lambda^2(\odot^k \mathbb{R}^2)$ and $\odot^2(\odot^k \mathbb{R}^2)$ have canonical skew-forms defined up to scale by the 1-dimensional representation $\Lambda^2\mathbb{R}^2 \otimes \ldots \otimes \Lambda^2\mathbb{R}^2$ appearing in the terminating summands above.

Now suppose that S is a rank-2 vector bundle. 3.1.12 and 3.1.18 along with 0.3.5 means we have:

Lemma 3.1.20 (Clebsch-Gordan for vector bundles)**.** *Let $k, l \in \mathbb{N}$ with $k \geq l$ and let S be a rank-2 vector bundle. There are canonical decompositions of vector bundles*

$$\begin{aligned} &\odot^k S \otimes \odot^l S \cong \\ &\odot^{k+l} S \oplus (\odot^{k+l-2} S \otimes \Lambda^2 S) \oplus \ldots \oplus (\odot^{k-l} S \otimes \Lambda^2 S \otimes \ldots \otimes \Lambda^2 S) \end{aligned} \tag{3.1.21}$$

and

$$\Lambda^2(\odot^k S) \cong (\odot^{2k-2} S \otimes \Lambda^2 S) \oplus (\odot^{2k-6} S \otimes \Lambda^2 S \otimes \Lambda^2 S \otimes \Lambda^2 S) \oplus \ldots \tag{3.1.22}$$

$$\odot^2(\odot^k S) \cong \odot^{2k} S \oplus (\odot^{2k-4} S \otimes \Lambda^2 S \otimes \Lambda^2 S) \oplus \ldots \tag{3.1.23}$$

The ease with which one can decompose tensor powers of rank-2 vector bundles will be an important computational tool.

3.2 Motivating the definition

Without going into much detail, or the general theory, we will try to motivate the definition of a G_2 contact geometry. We want to define "curved analogues" of the space G_2/P_2 where G_2 and P_2 are defined as in the introduction. So we should look at the geometric structure on this space.

From [LNS17] we have the following realisation of the Lie algebra $\mathfrak{g}_2$ of the Lie group G_2:

$$\mathfrak{g}_2 = \left\{ \begin{bmatrix} -c^1 - c^2 & -2c^3 & -12d^2 & -2d^3 & d^4 & -6e & 0 \\ \frac{1}{2}c^4 & -c^2 & 6d^1 & -6d^2 & \frac{1}{2}d^3 & 0 & 6e \\ -\frac{1}{12}b^4 & \frac{1}{3}b^3 & -c^1 & c^3 & 0 & -\frac{1}{2}d^3 & -d^4 \\ \frac{1}{3}b^2 & -\frac{1}{3}b^4 & 2c^4 & 0 & -c^3 & 6d^2 & 2d^3 \\ -b^1 & -\frac{2}{3}b^2 & 0 & -2c^4 & c^1 & -6d^1 & 12d^2 \\ \frac{1}{6}a & 0 & \frac{2}{3}b^2 & \frac{1}{3}b^4 & -\frac{1}{3}b^3 & c^2 & 2c^3 \\ 0 & -\frac{1}{6}a & b^1 & -\frac{1}{3}b^2 & \frac{1}{12}b^4 & \frac{1}{2}c^4 & c^1 + c^2 \end{bmatrix}, a, b^i, c^i, d^i, e \in \mathbb{R} \right\} \tag{3.2.1}$$

which stabilises the split signature metric $-2e^1e^7 - 2e^2e^6 - 2e^3e^5 - e^4e^4$ and the 3-form $2e^{147} + e^{156} + 8e^{237} - 2e^{246} - 2e^{345}$, which are related to the standard metric and the 3-form 0.0.1 given in the introduction by a change of basis. P_2 can therefore be defined as the stabiliser of the null-plane $\text{span}\{e_1, e_2\}$ and so

$$\mathfrak{p} = \left\{ \begin{bmatrix} -c^1 - c^2 & -2c^3 & -12d^2 & -2d^3 & d^4 & -6e & 0 \\ \frac{1}{2}c^4 & -c^2 & 6d^1 & -6d^2 & \frac{1}{2}d^3 & 0 & 6e \\ 0 & 0 & -c^1 & c^3 & 0 & -\frac{1}{2}d^3 & -d^4 \\ 0 & 0 & 2c^4 & 0 & -c^3 & 6d^2 & 2d^3 \\ 0 & 0 & 0 & -2c^4 & c^1 & -6d^1 & 12d^2 \\ 0 & 0 & 0 & 0 & 0 & c^2 & 2c^3 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}c^4 & c^1 + c^2 \end{bmatrix}, c^i, d^i, e \in \mathbb{R} \right\} \tag{3.2.2}$$

is the (evidently 9-dimensional) Lie subalgebra corresponding of P_2.

We have a decomposition $\mathfrak{g}_2 = \mathfrak{g}_{-II} \oplus \mathfrak{g}_{-I} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_I \oplus \mathfrak{g}_{II}$. This is best illustrated schematically:

$$\mathfrak{g}_{-II} =: \begin{bmatrix} & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ * & & & & & & \\ & * & & & & & \end{bmatrix}, \quad \mathfrak{g}_{-I} =: \begin{bmatrix} & & & & & & \\ & & & & & & \\ * & * & & & & & \\ * & * & & & & & \\ * & * & & & & & \\ & & * & * & * & & \\ & & * & * & * & & \end{bmatrix}, \quad \mathfrak{g}_0 =: \begin{bmatrix} * & * & & & & & \\ * & * & & & & & \\ & & * & * & * & & \\ & & * & * & * & & \\ & & * & * & * & & \\ & & & & & * & * \\ & & & & & * & * \end{bmatrix}$$

$$\mathfrak{g}_I =: \begin{bmatrix} & & * & * & * & & \\ & & * & * & * & & \\ & & & & & * & * \\ & & & & & * & * \\ & & & & & * & * \\ & & & & & & \\ & & & & & & \end{bmatrix}, \quad \mathfrak{g}_{II} = \begin{bmatrix} & & & & & * & \\ & & & & & & * \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \\ & & & & & & \end{bmatrix} \tag{3.2.3}$$

Then $\mathfrak{p} \cong \mathfrak{g}_0 \oplus \mathfrak{g}_I \oplus \mathfrak{g}_{II}$ and $\mathfrak{g}/\mathfrak{p} \cong \mathfrak{g}_{-II} \oplus \mathfrak{g}_{-I}$. Now given a homogeneous space G/P we get a canonical isomorphism $T(G/P) \cong G \times_P \mathfrak{g}/\mathfrak{p}$ where the representation on $\mathfrak{g}/\mathfrak{p}$ is given by the adjoint representation. See [ČS09] Example 1.4.3 for a construction of this isomorphism. In particular, this means there is a canonically defined rank 4-subbundle of $T(G_2/P_2)$ corresponding to $\mathfrak{g}_{-I}$ and furthermore by playing with the above realisation for $\mathfrak{g}_2$ it is possible to show $\mathfrak{g}_{-II} \oplus \mathfrak{g}_{-I}$ is isomorphic to the Heisenberg algebra of dimension 5. This means that G_2/P_2 is (locally) a contact manifold.

Lastly, $\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(2,\mathbb{R})$ and again from the above realisation one can check this acts irreducibly on $\mathfrak{g}_{-I}$ via the adjoint action (that is, acting via matrix commutation). Exponentiating the adjoint action, this induces a 4-dimensional irreducible representation of $GL(2,\mathbb{R})$ on $\mathfrak{g}_{-I}$, which is the subspace corresponding to the contact distribution.

In [ČS09] it is shown that the correct generalisation of the flat model to a family (G_2, P_2) of parabolic geometries is the following:

Definition 3.2.4 (G_2 contact structure, [ČS09] pg. 425)**.** *A G_2 contact structure is a contact manifold (M, H) of dimension 5 equipped with a rank-2 vector bundle $S \to M$, and an isomorphism, $\Lambda^1_H \cong \odot^3 S$ which preserves the Levi-form. By this we mean that $\mathcal{L} : L \to \Lambda^2_H$ has range in the second summand of the decomposition $\Lambda^2_H \cong (\odot^4 S \otimes \Lambda^2 S) \oplus (\Lambda^2 S \otimes \Lambda^2 S \otimes \Lambda^2 S)$.*

We will denote a G_2 contact structure like $S \to M$.

Definition 3.2.5 (Isomorphism of G_2 contact structures)**.** *Let $S \to M, T \to N$ be G_2 contact geometries. An isomorphism of G_2 contact geometries is a diffeomorphism $f : M \to N$ and an isomorphism $S \cong f^*T$ such that the following diagram commutes*

$$\begin{tikzcd} \odot^3 S \arrow[r] \arrow[d] & f^*(\odot^3 T) \arrow[d] \\ \Lambda^1_H(M) & f^*\Lambda^1_H(N) \arrow[l] \end{tikzcd}$$

where the bottom map is the vector bundle isomorphism induced by the pull-back of forms by a diffeomorphism. That is $(\omega, x) \mapsto f^*\omega$.

Remark 3.2.6. *In particular an isomorphism of G_2 contact structures on M over the identity is a vector bundle isomorphism $S \cong T$ such that*

$$\begin{tikzcd} \odot^3 S \arrow[rr] \arrow[dr] & & \odot^3 T \arrow[dl] \\ & \Lambda^1_H & \end{tikzcd}$$

commutes, where $\odot^3 S \to \odot^3 T$ is the map induced by $S \to T$. These are the "trivial" isomorphisms of G_2 contact geometries on M in the same sense that (M, cg) and (M, g) are isomorphic Riemannian structures when $c \in \mathbb{R}$.

3.3 G_2 contact geometry and spinor indices

On a G_2 contact structure we will alternately denote an element ϕ_a of $\Lambda^1_H \cong \odot^3 S$ like ϕ_{ABC}, with a triplet of uppercase symmetric Latin "spinor" indices corresponding to a single lower case Latin index.

The Levi bracket gives an isomorphism $L \cong \Lambda^2 S \otimes \Lambda^2 S \otimes \Lambda^2 S$. So $L^{1/3} \cong \Lambda^2 S$. Importantly, a choice of contact form $\alpha \in L$ distinguishes non-vanishing section ε_{AB} of $\Lambda^2 S$ defined by (abusing notation)

$$J_{ab} = \varepsilon_{A(D}\varepsilon_{B|E|}\varepsilon_{C|F)} \in \Lambda^2(\odot^3 S) \tag{3.3.1}$$

and conversely, a choice of non-vanishing section of $\Lambda^2 S$ gives a contact form.

If we define $\varepsilon^{AB} \in \Lambda^2 S^*$ by $\varepsilon_{BC}\varepsilon^{AC} = \delta_B{}^A$ then it is easy to calculate that

$$\varepsilon_{A(G}\varepsilon_{B|H|}\varepsilon_{C|I)}\varepsilon^{D(G|}\varepsilon^{E|H|}\varepsilon^{F|I)} = \delta_A{}^{(D}\delta_B{}^E\delta_C{}^{F)} \tag{3.3.2}$$

but the right-hand side is precisely how we would write $\delta_a{}^b$ in spinor indices and so

$$J^{ab} = \varepsilon^{A(D}\varepsilon^{B|E|}\varepsilon^{C|F)} \in \Lambda^2(\odot^3 S^*) \tag{3.3.3}$$

and everything is consistent. Since we would like the calculations in the following section to be manifestly invariant from now on, unless otherwise stated, we will denote the canonical section of $\Lambda^2 S \otimes \Lambda^2 S^*$ by ε_{AB} or ε^{AB} depending on the context, and use this to raise and lower indices. In order to then declutter notation, we will often omit indices corresponding to tensor powers of the line bundle $\Lambda^2 S$. For example given $\phi_A \in S$ we have

$$\phi^A := \varepsilon^{AB}\phi_B \in S^* \otimes \Lambda^2 S. \tag{3.3.4}$$

and given $\psi_{AB} \in \otimes^2 S$ we have.

$$\psi_C{}^C \in \Lambda^2 S \tag{3.3.5}$$

On a G_2 contact geometry it makes sense to denote a partial connection $E \to \odot^3 S \otimes E$ by ∇_{ABC}.

Given an affine partial connection on a G_2 contact geometry, the partial torsion $\Lambda^1_H \to \Lambda^2_{H\perp}$ defines a vector bundle homomorphism $\odot^3 S \to \odot^4 S \otimes \Lambda^2 S$. So the partial torsion is a section of $\odot^4 S \otimes \odot^3 S^* \otimes \Lambda^2 S$.

Using the projection $\Lambda^2 S(\odot^3 S) \to \odot^4 S \otimes \Lambda^2 S$ we write

$$\tau_{ABCD}{}^{EFG}\phi_{EFG} := (d_\perp \phi)_{ABCD} - \nabla_{K(AB}\phi_{CD)}{}^K \in \odot^4 S \otimes \Lambda^2 S. \tag{3.3.6}$$

Now lower indices to get a section

$$\tau_{ABCDEFG} \in \odot^4 S \otimes \odot^3 S \otimes \Lambda^2 S^* \otimes \Lambda^2 S^*. \tag{3.3.7}$$

We have a decomposition of bundles

$$\begin{aligned}&\odot^4 S \otimes \odot^3 S \otimes \Lambda^2 S^* \otimes \Lambda^2 S^* \cong \\ &\qquad (\odot^7 S \otimes \Lambda^2 S^* \otimes \Lambda^2 S^*) \oplus (\odot^5 S \otimes \Lambda^2 S^*) \oplus (\odot^3 S) \oplus (S \otimes \Lambda^2 S)\end{aligned} \tag{3.3.8}$$

In particular, we will now see the component $\tau_{(ABCDEFG)}$ in $\odot^7 S \otimes \Lambda^2 S^* \otimes \Lambda^2 S^*$ is an invariant of the G_2 contact structure. That is, it is independent of the chosen partial connection, and depends only on the isomorphism $\Lambda^1_H \cong \odot^3 S$ which defines the structure.

We now state a theorem analogous to the existence and uniqueness of the Levi-Civita connection on a Riemannian manifold.

Theorem 3.3.9 (Minimal partial torsion)**.** *Given a non-vanishing section (scale) $\varepsilon_{AB} \in \Lambda^2 S$, or equivalently a choice of contact form $\alpha \in L$, there is a unique partial connection ∇_{ABC} on S such that $\nabla_{ABC}\varepsilon_{DE} = 0$ and such that the induced partial connection on $\odot^3 S$ has $\tau_{ABCDEFG} = \tau_{(ABCDEFG)}$. Furthermore, the tensor $\tau_{ABCDEFG}$ is independent of the chosen contact form, and thus is an invariant of the G_2 contact structure.*

Proof. The freedom in choosing a partial connection on S is

$$\hat{\nabla}_{ABC}\phi_D = \nabla_{ABC}\phi_D + \Gamma_{ABCD}{}^E\phi_E \tag{3.3.10}$$

where $\Gamma_{ABCD}{}^E \in \odot^3 S \otimes \mathrm{End}(S)$ is arbitrary. Changing connection and using 3.3.6 the new torsion is

$$\hat{\tau}_{ABCD}{}^{EFG}\phi_{EFG} = \tau_{ABCD}{}^{EFG}\phi_{EFG} - \Gamma_{E(AB}{}^{EF}\phi_{CD)F} + 2\Gamma^E{}_{(ABC}{}^F\phi_{D)EF} \tag{3.3.11}$$

Since ϕ_{ABC} is arbitrary

$$\hat{\tau}_{ABCD}{}^{EFG} = \tau_{ABCD}{}^{EFG} - \Gamma_{H(AB}{}^{H(E}\delta_C{}^F\delta_{D)}{}^{G)} + 2\Gamma^{(E}{}_{(ABC}{}^F\delta_{D)}{}^{G)}. \tag{3.3.12}$$

In particular, lowering all indices and symmetrising gives $\hat{\tau}_{(ABCDEFG)} = \tau_{(ABCDEFG)}$.

On $\Lambda^2 S$, for a non-vanishing $\varepsilon_{AB} \in \Lambda^2 S$

$$\begin{aligned}\hat{\nabla}_{ABC}\varepsilon_{DE} &= \nabla_{ABC}\varepsilon_{DE} + \Gamma_{ABCE}{}^F\varepsilon_{DF} + \Gamma_{ABCD}{}^F\varepsilon_{FE} \\ &= \nabla_{ABC}\varepsilon_{DE} + 2\Gamma_{ABC[DE]}.\end{aligned} \tag{3.3.13}$$

So clearly we can arrange that $\hat{\nabla}_{ABC}\varepsilon_{DE} = 0$. The remaining freedom lies in $\Gamma_{ABC(DE)} \in \odot^3 S \otimes \odot^2 S \otimes \Lambda^2 S^*$.

We consider the kernel of the map $\odot^3 S \otimes \odot^2 S \otimes \Lambda^2 S^* \to \odot^4 S \otimes \odot^3 S^* \otimes \Lambda^2 S$

$$\Gamma_{ABCDE} \mapsto -\Gamma_{H(AB}{}^{H(E}\delta_C{}^F\delta_{D)}{}^{G)} + 2\Gamma^{(E}{}_{(ABC}{}^F\delta_{D)}{}^{G)}. \tag{3.3.14}$$

The domain breaks up like $\odot^3 S \otimes \odot^2 S \otimes \Lambda^2 S^* \cong (\odot^5 S \otimes \Lambda^2 S^*) \oplus \odot^3 S \oplus (S \otimes \Lambda^2 S)$ and we will check that for Γ_{ABCDE} in the kernel, each of its irreducible parts vanishes. Contract over indices to verify that

$$\begin{aligned} &- \Gamma_{H(AB}{}^{H(E}\delta_C{}^F\delta_{D)}{}^{G)} + 2\Gamma^{(E}{}_{(ABC}{}^F\delta_{D)}{}^{G)} = 0 \\ \implies &- \Gamma_{H(AB}{}^{H(E}\delta_{C)}{}^{F)} - \frac{1}{6}\Gamma_{HD(A}{}^{HD}\delta_B{}^{(E}\delta_{C)}{}^{F)} + \frac{7}{6}\Gamma^{(E}{}_{(ABC)}{}^{F)} \\ &+ \frac{1}{3}\Gamma^{(E}{}_{D(AB}{}^{|D|}\delta_{C)}{}^{F)} + \frac{1}{6}\Gamma^D{}_{(AB|D|}{}^{(E}\delta_{C)}{}^{F)} = 0. \end{aligned} \tag{3.3.15}$$

Lowering all unpaired indices and symmetrising yields $\Gamma_{(ABCDE)} = 0$. Contract again to get

$$-\frac{7}{6}\Gamma_{HAB}{}^{HE} + \frac{5}{9}\Gamma^E{}_{H(AB)}{}^H - \frac{2}{3}\Gamma_{HC(A}{}^{HC}\delta_{B)}{}^E = 0 \tag{3.3.16}$$

and lowering all indices and symmetrising yields $\Gamma_{E(ABC)}{}^E = 0$. Contracting a final time yields $\Gamma_{AEF}{}^{EF} = 0$ and hence $\Gamma_{ABCDE} = 0$ so that the map $\odot^3 S \otimes \odot^2 S \otimes \Lambda^2 S^* \to \odot^4 S \otimes \odot^3 S^* \otimes \Lambda^2 S$ is injective. The target bundle breaks up like

$$\begin{aligned} &\odot^4 S \otimes \odot^3 S^* \otimes \Lambda^2 S \\ &\quad \cong (\odot^7 S \otimes \Lambda^2 S^* \otimes \Lambda^2 S^*) \oplus (\odot^5 S \otimes \Lambda^2 S^*) \oplus \odot^3 S \oplus (S \otimes \Lambda^2 S) \end{aligned} \tag{3.3.17}$$

and the range is contained in the trailing bundles. The sum of the ranks of the these bundles is 12 which is exactly the rank of $\odot^3 S \otimes \odot^2 S \otimes \Lambda^2 S^*$ and hence the map is a bijection onto these bundles. In particular there is a unique Γ_{ABCDE} to eliminate all of the partial torsion except for the component in $\odot^7 S \otimes \Lambda^2 S^* \otimes \Lambda^2 S^*$. □

From the general theory of parabolic geometries it is known:

Theorem 3.3.18 ([EN20a] Theorem 5)**.** *A G_2 contact geometry $S \to M$ is locally isomorphic to G_2/P_2 as a G_2 contact geometry if and only if $\tau_{ABCDEFG}$ vanishes.*

Proposition 3.3.19 (Change of scale)**.** *Replacing $\hat{\varepsilon}_{AB} = \Omega^{\frac{1}{3}}\varepsilon_{AB}$ the partial connection on S corresponding to the scale $\hat{\varepsilon}_{AB}$ with minimal partial torsion is given by the formula*

$$\hat{\nabla}_{ABC}\phi_D = \nabla_{ABC}\phi_D + \frac{1}{3}\Upsilon_{ABC}\phi_D - \Upsilon_{D(AB}\phi_{C)}. \tag{3.3.20}$$

where $\Upsilon_{ABC} = \nabla_{ABC}\Omega$.

Proof. First we check that the torsion is invariant under a change of partial connection as given by the above formula. We have

$$\hat{\nabla}_{ABC}\phi_{DEF} = \nabla_{ABC}\phi_{DEF} + \Upsilon_{ABC}\phi_{DEF} - 3\Upsilon_{(D|(AB}\phi_{C)|EF)}. \tag{3.3.21}$$

Contracting with ε^{AF} and then symmetrising yields

$$\hat{\nabla}_{K(AB}\phi_{CD)}{}^{K} = \nabla_{K(AB}\phi_{CD)}{}^{K}. \tag{3.3.22}$$

Then we check that the new connection annihilates $\hat{\varepsilon}_{DE}$.

$$\begin{aligned}
&\hat{\nabla}_{ABC}\Omega^{\frac{1}{3}}\varepsilon_{DE} \\
&= \nabla_{ABC}(\Omega^{\frac{1}{3}}\varepsilon_{DE}) + \frac{2}{3}\Upsilon_{ABC}\Omega^{\frac{1}{3}}\varepsilon_{DE} - \Upsilon_{D(AB}\Omega^{\frac{1}{3}}\varepsilon_{C)E} + \Upsilon_{E(AB}\Omega^{\frac{1}{3}}\varepsilon_{C)D} \\
&= \frac{1}{3}\Upsilon_{ABC}\Omega^{\frac{1}{3}}\varepsilon_{DE} + \frac{2}{3}\Upsilon_{ABC}\Omega^{\frac{1}{3}}\varepsilon_{DE} - \Upsilon_{D(AB}\Omega^{\frac{1}{3}}\varepsilon_{C)E} + \Upsilon_{E(AB}\Omega^{\frac{1}{3}}\varepsilon_{C)D}.
\end{aligned} \tag{3.3.23}$$

The above vanishes after contraction with ε^{DE} which implies $\hat{\nabla}_{ABC}\hat{\varepsilon}_{DE} = 0$. □

Now on the line bundle $\Lambda^2 S$ the connection changes like

$$\hat{\nabla}_{ABC}\varepsilon_{DE} = \nabla_{ABC}\varepsilon_{DE} + \frac{2}{3}\Upsilon_{ABC}\varepsilon_{DE} - \Upsilon_{D(AB}\varepsilon_{C)E} + \Upsilon_{E(AB}\varepsilon_{C)D}. \tag{3.3.24}$$

Contracting yields

$$\hat{\nabla}_{ABC}\varepsilon_{DE} = \nabla_{ABC}\varepsilon_{DE} - \frac{1}{3}\Upsilon_{ABC}\varepsilon_{DE}. \tag{3.3.25}$$

So if we say $\phi_A \in S \otimes (\Lambda^2 S^*)^{3w}$ has weight w then, for future reference

$$\hat{\nabla}_{ABC}\phi_D = \nabla_{ABC}\phi_D + \left(\frac{1}{3} + w\right)\Upsilon_{ABC}\phi_D - \Upsilon_{D(AB}\phi_{C)}. \tag{3.3.26}$$

Enforcing the Leibniz rule gives the connection applied to $\psi^A \in S^* \otimes (\Lambda^2 S^*)^{3w}$

$$\hat{\nabla}_{ABC}\psi^D = \nabla_{ABC}\psi^D + \left(\frac{2}{3} + w\right)\Upsilon_{ABC}\psi^D - \Upsilon^D{}_{(AB}\psi_{C)}. \tag{3.3.27}$$

Now that we have formulae for how the preferred partial connection changes, we can use these to construct quantities that are intrinsic to the G_2 contact structure.

3.4 Partial curvature on a G_2 contact geometry

We now investigate the partial curvature of the preferred partial connections on G_2 contact geometries.

Translating 2.2.7 into spinor indices means we write the partial curvature of a partial connection $\nabla_{ABC} : S \to \odot^3 S \otimes S$ like $R_{ABCDE}{}^F \in \odot^4 S \otimes \mathrm{End(S)} \otimes \Lambda^2 S$, which is defined by

$$R_{ABCDE}{}^F \phi_F = \nabla_{K(AB} \nabla_{CD)}{}^K \phi_E + \tau_{ABCD}{}^{IJK} \nabla_{IJK} \phi_E. \tag{3.4.1}$$

We first examine its symmetries. Firstly, applying the Leibniz rule on simple tensors then extending by linearity gives

$$\begin{aligned}\nabla_{K(AB} \nabla_{CD)}{}^K T_{E_1 \ldots E_n} &= \\ R_{ABCDE_1}{}^K T_{K \ldots E_n} + \ldots + R_{ABCDE_N}{}^K T_{E_1 \ldots K} &- \tau_{ABCD}{}^{IJK} \nabla_{IJK} T_{E_1 \ldots E_n}\end{aligned} \tag{3.4.2}$$

for any $T_{E_1 \ldots E_n}$ a section of $\otimes^n S$. In particular, applying this identity to the section ε_{AB} of $\Lambda^2 S$ gives the first useful symmetry.

Proposition 3.4.3. *For the preferred partial connection $\nabla_{ABC} : S \to \otimes^3 S \otimes S$ annihilating ε_{AB} on a G_2 contact geometry we have $R_{ABCDEF} = R_{ABCD(EF)}$.*

Proof. $\nabla_{ABC} \varepsilon_{DE} = 0$ with 3.4.2 gives $R_{ABCDE}{}^K \varepsilon_{KF} + R_{ABCDF}{}^K \varepsilon_{EK} = 0$ which is to say $R_{ABCDEF} = R_{ABCDFE}$.

Proposition 3.4.4. *If the preferred partial connection $\nabla_{ABC} : S \to \odot^3 S \otimes S$ is partial torsion free, then $R_{(ABC|K|}{}^K{}_{D)} = 0$.*

Proof. Firstly we write down the curvature of the partial connection on $\Lambda^1_H \cong \odot^3 S$ in spinor indices. The Leibniz rule implies

$$\nabla_{K(AB} \nabla_{CD)}{}^K \phi_{EFG} = 3 R_{ABCD(E}{}^H \phi_{FG)H} = 3 R_{ABCD(E}{}^H \delta_F{}^I \delta_{G)}{}^J \phi_{HIJ}. \tag{3.4.5}$$

This is a consequence of the Bianchi symmetry 2.3.6, but it is not immediately obvious how to translate the symmetry operations in 2.3.6 to this situation. In order to make it clearer we will use the inclusion $\odot^4 S \otimes \Lambda^2 S \otimes \odot^3 S \otimes \odot^3 S^* \hookrightarrow \odot^3 S \wedge \odot^3 S \otimes \odot^3 S \otimes \odot^3 S^*$ and then raise indices to write the partial curvature as a section of $\odot^3 S \wedge \odot^3 S \otimes \odot^3 S \otimes \odot^3 S^*$. Given $\omega_{ABC\ DEF} \in \odot^3 S \wedge \odot^3 S$ we can uniquely write

$$\omega_{ABC\ DEF} = \lambda \varepsilon_{A(D|} \varepsilon_{B|E|} \varepsilon_{C|F)} - \rho_{(AB|(DE} \delta_{F)|C)} \tag{3.4.6}$$

for $\rho_{ABCD} \in \odot^4 S \otimes \Lambda^2 S$ and $\lambda \in \Lambda^2 S \otimes \Lambda^2 S \otimes \Lambda^2 S$. So the inclusion $\odot^4 S \otimes \Lambda^2 S \hookrightarrow \odot^3 S \wedge \odot^3 S$ is

$$\rho_{ABDE} \mapsto -\rho_{(AB|(DE}\delta_{F)|C)}. \tag{3.4.7}$$

Accordingly, if we apply this to the curvature and raise indices, the expression

$$3\delta_{(A}{}^{(D}R_{BC)}{}^{EF)}{}_{(G}{}^{(J}\delta_H{}^K\delta_{I)}{}^{L)} \tag{3.4.8}$$

represents $R_a{}^b{}_c{}^d$ with each lower case Latin index corresponding to a triple of upper case Latin indices (arranged in alphabetical order so that a corresponds to ABC and so on). We contract over triples to get the spinor expression for $R_d{}^b{}_c{}^d$.

$$\begin{aligned} 3\delta_{(J}{}^{(D}R_{KL)}{}^{EF)}{}_{(G}{}^J\delta_H{}^K\delta_{I)}{}^L &= R_{KL}{}^{(DE}{}_{(G}{}^{F)}\delta_H{}^K\delta_{I)}{}^L + 2\delta_{(G}{}^{(D}R_{H|J|}{}^{EF)}{}_{I)}{}^J \\ &= R_{(GH}{}^{(DE}{}_{I)}{}^{F)} + 2\delta_{(G}{}^{(D}R_{H|J|}{}^{EF)}{}_{I)}{}^J. \end{aligned} \tag{3.4.9}$$

So $R_{d[bc]}{}^d$ is given by lowering indices then antisymmetrising on exchange of symmetric triplets. The first term vanishes and so the identity $R_{d[ab]}{}^d = 0$ from 2.3.6 implies

$$\delta_{(D}{}^{(A}R_{E|G|}{}^{BC)}{}_{F)}{}^G + \delta_{(D}{}^{(A}R_{EF)G}{}^{BC)G} = 0. \tag{3.4.10}$$

Contract over indices C, F to get

$$R_{G(D}{}^{AB}{}_{E)}{}^G + R_{GDE}{}^{(AB)G} = 0. \tag{3.4.11}$$

In particular, lowering all indices then symmetrising gives the desired result. □

So we have obtained a necessary condition on a G_2 contact geometry being locally isomorphic to the flat model.

We also want a formula for the change of partial curvature.

Proposition 3.4.12 (Change of partial curvature)**.** *Given a change of scale by $\hat{\sigma} = \Omega\sigma$, the partial curvature of the distinguished connection changes according to the formula:*

$$\begin{aligned} \hat{R}_{ABCD}{}^{EF} &= R_{ABCD}{}^{EF} + \nabla_{(AB}{}^{(E}\Upsilon_{CD)}{}^{F)} - \frac{2}{3}\nabla_{(ABC}\Upsilon_{D)}{}^{EF} \\ &\quad + \frac{1}{3}\Upsilon_{(AB}{}^{(E}\Upsilon_{CD)}{}^{F)} - \frac{2}{3}\Upsilon_{(ABC}\Upsilon_{D)}{}^{EF} - \tau_{ABCD}{}^{IJ(E}\Upsilon^{F)}{}_{IJ}. \end{aligned} \tag{3.4.13}$$

Proof. Omitted. This is a simple but tedious calculation that may take a few pages. □

From 3.4.3 we have that $R_{ABCDEF} \in \odot^4 S \otimes \odot^2 S \otimes \Lambda^2 S$ decomposes according to 3.1.21 like

$$\odot^4 S \otimes \odot^2 S \cong \odot^6 S \oplus (\odot^4 S \otimes \Lambda^2 S) \oplus (\odot^2 S \otimes \Lambda^2 S \otimes \Lambda^2 S). \tag{3.4.14}$$

Before giving labels to each of these irreducible parts, it's convenient to work out how they change under change of scale and normalise accordingly. Firstly

$$\begin{aligned}\hat{R}_{(ABCDEF)} = R_{(ABCDEF)} + \frac{1}{3}\nabla_{(ABC}\Upsilon_{DEF)} - \frac{1}{3}\Upsilon_{(ABC}\Upsilon_{DEF)}\\ -\tau_{(ABCDE}{}^{IJ}\Upsilon_{F)IJ}.\end{aligned} \tag{3.4.15}$$

So define $Z_{ABCDEF} := 3R_{(ABCDEF)} \in \odot^6 S$. Next, contracting over D,E we have

$$\begin{aligned}\hat{R}_{ABC|E|}{}^{E}{}_{F} = R_{ABC|E|}{}^{E}{}_{F} + \frac{1}{4}\nabla^{E}{}_{(AB}\Upsilon_{C)FE} + \frac{1}{4}\nabla_{EF(A}\Upsilon_{BC)}{}^{E}\\ -\frac{1}{2}\nabla_{E(AB}\Upsilon_{C)F}{}^{E} + \frac{1}{6}\Upsilon^{E}{}_{(AB}\Upsilon_{C)FE} - \frac{1}{2}\Upsilon_{E(AB}\Upsilon_{C)F}{}^{E} + \frac{1}{2}\tau_{ABCF}{}^{IJK}\Upsilon_{IJK}.\end{aligned} \tag{3.4.16}$$

Hence, using the definition of the partial torsion, and noting that the terms quadratic in upsilon vanish under symmetrisation.

$$\hat{R}_{(ABC|K|}{}^{K}{}_{D)} = R_{(ABC|K|}{}^{K}{}_{D)} + \tau_{ABCF}{}^{IJK}\Upsilon_{IJK} \tag{3.4.17}$$

So define $W_{ABCD} := R_{(ABC|K|}{}^{K}{}_{D)} \in \odot^4 S \otimes \Lambda^2 S$. Note that this formula shows invariance under change of scale in the torsion free case, which is consistent with the earlier result that this vanishes with the torsion. In fact W_{ABCD} lies in the same bundle as

$$\nabla^{IJK}\tau_{IJKABCD} \in \odot^4 S \otimes \Lambda^2 S \tag{3.4.18}$$

and so we conjecture that if one were to repeat the proof of 2.3.6 while carefully handling the invariant partial torsion it would be possible to show that W_{ABCD} is proportional to the above.

