ANALYTIC PONTRYAGIN DUALITY

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For my parents.

Declaration

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[Updated August 2019] An article based on the content of this thesis has been published [35] prior to submission. Certain results and proofs may be duplicated.

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Contents

Abstract

Let X be a smooth compact manifold. We propose a geometric model for the group $K^0(X, \mathbb{R}/\mathbb{Z})$. We study a well-defined and non-degenerate analytic duality pairing between $K^0(X, \mathbb{R}/\mathbb{Z})$ and its Pontryagin dual group, the Baum-Douglas geometric K-homology $K_0(X)$, whose pairing formula comprises of an analytic term involving the Dai-Zhang eta-invariant associated to a twisted Dirac-type operator and a topological term involving a differential form and some characteristic forms. This yields a robust $\mathbb{R}/\mathbb{Z}$-valued invariant. We also study two special cases of the analytic pairing of this form in the cohomology groups $H^1(X, \mathbb{R}/\mathbb{Z})$ and $H^2(X, \mathbb{R}/\mathbb{Z})$.

Introduction

The purpose of this thesis is to introduce an $\mathbb{R}/\mathbb{Z}$-valued invariant defined by an analytic duality pairing between the even K-theory with coefficients in $\mathbb{R}/\mathbb{Z}$ and the even Baum-Douglas geometric K-homology [13],

$$K^0(X, \mathbb{R}/\mathbb{Z}) \times K_0(X, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z},$$

for a smooth compact manifold X. It is commonly known as the *Pontryagin duality* pairing. By the Universal Coefficient Theorem in K-theory [53], there is a short exact sequence

$$0 \to \mathrm{Ext}(K_{-1}(X), \mathbb{R}/\mathbb{Z}) \to K^0(X, \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(K_0(X), \mathbb{R}/\mathbb{Z}) \to 0.$$

Since $\mathbb{R}/\mathbb{Z}$ is divisible, the vanishing of the Ext group implies a natural isomorphism $K^0(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(K_0(X), \mathbb{R}/\mathbb{Z})$. We formulate an *analytic* pairing implementing the isomorphism. By 'analytic', we mean that the pairing involves the eta-invariant associated to a Dirac-type operator twisted by some pullback bundle over a smooth compact manifold.

This is inspired by the work of Lott [36] on the $\mathbb{R}/\mathbb{Z}$ index theory. As motivated by Karoubi's model of K-theory with coefficients [31] and the index theorem for flat bundles of Atiyah-Patodi-Singer [7], Lott formulated an analytic K^1-pairing $K^1(X, \mathbb{R}/\mathbb{Z}) \times K_1(X)$ in terms of the eta-invariant of Atiyah-Patodi-Singer [5]. In the physical aspect, such a pairing has been observed by Maldacena-Seiberg-Moore [37] as describing the Aharonov-Bohm effect of D-branes in Type IIA String theory. An extended discussion of such a manifestation in String theory was given by Warren [51]. Beyond this, there are several studies related to the $\mathbb{R}/\mathbb{Z}$ K-theory from different points of view. For instance, Basu [11] provided a model via bundles of von Neumann algebras, according to the suggestion in [7, Section 5, Remark 4]; Antonini *et al* [1] gave a construction of $\mathbb{R}/\mathbb{Z}$ K-theory via an operator algebraic approach; on the other hand, the strategy of Deeley [23] is rather different in that he studied the pairing between the usual K-theory and the K-homology with $\mathbb{R}/\mathbb{Z}$ coefficients.

However, there is no known work, to the author's knowledge, on the *direct* analog of the analytic K^1-pairing of Lott in the *even* case. This thesis is aimed to fill in this gap. We construct a geometric model of the group $K^0(X, \mathbb{R}/\mathbb{Z})$, whose cocycle is a triple consisting of an element g of $K^1(X)$, a pair of flat connections $(d, g^{-1}dg)$ on a trivial bundle and an even degree differential form μ on X satisfying a certain exactness condition on the odd Chern character of g. Its pairing with an even geometric K-cycle (M, E, f) can then be explicitly described by an (reduced) even eta-type invariant of some twisted Dirac-type operator on a cylinder $M \times [0, 1]$, which appears as one of the boundary correction terms in the Dai-Zhang Toeplitz index theorem on manifolds with boundary [21], and a topological term, whose integrand is the wedge product of the pullback of μ and some characteristic forms on M. The resulting $\mathbb{R}/\mathbb{Z}$-valued invariant is robust in the sense that it is independent of the geometry of the underlying manifold and the bundle. We also show that such an analytic pairing is non-degenerate, and thus it is a valid implementation of the isomorphism $K^0(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\sim} \mathrm{Hom}(K_0(X), \mathbb{R}/\mathbb{Z})$. As an intermediate step, we consider the special case of n-spheres. This provides a non-trivial example of the pairing. In terms of application, we believe that the analytic pairing in K^0 describes the Aharonov-Bohm effect of D-branes in Type IIB String theory, as explained in Section 5.3.

Then, we study two non-trivial special cases of the analytic pairing in the $\mathbb{R}/\mathbb{Z}$-cohomology of degree one and two. In the case of H^2, by the pullback via a smooth map, we investigate the pairing on $H^2(S^2, \mathbb{R}/\mathbb{Z})$, elements of which are represented by pure Hermitian local line bundles introduced by Melrose [41]. A local line bundle is *projective* in that it is defined locally over a neighbourhood of the diagonal. Thus, the corresponding twisted Dirac operator is projective *ala* Mathai, Melrose and Singer [39, 40]. These are projective differential operators with kernels whose supports are contained in the diagonal of S^2. The caveat is that these operators do not have a spectrum and thus do not have a well-defined eta-invariant. We make several assumptions and define a variant of the Dai-Zhang eta-invariant for twisted projective Dirac operators in the special case of S^2. On the other hand, the analytic pairing in H^1 is less complicated. The pairing consists of the Atiyah-Patodi-Singer eta-invariant of the Dirac operator on S^1 twisted by a flat bundle and the holonomy of a flat connection over S^1. This can be viewed as a special case of the analytic K^1-pairing.

Organisation. We discuss the necessary preliminaries in Chapter 1. This includes an overview of fundamental objects and notions such as vector bundles, K-theory, characteristic classes, and Dirac operators. With these established, we propose a geometric model of $\mathbb{R}/\mathbb{Z}$ K^0-theory and study its properties in Chapter 2. A $\mathbb{R}/\mathbb{Q}$-valued Chern character map will then be formulated. In Chapter 3, we revisit the classical notion of the Baum-Douglas geometric K-homology. In particular, we study the homological Chern character on the level of K-cycles and show its well-definedness under K-homology relations.

In Chapter 4, we discuss four crucial index theorems separately: the Atiyah-Singer (even dimensional, closed manifold), the Atiyah-Patodi-Singer (even dimensional, manifolds with boundary), the classical Toeplitz (odd dimensional, closed manifold) and the Dai-Zhang Toeplitz (odd dimensional, manifolds with boundary). For the purpose of this thesis, the Dai-Zhang Toeplitz index theorem plays a vital role and . In particular, we study the even Dai-Zhang eta-invariant associated to a certain Dirac-type operator on a cylinder.

In Chapter 5, we establish the main result in this thesis. Using results in Chapters 2,3 and 4, we formulate an explicit analytic K^0-pairing in terms of the Dai-Zhang eta-invariant. We show its well-definedness under various relations and its non-degeneracy. For completeness, we also cover the discussion of the odd case. In Chapter 6, we study the two aforementioned non-trivial cases of the analytic pairing in cohomology theory.

Chapter 1

Preliminaries

1.1 Vector bundles and K-theory

In this chapter, we review the bundle-theoretic framework of topological K-theory in the sense of Atiyah-Hirzebruch [2]. For the sake of self-containedness, we lay out elementary facts about vector bundles, connections, curvature and so on, as well as fixing terminology used throughout this thesis. These can be found in standard references such as [2, 16, 18, 19, 29]. Readers who are familiar with these notions can skip to the next chapter directly.

1.1.1 Vector bundles, connections and curvature

Definition 1.1.1. Let X and E be paracompact topological spaces. Let $\pi : E \to X$ be a continous surjective map. A *complex vector bundle* of rank n is a locally trivial fibration whose fiber is an n-dimensional complex vector space such that for an open cover $\{U_\alpha\}_{\alpha \in I}$ of the base space X, two conditions are satisfied.

1. *Local trivialisation*: There is a homeomorphism

$$\phi_\alpha : E|_{U_\alpha} = \pi^{-1}(U_\alpha) \xrightarrow{\cong} U \times \mathbb{C}^n$$

   that fits into the commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times \mathbb{C}^n \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle \pi_1} \\ U_\alpha & \xrightarrow{\quad = \quad} & U_\alpha \end{array}$$

which restricts to a linear isomorphism of vector spaces for each $x \in X$

$$\phi_x : \pi^{-1}(x) \cong \{x\} \times \mathbb{C}^n.$$

2. *Cocycle condition*: Over the double overlap $U_{\alpha\beta} = U_\alpha \cap U_\beta$, we have

$$U_{\alpha\beta} \times \mathbb{C}^n \xrightarrow{\phi_\beta^{-1}} E_{U_\beta}\big|_{U_{\alpha\beta}} = E_{U_\alpha}\big|_{U_{\alpha\beta}} \xrightarrow{\phi_\alpha} U_{\alpha\beta} \times \mathbb{C}^n$$

for all $\alpha, \beta \in I$. These are vector space automorphisms of $\mathbb{C}^n$ in each fiber.

Definition 1.1.2. Define the *transition functions* $g_{\alpha\beta} : U_{\alpha\beta} \to GL(n, \mathbb{C})$ by

$$g_{\alpha\beta}(x) = \phi_\alpha \circ \phi_\beta^{-1}\big|_{\{x\}\times\mathbb{C}^n}, \tag{1.1.1}$$

whose inverse over $U_{\alpha\beta}$ is $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and these satisfy the *cocycle* condition

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \quad \text{on } U_{\alpha\beta\gamma}. \tag{1.1.2}$$

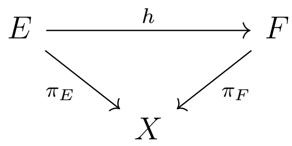
Definition 1.1.3. Let $\pi : E \to X$ be a complex vector bundle. A local section of E is a smooth map $s_\alpha : U_\alpha \to E|_{U_\alpha}$ such that $\pi \circ s = \mathrm{Id}_{U_\alpha}$. The space of all smooth sections of E is denoted by $\Gamma(E) = C^\infty(E)$.

Choose a good cover $\{U_\alpha\}$ of X so that there are non-vanishing local sections $s_\alpha : U_\alpha \to E|_{U_\alpha}$, together with local trivialisations $\phi_\alpha : E|_{U_\alpha} \cong U_\alpha \times \mathbb{C}^n$. Then, the composition $\phi_\alpha \circ s_\alpha : U_\alpha \to U_\alpha \times \mathbb{C}^n$ is equivalent to a map $U_\alpha \to \mathbb{C}^n$. We write s_α instead of $\phi_\alpha \circ s_\alpha$ to *implicitly* imply that a choice of local trivialisation is made. Then, equation (1.1.1) can be rewritten as

$$s_\alpha(x) = g_{\alpha\beta}(x) s_\beta(x) \tag{1.1.3}$$

for all $x \in U_{\alpha\beta}$. Note that, in general there is no *globally* non-vanishing section, i.e. there is no non–vanishing section s_α for *all* U_α. In fact, a bundle is trivial if and only if there exists such a globally non-vanishing section. One can however define a globally well-defined section by means of partition of unity subordinate to the open cover U_α which extends to X.

Definition 1.1.4. Let E, F be complex vector bundles over X. Then, E is said to be isomorphic to F, denoted by $E \cong F$, if there exists a bundle isomorphism $h : E \to F$ such that the following diagram



commutes.

Example 1.1.1. Let X be a smooth compact manifold.

1. We call $X \times \mathbb{C}^n \to X$ the complex trivial bundle of rank n. It is a globally trivial bundle over X with the obvious projection $(x, z) \mapsto x$.

2. A complex line bundle $L \to X$ is a non-trivial bundle of complex dimension one. Locally, the transition functions are given by $g_{\alpha\beta} : U_{\alpha\beta} \to U(1)$ which satisfy (1.1.2). The obstruction of L being trivial is encoded by the *first Chern class* $c_1(L) \in H^2(X, \mathbb{Z})$. In fact, there is an one-to-one correspondence between the isomorphism classes of complex line bundles over X and the isomorphism classes of principal $U(1)$-bundles over X.