Lastly

$$\hat{R}_{ABEF}{}^{EF} = R_{ABEF}{}^{EF} - \frac{2}{3}\nabla_{EF(A}\Upsilon_{B)}{}^{EF} - \frac{4}{9}\Upsilon_{EF(A}\Upsilon_{B)}{}^{EF}. \tag{3.4.19}$$

So define $Y_{AB} := -\frac{3}{2}R_{ABIJ}{}^{IJ} \in \odot^3 S \otimes \Lambda^2 S \otimes \Lambda^2 S$.

3.5 Twisted cubics and G_2 contact geometry

We will now present an equivalent definition for a G_2 contact geometry in terms of some basic algebraic geometry. A well known analogue of this alternate definition is the equivalence between a field of quadratic cones in the tangent bundle of a manifold M and a Lorentzian metric defined up to scale, hence a conformal structure $(M,[g])$. For example, the equivalence is explained in [MT13].

Recall the definition 3.2.4 in terms of an identification $\odot^3 S \cong \Lambda^1_H$, where S is some rank-2 vector bundle. Of course once we choose a contact form $\alpha \in L$ the Levi form $d\alpha|_H$ identifies $\Lambda^1_H \cong H$ and hence we might as well instead consider isomorphisms $\odot^3 S \cong H$. By rescaling S is easy to verify that the resulting G_2 contact geometry is independent of the choice of contact form up to isomorphism in the sense of 3.2.6. For an isomorphism $\odot^3 S \cong H$ to induce a G_2 contact geometry we require that the Levi form $H \wedge H \to \mathbb{R}$ must vanish on the subbundle in $H \wedge H$ identified with $\odot^4 S \otimes \Lambda^2 S$.

Given a rank-2 vector bundle S over M there is an obvious smooth map $t : S \to \odot^3 S$ defined by

$$t : s \mapsto s \odot s \odot s. \tag{3.5.1}$$

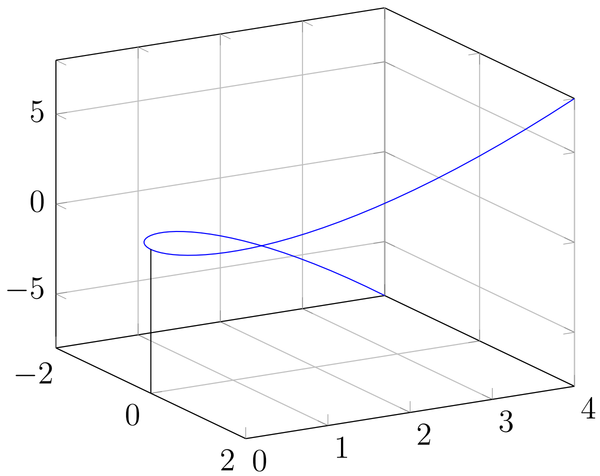
Pick locally trivialising sections e_1,e_2 for S then in the fibre over $p \in M$ this gives a map $\mathbb{R}^2 \to \odot^3 S_p$

$$\begin{aligned}(\lambda,\tau) \mapsto &(\lambda e_1 + \tau e_2) \odot (\lambda e_1 + \tau e_2) \odot (\lambda e_1 + \tau e_2)\\ &= (\lambda^3, \lambda^2\tau, \lambda\tau^2, \tau^3),\end{aligned} \tag{3.5.2}$$

where we are writing elements of $\odot^3 S_p$ with respect to the basis $\{e_1^3, 3e_1^2 e_2^1, 3e_1 e_2^2, e_2^3\}$. If we take the projectivisation $\mathbb{R}P^1 \to P(\odot^3 S_p) \cong \mathbb{R}P^3$ of this curve, using homogeneous coordinates it is, to borrow

$$[1 : \tau] \mapsto [1 : \tau : \tau^2 : \tau^3] \tag{3.5.3}$$

and in the corresponding coordinate chart of $\mathbb{R}P^3$ looks like the blue curve below.



Let U, V be vector spaces of dimension 2 and 4 respectively. We will say a curve $U \to V$ is called a twisted cubic if it can be written like $(\lambda, \tau) \mapsto (\lambda^3, \lambda^2\tau, \lambda\tau^2, \tau^3)$ with respect to some bases for U and V. Usually, the term "twisted cubic" would be reserved for the projectivisation, but we take the above definition for convenience. An isomorphism $\odot^3 S \cong H$ defines a twisted cubic in each fibre of H.

Recall the compatibility condition that the Levi form vanish on $\odot^4 S \otimes \Lambda^2 S$. The projection of $t(\lambda, \tau) \wedge t(\tilde{\lambda}, \tilde{\tau}) = (\lambda^3, \lambda^2\tau, \lambda\tau^2, \tau^3) \wedge (\tilde{\lambda}^3, \tilde{\lambda}^2\tilde{\tau}, \tilde{\lambda}\tilde{\tau}^2, \tilde{\tau}^3) \in \odot^3 S_p \wedge \odot^3 S_p$ onto $\Lambda^2 S_p \otimes \Lambda^2 S_p \otimes \Lambda^2 S_p$ is proportional $(\lambda\tilde{\tau} - \tilde{\lambda}\tau)^3$ and thus the Levi form restricted to the twisted cubic must be given by $t(\lambda, \tau) \wedge t(\tilde{\lambda}, \tilde{\tau}) \mapsto c(\lambda\tilde{\tau} - \tilde{\lambda}\tau)^3$ for some constant $c \in \mathbb{R} \neq 0$.

Now we show the data above gives a G_2 contact geometry. Suppose we have rank-2 vector bundle $S \to M$ and a smooth map $t : S \to H$ over M which gives a twisted cubic in each fibre. We will say this is a field of twisted cubics. Furthermore, we suppose the existence of local trivialisations for S such that the Levi form is written $t(\lambda, \tau) \wedge t(\tilde{\lambda}, \tilde{\tau}) \mapsto c(\lambda\tilde{\tau} - \tilde{\lambda}\tau)^3$ on the twisted cubic. If this is the case, we will say the curve is compatible with the Levi form. Starting with this data, take a local trivialisation $\{e_1, e_2\}$ for S. Given a point p in this trivialising neighbourhood U it is simple to check, using the fact points on the twisted cubic span $\mathbb{R}^4$, that the basis for H_p such that $t : S_p \to H_p$ takes the form

$$t(\lambda, \tau) = (\lambda^3, \lambda^2\tau, \lambda\tau^2, \tau^3) \tag{3.5.4}$$

is unique. Then, the smoothness of t implies there is a unique trivialisation $\{s_1, s_2, s_3, s_4\}$ of H such that the t takes this form as a bundle map over that trivialisation.

Define a local isomorphism $H|_U \to \odot^3 S|_U$ by

$$s_1 \mapsto e_1^3, \;\; s_2 \mapsto 3e_1^2 e_2, \;\; s_3 \mapsto 3e_1 e_2^2, \;\; s_4 \mapsto e_2^3. \tag{3.5.5}$$

It is not immediately clear that we get a well defined vector bundle homomorphism $H \to \odot^3 S$ by defining the map as above in each such trivialisation. However, we will show that 3.5.5 is invariantly defined as long as the change of trivialisation occurs in such a way t takes the form 3.5.4 in the new trivialisation. To see this suppose we take a differential trivialisation $\{\tilde{e}_1, \tilde{e}_2\}$. Then at a point we have $\lambda\tilde{e}_1 + \tau\tilde{e}_2 = (a\lambda + b\tau)e_1 + (c\lambda + d\tau)e_2$ where we define:

$$A := \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbb{R}). \tag{3.5.6}$$

The twisted cubic t is then written

$$t(\lambda, \tau) = \Big((a\lambda + b\tau)^3, (a\lambda + b\tau)^2(c\lambda + d\tau), (a\lambda + b\tau)(c\lambda + d\tau)^2, (c\lambda + d\tau)^3\Big) \tag{3.5.7}$$

with respect to trivialisations $\{\tilde{e}_1, \tilde{e}_2\}$ and $\{s_1, s_2, s_3, s_4\}$. Inspecting the right-hand side we see that the updated trivialisation for H which we are obliged to take in order to return t to the form 3.5.4 is gotten by acting by the matrix which is the representation of A on $\odot^3\mathbb{R}^2 \cong \mathbb{R}^4$, but this is precisely how the trivialisation $\{e_1^3, 3e_1^2e_2^1, 3e_1e_2^2, e_2^3\}$ transforms. All this means that the map

$$\tilde{s}_1 \mapsto \tilde{e}_1^3,\ \tilde{s}_2 \mapsto 3\tilde{e}_1^2\tilde{e}_2,\ \tilde{s}_3 \mapsto 3\tilde{e}_1\tilde{e}_2^2,\ \tilde{s}_4 \mapsto \tilde{e}_2^3. \tag{3.5.8}$$

agrees with 3.5.5.

Lastly that the Levi form is written $t(\lambda,\tau)\wedge t(\tilde{\lambda},\tilde{\tau}) \mapsto c(\lambda\tilde{\tau}-\tilde{\lambda}\tau)^3$ on the twisted cubic, implies that the Levi form $H \wedge H \to \mathbb{R}$ depends only on the component in the subspace identified with $\Lambda^2 S \otimes \Lambda^2 S \otimes \Lambda^2 S$, and hence vanishes on the subspace identified with $\odot^4 S \otimes \Lambda^2 S$, again using the fact points on the twisted cubic span the space. So the isomorphism $H \to \odot^3 S$ preserves the Levi form.

Thus we have an alternate characterisation of a G_2 contact geometry.

Definition 3.5.9 (G_2 contact structure II)**.** *A G_2 contact structure on a contact manifold M is a field of twisted cubic curves in H such that the curve is compatible with the Levi form.*

The abstract way of interpreting this equivalence, is that the presence of the field of twisted cubic reduces the structure group of the distribution H to $GL(2,\mathbb{R}) \hookrightarrow GL(4,\mathbb{R})$ acting irreducibly, and furthermore this reduction is done in such a way as to factor $GL(2,\mathbb{R}) \hookrightarrow CSp(4,\mathbb{R}) \hookrightarrow GL(4,\mathbb{R})$ where $CSp(4,\mathbb{R})$ is the conformal symplectic group defined as the subgroup of $GL(4,\mathbb{R})$ which preserves the canonical symplectic form up to scale.

Note that the above argument does not really use the fact (M, H) is a contact manifold, aside for the fact we have a symplectic form up to scale on H. Without the compatibility condition the above argument shows that a field of twisted cubics $S \to E$ reduces the structure group of a rank-4 vector bundle E to $GL(2,\mathbb{R})$ acting irreducibly.

In [Bry91], Bryant investigates the geometry associated with such reductions of the structure group of the tangent bundle of 4-manifolds. He gives the alternate definition as a field of twisted cubic curves. It may be a good direction for future investigation to see if the development in that article can be adapted to the study of G_2 contact geometries.

Chapter 4

Flying saucers

In this chapter we investigate the problem of constructing a G_2 contact geometry. The construction follows Eastwood and Nurowski's pair of articles [EN20a] [EN20b]. One starts with a projective structure on a 3-dimensional manifold M. The Grassmannian of oriented planes in the tangent bundle forms a fibre bundle $\pi : C \to M$ and it is a 5-dimensional contact manifold. It was an observation of Takeuchi [Tak94] that the projective structure on M induces a splitting of the contact distribution into a pair of rank-2 subbundles. Given a choice of two linearly independent 1-forms on M, the authors construct a G_2 contact geometry in a neighbourhood of any point on C. The construction involves the notion of an Ehresmann connection on a vector bundle, which is an interpretation of a linear connection as a splitting of the tangent bundle of the total space of a vector bundle, so we introduce these first. We also define projective structures.

4.1 Ehresmann connections

Firstly, we need a notion of those vectors tangent to the total space of a fibre bundle, which are velocities of curves lying in a fixed fibre.

Definition 4.1.1 (Vertical bundle)**.** *Let* $\mathrm{pr} : F \to M$ *be a smooth fibre bundle. We have the tangent map* $d\,\mathrm{pr} : TF \to TM$*. The subbundle* $VF := \ker d\,\mathrm{pr}$ *is called the vertical bundle of* F*.*

At the heart of the general theory of connections on fibre bundles is the problem of defining what it means for a curve to be "constant" with respect to fibre coordinates. See [KMS93] for a more comprehensive, technical treatment of this matter. We briefly summarise the story for vector bundles here.

Let $\mathrm{pr} : E \to M$ be a vector bundle. We have a canonical inclusion $VE \to TE$

and projection $TE \to TE/VE$ which gives a short exact sequence

$$0 \longrightarrow VE \longrightarrow TE \longrightarrow TE/VE \longrightarrow 0. \tag{4.1.2}$$

By writing out a linear connection ∇ in local coordinates as in 2.1.3 one can show that for each $x \in M$ and $y \in E_x$ there exists a unique 1-jet at x with target y (equivalence class of sections of E with $s(x) = y$ under the equivalence relation $s \sim s'$ if s and s' have the same partial derivatives to first order with respect to any coordinates) such that $\nabla s|_x = 0$. Define $H_y E = ds(T_x M)$. We have $d(\mathrm{pr} \circ s) = \mathrm{id}_{TM}$ and so $H_y E$ must be of dimension $\dim(M)$ and be complementary to $V_y E$. Since the first order differential equation defined by $\nabla s = 0$ varies smoothly across E these subspaces $H_y E$, pointwise tangent to E, together form a subbundle denoted HE, called the horizontal bundle of E. The linearity of ∇ also enforces that $H_{\mu y} E = d\mu(H_y E)$ where $\mu : E \to E$ is fibrewise multiplication by the real number μ.

In [KMS93] it is shown that such data defines a connection on E. So we give a second definition of a linear connection.

Definition 4.1.3 (Connection II)**.** *Let* $\mathrm{pr} : E \to M$ *be a vector bundle. A (linear) connection on* E *is a choice of complementary subbundle* HE *to* VE *satisfying* $H_{\mu y} E = d\mu(H_y E)$.

So a linear connection defines a splitting of the short exact sequence 4.1.2, alternatively a projection $TE \to VE$ called the vertical projection.

There are some convenient identifications for the bundles in this exact sequence that we will use. In particular we have canonical isomorphisms

$$\begin{aligned} TE/VE &\xrightarrow{\sim} \mathrm{pr}^* TM \\ [v] &\mapsto (p(v), d\,\mathrm{pr}(v)) \end{aligned} \tag{4.1.4}$$

where $p : TE \to E$ is the projection, and

$$\begin{aligned} \mathrm{pr}^* E &\xrightarrow{\sim} VE \\ (u, v) &\mapsto \frac{d}{dt}(u + tv)\Big|_{t=0}. \end{aligned} \tag{4.1.5}$$

Note the first identification works for an arbitrary fibre bundle, whereas the second only works for vector bundles and is a consequence of the fact the tangent spaces of a vector space can be canonically identified with that vector space.

So we see a linear connection defines a splitting $TE \xrightarrow{\sim} \mathrm{pr}^* TM \oplus \mathrm{pr}^* E$. Also note that the identification $\mathrm{pr}^* E \xrightarrow{\sim} VE$ defines a tautological section t^γ of VE whose value at $f \in E$ is the image of $(f, f) \in \mathrm{pr}^* E$ in VE.

Recall that the most general change of connection is given by

$$\hat{\nabla}_a s^\beta = \nabla_a s^\beta + \Gamma_{a\beta}{}^\gamma s^\beta \tag{4.1.6}$$

where $\Gamma_{a\beta}{}^\gamma \in \Lambda^1 \otimes \operatorname{End}(E)$.

Proposition 4.1.7. *Given such a change of connections, the splitting $TE \xrightarrow{\sim} \operatorname{pr}^* TM \oplus \operatorname{pr}^* E$ changes according to*

$$\widehat{\begin{bmatrix} X^b \\ r^\beta \end{bmatrix}} = \begin{bmatrix} X^b \\ r^\beta - X^a \Gamma_{a\gamma}{}^\beta t^\gamma \end{bmatrix} \tag{4.1.8}$$

Proof. Over a coordinate chart $(x^1, ..., x^n)$ and trivialisation $(e_1, ..., e_r)$, we get induced coordinates $(\tilde{x}^1, ..., \tilde{x}^n, \tilde{e}_1, ..., \tilde{e}_r)$ for E. Writing our connection

$$\nabla s = ds^i \otimes e_i + \Gamma_{ij}{}^k s^j dx^i \otimes e_k \tag{4.1.9}$$

for a local section $s = s^i e_i$, one sees that the 1-jet with $s(x) = f = f^i e_i(x) \in E$ annihilated by the connection has differential $T_x M \hookrightarrow T_f E$ given by

$$ds|_x\Big(X^i \frac{\partial}{\partial x^i}\Big) = -\Gamma_{ij}{}^k f^j X^i \frac{\partial}{\partial \tilde{e}^k} + X^i \frac{\partial}{\partial \tilde{x}^i}. \tag{4.1.10}$$

and the image of this map is $H_f E$. The vertical projection of $v \in T_f E$ is given by

$$v \mapsto v - ds|_x(d\pi|_f(v)). \tag{4.1.11}$$

Given a new connection with connection matrix $\hat{\Gamma}_{ij}{}^k$ the change is captured by

$$d\hat{s}|_x(d\pi|_f(v)) = ds|_x(d\pi|_f(v)) - (\hat{\Gamma}_{ij}{}^k - \Gamma_{ij}{}^k) f^j X^i \frac{\partial}{\partial \tilde{e}^k}. \tag{4.1.12}$$

Here we write $v = r^i \frac{\partial}{\partial \tilde{e}^i} + X^i \frac{\partial}{\partial \tilde{x}^i}$ so $d\pi(v) = X^i \frac{\partial}{\partial x^i}$. Finally since the difference between two connections is a tensorial quantity, $\{\hat{\Gamma}_{ij}{}^k - \Gamma_{ij}{}^k\}$ gives a well-defined tensor field $\Gamma_{a\gamma}{}^\beta$, so we can write everything in abstract indices as in 4.1.8.

4.2 Projective differential geometry

We briefly introduce some of the fundamentals of projective differential geometry. A projective structure is an equivalence class of torsion free connections with the same unparametrised geodesics. It turns out, see [Eas08] that the precise freedom that should be allowed is as follows:

Definition 4.2.1 (Projective structure)**.** *Let M be an oriented smooth manifold of dimension n with cotangent bundle Λ^1. A projective structure is an equivalence class of connections $[\nabla_a]$ on Λ^1 where $\hat{\nabla}_a \sim \nabla_a$ if and only if*

$$\hat{\nabla}_a \phi_b = \nabla_a \phi_b - \Upsilon_a \phi_b - \Upsilon_b \phi_a, \tag{4.2.2}$$

where Υ_a is some 1-form.

Note that the formula for the change of connection on Λ^n is

$$\hat{\nabla}_a \omega_{b_1 \dots b_n} = \nabla_a \omega_{b_1 \dots b_n} - n\Upsilon_a \omega_{b_1 \dots b_n} - \Upsilon_{b_1} \omega_{a \dots b_n} - \dots - \Upsilon_{b_n} \omega_{b_1 \dots a}. \tag{4.2.3}$$

If $\omega_{b_1 \dots b_n}$ is non-zero, there is a unique $\omega^{c_1 \dots c_n}$ such that $\omega_{ab_1 \dots b_{n-1}} \omega^{cb_1 \dots b_{n-1}} = \delta_a{}^c$. Contracting with $\omega^{b_1 \dots b_n}$ defines an isomorphism $\Lambda^1 \otimes \Lambda^n \to \Lambda^1$ so one can check

$$\hat{\nabla}_a \omega_{b_1 \dots b_n} = \nabla_a \omega_{b_1 \dots b_n} - (n+1)\Upsilon_a \omega_{b_1 \dots b_n}. \tag{4.2.4}$$

Thus, noting that Λ^n is a line bundle, there is a unique ∇_a in the projective class such that $\omega_{b_1 \dots b_n}$ is parallel. So a non-vanishing section of Λ^n distinguishes a connection in the projective class.

In light of this, it is useful to define projective densities, which allow calculus on a projective manifold to be written in a more manifestly invariant manner. Just like for a G_2 contact geometry, where one has the canonical section $\varepsilon_{AB} \in \Lambda^2 S \otimes \Lambda^2 S^*$ to play the role of the skew form while remaining agnostic to the choice of contact form, the machinery of projective densities allows one to have a canonical section $\varepsilon_{a_1 \dots a_n} \in \Lambda^n \otimes \Lambda^{-n}$ to play the role of the volume form.

Definition 4.2.5 (Projective densities)**.** *Define $\varepsilon(w) := (\Lambda^n)^{-\frac{w}{n+1}}$, called the bundle of densities of weight w. This root of the (trivial) determinant bundle is conveniently realised as the bundle whose sections are functions on Λ^n_+ of homogeneity $\frac{w}{n+1}$, where Λ^n_+ is the set of positively oriented points, (excluding zero) in the determinant bundle. That is $f : \Lambda^n_+ \to \mathbb{R}$ is a section if*

$$f(c\omega) = c^{\frac{w}{n+1}} f(\omega) \quad \omega \in \Lambda^n_+, \ c > 0 \tag{4.2.6}$$

Then we call $\Lambda^k \otimes \varepsilon(w)$ the bundle of k-forms of weight w and we write its sections like $\mu_{a_1 \dots a_k}$.

There is an isomorphism

$$\begin{aligned} &\Lambda^n \cong \varepsilon(-n-1) \\ &\Gamma(\Lambda^n) \ni \phi \mapsto f \text{ defined by } \phi = f(\psi)\psi, \ \forall \psi \in \Gamma(\Lambda^n_+) \end{aligned} \tag{4.2.7}$$

and isomorphisms

$$\begin{aligned}&\otimes^w \varepsilon(n) \cong \varepsilon(nw)\\ &f_1 \otimes ... \otimes f_w \mapsto f_1 ... f_w.\end{aligned} \tag{4.2.8}$$

These isomorphisms show that the projective densities of weight w are an associated bundle of M and accordingly, given a ∇_a in the projective class, we get a unique connection on the projective densities of weight w, computed by enforcing the Leibniz rule with respect to tensor products.

Now for $f \in \varepsilon(-n-1)$ we get, given a change of in connection in the projective class by the one form Υ_a,

$$\hat{\nabla}_a f = \nabla_a f - (n+1)\Upsilon_a f. \tag{4.2.9}$$

The change of induced connection on $\varepsilon(w)$ must then be

$$\hat{\nabla}_a f = \nabla_a f + w\Upsilon_a f \tag{4.2.10}$$

and hence for sections of $\Lambda^1 \otimes \varepsilon(w)$

$$\hat{\nabla}_a \mu_b = \nabla_a \mu_b + (w-1)\Upsilon_a \phi_b - \Upsilon_b \phi_a. \tag{4.2.11}$$

Note that the Killing equation for 1-forms,

$$\nabla_{(a}\phi_{b)} = 0, \tag{4.2.12}$$

is projectively invariant for 1-forms of weight 2.

See [CG18] for another treatment of densities, in the conformal case.

4.3 The configuration space C of a flying saucer

Let M be an oriented 3-dimensional manifold. Construct a smooth fibre bundle with typical fibre $\mathbb{R}^3/\{0\}$ by taking the cotangent bundle $\mathrm{pr} : T^*M \to M$ then deleting the zero section. Call this bundle $T^*M \setminus \{0\}$. Next, there is a smooth, free, proper $\mathbb{R}^+$ action, given by multiplication of cotangent vectors, and taking the quotient by this action, we get a bundle over M

$$\pi : \frac{T^*M \setminus \{0\}}{\mathbb{R}^+} \to M. \tag{4.3.1}$$

We briefly note that the total space is a good model for the configuration space of an alien 'flying saucer', with cockpit, as traditionally depicted in science-fiction.

A configuration is precisely the choice of a point $x \in M$ for the centre of mass, along with a choice of oriented plane (in which the saucer lies in) in T_pM. Such a choice of oriented plane is equivalent to a choice of ray in the cotangent space, by the right-hand rule. Informed by this novel observation we use the notation $\pi : C \to M$ for the above bundle.

It is well known that there is a contact structure on C, and we will show that here. The authors of [EN20a], [EN20b] point out that the contact distribution on C is given by the velocities of a flying saucer, as traditionally depicted. The velocity of the centre of mass is usually shown parallel to the oriented plane in which the saucer lies, while the orientation of that plane can change freely. See, for example, the Millennium Falcon's daring manoeuvres in Return of the Jedi [Mar83]. So this constraint may define a rank four distribution in TC.

Proposition 4.3.2. *There is a canonical filtration $V \subset H \subset TC$ of the tangent bundle with $V = \ker d\pi$ and H a rank 4 subbundle.*

Proof. Consider the pull-back bundle π^*TM. The elements are (p, v) for $p \in C$ and $v \in TM$ which lie above the same base point $x \in M$. There is a canonical bundle map $TC \to \pi^*TM$ given by $T_pC \ni u \mapsto (p, d\pi(u))$. A point $p \in C$ annihilates a plane in the tangent space of M. So there is a subbundle of π^*TM defined as those (p, v) where v lies in the plane annihilated by p. Define H to be the preimage in TC of this rank two subbundle. H is a rank four subbundle since the kernel of $TC \to \pi^*TM$ is the vertical bundle $V \subset H$ which is of rank two.

H here corresponds precisely to the velocity constraint explained above. One can interpret the vertical bundle V as velocities changing the orientation of the of the saucer, leaving the centre of mass (a point in M) stationary.

Before we show H is a contact form we need to define a distinguished 1-form on the cotangent bundle. In what follows in this chapter reserve the notation $\mathrm{pr} : T^*M \to M$ for the cotangent bundle of a manifold M. Letting $(x^1, ..., x^n)$ be local coordinates on M then $(q^1, .., q^n, p_1, ..., p_n)$ give coordinates on T^*M where $q^i = x^i \circ \mathrm{pr}$ and if $\omega = \omega_i dx^i$ then $p_j(\omega|_p) = \omega_j(p)$ for $p \in M$.

Definition 4.3.3 (Tautological 1-form)**.** *Writing $u \in T^*M$ define*

$$\theta|_u := \mathrm{pr}^* u. \tag{4.3.4}$$

Then $\theta \in \Lambda^1(T^*M)$ is the globally defined tautological 1-form.

More conveniently over a coordinate patch $(x^1, ..., x^n)$, given $u \in T_p^*M$ one can write $u = p_i(u)dx^i|_p$. Now $\mathrm{pr}^* dx^i|_p = dx^i|_p \circ d\,\mathrm{pr}\,|_u = dq^i|_u$, so

$$\theta|_u = p_i(u)dq^i|_u. \tag{4.3.5}$$

So over this coordinate patch the tautological one form is $\theta = p_i dq^i$.

The kernel of θ at $u \in T^*M$ consists of those $v \in T_u(T^*M)$ such that $d\,\mathrm{pr}\,|_u(v) \in \ker u$. So

$$v = Y\frac{\partial}{\partial p_i} + X^i\frac{\partial}{\partial q^i} \in \ker\theta \iff p_i X^i = 0. \tag{4.3.6}$$

Note that θ annihilates the vertical bundle of $T(T^*M) \to T^*M$.

Proposition 4.3.7. *C is a contact manifold as defined in 1.1.2.*

Proof. Given local coordinates (x^1, x^2, x^3) on M and accordingly local coordinates $(q^1, q^2, q^3, p_1, p_2, p_3)$ for T^*M, on a suitably small neighbourhood U around each point we get induced homogeneous coordinates (q^1, q^2, q^2, a, b) for C so that the map sending each point to its ray $r : T^*M \setminus \{0\} \to C$ has coordinate expression

$$(q^1, q^2, q^3, p_1, p_2, p_3) \mapsto (q^1, q^2, q^3, p_2/p_1, p_3/p_1). \tag{4.3.8}$$

It is evident that $H = dr \ker\theta$ where θ is the tautological one-form 4.3.3 but we need to construct a contact form on C, at least locally. Take a section of r, that is, a smooth embedding $\iota : C \hookrightarrow T^*M \setminus \{0\}$ with $r \circ \iota = \mathrm{id}_C$ and this implies $\mathrm{pr} \circ \iota = \pi : C \to M$. In U the map must have the coordinate expression

$$\iota : (q^1, q^2, q^2, a, b) \mapsto (q^1, q^2, q^3, f, af, bf). \tag{4.3.9}$$

for some non-vanishing smooth function f on U. Let us pull back $\theta = p_i dq^i$ by this embedding. We have $\iota^* dq^i = dq^i$ and so

$$\iota^*\theta = (p_i \circ \iota)dq^i = f dq^1 + fadq^2 + fbdq^3 = f(dq^1 + adq^1 + bdq^2) \tag{4.3.10}$$

So $\iota^*\theta$ is obviously a contact form defining some contact distribution. Next, if $\pi : C \to M$ is the projection then $d\pi(X^i\frac{\partial}{\partial q^i} + A\frac{\partial}{\partial a} + B\frac{\partial}{\partial b}) = X^i\frac{\partial}{\partial x^i}$. In particular $v = X^i\frac{\partial}{\partial q^i} + A\frac{\partial}{\partial a} + B\frac{\partial}{\partial b} \in T_pC$ is in H if and only if its image under $d\pi$ is in the plane annihilated by p, which is precisely when

$$X^1 + aX^2 + bX^3 = 0, \tag{4.3.11}$$

which occurs if and only if $v \in \ker \iota^*\theta$. So $H = \ker \iota^*\theta$. □

Note that since there is no canonical choice of embedding ι there is no canonical contact form on C.

Next, reformulating the above construction in the language of Ehresmann connections reveals more structure. Pick a connection ∇ for $\mathrm{pr} : T^*M \to M$, so that we split

$$T(T^*M \setminus \{0\}) \cong \mathrm{pr}^* TM \oplus \mathrm{pr}^* T^*M, \tag{4.3.12}$$

where we are abusing notation slightly and writing pr for the map $T^*M \setminus \{0\} \to M$. Pulling back by our embedding ι and using $\pi^* = \iota^* \circ \mathrm{pr}^*$ splits

$$\iota^* T(T^*M \setminus \{0\}) \cong \pi^* TM \oplus \pi^* T^* M. \tag{4.3.13}$$

We will write elements like

$$\begin{bmatrix} X^a \\ \omega_a \end{bmatrix}. \tag{4.3.14}$$

There is an obvious vector bundle map $\iota^* T(T^*M \setminus \{0\}) \to TC$ over C given by

$$(p, v) \mapsto dr(v) \tag{4.3.15}$$

and the kernel in the fibre above p consists precisely of elements of the form

$$\begin{bmatrix} 0 \\ tp_a \end{bmatrix}, t \in \mathbb{R}. \tag{4.3.16}$$

where, we denote by p_a the tautological section of $\mathrm{pr}^* T^*M$ pulled back over C by ι. The condition $r \circ \iota = \mathrm{id}_C$ ensures is p_a independent of ι up to scale, and accordingly we have a splitting

$$TC \cong \pi^* TM \oplus (\pi^* T^* M / \{\omega_a \sim \omega_a + tp_a\}) \tag{4.3.17}$$

and this splitting is independent of ι. The contact distribution H is

$$H = \left\{ \begin{bmatrix} X^b \\ [\omega_b] \end{bmatrix} \in TC \;\middle|\; X^b p_b = 0 \right\}. \tag{4.3.18}$$

We therefore have a contact form

$$\begin{bmatrix} p_b & 0 \end{bmatrix} \in T^*C. \tag{4.3.19}$$

Taking coordinates as before, given $v = X^i \frac{\partial}{\partial q^i} + A \frac{\partial}{\partial a} + B \frac{\partial}{\partial b} \in TC$ we have $\iota^*\theta(v) = \theta(d\iota(v)) = p_i X^i$ and so the above contact form is precisely $\iota^*\theta$ that we used previously.

Choosing any other connection $\hat{\nabla}_a \omega_b = \nabla_a \omega_b - \Gamma_{ab}{}^c \omega_c$ would mean changing the splitting of TC by

$$\widehat{\begin{bmatrix} X^b \\ [\omega_b] \end{bmatrix}} = \begin{bmatrix} X^b \\ [\omega_b - X^a \Gamma_{ab}{}^c p_c] \end{bmatrix} \tag{4.3.20}$$

In particular, pick a connection in a projective structure on M. Changing connection in the projective class means we have $\Gamma_{ab}{}^c = \Upsilon_a \delta_b{}^c + \Upsilon_b \delta_a{}^c$ as per 4.2.2. Then for elements of H

$$\widehat{\begin{bmatrix} X^b \\ [\omega_b] \end{bmatrix}} = \begin{bmatrix} X^b \\ [\omega_b - X^a \Upsilon_a p_b - X^a \Upsilon_b p_a] \end{bmatrix} = \begin{bmatrix} X^b \\ [\omega_b] \end{bmatrix}. \tag{4.3.21}$$

So we have shown the first part of the following, due to Takeuchi in [Tak94] as recently as 1994.

Proposition 4.3.22. *A projective structure on M defines a splitting of the contact distribution $H = P \oplus V$ where*

$$\begin{aligned} P &= \left\{ \begin{bmatrix} X^b \\ [0] \end{bmatrix} \in TC \;\middle|\; X^b p_b = 0 \right\} \\ V &= \left\{ \begin{bmatrix} 0 \\ [\omega_b] \end{bmatrix} \in TC \right\} \end{aligned} \tag{4.3.23}$$

and the Levi form restricts to 0 *on* P *and* V*.*

Proof. All that is left to show is that the Levi form restricts to 0 on P and V. After picking coordinates on M, and using the induced coordinates on T^*M we can write $d\theta = dp_i \wedge dq^i$. To rephrase this in the notation introduced above, taking the locally defined flat connection in our coordinate chart, hence splitting $T(T^*M \setminus \{0\}) \cong \mathrm{pr}^* TM \oplus \mathrm{pr}^* T^*M$ over that chart, $d\theta$ is written

$$\left(\begin{bmatrix} X^b \\ \omega_b \end{bmatrix}, \begin{bmatrix} Y^b \\ \mu_b \end{bmatrix} \right) \mapsto X^b \mu_b - Y^b \omega_b. \tag{4.3.24}$$

Over each such coordinate chart, any connection in our projective class will differ from the flat connection by some locally defined tensor field $\Gamma_{ab}{}^c$, but since the locally defined flat connection and the connection in our projective class are torsion free, $\Gamma_{ab}{}^c$ is symmetric. Then

$$X^b(\mu_b - Y^a \Gamma_{ab}{}^c p_c) - Y^b(\omega_b - X^a \Gamma_{ab}{}^c p_c) = X^b \mu_b - Y^b \omega_b. \tag{4.3.25}$$

So $d\theta$ is written like 4.3.24 with respect to the splitting defined by the connection in our projective class.