3. Since X is smooth, the tangent bundle $TX \to X$ arises naturally, whose fiber at each $x \in X$ is the space T_xX of vector fields at x. If X is a Riemannian manifold with a given Riemannian metric, then there is a natural isomorphism between TX and its 'dual' the cotangent bundle T^*X.

Definition 1.1.5. Let $\bigwedge^* T^*X$ be the exterior algebra bundle of T^*X. For $0 \leq p \leq \dim(X)$, denote by $\Omega^p(X) := \Gamma(\bigwedge^p T^*X)$ the space of all smooth sections of $\bigwedge^* T^*X$. Equivalently, this is the space of all smooth differential p-forms on X. Let E be a complex vector bundle over X. Denote by $\Omega^*(X, E) = \Gamma(\bigwedge^p T^*X \otimes E)$ the space of all smooth differential forms with values in E.

Definition 1.1.6 (Operations on vector bundles)**.** Let E and F be complex vector bundles over X.

1. **Direct sum.** Define $E \oplus F$ to be the fiberwise direct sum of E and F, i.e. $(E \oplus F)_x := E_x \oplus F_x$, for $x \in X$. The space of sections $\Gamma(E \oplus F)$ coincides with $\Gamma(E) \oplus \Gamma(F)$.
2. **Dual.** Define E^* to be the fiberwise dual of E, i.e. $(E^*)_x := (E_x)^*$, for $x \in X$.
3. **Homomorphism.** Define $\mathrm{Hom}(E, F)$ to be the fibrewise homomorphism bundle, i.e. $\mathrm{Hom}(E, F)_x := \mathrm{Hom}(E_x, F_x)$, for $x \in X$.
4. **Tensor product.** Define $E \otimes F$ to be the fibrewise tensor product $E_x \otimes F_x$, for $x \in X$. Alternatively, $E \otimes F$ can be viewed as $\mathrm{Hom}(E^*, F)$, i.e. for $v \in E$ and $u \in F$, the tensor product $v \otimes u$ is a linear map $E^* \to F; \xi \mapsto \xi(v)u$.
5. **Pullback.** Let F be a complex vector bundle over a smooth compact manifold Y. Let $f : X \to Y$ be a smooth map. Define the pullback bundle $f^*F \to X$ by $\{(a, x) \in F \times X \mid f(x) = \pi(a)\}$, whose fiber is $(f^*F)_x := F_{f(x)}$ for $x \in X$.

1.1.2 Connections and Curvature

Definition 1.1.7. Let E be a complex vector bundle over a smooth compact manifold X. A connection ∇^E on E is a $\mathbb{R}$-linear first order differential operator

$$\nabla^E : \Gamma(E) \longrightarrow \Omega^1(M, E)$$

satisfying the Leibniz rule

$$\nabla^E(fs) = df \otimes s + f\nabla^E s \tag{1.1.4}$$

for $f \in C^\infty(X)$ and $s \in \Gamma(E)$.

Proposition 1.1.8. *The space $\mathscr{A}$ of all connections is an infinite dimensional affine space modelled on $\Omega^1(X, End(E))$ where $End(E)$ is the endomorphism bundle over X.*

Remark 1.1.9. Connections exist on every vector bundle and their existence can be shown by means of partition of unity. Moreover, a connection extends canonically to a differential operator $\nabla^E : \Omega^p(X,E) \longrightarrow \Omega^{p+1}(X,E)$ such that

$$\nabla^E(\omega s) = (d\omega)s + (-1)^{\deg \omega}\omega \wedge \nabla^E s$$

for any $\omega \in \Omega^p(X)$ and $s \in \Gamma(E)$.

Remark 1.1.10. Let $E \to X$ be a complex vector bundle with a connection ∇. Let $\{U_\alpha\}$ be a good cover of X. Locally over U_α, a local 1-form A_α is given by $\nabla s_\alpha = A_\alpha \otimes s_\alpha$, where s_α is a non-vanishing local section. Then, from (1.1.3), two local 1-forms A_α and A_β over the overlap $U_{\alpha\beta}$ are related by

$$A_\alpha = g_{\alpha\beta}^{-1} A_\beta g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}. \tag{1.1.5}$$

Let $f_\alpha \in C^\infty(U_\alpha)$. By the Leibniz rule,

$$\begin{aligned}\nabla_\alpha(f_\alpha s_\alpha) &= df_\alpha \otimes s_\alpha + f_\alpha(A_\alpha \otimes s_\alpha)\\ &= (d + A_\alpha)(f_\alpha s_\alpha).\end{aligned}$$

Then, a connection ∇ has the local form $\nabla_\alpha = \nabla|_{U_\alpha} = d + A_\alpha$.

Definition 1.1.11. The curvature F_∇ of a connection ∇ is a map $F_\nabla : \Gamma(E) \to \Omega^2(X,E)$ defined by $F_\nabla = \nabla^2 = \nabla \circ \nabla$.

By a direct computation we verify that the curvature F_∇ is $C^\infty(X)$-linear, i.e. $F_\nabla(fs) = fF_\nabla s$ for any $f \in C^\infty(X)$ and $s \in \Gamma(E)$, but a connection is *not*, by (1.1.4). Hence, the curvature F_∇ can be viewed as a section of $\mathrm{End}(E)$ with coefficients in $\Omega^2(X)$, i.e. $F_\nabla \in \Omega^2(X, \mathrm{End}(E))$.

Example 1.1.2. Let $L \to X$ be a complex line bundle. A transition function takes the form $g_{\alpha\beta} : U_{\alpha\beta} \to U(1)$ and the relation (1.1.5) reduces to $A_\alpha = A_\beta + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$. Then, over $U_{\alpha\beta}$, we have

$$dA_\alpha = dA_\beta + d(g_{\alpha\beta}^{-1} dg_{\alpha\beta}) = dA_\beta - g_{\alpha\beta}^{-1} dg_{\alpha\beta} g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} d^2 g_{\alpha\beta} = dA_\beta.$$

Since $U_{\alpha\beta}$ is arbitrary, this defines a global 2-form over X, which is exactly the curvature F_∇ of the connection ∇ on the line bundle L.

Definition 1.1.12. Let E and E' be complex vector bundles over X, equipped with connections ∇^E and $\nabla^{E'}$ respectively. Let $E \otimes E'$ be the tensor product bundle. Then,

$$\nabla^{E\otimes E'} = 1 \otimes \nabla^{E'} + \nabla^E \otimes 1 \tag{1.1.6}$$

$$F_{\nabla^{E\otimes E'}} = 1 \otimes F_{\nabla^{E'}} + F_{\nabla^E} \otimes 1. \tag{1.1.7}$$

Definition 1.1.13. A connection ∇ is *flat* if its curvature vanishes, i.e. $F_\nabla = 0$.

1.1.3 Characteristic classes

By (1.1.5) and a direct computation one can verify that the local curvature matrix F_α satisfies the relation

$$F_\alpha = g_{\alpha\beta}^{-1} F_\beta g_{\alpha\beta}. \tag{1.1.8}$$

Equation (1.1.8) tells us that F_α does not depend on the local frames up to conjugation. The trace of F_α is then given by

$$\mathrm{Tr}(F_\alpha) = \sum_i (F_\alpha)_{ii} \in \Omega^2(U_\alpha).$$

It is invariant under conjugation, i.e. by the multiplicative property of trace,

$$\mathrm{Tr}(F_\alpha) = \mathrm{Tr}(g_\alpha^{-1} F_\beta g_{\alpha\beta}) = \mathrm{Tr}(F_\beta).$$

By gluing along all overlaps $U_{\alpha\beta}$, we obtain a globally defined 2-form $\mathrm{Tr}(F)$. Given a bundle E with a connection ∇ and its curvature F_∇, the trace map becomes

$$\mathrm{Tr} : \Omega^2(X, \mathrm{End}(E)) \to \Omega^2(X)$$

with an "evaluation" map $\mathrm{End}(E) \to \mathbb{C}$.

Chern class. The total Chern form of E associated to ∇ is defined by

$$c(E, \nabla) = \det\Big(\mathrm{Id} + \frac{i}{2\pi} F_\nabla\Big). \tag{1.1.9}$$

Here, Id is the identity endomorphism of E. It has the total decomposition

$$c(E,\nabla) = 1 + c_1(E,\nabla) + \cdots + c_k(E,\nabla) + \cdots \tag{1.1.10}$$

where $c_k(E,\nabla) \in \Omega^{2k}(X)$ is the k-th Chern form. By the Chern-Weil theorem 1.1.15, all of these k-forms are closed, so they represent the k-th Chern class $c_k(E)$ in $H^{2k}(X,\mathbb{Z})$. The total Chern class $c(E) = \sum_k c_k(E)$ is a finite sum and thus is well-defined since $c_k(E) = 0$ for $k > \dim(X)$.

On the other hand, we can rewrite the RHS of (1.1.9) as

$$\det\Big(\mathrm{Id} + \frac{i}{2\pi}F\Big) = \exp\Big(\mathrm{Tr}\Big(\log\Big(\mathrm{Id} + \frac{i}{2\pi}F\Big)\Big)\Big). \tag{1.1.11}$$

By the Taylor expansion of log, we have

$$\log\Big(\mathrm{Id} + \frac{i}{2\pi}F\Big) = \frac{i}{2\pi}F - \frac{1}{2(2\pi)^2}F^2 + \cdots . \tag{1.1.12}$$

Then,

$$\mathrm{Tr}\Big(\log\big(\mathrm{Id} + \frac{i}{2\pi}F\big)\Big) = \frac{i}{2\pi}\mathrm{Tr}(F) - \frac{1}{2(2\pi)^2}\mathrm{Tr}(F)^2 + \cdots . \tag{1.1.13}$$

By substituting (1.1.13) into (1.1.11), we obtain

$$\exp\Big(\mathrm{Tr}\Big(\log(\mathrm{Id} + \frac{i}{2\pi}F)\Big)\Big) = 1 + \frac{i}{2\pi}\mathrm{Tr}(F) - \frac{1}{(2\pi)^2}\mathrm{Tr}(F)^2 + \cdots . \tag{1.1.14}$$

Definition 1.1.14. From (1.1.10) and (1.1.14), the *first Chern class* of E is given by $c_1(E) := \Big[\frac{i}{2\pi}\mathrm{Tr}(F)\Big] \in H^2(X,\mathbb{Z})$.

Example 1.1.3. The first Chern class of a line bundle L is $c_1(L) = \frac{i}{2\pi}F$, where $\mathrm{End}(L) = L^* \otimes L$ is trivial and the trace Tr is the identity map on F. Hence, line bundles over X are fully classified by $c_1 \in H^2(X,\mathbb{Z})$. In particular, L is trivial if and only if $c_1(L) = 0$.

Example 1.1.4. Assume that a complex vector bundle E with connection ∇ has a Hermitian metric, so that its structure group $GL(n,\mathbb{C})$ can be reduced to the unitary group $U(n)$. If it can be further reduced to $SU(n)$, then its curvature is a $\mathfrak{su}(n)$-valued 2-form. Hence, $\mathrm{Tr}(F_\nabla) = 0$ and $c_1(E) = 0$.

Let $f(x) = a_0 + a_1 x + \cdots a_n x^n + \cdots$ be a power series in one variable. By replacing x with the curvature F_∇ of a connection ∇ on E, the polynomial $f(F_\nabla)$ is a sum of even forms, which is again an even form in $\Omega^{2\mathrm{k}}(X)$.

Theorem 1.1.15 (Chern-Weil theorem)**.** *The trace $Tr(f(F_\nabla))$ is a closed form, i.e. $dTr(f(F_\nabla)) = 0$. If ∇' is another connection on E with curvature $F_{\nabla'}$, then there exists a differential form $\omega \in \Omega^*(X)$ such that*

$$Tr(f(F_\nabla)) - Tr(f(F_{\nabla'})) = d\omega.$$

Pontryagin class. Let V be a real vector bundle. The total Pontryagin form associated to ∇^V is given by

$$\begin{aligned} p(V, \nabla^V) &= \det\left(\left(\mathrm{Id} - \left(\frac{F_\nabla}{2\pi}\right)^2\right)^{\frac{1}{2}}\right) \\ &= 1 + p_1(V, \nabla^V) + \cdots + p_k(V, \nabla^V) + \cdots \end{aligned}$$

where $p_j(V, \nabla^V) \in \Omega^{4j}(X)$. The sum is well-defined since $p_j(V, \nabla^V) = 0$ when its degree is larger than the dimension of X. Let $V \otimes \mathbb{C}$ be the complexified bundle, which is now a complex vector bundle, then its relation with c_k is given by

$$p_k(V) = (-1)^k c_{2k}(V \otimes \mathbb{C}).$$

The associated characteristic class is the Pontryagin class $p(V) \in H^{4k}(X, \mathbb{Z})$.

$\hat{A}$-class. It turns out that the Pontryagin class is also deeply related to the A-hat class $\hat{A}(X)$ of a smooth compact oriented manifold X. The A-hat function, in terms of polynomials, is given by the formula

$$\hat{A}(x) = \frac{x/2}{\sinh(x/2)}.$$

Let TX be the tangent bundle of X with a connection ∇^{TX} and $F_{\nabla^{TX}}$ its curvature. Then, $\hat{A}(TX, \nabla^{TX})$ is explicitly given by

$$\hat{A}(TX, \nabla^{TX}) = \det\left(\left(\frac{\frac{i}{4\pi} F_{\nabla^{TX}}}{\sinh(\frac{i}{4\pi} F_{\nabla^{TX}})}\right)^{\frac{1}{2}}\right) \in \Omega^*(X).$$

Let $\hat{A}(TX)$ be its associated cohomology class. In terms of Pontryagin classes,

$$\hat{A}(TX) = 1 - \frac{1}{24}p_1(TX) + \frac{7p_1^2(TX) - 4p_2(TX)}{5760} + \cdots .$$

Let $[X]$ be the fundamental class of X, then the A-hat genus of X is defined by the pairing

$$\hat{A}(X) = \langle \hat{A}(TX), [X] \rangle = \int_X \hat{A}(TX, \nabla^{TX}). \tag{1.1.15}$$

Todd class. Consider the formal power series and the product of r factors

$$td(t) = \frac{t}{1 - e^{-t}} = 1 + \frac{t}{2} + \frac{t^2}{12} - \frac{t^4}{120} + \cdots$$

$$\begin{aligned} \mathrm{Td}(t_1, \ldots, t_r) &= \prod_{i=1}^{r} td(t_i) = 1 + td_1(t_1) + td_2(t_1, t_2) + \cdots \\ &= 1 + \mathrm{Td}_1(x_1) + \mathrm{Td}_2(x_1, x_2) + \cdots . \end{aligned}$$

Here, each x_i is the elementary symmetric polynomial in t_i. Let $x_i = c_1(\ell_i)$ where ℓ_i is the line bundle from the decomposition $E = \oplus \ell_i$ by the Splitting Principle. Then, the first few terms of $\mathrm{Td}_n(x_1, \ldots, x_n)$ are

$$\mathrm{Td}_1(c_1) = \frac{1}{2}c_1, \quad \mathrm{Td}_2(c_1, c_2) = \frac{1}{12}(c_2 + c_1^2), \quad \mathrm{Td}_3(c_1, c_2, c_3) = \frac{1}{24}(c_2 c_1).$$

Now, when X is a smooth compact spin^c manifold with $E = TX$, the Todd genus is the rational number

$$\mathrm{Td}(X) = \langle \mathrm{Td}_k(TX), [X] \rangle \tag{1.1.16}$$

where $[X] \in H_{2k}(X, \mathbb{Q})$ is the fundamental class of X. The Todd forms are related to the A-hat forms by

$$\mathrm{Td}(TX, \nabla^{TX}) = e^{\frac{c_1(L, \nabla^L)}{2}} \wedge \hat{A}(TX, \nabla^{TX}) \tag{1.1.17}$$

where L is the line bundle associated to the spin^c structure. The Todd forms

(1.1.17) are closed and represent Todd classes

$$\mathrm{Td}(TX) = [\mathrm{Td}(TX, \nabla^{TX})] \in H^{2k}(X, \mathbb{Q}). \tag{1.1.18}$$

Theorem 1.1.16 ([3])**.** *Let X be a compact spin manifold of dimension $4k$. Then $\hat{A}(X)$ is an integer. If dim$(X) \equiv 4(mod\ 8)$, then $\hat{A}(X)$ is an even integer.*

Remark 1.1.17. When X is a compact manifold which is not spin, the $\hat{A}$-genus is often not an integer. This leads to the study of projective analytic indices of projective differential operators, cf. [38–40].

Theorem 1.1.18 ([3])**.** *Let X be a compact spin^c manifold (i.e. an orientable manifold with $c \equiv w_2(X) mod\ 2$ for $c \in H^2(X, \mathbb{Z})$). Then, Td(X) is an integer.*

1.1.4 Topological K-theory

By Definition 1.1.4, we denote by $[E]$ the isomorphism class of a complex vector bundle E. Let $[E]$ and $[F]$ be two isomorphism classes of vector bundles. The addition is given by the direct sum of the underlying representatives

$$[E] + [F] := [E \oplus F].$$

The formal inverse of $[E]$ is $-[E]$, which is the isomorphism class $[\ominus E]$ of the formal inverse of E. Then, the formal difference is $[E] - [F] = [E \ominus F]$.

Definition 1.1.19. Let X be a smooth compact manifold. Let $\mathrm{Vect}_{\mathbb{C}}(X)$ be the monoid of all isomorphism classes of complex vector bundles over X. The K-theory of X, denoted by $K^0(X)$, is defined as the group given by the Grothendieck completion of $\mathrm{Vect}_{\mathbb{C}}(X)$. Every element in $K^0(X)$ can be written as $[E] - [F]$. The stabilization relation is $[E] - [F] = [E'] - [F']$ if and only if there exists a bundle G over X such that $E \oplus F' \oplus G \cong E' \oplus F \oplus G$.

The K^0-group is abelian. It forms a commutative ring if we take

$$([E] - [F]) \cdot ([E'] - [F']) = [E \otimes E'] + [F \otimes F'] - [E \otimes F'] - [F \otimes E'].$$

Since every bundle E has a complement E^c such that $E \oplus E^c \cong X \times \mathbb{C}^n$, the sum $[E]+[E^c]$ defines a trivial class $1=[X\times\mathbb{C}^n]\in K^0(X)$. Alternatively, every element $[E]-[F]$ can be rewritten as $[H]-1\in K^0(X)$ for some bundle H.

Example 1.1.5.
- Let $X=\mathrm{pt}$. Then, $K^0(X)=K^0(\mathrm{pt})=\mathbb{Z}$. Informally, an associated integer indicates the rank of a trivial bundle over a point.
- Let $X=S^1$. Then, $K^0(S^1)=\mathbb{Z}$, also given by the rank of a trivial bundle, since every complex vector bundle over S^1 is necessarily trivial.
- Let $X=S^2$. Then, $K^0(S^2)=\mathbb{Z}\oplus\mathbb{Z}$, in which the first factor is generated by trivial bundles and the second factor is generated by the Bott bundle $\beta_0=\mathcal{L}_0-1$, where $\mathcal{L}_0$ is the canonical Hopf line bundle over $S^2\cong\mathbb{C}P^1$.
- Let $L_k=L(3;k,1)=S^3/\mathbb{Z}_k$ be a three dimensional Lens space. By [31, pp. IV, 1.14], L_k can be identified with the sphere bundle of the k-tensor product of $\beta_0\to S^2$, i.e. $L_k\cong S(\beta_0^{\otimes k})$. Let $\pi:L_k\to S^2$ be the projection. Then, $\pi^*\beta_0^*\to L_k$, the pullback of the dual of β_0, is a non-trivial bundle and it generates a class in $K^0(L_k)$.

Proposition 1.1.20. *K-theory is contravariant. Let $f:X\to Y$ be a continuous map of smooth manifolds. Let E be a complex vector bundle over Y with $[E]\in K^0(Y)$. Then, the pullback bundle $f^*F\to X$ represents the class $f^*[E]:=[f^*E]\in K^0(X)$. This defines a map $f^*:K^0(Y)\to K^0(X)$.*

Lemma 1.1.21 (Homotopy invariant)**.** *For $t\in[0,1]$, let $\varphi(t)$ be a homotopy between continuous maps $f_0,f_1:X\to Y$ such that $\varphi(0)=f_0$ and $\varphi(1)=f_1$. Then, f_0 and f_1 induces the same map $f_0^*=f_1^*:K^0(Y)\to K^0(X)$.*

Definition 1.1.22. The reduced K-theory $\widetilde{K}^0(X)$ of X is defined by the kernel of the map ι^* in the following exact sequence

$$0\to\widetilde{K}^0(X)\to K^0(X)\xrightarrow{\iota^*}K^0(\mathrm{pt})\to 0, \tag{1.1.19}$$

where $\iota:\mathrm{pt}\to X$ denotes the inclusion map. That is,

$$\widetilde{K}^0(X)\cong\ker\{K^0(X)\longrightarrow K^0(\mathrm{pt})\cong\mathbb{Z}\}.$$

So, $K^0(X) \cong \mathbb{Z} \oplus \widetilde{K}^0(X)$. On the other hand, if X is locally compact, the K-theory $K^0(X)$ can be defined by

$$K^0(X) = \widetilde{K}^0(X^+) \tag{1.1.20}$$

where $X^+ = X \sqcup \{\text{pt}\}$ denotes the one-point compactification of X.

Example 1.1.6. Let $S^n = \mathbb{R}^n \sqcup \{\text{pt}\}$ be the n-sphere obtained from the one-point compactification of $\mathbb{R}^n$. That is, there is a short exact sequence

$$0 \to \text{pt} \xrightarrow{\iota} S^n \to \mathbb{R}^n \to 0.$$

The induced short exact sequence of K-theory is

$$0 \to K^0(\mathbb{R}^n) \to K^0(S^n) \xrightarrow{\iota^*} K^0(\text{pt}) \to 0. \tag{1.1.21}$$

By Example 1.1.5, (1.1.19) and (1.1.21), there is an isomorphism $K^0(\mathbb{R}^n) \cong \widetilde{K}^0(S^n)$. This verifies (1.1.20) since $\mathbb{R}^n$ is locally compact.

Definition 1.1.23. Define the higher K-groups $K^{-n}(X)$ $(n \in \mathbb{N})$ by

$$K^{-n}(X) := K^0(X \times \mathbb{R}^n).$$

Theorem 1.1.24 (Bott Periodicity)**.** *For any locally compact space X, there is a natural isomorphism $K^*(X) \to K^*(X \times \mathbb{R}^2)$.*

Bott Periodicity is one of the fundamental results in K-theory. The proof can be found in any standard references on K-theory cf. [2],[29],[31]. We illustrate an example of this instead.

Example 1.1.7. Recall from Example 1.1.5 and (1.1.19) that $\widetilde{K}^0(S^2) \cong \mathbb{Z}$ is generated by the Bott bundle β. Since β has virtual dimension 0, it defines a generator of $K^0_c(\mathbb{R}^2)$, which coincides with the relative K-theory $K^0(S^2, \{\infty\})$. Then, the map $K^0(X) \to K^0(X \times \mathbb{R}^2)$ is given by the external product with β, i.e. $E \mapsto E \boxtimes \beta$. In the special case when $X = \text{pt}$ and $E = \tau$ is a trivial bundle, this map is an isomorphism. It can then be shown (the hard part) that this holds for more general spaces.

1.1.5 In relation to cohomology theory

Even Chern character. Let E be a complex vector bundle over a smooth compact manifold X. Assume E is equipped with a Hermitian metric h^E and a compatible Hermitian connection ∇^E. Define the Chern character of E by

$$ch(E) := [ch(E, \nabla^E)] = [\mathrm{Tr}(e^{F^2_\nabla})] \in H^*(X, \mathbb{Q}). \tag{1.1.22}$$

Let $ch(x_1, \ldots, x_n) = \sum_{i=1}^n e^{t_i}$ be a symmetric polynomial in $t_1, \ldots, t_n$ in each degree. It can be expressed as a polynomial in the elementary symmetric functions $x_1, \ldots, x_n$. In particular,

$$ch(x_1, \ldots, x_n) = n + ch_1(x_1, \ldots, x_n) + ch_2(x_1, \ldots, x_n) + \cdots \tag{1.1.23}$$

whose k-th term is given by

$$ch_k(x_1, \ldots, x_n) = \sum_{i=1}^n \frac{t_i^k}{k!}.$$

It is readily checked that the first two terms are respectively

$$\begin{aligned} ch(x_1, \ldots, x_n) &= \sum t_i = x_1, \\ ch_2(x_1, \ldots, x_n) &= \sum \frac{t_1^2}{2} = \frac{(\sum t_i)^2 - 2\sum_{i<j} t_i t_j}{2} = \frac{x_1^2 - x_2}{2}. \end{aligned}$$

Define the Chern character of E by

$$ch(E, \nabla) := ch(c_1(E, \nabla), \ldots, c_n(E, \nabla))$$

where $c_i(E, \nabla)$ is the i-th Chern class of E. Then, (1.1.23) can be rewritten as

$$ch(E) = rk(E) + c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E)) + \text{higher order terms.} \tag{1.1.24}$$

Let E and F be complex vector bundles over X. Let $E \oplus F$ and $E \otimes F$ be the corresponding direct sum and tensor product bundles respectively.