Now the Levi form is $d(\iota^*\theta)|_H = (\iota^* d\theta)|_H$. Note that V is just the vertical bundle of $\pi : C \to M$ and using the fact $\mathrm{pr} \circ \iota = \pi$ we have $d\,\mathrm{pr} \circ d\iota = d\pi$ which shows that $d\iota(V) \subset V(T^*M \setminus \{0\}) \cong \{0\} \oplus \mathrm{pr}^* T^*M$. So from 4.3.24 it is clear $\iota^* d\theta$ restricts to 0 on V. Finding $d\iota(P)$ is a little more complicated. Using $r \circ \iota = \mathrm{id}_C$ we get $dr \circ d\iota = \mathrm{id}_{TC}$. So for vectors in P

$$dr\left(d\iota \begin{bmatrix} X^b \\ [0] \end{bmatrix}\right) = \begin{bmatrix} X^b \\ [0] \end{bmatrix} \in TC. \tag{4.3.26}$$

We claim that this implies

$$d\iota \begin{bmatrix} X^b \\ [0] \end{bmatrix} = \begin{bmatrix} X^b \\ tp_b \end{bmatrix}, t \in \mathbb{R}. \tag{4.3.27}$$

To see this, first note that $d\iota : TC \to T(T^*M \setminus \{0\})$ has image in the image of the map $\mathrm{pr}_2 : \iota^* T(T^*M \setminus \{0\}) \to T(T^*M \setminus \{0\})$ and pr_2 obviously preserves the splitting. Next, $dr \circ pr_2$ which is the map 4.3.15, preserves the splitting, by definition of the splitting induced on TC. So dr preserves the splitting on the image of $d\iota$ and so 4.3.26 only makes sense if 4.3.27 holds. Then, looking at 4.3.24 and using the fact $X^b p_b = 0$ shows that $\iota^* d\theta$ restricts to 0 on P. □

A splitting $H = P \oplus V$ such that P, V are of equal rank and the Levi form is null on P and V is called a Legendrean contact structure. We will study such structures in generality in the next chapter.

4.4 The G_2 contact structure on C

We now follow [EN20b] in constructing the G_2 contact structure on C. We start with a projective structure on M, and two 1-forms $\phi \in T^*M \otimes \varepsilon(2), \psi \in T^*M \otimes \varepsilon(1)$ on M which are linearly independent on some neighbourhood (linear independence makes sense since we can choose a volume form to map $\Lambda^1 \otimes \varepsilon(k) \xrightarrow{\sim} \Lambda^1$). Define $\varepsilon_C(w) := \pi^* \varepsilon(w)$, the bundle of projective weights pulled back over C. A section is precisely a function $\pi^* \Lambda^n_+ \to \mathbb{R}$ with homogeneity $\frac{w}{n+1}$. We can pull a weight $f \in \varepsilon(w)$ back to $\pi^* f \in \varepsilon_C(w)$ by defining $\pi^* f(p, \omega) = f(\omega)$.

There is a canonical injection $P \hookrightarrow \pi^* TM$. Accordingly we have surjections $\Lambda^k(\pi^* T^*M) = \pi^* \Lambda^k T^*M \to \Lambda^k P^*$ given by restriction of linear maps to the image of P in $\pi^* TM$. For some ω, a section of $\Lambda^k T^*M$ define $\pi^! \omega \in \Lambda^k P^*$ to be this surjection applied to the pullback $\pi^* \omega$. If ω is a section of $\Lambda^k T^*M \otimes \varepsilon(w)$ then define $\pi^! \omega$ as the obvious section of $\Lambda^k P^* \otimes \varepsilon_C(w)$, having also pulled back the density.

We have the exact sequence

$$0 \longrightarrow P \longrightarrow \pi^* TM \longrightarrow L^* \longrightarrow 0$$

where the second map sends a vector $v \in \pi^* TM \hookrightarrow TC$ to its evaluation map $L \to \mathbb{R}$.

Dually we have the exact sequence

$$0 \longrightarrow L \longrightarrow \pi^* T^*M \longrightarrow P^* \longrightarrow 0$$

where the second map is the restriction. In particular, as per 1.1.30 we get short exact sequences

$$0 \longrightarrow L \otimes \Lambda^{k-1} P^* \longrightarrow \pi^* \Lambda^k T^* M \longrightarrow \Lambda^k P^* \longrightarrow 0$$

which for $k = 3$ yields a (canonical) isomorphism $L \otimes \Lambda^2 P^* \cong \pi^* \Lambda^3 T^* M$ and this last bundle can be canonically identified with $\varepsilon_C(-4)$ as per 4.2.7. Thus we have found an important relation between projective weights on M and contact weights on C. Without choosing a contact form we get $P \otimes V \to L^*$, which is the Levi form defined up to scale. We will write this $J_{\bar{\alpha}\alpha}$, where we denote sections of P, V with a barred and unbarred indices from the lower-case of the Greek alphabet respectively. So we will write and $\pi^! \phi_{\bar{\alpha}}$ for $\pi^! \phi$ et cetera. We also have $J^{\bar{\alpha}\alpha} \in P \otimes V \otimes L$ defined as the unique section with $J^{\bar{\alpha}\gamma} J_{\bar{\beta}\gamma} = \delta^{\bar{\alpha}}{}_{\bar{\beta}}$.

Next, define

$$\Theta_{\bar{\alpha}\bar{\beta}} := \pi^! \phi_{\bar{\alpha}} \wedge \pi^! \psi_{\bar{\beta}} \in \Lambda^2 P^* \otimes \varepsilon_C(3). \tag{4.4.1}$$

Lemma 4.4.2. *Θ is non-vanishing on an open subset U of C.*

Proof. We just need to show it is non-vanishing at a point. Suppressing indices, $(\pi^*\phi \wedge \pi^*\phi)|_p = (p, (\phi \wedge \psi)|_x)$ $\forall p$ with $\pi(p) = x$. Now P_p consists of those $(p, X) \in \pi^* T_x M$ such that X is in the plane annihilated by p. In particular given $X \in T_x M$ there exists p with $\pi(p) = x$ such that $(p, X) \in P_p$. This is because each fibre of C above x contains all oriented planes in $T_x M$. So if $(\pi^*\phi \wedge \pi^*\phi)|_p = 0$ $\forall p, \pi(p) = x$ then $\phi \wedge \psi(X, Y) = 0$ $\forall X, Y \in T_x M$ which is a contradiction since ϕ, ψ are linearly independent.

We will now construct a G_2 contact geometry on U. On U we can define $\Theta^{\bar{\alpha}\bar{\beta}} \in \Lambda^2 P \otimes \varepsilon_C(-3)$ by $\Theta^{\bar{\alpha}\bar{\gamma}} \Theta_{\bar{\beta}\bar{\gamma}} = \frac{1}{2} \delta_{\bar{\beta}}{}^{\bar{\alpha}}$.

On U let us define non-vanishing

$$\begin{aligned} s_1 &= -\Theta^{\bar{\alpha}\bar{\beta}} \pi^! \psi_{\bar{\beta}} \in P \otimes \varepsilon_C(-2) \\ s_2 &= \Theta^{\bar{\alpha}\bar{\beta}} \pi^! \phi_{\bar{\beta}} \in P \otimes \varepsilon_C(-1) \\ t_1 &= J^{\bar{\alpha}\alpha} \Theta_{\bar{\beta}\bar{\gamma}} \pi^! \psi_{\bar{\alpha}} \in V \\ t_2 &= J^{\bar{\alpha}\alpha} \Theta_{\bar{\beta}\bar{\gamma}} \pi^! \phi_{\bar{\alpha}} \in V \otimes \varepsilon_C(1). \end{aligned} \tag{4.4.3}$$

Now define a smooth map $S := \varepsilon_C(2/3) \oplus \varepsilon_C(-1/3) \to H$ by

$$\varepsilon_C(2/3) \oplus \varepsilon_C(-1/3) \ni (f, g) \mapsto f^3 s_1 + f^2 g s_2 - 3 f g^2 t_1 + g^3 t_2 \in H. \tag{4.4.4}$$

Proposition 4.4.5. *The above smooth map defines a field of twisted cubic curves compatible with the Levi form and hence a G_2 contact geometry.*

Proof. Note that each of terms end up in the right bundle. The reason for the mysterious choice of constants will soon become clear. Pick some locally non-vanishing

$h \in \varepsilon_C(1/3)$ which gives trivialisations h^2 and h^{-1} for $\varepsilon_C(2/3), \varepsilon_C(-1/3)$ respectively. Then with respect to the trivialisations $\{h^2, h^{-1}\}$, and $\{h^6 s_1, h^3 s_2, -3t_1, h^{-3}t_2\}$ for S and $H|_U$ respectively, the above smooth map is

$$(\lambda, \tau) \mapsto (\lambda^3, \lambda^2\tau, \lambda\tau^2, \tau^3) \tag{4.4.6}$$

as a map $\mathbb{R}^2 \to \mathbb{R}^4$ in each fibre, as required to define a field of twisted cubics.

To check the compatibility, we need to compute, given a choice of contact form:

$$\begin{aligned} &d\alpha(\lambda^3 h^6 s_1 + \lambda^2\tau h^3 s_2 - 3\lambda\tau^2 t_1 + \tau^3 h^{-3}t_2,\ \tilde{\lambda}^3 h^6 s_1 + \tilde{\lambda}^2\tilde{\tau} h^3 s_2 - 3\tilde{\lambda}\tilde{\tau}^2 t_1 + \tilde{\tau}^3 h^{-3} t_2) \\ =&(\lambda^3\tilde{\tau}^3 - \tilde{\lambda}^3\tau^3) d\alpha(h^6 s_1, h^{-3}t_2) - (3\lambda^2\tau\tilde{\lambda}\tilde{\tau}^2 - 3\lambda\tau^2\tilde{\lambda}^2\tilde{\tau}) d\alpha(h^3 s_2, t_1) \\ -&(3\lambda^3\tilde{\lambda}\tilde{\tau}^2 - 3\tilde{\lambda}^3\lambda\tau^2) d\alpha(h^6 s_1, t_1) + (\lambda^2\tau\tilde{\tau}^3 - \tilde{\lambda}^2\tilde{\tau}\tau^3) d\alpha(h^3 s_2, h^{-3}t_2) \end{aligned} \tag{4.4.7}$$

where we've used the fact $d\alpha$ restricts to 0 on P, V to eliminate some terms. In indices

$$-J_{\bar{\alpha}\nu}\Theta^{\bar{\alpha}\bar{\gamma}}\pi^! \psi_{\bar{\gamma}} J^{\bar{\beta}\nu}\pi^! \psi_{\bar{\beta}} = -\Theta^{\bar{\beta}\bar{\gamma}}\pi^! \psi_{\bar{\gamma}}\psi_{\bar{\beta}} = 0 \tag{4.4.8}$$

so $d\alpha(h^6 s_1, t_1) = 0$. Similarly $d\alpha(h^3 s_2, h^{-3}t_2) = 0$. On the other hand a similar calculations checks that $d\alpha(h^3 s_2, t_1) = d\alpha(h^6 s_1, h^{-3}t_2) =: c$ so 4.4.7 factorises as

$$c(\lambda\tilde{\tau} - \tilde{\lambda}\tau)^3 \tag{4.4.9}$$

as required for compatibility. □

This is the construction as in [EN20b].

The reason for carefully handling the densities is as follows. Recalling, 4.2.12, since ϕ is a section of $T^*M \otimes \varepsilon(2)$ there is a well-defined notion of it being a solution to the Killing equation. Given this assumption that ϕ is a solution, the authors go on to show, using Cartan's method of equivalence (see [IL16]), that the G_2 contact structure defined above is locally isomorphic to the flat model if

- $[\nabla_a]$ is projectively flat. (The projectively invariant Weyl component of the curvature of a representative connection vanishes see).
- $\nabla_a(\phi_{[b}\psi_{c]}) = \nabla_{[a}(\phi_b\psi_{c]})$
- $\psi_{[a}\nabla_{b]}\psi_c = 0$

We will not follow that route here, but the flying saucers construction gives us a clue of where to look next if we want to construct G_2 contact geometries.

Chapter 5

Legendrean contact structures

In the previous section we showed that a projective structure on a 3-dimensional manifold M induces an invariant splitting of the contact distribution H on the "flying saucer" space C constructed from M. We chose M to be of dimension 3 for the purposes of constructing a G_2 contact geometry on an open subset of C. However, it is obvious that one can repeat the construction on a $2n+1$-dimensional manifold M to get an invariant splitting of the rank-$4n$ contact distribution into two rank-$2n$ subbundles on the $4n+1$-dimensional "flying saucer" space C.

As previously mentioned in passing, the splitting of the contact distribution of C is called a Legendrean contact structure, and this particular construction first appeared in [Tak94].

The study of splittings of the contact distribution into a pair of subbundles on which the Levi form vanishes can be thought of as the contact analogue of the study of Lagrangian subbundles in symplectic geometry, which are rank-n subbundles of the tangent bundle on a $2n$-dimensional symplectic manifold for which the sympelctic form vanishes. Usually, the concern is finding Lagrangian submanifolds, which are n-dimensional submanifolds with tangent spaces on which the symplectic form vanishes. So an important invariant of a Lagrangian subbundle is its integrability. Because of this analogue, many authors, for example [ČS09] and indeed [Tak94], use the terminology Lagrangian or Lagrangean contact instead of Legendrean contact [1].

Legendrean contact geometry is sometimes called (see for instance [Slo97]) the "real analogue" of CR geometry, or, to be more specific, the geometry of non-degenerate almost-CR structures of hypersurface type. The reasons for this is the following: Such CR structures (see [ČS09] Section 4.2.4) consist of a contact manifold (M, H) with a complex structure on H which implies that the complexification of H splits into two subbundles of complex rank n.

[1] [DMT19] writes Legendrian contact, to round out a quartet of names for these structures.

Returning to C, we see that the vertical subbundle $V \hookrightarrow TC$ of the bundle $\pi : C \to M$ is always integrable. Its integral submanifolds are the spheres of possible orientations of the flying saucer at each point in M. So the Legendrean contact structure on C is less than generic. In fact, in [Tak94] Takeuchi shows that P is integrable precisely when the underlying projective structure on M is flat.

With the motivation of constructing more G_2 contact geometries in the back of our minds, this chapter is dedicated to defining and understanding Legendrean contact geometries. Most importantly for the sequel, as in the G_2 contact case, we give a construction of the preferred partial connection on a Legendrean contact geometry that arises in the presence of a contact form. We also realise the flat model as a space of lines inside codimension-1 hyperplanes in Euclidean space.

5.1 Legendrean contact structures

Definition 5.1.1 (Legendrean contact structure)**.** *A Legendrean contact structure is a contact manifold M of dimension $2n+1$ with a splitting $H = P \oplus V$ of the contact distribution such that P, V are each of rank n and the Levi form restricts to 0 on P and V.*

As mentioned before, a consequence of the Levi form vanishing is that there is a canonical non-degenerate pairing $P \otimes V \to L^*$.

Definition 5.1.2 (Isomorphism of Legendrean contact structures)**.** *Let (M, H_M) and (N, H_N) be contact manifolds equipped with Legendrean contact structures $H_M = P_M \oplus V_M$ and $H_N = P_N \oplus V_N$ respectively. Then an isomorphism of $\phi : M \to N$ of Legendrean contact structure is a diffeomorphism such that $d\phi(V_M) = V_N$ and $d\phi(P_M) = P_N$.*

From the standard contact structure on $\mathbb{R}^{2n+1}$ we can easily find an example of a Legendrean contact structure.

Example 5.1.3 (Standard Legendrean contact structure on $\mathbb{R}^{2n+1}$)**.** *Given $\mathbb{R}^{2n+1}$ with global coordinates $(t, p_1, ..., p_n, q^1, ..., q^n)$ and contact form*

$$\alpha = dt - p_i dq^i \tag{5.1.4}$$

define

$$P = \operatorname{span}\left\{p_i \frac{\partial}{\partial t} + \frac{\partial}{\partial q_i}\right\},\ V = \operatorname{span}\left\{\frac{\partial}{\partial p_i}\right\} \tag{5.1.5}$$

This is a Legendrean contact structure and P and V are integrable.

Proof. The Levi form $d\alpha = dq^i \wedge dp_i$ clearly vanishes on P and V. Next, V is clearly integrable and

$$\Big[p_i \frac{\partial}{\partial t} + \frac{\partial}{\partial q_i}, p_j \frac{\partial}{\partial t} + \frac{\partial}{\partial q_j}\Big] = 0 \tag{5.1.6}$$

since none of the component functions depend on t, q^i, q^j, and so P is integrable too. $\square$.

Remark 5.1.7. *We can construct examples of Legendrean contact structures to our hearts content by instead writing*

$$P = \operatorname{span}\Big\{f^i\big(p_i \frac{\partial}{\partial t} + \frac{\partial}{\partial q^i}\big)\Big\},\ V = \operatorname{span}\Big\{g_i \frac{\partial}{\partial p_i}\Big\} \tag{5.1.8}$$

for some smooth functions f^i, g_i, $i = 1, ..., n$. In particular, P and V are generically non-integrable. So Legendrean contact geometry, unlike contact geometry, does have local invariants. In Chapter 7 will see that the obstructions to integrability are not the only local invariants.

5.2 The flat model

There is a more geometrically satisfying realisation of the standard Legendrean contact structure. Here we take $n = 2$ but the construction of the flat model to follow can readily be adapted for arbitrary n.

Consider $F_{1,3}(\mathbb{R}^4) \hookrightarrow \mathbb{R}P^3 \times (\mathbb{R}P^3)^*$ consisting of

$$(l, \omega) = ([x^0 : x^1 : x^2 : x^3], [y_0 : y_1 : y_2 : y_3]) \in \mathbb{R}P^3 \times (\mathbb{R}P^3)^* \tag{5.2.1}$$

with

$$x^0 y_0 + x_1 y^1 + x_2 y^2 + x_3 y^3 = 0. \tag{5.2.2}$$

This space can be thought of as the space of lines inside 3-planes inside $\mathbb{R}^4$ if we identify an element of $(\mathbb{R}P^3)^*$ with the plane it annihilates.

We have the usual coordinate charts on $\mathbb{R}P^3 \times (\mathbb{R}P^3)^*$. For example:

$$\begin{aligned}\mathbb{R}P^3 \times (\mathbb{R}P^3)^* \supset U \ni ([x^0 : x^1 : x^2 : 1], [1 : y_1 : y_2 : y_3])\\ \mapsto (x_0, x_1, x_1, y_1, y_2, y_3)\end{aligned} \tag{5.2.3}$$

with other charts for an atlas being given shifting around the 1s in the obvious way.

Note that for $(l,\omega) \in F_{1,3}(\mathbb{R}^4)$, we have $x_i = y^i = 0$ for at most two out of $i = 0, 1, 2, 3$ so every $(l,\omega) \in F_{1,3}(\mathbb{R}^4)$ lies in a coordinate chart $U \subseteq \mathbb{R}P^3 \times (\mathbb{R}P^3)^*$ of the form

$$\mathbb{R}P^3 \times (\mathbb{R}P^3)^* \supset U \ni (l,\omega) \mapsto (t, p_1, p_2, q^1, q^2, s). \tag{5.2.4}$$

with the pairing $\mathbb{R}P^3 \times (\mathbb{R}P^3)^* \to \mathbb{R}$ having coordinate expression

$$(t, p_1, p_2, q^1, q^2, s) \mapsto s - t + p_1 q^1 + p_2 q^2 \tag{5.2.5}$$

such that t, q_1, q_2 determine the projection to $\mathbb{R}P^3$ and s, p^1, p^2 determine the projection to $(\mathbb{R}P^3)^*$. Now

$$\mathbb{R}P^3 \times (\mathbb{R}P^3)^* \supset U \ni (l,\omega) \mapsto (t, p_1, p_2, q^1, q^2, s - t + p_1 q^1 + p_2 q^2). \tag{5.2.6}$$

gives coordinates in U which realises $F_{1,3}(\mathbb{R}^4)$ as a regular smooth submanifold of $\mathbb{R}P^3 \times (\mathbb{R}P^3)^*$. This submanifold is sometimes called the incidence variety.

Proposition 5.2.7 (The flat model of Legendrean contact geometry)**.** *Let $V \subseteq TF_{1,3}(\mathbb{R}^4)$ consist of the vertical subbundle of the projection to $F_{1,3}(\mathbb{R}^4) \to \mathbb{R}P^3$ and P consist of the vertical subbundle of the projection to $F_{1,3}(\mathbb{R}^4) \to (\mathbb{R}P^3)^*$. Then setting $H = P \oplus V \subseteq TM$ gives a Legendrean contact structure on $F_{1,3}(\mathbb{R}^4)$ and furthermore it is locally isomorphic to the standard structure 5.1.3.*

Proof. The geometric interpretation here is that V and P consist of velocities of curves moving through a family of lines inside a fixed hyperplane, and curves moving through a family of hyperplanes containing a fixed line, respectively.

We have coordinates (t, p_1, p_2, q^1, q^2) such that t, q^1, q^2 completely determine the projection to $\mathbb{R}P^3$. On the other hand inspecting 5.2.6 we can see s, p^1, p^2 determine the projection to $(\mathbb{R}P^3)^*$ where $s(t, q^1, q^2, p^1, p^2) = t - p_1 q^1 - p_2 q^2$.

The vertical subbundle V of the projection $F_{1,3}(\mathbb{R})^4 \to \mathbb{R}P^3$ is given by

$$V = \operatorname{span}\left\{\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}\right\}. \tag{5.2.8}$$

If we define

$$\alpha = dt - p_1 dq^1 - p_2 dq^2 \tag{5.2.9}$$

and then define

$$P = \left\{p_1 \frac{\partial}{\partial t} + \frac{\partial}{\partial q^1}, p_2 \frac{\partial}{\partial t} + \frac{\partial}{\partial q^2}\right\} \tag{5.2.10}$$

then $H = P \oplus V$ locally equips $F_1, 3(\mathbb{R}^4)$ with the standard Legendrean contact structure.

All that is left to show is that P is the vertical subbundle of the projection to $(\mathbb{R}P^3)^*$. To do this make the coordinate transformation

$$\begin{aligned} a_1 &= -q^1, & a_2 &= -q^2, \\ x^1 &= p_1, & x^2 &= p_2, \\ z &= t - p_1 q^1 - p_2 q^2. \end{aligned} \tag{5.2.11}$$

Given the above coordinate transformation we have $s = t - p_1 q^1 - p_2 q^2 = z$ so that the projection to $(\mathbb{R}P^3)^*$ is determined by z, x^1, x^2 and furthermore we have

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial z}{\partial t}\frac{\partial}{\partial z} + \frac{\partial x^i}{\partial t}\frac{\partial}{\partial x^i} + \frac{\partial a_i}{\partial t}\frac{\partial}{\partial a_i} = \frac{\partial}{\partial z} \\ \frac{\partial}{\partial q^j} &= \frac{\partial z}{\partial q^j}\frac{\partial}{\partial z} + \frac{\partial x^i}{\partial q^j}\frac{\partial}{\partial x^i} + \frac{\partial a_i}{\partial q^j}\frac{\partial}{\partial a_i} = -x^j\frac{\partial}{\partial z} + \frac{\partial}{\partial a_i} \end{aligned} \tag{5.2.12}$$

which means that

$$P = \operatorname{span}\left\{\frac{\partial}{\partial a_1}, \frac{\partial}{\partial a_2}\right\} \tag{5.2.13}$$

is evidently the vertical subbundle of $F_{1,3}(\mathbb{R}^4) \to (\mathbb{R}P^3)^*$. □

There is a transitive Lie group action of $SL(n+2,\mathbb{R})$ on the configuration space $F_{1,n+1}(\mathbb{R}^{n+2})$. Considering what matrices stabilise the configuration of the line spanned by the standard basis vector lying inside the hyperplane spanned by the first $2n-1$ standard basis vectors, we see that we can realise $F_{1,n+1}(\mathbb{R}^{n+2})$ as the homogeneous space $SL(n+2,\mathbb{R})/P$ where P is the subgroup consisting of matrices in $SL(n+2,\mathbb{R})$ of the form (in the $n=2$ case)

$$\begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \tag{5.2.14}$$

or in general, block upper-triangular matrices with block sizes $1, n$ and 1.

However, for technical reasons (explained in [Slo97] Section 5.) Legendrean contact structures are thought of as parabolic geometries of type (G, P), not with G being $SL(n+2,\mathbb{R})$ as above, but with

$$G = \{A \in GL(n+2,\mathbb{R}), |\det A| = 1\}/\{A \sim -A\}, \tag{5.2.15}$$

and P the subgroup consisting of block upper-triangular matrices with block sizes $1, n$ and 1. $G \cong SL(n+2,\mathbb{R})$ only for odd n. Either way $F_{1,n+1}(\mathbb{R}^{n+2}) = G/P$.

5.3 Preferred partial connections on Legendrean contact structures

The splitting $H = P \oplus V$ naturally splits $\Lambda^1_H \cong P^* \oplus V^*$ where we identify P^* and V^* as those elements which restrict to 0 on V and P respectively. With this identification the Rumin operator breaks up into six (a priori) first order differential operators:

$$\begin{tikzcd}
P^* \arrow[r] \arrow[dr] \arrow[ddr, dashed] & \Lambda^2 P^* \\
 & (P \oplus V)_\perp \\
V^* \arrow[r] \arrow[ur] \arrow[uur, dashed] & \Lambda^2 V^*
\end{tikzcd}$$

Inspecting the formula 1.2.19 it is clear that $V^* \to \Lambda^2 P^*$ and $P^* \to \Lambda^2 V^*$ are vector bundle homomorphisms, and also that P and V are integrable precisely when these homomorphisms vanish.

We will find it convenient to use the notation for Legendrean contact structures introduce in 4.4. Denote sections of V^* like $\omega_\alpha \in V^*$ and sections of P^* with barred indices like $\rho_{\bar\alpha}$. Accordingly we write elements of Λ^1_H like $(\rho_{\bar\alpha}, \omega_\alpha)$.

There is an important distinguished partial connection that arises on a Legendrean contact manifold given a choice of contact form. This basically corresponds to the partial connection in [ČS09] proposition 5.2.14.

Theorem 5.3.1. *Given a choice of contact form $\alpha \in L$ there is a unique partial connection $V^* \to \Lambda^1_H \otimes V^*$ with minimal partial torsion in the sense that the induced operator $P^* \oplus V^* = \Lambda^1_H \to \Lambda^2_{H\perp} = \Lambda^2 P^* \oplus (P^* \oplus V^*)_\perp \oplus \Lambda^2 V^*$ is equal to the Rumin operator modulo the homomorphisms $V^* \to \Lambda^2 P^*$ and $P^* \to \Lambda^2 V^*$, which are the obstructions to integrability of P, V respectively.*

Proof. The freedom for choosing a partial connection $(\bar\nabla_{\bar\alpha}, \nabla_\alpha)$ on V^* is

$$(\hat{\bar\nabla}_{\bar\alpha}\omega_\beta, \hat\nabla_\alpha\omega_\beta) = (\bar\nabla_{\bar\alpha}\omega_\beta + \Gamma_{\bar\alpha\beta}{}^{\gamma}\omega_\gamma, \nabla_\alpha\omega_\beta + \Phi_{\alpha\beta}{}^{\gamma}\omega_\gamma) \tag{5.3.2}$$

with $\Gamma_{\bar\alpha\beta}{}^{\gamma}\omega_\gamma \in P^* \otimes \mathrm{End}(V)$ and $\Phi_{\alpha\beta}{}^{\gamma} \in V^* \otimes \mathrm{End}(V)$. The freedom is rank $2n^3$. Of course, picking a partial connection on V^* induces one on P^* thanks to our isomorphism $J^{\bar\alpha\alpha}$. That is

$$(\bar\nabla_{\bar\alpha}\omega_{\bar\beta}, \nabla_\alpha\omega_{\bar\beta}) := (J_{\bar\beta\beta}\bar\nabla_{\bar\alpha}(J^{\bar\gamma\beta}\omega_{\bar\gamma}), J_{\bar\beta\beta}\nabla_\alpha(J^{\bar\gamma\beta}\omega_{\bar\gamma})) \tag{5.3.3}$$

and, using $J^{\bar{\alpha}\alpha}$ to raise and lower indices, this changes by

$$(\hat{\bar{\nabla}}_{\bar{\alpha}}\omega_{\bar{\beta}}, \hat{\nabla}_{\alpha}\omega_{\bar{\beta}}) = (\bar{\nabla}_{\bar{\alpha}}\omega_{\bar{\beta}} - \Gamma_{\bar{\alpha}}{}^{\bar{\gamma}}{}_{\bar{\beta}}\omega_{\bar{\gamma}}, \nabla_{\alpha}\omega_{\bar{\beta}} - \Phi_{\alpha}{}^{\bar{\gamma}}{}_{\bar{\beta}}\omega_{\bar{\gamma}}). \tag{5.3.4}$$

Under what conditions will the induced connection on $\Lambda^1_H = P^* \oplus V^*$ be partial torsion free? We need to work out the induced operator with the same symbol as the Rumin operator. Firstly the direct sum partial connection is

$$\begin{aligned}(\bar{\nabla}_{\bar{\alpha}}, \nabla_{\alpha})(\rho_{\bar{\beta}}, \omega_{\beta}) &\mapsto (\bar{\nabla}_{\bar{\alpha}}\rho_{\bar{\beta}}, \bar{\nabla}_{\bar{\alpha}}\omega_{\beta}, \nabla_{\alpha}\rho_{\bar{\beta}}, \nabla_{\alpha}\omega_{\beta}) \\ &\in \otimes^2 P^* \oplus (P^* \otimes V^*) \oplus (V^* \otimes P^*) \oplus \otimes^2 V^*\end{aligned} \tag{5.3.5}$$

and the projection is

$$(\bar{\nabla}_{[\bar{\alpha}}\mu_{\bar{\beta}]}, \bar{\nabla}_{\bar{\alpha}}\nu_{\beta} - \nabla_{\beta}\mu_{\bar{\alpha}}, \nabla_{[\alpha}\nu_{\beta]}) \in \Lambda^2 P^* \oplus (P^* \otimes V^*) \oplus \Lambda^2 V^* \cong \Lambda^2_H. \tag{5.3.6}$$

Then we take out the trace to get the induced operator

$$\begin{aligned}(\bar{\nabla}_{[\bar{\alpha}}\mu_{\bar{\beta}]}, &\bar{\nabla}_{\bar{\alpha}}\nu_{\beta} - \nabla_{\beta}\mu_{\bar{\alpha}} - \frac{1}{n}J_{\bar{\alpha}\beta}(\bar{\nabla}_{\bar{\gamma}}\nu^{\bar{\gamma}} - \nabla_{\gamma}\mu^{\gamma}), \nabla_{[\alpha}\nu_{\beta]}) \\ &\in \Lambda^2 P^* \oplus (P^* \otimes V^*)_{\perp} \oplus \Lambda^2 V^* \cong \Lambda^2_{H\perp}.\end{aligned} \tag{5.3.7}$$

We can see that we can't hope to eliminate the pieces of partial torsion $P^* \to \Lambda^2 V^*$, $V^* \to \Lambda^2 P^*$, which are the obstructions to integrability, but can hope to eliminate the pieces $P^* \to \Lambda^2 P^*$, $V^* \to \Lambda^2 V^*$ and $P^* \oplus V^* \to (P \otimes V)_{\perp}$. The component of the induced operator mapping into $\Lambda^2 V^*$ changes by

$$\hat{\nabla}_{[\alpha}\nu_{\beta]} = \nabla_{[\alpha}\nu_{\beta]} + \Phi_{[\alpha\beta]}{}^{\gamma}\nu_{\gamma}. \tag{5.3.8}$$

So we can arrange that $\hat{\nabla}_{[\alpha}\nu_{\beta]}$ is the part of the Rumin operator $V^* \to \Lambda^2 V^*$ and fixing this means we are restricted to change connections by $\Phi_{(\alpha\beta)}{}^{\gamma} = \Phi_{\alpha\beta}{}^{\gamma}$.

Similarly

$$\hat{\bar{\nabla}}_{[\bar{\alpha}}\mu_{\bar{\beta}]} = \bar{\nabla}_{[\bar{\alpha}}\mu_{\bar{\beta}]} - \Gamma_{[\bar{\alpha}}{}^{\bar{\gamma}}{}_{\bar{\beta}]}\omega_{\bar{\gamma}} \tag{5.3.9}$$

so we can also arrange that $\hat{\bar{\nabla}}_{[\bar{\alpha}}\mu_{\bar{\beta}]}$ agrees with the part of the Rumin operator $P^* \to \Lambda^2 P^*$ and to fix this means $\Gamma_{\bar{\alpha}}{}^{\bar{\gamma}}{}_{\bar{\beta}} = \Gamma_{(\bar{\alpha}}{}^{\bar{\gamma}}{}_{\bar{\beta})}$. We still have freedom of rank $2n^3 - 2n\binom{n}{2}$ remaining in picking the partial connection.