Lemma 1.1.25. *The map ch respects the direct sum and the tensor product of* *vector bundles*

$$ch(E \oplus F) = ch(E) + ch(E), \quad ch(E \otimes F) = ch(E) \wedge ch(F)$$

where $\wedge$ denotes the wedge product on the level of forms and cup product on the level of characteristic classes. In particular, $ch(E^*) = -ch(E)$.

By Lemma 1.1.25, the even Chern character map

$$ch : K^0(X) \to H^{\text{even}}(X, \mathbb{Q})$$

is a ring homomorphism. By [4], it is a *rational* isomorphism

$$ch : K^0(X) \otimes \mathbb{Q} \xrightarrow{\sim} H^{\text{even}}(X, \mathbb{Q}).$$

Lemma 1.1.26 (Naturality)**.** *Let $f : X \to Y$ be a smooth map. Let E be a complex vector bundle over X. Then, the map ch respects the pullback of E via f, i.e. $ch(f^*E) = f^*ch(E)$.*

Example 1.1.8.

- Let τ be the trivial bundle over X. Then, $ch(\tau) = \mathrm{rk}(\tau)$ since $c_i(\tau, d) = 0$ for all $i \geq 1$.

- Let $\beta_0 \to S^2$ be the Bott bundle as in Example 1.1.5. Then,

$$ch(\beta_0) = ch(\mathcal{L}_0 - 1) = c_1(\mathcal{L}_0) \in H^2(S^2, \mathbb{Z}).$$

- Let $\pi^*\beta_0 \to L_k$ be the non-trivial line bundle over a Lens space as in Example 1.1.5. Then, by Lemma 1.1.26 and Lemma 1.1.25,

$$ch(\pi^*\beta_0) = -\pi^* ch(\beta_0) \in H^2(L_k, \mathbb{Z}) \cong \mathbb{Z}_k.$$

Chern-Simons form. Let E be a complex vector bundle over X. Let ∇_1, ∇_2 be two connections on E. For $t \in [0,1]$, consider a path of connections ∇_t on E defined by

$$\nabla_t = (1-t)\nabla_1 + t\nabla_2.$$

It is clear that

$$\frac{d\nabla_t}{dt} = \nabla_2 - \nabla_1 \in \Omega^1(X, \mathrm{End}(E)). \tag{1.1.25}$$

Then, one computes

$$\frac{d}{dt}\mathrm{Tr}(e^{F_{\nabla_t}}) = \mathrm{Tr}\Big(\frac{d(\nabla_t)^2}{dt}e^{F_{\nabla_t}}\Big) = \mathrm{Tr}\Big(\Big[\nabla_t, \frac{d\nabla_t}{dt}\Big]e^{F_{\nabla_t}}\Big) = \mathrm{Tr}\Big(\Big[\nabla_t, \frac{d\nabla_t}{dt}e^{F_{\nabla_t}}\Big]\Big). \tag{1.1.26}$$

By (1.1.25), it is a basic fact that

$$d\mathrm{Tr}\Big(\frac{d\nabla_t}{dt}e^{F_{\nabla_t}}\Big) = \mathrm{Tr}\Big(\Big[\nabla_t, \frac{d\nabla_t}{dt}e^{F_{\nabla_t}}\Big]\Big).$$

It is immediate from (1.1.26) that

$$d\mathrm{Tr}\Big(\frac{d\nabla_t}{dt}e^{F_{\nabla_t}}\Big) = \frac{d}{dt}\mathrm{Tr}(e^{F_{\nabla_t}}).$$

Upon integrating over $t \in [0,1]$, we obtain the *transgression form*

$$\mathrm{Tr}(e^{F_{\nabla_2}}) - \mathrm{Tr}(e^{F_{\nabla_1}}) = d\int_0^1 \mathrm{Tr}\Big(\frac{d\nabla_t}{dt}e^{F_{\nabla_t}}\Big)dt \tag{1.1.27}$$

of Chern character.

Definition 1.1.27. The term on the RHS of (1.1.27) is the *Chern-Simons form*

$$CS(\nabla_1, \nabla_2) := \int_0^1 \mathrm{Tr}\Big(\frac{d\nabla_t}{dt}e^{F_{\nabla_t}}\Big)dt \tag{1.1.28}$$

satisfying the property

$$dCS(\nabla_1, \nabla_2) = ch(\nabla_2) - ch(\nabla_1). \tag{1.1.29}$$

Let $\nabla_1, \nabla_2, \nabla_3$ be three connections on E, then the Chern-Simons forms satisfy the relation

$$CS(\nabla_1, \nabla_2) + CS(\nabla_2, \nabla_3) = CS(\nabla_1, \nabla_3).$$

Definition 1.1.28. Given a short exact sequence of complex vector bundles

$0 \to E_1 \to E_2 \to E_3 \to 0$. Choose a splitting map $j : E_3 \to E_2$. Let $i : E_1 \to E_2$ be an inclusion map such that the pullback $(i \oplus j)^* E_2 \cong E_1 \oplus E_3$. Define

$$CS(\nabla_1^E, \nabla_2^E, \nabla_3^E) := CS((i \oplus j)^* \nabla_2^E, \nabla_1^E \oplus \nabla_3^E). \tag{1.1.30}$$

It is independent of the choice of splitting j. Then, relation (1.1.29) becomes

$$dCS(\nabla_1^E, \nabla_2^E, \nabla_3^E) = ch(\nabla_2^E) - ch(\nabla_1^E) - ch(\nabla_3^E). \tag{1.1.31}$$

Odd Chern Character. Let X be a smooth closed manifold. An element of $K^1(X)$ can be represented by a differentiable map $g : X \to GL(N, \mathbb{C})$ for a sufficiently large positive integer N. Let $\tau = X \times \mathbb{C}^N$ be the trivial complex vector bundle of rank N over X, upon which g acts on it as an automorphism i.e. $g \in \Gamma(\mathrm{Aut}(\tau))$. Suppose τ is endowed with a Hermitian metric. Without loss of generality, we assume that its structure group $GL(N, \mathbb{C})$ is reduced to the unitary group $U(N) = U(N, \mathbb{C})$ via the metric. Thus, equivalently we write

$$g : X \to U(N), \quad [g] \in K^1(X). \tag{1.1.32}$$

Let $\nabla_t = d + t\omega$ be a path of connections on τ, connecting the trivial connection d to $\omega = g^{-1}dg \in \Omega^1(X, \mathrm{End}(\tau))$. In particular, $\nabla_1 = d + g^{-1}dg$ is gauge equivalent to d. By (1.1.29), $CS(d, d + g^{-1}dg)$ is a closed form of odd degree.

Definition 1.1.29 ([26])**.** Let $g : X \to U(N)$ be a K^1-representative. Define the odd Chern character of g by

$$ch(g, d) = CS(d, d + g^{-1}dg). \tag{1.1.33}$$

Explicitly, it is given by the formula

$$ch(g, d) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \mathrm{Tr}\big((g^{-1}dg)^{2k+1}\big). \tag{1.1.34}$$

Lemma 1.1.30. *Let $[g]$ be the homotopy class represented by g. The cohomology class of $ch([g])$ depends only on the class $[g]$. Moreover, the odd Chern character respects the addition $ch(g \oplus h) = ch(g) + ch(h)$. Thus, it defines a group* *homomorphism*

$$ch : K^1(X) \longrightarrow H^{odd}(X, \mathbb{C})$$

which is an isomorphism after tensoring $K^1(X)$ *with* $\mathbb{C}$.

Remark 1.1.31. Another similar formula to (1.1.34) is given by Zhang in [54, Chapter 1,(1.50)]. His formulation includes the normalisation factor $\left(\frac{1}{2\pi i}\right)^{\frac{k+1}{2}}$, so that the k-th Chern form associated to (g, d) is

$$c_{2k+1}(g, d) = \left(\frac{1}{2\pi i}\right)^{\frac{k+1}{2}} \operatorname{Tr}\big((g^{-1}dg)^{2k+1}\big).$$

Thus, $ch(g, d) = \sum(n!/(2n + 1)!)c_{2n+1}(g, d)$.

Facts and properties of odd Chern character [20, 21, 54].

- $\operatorname{Tr}((g^{-1}dg)^k) = 0$ when $k > 0$ is an even integer. It is closed, i.e. $d\operatorname{Tr}((g^{-1}dg)^k) = 0$ when $k > 0$ is an odd integer.
- (Variation) If $g_t : X \to U(N)$ is a smooth family which depends only on $t \in [0, 1]$, then for any odd integer $k > 0$, the identity holds

$$\frac{\partial}{\partial t}\operatorname{Tr}\big((g_t^{-1}dg_t)^k\big) = nd\operatorname{Tr}\Big(g_t^{-1}\frac{\partial g_t}{\partial t}(g_t^{-1}dg_t)^k\Big).$$

- (Product) Let $f, g : X \to U(N)$ be representatives in $K^1(X)$. For any odd integer $k > 0$, there exists $\theta_k \in \Omega^{k-1}(X)$ satisfying

$$\operatorname{Tr}\big((fg)^{-1}d(fg))^k\big) = \operatorname{Tr}\big((f^{-1}df)^k\big) + \operatorname{Tr}\big((g^{-1}dg)^k\big) + d\theta_k.$$

- (Independent of the choice of d) Let $g \in \Gamma(\operatorname{Aut}(\tau))$. Let d' be another trivial connection on τ. Then, for any odd integer $k > 0$, there exists $\theta_k \in \Omega^{k-1}(X)$ such that

$$\operatorname{Tr}\big(g^{-1}dg)^k\big) = \operatorname{Tr}\big((g^{-1}d'g)^k\big) + d\theta_k.$$

- There is a transgression form $\mathrm{T}ch(g_t, d)$ of the odd Chern character $ch(g_t)$ for a path g_t connecting g_0 and g_1. In particular, for $t \in [0, 1]$, define

$$\mathrm{T}ch(g_t, d) = \sum_{k=0}^{\infty} \frac{k!}{(2k)!} \int_0^1 \operatorname{Tr}\Big(g_t^{-1}\frac{\partial g_t}{\partial t}(g^{-1}dg_t)^{2k}\Big)dt \tag{1.1.35}$$

which satisfies the odd analog of the transgression formula

$$ch(g_1, d) - ch(g_0, d) = d\mathrm{T}ch(g_t, d). \tag{1.1.36}$$

1.2 spin^c structures and Dirac operators

1.2.1 spin and spin^c structures

Let X be a smooth oriented Riemannian manifold. Let $\{U_\alpha\}$ be a good cover of X. Suppose $E \to X$ is an oriented real vector bundle of rank n equipped with a fibrewise metric. Let $F(E)$ be its frame bundle whose fiber at x is the set of oriented orthonormal frames of E_x for all $x \in X$. Then, $F(E)$ is a principal $SO(n)$-bundle over X. In particular, locally over $U_{\alpha\beta}$ the transition functions are smooth maps

$$g_{\alpha\beta} : U_{\alpha\beta} \longrightarrow SO(n). \tag{1.2.1}$$

By a standard reconstruction procedure, E can be constructed from its frame bundle as an associated bundle. Thus, E has structure group the general linear group $GL(n, \mathbb{R})$. Via the equipped fiberwise metric, the structure group $GL(n, \mathbb{R})$ can be reduced to $SO(n)$. A collection of maps (1.2.1) defines a Cěch cocycle $[g_{\alpha\beta}] \in H^1_{\text{Cěch}}(X, SO(n))$. By standard bundle theory, the cohomology group $H^1_{\text{Cěch}}(X, SO(n))$ classifies all principal $SO(n)$-bundles.

Given a Riemannian metric on X, the (real) Clifford algebra is defined by

$$Cl(X) = \bigoplus_{x\in X} Cl_x(X), \quad Cl_x(X) = \left(\bigoplus_{k=0}^{\infty} (T_x^*X)^k\right) / \sim \tag{1.2.2}$$

where $\sim$ is given by $\langle a \otimes b + b \otimes a - 2(a, b)_g \rangle$ for $a, b \in T_x^*X$.

Definition 1.2.1. Let $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ be the Euclidean space equipped with the standard inner product. Let $Cl(n) = Cl(\mathbb{R}^n)$ be the Euclidean Clifford algebra generated by an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$ subjected to the (Clifford) relation

$$e_i e_j + e_j e_i = -2\delta_{ij} = -2\langle e_i, e_j \rangle. \tag{1.2.3}$$

Definition 1.2.2. The Euclidean Spin group is defined by

$$\text{Spin}(n) = \{v_1 v_2 \cdots v_n \in Cl(n) \mid v_i \in \mathbb{R}^n, |v_i| = 1\}. \tag{1.2.4}$$

Proposition 1.2.3. *There is a short exact sequence of groups*

$$1 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{\rho} SO(n) \to 1. \tag{1.2.5}$$

The group Spin(n) is the non-trivial double cover of SO(n) for $n = 1$ and is a universal cover for $n \geq 2$. Moreover, Spin(n) is a compact Lie group.

For a contractible open set $U_{\alpha\beta}$ of a good cover, there exists a lift $\tilde{g}_{\alpha\beta} : U_{\alpha\beta} \to \text{Spin}(n)$ of $g_{\alpha\beta}$. In particular, the lift of the cocycle condition (1.1.2) becomes

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = \pm 1. \tag{1.2.6}$$

Definition 1.2.4. A spin structure on E is a collection of lifts $\{\tilde{g}_{\alpha\beta}\}$ such that the condition (1.2.6) is

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = 1. \tag{1.2.7}$$

A complex vector bundle E is spin if there exists a spin structure on E.

Now, a collection of double covering maps

$$\theta_{\alpha\beta\gamma} = \tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} : U_{\alpha\beta\gamma} \to \mathbb{Z}_2 \tag{1.2.8}$$

represents a class $[\theta] \in H^2_{\check{\text{C}}\text{ech}}(X, \mathbb{Z}_2)$. Equivalently, the bundle E is spin if and only if $[\theta]$ is trivial. Such a class is called the second Stiefel-Whitney class of E, denoted by $w_2(E) := [\theta]$.

Proposition 1.2.5 ([33])**.** *A bundle E admits a spin structure if and only if there is a lift of the classifying map $X \to BSO(n)$ to a map $X \to BSpin(n)$. Equivalently, E admits a spin structure if and only if $w_2(E) = 0$. Moreover, the class $w_2(E)$ is independent of the choice of lifts. Spin structures are not unique, if they exist. They are determined by the group $H^1(X, \mathbb{Z}_2)$.*

Alternatively, a spin structure can be defined in terms of principal bundles.

Definition 1.2.6. Let $E \to X$ be a complex vector bundle. A spin structure on E is a principal Spin(n)-bundle, together with a double-sheeted covering map

$$\pi : P_{\mathrm{Spin}(n)} \longrightarrow P_{SO(n)}$$

such that $\pi(p \cdot g) = \pi(p) \cdot \rho(g)$ for $p \in P_{\mathrm{Spin}(n)}$, $g \in \mathrm{Spin}(n)$ and $\rho : \mathrm{Spin}(n) \to SO(n)$ is the non-trivial double covering map.

Definition 1.2.7. A smooth oriented Riemannian manifold X is spin if there exists a spin structure on its tangent bundle TX, i.e. $w_2(X) := w_2(TX) = 0$.

It is natural to ask the question: can we extend the condition (1.2.7) of a spin structure to one which is still well-defined when $w_2(X) \neq 0$? The answer is affirmative. Let $\{\tilde{g}_{\alpha\beta}\}$ be the lifting of $\{g_{\alpha\beta}\}$ satisfying (1.2.8). Such a map $\theta_{\alpha\beta\gamma}$ can be viewed as the coboundary of a collection $\{f_{\alpha\beta}\}$ of functions

$$f_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{Z}_2.$$

These 'signed' functions relate $g_{\alpha\beta}$ to another transition function $\tilde{g}'_{\alpha\beta}$ by

$$\tilde{g}'_{\alpha\beta} = f_{\alpha\beta}\tilde{g}_{\alpha\beta}.$$

Now, let $\tilde{f}_{\alpha\beta} : U_{\alpha\beta} \to U(1)$ be an extension of $f_{\alpha\beta}$.

Definition 1.2.8. A spin^c structure on a bundle E consists of the lifts $\{\tilde{g}_{\alpha\beta}\}$ and $U(1)$-valued functions $\{\tilde{f}_{\alpha\beta}\}$ such that

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} = \tilde{f}_{\alpha\beta}\tilde{f}_{\beta\gamma}\tilde{f}_{\gamma\alpha}.$$

Note that the functions $\tilde{f}_{\alpha\beta}$ include $f_{\alpha\beta}$ by $f_{\alpha\beta} = \tilde{f}^2_{\alpha\beta}$. This defines a complex line bundle $L \to X$ whose first Chern class is $c_1(L)$ mod 2. Here, modulo 2 corresponds to the cocycle condition $\tilde{f}_{\alpha\beta}\tilde{f}_{\beta\gamma}\tilde{f}_{\gamma\alpha}$ being $\mathbb{Z}_2$-valued.

Definition 1.2.9. The Euclidean Spin^c group is defined by

$$\begin{aligned}\mathrm{Spin}^c(n) &= \{cv_1v_2\cdots v_{2k} \in Cl(n) \mid v_i \in \mathbb{R}^n \text{ s.t. } |v_i| = 1, c \in \mathbb{C} \text{ s.t. } |c| = 1\} \\ &= \mathrm{Spin}(n) \times_{\mathbb{Z}_2} U(1).\end{aligned}$$

This is a subgroup of the complexified Clifford algebra $\mathbb{C}l(n) = Cl(n) \otimes_{\mathbb{R}} \mathbb{C}$.

Remark 1.2.10. There is an associated short exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}^c(n) \xrightarrow{\rho^c} SO(n) \times U(1) \to 1 \tag{1.2.9}$$

with $\rho^c([x, z]) = (\rho(x), z^2)$. In particular, the Spin^c group fits into the following diagram

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & U(1) & & & & \\
 & & & & \downarrow{\scriptstyle \gamma} & \overset{\theta}{\searrow} & & & \\
1 & \longrightarrow & \mathrm{Spin}(n) & \xrightarrow{\alpha} & \mathrm{Spin}^c(n) & \xrightarrow{\beta} & U(1) & \longrightarrow & 1 \\
 & & & \underset{\lambda}{\searrow} & \downarrow{\scriptstyle \lambda} & & & & \\
 & & & & SO(n) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 1 & & & &
\end{array}$$

where both of the row and the column are short exact sequences. Here, the maps α and γ are natural inclusions of Spin(n) and $U(1)$ into Spin$^c(n)$ respectively; λ is the universal cover Spin$(n) \to SO(n)$; θ is the squaring map $z \mapsto z^2$; and β is the map $(a, z) \mapsto z^2$.

As an analog to spin structure, we can alternatively define spin^c structure in terms of principal bundles.

Definition 1.2.11. A spin^c structure on $E \to X$ consists of a principal Spin$^c(n)$-bundle $P_{\mathrm{Spin}^c(n)}$ and a principal $U(1)$-bundle $P_{U(1)}$ together with an equivariant map

$$\pi^c : P_{\mathrm{Spin}^c(n)} \longrightarrow P_{SO(n)} \times P_{U(1)}$$

i.e. $\pi^c(p \cdot g) = \pi^c(p) \cdot \rho^c(g)$ for $p \in P_{\mathrm{Spin}(n)}$, $g \in \mathrm{Spin}(n)$.

Definition 1.2.12. Let X be an oriented Riemannian manifold. Then, X is spin^c if the tangent bundle TX admits a spin^c structure. The vanishing condition of the Stiefel-Whitney class is given by $W_3(X) := W_3(TX) = 0$.

Remark 1.2.13. Equivalently, a smooth oriented manifold X is spin^c if and only if there is a complex line bundle $L \to X$ such that $TX \oplus L$ admits a spin structure.

Example 1.2.1.

1. The Euclidean space $\mathbb{R}^n$, contractible manifolds, orientable surfaces and all oriented 3-manifolds are spin.
2. It is well-known that $\mathbb{RP}^2$ is not orientable, and thus not spin. However, for $n > 2$, $\mathbb{RP}^n$ is spin if and only if $n \equiv 3 \bmod 4$. The complex projective space $\mathbb{CP}^n$ is spin if and only if n is odd.
3. Almost complex manifolds and all orientable 4-manifolds are spin^c.

1.2.2 Spinors and Dirac operators

Definition 1.2.14. Let Δ_n be the space of complex n-spinors defined by $\Delta_n := \mathbb{C}^{2^r}$ where $r = [n/2]$ is the integer part of $n/2$.

Proposition 1.2.15.

$$\mathbb{C}l(n) \cong \begin{cases} End(\Delta_n) & for\ n = 2k; \\ End(\Delta_n) \oplus End(\Delta_n) & for\ n = 2k+1. \end{cases}$$

By Proposition 1.2.15, one can view elements of $\mathbb{C}l(n)$ as acting on complex spinors, which leads to the following definition.

Definition 1.2.16. Let $v \in \mathbb{R}^n$. The Clifford multiplication by v on spinors is defined by a map $c(v) : \Delta_n \to \Delta_n$ which satisfies the Clifford relation

$$c(v)c(w) + c(w)c(v) = -2\langle v, w\rangle. \tag{1.2.10}$$

Definition 1.2.17. Let X be a spin manifold of dimension n. Define the complex spinor bundle $S \to X$ as the associated bundle

$$S = P_{\mathrm{Spin}(n)} \times_{\rho_n} \Delta_n \tag{1.2.11}$$

where the (complex) spin representation

$$\rho_n : \mathrm{Spin}(n) \to \mathrm{End}(\Delta_n) \tag{1.2.12}$$

is given by the restriction of an irreducible complex representation $\mathbb{C}l(n) \to \mathrm{End}(\Delta_n)$ to $\mathrm{Spin}(n) \subset Cl(n) \subset \mathbb{C}l(n)$. A section $\psi \in \Gamma(S)$ is called a spinor field.

$\mathbb{Z}_2$-grading. In the case $n = 2k$, the spin representation (1.2.12) is $\mathbb{Z}_2$-graded

$$\rho^{\pm}_{2k} : \mathrm{Spin}(2k) \to \mathrm{End}(\Delta^{\pm}_{2k}), \tag{1.2.13}$$

where $\Delta_{2k} = \Delta^{+}_{2k} \oplus \Delta^{-}_{2k}$. Hence, the spinor bundle is also $\mathbb{Z}_2$-graded, i.e. $S = S^{+} \oplus S^{-}$. To see this, let $\{e_i\}$ be a positively oriented orthonormal frame of TX. Then, define the complex volume form as

$$\omega_{\mathbb{C}} = i^k e_1 \cdots e_{2k}, \tag{1.2.14}$$

where $i = \sqrt{-1}$. It is readily checked that (1.2.14) is independent of the choice of orthonormal frame and $\omega_{\mathbb{C}}^2 = 1$. Thus, $S^{\pm}$ is the ± 1-eigenbundle of $c(\omega_{\mathbb{C}}) = i^k c(e_1) \cdots c(e_{2k})$, given by

$$S^{\pm} = (1 \pm \omega_{\mathbb{C}})S. \tag{1.2.15}$$

Remark 1.2.18. More generally, for a spin^c manifold X of dimension n, we can generalise (1.2.11) to an associated bundle

$$S = P_{\mathrm{Spin}^c(n)} \times_{\rho^c_n} \Delta_n \tag{1.2.16}$$

where $\rho^c_n : \mathrm{Spin}^c(n) \subset \mathbb{C}l(n) \to \mathrm{End}(\Delta_n)$ is the (complex) spin^c representation. When $n = 2k$, there is only one fundamental (complex Clifford representations are irreducible) $\mathbb{Z}_2$-graded spinor bundle over X, given by a similar formula to (1.2.15). When $n = 2k + 1$, there are two irreducible complex Clifford representations but they are equivalent when restricted to $\mathrm{Spin}^c(n)$. The intertwining relation between a spin^c spinor bundle and a spin spinor bundle is given by

$$S_{\mathrm{Spin}^c} = S_{\mathrm{Spin}} \otimes L^{\frac{1}{2}} \tag{1.2.17}$$

where L is the complex line bundle associated to the spin^c structure and $L^{\frac{1}{2}}$ is its square root. Note that it is a non-trivial fact that even when S_{Spin} and $L^{1/2}$ in (1.2.17) cannot be constructed individually, their product determines a globally defined bundle S_{Spin^c}, cf. [33, Appendix D]. It is clear from (1.2.17) that when X is spin, then X is spin^c with a canonical trivial class of L, i.e. $S_{\mathrm{Spin}^c} = S_{\mathrm{Spin}}$.