Finally

$$\begin{aligned}\hat{\bar{\nabla}}_{\bar{\alpha}}\nu_{\beta} - \hat{\nabla}_{\beta}\mu_{\bar{\alpha}} - \frac{1}{n}J_{\bar{\alpha}\beta}(\hat{\bar{\nabla}}_{\bar{\gamma}}\nu^{\bar{\gamma}} - \hat{\nabla}_{\gamma}\mu^{\gamma}) &= \bar{\nabla}_{\bar{\alpha}}\nu_{\beta} - \nabla_{\beta}\mu_{\bar{\alpha}} - \frac{1}{n}J_{\bar{\alpha}\beta}(\bar{\nabla}_{\bar{\gamma}}\nu^{\bar{\gamma}} - \nabla_{\gamma}\mu^{\gamma}) \\ + \Gamma_{\bar{\alpha}\beta}{}^{\gamma}\nu_{\gamma} + \Phi_{\beta}{}^{\bar{\gamma}}{}_{\bar{\alpha}}\mu_{\bar{\gamma}} - \frac{1}{n}J_{\bar{\alpha}\beta}(\Gamma_{\bar{\nu}}{}^{\bar{\nu}\gamma}\nu_{\gamma} + \Phi_{\nu}{}^{\bar{\gamma}\nu}\mu_{\bar{\gamma}}).\end{aligned} \tag{5.3.10}$$

Since $\mu_{\bar{\alpha}}, \nu_\beta$ are arbitrary the last line is zero if and only if

$$\Gamma_{\bar{\alpha}\beta}{}^{\gamma} - \frac{1}{n} J_{\bar{\alpha}\beta} \Gamma_{\bar{\nu}}{}^{\bar{\nu}\gamma} = 0 \tag{5.3.11}$$

and

$$\Phi_{\beta}{}^{\bar{\gamma}}{}_{\bar{\alpha}} - \frac{1}{n} J_{\bar{\alpha}\beta} \Phi_{\nu}{}^{\bar{\gamma}\nu} = 0. \tag{5.3.12}$$

Considering the first condition 5.3.11, if

$$\Gamma_{\bar{\alpha}\beta}{}^{\gamma} = J_{\bar{\alpha}\beta} \xi^{\gamma} \tag{5.3.13}$$

enforcing $\Gamma_{\bar{\alpha}}{}^{\bar{\gamma}}{}_{\bar{\beta}} = \Gamma_{(\bar{\alpha}}{}^{\bar{\gamma}}{}_{\bar{\beta})}$ gives

$$\delta^{\bar{\gamma}}{}_{\bar{\alpha}} \xi_{\bar{\beta}} = \delta^{\bar{\gamma}}{}_{(\bar{\alpha}} \xi_{\bar{\beta})} \implies \xi^{\gamma} = 0 \implies \Gamma_{\bar{\alpha}\beta}{}^{\gamma} = 0. \tag{5.3.14}$$

Similarly 5.3.12 implies that $\Phi_{\beta\gamma}{}^{\alpha} = 0$ using the fact $\Phi_{(\alpha\beta)}{}^{\gamma} = \Phi_{\alpha\beta}{}^{\gamma}$. In particular the map

$$(\Gamma_{\bar{\alpha}\beta}{}^{\gamma}, \Phi_{\alpha\beta}{}^{\gamma}) \mapsto \left\{ \Gamma_{\bar{\alpha}\beta}{}^{\gamma} - \frac{1}{n} J_{\bar{\alpha}\beta} \Gamma_{\bar{\nu}}{}^{\bar{\nu}\gamma},\ \Phi_{\beta}{}^{\bar{\gamma}}{}_{\bar{\alpha}} - \frac{1}{n} J_{\bar{\alpha}\beta} \Phi_{\nu}{}^{\bar{\gamma}\nu} \right\} \tag{5.3.15}$$

for $\Gamma_{\bar{\alpha}\beta}{}^{\gamma}, \Phi_{\alpha\beta}{}^{\gamma}$ satisfying $\Gamma_{\bar{\alpha}}{}^{\bar{\gamma}}{}_{\bar{\beta}} = \Gamma_{(\bar{\alpha}}{}^{\bar{\gamma}}{}_{\bar{\beta})}$ and $\Phi_{(\alpha\beta)}{}^{\gamma} = \Phi_{\alpha\beta}{}^{\gamma}$ and is injective, which shows uniqueness. Note that the rank of the bundle $\mathrm{Hom}(P^* \oplus V^*, (P^* \otimes V^*)_o)$ is $2n(n^2-1)$ which coincides with the freedom $2n^3 - 2n\binom{n}{2}$ for $n = 2$. Thus the map 5.3.15 is a surjection onto $\mathrm{Hom}(P^* \oplus V^*, (P^* \otimes V^*)_o)$ for $n = 2$ and thus we have also shown existence for $n = 2$. We are mostly concerned with 5-dimensional Legendrean contact manifolds, so the proof of existence in the general case is in appendix B. □

Note that we have completely accounted for the Rumin operator if and only if the components of the Rumin operator $P^* \to \Lambda^2 V^*$ and $V^* \to \Lambda^2 P^*$ vanish which happens precisely when the contact Legendrean structure is integrable. Accordingly, we will use the terms partial torsion free and integrable interchangeably when referring to a Legendrean contact structure. If we write $\Sigma_{\alpha\beta}{}^{\bar{\gamma}}$ and $\Pi_{\bar{\alpha}\bar{\beta}}{}^{\gamma}$ for these homomorphisms, respectively, then in the case of a general contact Legendrean structure we can write the Rumin operator

$$\begin{bmatrix} \mu_{\bar{\beta}} \\ \nu_{\beta} \end{bmatrix} \mapsto \begin{bmatrix} \bar{\nabla}_{[\bar{\alpha}} \mu_{\bar{\beta}]} + \Pi_{\bar{\alpha}\bar{\beta}}{}^{\gamma} \nu_{\gamma} \\ \bar{\nabla}_{\bar{\alpha}} \nu_{\beta} - \nabla_{\beta} \mu_{\bar{\alpha}} - \frac{1}{n} J_{\bar{\alpha}\beta} (\bar{\nabla}_{\bar{\gamma}} \nu^{\bar{\gamma}} - \nabla_{\gamma} \mu^{\gamma}) \\ \nabla_{[\alpha} \nu_{\beta]} + \Sigma_{\alpha\beta}{}^{\bar{\gamma}} \mu_{\bar{\gamma}} \end{bmatrix}. \tag{5.3.16}$$

Lastly note that we get partial connections induced on $\Lambda^2 P^*$ and L, the latter because $(\bar{\nabla}_{\bar{\alpha}}, \nabla_\alpha)$ preserves the image of L under the Levi form. In particular, the canonical isomorphism $P \to V^* \otimes L^*$ is parallel.

Using the existence and uniqueness, it is not hard to verify the following formulae for the change of partial connection given a change of contact form $\hat{\alpha} = \Omega\alpha$:

$$(\hat{\bar{\nabla}}_{\bar{\alpha}}\omega_\beta, \hat{\nabla}_\alpha\omega_\beta) = (\bar{\nabla}_{\bar{\alpha}}\omega_\beta + J_{\bar{\alpha}\beta}\bar{\Upsilon}_{\bar{\gamma}}\omega^{\bar{\gamma}}, \nabla_\alpha\omega_\beta - 2\Upsilon_{(\alpha}\omega_{\beta)}). \tag{5.3.17}$$

Here $\Upsilon_\alpha = \nabla_\alpha \log\Omega$ and $\bar{\Upsilon}_{\bar{\alpha}} = \bar{\nabla}_{\bar{\alpha}} \log\Omega$. For sections of P^*

$$(\hat{\bar{\nabla}}_{\bar{\alpha}}\omega_{\bar{\beta}}, \hat{\nabla}_\alpha\omega_{\bar{\beta}}) = (\bar{\nabla}_{\bar{\alpha}}\omega_{\bar{\beta}} - 2\bar{\Upsilon}_{(\bar{\alpha}}\omega_{\bar{\beta})}, \nabla_\alpha\omega_{\bar{\beta}} + J_{\bar{\beta}\alpha}\Upsilon_\gamma\omega^\gamma). \tag{5.3.18}$$

For sections of L we have

$$(\hat{\bar{\nabla}}_{\bar{\alpha}}f, \hat{\nabla}_\alpha f) = (\bar{\nabla}_{\bar{\alpha}}f - \bar{\Upsilon}_{\bar{\alpha}}f, \nabla_\alpha f - \Upsilon_\alpha f). \tag{5.3.19}$$

Lastly, in 5-dimensions we will need the formula for the change of connection on the line bundle $\Lambda^2 P^*$, which is

$$(\hat{\bar{\nabla}}_{\bar{\alpha}}f, \hat{\nabla}_\alpha f) = (\bar{\nabla}_{\bar{\alpha}}f - 3\bar{\Upsilon}_{\bar{\alpha}}f, \nabla_\alpha f + \Upsilon_\alpha f). \tag{5.3.20}$$

Using these it is then straightforward to write down formulas for then change of connection on all the bundles of interest.

It is worth noting that the formulae 5.3.17 and 5.3.18 are almost identical to the formulae in [GG05] 2.7, for the change of distinguished partial connection associated with a choice of contact form for a CR-structure of hypersurface type.

Chapter 6

A bridge between Legendrean and G_2 contact structures

In this chapter we demonstrate a new link between G_2 contact geometries and Legendrean contact structures. The main idea is inspired by the flying saucers construction. We start with a Legendrean contact structure, and with some extra choice of input data, a choice of sections, construct a G_2 contact geometry.

We calculate the partial torsion of this G_2 contact geometry. The formula is in terms of derivatives of these sections with respect to the preferred partial connection constructed in Chapter 5, and the formula also features the two obstructions to integrability. Thus we have a way of determining when the resulting G_2 contact structure is locally isomorphic to G_2/P_2.

Lastly, we show that locally, all G_2 contact geometries arise via this construction. The idea is simple: We use a G_2 contact geometry plus a choice of two non-vanishing sections to generate a Legendrean contact structure. The resulting Legendrean contact structure also comes equipped with appropriate input data to generate an isomorphic G_2 contact geometry via the construction above.

6.1 G_2 contact geometry from a Legendrean contact structure

Firstly we will need to fix some notation. We have a canonical section of $\Lambda^2 V \otimes \Lambda^2 V^*$ that we will use to raise and lower indices, written either $e^{\alpha\beta}$ or $e_{\alpha\beta}$ depending on the context. We specify

$$e_{\alpha\beta}e^{\alpha\gamma} = \delta_\alpha{}^\gamma. \tag{6.1.1}$$

For the following section let $J_{\bar{\alpha}\beta}$ denote the canonical section of $P^* \otimes V^* \otimes L^*$ associated with a Legendrean contact structure. More specifically given $\Phi_\alpha \in V^*$

we define

$$\begin{aligned}\Phi^{\bar\alpha} &:= J^{\bar\alpha\alpha}\Phi_\alpha \in P\otimes L\\ \Phi^{\alpha} &:= e^{\alpha\beta}\Phi_\beta \in V\otimes \Lambda^2 V^*\\ \Phi_{\bar\alpha} &:= J_{\bar\alpha\alpha}\Phi^\alpha = e_{\bar\alpha\bar\beta}\Phi^{\bar\beta} \in P^*\otimes\Lambda^2 P\otimes L \qquad (6.1.2)\end{aligned}$$

so note that $e_{\bar\alpha\bar\beta}$ lowers indices in the 'opposite' way to $e_{\alpha\beta}$ and similarly we have $\Phi^{\bar\alpha} = e^{\bar\beta\bar\alpha}\Phi_{\bar\beta}$. The notation is consistent because the following diagram commutes by construction:

$$\begin{array}{ccc} V^* & \xrightarrow{e^{\alpha\beta}} & V\otimes\Lambda^2V^* \\ \big\downarrow{\scriptstyle J^{\bar\alpha\beta}} & & \big\downarrow{\scriptstyle J^{\bar\alpha\beta}} \\ P\otimes L & \xrightarrow{e_{\bar\alpha\bar\beta}} & P^*\otimes\Lambda^2P\otimes L \end{array}$$

Inspired by the flying saucers construction, we construct a G_2 contact structure from a Legendrean contact structure and what amounts to a choice of frame on one of the rank-2 subbundles.

Theorem 6.1.3. *Two sections $\Phi_{\bar\alpha} \in P^*\otimes(\Lambda^2P^*)^{-3/4}\otimes L^{1/4}$ and $\Psi_\beta \in V^*\otimes(\Lambda^2P^*)^{3/4}\otimes L^{-5/4}$ satisfying $\Phi_{\bar\alpha}J^{\bar\alpha\beta}\Psi_\beta = \Phi_{\bar\alpha}\Psi^{\bar\alpha} = 1$ define a G_2 contact geometry.*

Proof. Firstly note

$$\Phi_{\bar\alpha}\Psi^{\bar\alpha} = -\Phi^{\bar\alpha}\Psi_{\bar\alpha} = 1 \qquad (6.1.4)$$

while

$$\Phi^{\bar\alpha}\Phi_{\bar\alpha} = 0. \qquad (6.1.5)$$

Define $E := ((\Lambda^2P^*)^{1/4}\otimes L^{-1/12})$ and $F := ((\Lambda^2P^*)^{-1/4}\otimes L^{5/12})$, then $S = E\oplus F$. Then we get the usual isomorphism

$$\begin{aligned}\odot^3 S &\cong E^3\oplus E^2F\oplus EF^2\oplus F^3\\ (e,f)\odot(e,f)\odot(e,f) &\mapsto (e^3, 3e^2f, 3ef^2, f^3). \qquad (6.1.6)\end{aligned}$$

Define an isomorphism $E^3\oplus E^2F\oplus EF^2\oplus F^3 \cong P^*\oplus V^*$ by

$$(x,y,z,w)\mapsto (x\Phi_{\bar\alpha} - \frac{1}{\sqrt3}y\Psi_{\bar\alpha},\ -\frac{1}{\sqrt3}z\Phi_\beta + w\Psi_\beta) \qquad (6.1.7)$$

where we pair factors of L and Λ^2P^* with their duals naturally. One can verify that the inverse $P^*\oplus V^*\to\odot^3S$ is

$$(\mu_{\bar\alpha},\nu_\beta)\mapsto(\mu_{\bar\alpha}\Psi^{\bar\alpha}, \sqrt3\mu_{\bar\alpha}\Phi^{\bar\alpha}, \sqrt3\nu_\beta\Psi^\beta, \nu_\beta\Phi^\beta). \qquad (6.1.8)$$

Noting that $E^3F^3 \cong L$, we use the isomorphism

$$\begin{aligned}
&\Lambda^2(E^3 \oplus E^2F \oplus EF^2 \oplus F^3) \cong E^5F \oplus E^4F^2 \oplus L \oplus L \oplus E^2F^4 \oplus EF^5 \\
&(x_1, y_1, z_1, w_1) \wedge (x_2, y_2, z_2, w_2) \\
&\mapsto (x_1y_2 - x_2y_1,\ x_1z_2 - z_1x_2,\ x_1w_2 - w_1x_1, \\
&\qquad y_1z_2 - z_1y_2,\ y_1w_2 - y_2w_1,\ z_1w_2 - w_1z_2)
\end{aligned} \tag{6.1.9}$$

and use the same convention to decompose $\Lambda^2(P^* \oplus V^*) \cong \Lambda^2P^* \oplus (P^* \otimes V^*) \oplus \Lambda^2V^*$ so that the induced map

$$\Lambda^2P^* \oplus (P^* \otimes V^*) \oplus \Lambda^2V^* \to E^5F \oplus E^4F^2 \oplus L \oplus L \oplus E^2F^4 \oplus EF^5$$

is written

$$\begin{aligned}
&(\mu_{\bar\alpha\bar\beta}, \kappa_{\bar\alpha\beta}, \nu_{\alpha\beta}) \mapsto \\
&\Big(\frac{1}{\sqrt{3}}\Psi^{\bar\alpha}\mu_{\bar\alpha\bar\beta}\Phi^{\bar\beta}, \frac{1}{\sqrt{3}}\Psi^{\bar\alpha}\kappa_{\bar\alpha\beta}\Psi^{\beta}, \Psi^{\bar\alpha}\kappa_{\bar\alpha\beta}\Phi^{\beta}, \frac{1}{3}\Phi^{\bar\alpha}\kappa_{\bar\alpha\beta}\Psi^{\beta}, \frac{1}{\sqrt{3}}\Phi^{\bar\alpha}\kappa_{\bar\alpha\beta}\Phi^{\beta}, \frac{1}{\sqrt{3}}\Psi^{\alpha}\nu_{\alpha\beta}\Phi^{\beta}\Big).
\end{aligned} \tag{6.1.10}$$

We need to check that the map preserves the Levi form. The canonical subbundle

$$\Lambda^2(E \oplus F) \otimes \Lambda^2(E \oplus F) \otimes \Lambda^2(E \oplus F) \hookrightarrow \Lambda^2(\odot^3(E \oplus F)) \tag{6.1.11}$$

is spanned by

$$e^3 \wedge f^3 - 3e^2f \wedge ef^2 \tag{6.1.12}$$

where e and f are trivialisations for E and F respectively. It is easy to check that given a choice of contact form $\alpha \in L$

$$(0, J_{\bar\alpha\beta}\alpha, 0) \mapsto (0, 0, \alpha, -3\alpha, 0, 0), \alpha \in L \tag{6.1.13}$$

which shows the compatibility. □

6.2 Calculating the minimal partial torsion

Consider partial connections on the bundle $S = E \oplus F$. We will use matrix notation, with entries taken in sections of powers of the line bundles L and Λ^2P^*. We have the following:

Proposition 6.2.1. *Fixing connections on L and $\Lambda^2 P^*$, then we can write any connection on $S = E \oplus F$*

$$\nabla_a \begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} \nabla_a e + \kappa_a e + \lambda_a f \\ \nabla_a f + \mu_a e + \nu_a f \end{bmatrix} \tag{6.2.2}$$

for appropriately weighted 1-forms $\kappa_a, \lambda_a, \mu_a, \nu_a$, where the connections on E and F are the ones induced by the choice of connections L, $\Lambda^2 P^*$.

Choose a contact form $\alpha \in L$, then recall we get a distinguished partial connection $\nabla : V^* \to \Lambda^1_H \otimes V^*$ which induces partial connections on $L \hookrightarrow \Lambda^2_H$ and $\Lambda^2 P^*$ and furthermore $\alpha \in L$ is parallel. Take the connections on L and $\Lambda^2 P^*$ in 6.2.1 to be these induced partial connections.

If $\kappa_a, \lambda_a, \mu_a, \nu_a$ vanish, the induced partial connection on $\odot^3 S \cong E^3 \oplus E^2 F \oplus EF^2 \oplus F^3$ is manifestly contact since the Levi form 6.1.13 is annihilated by the induced partial connection on the bundle $E^5 F \oplus E^4 F^2 \oplus L \oplus L \oplus E^2 F^4 \oplus EF^5$.

This is equivalent to 6.2.2 annihilating a non-zero section of $\Lambda^2(E \oplus F)$ (the cube root of the Levi form) and this condition still holds true for non-zero $\kappa_a, \lambda_a, \mu_a, \nu_a$ so long as $\kappa_a = -\nu_a$. So a connections on S as in 6.2.2 will induce a contact connection on $\odot^3 S$ when the connections on E, F are induced by the choice of contact form and $\kappa_a = -\nu_a$.

Then, one calculates that the induced connection on $\odot^3 S \cong E^3 \oplus E^2 F \oplus EF^2 \oplus F^3$ is

$$\nabla_a \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} \nabla_a x \\ \nabla_a y \\ \nabla_a z \\ \nabla_a w \end{bmatrix} + \begin{bmatrix} 3\kappa_a & \lambda_a & 0 & 0 \\ 3\mu_a & \kappa_a & 2\lambda_a & 0 \\ 0 & 2\mu_a & -\kappa_a & 3\lambda_a \\ 0 & 0 & \mu_a & -3\kappa_a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \tag{6.2.3}$$

Rewriting the above using our isomorphism $E^3 \oplus E^2 F \oplus EF^2 \oplus F^3 \cong P^* \oplus V^*$

$$\begin{aligned} \nabla_a(x,y,z,w) = \Big(&(\nabla_a x + 3x\kappa_a + y\lambda_a)\Phi_{\bar{\alpha}} - \frac{1}{\sqrt{3}}(\nabla_a y + 3x\mu_a + y\kappa_a + 2z\lambda_a)\Psi_{\bar{\alpha}}, \\ &-\frac{1}{\sqrt{3}}(\nabla_a z + 2y\mu_a - z\kappa_a + 3w\lambda_a)\Phi_\beta + (\nabla_a w + z\mu_a - 3w\kappa_a)\Psi_\beta \Big) \\ &\in \Lambda^1_H \otimes (P^* \oplus V^*) \end{aligned} \tag{6.2.4}$$

To calculate the induced operator $\wedge_\perp \circ \nabla$ we project the above onto the direct sum decomposition $\Lambda^2 P^* \oplus (P^* \oplus V^*)_\perp \oplus \Lambda^2 V^*$. We write $\kappa_a = (\bar{\kappa}_{\bar{\alpha}}, \kappa_\alpha) \in P^* \oplus V^*$ and so on.

$$(\wedge_\perp \circ \nabla)(x,y,z,w) = \begin{bmatrix} (\bar\nabla_{[\bar\alpha}x + 3x\bar\kappa_{[\bar\alpha} + y\bar\lambda_{[\bar\alpha]})\Phi_{\bar\beta]} - \frac{1}{\sqrt{3}}(\bar\nabla_{[\bar\alpha}y + 3x\bar\mu_{[\bar\alpha} + y\bar\kappa_{[\bar\alpha} + 2z\bar\lambda_{[\bar\alpha]})\Psi_{\bar\beta]} \\ \Big\{ -\frac{1}{\sqrt{3}}(\bar\nabla_{\bar\alpha}z + 2y\bar\mu_{\bar\alpha} - z\bar\kappa_{\bar\alpha} + 3w\bar\lambda_{\bar\alpha})\Phi_\beta + (\bar\nabla_{\bar\alpha}w + z\bar\mu_{\bar\alpha} - 3w\bar\kappa_{\bar\alpha})\Psi_\beta \\ -(\nabla_\beta x + 3x\kappa_\beta + y\lambda_\beta)\Phi_{\bar\alpha} + \frac{1}{\sqrt{3}}(\nabla_\beta y + 3x\mu_\beta + y\kappa_\beta + 2z\lambda_\beta)\Psi_{\bar\alpha}\Big\}_\perp \\ -\frac{1}{\sqrt{3}}(\nabla_{[\alpha}z + 2y\mu_{[\alpha} - z\kappa_{[\alpha} + 3w\lambda_{[\alpha]})\Phi_{\beta]} + (\nabla_{[\alpha}w + z\mu_{[\alpha} - 3w\kappa_{[\alpha]})\Psi_{\beta]} \end{bmatrix}.$$

On the other hand, the Rumin operator can be written in terms of the distinguished partial connection. Feeding 6.1.7 into 5.3.16 we can write the Rumin operator

$$d_\perp(x,y,z,w) = \begin{bmatrix} \bar\nabla_{[\bar\alpha}(x\Phi_{\bar\beta]}) - \frac{1}{\sqrt{3}}\bar\nabla_{[\bar\alpha}(y\Psi_{\bar\beta]}) + \Pi_{\bar\alpha\bar\beta}{}^{\gamma}(-\frac{z}{\sqrt{3}}\Phi_\gamma + w\Psi_\gamma) \\ \{\bar\nabla_{\bar\alpha}(-\frac{z}{\sqrt{3}}\Phi_\beta + w\Psi_\beta) - \nabla_\beta(x\Phi_{\bar\alpha} - \frac{y}{\sqrt{3}}\Psi_{\bar\alpha})\}_\perp \\ -\frac{1}{\sqrt{3}}\nabla_{[\alpha}(z\Phi_{\beta]}) + \nabla_{[\alpha}(w\Psi_{\beta]}) + \Sigma_{\alpha\beta}{}^{\bar\gamma}(x\Phi_{\bar\gamma} - \frac{y}{\sqrt{3}}\Psi_{\bar\gamma}) \end{bmatrix}. \tag{6.2.5}$$

Enforce $d_\perp - (\wedge_\perp \circ \nabla) = 0$ to obtain a system of 12 equations that must be satisfied for the connection on S to be partial torsion free.

$$\bar\nabla_{[\bar\alpha}\Phi_{\bar\beta]} = 3\bar\kappa_{[\bar\alpha}\Phi_{\bar\beta]} - \sqrt{3}\bar\mu_{[\bar\alpha}\Psi_{\bar\beta]} \tag{6.2.6}$$
$$\bar\nabla_{[\bar\alpha}\Psi_{\bar\beta]} = -\sqrt{3}\bar\lambda_{[\bar\alpha}\Phi_{\bar\beta]} + \bar\kappa_{[\bar\alpha}\Psi_{\bar\beta]} \tag{6.2.7}$$
$$\Pi_{\bar\alpha\bar\beta}{}^{\gamma}\Phi_\gamma = 2\bar\lambda_{[\bar\alpha}\Psi_{\bar\beta]} \tag{6.2.8}$$
$$\Pi_{\bar\alpha\bar\beta}{}^{\gamma}\Psi_\gamma = 0 \tag{6.2.9}$$
$$(\nabla_\beta\Phi_{\bar\alpha})_\perp = (3\kappa_\beta\Phi_{\bar\alpha} - \sqrt{3}\mu_\beta\Psi_{\bar\alpha})_\perp \tag{6.2.10}$$
$$(\nabla_\beta\Psi_{\bar\alpha})_\perp = (-2\bar\mu_{\bar\alpha}\Phi_\beta - \sqrt{3}\lambda_\beta\Phi_{\bar\alpha} + \kappa_\beta\Psi_{\bar\alpha})_\perp \tag{6.2.11}$$
$$(\bar\nabla_{\bar\alpha}\Phi_\beta)_\perp = (-\bar\kappa_{\bar\alpha}\Phi_\beta - \sqrt{3}\bar\mu_{\bar\alpha}\Psi_\beta - 2\lambda_\beta\Psi_{\bar\alpha})_\perp \tag{6.2.12}$$
$$(\bar\nabla_{\bar\alpha}\Psi_\beta)_\perp = (-\sqrt{3}\bar\lambda_{\bar\alpha}\Phi_\beta - 3\bar\kappa_{\bar\alpha}\Psi_\beta)_\perp \tag{6.2.13}$$
$$\nabla_{[\alpha}\Phi_{\beta]} = -\kappa_{[\alpha}\Phi_{\beta]} - \sqrt{3}\mu_{[\alpha}\Psi_{\beta]} \tag{6.2.14}$$
$$\nabla_{[\alpha}\Psi_{\beta]} = -\sqrt{3}\lambda_{[\alpha}\Phi_{\beta]} - 3\kappa_{[\alpha}\Psi_{\beta]} \tag{6.2.15}$$
$$\Sigma_{\alpha\beta}{}^{\bar\gamma}\Psi_{\bar\gamma} = 2\mu_{[\alpha}\Phi_{\beta]} \tag{6.2.16}$$
$$\Sigma_{\alpha\beta}{}^{\bar\gamma}\Phi_{\bar\gamma} = 0 \tag{6.2.17}$$

Solving the above system of equations for $\kappa_a, \lambda_a, \mu_a$, then substituting back (see appendix C for more details) we find the eight components of the minimal partial torsion.

$$\tau_1 = \Phi^{\alpha}\Pi_{\bar{\alpha}\bar{\beta}}{}^{\gamma}\Psi_{\gamma}\Psi^{\beta} \tag{6.2.18}$$

$$\tau_2 = \Phi^{\bar{\alpha}}\Pi_{\bar{\alpha}\bar{\beta}}{}^{\gamma}\Phi_{\gamma}\Psi^{\bar{\beta}} - \frac{1}{\sqrt{3}}\Psi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Psi_{\beta})\Psi^{\beta} \tag{6.2.19}$$

$$\tau_3 = \Psi^{\beta}(\nabla_{\beta}\Phi_{\bar{\alpha}})\Phi^{\bar{\alpha}} + \Phi^{\beta}(\nabla_{\beta}\Phi_{\bar{\alpha}})\Psi^{\bar{\alpha}} - 2\Phi^{\alpha}(\nabla_{[\alpha}\Phi_{\beta]})\Psi^{\beta} + 2\Phi^{\beta}(\nabla_{\beta}\Psi_{\bar{\alpha}})\Phi^{\bar{\alpha}} \tag{6.2.20}$$

$$\begin{aligned}\tau_4 = {} & \Psi^{\beta}(\nabla_{\beta}\Psi_{\bar{\alpha}})\Phi^{\bar{\alpha}} + \Phi^{\beta}(\nabla_{\beta}\Psi_{\bar{\alpha}})\Psi^{\bar{\alpha}} - \frac{2}{3}\Phi^{\alpha}(\nabla_{[\alpha}\Psi_{\beta]})\Psi^{\beta} + \frac{2}{3}\Psi^{\beta}(\nabla_{\beta}\Phi_{\bar{\alpha}})\Psi^{\bar{\alpha}} \\ & + \frac{2}{\sqrt{3}}\Phi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Phi_{\beta})\Phi^{\beta}\end{aligned} \tag{6.2.21}$$

$$\begin{aligned}\tau_5 = {} & \Phi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Phi_{\beta})\Psi^{\beta} + \Psi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Phi_{\beta})\Phi^{\beta} + \frac{2}{3}\Phi^{\bar{\alpha}}(\bar{\nabla}_{[\bar{\alpha}}\Phi_{\bar{\beta}]})\Psi^{\bar{\beta}} + \frac{2}{3}\Phi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Psi_{\beta})\Phi^{\beta} \\ & + \frac{2}{\sqrt{3}}\Psi^{\beta}(\nabla_{\beta}\Psi_{\bar{\alpha}})\Psi^{\bar{\alpha}}\end{aligned} \tag{6.2.22}$$

$$\tau_6 = \Phi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Psi_{\beta})\Psi^{\beta} + \Psi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Psi_{\beta})\Phi^{\beta} + 2\Phi^{\bar{\alpha}}(\bar{\nabla}_{[\bar{\alpha}}\Psi_{\bar{\beta}]})\Psi^{\beta} + 2\Psi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Phi_{\beta})\Psi^{\beta} \tag{6.2.23}$$

$$\tau_7 = \Phi^{\alpha}\Sigma_{\alpha\beta}{}^{\bar{\gamma}}\Psi_{\bar{\gamma}}\Psi^{\beta} + \frac{1}{\sqrt{3}}\Phi^{\beta}(\nabla_{\beta}\Phi_{\bar{\alpha}})\Phi^{\bar{\alpha}} \tag{6.2.24}$$

$$\tau_8 = \Phi^{\alpha}\Sigma_{\alpha\beta}{}^{\bar{\gamma}}\Phi_{\bar{\gamma}}\Psi^{\beta} \tag{6.2.25}$$

These are invariants, as can be checked by using 5.3.17 through 5.3.19.

The first observation to be made is that for integrable Legendrean contact structures, the resulting G_2 contact geometry has minimal partial torsion lying in a bundle of rank 6.

Theorem 6.2.26 (Flat G_2 contact structure)**.** *Given a 5-dimensional flat Legendrean contact structure $(M, H = P \oplus V)$ one can construct a flat G_2 contact structure in a neighbourhood of any point.*

Proof. Given a flat Legendrean contact structure we can assume locally we are in the setting of 5.1.3 with $n = 2$. This comes equipped with a contact form and it is straightforward to check that the preferred partial connection on the bundle $\Lambda^1_H \cong P^* \oplus V^*$ in the presence of this contact form is given by the canonical flat partial connection with respect to the frame $\{d_{\perp}p_1, d_{\perp}p_1, d_{\perp}q^1, d_{\perp}q^2\}$. Define $\lambda = d_{\perp}p_1 \wedge d_{\perp}p_2 \in \Lambda^2 P^*$ and then define

$$\Phi = \lambda^{-3/4}\alpha^{1/4}d_{\perp}p_1 \in P^* \otimes (\Lambda^2 P^*)^{-3/4} \otimes L^{1/4} \tag{6.2.27}$$

$$\Psi = \lambda^{3/4}\alpha^{-5/4}d_{\perp}q_1 \in V^* \otimes (\Lambda^2 P^*)^{3/4} \otimes L^{-5/4} \tag{6.2.28}$$

These satisfy the hypotheses of 6.1.3. Since these are parallel with respect to the preferred connection and the Legendrean contact geometry is integrable, it is clear that $\tau_1, \ldots, \tau_8$ listed above all vanish. □

Interestingly, the formula does not seem to preclude a flat G_2 contact geometry being generated from a non-integrable Legendrean contact structure.

6.3 Every G_2 contact structure arises from a Legendrean contact structure

Let $S \to M$ where M is a contact manifold be a G_2 contact geometry so that we have an isomorphism $\Lambda^1_H \cong \odot^3 S$ which preserves the Levi form. As we can always do locally, take two sections

$$\phi_A, \psi_A \in S \otimes L^{-1/6} \tag{6.3.1}$$

with $\phi_A \psi^A = 1$. Choosing a contact form α we can trivialise $\odot^3 S$ by

$$\alpha^{1/2}\phi_A\phi_B\phi_C,\ \alpha^{1/2}\phi_{(A}\phi_B\psi_{C)},\ \alpha^{1/2}\phi_{(A}\psi_B\psi_{C)},\ \alpha^{1/2}\psi_A\psi_B\psi_C. \tag{6.3.2}$$

Define

$$P^* = \mathrm{span}\{\alpha^{1/2}\phi_A\phi_B\phi_C, \alpha^{1/2}\phi_{(A}\phi_B\psi_{C)}\}, \tag{6.3.3}$$

$$V^* = \mathrm{span}\{\alpha^{1/2}\phi_{(A}\psi_B\psi_{C)}, \alpha^{1/2}\psi_A\psi_B\psi_C\} \tag{6.3.4}$$

and by contracting with the Levi form it is easy to check this defines a Legendrean contact structure $\odot^3 S = P^* \oplus V^*$ on M. This is obviously independent of the chosen contact form.