Definition 1.2.19. A spinor connection ∇^S on the spinor bundle S is a covariant derivative

$$\nabla^S : \Gamma(S) \longrightarrow \Gamma(T^*X \otimes S)$$

satisfying the compatibility condition

$$\nabla^S(c(v)\psi) = c(\nabla v)\psi + c(v)\nabla^S\psi \tag{1.2.18}$$

where c is the Clifford multiplication, $v \in T^*X$ and $\psi \in \Gamma(S)$.

Definition 1.2.20. Let X be a spin manifold. Let S be the complex spinor bundle. The Dirac operator $D : \Gamma(S) \to \Gamma(S)$ is the first order differential operator defined by the composition of maps

$$\Gamma(S) \xrightarrow{\nabla^S} \Gamma(T^*X \otimes S) \xrightarrow{c} \Gamma(S)$$

where c is the Clifford multiplication $v \otimes s \mapsto c(v)s$. Locally, with respect to an orthonormal basis $\{e_i\}$, the Dirac operator can be expressed as

$$D = \sum_i c(e_i)\nabla^S_{e_i}. \tag{1.2.19}$$

Remark 1.2.21. The setting above can be slightly extended to the case of Clifford modules. Let X be a spin manifold. Let $E \to X$ be a complex vector bundle. Then, the tensor product bundle $S \otimes E$ is a Clifford module over X, i.e. there is a Clifford multiplication

$$\begin{aligned} c : \Gamma(T^*X \otimes (S \otimes E)) &\longrightarrow \Gamma(S \otimes E) \\ v \otimes a &\longmapsto c(v)a \end{aligned}$$

which satisfies the Clifford relation (1.2.10). Let $\nabla^{S\otimes E}$ be the induced tensor product connection on $S \otimes E$. Then, $\nabla^{S\otimes E}$ satisfies the relation (1.2.18) for $\phi \in \Gamma(S \otimes E)$. The Dirac operator twisted by E is given by

$$D^E = c \circ \nabla^{S\otimes E}. \tag{1.2.20}$$

It is well known that Dirac operators are *elliptic*. Let E, F be two complex vector bundles over a Riemannian manifold X. Let $D : \Gamma(E) \to \Gamma(F)$ be a differential operator of order m, which takes the form

$$D = \sum_{|\alpha| \leq m} A_\alpha(x) \frac{\partial^{|\alpha|}}{\partial x^\alpha} \tag{1.2.21}$$

in local coordinates, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index, $\partial_x^{|\alpha|} = \partial_x^{\alpha_1} \cdots \partial_x^{\alpha_n}$ and $A_\alpha(x) : E_x \to F_x$ is a fiberwise linear map. Then, the principal symbol σ_ξ is a linear map defined by

$$\sigma_\xi(D) = i^m \sum_{|\alpha|=m} A_\alpha(x) \xi^\alpha : E_x \to F_x$$

for $\xi = \xi_k dx^k \in T_x^* X$ and $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$.

Definition 1.2.22. A differential operator D is called elliptic if $\sigma_\xi(D)$ is an isomorphism for all $\xi \neq 0 \in T_x^* X, x \in X$.

Proposition 1.2.23. *Let D be the Dirac operator on S. Then, for any $\xi \in T^* X$ we have*

$$\sigma_\xi(D) = ic(\xi), \quad \sigma_\xi(D^2) = \sigma_\xi(D) \circ \sigma_\xi(D) = ||\xi||^2,$$

where $c(\xi)$ denotes the Clifford multiplication by ξ. In particular, both of the Dirac operators D and D^2 are elliptic.

Proposition 1.2.24. *There is a Hermitian inner product $\langle \cdot, \cdot \rangle$ on Δ_n such that*

$$\langle c(e)v, c(e)w \rangle = \langle v, w \rangle \quad \text{or equivalently} \quad \langle c(e)v, w \rangle = -\langle v, c(e)w \rangle$$

for a unit vector $e \in \mathbb{R}^n$ and spinors $v, w \in \Delta_n$. This inner product induces a Hermitian metric $(\cdot, \cdot)$ on the spinor bundle S such that

$$(c(e)s, s') = -(s, c(e)s') \tag{1.2.22}$$

and satisfies the compatibility condition $\nabla(s, s') = (\nabla_X^S s, s') + (s, \nabla_X^S s')$. The metric $(\cdot, \cdot)$ is understood to be $(s, s') = \int_X (s, s')_x$ where $(\cdot, \cdot)_x$ denotes the fiberwise inner product at $x \in X$.

Proposition 1.2.25. *Let S be the complex spinor bundle over a Riemannian manifold X. Assume S is equipped with a connection and a metric as in* (1.2.22). *Then, the Dirac operator D is formally self-adjoint with respect to the metric, i.e.*

$$(Ds, s') = (s, Ds') \tag{1.2.23}$$

for all compactly supported sections s, s' of S.

Definition 1.2.26. Let X be a Riemannian manifold. A Dirac bundle is a complex vector bundle $\mathcal{E}$ which is a left module over $Cl(X)$, whose connection $\nabla^{\mathcal{E}}$ satisfies (1.2.18) and whose Riemannian metric satisfies (1.2.22).

Remark 1.2.27. If X has a boundary ∂X, then (1.2.23) becomes

$$(Ds, s') - (s, Ds') = \int_{\partial X} (v \cdot s, s')_x$$

where v is an outer normal vector to ∂X and $\cdot$ is the Clifford multiplication.

Proposition 1.2.28. *Let $S \to X$ be a Dirac bundle. Let D be the Dirac operator on S. Then, for any $f \in C^\infty(X)$ and $s \in \Gamma(S)$, we have*

$$D(fs) = grad(f) \cdot s + fDs.$$

We end this subsection by discussing two classical examples.

Example 1.2.2.

1. Let $X = \mathbb{R}^n$. The classical Dirac operator is $D = \sum_{j=1}^n E_j d/dx_j$ acting on (compactly supported) smooth sections of the trivial bundle $\mathbb{R}^n \times \mathbb{C}^{2^r}$. Here, E_j are matrices which satisfy certain properties, cf. [14]. For instance, when $n = 3$, these $E_j's$ are the renowned Pauli matrices [25].

2. Let $X = S^1$. There are two distinct spin structures on S^1 : the disconnected-cover and the connected-cover spin structures. The space of complex spinors of the former is $\Delta_1^{dc} = S^1 \times \mathbb{C} = \mathbb{R} \times \mathbb{C}/\sim$, where $(x, z) \sim (x', z')$ if and only if $x - x' \in \mathbb{Z}$ and $z = z'$; whilst for the latter it is $\Delta_1^c = \mathbb{R} \times \mathbb{C}/\sim$, where $(x, z) \sim (x', z')$ if and only if $x - x' \in \mathbb{Z}$ and $z = e^{i\pi(x-x')}z'$. The Dirac operators for both cases are $D^{dc} = D^c = -id/dt$ but they are different operators. In particular, $\ker(D^{dc})$ is non-trivial whereas $\ker(D^c)$ is trivial.

Chapter 2

$\mathbb{R}/\mathbb{Z}$ K-theory

2.1 $\mathbb{R}/\mathbb{Z}$ K^0-theory

2.1.1 The $K^0(X, \mathbb{R}/\mathbb{Z})$ group

In this section, we propose a geometric model of *even* K-theory with coefficients in $\mathbb{R}/\mathbb{Z}$. We show that this model is a $K^0(X)$-module and has a well-defined $\mathbb{R}/\mathbb{Q}$ Chern character map. We clarify that there is a rather different model of $\mathbb{R}/\mathbb{Z}$ K^0-theory in the literature. In [11], Basu gave a model of this group in terms of the suspension of Lott's $\mathbb{R}/\mathbb{Z}$ K^1-theory [36], the cocycle of which is a pair of vector bundles (E_1, E_2) over the suspension SX, together with an isomorphism $\phi : E_1 \otimes V \cong E_2 \otimes V$, where V is a von-Neumann algebra bundle.

However, this is not an appropriate model in which to formulate the analytic K^0-pairing (5.1.2). The main reason is that differential forms are used in a fundamental way, but the suspension SX may not be smooth even if X is smooth. Moreover, the model we propose is compatible with the construction of the Dai-Zhang eta-invariant, which forms a crucial term of the pairing formula (5.1.2). As we will see later, this proposal *is* the direct analog of Lott's $\mathbb{R}/\mathbb{Z}$ K^1-theory.

Definition 2.1.1. Let X be a smooth compact manifold. An $\mathbb{R}/\mathbb{Z}$ K^0-cocycle over X is a triple

$$(g, (d, g^{-1}dg), \mu) \tag{2.1.1}$$

- $g : X \to U(N)$ is a smooth map, i.e. a $K^1(X)$-representative,

- $(d, g^{-1}dg)$ is a pair of flat connections on the trivial bundle acted on by g,
- $\mu \in \Omega^{\text{even}}(X)/d\Omega$ satisfying the *exactness* condition

$$d\mu = ch(g,d) - \mathrm{Tr}(g^{-1}dg). \tag{2.1.2}$$

Here, the odd Chern character of g with flat connections $(d, g^{-1}dg)$ is explicitly given by

$$ch(g,d) := \sum_{n=0}^{\infty} \frac{n!}{(2n+1)!} \mathrm{Tr}(g^{-1}dg)^{2n+1} \tag{2.1.3}$$

cf. [26] and [54].

Definition 2.1.2. For $i = 1,2,3$, let $g_i : X \to U(N_i)$ be smooth maps for some large $N_i \in \mathbb{Z}$. Let $\mathcal{E}_i$ be the $\mathbb{R}/\mathbb{Z}$ K^0-cocycles corresponding to g_i. Then, the $\mathbb{R}/\mathbb{Z}$ K^0-relation is given by

$$\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3, \tag{2.1.4}$$

i.e. whenever there is a sequence of maps $g_1 \longrightarrow g_2 \longrightarrow g_3$ such that

$$g_2 \simeq g_1 \oplus g_3, \tag{2.1.5}$$

i.e. g_2 is homotopic to $\mathrm{Diag}(g_1, g_3)$ as unitary matrices, then

$$\mu_2 = \mu_1 + \mu_3 - \mathrm{T}ch(g_1, g_2, g_3). \tag{2.1.6}$$

Here, $\mathrm{T}ch(g_1, g_2, g_3)$ denotes the transgression form of the odd Chern character satisfying

$$d\mathrm{T}ch(g_1, g_2, g_3) = ch(g_1) - ch(g_2) + ch(g_3). \tag{2.1.7}$$

Remark 2.1.3. The transgression $\mathrm{T}ch(g_1, g_2, g_3)$ is taken as $\mathrm{T}ch((i \oplus j)^* g_2, g_1 \oplus g_3)$ where $i : g_1 \to g_2$ is an inclusion map and $j : g_3 \to g_2$ is a splitting map. The term with two entries is the transgression form (1.1.35) of the odd Chern character of g_t given by

$$\mathrm{T}ch(g_t, d) = \sum_{n=0}^{\infty} \frac{n!}{(2n)!} \int_0^1 \mathrm{Tr}\Big(g_t^{-1} \frac{\partial g_t}{\partial t} (g^{-1} dg_t)^{2n}\Big) dt \tag{2.1.8}$$

where g_t a path of smooth maps connecting $(i \oplus j)^* g_2$ to $g_1 \oplus g_3$, for $0 < t < 1$. One can show that $\mathrm{T}ch((i \oplus j)^* g_2, g_1 \oplus g_3)$ is independent of the choice of j.