Next, let us construct a G_2 contact geometry associated with this Legendrean contact structure. Define a rank-2 vector bundle and then $T := E \oplus F$ with:

$$E = (\Lambda^2 P^*)^{1/4} \otimes L^{-1/12}, \quad F = (\Lambda^2 P^*)^{-1/4} \otimes L^{5/12}. \tag{6.3.5}$$

We have a distinguished section

$$\gamma := \sqrt{3/2}(\phi_A\phi_B\phi_C\phi_{(D}\phi_E\psi_{F)} - \phi_{(A}\phi_B\psi_{C)}\phi_D\phi_E\phi_F) \tag{6.3.6}$$

of $\Lambda^2 P^* \otimes L^{-1}$ which we will denote γ and then

$$\gamma^{-1} = \sqrt{3/2}(\psi^A\psi^B\psi^C\phi^{(D}\psi^E\psi^{F)} - \phi^{(A}\psi^B\psi^{C)}\psi^D\psi^E\psi_F). \tag{6.3.7}$$

Then if we define

$$\Phi_{ABC} = \gamma^{-3/4}\phi_A\phi_B\phi_C \in P^* \otimes (\Lambda^2 P^*)^{-3/4} \otimes L^{1/4} \tag{6.3.8}$$

$$\Psi_{ABC} = \gamma^{3/4}\psi_A\psi_B\psi_C \in V^* \otimes (\Lambda^2 P^*)^{3/4} \otimes L^{-5/4}, \tag{6.3.9}$$

we have

$$\Phi_{ABC}\Psi^{ABC} = 1 \tag{6.3.10}$$

and therefore the data to construct a G_2 contact structure as per Theorem 6.1.3.

Recall that in 5-dimensional Legendrean contact geometry we have a canonical section of $\Lambda^2 P^* \otimes \Lambda^2 P$ which gives an isomorphism $P \to P^* \otimes \Lambda^2 P$ written e_{ab} or e^{cd}. This section is characterised by being identically equal to 2 when paired with itself $(\Lambda^2 P^* \otimes \Lambda^2 P) \otimes (\Lambda^2 P \otimes \Lambda^2 P^*) \to \mathbb{R}$ and therefore it is realised as

$$\frac{3}{\sqrt{2}}(\phi_A\phi_B\phi_C\phi_{(D}\phi_E\psi_{F)} - \phi_{(A}\phi_B\psi_{C)}\phi_D\phi_E\phi_F)$$
$$(\psi^G\psi^H\psi^I\phi^{(L}\psi^M\psi^{N)} - \phi^{(G}\psi^H\psi^{I)}\psi^L\psi^M\psi^N) \tag{6.3.11}$$

and suppressing indices we will alternately write it as e_{ABCDEF} or e^{GHILMN}. Likewise we have a canonical section of $\Lambda^2 V^* \otimes \Lambda^2 V$ which can be realised as

$$\frac{3}{\sqrt{2}}(\phi^A\phi^B\phi^C\phi^{(D}\phi^E\psi^{F)} - \phi^{(A}\phi^B\psi^{C)}\phi^D\phi^E\phi^F)$$
$$(\psi_G\psi_H\psi_I\phi_{(L}\psi_M\psi_{N)} - \phi_{(G}\psi_H\psi_{I)}\psi_L\psi_M\psi_N) \tag{6.3.12}$$

which we will write as f^{ABCDEF} or f_{ABCDEF}. Then if we define

$$\Theta_{ABC} = e_{ABCDEF}\Psi^{DEF} \tag{6.3.13}$$
$$\Omega_{ABC} = f_{ABCDEF}\Phi^{DEF} \tag{6.3.14}$$

the isomorphism 6.1.7, $E^3 \oplus E^2F \oplus EF^2 \oplus F^3 \cong \odot^3 S$ defining the G_2 contact geometry is given by

$$(x, y, z, w) \mapsto (x\Phi_{ABC} - \frac{y}{\sqrt{3}}\Theta_{ABC} - \frac{z}{\sqrt{3}}\Omega_{ABC} + w\Psi_{ABC}). \tag{6.3.15}$$

We now show there is an isomorphism $T \cong S$ such that the induced map $\odot^3 T \cong \odot^3 S$ agrees with above. If so, we have isomorphism of G_2 contact geometries $T \to M$ and $S \to M$ as in 3.2.6. Define an isomorphism $T = E \oplus F \cong S$

$$(e, f) \mapsto e\gamma^{-1/4}\phi_A + f\gamma^{1/4}\psi_A. \tag{6.3.16}$$

The induced map $E^3 \oplus E^2F \oplus EF^2 \oplus F^3 \cong \odot^3 S$ is

$$(x, y, z, w) \mapsto (x\gamma^{-3/4}\phi_A\phi_B\phi_C + y\gamma^{-1/4}\phi_{(A}\phi_B\psi_{C)} \tag{6.3.17}$$
$$+ z\gamma^{1/4}\phi_{(A}\psi_B\psi_{C)} + w\gamma^{3/4}\psi_A\psi_B\psi_C) \tag{6.3.18}$$
$$= (x\Phi_{ABC} - \frac{y}{\sqrt{3}}\Theta_{ABC} - \frac{z}{\sqrt{3}}\Omega_{ABC} + w\Psi_{ABC}) \tag{6.3.19}$$

which is precisely the map defining our new G_2 contact geometry. So the new G_2 contact geometry is isomorphic to the old one. By bootstrapping a Legendrean contact structure to a G_2 contact structure we have proven:

Theorem 6.3.20. *Locally, every G_2 contact structure arises via Theorem 6.1.3.*

Chapter 7

Legendrean contact tractors

Some geometric structures, for example, Riemannian manifolds, are equipped with canonical affine connections. It is known, see [ČS09] proposition 3.1.4, that parabolic geometries of type (G, P) do not admit such connections. The reason is that there is no G invariant connection on $T(G/P)$, a fact which can be proved in general by Lie algebra structure theory. Riemannian geometry is not a parabolic geometry, but it is a Cartan geometry modelled on the homogeneous space $\mathrm{Euc}(n)/O(n) \cong \mathbb{R}^n$. The canonical flat affine connection on $\mathbb{R}^n$ is $O(n)$ invariant and so we see there is no contradiction with what we know about the Levi-Civita connection. On the other hand we have seen examples of parabolic geometries: projective structures, Legendrean contact structures and G_2 contact structures, each with a class of connections that are compatible with the structure. In the above examples there was a choice required, a volume form or a contact form, before a distinguished connection arose. A problem is then that quantities derived from such a distinguished connection, for example, the curvature, are not necessarily intrinsic to the geometry.

Tractor calculus as it is known today was introduced in [BEG94] although the methods themselves originate in the work [Tho26] of Tracy Yerkes Thomas. The term “tractor” is in fact a blend of “Tracy” with “twistor” or “tensor”. In light of the above discussion about canonical connections, tractor calculus is well motivated. It turns out that on Cartan geometries, there are always what are called tractor bundles, which are bundles which necessarily admit canonical linear connections called tractor connections.

Tractor bundles have a precise definition as a type of associated bundle of the parabolic geometry. The general theory is beyond the scope of the thesis. However, in [BEG94], a particular tractor bundle on conformal structures, (a conformal structure is a Riemannian metric defined up to scale), called the standard tractor bundle, along with its canonical tractor connection, is obtained via consideration of the solution space of the conformal to Einstein equation. This is a conformally invariant partial differential equation in σ, a section of an appropriate density bundle

$$(\nabla_a \nabla_b + P_{ab})_\circ \sigma = 0 \tag{7.0.1}$$

where ∇_a is the Levi-Civita connection associated with a choice of metric in the conformal class, P_{ab} is some suitable contraction of the curvature, and $\circ$ denotes taking the trace-free part with respect to the metric. Solutions correspond to rescalings of the metric such that the resulting new metric satisfies Einstein's equation in a vacuum.

The authors obtain a bundle with connection via prolongation. By this we mean (omitting all the details) they define $\mu_a := \nabla_a \sigma$ and attempt to solve for $\nabla_a \mu_b$ in terms of known-quantities. The irreducible part $\nabla_a \mu^a$ which is unspecified is then given a new name ρ and after some algebraic manipulation it is shown that $\nabla_a \rho = -P_{ab} \mu^b$. So a system of necessary conditions on the derivatives of σ a solution of 7.0.1 is obtained. Then however, there is a system of first order differential equations

$$\nabla_a \sigma - \mu_a = 0 \tag{7.0.2}$$
$$\nabla_a \mu_b + g_{ab} \rho + P_{ab} \sigma = 0 \tag{7.0.3}$$
$$\nabla_a \rho - P_{ab} \mu^b = 0 \tag{7.0.4}$$

for which solutions (σ, μ_a, ρ) correspond to solutions to 7.0.1. Defining a connection (on the appropriate direct sum) by

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_a \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab} \rho + P_{ab} \sigma \\ \nabla_a \rho - P_{ab} \mu^b \end{bmatrix}, \tag{7.0.5}$$

its parallel sections are in 1-1 correspondence with solutions to 7.0.1 and furthermore, by virtue of 7.0.1 being invariant, the connection is conformally invariant. Accordingly its curvature will be an invariant of the conformal structure, and in fact it turns out to vanish if and only if the conformal structure is flat.

More examples of prolongation procedures are to be found in [BČEG07]. It is worth noting that we can only obtain a finite-rank vector bundle if the prolonged PDE has a finite dimensional solution space. If the prolongation produces a finite-rank vector bundle the partial differential equation is said to be of finite-type.

In this chapter, we carry out a similar prolongation procedure to obtain the standard tractor bundle in the 5-dimensional integrable Legendrean contact case. We also compute the tractor connection's partial curvature, and show that vanishing of this partial curvature is equivalent to the Legendrean contact structure being locally isomorphic to the flat model. First however, for sections 7.1 to 7.3 the task is to set up notation and rewrite many of the identities in Chapter 2 in this notation.

7.1 Notation for Legendrean contact structures

To facilitate future calculations it is helpful to introduce projectors $\bar{\Pi}^a_{\bar{\alpha}}, \Pi^a_\alpha$ to denote the projections $\Lambda^1_H \to P^*$ and $\Lambda^1_H \to V^*$ respectively. We transpose the position of the indices $\bar{\Pi}^{\bar{\alpha}}_a, \Pi^\alpha_a$ in order to denote the inclusion maps $P^* \to \Lambda^1_H$ and $V^* \to \Lambda^1_H$.

We have

$$\bar{\Pi}^a_{\bar{\beta}}\bar{\Pi}^{\bar{\alpha}}_a = \delta_{\bar{\beta}}{}^{\bar{\alpha}}, \quad \Pi^a_\beta \Pi^\alpha_a = \delta_\beta{}^\alpha. \tag{7.1.1}$$

Dually we can think of $\bar{\Pi}^a_{\bar{\alpha}}$ as the inclusion $P \to H$ and so on. Further identities are

$$\Pi^\gamma_b \Pi^a_\gamma + \bar{\Pi}^{\bar{\gamma}}_b \bar{\Pi}^a_{\bar{\gamma}} = \delta_b{}^a. \tag{7.1.2}$$

Suppose we have a 2-form $\omega_{ab} \in \Lambda^2_H$. Then we write the decomposition of 2-forms $\Lambda^2_H \cong \Lambda^2 P^* \oplus (P \otimes V) \oplus \Lambda^2 V^*$ like

$$\omega_{ab} \mapsto (\bar{\Pi}^a_{\bar{\alpha}}\bar{\Pi}^b_{\bar{\beta}}\omega_{ab}, 2\bar{\Pi}^a_{\bar{\alpha}}\Pi^b_\beta\omega_{ab}, \Pi^a_\alpha\Pi^b_\beta\omega_{ab}). \tag{7.1.3}$$

One can check that the inverse is

$$(\omega_{\bar{\alpha}\bar{\beta}}, \mu_{\bar{\alpha}\beta}, \nu_{\alpha\beta}) \mapsto \bar{\Pi}^{\bar{\alpha}}_a \bar{\Pi}^{\bar{\beta}}_b \omega_{\bar{\alpha}\bar{\beta}} + \bar{\Pi}^{\bar{\alpha}}_{[a}\Pi^\beta_{b]}\mu_{\bar{\alpha}\beta} + \Pi^\alpha_a \Pi^\beta_b \nu_{\alpha\beta}. \tag{7.1.4}$$

In this chapter we will write $J_{\bar{\alpha}\beta}$ for the canonical section of $P^* \otimes V^* \otimes L^*$, and use this to raise and lower indices. If we write J_{ab} for the canonical section of $\Lambda^2_H \otimes L^*$, then $J_{\bar{\alpha}\beta} = \bar{\Pi}^a_{\bar{\alpha}} J_{ab} \Pi^b_\beta$ so that

$$J_{ab} = 2\bar{\Pi}^{\bar{\alpha}}_{[a}\Pi^\beta_{b]} J_{\bar{\alpha}\beta} \tag{7.1.5}$$

and

$$J^{ab} = 2\bar{\Pi}^{[a}_{\bar{\alpha}}\Pi^{b]}_\beta J^{\bar{\alpha}\beta}. \tag{7.1.6}$$

This means

$$J_{ab}J^{ac} = \delta_b{}^c, \tag{7.1.7}$$

so that everything is consistent.

Remark 7.1.8. *The notation introduced here imitates the "barred and unbarred" indices in almost complex geometry. The above is similar to the decomposition of differential forms into type with* $\omega_{\bar{\alpha}\bar{\beta}}, \omega_{\bar{\alpha}\beta}, \omega_{\alpha\beta}$ *denoting forms of type* $(2,0), (1,1), (0,2)$ *respectively. The notation for the projectors is basically the same as [CEMN20], although in that article the roles of the Greek and Latin indices are swapped and the projection onto* 2*-forms of mixed type omits the factor of* 2*.*

7.2 Curvature on Legendrean contact structures

The partial curvature $V^* \to \Lambda^2_{H\perp} \otimes V^*$ of the preferred partial connection 5.3.1 $V^* \to \Lambda^1_H \otimes V^*$ breaks up into three homomorphisms:

$$\begin{tikzcd} & \Lambda^2 P^* \otimes V^* \\ V^* \arrow[ur] \arrow[r] \arrow[dr] & (P^* \otimes V^*)_\perp \otimes V^* \\ & \Lambda^2 V^* \otimes V^* \end{tikzcd}$$

We will define the components by

$$X_{\bar\alpha\bar\beta\gamma}{}^{\nu}\phi_\nu := \bar\nabla_{[\bar\alpha}\bar\nabla_{\bar\beta]}\phi_\gamma + \Pi_{\bar\alpha\bar\beta}{}^{\nu}\nabla_\nu\phi_\gamma \tag{7.2.1}$$

$$Y_{\bar\alpha\beta\gamma}{}^{\nu}\phi_\nu := \frac{1}{2}(\bar\nabla_{\bar\alpha}\nabla_\beta\phi_\gamma - \nabla_\beta\bar\nabla_{\bar\alpha}\phi_\gamma)_\perp \tag{7.2.2}$$

$$Z_{\alpha\beta\gamma}{}^{\nu}\phi_\nu := \nabla_{[\alpha}\nabla_{\beta]}\phi_\gamma + \Sigma_{\alpha\beta}{}^{\bar\nu}\bar\nabla_{\bar\nu}\phi_\gamma. \tag{7.2.3}$$

By enforcing that $\bar\nabla_{[\bar\alpha}\bar\nabla_{\bar\beta]}(\phi_{\bar\gamma}\psi^{\bar\gamma}) = -\Pi_{\bar\alpha\bar\beta}{}^{\gamma}\nabla_\gamma(\phi_{\bar\gamma}\psi^{\bar\gamma})$ as required for the Rumin operator 5.3.16 to annihilate an exact 1-form we get

$$-X_{\bar\alpha\bar\beta}{}^{\bar\nu}{}_{\bar\gamma}\phi_{\bar\nu} = \bar\nabla_{[\bar\alpha}\bar\nabla_{\bar\beta]}\phi_{\bar\gamma} + \Pi_{\bar\alpha\bar\beta}{}^{\nu}\nabla_\nu\phi_{\bar\gamma} \tag{7.2.4}$$

$$-Y_{\bar\alpha\beta}{}^{\bar\nu}{}_{\bar\gamma}\,\phi_{\bar\nu} = \frac{1}{2}(\bar\nabla_{\bar\alpha}\nabla_\beta\phi_{\bar\gamma} - \nabla_\beta\bar\nabla_{\bar\alpha}\phi_{\bar\gamma})_\perp \tag{7.2.5}$$

$$-Z_{\alpha\beta}{}^{\bar\nu}{}_{\bar\gamma}\phi_{\bar\nu} = \nabla_{[\alpha}\nabla_{\beta]}\phi_{\bar\gamma} + \Sigma_{\alpha\beta}{}^{\nu}\nabla_\nu\phi_{\bar\gamma}. \tag{7.2.6}$$

So the curvature of the induced partial connection $\Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ preserves the decomposition $\Lambda^1_H = P^* \oplus V^*$. Writing the curvature of the induced connection on $\Lambda^1_H = P^* \oplus V^*$ as $R_{abc}{}^d$ as usual, this means:

$$R_{abcd}\bar\Pi^c_{\bar\gamma}\bar\Pi^d_{\bar\nu} = 0,\ R_{abcd}\Pi^c_\gamma\Pi^d_\nu = 0. \tag{7.2.7}$$

Now the partial connection $(\nabla_{\bar\alpha}, \nabla_\alpha)$ respects the projectors, so we can write the above pieces of partial curvature in terms of projectors acting on the full partial curvature. We have

$$X_{\bar\alpha\bar\beta\gamma\bar\nu} = -R_{abcd}\bar\Pi^a_{\bar\alpha}\bar\Pi^b_{\bar\beta}\Pi^c_\gamma\bar\Pi^d_{\bar\nu}, \tag{7.2.8}$$

$$Y_{\bar\alpha\beta\gamma\bar\nu} = -R_{abcd}\bar\Pi^a_{\bar\alpha}\Pi^b_\beta\Pi^c_\gamma\bar\Pi^d_{\bar\nu}, \tag{7.2.9}$$

$$Z_{\alpha\beta\gamma\bar\nu} = -R_{abcd}\Pi^a_\alpha\Pi^b_\beta\Pi^c_\gamma\bar\Pi^d_{\bar\nu}, \tag{7.2.10}$$

and can work out how other combinations of the projectors act on curvature by using the fact R_{abcd} is antisymmetric on its first two indices and symmetric on its last two. The minus signs in the above equations are an unfortunate consequence of defining the initial partial connection on V^* instead of P^* or V.

We define "Ricci" tensors

$$X_{\bar{\alpha}\bar{\beta}} := X_{\bar{\alpha}\bar{\gamma}}{}^{\bar{\gamma}}{}_{\bar{\beta}} \tag{7.2.11}$$
$$Z_{\alpha\beta} := Z^{\bar{\gamma}}{}_{\alpha\beta\bar{\gamma}}. \tag{7.2.12}$$

Also define

$$Y_{\bar{\alpha}\beta} := Y_{\bar{\alpha}\beta\gamma}{}^{\gamma}. \tag{7.2.13}$$

Recall 2.3.6, that ∇_a a contact and partial torsion free partial connection on Λ^1_H satisfies:

$$R_{c[ab]}{}^{c} = 0 \tag{7.2.14}$$

We will write out the consequences for Legendrean contact partial curvature.

Proposition 7.2.15. *The curvature of the distinguished partial connection 5.3.1 on a Legendrean contact structure $H = P \oplus V$ with P and V integrable satisfies:*

$$X_{\bar{\alpha}\bar{\beta}} = X_{(\bar{\alpha}\bar{\beta})} \tag{7.2.16}$$
$$Y_{\bar{\alpha}}{}^{\bar{\gamma}}{}_{\beta\bar{\gamma}} = Y_{\bar{\gamma}\beta}{}^{\bar{\gamma}}{}_{\bar{\alpha}} \tag{7.2.17}$$
$$Z_{\alpha\beta} = Z_{(\alpha\beta)} \tag{7.2.18}$$

Proof. First we rewrite the identity

$$R_{c[ab]}{}^{c} = R_{c[ab]d}J^{cd} = 2R_{c[ab]d}\bar{\Pi}^{[c}_{\bar{\alpha}}\Pi^{d]}_{\beta}J^{\bar{\alpha}\beta} = 0. \tag{7.2.19}$$

We apply projectors to extract the $(2,0)$ part

$$\begin{aligned} 2R_{c[ab]d}\bar{\Pi}^{[c}_{\bar{\alpha}}\Pi^{d]}_{\beta}J^{\bar{\alpha}\beta}\bar{\Pi}^{a}_{\bar{\gamma}}\bar{\Pi}^{b}_{\bar{\nu}} &= 0 \\ \implies R_{c[ab]d}\bar{\Pi}^{c}_{\bar{\alpha}}\Pi^{d}_{\beta}J^{\bar{\alpha}\beta}\bar{\Pi}^{a}_{\bar{\gamma}}\bar{\Pi}^{b}_{\bar{\nu}} - R_{c[ab]d}\bar{\Pi}^{d}_{\bar{\alpha}}\Pi^{c}_{\beta}J^{\bar{\alpha}\beta}\bar{\Pi}^{a}_{\bar{\gamma}}\bar{\Pi}^{b}_{\bar{\nu}} &= 0. \end{aligned} \tag{7.2.20}$$

The second term vanishes by 7.2.7. So

$$X_{\bar{\alpha}[\bar{\gamma}|\beta|\bar{\nu}]}J^{\bar{\alpha}\beta} = X_{\bar{\alpha}[\bar{\gamma}}{}^{\bar{\alpha}}{}_{\bar{\nu}]} = 0. \tag{7.2.21}$$

Similarly for the $(0,2)$ part we get

$$\begin{aligned} 2R_{c[ab]d}\bar{\Pi}^{[c}_{\bar{\alpha}}\Pi^{d]}_{\beta}J^{\bar{\alpha}\beta}\Pi^{a}_{\gamma}\Pi^{b}_{\nu} &= 0 \\ \implies R_{c[ab]d}\bar{\Pi}^{c}_{\bar{\alpha}}\Pi^{d}_{\beta}J^{\bar{\alpha}\beta}\Pi^{a}_{\gamma}\Pi^{b}_{\nu} - R_{c[ab]d}\bar{\Pi}^{d}_{\bar{\alpha}}\Pi^{c}_{\beta}J^{\bar{\alpha}\beta}\Pi^{a}_{\gamma}\Pi^{b}_{\nu} &= 0 \\ \implies Z_{\beta[\gamma\nu]\bar{\alpha}}J^{\bar{\alpha}\beta} = Z^{\bar{\alpha}}{}_{[\gamma\nu]\bar{\alpha}} &= 0. \end{aligned} \tag{7.2.22}$$

The $(1,1)$ part is a little more complicated to deal with

$$
\begin{aligned}
2R_{c[ab]d}\bar{\Pi}^{[c}_{\bar{\alpha}}\Pi^{d]}_{\beta}J^{\bar{\alpha}\beta}\bar{\Pi}^{a}_{\bar{\gamma}}\Pi^{b}_{\nu} &= 0\\
\implies R_{c[ab]d}\bar{\Pi}^{c}_{\bar{\alpha}}\Pi^{d}_{\beta}J^{\bar{\alpha}\beta}\bar{\Pi}^{a}_{\bar{\gamma}}\Pi^{b}_{\nu} - R_{c[ab]d}\bar{\Pi}^{d}_{\bar{\alpha}}\Pi^{c}_{\beta}J^{\bar{\alpha}\beta}\bar{\Pi}^{a}_{\bar{\gamma}}\Pi^{b}_{\nu} &= 0\\
\implies -Y_{\bar{\alpha}\nu\beta\bar{\gamma}}\,J^{\bar{\alpha}\beta} + Y_{\bar{\gamma}\beta\nu\bar{\alpha}}\,J^{\bar{\alpha}\beta} = 0 \implies Y_{\bar{\alpha}\nu}{}^{\bar{\alpha}}{}_{\bar{\gamma}} = Y_{\bar{\gamma}}{}^{\bar{\alpha}}{}_{\nu\bar{\alpha}}\,. &
\end{aligned}
\tag{7.2.23}
$$

□

Corollary 7.2.24. *On a Legendrean contact structure $H = P \oplus V$ with $n = 2$ and P and V integrable satisfies:*

$$X_{\bar{\alpha}\bar{\beta}\gamma}{}^{\gamma} = 0 \tag{7.2.25}$$

$$Z_{\alpha\beta\gamma}{}^{\gamma} = 0 \tag{7.2.26}$$

Proof. In this dimension skewing over three indices belonging to P^* is the zero map, so 7.2.15 yields

$$X_{[\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}}{}_{\bar{\gamma}]} = 0 \implies X_{\bar{\alpha}\bar{\beta}\gamma}{}^{\gamma} = 0, \tag{7.2.27}$$

and similarly

$$Z_{[\alpha\beta\gamma]}{}^{\gamma} = 0 \implies Z_{\alpha\beta\gamma}{}^{\gamma} = 0. \tag{7.2.28}$$

□

The next task is to compute formulae for the change of partial curvature given a change of contact form. We use 5.3.17 and 5.3.18 to calculate the following:

Proposition 7.2.29 (Legendrean contact change of partial curvature)**.** *Given a change of contact form $\hat{\alpha} = \Omega\alpha$ the components of the partial curvature of the distinguished partial connection change by*

$$\hat{X}_{\bar{\alpha}\bar{\beta}\gamma}{}^{\nu}\phi_{\nu} = X_{\bar{\alpha}\bar{\beta}\gamma}{}^{\nu}\phi_{\nu} + J_{[\bar{\beta}|\gamma|}(\bar{\nabla}_{\bar{\alpha}]}\bar{\Upsilon}_{\bar{\gamma}})\phi^{\bar{\gamma}} + J_{[\bar{\alpha}|\gamma|}\bar{\Upsilon}_{\bar{\beta}]}\bar{\Upsilon}_{\bar{\gamma}}\phi^{\bar{\gamma}} - 2\Pi_{\bar{\alpha}\bar{\beta}}{}^{\nu}\Upsilon_{(\nu}\phi_{\gamma)} \tag{7.2.30}$$

$$\hat{Y}_{\bar{\alpha}\beta\gamma}{}^{\nu}\phi_{\nu} = Y_{\bar{\alpha}\beta\gamma}{}^{\nu}\phi_{\nu} - ((\bar{\nabla}_{\bar{\alpha}}\Upsilon_{(\beta})\phi_{\gamma)} + \frac{1}{2}J_{\bar{\alpha}\gamma}(\nabla_{\beta}\bar{\Upsilon}_{\bar{\nu}})\phi^{\bar{\nu}} + \frac{1}{2}J_{\bar{\alpha}\gamma}\bar{\Upsilon}^{\nu}\Upsilon_{\nu}\phi_{\beta})_{\perp} \tag{7.2.31}$$

$$\hat{Z}_{\alpha\beta\gamma}{}^{\nu}\phi_{\nu} = Z_{\alpha\beta\gamma}{}^{\nu}\phi_{\nu} - (\nabla_{[\alpha}\Upsilon_{\beta]})\phi_{\gamma} - \phi_{[\beta}\nabla_{\alpha]}\Upsilon_{\gamma} + \Upsilon_{\gamma}\Upsilon_{[\alpha}\phi_{\beta]} + \Sigma_{\alpha\beta\gamma}\bar{\Upsilon}_{\bar{\gamma}}\phi^{\bar{\gamma}} \tag{7.2.32}$$

where $\bar{\Upsilon}_{\bar{\alpha}}$ and Υ_{α} are as in 5.3.17 and 5.3.18.

In five dimensions we have

$$\hat{X}_{\bar{\alpha}\bar{\beta}} = X_{\bar{\alpha}\bar{\beta}} + \frac{1}{2}\bar{\nabla}_{\bar{\alpha}}\bar{\Upsilon}_{\bar{\beta}} - \frac{1}{2}\bar{\Upsilon}_{\bar{\alpha}}\bar{\Upsilon}_{\bar{\beta}} - \Pi_{\bar{\alpha}\bar{\beta}}{}^{\gamma}\Upsilon_{\gamma} - \Pi_{\bar{\alpha}\bar{\gamma}\bar{\beta}}\Upsilon^{\bar{\gamma}} \tag{7.2.33}$$

$$\hat{Y}_{\bar{\alpha}\beta} = Y_{\bar{\alpha}\beta} - 2(\bar{\nabla}_{\bar{\alpha}}\Upsilon_{\beta}) + J_{\bar{\alpha}\beta}(\bar{\nabla}_{\bar{\gamma}}\Upsilon^{\bar{\gamma}}) \tag{7.2.34}$$

$$\hat{Y}_{\bar{\alpha}\gamma\beta}{}^{\gamma} = Y_{\bar{\alpha}\gamma\beta}{}^{\gamma} - \bar{\nabla}_{\bar{\alpha}}\Upsilon_{\beta} - \frac{3}{4}J_{\bar{\alpha}\beta}\bar{\Upsilon}_{\bar{\gamma}}\Upsilon^{\bar{\gamma}} - \frac{3}{8}J_{\bar{\alpha}\beta}\nabla_{\gamma}\Upsilon^{\gamma} + \frac{1}{8}J_{\bar{\alpha}\beta}\bar{\nabla}_{\bar{\gamma}}\bar{\Upsilon}^{\bar{\gamma}} \tag{7.2.35}$$

$$\hat{Z}_{\alpha\beta} = Z_{\alpha\beta} + \nabla_{[\alpha}\Upsilon_{\beta]} - \frac{1}{2}\nabla_{\alpha}\Upsilon_{\beta} - \frac{1}{2}\Upsilon_{\alpha}\Upsilon_{\beta} + \Sigma_{\gamma\alpha\beta}\bar{\Upsilon}^{\gamma}. \tag{7.2.36}$$

Proof. The first three can be calculated directly from 5.3.17 and 5.3.18. The next follow by contracting the first three expressions. The only trick is in the calculation of 7.2.34 where we need to use that $\nabla_\beta \bar{\Upsilon}_{\bar{\alpha}} = \bar{\nabla}_{\bar{\alpha}} \Upsilon_\beta - \frac{1}{2} J_{\bar{\alpha}\beta}(\bar{\nabla}_{\bar{\gamma}} \Upsilon^\gamma + \nabla_{\bar{\gamma}} \bar{\Upsilon}^\gamma)$, which follows from 5.3.16. $\square$

We will also need some formulae relating to commuting derivatives on densities in the five dimensional integrable case. For a smooth function f the formula 5.3.16 and integrability give that

$$\bar{\nabla}_{[\bar{\alpha}} \bar{\nabla}_{\bar{\beta}]} f = 0 \tag{7.2.37}$$

$$(\bar{\nabla}_{\bar{\alpha}} \nabla_\beta f - \nabla_\beta \bar{\nabla}_{\bar{\alpha}} f)_\perp = 0 \tag{7.2.38}$$

$$\nabla_{[\alpha} \nabla_{\beta]} f = 0. \tag{7.2.39}$$

and we want similar expressions for f a section of $L^p \otimes (\Lambda^2 P^*)^q$, $p, q \in \mathbb{Q}$. Since the partial connection is flat on L (it annihilates a chosen contact form), the above equations still hold for f a section of L, and accordingly this is true for f a section of L^p.

The partial connection is not automatically flat on $\Lambda^2 P^*$. For a moment we will stop suppressing indices on sections of $\Lambda^2 P^*$. Pick some locally non-vanishing $e_{\bar{\alpha}\bar{\beta}} \in \Lambda^2 P^*$ in order that we can write a section as $f = \tilde{f} e_{\bar{\alpha}\bar{\beta}}$ locally, for a smooth function $\tilde{f}$. Then we calculate

$$\bar{\nabla}_{[\bar{\alpha}} \bar{\nabla}_{\bar{\beta}]} \tilde{f} e_{\bar{\gamma}\bar{\nu}} = -X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\rho}}{}_{\bar{\gamma}} \tilde{f} e_{\bar{\rho}\bar{\nu}} + X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\rho}}{}_{\bar{\nu}} \tilde{f} e_{\bar{\rho}\bar{\gamma}} \tag{7.2.40}$$

$$(\bar{\nabla}_{\bar{\alpha}} \nabla_\beta \tilde{f} e_{\bar{\gamma}\bar{\nu}} - \nabla_\beta \bar{\nabla}_{\bar{\alpha}} \tilde{f} e_{\bar{\gamma}\bar{\nu}})_\perp = -2Y_{\bar{\alpha}\beta}{}^{\bar{\rho}}{}_{\bar{\gamma}} \tilde{f} e_{\bar{\rho}\bar{\nu}} + 2Y_{\bar{\alpha}\beta}{}^{\bar{\rho}}{}_{\bar{\nu}} \tilde{f} e_{\bar{\rho}\bar{\gamma}} \tag{7.2.41}$$

$$\nabla_{[\alpha} \nabla_{\beta]} \tilde{f} e_{\bar{\gamma}\bar{\nu}} = -Z_{\alpha\beta}{}^{\bar{\rho}}{}_{\bar{\gamma}} \tilde{f} e_{\bar{\rho}\bar{\nu}} + Z_{\alpha\beta}{}^{\bar{\rho}}{}_{\bar{\nu}} \tilde{f} e_{\bar{\rho}\bar{\gamma}}. \tag{7.2.42}$$

Contracting the above with $e^{\bar{\gamma}\bar{\nu}}$ and using the Leibniz rule to calculate the induced partial connection on tensor powers gives:

Proposition 7.2.43. *Let $H = P \oplus V$ be an integrable Legendrean contact structure on a five dimensional manifold. Let f be a section of $L^p \otimes (\Lambda^2 P^*)^q$ then*

$$\bar{\nabla}_{[\bar{\alpha}} \bar{\nabla}_{\bar{\beta}]} f = 0 \tag{7.2.44}$$

$$(\bar{\nabla}_{\bar{\alpha}} \nabla_\beta f - \nabla_\beta \bar{\nabla}_{\bar{\alpha}} f)_\perp = -2q Y_{\bar{\alpha}\beta} f \tag{7.2.45}$$

$$\nabla_{[\alpha} \nabla_{\beta]} f = 0. \tag{7.2.46}$$

These are reminiscent of the formulae in [GG05] proposition 2.2 in the CR case.

7.3 Bianchi identities for Legendrean contact structures

Here we collect some (differential) Bianchi-like identities for later.

First we need:

Lemma 7.3.1. *Let $H = P \oplus V$ be a Legendrean contact structure. If V is integrable, given some $\phi_{\bar{\alpha}} \in P^*$ we can write, at any point p, $\phi_{\bar{\alpha}}|_p = \bar{\nabla}_{\bar{\alpha}} f|_p$ for some smooth function f with $\nabla_\alpha f = 0$ in a neighbourhood. Similarly, if P is an integrable subbundle, given some $\psi_\alpha \in V^*$ we can write, at any point p, $\psi_\alpha|_p = \nabla_\alpha g|_p$ for some smooth function g with $\bar{\nabla}_{\bar{\alpha}} g = 0$ in a neighbourhood.*

Proof. Integrability of V implies we can find coordinates (x, y, z, a, b) so that

$$V = \operatorname{span}\left\{\frac{\partial}{\partial a}, \frac{\partial}{\partial b}\right\}. \tag{7.3.2}$$

Note that $\operatorname{span}\{d_\perp x, d_\perp y, d_\perp z\} = P^* \hookrightarrow \Lambda^1_H$ since P^* is identified as the rank 2 subbundle of Λ^1_H which vanishes when restricted to V. For a smooth function f

$$d_\perp f = \frac{\partial f}{\partial x} d_\perp x + \frac{\partial f}{\partial y} d_\perp y + \frac{\partial f}{\partial z} d_\perp z + \frac{\partial f}{\partial a} d_\perp a + \frac{\partial f}{\partial b} d_\perp b \tag{7.3.3}$$

$$\implies \nabla_\alpha f = df|_V = \frac{\partial f}{\partial a} da|_V + \frac{\partial f}{\partial b} db|_V \tag{7.3.4}$$

$$\bar{\nabla}_{\bar{\alpha}} f = df|_P = \frac{\partial f}{\partial x} dx|_P + \frac{\partial f}{\partial y} dy|_P + \frac{\partial f}{\partial z} dz|_P + \frac{\partial f}{\partial a} da|_P + \frac{\partial f}{\partial b} db|_P. \tag{7.3.5}$$

The condition $\nabla_\alpha f = 0$ in a neighbourhood, in other words $\frac{\partial f}{\partial a} = 0$ and $\frac{\partial f}{\partial b} = 0$, is just to say f is a function of x, y, z, and given this constraint, at any point p, $\bar{\nabla}_{\bar{\alpha}} f|_p$ remains completely arbitrary. The same works, interchanging P and V. □

Proposition 7.3.6. *For a 5-dimensional Legendrean contact structure with P and V integrable*

$$\bar{\nabla}_{[\bar{\alpha}} X_{\bar{\beta}]\bar{\gamma}}{}^{\bar{\gamma}\nu} = 0 \quad \textit{and} \quad \bar{\nabla}_{\bar{\gamma}} X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\nu} = 0. \tag{7.3.7}$$

Similarly

$$\nabla_{[\alpha} Z_{\beta]\nu\gamma}{}^{\nu} = 0 \quad \textit{and} \quad \nabla_{[\alpha} Z_{\beta]\nu\gamma}{}^{\nu} = 0. \tag{7.3.8}$$

Proof. Due to the dimension we have the trivial index identity

$$\bar{\nabla}_{[\bar{\alpha}} X_{\bar{\beta}\bar{\gamma}]\gamma}{}^{\nu} = 0. \tag{7.3.9}$$

Contracting gives

$$\bar{\nabla}_{[\bar{\alpha}}X_{\bar{\beta}]\bar{\gamma}}{}^{\bar{\gamma}\nu} + \frac{1}{2}\bar{\nabla}_{\bar{\gamma}}X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\nu} = 0 \tag{7.3.10}$$

and so the two statements 7.3.7 are equivalent.

We will show

$$\bar{\nabla}_{\bar{\gamma}}X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\nu} = 0. \tag{7.3.11}$$

At a point p we can replace

$$\bar{\nabla}_{\bar{\gamma}}X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\nu}\phi_\nu = \bar{\nabla}_{\bar{\gamma}}X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\nu}\nabla_\nu f \tag{7.3.12}$$

for some smooth function f with $\bar{\nabla}_{\bar{\alpha}}f = 0$ on a neighbourhood.

Now use the Leibniz rule

$$\begin{aligned}\bar{\nabla}_{\bar{\gamma}}X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\nu}\nabla_\nu f &= \bar{\nabla}_{\bar{\gamma}}(X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\nu}\nabla_\nu f) - X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\nu}\bar{\nabla}_{\bar{\gamma}}\nabla_\nu f & (7.3.13)\\ &= (\bar{\nabla}_{\bar{\gamma}}\bar{\nabla}_{[\bar{\alpha}}\bar{\nabla}_{\bar{\beta}]}\nabla^{\bar{\gamma}}f) - X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\nu}\bar{\nabla}_{\bar{\gamma}}\nabla_\nu f. & (7.3.14)\end{aligned}$$

Using the fact our function is constant in the P directions combined with 5.3.16 gives

$$\bar{\nabla}_{\bar{\alpha}}\nabla_\beta f = \frac{1}{2}J_{\bar{\alpha}\beta}\bar{\nabla}_{\bar{\gamma}}\nabla^{\bar{\gamma}}f. \tag{7.3.15}$$

The key point is that differentiating f in the V direction and then differentiating in the P direction produces something that is totally trace. Returning to the expression

$$(\bar{\nabla}_{\bar{\gamma}}\bar{\nabla}_{[\bar{\alpha}}\bar{\nabla}_{\bar{\beta}]}\nabla^{\bar{\gamma}}f) - X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}\nu}\bar{\nabla}_{\bar{\gamma}}\nabla_\nu f, \tag{7.3.16}$$

we showed previously that $X_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}}{}_{\bar{\gamma}} = 0$ and so the last terms vanishes. Using 7.3.15 we can rewrite the remaining term as

$$\frac{1}{2}(\bar{\nabla}^{\gamma}\bar{\nabla}_{[\bar{\alpha}}J_{\bar{\beta}]\gamma}\bar{\nabla}_{\bar{\nu}}\nabla^{\bar{\nu}}f) = \frac{1}{2}(\bar{\nabla}_{[\bar{\beta}}\bar{\nabla}_{\bar{\alpha}]}\bar{\nabla}_{\bar{\nu}}\nabla^{\bar{\nu}}f) = 0 \tag{7.3.17}$$

at the arbitrary point p. This completes the proof of 7.3.7. The proof of 7.3.8 is exactly analogous. □

Remark 7.3.18. *It is interesting to note that the respective calculi with the partial connections $\bar{\nabla}_{\bar{\alpha}}$ and ∇_α on the integrable subbundles P and V resemble the calculus on flat 2-dimensional projective structures. See [Eas17] for 2-dimensional projective structures. In particular the tensors $\bar{\nabla}_{[\bar{\alpha}}X_{\bar{\beta}]\bar{\gamma}}{}^{\bar{\gamma}}{}_{\bar{\nu}}$ and $\nabla_{[\alpha}Z_{\beta]\nu\gamma}{}^{\nu}$ are calculated analogously to the Cotton tensor, which is the obstruction to projective flatness. Furthermore, observing 5.3.17, one obtains a distinguished projective class of connections by pulling the distinguished partial connections $V^* \to V^* \otimes V^*$ back to proper connections on an integral submanifold of V. The same argument works for an integral submanifold of P.*

It will turn out later that it is a good idea to write down some of the consequences of the second order identity 2.5.30. Recall that for a contact, partial torsion free partial connection $\Lambda^1_H \to \Lambda^1_H \otimes \Lambda^1_H$ we have

$$\nabla_{[a}\nabla^e R_{b]ecd} - \frac{1}{4}\nabla_e\nabla^e R_{abcd} + 2R_{e[a|cd}R_{|b]d}{}^{ed} = 0. \tag{7.3.19}$$

Applying barred projectors to the first term yields

$$\begin{aligned} &2\nabla_{[a|}\nabla_f R_{|b]ecd}\bar{\Pi}^{[e}_{\bar{\alpha}}\Pi^{f]}_{\beta}J^{\bar{\alpha}\beta}\bar{\Pi}^a_{\bar{\beta}}\bar{\Pi}^b_{\bar{\gamma}}\Pi^c_{\gamma}\bar{\Pi}^d_{\bar{\nu}} \\ =&\bar{\nabla}_{[\bar{\beta}|}\bar{\nabla}_{\bar{\alpha}}Y_{|\bar{\gamma}]}{}^{\bar{\alpha}}{}_{\gamma\bar{\nu}} - \bar{\nabla}_{[\bar{\beta}|}\bar{\nabla}^{\bar{\alpha}}X_{|\bar{\gamma}]\bar{\alpha}\gamma\bar{\nu}}. \end{aligned} \tag{7.3.20}$$

The second term gives

$$\begin{aligned} &-\frac{1}{2}\nabla_e\nabla_f R_{abcd}\bar{\Pi}^{[e}_{\bar{\alpha}}\Pi^{f]}_{\beta}J^{\bar{\alpha}\beta}\bar{\Pi}^a_{\bar{\beta}}\bar{\Pi}^b_{\bar{\gamma}}\Pi^c_{\gamma}\bar{\Pi}^d_{\bar{\nu}} \\ =&\frac{1}{4}(\bar{\nabla}_{\bar{\alpha}}\nabla^{\bar{\alpha}} - \nabla^{\bar{\alpha}}\bar{\nabla}_{\bar{\alpha}})X_{\bar{\beta}\bar{\gamma}\gamma\bar{\nu}}. \end{aligned} \tag{7.3.21}$$

Finally the third term is

$$\begin{aligned} &8R_{e[a|cd}R_{|b]hfg}\bar{\Pi}^{[e}_{\bar{\alpha}}\Pi^{f]}_{\beta}J^{\bar{\alpha}\beta}\bar{\Pi}^{[h}_{\bar{\rho}}\Pi^{g]}_{\rho}J^{\bar{\rho}\rho}\bar{\Pi}^a_{\bar{\beta}}\bar{\Pi}^b_{\bar{\gamma}}\Pi^c_{\gamma}\bar{\Pi}^d_{\bar{\nu}} \\ =&(4R_{eacd}R_{bhfg} - 4R_{ebcd}R_{ahfg})\bar{\Pi}^{[e}_{\bar{\alpha}}\Pi^{f]}_{\beta}J^{\bar{\alpha}\beta}\bar{\Pi}^{[h}_{\bar{\rho}}\Pi^{g]}_{\rho}J^{\bar{\rho}\rho}\bar{\Pi}^a_{\bar{\beta}}\bar{\Pi}^b_{\bar{\gamma}}\Pi^c_{\gamma}\bar{\Pi}^d_{\bar{\nu}} \\ =&X_{\bar{\alpha}\bar{\beta}\gamma\bar{\nu}}Y_{\bar{\gamma}\rho}{}^{\bar{\alpha}\rho} - Y_{\bar{\beta}}{}^{\bar{\alpha}}{}_{\gamma\nu}X_{\bar{\gamma}\bar{\rho}}{}^{\bar{\rho}}{}_{\bar{\alpha}} - X_{\bar{\alpha}\bar{\gamma}\gamma\bar{\nu}}Y_{\bar{\beta}\rho}{}^{\bar{\alpha}\rho} + Y_{\bar{\gamma}}{}^{\bar{\alpha}}{}_{\gamma\nu}X_{\bar{\beta}\bar{\rho}}{}^{\bar{\rho}}{}_{\bar{\alpha}}. \end{aligned} \tag{7.3.22}$$

Contracting the three terms with $J^{\bar{\nu}\gamma}$ gives

$$\bar{\nabla}_{[\bar{\beta}|}\bar{\nabla}_{\bar{\alpha}}Y_{|\bar{\gamma}]}{}^{\bar{\alpha}} - 2Y_{[\bar{\beta}}{}^{\bar{\alpha}}X_{\bar{\gamma}]\bar{\alpha}} = 0. \tag{7.3.23}$$

Similarly we apply unbarred projectors to get

$$\nabla_{[\lambda|}\nabla^{\bar{\alpha}}Y_{\bar{\alpha}|\mu]} + 2Y_{\bar{\alpha}[\lambda}Z_{\mu]}{}^{\bar{\alpha}} = 0. \tag{7.3.24}$$

7.4 Legendrean contact standard tractors via invariant prolongation

Without going into detail, the general theory provides some clues as to what partial differential equation we should prolong to construct the standard tractor bundle for Legendrean contact geometry in 5-dimensions. In [ČS09] tractor bundles on parabolic geometries $\mathcal{P} \to M$ of type (G, P) are defined to be the associated vector bundles $\mathcal{P} \times_\alpha V$ where α is a representation of P on V corresponding to a restriction of a representation of G on V. The standard tractor bundle is simply this construction where we take α to be the restricted standard representation of G acting as a matrix group. Recalling that the flat model is

$$G \Bigg/ \left\{ \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \right\} \tag{7.4.1}$$

with G as in 5.2.15, we see that fibres of the standard tractor bundle, which are vector spaces with a representation of P, should be canonically filtered by a rank-1 subbundle and then a corank-1 subbundle. Thinking about the conformal-to-Einstein prolongation, and how starting with an operator on a density bundle gave rise to a canonical line subbundle, this suggests the partial differential equation to prolong should act on sections of a line bundle $L^p \otimes (\Lambda^2 P^*)^q$. If we want

$$\bar{\nabla}_{\bar{\alpha}} f = 0, \;\; f \in L^p \otimes (\Lambda^2 P^*)^q \tag{7.4.2}$$

to be invariant, 5.3.20 and 5.3.19 show that we need to take $p = -3q$. Unfortunately $\nabla_\alpha f$ is then non-invariant and there is seemingly no way to correct this. So we instead consider $\nabla_\alpha \nabla_\beta f$. This is again non-invariant, but there is hope we can find an appropriate curvature correction.

In particular

Proposition 7.4.3. *Let $H = P \oplus V$ be a Legendrean contact structure with $n = 2$ and P and V integrable. There is an invariant differential operator*

$$(\Lambda^2 P^*)^{1/4} \otimes L^{-3/4} \to (P^* \otimes (\Lambda^2 P^*)^{1/4} \otimes L^{-3/4}) \oplus (V^* \otimes V^* \otimes (\Lambda^2 P^*)^{1/4} \otimes L^{-3/4})$$

given by

$$f \mapsto \left(\bar{\nabla}_{\bar{\alpha}} f, \nabla_\alpha \nabla_\beta f - 2 Z_{\alpha\beta} f \right). \tag{7.4.4}$$

Proof. From 5.3.17 through 5.3.19

$$\begin{aligned} \hat{\nabla}_\alpha \hat{\nabla}_\beta f &= \hat{\nabla}_\alpha (\nabla_\beta f + \Upsilon_\beta f) \\ &= \nabla_\alpha \nabla_\beta f - 2\Upsilon_{(\alpha} \nabla_{\beta)} f + \Upsilon_\alpha \nabla_\beta f + \nabla_\alpha (\Upsilon_\beta f) + \Upsilon_{(\alpha} \Upsilon_{\beta)} f + \Upsilon_\alpha \Upsilon_\beta \\ &= \nabla_\alpha \nabla_\beta f + \nabla_\alpha \Upsilon_\beta f + 2 \Upsilon_\alpha \Upsilon_\beta, \end{aligned} \tag{7.4.5}$$

while 7.2.36 in the partial torsion free (equivalently, integrable) case gives

$$\hat{Z}_{\alpha\beta} = Z_{\alpha\beta} - \frac{1}{2} \nabla_\alpha \Upsilon_\beta - \frac{1}{2} \Upsilon_\alpha \Upsilon_\beta \tag{7.4.6}$$

since $\nabla_{[\alpha} \Upsilon_{\beta]} = 0$ in the partial torsion free case. □

Remark 7.4.7. *We prolong in the integrable case in light of the simplifications we obtain in section 7.2. We expect a similar procedure can be carried out without this assumption, but one has to carry the two components of torsion and their derivatives throughout the calculation. The operator to prolong also needs some adjustment. In the non-integrable case*

$$\left(\bar{\nabla}_{\bar{\alpha}} f, \nabla_\alpha \nabla_\beta f - 2Z_{\alpha\beta} f + \frac{2}{3} Z_{\alpha\beta\gamma}{}^{\gamma} f + 2\bar{\nabla}_{\bar{\gamma}} \Sigma^{\bar{\gamma}}{}_{\alpha\beta} - \frac{4}{9} \bar{\nabla}_{\bar{\gamma}} \Sigma_{\alpha\beta}{}^{\bar{\gamma}}\right) \tag{7.4.8}$$

is invariant.

f being in the kernel of the 7.4.4 is equivalent to

$$\bar{\nabla}_{\bar{\alpha}} f = 0 \tag{7.4.9}$$
$$\nabla_\alpha f = \phi_\alpha \tag{7.4.10}$$

for some $\phi_\beta \in V^* \otimes (\Lambda^2 P^*)^{1/4} \otimes L^{-3/4}$ such that

$$\bar{\nabla}_{\bar{\alpha}} \phi_\beta = J_{\bar{\alpha}\beta}\, g - \frac{1}{2} Y_{\bar{\alpha}\bar{\beta}\gamma}{}^{\gamma} f \tag{7.4.11}$$
$$\nabla_\alpha \phi_\beta = 2Z_{\alpha\beta} f \tag{7.4.12}$$

for some $g \in (\Lambda^2 P^*)^{1/4} \otimes L^{1/4}$ with

$$\bar{\nabla}_{\bar{\alpha}} g = 2X_{\bar{\alpha}}{}^{\beta} \phi_\beta + \frac{1}{2} \bar{\nabla}_{\bar{\gamma}} Y_{\bar{\alpha}}{}^{\bar{\gamma}} f \tag{7.4.13}$$
$$\nabla_\alpha g = \frac{2}{3} \nabla_{\bar{\beta}} Z_\alpha{}^{\bar{\beta}} f - \frac{4}{3} Y_{\bar{\beta}\alpha}{}^{\bar{\beta}\gamma} \phi_\gamma + \frac{1}{6} Y_{\bar{\beta}\alpha}\, \phi^{\bar{\beta}} - \frac{1}{6} \nabla^{\bar{\beta}} Y_{\bar{\beta}\alpha}\,. \tag{7.4.14}$$

Now (f, ϕ_α, g) are our prolongation variables and (f, ϕ_α, g) satisfying 7.4.9 through 7.4.14 are in bijective correspondence with solutions of 7.4.4.

We define $\mathbb{T} \to M$ as the vector bundle which given a choice of contact form splits as $\mathbb{T} \cong ((\Lambda^2 P^*)^{1/4} \otimes L^{-3/4}) \oplus (V^* \otimes (\Lambda^2 P^*)^{1/4} \otimes L^{-3/4}) \oplus ((\Lambda^2 P^*)^{1/4} \otimes L^{1/4})$ such that the splitting changes by

$$\widehat{\begin{bmatrix} f \\ \phi_\alpha \\ g \end{bmatrix}} = \begin{bmatrix} f \\ \phi_\alpha + \Upsilon_\alpha f \\ g + \bar{\Upsilon}_{\bar{\gamma}} \phi^{\bar{\gamma}} + \frac{1}{2}(\bar{\nabla}_{\bar{\gamma}} \Upsilon^{\bar{\gamma}}) f + \bar{\Upsilon}_{\bar{\gamma}} \Upsilon^{\bar{\gamma}} f \end{bmatrix} \tag{7.4.15}$$

exactly as the definitions for f, ϕ_α, g prescribe. We can check this is well defined vector bundle by checking that the matrix associated with the change of contact form $\hat{\hat{\alpha}} = \Omega_1 \Omega_2 \alpha$ is given by composing the matrix for the transformation corresponding with $\hat{\alpha} = \Omega_1 \alpha$ with the matrix corresponding to the transformation $\hat{\hat{\alpha}} = \Omega_2 \hat{\alpha}$.

The change of splitting above also induces a change of the splitting of the dual $\mathbb{T}^* \cong ((\Lambda^2 P^*)^{-1/4} \otimes L^{3/4}) \oplus (V \otimes (\Lambda^2 P^*)^{-1/4} \otimes L^{3/4}) \oplus ((\Lambda^2 P^*)^{-1/4} \otimes L^{-1/4})$ if we enforce that the natural pairing $\mathbb{T} \otimes \mathbb{T}^* \to \mathbb{R}$, which is given by pairing each summand with its dual and summing, is invariant. In practice this means we take the transpose of the inverse of the transformation 7.4.15 and the splitting of $\mathbb{T}^*$ changes like

$$\widehat{[\tilde{f} \quad \tilde{\phi}^\alpha \quad \tilde{g}]} = [\tilde{f} - \Upsilon_\gamma \tilde{\phi}^\gamma - \tilde{g}\tfrac{1}{2}(\bar{\nabla}_{\bar{\gamma}} \Upsilon^{\bar{\gamma}}) \quad \tilde{\phi}^\alpha - \tilde{g}\bar{\Upsilon}^\alpha \quad \tilde{g}] \tag{7.4.16}$$

where we will write elements of the bundle $\mathbb{T}^*$ as horizontal block matrices to emphasise the pairing.

By construction the differential operator

$$\bar{\nabla}_{\bar{\alpha}} \begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} = \begin{bmatrix} \bar{\nabla}_{\bar{\alpha}} f \\ \bar{\nabla}_{\bar{\alpha}} \phi_\beta - J_{\bar{\alpha}\beta} g + \frac{1}{2} Y_{\bar{\alpha}\beta} f \\ \bar{\nabla}_{\bar{\alpha}} g - 2X_{\bar{\alpha}}{}^\beta \phi_\beta + \frac{1}{2} \bar{\nabla}_{\bar{\gamma}} Y_{\bar{\alpha}}{}^{\bar{\gamma}} f \end{bmatrix} \tag{7.4.17}$$

is invariant and this can be checked explicitly by checking that

$$\begin{bmatrix} \hat{\bar{\nabla}}_{\bar{\alpha}} \hat{f} \\ \hat{\bar{\nabla}}_{\bar{\alpha}} \hat{\phi}_\beta - J_{\bar{\alpha}\beta} \hat{g} + \frac{1}{2} \hat{Y}_{\bar{\alpha}\beta} \hat{f} \\ \hat{\bar{\nabla}}_{\bar{\alpha}} \hat{g} - 2\hat{X}_{\bar{\alpha}}{}^\beta \hat{\phi}_\beta + \frac{1}{2} \hat{\bar{\nabla}}_{\bar{\gamma}} \hat{Y}_{\bar{\alpha}}{}^{\bar{\gamma}} \hat{f} \end{bmatrix} - \begin{bmatrix} \bar{\nabla}_{\bar{\alpha}} f \\ \bar{\nabla}_{\bar{\alpha}} \phi_\beta - J_{\bar{\alpha}\beta} g + \frac{1}{2} Y_{\bar{\alpha}\beta} f \\ \bar{\nabla}_{\bar{\alpha}} g - 2X_{\bar{\alpha}}{}^\beta \phi_\beta + \frac{1}{2} \bar{\nabla}_{\bar{\gamma}} Y_{\bar{\alpha}}{}^{\bar{\gamma}} f \end{bmatrix} \tag{7.4.18}$$
$$= \begin{bmatrix} 0 \\ \Upsilon_\beta \bar{\nabla}_{\bar{\alpha}} f \\ \bar{\Upsilon}_{\bar{\gamma}}(\bar{\nabla}_{\bar{\alpha}} \phi^{\bar{\gamma}} - \delta_{\bar{\alpha}}{}^{\bar{\gamma}} g + \frac{1}{2} Y_{\bar{\alpha}}{}^{\bar{\gamma}} f) + \frac{1}{2}(\bar{\nabla}_{\bar{\gamma}} \Upsilon^{\bar{\gamma}}) \bar{\nabla}_{\bar{\alpha}} f + \bar{\Upsilon}_{\bar{\gamma}} \Upsilon^{\bar{\gamma}} \bar{\nabla}_{\bar{\alpha}} f \end{bmatrix}.$$

For the derivative in the V direction the naive prescription is

$$\nabla_\alpha \begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} = \begin{bmatrix} \nabla_\alpha f - \phi_\alpha \\ \nabla_\alpha \phi_\beta - 2Z_{\alpha\beta} f \\ \nabla_\alpha g - \frac{2}{3} \bar{\nabla}_{\bar{\beta}} Z_\alpha{}^{\bar{\beta}} f + \frac{4}{3} Y_{\bar{\beta}\alpha}{}^{\bar{\beta}\gamma} \phi_\gamma + \frac{1}{6}(\nabla^{\bar{\gamma}} Y_{\bar{\gamma}\alpha}) f - \frac{1}{6} Y_{\bar{\gamma}\alpha} \phi^{\bar{\gamma}} \end{bmatrix} \tag{7.4.19}$$

and similarly to above it can be verified that this is an invariant differential operator $\mathbb{T} \to V^* \otimes \mathbb{T}$. Together, the two invariant differential operators form an invariant partial connection $\mathbb{T} \to (P^* \oplus V^*) \otimes \mathbb{T}$. Furthermore, parallel sections of this partial connection are in bijective correspondence with 7.4.4.

We also will compute the connection on the dual bundle. Enforcing the Leibniz rule so the partial connection preserves the natural pairing gives

$$\bar{\nabla}_{\bar{\alpha}} \begin{bmatrix} \tilde{f} \\ \tilde{\phi}^\beta \\ \tilde{g} \end{bmatrix}^T = \begin{bmatrix} \bar{\nabla}_{\bar{\alpha}} \tilde{f} - \frac{1}{2} Y_{\bar{\alpha}\beta} \tilde{\phi}^\beta - \frac{1}{2} \bar{\nabla}_{\bar{\gamma}} Y_{\bar{\alpha}}{}^{\bar{\gamma}} \tilde{g} \\ \bar{\nabla}_{\bar{\alpha}} \tilde{\phi}^\beta + 2X_{\bar{\alpha}}{}^\beta \tilde{g} \\ \bar{\nabla}_{\bar{\alpha}} \tilde{g} + \tilde{\phi}_{\bar{\alpha}} \end{bmatrix}^T \tag{7.4.20}$$

and

$$\nabla_\alpha \begin{bmatrix} \tilde{f} \\ \tilde{\phi}^\beta \\ \tilde{g} \end{bmatrix}^T = \begin{bmatrix} \nabla_\alpha \tilde{f} + \frac{2}{3}\nabla_{\bar\beta} Z_\alpha{}^{\bar\beta}\tilde{g} + \frac{1}{6}(\nabla^{\bar\gamma} Y_{\bar\gamma\alpha}\,)\tilde{g} + 2Z_{\alpha\beta}\tilde{\phi}^\beta \\ \nabla_\alpha \tilde{\phi}^\beta + \delta_\alpha{}^\beta \tilde{f} - \frac{4}{3}Y_{\bar\beta\alpha}{}^{\bar\beta\beta}\tilde{g} - \frac{1}{6}Y^{\bar\beta}{}_\alpha\,\tilde{g} \\ \nabla_\alpha \tilde{g} \end{bmatrix}^T. \tag{7.4.21}$$

We should compare the construction of the bundle here with constructions of the standard tractor bundle which appear in [ČS09] and also in [EZ20]. In the presence of a contact form, the bundle splits as the appropriate direct sum. However, at this point there appears to be a major discrepancy with [ČS09] proposition 5.2.15. In the formula presented there (having converted notation appropriately) the change of splitting of the standard co-tractor bundle given a change of contact form is

$$\widehat{\begin{bmatrix} f \\ \phi_\alpha \\ g \end{bmatrix}} = \begin{bmatrix} f \\ \phi_\alpha + \Upsilon_\alpha f \\ g + \bar\Upsilon_{\bar\gamma}\phi^{\bar\gamma} + \frac{1}{4}(\bar\nabla_{\bar\gamma}\Upsilon^{\bar\gamma} - \nabla_\gamma \bar\Upsilon^\gamma)f + \frac{1}{2}\bar\Upsilon_{\bar\gamma}\Upsilon^{\bar\gamma} f \end{bmatrix}. \tag{7.4.22}$$

Accordingly, the formula for the splitting of the dual bundle is also at odds with the splitting here.

This problem has a nice resolution. It turns out that the bundle $\mathbb{T}$ with change of splitting 7.4.15, prescribed by the choice of prolongation variables, can be identified with the standard tractor bundle with the splitting that appears in the literature. The relation between the splitting presented here and the usual splitting depends on a choice of scale. In particular we can apply the following gauge transformation

$$\begin{bmatrix} f \\ \phi_\alpha \\ h \end{bmatrix} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{3}Y_{\bar\gamma\gamma}{}^{\bar\gamma\gamma} & 0 & 1 \end{bmatrix} \begin{bmatrix} f \\ \phi_\alpha \\ g \end{bmatrix} \tag{7.4.23}$$

and from 7.2.35 we have $\frac{1}{3}\hat{Y}_{\bar\gamma\gamma}{}^{\bar\gamma\gamma} = \frac{1}{3}Y_{\bar\gamma\gamma}{}^{\bar\gamma} - \frac{1}{4}\bar\nabla_{\bar\gamma}\Upsilon^{\bar\gamma} - \frac{1}{4}\nabla_\gamma\bar\Upsilon^\gamma - \frac{1}{2}\bar\Upsilon_{\bar\gamma}\Upsilon^{\bar\gamma}$. So that $\hat{h} = h + \bar\Upsilon_{\bar\gamma}\phi^{\bar\gamma} + \frac{1}{4}(\bar\nabla_{\bar\gamma}\Upsilon^{\bar\gamma} - \nabla_\gamma\bar\Upsilon^\gamma)f + \frac{1}{2}\bar\Upsilon_{\bar\gamma}\Upsilon^{\bar\gamma} f$ as desired. So we can identify $\mathbb{T}$ as the standard co-tractor bundle in [ČS09].

With the benefit of hindsight we could have defined h earlier, during the prolongation procedure, by rewriting 7.4.11 as

$$\bar\nabla_{\bar\alpha}\phi_\beta = J_{\bar\alpha\beta}h - \frac{1}{2}Y_{\bar\alpha\beta\gamma}{}^\gamma f - \frac{1}{3}J_{\bar\alpha\beta}Y_{\bar\gamma\gamma}{}^{\bar\gamma\gamma}f, \tag{7.4.24}$$

and letting this define h.

7.5 Tractor partial curvature

In this section we compute the partial curvature of the invariant partial connection $\mathbb{T} \to P^* \oplus V^*$. While the computations are involved, the result is a simple formula for a second order differential invariant of the Legendrean contact structure.

As with the partial connection on the bundle V^* we can decompose it like:

$$\begin{tikzcd} & \Lambda^2 P^* \otimes \mathbb{T} \\ \mathbb{T} \arrow[ur] \arrow[r] \arrow[dr] & (P^* \otimes V^*)_\perp \otimes \mathbb{T} \\ & \Lambda^2 V^* \otimes \mathbb{T} \end{tikzcd}$$

Proposition 7.5.1. *The $\Lambda^2 P^* \otimes \mathrm{End}(\mathbb{T})$ and $\Lambda^2 V^* \otimes \mathrm{End}(\mathbb{T})$ components of the partial curvature of the invariant connection $\mathbb{T} \to P^* \oplus V^*$ given by 7.4.17 and 7.4.19 vanish.*

Proof. Applying $\bar\nabla_{\bar\alpha}$ twice then skew-symmetrising in the P^* indices, yields that the curvature $\mathbb{T} \to \Lambda^2 P^* \otimes \mathbb{T}$ is

$$\begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{2}\bar\nabla_{[\bar\alpha}Y_{\bar\beta]\gamma} - \frac{1}{2}J_{[\bar\alpha|\gamma}\bar\nabla_{\bar\gamma}Y_{|\bar\beta]}{}^{\bar\gamma} & X_{\bar\alpha\bar\beta\gamma}{}^{\nu} + 2J_{[\bar\alpha|\gamma}X_{|\bar\beta]}{}^{\nu} & 0 \\ \frac{1}{2}\bar\nabla_{[\bar\alpha|}\bar\nabla_{\bar\gamma}Y_{|\bar\beta]}{}^{\bar\gamma} - X_{[\bar\alpha}{}^{\gamma}Y_{\bar\beta]\gamma} & -2\bar\nabla_{[\bar\alpha}X_{\bar\beta]}{}^{\nu} & 0 \end{bmatrix} \begin{bmatrix} f \\ \phi_\nu \\ g \end{bmatrix} \qquad (7.5.2)$$

Using the fact that skew-symmetrising over three indices is the zero map we see that $\frac{1}{2}\bar\nabla_{[\bar\alpha}Y_{\bar\beta]\gamma} - \frac{1}{2}J_{[\bar\alpha|\gamma}\bar\nabla_{\bar\gamma}Y_{|\bar\beta]}{}^{\bar\gamma} = 0$ and $X_{\bar\alpha\bar\beta\gamma}{}^{\nu} + 2J_{[\bar\alpha|\gamma}X_{|\bar\beta]}{}^{\nu} = 0$ are trivial index identities. Then the other two components vanish by 7.3.7 and 7.3.23 respectively.