Definition 2.1.4. Let X be a smooth compact manifold. The $\mathbb{R}/\mathbb{Z}$ K^0-theory of X, denoted by $K^0(X, \mathbb{R}/\mathbb{Z})$, is the free abelian group generated by $\mathbb{R}/\mathbb{Z}$ K^0-cocycles with zero virtual trace in the lowest degree, modulo the $\mathbb{R}/\mathbb{Z}$ K^0-relation. The group operation is given by the addition of $\mathbb{R}/\mathbb{Z}$ K^0-cocycles

$$(g, (d, g^{-1}dg), \mu) + (h, (d, h^{-1}dh), \theta) = (g \oplus h, (d \oplus d, g^{-1}dg \oplus h^{-1}dh), \mu \oplus \theta). \tag{2.1.9}$$

Remark 2.1.5. There is another equivalent definition of $K^1(X)$, in which a class is represented by a pair (E, h) where E is a complex vector bundle over X and h is a smooth automorphism of E. One way to see the equivalence between these two definitions of $K^1(X)$ is by first complementing E to a trivial bundle τ by a complementary bundle E^c, which is always exists. Let T be an isomorphism $E \oplus E^c \cong \tau$. Let $\tilde{g} := T^{-1}(h \oplus \mathrm{Id}_{E^c})T$ be an automorphism of τ. Then, $\tilde{g}$ and h define the same class in $K^1(X)$.

Note that the second entry of (2.1.1) is uniquely determined by g. However, if another definition of $K^1(X)$ is used, then a choice of a pair of connections comes into the picture. In particular, the cocycle $(g, (d, g^{-1}dg), \mu)$ can be equivalently modified to $(h, (\nabla^E, h^{-1} \circ \nabla^E \circ h), \mu)$ for a pair (E, h) where E is a complex vector bundle with connection ∇^E, h is an automorphism of E, $(\nabla^E, h^{-1}\nabla^E h)$ is a pair of connections on E and μ is an even degree form on X satisfying the exactness condition. The relation is similar: whenever there is a short exact sequence of maps $0 \to h_1 \to h_2 \to h_3 \to 0$, the relation is given by $\xi_2 = \xi_1 + \xi_3$. The corresponding odd Chern character of (E, h) is defined by

$$ch(h) := CS(\nabla^E, h^{-1} \circ \nabla^E \circ h). \tag{2.1.10}$$

Its explicit formula is now in the general setting and becomes much more complicated, see [24].

$K^0(X)$**-module structure:** We show that the group $K^0(X, \mathbb{R}/\mathbb{Z})$ is a $K^0(X)$-module. For clarity, we use the second definition of $K^1(X)$ as in Remark 2.1.5.

Let (E, g) be a K^1-representative. The module multiplication is given by

$$\begin{aligned} &K^0(X) \times K^0(X, \mathbb{R}/\mathbb{Z}) \longrightarrow K^0(X, \mathbb{R}/\mathbb{Z}) \\ &V \hat{\otimes} (g, (\nabla^E, g^{-1}\nabla^E g), \mu) \\ &\quad = \big(V \otimes E, (\nabla^V \otimes \nabla^E, h^{-1}\nabla^V h \otimes g^{-1}\nabla^E g), ch(\nabla^V) \wedge \mu\big) \end{aligned} \tag{2.1.11}$$

where h is a chosen automorphism of V.

The tensor product (2.1.11) requires a choice of automorphism h of V, which is always exists from the viewpoint of the complementary bundle and the automorphism of the trivial bundle as a global trivialisation. Here, $\nabla^V \otimes \nabla^E := \nabla^V \otimes 1 + 1 \otimes \nabla^E$. Fix g, consider two choices h_1 and h_2 so that

$$ch(h_1 \otimes g) = CS(\nabla^V \otimes \nabla^E, h_1^{-1}\nabla^V h_1 \otimes g^{-1}\nabla^E g) \tag{2.1.12}$$

$$ch(h_2 \otimes g) = CS(\nabla^V \otimes \nabla^E, h_2^{-1}\nabla^V h_2 \otimes g^{-1}\nabla^E g). \tag{2.1.13}$$

By taking the difference (2.1.12) – (2.1.13), we get

$$\begin{aligned} &ch(h_1 \otimes g) - ch(h_2 \otimes g) \\ &\qquad = ch(g^{-1}\nabla^E g) \wedge \big(CS(\nabla^V, h_1^{-1}\nabla^V h_1) - CS(\nabla^V, h_2^{-1}\nabla^V h_2)\big) \\ &\qquad = ch(g^{-1}\nabla^E g) \wedge CS(h_2^{-1}\nabla^V h_2, h_1^{-1}\nabla^V h_1) \end{aligned}$$

If h_1 and h_2 represent the same class, then $h_2 h_1^{-1}$ is homotopic to the identity. The Chern-Simons form reduces to $CS(\nabla^V, \nabla^V)$. For $t \in [0, 1]$, let $\gamma(t)$ be a path of connections joining ∇^V back to itself, which is a closed curve. Let $A_t \in \Omega^1(X, \mathrm{End}(V))$ and R_t be the curvature of ∇^V_t. Consider

$$cs(\gamma) = \int_0^1 \sum_{j=1} \frac{1}{(j-1)!} \Big(\frac{1}{2\pi i}\Big)^j \mathrm{Tr}(A_t \wedge (R_t)^{j-1}).$$

By [49, Proposition 1.6], the odd form $cs(\gamma)$ is exact since γ is a closed curve. Together with [49, (1.7)], we have

$$CS(\nabla^V, \nabla^V) = cs(\gamma) \text{ mod exact} \equiv 0.$$

The above argument shows the following corollary.

Corollary 2.1.6. *For a fixed K^0-cocycle in $K^0(X, \mathbb{R}/\mathbb{Z})$, the module multiplication given by* (2.1.11) *only depends on the homotopy class of h.*

Moreover, since $ch(\nabla^V)$ is closed, it is straightforward that

$$d(ch(\nabla^V) \wedge \mu) = ch(\nabla^V) \wedge d\mu. \tag{2.1.14}$$

Remark 2.1.7. There is also a description using $\mathbb{Z}_2$-graded cocycles. A $\mathbb{Z}_2$-graded K^0-cocycle consists of $(g^{\pm}, (d, g^{\pm} dg^{\pm}), \mu)$ where $g^{\pm} = g^+ \oplus g^-$ is a $\mathbb{Z}_2$-graded K^1-representative and $\mu \in \Omega^{\text{even}}(X)/d\Omega$ such that

$$d\mu = ch(g^{\pm}, d) = ch(g^+, d) - ch(g^-, d). \tag{2.1.15}$$

Explicit maps and exactness of (part of) sequence : Consider the following sequence

$$\cdots \to K^0(X, \mathbb{R}) \xrightarrow{\alpha} K^0(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} K^1(X, \mathbb{Z}) \xrightarrow{ch} K^1(X, \mathbb{R}) \to \cdots \tag{2.1.16}$$

associated to the short exact sequence of coefficients $1 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 1$, where the horizontal maps are respectively

(2.1.17) $\alpha(\mu) = (\mathrm{Id}, (d, d), \mu) - (\mathrm{Id}, (d, d), 0) = (0, 0, \mu)$ is the inclusion map,

(2.1.18) $\beta(g, (d, d + g^{-1}dg), \theta) = [g]$ is the forgetful map,

(2.1.19) $ch(g)$ is the odd Chern character map given by (2.1.3).

Lemma 2.1.8. *With respect to the sequence* (2.1.16)*, it is exact at $K^0(X, \mathbb{R}/\mathbb{Z})$ and at $K^1(X, \mathbb{Z})$.*

Proof. For

$$K^0(X, \mathbb{R}) \xrightarrow{\alpha} K^0(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} K^1(X, \mathbb{Z}),$$

note that $\mathrm{im}(\alpha) \subseteq \ker(\beta)$ follows from the definition. We need to show $\ker(\beta) \subseteq \mathrm{im}(\alpha)$. Let $\mathcal{E}_1 - \mathcal{E}_2 = (g_1, (d, g_1^{-1}dg_1^{-1}), \mu_1) - (g_2, (d, g_2^{-1}dg_2^{-1}), \mu_2) \in \ker(\beta)$ so that $\beta(\mathcal{E}_1 - \mathcal{E}_2) = [g_1] - [g_2] = 0$. In particular, $[g_1] = [g_2]$ if and only if there exists an identity matrix Id of suitable rank in the unitary group, such that $g_1 \oplus \mathrm{Id}$ is homotopic to $g_2 \oplus \mathrm{Id}$. The direct sum means that they sit along the diagonal of a suitably large matrix h. This defines an element $(h, (d, h^{-1}dh), \mu_1) - (h, (d, h^{-1}dh), \mu_2) = (0, 0, \mu_1 - \mu_2)$ of $\mathrm{im}(\alpha)$, as the image of $\mu_1 - \mu_2$ under α. This shows $ker(\beta) \subseteq im(\alpha)$ and thus is exact at $K^0(X, \mathbb{R}/\mathbb{Z})$.

On the other hand, consider

$$K^0(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} K^1(X, \mathbb{Z}) \xrightarrow{ch} K^1(X, \mathbb{R}).$$

Since any $[g]$ in $K^1(X)$ with vanishing Chern character lies in the torsion subgroup $K^1_{\mathrm{Tors}}(X)$, the sequence reduces to

$$K^0(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} K^1_{\mathrm{Tors}}(X) \to 0. \tag{2.1.20}$$

Hence, an element of $K^1_{\mathrm{Tors}}(X)$ lifts to an element of $K^0(X, \mathbb{R}/\mathbb{Z})$ such that it is the image under β. This shows $ker(ch) \subseteq im(\beta)$. To show the opposite direction, consider two elements $\mathcal{E}_1$ and $\mathcal{E}_2$ of $K^0(X, \mathbb{R}/\mathbb{Z})$. By applying the odd Chern character to the image of β, together with the exactness condition, we get $ch([g_1] - [g_2]) = [d(\mu_1 - \mu_2)] = 0$. So, $[g_1] - [g_2]$ lies in the kernel of ch. This shows $im(\beta) \subseteq ker(ch)$ and thus is exact at $K^0(X, \mathbb{Z})$. □

2.1.2 The $\mathbb{R}/\mathbb{Q}$ Chern character $ch_{\mathbb{R}/\mathbb{Q}}$

Next, we define the $\mathbb{R}/\mathbb{Q}$ Chern character map $ch_{\mathbb{R}/\mathbb{Q}}$ between $K^0(X, \mathbb{R}/\mathbb{Z})$ and $H^{\mathrm{even}}(X, \mathbb{R}/\mathbb{Q})$ such that the following diagram commutes.

$$\begin{array}{ccccccccc}
\cdots \to K^0(X,\mathbb{R}) & \xrightarrow{\alpha} & K^0(X,\mathbb{R}/\mathbb{Z}) & \xrightarrow{\beta} & K^1(X,\mathbb{Z}) & \xrightarrow{ch} & K^1(X,\mathbb{R}) \to \cdots \\
\downarrow \cong & & \downarrow \mathrm{ch}_{\mathbb{R}/\mathbb{Q}} & & \downarrow \mathrm{ch}_{\mathbb{Q}} & & \downarrow \cong \\
\cdots \to H^{even}(X,\mathbb{R}) & \xrightarrow{r} & H^{even}(X,\mathbb{R}/\mathbb{Q}) & \xrightarrow{\widetilde{\beta}} & H^{odd}(X,\mathbb{Q}) & \xrightarrow{i} & H^{odd}(X,\mathbb{R}) \to \cdots
\end{array}$$

The upper (resp. bottom) row is the long exact sequence of K-theory (resp. cohomology) associated to the short exact sequence of the coefficients. Here r, i

and $\widetilde{\beta}$ are the reduction, inclusion and Bockstein maps in cohomology respectively. The maps in the upper row are given by (2.1.17), (2.1.18) and the odd Chern character. By tensoring the upper row by $\mathbb{Q}$ and by applying the Five lemma, $ch_{\mathbb{R}/\mathbb{Q}}$ is a rational isomorphism.