One can use the same strategies to show the component of partial curvature $\mathbb{T} \to \Lambda^2 V^* \otimes \mathbb{T}$ vanishes, but the calculations are much longer due to the more complicated derivative in the V directions and so we leave them to appendix D. $\square$

Remark 7.5.3. *In the non-integrable case, we expect the $\Lambda^2 P^* \otimes \mathrm{End}(\mathbb{T})$ and $\Lambda^2 V^* \otimes \mathrm{End}(\mathbb{T})$ components to be expressions in $\Pi_{\bar\alpha\bar\beta}{}^{\gamma}$ and $\Sigma_{\bar\alpha\bar\beta}{}^{\gamma}$, respectively.*

We now consider the component in the bundle $(P^* \otimes V^*)_\perp \otimes \mathrm{End}(\mathbb{T})$. We anticommute the tractor connection in the P and V directions and then take the trace-free part to get

$$\begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 0 \\ A_{\bar\alpha\alpha\beta} & W_{\bar\alpha\alpha\beta}{}^{\gamma} & 0 \\ B_{\bar\alpha\alpha} & C_{\bar\alpha\alpha}{}^{\gamma} & D_{\bar\alpha\alpha} \end{bmatrix} \begin{bmatrix} f \\ \phi_\nu \\ g \end{bmatrix}, \qquad (7.5.4)$$

where

$$\begin{aligned}
A_{\bar\alpha\alpha\beta} &= -\bar\nabla_{\bar\alpha} Z_{\alpha\beta} + \frac{1}{3} J_{\bar\alpha(\alpha|}\bar\nabla_{\bar\beta} Z_{|\beta)}{}^{\bar\beta} - \frac{1}{4}\nabla_\alpha Y_{\bar\alpha\beta}\, f - \frac{1}{6} J_{\bar\alpha\alpha}\nabla^{\bar\gamma} Y_{\bar\gamma\beta} \\
&\quad - \frac{1}{12} J_{\bar\alpha\beta}\nabla^{\bar\gamma} Y_{\bar\gamma\alpha}\, f && (7.5.5)\\
B_{\bar\alpha\alpha} &= -\bar\nabla_{\bar\alpha}\bar\nabla_{\bar\beta} Z_\alpha{}^{\bar\beta} + \frac{1}{12}\bar\nabla_{\bar\alpha}\nabla^{\bar\gamma} Y_{\bar\gamma\alpha} - 2X_{\bar\alpha}{}^{\beta} Z_{\alpha\beta} - \frac{1}{4}\nabla_\alpha\bar\nabla_{\bar\gamma} Y_{\bar\alpha}{}^{\bar\gamma} - \frac{1}{3} Y_{\bar\beta\alpha}{}^{\bar\beta\gamma} Y_{\bar\alpha\gamma} \\
&\quad + \frac{1}{24} Y^\gamma{}_\alpha Y_{\bar\alpha\gamma} + \frac{1}{6} J_{\bar\alpha\alpha}\bar\nabla_{\bar\gamma}\bar\nabla_{\bar\beta} Z^{\bar\gamma\bar\beta} - \frac{1}{48} J_{\bar\alpha\alpha} Y^\gamma{}_\nu Y^\nu{}_\gamma + \frac{1}{6} J_{\bar\alpha\alpha} Y_{\bar\beta\nu}{}^{\bar\beta\gamma} Y^\nu{}_\gamma \\
&\quad - \frac{1}{6} J_{\bar\alpha\alpha}\bar\nabla_{\bar\nu}\nabla^{\bar\gamma} Y_{\bar\gamma}{}^{\bar\nu} + J_{\bar\alpha\alpha} X^{\gamma\beta} Z_{\gamma\beta} + \frac{1}{8} J_{\bar\alpha\alpha}\nabla_\nu\bar\nabla_{\bar\gamma} Y^{\nu\bar\gamma} && (7.5.6)\\
C_{\bar\alpha\alpha}{}^\gamma &= \frac{2}{3}\bar\nabla_{\bar\alpha} Y_{\bar\beta\alpha}{}^{\bar\beta\gamma} - \frac{1}{12}\bar\nabla_{\bar\alpha} Y^\gamma{}_\alpha - \frac{1}{2}\bar\nabla_{\bar\gamma} Z_{\bar\alpha}{}^{\bar\gamma}\delta_\alpha{}^\gamma + \nabla_\alpha X_{\bar\alpha}{}^\gamma \\
&\quad - \frac{1}{3} J_{\bar\alpha\alpha}\bar\nabla_{\bar\gamma} Y_{\bar\beta}{}^{\bar\gamma\bar\beta\gamma} + \frac{1}{24} J_{\bar\alpha\alpha}\bar\nabla_{\bar\beta} Y^{\gamma\bar\beta} + \frac{1}{4} J_{\bar\alpha\alpha}\bar\nabla_{\bar\gamma} Z^{\gamma\bar\gamma} - \frac{1}{2} J_{\bar\alpha\alpha}\nabla_\beta X^{\beta\gamma} && (7.5.7)\\
D_{\bar\alpha\alpha} &= -\frac{1}{3} Y_{\bar\alpha\alpha} + \frac{2}{3} Y_{\bar\beta\alpha}{}^{\bar\beta}{}_{\bar\alpha} - \frac{1}{3} J_{\bar\alpha\alpha} Y_{\bar\beta}{}^{\bar\gamma\bar\beta}{}_{\bar\gamma} && (7.5.8)
\end{aligned}$$

and

$$\begin{aligned}
W_{\bar\alpha\alpha\beta}{}^\gamma &= Y_{\bar\alpha\alpha\beta}{}^\gamma - \frac{2}{3} J_{\bar\alpha\beta} Y_{\bar\beta\alpha}{}^{\bar\beta\gamma} + \frac{1}{3} J_{\bar\alpha\alpha} Y_{\bar\beta\beta}{}^{\bar\beta\gamma} + \frac{1}{12} J_{\bar\alpha\beta} Y^\gamma{}_\alpha + \frac{1}{12} J_{\bar\alpha\alpha} Y^\gamma{}_\beta \\
&\quad - \frac{1}{4} Y_{\bar\alpha\beta}\,\delta_\alpha{}^\gamma - \frac{1}{4} Y_{\bar\alpha\alpha}\,\delta_\beta{}^\gamma. && (7.5.9)
\end{aligned}$$

There are some simplifications we can immediately make. Using the fact that skewing over three indices gives the zero map we show that $D_{\bar\alpha\alpha} = 0$, and $W_{\bar\alpha\alpha\beta}{}^\gamma = W_{\bar\alpha(\alpha\beta)}{}^\gamma$. In particular

$$W_{\bar\alpha\alpha\beta}{}^\gamma = Y_{\bar\alpha(\alpha\beta)}{}^\gamma - \frac{1}{3} J_{\bar\alpha(\alpha} Y_{|\bar\beta|\beta)}{}^{\bar\beta\gamma} + \frac{1}{6} J_{\bar\alpha(\alpha} Y^\gamma{}_{\beta)\nu}{}^\nu - \frac{1}{2} Y_{\bar\alpha(\alpha|\nu}{}^\nu\delta_{\beta)}{}^\gamma \qquad (7.5.10)$$

is the totally trace-free part of $Y_{\bar\alpha\alpha\beta}{}^\gamma$.

Proposition 7.5.11. *$W_{\bar\alpha\alpha\beta}{}^\gamma$ is an invariant of the Legendrean contact structure.*

Proof. By construction it must be invariant since it appears along the diagonal of the endomorphism part of the tractor curvature. Otherwise, this can also be checked by direct calculation in a page or so using the formulae provided. □

The above result is consistent with [DMT19], in which the authors take an adapted coframe and show that the curvature invariant of an integrable Legendrean contact structure can be calculated as the totally trace-free part of a tensor of curvature coefficients.

The general theory of parabolic geometries ([ČS09] pg. 412) says that a complete set of tensor invariants on a Legendrean contact structure consists of the two obstructions to integrability and this curvature invariant. Accordingly, it should be that $A_{\bar{\alpha}\alpha\beta}, B_{\bar{\alpha}\alpha}, C_{\bar{\alpha}\alpha}{}^{\gamma}$ vanish with $W_{\bar{\alpha}\alpha\beta}{}^{\gamma}$. Unfortunately, the rather complicated expressions above resisted simplifications and so this could not be shown explicitly. Once this is done the situation will mirror the conformal case, [BEG94]. There, the tensor invariant of conformal geometry, the Weyl curvature, appears along the diagonal of the tractor curvature matrix, and the lower triangular off-diagonal entries are Cotton-York tensors, which turn out to vanish with the Weyl curvature via a Bianchi-like identity.

7.6 Vanishing tractor partial curvature

In this section we show that an integrable 5-dimensional Legendrean contact structure is locally isomorphic to the flat model if and only if the tractor curvature calculated above vanishes. This is immediate from the general theory of parabolic geometry, and partly motivates the construction of tractor bundles and tractor connections [ČS09]. Nevertheless we give a proof here using relatively elementary machinery.

The idea of the proof comes from [Eas17]. In fact we will clarify some points made in that article as a test bed for how to construct the isomorphism in the Legendrean contact case.

First however, consider the following problem. Let $E \to M$ be a vector bundle and ∇ a connection on E. If ∇ is flat, parallel transport does not depend on the curve and so we get canonical isomorphisms $P^{x}_{x_0}$ between fibres of E. See [Lee09] for the background on parallel transport.

Given a section σ we have map $M \to E_{x_0}$ given by

$$x \mapsto P^{x_0}_{x}\sigma(x). \tag{7.6.1}$$

What is the derivative $T_xM \to T_{\phi(x)}E_{x_0} \cong E_{x_0}$? We take a curve γ with $\gamma(0) = x$ and calculate

$$\frac{d}{dt}(P^{x_0}_{\gamma(t)}\sigma(\gamma(t)))|_{t=0}. \tag{7.6.2}$$

Let $\{e_i\}_{i=1}^{k}$ be a local, parallel trivialisation. Then we can write $\sigma = \sigma^i e_i$ for some smooth functions σ^i and in particular $\sigma(\gamma(t)) = \sigma^i(\gamma(t))e_i|_{\gamma(t)}$. Then

$$\begin{aligned} &\frac{d}{dt}(P^{x_0}_{\gamma(t)}\sigma(\gamma(t)))|_{t=0} \\ &= \frac{d}{dt}\Big(\sigma^i(\gamma(t))P^{x_0}_{\gamma(t)}(e_i|_{\gamma(t)})\Big)\Big|_{t=0} \end{aligned} \tag{7.6.3}$$

Now since the sections $\{e_i\}_{i=1}^k$ are parallel, parallel transporting the section at $\gamma(t)$ to x_0 simply gives the section at x_0.

$$\begin{aligned}
&= \frac{d}{dt}(\sigma^i(\gamma(t))e_i|_{x_0})|_{t=0} \\
&= d\sigma^i|_x(\dot\gamma(0))e_i|_{x_0} \\
&= P_x^{x_0}\big(d\sigma^i|_x(\dot\gamma(0))e_i|_x\big) \\
&= P_x^{x_0}\big(\nabla_{\dot\gamma}\sigma\big) \qquad (7.6.4)
\end{aligned}$$

So the derivative at x is just

$$T_xM \ni v \mapsto P_x^{x_0}\big(\nabla_v\sigma\big). \qquad (7.6.5)$$

The article [Eas17] shows that a 2-dimensional projective manifold is projectively flat when the curvature of the projective tractor bundle vanishes (this implies a classical result known as Beltrami's theorem). In a neighbourhood of any point, the author uses the vanishing tractor curvature to define a local diffeomorphism with $\mathbb{R}P^2$, although a proof that this is a diffeomorphism is not in the article. Something like this is also done in the conformal case in [PR86] theorem 6.9.23, although the modern tractor machinery had not been introduced at that time.

Proposition 7.6.6. *Let M be a manifold of dimension 2. Suppose we have an affine connection $\nabla_a : TM \to \Lambda^1 \otimes TM$ and the connection*

$$\nabla_a \begin{bmatrix} X^b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a X^b - \delta_a^b \rho \\ \nabla_a \rho + R_{ab}X^b \end{bmatrix} \qquad (7.6.7)$$

on the bundle $\mathbb{T} = TM \oplus \Lambda^0$ is flat. The map $M \to \mathbb{R}P^2$ given by

$$x \mapsto L(x) \qquad (7.6.8)$$

where $L(x)$ is the subspace of $\mathbb{T}_{x_0}$ (identified with $\mathbb{R}^3$) given by parallel translating the subspace

$$\begin{bmatrix} 0 \\ * \end{bmatrix} \leq \mathbb{T}_x \qquad (7.6.9)$$

to x_0, is a local diffeomorphism about x_0.

Proof. Firstly, 2.2.32 ensures that $L(x)$ has the correct dimension. We can write this map as the composition $M \to \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2$ where $M \to \mathbb{R}^3 \setminus \{0\}$ is the map which maps

$$x \mapsto P_x^{x_0} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad (7.6.10)$$

and then $\mathbb{R}^3 \setminus \{0\} \to \mathbb{R}P^2$ is the canonical projectivisation. We have

$$\nabla_a \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\delta_a^b \\ 0 \end{bmatrix} \tag{7.6.11}$$

and so using 7.6.5 with $x = x_0$ we see the derivative $T_{x_0}M \to \mathbb{T}_{x_0}$ is just

$$X^a \mapsto \begin{bmatrix} -X^a \\ 0 \end{bmatrix}. \tag{7.6.12}$$

Finally, the kernel of $\mathbb{T}_{x_0} \to T_{L(x_0)}P(\mathbb{T}_{x_0})$ consists of vectors along $L(x_0)$ which is disjoint to the image of $T_{x_0}M \to \mathbb{T}_{x_0}$ and so the derivative $T_{x_0}M \to T_{L(x_0)}P(\mathbb{T}_{x_0})$ is of full rank. □

The idea of the proof is similar in the Legendrean contact case, just quite a bit more involved.

Theorem 7.6.13. *Let $(M, H = P \oplus V)$ be a 5-dimensional Legendrean contact structure with P and V integrable, suppose that the partial curvature of the invariant partial connection given by 7.4.17 in the P direction and 7.4.19 in the V direction vanishes. In the presence of a contact form so that $\mathbb{T}$ splits as in 7.4.15, there is a diffeomorphism $\phi : U \to V \subset F_{1,3}(\mathbb{R}^4)$ for some suitable open sets U, V with $x_0 \in U$ defined by*

$$U \ni x \mapsto (L(x), U(x)) \in F_{1,3}(\mathbb{R}^4), \tag{7.6.14}$$

where

$$L(x) = \left\{ \left. \begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} \right|_{x_0} \in \mathbb{T}_{x_0} \mid (\bar{\nabla}_{\bar{\alpha}}, \nabla_\alpha) \begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} = 0 \ , f|_x = 0, \phi_\beta|_x = 0 \right\} \tag{7.6.15}$$

and

$$U(x) = \left\{ \left. \begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} \right|_{x_0} \in \mathbb{T}_{x_0} \mid (\bar{\nabla}_{\bar{\alpha}}, \nabla_\alpha) \begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} = 0 \ , f|_x = 0 \right\} \tag{7.6.16}$$

where we have identified $\mathbb{T}_{x_0} \cong \mathbb{R}^4$. Furthermore, this map is an isomorphism of Legendrean contact structures.

Proof. Firstly, 2.2.32 ensures that $L(x)$ and $U(x)$ have the correct dimension. We recover the notion of parallel transport thanks to the canonical extension 2.2.16 of $(\bar{\nabla}_{\bar{\alpha}}, \nabla_\alpha)$ to an connection. Furthermore by 2.2.25 this extension is flat, so parallel transport is path independent.

Consider the map $(\psi_1, \psi_2) : M \supset U \to \mathbb{T}_{x_0} \times \mathbb{T}^*_{x_0}$ which maps x to the parallel translation from x to x_0 of the value at x of some choice of non-vanishing sections of $\mathbb{T}, \mathbb{T}^*$ defined in a sufficiently small U. That is, given non-vanishing

$$\begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} \in \mathbb{T}, \begin{bmatrix} \tilde{f} & 0 & 0 \end{bmatrix} \in \mathbb{T}^* \tag{7.6.17}$$

map

$$x \mapsto (\psi_1(x), \psi_2(x)) := \left(P^x_{x_0} \begin{bmatrix} 0 \\ 0 \\ g(x) \end{bmatrix}, P^x_{x_0} \begin{bmatrix} \tilde{f}(x) & 0 & 0 \end{bmatrix} \right). \tag{7.6.18}$$

Projectivise each factor to get $M \to \mathbb{R}P^3 \times (\mathbb{R}P^3)^*$. Since the connection, and hence parallel transport, preserve the natural pairing, the image factors through to $F_{1,3}(\mathbb{R}^4)$ identified as a smooth submanifold of $\mathbb{R}P^3 \times (\mathbb{R}P^3)^*$ in the obvious way, recovering ϕ. If $\phi(x) = (L(x), U(x))$ it then makes sense to think of the derivative as a linear map $d\phi|_x : T_xM \to T_{L(x)}\mathbb{R}P^3 \oplus T_{U(x)}(\mathbb{R}P^3)^*$. Note here that the double fibration of $F_{1,3}(\mathbb{R}^4)$ over $\mathbb{R}P^3$ and $(\mathbb{R}P^3)^*$ is just given by the projection in each factor. In particular, the vertical bundles of these projections are just the kernels of the projections onto each summand of the tangent bundle.

We want to calculate $dL|_x : T_xM \to T_{L(x)}\mathbb{R}^3P$ and $dU|_x : T_xM \to T_{U(x)}(\mathbb{R}^3P)^*$. To do this we will first calculate the derivatives of the maps $\psi_1 : M \to \mathbb{T}_{x_0}$ and $\psi_2 : M \to \mathbb{T}^*_{x_0}$ using the same trick as in the projective case.

We have,

$$\bar{\nabla}_{\bar{\alpha}} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ -J_{\bar{\alpha}\beta}g \\ \bar{\nabla}_{\bar{\alpha}}g \end{bmatrix} \tag{7.6.19}$$

$$\nabla_{\alpha} \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \nabla_{\alpha}g, \end{bmatrix} \tag{7.6.20}$$

$$-\frac{1}{4}(\bar{\nabla}_{\bar{\alpha}}\nabla^{\bar{\alpha}} - \nabla_{\alpha}\bar{\nabla}^{\alpha}) \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix} = \begin{bmatrix} \frac{1}{2}g \\ 0 \\ -\frac{1}{4}\nabla_{\bar{\alpha}}\nabla^{\bar{\alpha}}g + \frac{1}{4}\bar{\nabla}_{\bar{\alpha}}\nabla^{\bar{\alpha}}g - \frac{1}{3}Y_{\bar{\beta}\alpha}{}^{\bar{\beta}\alpha}g \end{bmatrix}. \tag{7.6.21}$$

Using 7.6.5, the first two can be used to calculate the blocks of the derivative $P_x \to \mathbb{T}_{x_0}$, $V_x \to \mathbb{T}_{x_0}$ while using our characterisation 2.4.26 of the canonical extension the third can be used to calculate the derivative $\langle T \rangle_x \to \mathbb{T}_{x_0}$ where $\langle T \rangle$ is the line subbundle of TM spanned by the Reeb field. Since $TM = P \oplus V \oplus \langle T \rangle$ these account for the blocks of the derivative $d\psi_1|_x$.

Now the projectivisation $\mathbb{T}_{x_0} \to \mathbb{R}^3P$ has derivative at $\psi_1(x) \in \mathbb{T}_{x_0}$ which is the map $\mathbb{T}_{x_0} \to T_{L(x)}\mathbb{R}^3P$ with kernel consisting of those tangent vectors parallel to $\psi_1(x)$, that is, precisely elements of the form

$$P_x^{x_0}\begin{bmatrix}0\\0\\*\end{bmatrix}. \tag{7.6.22}$$

Now from plugging an element of V_x into 7.6.20 we see, using 7.6.5 that the image of V_x under $d\psi_1|_x$ consists of elements of the form

$$P_x^{x_0}\begin{bmatrix}0\\0\\*\end{bmatrix} \tag{7.6.23}$$

and so $V_x \subseteq \ker dL|_x$. Furthermore it is clear from 7.6.19, 7.6.20, 7.6.21 that $V_{x_0} = \ker : T_{x_0}M \to T_{L(x_0)}\mathbb{R}^3P$.

On the other hand the induced connection on $\mathbb{T}^*$ gives

$$\bar{\nabla}_{\bar{\alpha}}\begin{bmatrix}\tilde{f} & 0 & 0\end{bmatrix} = \begin{bmatrix}\bar{\nabla}_{\bar{\alpha}}\tilde{f} & 0 & 0\end{bmatrix} \tag{7.6.24}$$

$$\nabla_{\alpha}\begin{bmatrix}\tilde{f} & 0 & 0\end{bmatrix} = \begin{bmatrix}\nabla_{\alpha}\tilde{f} & -\delta_{\alpha}{}^{\beta}\tilde{f} & 0\end{bmatrix} \tag{7.6.25}$$

$$(\bar{\nabla}_{\bar{\alpha}}\nabla^{\bar{\alpha}} - \nabla_{\alpha}\bar{\nabla}^{\alpha})\begin{bmatrix}\tilde{f} & 0 & 0\end{bmatrix} = \begin{bmatrix}(\bar{\nabla}_{\bar{\alpha}}\nabla^{\bar{\alpha}} - \nabla_{\alpha}\bar{\nabla}^{\alpha})\tilde{f} & 0 & -2\tilde{f}\end{bmatrix} \tag{7.6.26}$$

which, similarly, can be used to calculate the blocks of the derivative $d\psi_2|_x$ which are, $P_x \to \mathbb{T}^*_{x_0}$, $V_x \to \mathbb{T}^*_{x_0}$, $\langle T\rangle_x \to \mathbb{T}^*_{x_0}$. The projectivisation $\mathbb{T}_{x_0} \to (\mathbb{R}P^3)^*$ has derivative at $\psi_2(x)$ which as a map $\mathbb{T}_{x_0} \to T_{U(x_0)}(\mathbb{R}P^3)^*$ with kernel consisting of elements of the form

$$P_x^{x_0}\begin{bmatrix}* & 0 & 0\end{bmatrix}. \tag{7.6.27}$$

Then using the same argument as above, 7.6.24 shows that $P_x \subseteq \ker dU|_x$ while at the base point 7.6.24, 7.6.25 , 7.6.26 show $P_{x_0} = \ker T_{x_0}M \to T_{U(x_0)}(\mathbb{R}P^3)^*$.

Putting the two pieces together we have shown $\ker d\phi|_{x_0} = P_{x_0} \cap V_{x_0} = \{0\}$, so ϕ has full rank derivative at x_0.

Moreover we have also shown

$$d\phi(P) \subseteq T\mathbb{R}P^3 \oplus \{0\} \subseteq T(\mathbb{R}P^3 \times \mathbb{R}P^3)^* \tag{7.6.28}$$

$$d\phi(V) \subseteq \{0\} \oplus T(\mathbb{R}P^3)^* \subseteq T(\mathbb{R}P^3 \times \mathbb{R}P^3)^*. \tag{7.6.29}$$

So on the neighbourhood of x_0 where $d\phi$ is of full rank, P and V are each sent to the vertical bundles of the double fibration over $(\mathbb{R}P^3)^*$ and $\mathbb{R}P^3$. Thus ϕ preserves the Legendrean contact structure. □

Corollary 7.6.30. *Let $(M, H = P \oplus V)$ be a 5-dimensional Legendrean contact structure with P and V integrable. The Legendrean contact structure is locally isomorphic to the flat model if and only if 7.4.4 has a 4-dimensional space of local solutions.*

Chapter 8

Towards G_2 contact tractors

In this chapter we discuss how one might construct the standard tractor bundle for G_2 contact geometries.

8.1 Another invariant PDE

The aim is to write down a G_2 invariant connection on the standard tractor bundle. We will attempt to do this by the method of invariant prolongation as in 7.4. Refer to that section for how standard tractors are defined abstractly. We know that $G_2 \hookrightarrow SO(3,4)$ and so in practice we want to construct a rank-7 bundle equipped with an invariantly defined 3-form and invariantly defined metric of signature-$(3,4)$ and, thinking about how P_2 is defined, the bundle should have an invariantly defined rank-2 subbundle which is null with respect to both the metric and 3-form. So the invariant operator should start on a bundle of rank 2, some weighted version of the bundle S. For ϕ_D a section of the bundle $S \otimes \Lambda^2 S^* \otimes \Lambda^2 S^*$ we have from 3.3.26 that

$$\nabla_{(ABC}\phi_{D)} = 0 \tag{8.1.1}$$

is invariant. Note that superficially, this is reminiscent of the invariant Killing equation 4.2.12.

Thinking about our decompositions 3.1.20 we see that ϕ_A solves 8.1.1 if and only if

$$\nabla_{ABC}\phi_D = \mu_{(AB}\varepsilon_{C)D}. \tag{8.1.2}$$

for some $\mu_{AB} \in \odot^2 S \otimes \Lambda^2 S^*$ and we calculate $\hat{\mu}_{AB} = \mu_{AB} + \Upsilon_{ABC}\phi^C$. The equation 8.1.2 imposes differential constraints on μ_{AB}. Differentiating gives

$$\nabla_{ABC}\nabla_{DEF}\phi_G = \nabla_{ABC}\mu_{(DE}\varepsilon_{F)G} \tag{8.1.3}$$

and then contracting with ε^{CF} before symmetrising gives

$$\nabla_{I(AB}\nabla_{DE)}{}^{I}\phi_G = \frac{2}{3}\nabla_{I(AB}\mu_D{}^{I}\varepsilon_{E)G} - \frac{1}{3}\nabla_{G(AB}\mu_{DE)} \tag{8.1.4}$$

The left-hand side is just curvature minus the torsion contracted with ϕ and $\nabla\phi$ respectively, but note $\tau_{ABCD}{}^{IJK}\nabla_{IJK}\phi_E = \tau_{ABCD}{}^{IJK}\mu_{(IJ}\varepsilon_{K)E} = \tau_{ABCDE}{}^{IJ}\mu_{IJ}$. So

$$R_{ABCDE}{}^{I}\phi_I - \tau_{ABCDE}{}^{IJ}\mu_{IJ} = \frac{2}{3}\nabla_{I(AB}\mu_C{}^{I}\varepsilon_{D)E} - \frac{1}{3}\nabla_{E(AB}\mu_{CD)}. \tag{8.1.5}$$

The Clebsch-Gordan decomposition implies we can write

$$\nabla_{ABC}\mu^{DE} = \alpha_{ABC}{}^{DE} + \beta_{(AB}{}^{(D}\delta_{C)}{}^{E)} + \sigma_{(A}\delta_B{}^{(D}\delta_{C)}{}^{E)} \tag{8.1.6}$$

for $\alpha_{ABCDE} \in \odot^5 S \otimes \Lambda^2 S^*$ and $\beta_{ABD} \in \odot^3 S$ and $\sigma_A \in S \otimes \Lambda^2 S$. Clearly

$$\begin{aligned} \alpha_{ABCDE} &= 3\tau_{(ABCDE)}{}^{IJ}\mu_{IJ} - 3R_{(ABCDE)}{}^{I}\phi_I \\ \Longrightarrow \quad \alpha_{ABC}{}^{DE} &= 3\tau_{ABC}{}^{DEIJ}\mu_{IJ} - \frac{9}{5}R^{DE}{}_{(ABC)}{}^{I}\phi_I - \frac{6}{5}R^{(D}{}_{(ABC)}{}^{E)I}\phi_I \end{aligned} \tag{8.1.7}$$

while contracting 8.1.5 with ε^{DE} and using the fact the partial torsion is symmetric on all its indices yields

$$\begin{aligned} R_{ABCI}{}^{IJ}\phi_J &= \frac{1}{3}\nabla_{I(AB}\mu_{C)}{}^{I} + \frac{1}{6}\nabla_{I(AB}\mu_{|D|}{}^{I}\delta_{C)}{}^{D} + \frac{1}{3}\nabla_{ID(A}\mu_B{}^{I}\delta_{C)}{}^{D} + \frac{1}{6}\nabla_{I(AB}\mu_{C)}{}^{I} \\ &= \nabla_{I(AB}\mu_{C)}{}^{I} \end{aligned} \tag{8.1.8}$$

On the other hand, contract 8.1.6 over a pair of indices and symmetrise over the remaining three to obtain $\beta_{ABC} = \frac{6}{5}\nabla_{I(AB}\mu_{C)}{}^{I}$. Putting it together we have

$$\begin{aligned} \nabla_{ABC}\mu^{DE} =& 3\tau_{ABC}{}^{DEIJ}\mu_{IJ} - \frac{9}{5}R^{DE}{}_{(ABC)}{}^{I}\phi_I - \frac{6}{5}R^{(D}{}_{(ABC)}{}^{E)I}\phi_I \\ &+ \frac{6}{5}R_{(AB}{}^{(D}{}_{|I}{}^{|IJ|}\phi_{J|}\delta_{C)}{}^{E)} + \sigma_{(A}\delta_B{}^{(D}\delta_{C)}{}^{E)}. \end{aligned} \tag{8.1.9}$$

Now $\sigma_A \in S \otimes \Lambda^2 S$ cannot be obtained from the above but in any case 8.1.2 and 8.1.9 are equivalent to 8.1.1.

If, as in the Legendrean contact case, we let the invariance of 8.1.2 and 8.1.9 prescribe a change of splitting formula for the bundle which, in the presence of a contact form, can be identified

$$\mathbb{T} \cong (S \otimes \Lambda^2 S^* \otimes \Lambda^2 S^*) \oplus (\odot^2 S \otimes \Lambda^2 S^*) \oplus (S \otimes \Lambda^2 S), \tag{8.1.10}$$

then one can check that the obvious signature $(3, 4)$ metric, namely

$$\begin{bmatrix} \phi_D \\ \mu_{DE} \\ \sigma_D \end{bmatrix} \otimes \begin{bmatrix} \psi_D \\ \nu_{DE} \\ \lambda_D \end{bmatrix} \mapsto \phi_D \lambda^D + \mu_{DE} \nu^{DE} - \sigma_D \psi^D \tag{8.1.11}$$

is not invariant. Since we would like the metric to have a simple form, we can perform a similar trick to the Legendrean contact case and let ρ_A be defined by

$$\begin{aligned} \nabla_{ABC}\mu^{DE} = 3\tau_{ABC}{}^{DEIJ}\mu_{IJ} - \frac{9}{5}R^{DE}{}_{(ABC)}{}^{I}\phi_I - \frac{6}{5}R^{(D}{}_{(ABC)}{}^{E)I}\phi_I \\ + \frac{6}{5}R_{(AB}{}^{(D}{}_{|I}{}^{|IJ|}\phi_{J|}\delta_{C)}{}^{E)} + \rho_{(A}\delta_B{}^{(D}\delta_{C)}{}^{E)} + \frac{1}{2}Y_{F(A}\phi^F\delta_B{}^{(D}\delta_{C)}{}^{E)} \end{aligned} \tag{8.1.12}$$

8.1.2 and 8.1.12 are still equivalent to 8.1.1, we have just made a change of variables like in the Legendrean contact case. Then,

$$\hat{\rho}_D = \rho_D + \Upsilon_{DEF}\mu^{EF} + \frac{1}{2}\Upsilon_{DEF}\Upsilon^{EF}{}_G\phi^G - \frac{1}{4}\nabla_{EFG}\Upsilon^{EFG}\phi_D. \tag{8.1.13}$$

which ensures that 8.1.11 is invariant.

Furthermore, it is possible to check via a few pages of calculations that this change makes the 3-form

$$\begin{bmatrix} \phi_D \\ \mu_{DE} \\ \rho_D \end{bmatrix} \otimes \begin{bmatrix} \psi_D \\ \nu_{DE} \\ \lambda_D \end{bmatrix} \otimes \begin{bmatrix} \gamma_D \\ \kappa_{DE} \\ \zeta_D \end{bmatrix} \mapsto$$

$$\begin{aligned} \rho_D\nu^{DE}\gamma_E - \lambda_D\mu^{DE}\gamma_E + \phi_D\kappa^{DE}\lambda_E - \psi_D\kappa^{DE}\rho_E \\ -\phi_E\nu^{DE}\zeta_D + \psi_D\mu^{DE}\zeta_E + \mu_{DE}\nu^{DF}\kappa^E{}_F \end{aligned} \tag{8.1.14}$$

invariant. Given the change of splitting of the direct sum 8.1.10 we see that $\{0\} \oplus \{0\} \oplus (S \otimes \Lambda^2 S)$ is an invariantly defined rank-2 subbundle on which the invariant 3-form and metric are null, as we expect for the G_2 contact standard tractors.

To check that the prolongation actually closes up, we just need to check we can completely obtain $\nabla_{ABC}\rho_D$ from the above expression. Applying ∇_{ABC} again we have

$$\begin{aligned} \nabla_{ABC}\nabla_{DEF}\mu_{GH} =& 3\tau_{DEFGH}{}^{IJ}\nabla_{ABC}\mu_{IJ} + \nabla_{ABC}(\rho_{(D}\varepsilon_{E|G|}\varepsilon_{F)H}) + \ldots \\ =& 3\tau_{DEFGH(AB}\rho_{C)} + \nabla_{ABC}(\rho_{(D}\varepsilon_{E|G|}\varepsilon_{F)H}) + \ldots \end{aligned} \tag{8.1.15}$$

where we will omit terms which are contractions of R_{ABCDEF}, $\nabla_{ABC}R_{DEFGHI}$, $\tau_{ABCDEGH}$, ϕ_A, μ_{AB}, which are known quantities. Contact with ε^{AF} to get

$$\nabla_{KBC}\nabla_{DE}{}^{K}\mu_{GH} = \tau_{BCDEGH}{}^{K}\rho_K + \frac{1}{3}\nabla_{KBC}\rho^K\varepsilon_{(D|G}\varepsilon_{|E)H} + \frac{2}{3}\nabla_{BC(G}\rho_{|(D}\varepsilon_{E)|H)} + \ldots \quad . \tag{8.1.16}$$

Symmetrising on $BCDE$ gives

$$2R_{BCDE(G}{}^{I}\mu_{H)I} - \tau_{BCDE}{}^{IJK}\nabla_{IJK}\mu_{GH} = \tau_{BCDEGH}{}^{K}\rho_K + \frac{1}{3}\nabla_{K(BC|}\rho^K\varepsilon_{|D|G}\varepsilon_{|E)H} + \frac{2}{3}\nabla_{(G|(BC}\rho_D\varepsilon_{E)|H)} + \ldots \quad . \tag{8.1.17}$$

Again using the definition of $\nabla_{ABC}\mu_{DE}$

$$2R_{BCDE(G}{}^{I}\mu_{H)I} = 2\tau_{BCDEGH}{}^{K}\rho_K + \frac{1}{3}\nabla_{K(BC|}\rho^K\varepsilon_{|D|G}\varepsilon_{|E)H} + \frac{2}{3}\nabla_{(G|(BC}\rho_D\varepsilon_{E)|H)} + \ldots \quad . \tag{8.1.18}$$

Contracting with ε^{EH}

$$-R_{BCD}{}^{E}{}_{G}{}^{I}\mu_{EI} = \frac{5}{12}\nabla_{K(BC|}\rho^K\varepsilon_{|D)G} + \frac{5}{6}\nabla_{G(BC}\rho_{D)} + \ldots \quad . \tag{8.1.19}$$

We have $\odot^3 S \otimes S \otimes \Lambda^2 S \cong (\odot^4 S \otimes \Lambda^2 S) \oplus (\odot^2 S \otimes \Lambda^2 S \otimes \Lambda^2 S)$. Symmetrising

$$\frac{5}{6}\nabla_{(GBC}\rho_{D)} = -R_{(BCD}{}^{E}{}_{G)}{}^{I}\mu_{EI} + \ldots \quad . \tag{8.1.20}$$

So we can solve for the component of $\nabla_{ABC}\rho_D$ in $\odot^4 S \otimes \Lambda^2 S$ as a linear combination of contractions of R_{ABCDEF}, $\nabla_{ABC}R_{DEFGHI}$,
$\tau_{ABCDEGH}$, ϕ_A, μ_{AB}. Similarly contracting 8.1.19 with ε^{DG} gives

$$\frac{5}{6}\nabla_{DBC}\rho^D = -R_{BCD}{}^{EDI}\mu_{EI} + \ldots \quad . \tag{8.1.21}$$

which gives the component in $\odot^2 S \otimes \Lambda^2 S \otimes \Lambda^2 S$ in terms of known tensors. Thus the prolongation terminates and 8.1.1 is of finite type. Furthermore we have a invariantly defined connection on the rank-7 bundle $\mathbb{T}$ which in the presence of a scale splits

$$\mathbb{T} \cong (S \otimes \Lambda^2 S^* \otimes \Lambda^2 S^*) \oplus (\odot^2 S \otimes \Lambda^2 S^*) \oplus (S \otimes \Lambda^2 S) \tag{8.1.22}$$

Such that the components transform according to

$$\widehat{\begin{bmatrix} \phi_D \\ \mu_{DE} \\ \rho_D \end{bmatrix}} = \begin{bmatrix} \phi_D \\ \mu_{DE} + \Upsilon_{DEF}\phi^F \\ \rho_D + \Upsilon_{DEF}\mu^{EF} + \frac{1}{2}\Upsilon_{DEF}\Upsilon^{EF}{}_G\phi^G + \frac{1}{4}\nabla_{EFG}\Upsilon^{EFG}\phi_D \end{bmatrix}. \tag{8.1.23}$$

Explicitly calculating $\nabla_{ABC}\rho_D$ in the prolongation leads to an exceedingly messy expression that we will not reproduce here. However, from the argument above we can see the invariant connection (raising some indices to write down the expression in a manageable way) takes the form

$$\nabla_{ABC}\begin{bmatrix} \phi_D \\ \mu^{DE} \\ \rho_D \end{bmatrix} = \begin{bmatrix} \nabla_{ABC}\phi_D - \mu_{(AB}\varepsilon_{C)D} \\ \\ \nabla_{ABC}\mu^{DE} + \frac{9}{5}R^{DE}{}_{(ABC)}{}^I\phi_I + \frac{6}{5}R^{(D}{}_{(ABC)}{}^{E)I}\phi_I \\ -\frac{6}{5}R_{(AB}{}^{(D}{}_{|I}{}^{|IJ|}\phi_{J|}\delta_{C)}{}^{E)} - \rho_{(A}\delta_B{}^{(D}\delta_{C)}{}^{E)} - 3\tau_{ABC}{}^{DEIJ}\mu_{IJ} \\ \\ \nabla_{ABC}\rho_D + X_{ABC}{}^D\phi_D + Y_{ABC}{}^{DE}\mu_{DE} \end{bmatrix} \tag{8.1.24}$$

for some appropriate $X_{ABC}{}^D$ and $Y_{ABC}{}^{DE}$.

Now that we have such an expression, 8.1.24, the obvious thing to do is to calculate the partial curvature of this partial connection and check that it vanishes with $\tau_{ABCDEFG}$. We would then be able to see by relatively elementary means, the fact from the general theory of parabolic geometry that the sole invariant of a G_2 contact structure is contained in the bundle $\odot^7 S \otimes \Lambda^2 S^* \otimes \Lambda^2 S^*$. Furthermore, we would have found a simple partial differential equation: $\nabla_{(ABC}\phi_{D)} = 0$, which has 7-dimensional solution space if and only if the G_2 contact geometry is flat. Unfortunately such calculations seem difficult by hand but we hope to set up the problem in Maple in the future.

Appendices

A Cohomology groups of the Rumin complex

To show that the maps 1.3.3, 1.3.4 and 1.3.5 defined on cohomology are well-defined, we will need to consider various cases. First note that in the case $k = 1, ..., n$ the map

$$\begin{aligned} \{\omega \in \Lambda_H^k \mid d\tilde{\omega}|_H \in \mathcal{L}(L \otimes \Lambda_H^{k-1})\} &\to \ker d^{(k+1)} \\ \omega &\mapsto \tilde{\omega} - (i_k \circ \mathcal{L}^{-1})(d\tilde{\omega}|_H) \end{aligned} \tag{A.1}$$

is well-defined as one can readily verify by noting that two lifts of $\omega \in \Lambda_H^k$ differ by $\alpha \wedge \tilde{\nu}$ and then checking

$$\alpha \wedge \tilde{\nu} - (i_k \circ \mathcal{L}^{-1})(d(\alpha \wedge \tilde{\nu})|_H) = 0. \tag{A.2}$$

Now we check that A.1 descends to 1.3.4 on equivalence classes for $k = 2, ..., n$. For two representatives to lie in the same equivalence class

$$\begin{aligned} & [\omega + \mathcal{L}(L \otimes \Lambda_H^{k-2})] = [\mu + \mathcal{L}(L \otimes \Lambda_H^{k-2})] \\ \iff & \omega - \mu + \mathcal{L}(L \otimes \Lambda_H^{k-2}) = d\tilde{\rho}|_H + \mathcal{L}(L \otimes \Lambda_H^{k-2}) \\ \iff & \omega - \mu = d\tilde{\rho}|_H + d\alpha|_H \wedge \nu \end{aligned} \tag{A.3}$$

for some $\rho \in \Lambda_H^k, \nu \in \Lambda_H^{k-1}$. Now

$$d\tilde{\rho} + d\alpha \wedge \tilde{\nu} - (i_n \circ \mathcal{L}^{-1})(d\alpha|_H \wedge d\nu|_H) = d\tilde{\rho} \tag{A.4}$$

which shows the map descends. Now if

$$\tilde{\omega} - (i_k \circ \mathcal{L}^{-1})(d\tilde{\omega}|_H) = d\mu \tag{A.5}$$

we have $\omega = d\mu|_H$ and $\omega + \mathcal{L}(L \otimes \Lambda_H^{k-2})$ is in the range of $d_\perp^{(k-1)}$ which shows injectivity of 1.3.4. If $\tilde{\omega}$ is a closed form then

$$[\omega + \mathcal{L}(L \otimes \Lambda_H^{k-2})] \mapsto [\tilde{\omega}] \tag{A.6}$$

which shows surjectivity of 1.3.4. So 1.3.4 is an isomorphism. This argument is readily modifiable to show A.1 descends to an isomorphism 1.3.3 when $k = 1$.

Next we will deal with cases $k = n + 1, ..., 2n$. For $k = n + 1$ the canonical inclusion applied to an element in the image of the nth Rumin operator is clearly exact, by definition. For $k > n$ it is the same story, and so the map 1.3.5 map is well-defined.

Suppose that for $k = n + 1$ we had

$$\alpha \wedge \tilde{\omega} = d\tilde{\mu}. \tag{A.7}$$

Then $d\tilde{\mu}|_H = 0$ and so $\alpha \otimes \omega = d_\perp^{(n)}\mu$ which shows injectivity of 1.3.5.

Then for $k > n + 1$ suppose we have

$$\alpha \wedge \tilde{\omega} = d\tilde{\mu}. \tag{A.8}$$

Using the surjectivity of $\mathcal{L} : L \otimes \Lambda_H^{k-2} \to \Lambda_H^k$ we can write $\tilde{\mu} = d\alpha \wedge \tilde{\rho} + \alpha \wedge \tilde{\kappa}$. Taking the exterior derivative we have $\alpha \wedge \tilde{\omega} = d\alpha \wedge d\tilde{\rho} + d\alpha \wedge \tilde{\kappa} - \alpha \wedge d\tilde{\kappa}$ and furthermore $d\alpha|_H \wedge (d\tilde{\rho}|_H + \kappa) = 0$. Finally

$$d(\alpha \wedge (d\tilde{\rho} + \kappa)) = \alpha \wedge \tilde{\omega} \tag{A.9}$$

so

$$\alpha \otimes \omega = d_\perp^{(k-1)}(\alpha \otimes (d\tilde{\rho}|_H + \kappa)) \tag{A.10}$$

which shows injectivity of 1.3.5.

To see surjectivity for $k \geq n + 1$ suppose that $\tilde{\mu} \in \Lambda^{k+1}$ was a closed form. Again use the surjectivity of $\mathcal{L} : L \otimes \Lambda_H^{k-1} \to \Lambda_H^{k+1}$ to write $\tilde{\mu} = d\alpha \wedge \tilde{\rho} + \alpha \wedge \tilde{\kappa}$. Using the fact $\tilde{\mu}$ is closed one can check that $\alpha \otimes (d\tilde{\rho}|_H \wedge \kappa) \in \ker d_\perp^{(k)}$ and then

$$[\alpha \otimes (d\tilde{\rho}|_H \wedge \kappa)] \mapsto \tag{A.11}$$
$$[\alpha \wedge d\tilde{\rho} + \alpha \wedge \tilde{\kappa}] = [d\alpha \wedge \tilde{\rho} - d(\alpha \wedge \tilde{\rho}) + \alpha \wedge \tilde{\kappa}] = [\tilde{\mu}] \tag{A.12}$$

which shows surjectivity 1.3.5.

So we have written down isomorphisms between the cohomology groups of the Rumin complex and the de Rham cohomology groups. $\square$.

B Existence of the preferred partial connection

For existence in the general case, it is more convenient to discard the conditions ${\Gamma_{\bar{\alpha}}}^{\bar{\gamma}}{}_{\bar{\beta}} = {\Gamma_{(\bar{\alpha}}}^{\bar{\gamma}}{}_{\bar{\beta})}$ and ${\Phi_{(\alpha\beta)}}^{\gamma} = {\Phi_{\alpha\beta}}^{\gamma}$ and instead utilise the full rank $2n^3$ in picking ${\Gamma_{\bar{\alpha}\beta}}^{\gamma}, {\Phi_{\alpha\beta}}^{\gamma}$ to insist that $(\mu_{\bar{\alpha}}, \nu_{\beta}) \mapsto \hat{\hat{\nabla}}_{\bar{\alpha}}\nu_{\beta} - \hat{\nabla}_{\beta}\mu_{\bar{\alpha}}$ agrees with the differential operator $P^* \oplus V^* \to P^* \otimes V^*$ induced by the exterior derivative in the presence of a contact form. This normalisation is possible as the rank of $\mathrm{Hom}(P^* \oplus V^*, P^* \otimes V^*)$ is $2n^3$. If we arrange this then the partial torsion $P^* \oplus V^* \to (P \otimes V)_{\perp}$ automatically vanishes since the corresponding part of the Rumin operator is just the above operator with the trace taken out.

Lastly, we will show that the components of partial torsion $P^* \to \Lambda^2 P^*$ and similarly $V^* \to \Lambda^2 V^*$ also automatically vanish given the above normalisation. The calculation is as follows:

$$\begin{aligned} & X^{\bar{\alpha}} \hat{\hat{\nabla}}_{[\bar{\alpha}} \mu_{\bar{\beta}]} Y^{\bar{\beta}} \\ =& \frac{1}{2} X^{\bar{\alpha}} \hat{\hat{\nabla}}_{\bar{\alpha}} (\mu_{\bar{\beta}} Y^{\bar{\beta}}) - \frac{1}{2} Y^{\bar{\alpha}} \hat{\hat{\nabla}}_{\bar{\alpha}} (\mu_{\bar{\beta}} X^{\bar{\beta}}) - \frac{1}{2} X^{\bar{\alpha}} \mu^{\beta} (\hat{\hat{\nabla}}_{\bar{\alpha}} Y_{\beta}) + \frac{1}{2} Y^{\bar{\alpha}} \mu^{\beta} (\hat{\hat{\nabla}}_{\bar{\alpha}} X_{\beta}). \end{aligned} \tag{B.1}$$

Now the assumed normalisation means this is (using index free notation)

$$= \frac{1}{2} X(\mu(Y)) - \frac{1}{2} Y(\mu(X)) - d(d\alpha(Y, \cdot))(X, M) + d(d\alpha(X, \cdot))(Y, M). \tag{B.2}$$

Where $M \in V$ is defined by $M = J(\cdot, \mu)$ where J is the non-degenerate skew form on H distinguished by α. Using the coordinate free formula for the exterior derivative this simplifies to

$$\begin{aligned} = \frac{1}{2} X(\mu(Y)) - \frac{1}{2} Y(\mu(X)) - \frac{1}{2} X(d\alpha(Y, M)) + \frac{1}{2} d\alpha(Y, [X, M]) \\ + \frac{1}{2} Y(d\alpha(X, M)) - \frac{1}{2} d\alpha(X, [Y, M]). \end{aligned} \tag{B.3}$$

Now $d\alpha(Y, M) = \mu(Y)$ and $d\alpha(X, M) = \mu(X)$ and again using the coordinate free formula this is

$$\begin{aligned} =& \frac{1}{2} d\alpha(Y, [X, M]) - \frac{1}{2} d\alpha(X, [Y, M]) \\ =& \frac{1}{4} Y(\alpha([X, M])) - \frac{1}{4} \alpha([Y, [X, M]]) - \frac{1}{4} X(\alpha([Y, M])) + \frac{1}{4} \alpha([X, [Y, M]]). \end{aligned} \tag{B.4}$$

The coordinate free formula once more gives $d\alpha(U, V) = -\alpha([U, V])$ for U, V sections of H and, also using the Jacobi identity we get

$$= \frac{1}{2} X(d\alpha(Y, M)) - \frac{1}{2} Y(d\alpha(X, M)) + \frac{1}{4} \alpha([M, [X, Y]]). \tag{B.5}$$

Now $2d\alpha(X,Y) = -\alpha([X,Y])$, so $[X,Y] \in H$ and therefore $\alpha([M,[X,Y]]) = -2d\alpha(M,[X,Y])$ and so finally, the above is just

$$= \frac{1}{2}X(\mu(Y)) - \frac{1}{2}Y(\mu(X)) - \frac{1}{2}\mu([X,Y]) \tag{B.6}$$

Comparing this to 1.2.19 we see that $\hat{\hat{\nabla}}_{[\bar{\alpha}}\mu_{\bar{\beta}]} = d\mu|_{\Lambda^2 P}$ and the torsion $P^* \to \Lambda^2 P^*$ automatically vanishes given the assumed normalisation. Similarly the component $V^* \to \Lambda^2 V^*$ automatically vanishes. Thus we have shown existence for Legendrean contact manifolds with $n \geq 2$. $\square$

C Calculation of partial torsion in the G_2 contact construction

By contracting each of 6.2.6 to 6.2.17 with the appropriate combination of trivialising sections $\Phi^{\bar\alpha}, \Psi^{\bar\alpha}, \Phi^{\alpha}, \Psi^{\alpha}$ the system, equivalent to vanishing torsion, produces an equivalent system of 20 linear equations over $\mathbb{R}$ with 12 unknowns, the unknowns being the arbitrary components $\bar\kappa_{\bar\alpha}\Psi^{\bar\alpha}, \bar\kappa_{\bar\alpha}\Phi^{\bar\alpha}$ et cetera. 12 of these are

$$\Phi^{\bar\alpha}(\bar\nabla_{[\bar\alpha}\Phi_{\bar\beta]})\Psi^{\bar\beta} = \frac{3}{2}\bar\kappa_{\bar\alpha}\Phi^{\bar\alpha} - \frac{\sqrt{3}}{2}\mu_{\bar\alpha}\Psi^{\bar\alpha} \tag{C.1}$$
$$\Phi^{\bar\alpha}(\bar\nabla_{[\bar\alpha}\Psi_{\bar\beta]})\Psi^{\bar\beta} = -\frac{\sqrt{3}}{2}\lambda_{\bar\alpha}\Phi^{\bar\alpha} + \frac{1}{2}\kappa_{\bar\alpha}\Phi^{\bar\alpha} \tag{C.2}$$
$$\Phi^{\beta}(\nabla_{\beta}\Phi_{\bar\alpha})\Phi^{\bar\alpha} = \sqrt{3}\Phi^{\beta}\nu_{\beta} \tag{C.3}$$
$$\Psi^{\beta}(\nabla_{\beta}\Phi_{\bar\alpha})\Psi^{\bar\alpha} = 3\Psi^{\beta}\omega_{\beta} \tag{C.4}$$
$$\Phi^{\beta}(\nabla_{\beta}\Psi_{\bar\alpha})\Phi^{\bar\alpha} = -\Phi^{\beta}\omega_{\beta} \tag{C.5}$$
$$\Psi^{\beta}(\nabla_{\beta}\Psi^{\bar\alpha})\Psi^{\bar\alpha} = 2\bar\nu_{\bar\alpha}\Psi^{\bar\alpha} - \sqrt{3}\mu_{\beta}\Psi^{\beta} \tag{C.6}$$
$$\Phi^{\bar\alpha}(\bar\nabla_{\bar\alpha}\Phi_{\beta})\Phi^{\beta} = -\sqrt{3}\bar\nu_{\bar\alpha}\Phi^{\bar\alpha} + 2\mu_{\beta}\Phi^{\beta} \tag{C.7}$$
$$\Psi^{\bar\alpha}(\bar\nabla_{\bar\alpha}\Phi_{\beta})\Psi^{\beta} = \omega_{\bar\alpha}\Psi^{\bar\alpha} \tag{C.8}$$
$$\Phi^{\bar\alpha}(\bar\nabla_{\bar\alpha}\Psi_{\beta})\Phi^{\beta} = -3\bar\omega_{\bar\alpha}\Phi^{\bar\alpha} \tag{C.9}$$
$$\Psi^{\bar\alpha}(\bar\nabla_{\bar\alpha}\Psi_{\beta})\Psi^{\beta} = \sqrt{3}\bar\mu_{\bar\alpha}\Psi^{\bar\alpha} \tag{C.10}$$
$$\Phi^{\alpha}(\nabla_{[\alpha}\Phi_{\beta]})\Psi^{\beta} = \frac{\sqrt{3}}{2}\nu_{\alpha}\Psi^{\alpha} + \frac{1}{2}\omega_{\alpha}\Phi^{\alpha} \tag{C.11}$$
$$\Phi^{\alpha}(\nabla_{[\alpha}\Psi_{\beta]})\Psi^{\beta} = \frac{\sqrt{3}}{2}\mu_{\alpha}\Phi^{\alpha} + \frac{3}{2}\omega_{\alpha}\Psi^{\alpha} \tag{C.12}$$

then solve to yield

$$\bar\kappa_{\bar\alpha}\Phi^{\bar\alpha} = -\frac{1}{3}\Phi^{\bar\alpha}(\bar\nabla_{\bar\alpha}\Psi_{\beta})\Phi^{\beta} \tag{C.13}$$
$$\bar\kappa_{\bar\alpha}\Psi^{\bar\alpha} = \Psi^{\bar\alpha}(\bar\nabla_{\bar\alpha}\Phi_{\beta})\Psi^{\beta} \tag{C.14}$$
$$\kappa_{\alpha}\Phi^{\alpha} = -\Phi^{\beta}(\nabla_{\beta}\Psi_{\bar\alpha})\Phi^{\bar\alpha} \tag{C.15}$$
$$\kappa_{\alpha}\Psi^{\alpha} = \frac{1}{3}\Psi^{\beta}(\nabla_{\beta}\Phi_{\bar\alpha})\Psi^{\bar\alpha} \tag{C.16}$$

$$\bar{\lambda}_{\bar{\alpha}}\Phi^{\bar{\alpha}} = -\frac{2}{\sqrt{3}}\Phi^{\bar{\alpha}}(\bar{\nabla}_{[\bar{\alpha}}\Psi_{\bar{\beta}]})\Psi^{\bar{\beta}} + \frac{1}{\sqrt{3}}\Psi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Phi_{\beta})\Psi^{\beta} \tag{C.17}$$

$$\bar{\lambda}_{\bar{\alpha}}\Psi^{\bar{\alpha}} = \frac{1}{\sqrt{3}}\Psi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Psi_{\beta})\Psi^{\beta} \tag{C.18}$$

$$\lambda_{\alpha}\Phi^{\alpha} = -\frac{1}{\sqrt{3}}\Psi^{\beta}(\nabla_{\beta}\Phi_{\bar{\alpha}})\Psi^{\bar{\alpha}} + \frac{2}{\sqrt{3}}\Phi^{\alpha}(\nabla_{[\alpha}\Psi_{\beta]})\Psi^{\beta} \tag{C.19}$$

$$\lambda_{\alpha}\Psi^{\alpha} = -\frac{1}{\sqrt{3}}\Psi^{\beta}(\nabla_{\beta}\Psi_{\bar{\alpha}})\Psi^{\bar{\alpha}} - \frac{2}{3}\Phi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Psi_{\beta})\Phi^{\beta} - \frac{4}{3}\Phi^{\bar{\alpha}}(\bar{\nabla}_{[\bar{\alpha}}\Phi_{\bar{\beta}]})\Psi^{\bar{\beta}} \tag{C.20}$$

$$\bar{\mu}_{\bar{\alpha}}\Phi^{\bar{\alpha}} = -\frac{1}{\sqrt{3}}\Phi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Phi_{\beta})\Phi^{\beta} - \frac{2}{3}\Psi^{\beta}(\nabla_{\beta}\Phi_{\bar{\alpha}})\Psi^{\bar{\alpha}} + \frac{4}{3}\Phi^{\alpha}(\nabla_{[\alpha}\Psi_{\beta]})\Psi^{\beta} \tag{C.21}$$

$$\bar{\mu}_{\bar{\alpha}}\Psi^{\bar{\alpha}} = -\frac{1}{\sqrt{3}}\Phi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Psi_{\beta})\Phi^{\beta} - \frac{2}{\sqrt{3}}\Phi^{\bar{\alpha}}(\bar{\nabla}_{[\bar{\alpha}}\Phi_{\bar{\beta}]})\Psi^{\bar{\beta}} \tag{C.22}$$

$$\mu_{\alpha}\Phi^{\alpha} = \frac{1}{\sqrt{3}}\Phi^{\beta}(\nabla_{\beta}\Phi_{\bar{\alpha}})\Phi^{\bar{\alpha}} \tag{C.23}$$

$$\mu_{\alpha}\Psi^{\alpha} = \frac{2}{\sqrt{3}}\Phi^{\alpha}(\nabla_{[\alpha}\Phi_{\beta]})\Psi^{\beta} + \frac{1}{\sqrt{3}}\Phi^{\beta}(\nabla_{\beta}\Psi_{\bar{\alpha}})\Phi^{\bar{\alpha}} \tag{C.24}$$

which fixes the 12 degrees of freedom. Now the system 6.2.6 to 6.2.17 is equivalent to C.13 to C.24 in addition to the 8 equations over $\mathbb{R}$ that were not used

$$0 = \Phi^{\bar{\alpha}}\Pi_{\bar{\alpha}\bar{\beta}}{}^{\gamma}\Psi_{\gamma}\Psi^{\bar{\beta}} \tag{C.25}$$

$$\mu_{\alpha}\Phi^{\alpha} = -\Phi^{\alpha}\Sigma_{\alpha\beta}{}^{\bar{\gamma}}\Phi_{\bar{\gamma}}\psi^{\beta} \tag{C.26}$$

$$\sqrt{3}\mu_{\beta}\Psi^{\beta} + 3\kappa_{\beta}\Phi^{\beta} = \Psi^{\beta}(\nabla_{\beta}\Phi_{\bar{\alpha}})\Phi^{\bar{\alpha}} + \Phi^{\beta}(\nabla_{\beta}\Phi_{\bar{\alpha}})\Psi^{\bar{\alpha}} \tag{C.27}$$

$$2\bar{\mu}_{\bar{\alpha}}\Phi^{\bar{\alpha}} - \kappa_{\beta}\Psi^{\beta} - \sqrt{3}\mu_{\beta}\Phi^{\beta} = \Psi^{\beta}(\nabla_{\beta}\Psi_{\bar{\alpha}})\Phi^{\bar{\alpha}} + \Phi^{\beta}(\nabla_{\beta}\Psi_{\bar{\alpha}})\Psi^{\bar{\alpha}} \tag{C.28}$$

$$2\lambda_{\beta}\Psi^{\beta} + \bar{\kappa}_{\bar{\alpha}}\Phi^{\bar{\alpha}} - \sqrt{3}\bar{\mu}_{\bar{\alpha}}\Psi^{\bar{\alpha}} = \Phi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Phi_{\beta})\Psi^{\beta} + \Psi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Phi_{\beta})\Phi^{\beta} \tag{C.29}$$

$$\sqrt{3}\Phi^{\bar{\alpha}}\bar{\lambda}_{\bar{\alpha}} - 3\bar{\kappa}_{\bar{\alpha}}\Psi^{\bar{\alpha}} = \Phi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Psi_{\beta})\Psi^{\beta} + \Psi^{\bar{\alpha}}(\bar{\nabla}_{\bar{\alpha}}\Psi_{\beta})\Phi^{\beta} \tag{C.30}$$

$$\bar{\lambda}_{\bar{\alpha}}\Psi^{\bar{\alpha}} = \Phi^{\bar{\alpha}}\Pi_{\bar{\alpha}\bar{\beta}}{}^{\nu}\Phi_{\nu}\Psi^{\bar{\beta}} \tag{C.31}$$

$$0 = \Phi^{\alpha}\Sigma_{\alpha\beta}{}^{\bar{\gamma}}\Phi_{\bar{\gamma}}\Psi^{\beta} \tag{C.32}$$

Substituting C.13 to C.24 into the above, we see that the overall system is consistent if and only if 6.2.18 to 6.2.25 all vanish.

D Further Legendrean contact tractor curvature calculations

Here we calculate the component of tractor curvature $\mathbb{T} \to \Lambda^2 V \otimes \mathbb{T}$. Applying the connection twice then skewing gives the unfortunately complicated expression

$$\begin{bmatrix} f \\ \phi_\beta \\ g \end{bmatrix} \mapsto \begin{bmatrix} 0 & 0 & 0 \\ -2\nabla_{[\alpha}Z_{\beta]\gamma} & Z_{\alpha\beta\gamma}{}^{\nu} + 2Z_{[\alpha|\gamma}\delta^{\nu}_{|\beta]} & 0 \\ U_{\alpha\beta} & V_{\alpha\beta}{}^{\nu} & 0 \end{bmatrix} \begin{bmatrix} f \\ \phi_\nu \\ g \end{bmatrix} \tag{D.1}$$

where

$$U_{\alpha\beta} = -\frac{2}{3}\nabla_{[\alpha|}\bar{\nabla}_{\bar{\beta}}Z_{|\beta]}{}^{\bar{\beta}} - \frac{8}{3}Y_{\bar{\beta}[\alpha}{}^{\bar{\beta}\gamma}Z_{\beta]\gamma} + \frac{1}{6}\nabla_{[\alpha|}\nabla^{\bar{\gamma}}Y_{\bar{\gamma}|\beta]} + \frac{1}{3}Y_{\bar{\gamma}[\alpha}\, Z_{\beta]}{}^{\bar{\gamma}} \tag{D.2}$$

and

$$V_{\alpha\beta\bar{\gamma}} = -\frac{1}{6}\nabla_{[\alpha|}Y_{\bar{\gamma}|\beta]} - \frac{1}{6}\nabla^{\bar{\nu}}Y_{\bar{\nu}[\alpha|}\, J_{\bar{\gamma}|\beta]} + \frac{4}{3}\nabla_{[\alpha|}Y_{\bar{\beta}|\beta]}{}^{\bar{\beta}}{}_{\bar{\gamma}} + \frac{2}{3}\bar{\nabla}_{\bar{\beta}}Z_{[\alpha|}{}^{\bar{\beta}}J_{\bar{\gamma}|\beta]}. \tag{D.3}$$

We can use the fact skew-symmetrising over three indices gives the zero map to show $Z_{\alpha\beta\gamma}{}^{\nu} + 2Z_{[\alpha|\gamma}\delta^{\nu}_{|\beta]} = 0$ while $-2\nabla_{[\alpha}Z_{\beta]\gamma} = 0$ thanks to 7.3.8.

The third row of the matrix is more complicated.

We will deal with $U_{\alpha\beta}$ first. 7.3.24 gives $\frac{1}{6}\nabla_{[\alpha|}\nabla^{\bar{\gamma}}Y_{\bar{\gamma}|\beta]} + \frac{1}{3}Y_{\bar{\gamma}[\alpha}\, Z_{\beta]}{}^{\bar{\gamma}} = 0$ which deals with the last two terms. Again using 7.3.8, it is true, by commuting derivatives, that

$$-\frac{2}{3}\nabla_{[\alpha|}\bar{\nabla}_{\bar{\beta}}Z_{|\beta]}{}^{\bar{\beta}} = \frac{4}{3}Y_{\bar{\beta}[\alpha\beta]}{}^{\nu}Z_{\nu}{}^{\bar{\beta}} + \frac{4}{3}Y_{\bar{\beta}[\alpha}{}^{\bar{\beta}\nu}Z_{\beta]\nu} = \frac{8}{3}Y_{\bar{\beta}[\alpha}{}^{\bar{\beta}\nu}Z_{\beta]\nu} \tag{D.4}$$

where the last equality again results from $\Lambda^3 V^* = 0$. So $U_{\alpha\beta} = 0$.

To deal with $V_{\alpha\beta}{}^{\nu}$ first note that the first two terms together vanish using the fact skew-symmetrising over three indices giving the zero map. To deal with the last two terms recall that we can, at a point p, assume $\phi_\nu|_p = (\nabla_\nu f)|_p$ for some smooth function f with $\bar{\nabla}_{\bar{\nu}}f = 0$ in a neighbourhood. We skew over three indices and substitute for ϕ_ν to calculate (in the fibre over p)

$$\frac{2}{3}\bar{\nabla}_{\bar{\beta}}Z_{[\alpha|}{}^{\bar{\beta}}\phi_{\beta]} = -\frac{1}{3}\bar{\nabla}_{\bar{\beta}}Z_{\alpha\beta}{}^{\bar{\beta}\gamma}\nabla_\gamma f = -\frac{1}{3}\bar{\nabla}^{\gamma}\nabla_{[\bar{\alpha}}\nabla_{\beta]}\nabla_\gamma f. \tag{D.5}$$

Commuting the first two derivatives yields

$$\frac{2}{3}\bar{\nabla}_{\bar{\beta}}Z_{[\alpha|}{}^{\bar{\beta}}\phi_{\beta]} = -\frac{2}{3}Y^{\gamma}{}_{[\alpha\beta]}{}^{\nu}\nabla_\nu\nabla_\gamma f - \frac{2}{3}Y^{\gamma}{}_{[\alpha|\gamma}{}^{\nu}\nabla_{|\beta]}\nabla_\nu f - \frac{1}{3}\nabla_{[\alpha}\bar{\nabla}^{\gamma}\nabla_{\beta]}\nabla_\gamma f. \tag{D.6}$$

Now

$$
\begin{aligned}
&-\frac{2}{3}Y^{\gamma}_{[\alpha\beta]}{}^{\nu}\nabla_\nu\nabla_\gamma f \\
&= -\frac{2}{3}Y^{\gamma}_{[\alpha\beta]}{}^{\nu}\nabla_\gamma\nabla_\nu f \\
&= \frac{2}{3}\nabla_\gamma Y^{\gamma}_{[\alpha\beta]}{}^{\nu}\nabla_\nu f - \frac{2}{3}\nabla_\gamma(Y^{\gamma}_{[\alpha\beta]}{}^{\nu}\nabla_\nu f) \\
&= \frac{1}{3}\nabla_\gamma\nabla_{[\alpha}\bar{\nabla}^\gamma\nabla_{\beta]}f + \frac{2}{3}\nabla_\gamma Y^{\gamma}_{[\alpha\beta]}{}^{\nu}\nabla_\nu f - \frac{1}{3}\nabla_\gamma\bar{\nabla}^\gamma\nabla_{[\alpha}\nabla_{\beta]}f \\
&= \frac{2}{3}\nabla_\gamma Y^{\gamma}_{[\alpha\beta]}{}^{\nu}\nabla_\nu f
\end{aligned}
\tag{D.7}
$$

where the first term vanished since $\bar{\nabla}_{\bar{\alpha}}\nabla_\beta f$ is totally trace by our assumption on f, and the last term vanished using the fact derivatives in the V direction commute on functions.

Similarly

$$
\begin{aligned}
&-\frac{2}{3}Y^{\gamma}_{[\alpha|\gamma}{}^{\nu}\nabla_{|\beta]}\nabla_\nu f \\
&= \frac{2}{3}\nabla_{[\beta}Y^{\gamma}_{\alpha]\gamma}{}^{\nu}\nabla_\nu f - \frac{1}{3}\nabla_{[\beta}\bar{\nabla}^\gamma\nabla_{\alpha]}\nabla_\gamma f - \frac{1}{3}\nabla_{[\beta}\nabla_{\alpha]}\bar{\nabla}^\gamma\nabla_\gamma f \\
&= \frac{2}{3}\nabla_{[\beta}Y^{\gamma}_{\alpha]\gamma}{}^{\nu}\nabla_\nu f - \frac{1}{3}\nabla_{[\beta}\bar{\nabla}^\gamma\nabla_{\alpha]}\nabla_\gamma f.
\end{aligned}
\tag{D.8}
$$

Combining D.6 with D.7 and D.8, and remembering that $\phi_\nu|_p = (\nabla_\nu f)|_p$, yields

$$
\frac{2}{3}\bar{\nabla}_{\bar{\beta}}Z_{[\alpha|}{}^{\bar{\beta}}\phi_{\beta]} = \frac{2}{3}\nabla_\gamma Y^{\gamma}_{[\alpha\beta]}{}^{\nu}\phi_\nu + \frac{2}{3}\nabla_{[\beta}Y^{\gamma}_{\alpha]\gamma}{}^{\nu}\phi_\nu = \frac{4}{3}\nabla_{[\beta}Y^{\gamma}_{\alpha]\gamma}{}^{\nu}\phi_\nu \tag{D.9}
$$

where again the last equality follows for indices in the rank-2 vector bundle V. Since p was an arbitrary point we get $V_{\alpha\beta}{}^{\nu} = 0$ everywhere and so the component of partial curvature $\mathbb{T} \to \Lambda^2 V^* \otimes \mathbb{T}$ also vanishes. □

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