Now, the existence of μ in $K^0(X, \mathbb{R})$ implies that the odd Chern character $ch(g - \mathrm{Id}_N) = 0$, where Id_N denotes the identity matrix of size $N \times N$ with respect to $g : X \to U(N)$, for some large $N \in \mathbb{Z}$. So, $g - \mathrm{Id}_N$ is torsion in $K^1(X)$ and there exists some positive k such that $kg \cong \mathrm{Id}_{kN}$, i.e. $g \oplus \cdots \oplus g = \mathrm{Diag}(g, ..., g)$ is homotopic to the identity matrix. Using the second definition of K^1, i.e. by viewing g as a smooth automorphism of a complex vector bundle E, the unitary map kg corresponds to an automorphism on $kE = E \oplus \cdots \oplus E$. Let $k\nabla^E$ be its Hermitian connection and ∇_0^{kE} be a connection with trivial holonomy. Then, we obtain the conjugation $h^{-1}k\nabla^E h$ and $h^{-1}\nabla_0^{kE}h$ by $h = kg$ of these two connections. For $t \in [0, 1]$, fix $k\nabla^E$ and ∇_0^{kE} and vary h within the homotopy class of g, giving a path $h(t)$ connecting $h(t)^{-1}k\nabla^E h(t)$ and $h(t)^{-1}\nabla_0^{kE}h(t)$. This defines

$$ch(h(t), t \in [0, 1]) \in \Omega^{\mathrm{odd}}(X \times [0, 1]).$$

By the standard construction in [26, 54], the respective transgression form is

$$\mathrm{T}ch(h(t), [0, 1]) = \varphi \int_0^1 \mathrm{Tr}\Big(h(t)^{-1}\frac{\partial(h(t))}{\partial t}\Big(h(t)^{-1}(k\nabla^E)h(t)\Big)^{2n}\Big)dt.$$

This is an analog of (1.1.35). Then,

$$\frac{1}{k}\mathrm{T}ch(h(t), t \in [0, 1]) - \mu \tag{2.1.21}$$

defines an element of $H^{\mathrm{even}}(X, \mathbb{R})$.

Definition 2.1.9. Let $ch^0_{\mathbb{R}/\mathbb{Q}}(g, (d, g^{-1}dg), \mu)$ be the image of $\frac{1}{k}\mathrm{T}ch(h(t), t \in [0, 1]) - \mu$ under the map $H^{\mathrm{even}}(X, \mathbb{R}) \to H^{\mathrm{even}}(X, \mathbb{R}/\mathbb{Q})$.

Next, we show that $ch^0_{\mathbb{R}/\mathbb{Q}}(g, (d, g^{-1}dg), \mu)$ is well-defined.

Lemma 2.1.10. *Let $\mathcal{E} = (g, (d, g^{-1}dg), \mu)$. As an image in $H^{even}(X, \mathbb{R}/\mathbb{Q})$, $ch^0_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$ is independent of the choice of the homotopy class of h and the choice of k.*

Proof. Let g_1, g_2 be two K^1-representatives. Let $h_1(t)$ and $h_2(t)$ be the respective paths as constructed above. That is, $h_i(t)$ connect $h_i^{-1}k\nabla^E h_i$ to $h_i^{-1}\nabla_0^{kE}h_i$ for $i = 1, 2$. Let $\mathrm{T}ch(h_1(t))$ and $\mathrm{T}ch(h_2(t))$ be their transgression forms respectively. Note that in general $h_1(t)$ and $h_2(t)$ may not coincide, in which case both paths lie within their homotopy class. However, it is possible to connect $h_1(t)$ and $h_2(t)$ at the left endpoint. Since both h_1 and h_2 are unitary, we consider the multiplication $h_2^{-1}h_1$ for a fixed k. Then, the two left endpoints can be joined by the conjugation of $h_2^{-1}h_1$ since

$$(h_2^{-1}h_1)^{-1}(h_2^{-1}\circ(k\nabla^E)\circ h_2)(h_2^{-1}h_1) = h_1^{-1}h_2h_2^{-1}\circ(k\nabla^E)\circ h_2h_2^{-1}h_1 = h_1^{-1}\circ(k\nabla^E)\circ h_1.$$

Let $r(h_2^{-1}h_1)$ be the conjugation action. For $t \in [0, 1]$, define

$$(h_1h_2^{-1})(t) := h_1(t) \circ r(h_2^{-1}h_1) \circ h_2(t)^{-1}.$$

Then, the difference

$$\frac{1}{k}\mathrm{T}ch(h_1(t)) - \frac{1}{k}\mathrm{T}ch(h_2(t)) = \frac{1}{k}\mathrm{T}ch((h_1h_2^{-1})(t)) + d\omega_n \tag{2.1.22}$$

holds, where the second term of the RHS of (2.1.22) is some exact form independent of h_i, cf. [54, Corollary 1.18]. In particular, the difference (2.1.22) is the same up to multiplication by a rational number, as the image of $ch([h_1][h_2^{-1}]) = ch([h_1]) \wedge ch([h_2^{-1}]) \in H^{\mathrm{even}}(X, \mathbb{Q})$ in $H^{\mathrm{even}}(X, \mathbb{R})$, so it vanishes when mapped into $H^{\mathrm{even}}(X, \mathbb{R}/\mathbb{Q})$. This shows that $ch^0_{\mathbb{R}/\mathbb{Q}}(\mathcal{E})$ is independent of the homotopy class of h.

Next, for two different positive integers k and k', while keeping the choice of g fixed, we get $h(t) = kg_t$ and $h'(t) = k'g_t$. Then, the difference is

$$\frac{1}{k}\mathrm{T}ch(h(t)) - \frac{1}{k'}\mathrm{T}ch(h'(t)) = \frac{1}{kk'}\mathrm{T}ch((h'h^{-1})(t)) + d\omega_n.$$

By a similar argument as in the previous paragraph, the difference is the same up to multiplication by a rational number, as the image of the odd Chern character in $H^{\mathrm{even}}(X, \mathbb{R}/\mathbb{Q})$ vanishes. So, the image of $ch^0_{\mathbb{R}/\mathbb{Q}}$ is independent of the positive integer k. □

2.2 $\mathbb{R}/\mathbb{Z}$ K^1-theory

As inspired by the work of Karoubi [31] on K-theory with local coefficient systems, Lott proposed an explicit geometric model of $K^1(X, \mathbb{R}/\mathbb{Z})$, elements of which are cocycles defined in terms of vector bundle data and smooth differential forms. These cocycles are the fundamental objects in the analytic duality pairing of Lott (5.2.3), as we will see in chapter 5.

Definition 2.2.1. Let X be a smooth compact manifold. A $\mathbb{R}/\mathbb{Z}$ K^1-cocycle is a triple

$$\mathcal{V} = (V, \nabla^V, \omega) \tag{2.2.1}$$

- (V, ∇^V) is a complex vector bundle with a Hermitian connection ∇^V on X,
- $\omega \in \Omega^{\text{odd}}(X)/\text{im}(d)$ such that

$$d\omega = ch(V, \nabla^V) - rk(V). \tag{2.2.2}$$

Definition 2.2.2. Let $\mathcal{V}_i = (V_i, \nabla^V_i, \omega_i)$ be $\mathbb{R}/\mathbb{Z}$ K^1-cocycles for $i = 1, 2, 3$. The $\mathbb{R}/\mathbb{Z}$ K^1-relation on $\mathcal{V}_i$ is given by

$$\mathcal{V}_2 = \mathcal{V}_1 + \mathcal{V}_3 \tag{2.2.3}$$

whenever there is a sequence of bundles $0 \to V_1 \to V_2 \to V_3 \to 0$ such that

$$V_2 \cong V_1 \oplus V_3 \quad \text{and} \quad \omega_2 = \omega_1 + \omega_3 + CS(\nabla^{V_1}, \nabla^{V_2}, \nabla^{V_3}) \tag{2.2.4}$$

where $\cong$ denotes the vector bundle isomorphism (cf. Definition 1.1.4) and $CS(\cdot, \cdot, \cdot)$ denotes the Chern-Simons form (1.1.30) of connections which satisfy the equation (1.1.31).

Definition 2.2.3. Let X be a smooth compact manifold. The $\mathbb{R}/\mathbb{Z}$ K^1-theory of X, denoted by $K^1(X, \mathbb{R}/\mathbb{Z})$ is the free abelian group generated by $\mathbb{R}/\mathbb{Z}$ K^1-cocycles with zero virtual rank, modulo the $\mathbb{R}/\mathbb{Z}$ K^1-relation. The group operation is given by the addition of $\mathbb{R}/\mathbb{Z}$ K^1-cocycles

$$(V_1, \nabla^{V_1}, \omega_1) + (V_2, \nabla^{V_2}, \omega_2) = (V_1 \oplus V_2, \nabla^{V_1} \oplus \nabla^{V_2}, \omega_1 \oplus \omega_2). \tag{2.2.5}$$

Remark 2.2.4. The identity element of $K^1(X, \mathbb{R}/\mathbb{Z})$ is given by (τ, d, ω) where τ is a trivial bundle with trivial connection d and ω is a closed form on X. There is also a description of elements of $K^1(X, \mathbb{R}/\mathbb{Z})$ by $\mathbb{Z}_2$-graded cocycles:

$$\mathcal{V}^{\pm} = (V^{\pm}, \nabla^{V^{\pm}}, \omega)$$

where $(V^{\pm}, \nabla^{V^{\pm}})$ are $\mathbb{Z}_2$-graded vector bundles with connections and such that

$$d\omega = ch(V^+, \nabla^{V^+}) - ch(V^-, \nabla^{V^-}).$$

K^0-module structure. Let (E, ∇^E) be a complex vector bundle with connection ∇^E over X. Define the module multiplication by (E, ∇^E) on $\mathcal{V}$ by

$$(E, \nabla^E) \cdot \mathcal{V} = (E \otimes V, \nabla^E \otimes \nabla^V, ch(E, \nabla^E) \wedge \omega) \tag{2.2.6}$$

where $\nabla^E \otimes \nabla^V$ is the usual tensor product connection (1.1.6).

Explicit maps and exactness of (part of) sequence : Consider the following sequence

$$\cdots \to K^1(X, \mathbb{R}) \xrightarrow{\alpha} K^1(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} K^0(X, \mathbb{Z}) \xrightarrow{ch} K^0(X, \mathbb{R}) \to \cdots \tag{2.2.7}$$

associated to the short exact sequence of coefficients $1 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 1$, where the horizontal maps are respectively

(2.2.8) $\alpha(\omega) = (\tau, d, \omega) - (\tau, d, 0) = (0, 0, \omega)$ is the inclusion map,

(2.2.9) $\beta(V, \nabla^V, \omega) = [V]$ is the forgetful map,

(2.2.10) $ch(V)$ is the even Chern character map given by (1.1.22).

The exactness of the sequence (2.2.7) is implicit in [36]. For the sake of completeness, we give a full elementary proof.

Lemma 2.2.5. *With respect to the sequence* (2.2.7)*, it is exact at* $K^1(X, \mathbb{R}/\mathbb{Z})$

and at $K^0(X,\mathbb{Z})$.

Proof. For

$$K^1(X,\mathbb{R}) \xrightarrow{\alpha} K^1(X,\mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} K^0(X,\mathbb{Z}),$$

note that $\text{im}(\alpha) \subseteq \ker(\beta)$ is clear from (2.2.8). We need to show $\ker(\beta) \subseteq \text{im}(\alpha)$. Let $\mathcal{V}-\mathcal{W} = (V,\nabla^V,\omega)-(W,\nabla^W,\theta) \in \ker(\beta)$ so that $\beta(\mathcal{V}-\mathcal{W}) = [V]-[W] = 0$. Recall that $[V]=[W]$ if and only if V and W are stably equivalent. Without loss of generality, there is an isomorphism $V \oplus \tau \cong W \oplus \tau$ for some trivial bundle τ. Formally, $V - W \cong 0$. This defines an element $(0,0,\omega_1-\omega_2)$ of $\text{im}(\alpha)$, where ω_1 (resp. ω_2) is the odd form in $\mathcal{V}$ (resp. $\mathcal{W}$). This shows that $ker(\beta) \subseteq im(\alpha)$ and hence the exactness at $K^1(X,\mathbb{R}/\mathbb{Z})$.

On the other hand, we consider the part

$$K^1(X,\mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} K^0(X,\mathbb{Z}) \xrightarrow{ch} K^0(X,\mathbb{R}).$$

Since any $[V]$ in $K^0(X)$ with vanishing Chern character lies in the torsion subgroup $K^0_{\text{Tors}}(X,\mathbb{Z})$, the sequence reduces to

$$K^1(X,\mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} K^0_{\text{Tors}}(X,\mathbb{Z}) \to 0, \tag{2.2.11}$$

and hence $[V]$ is the image of some element of $K^1(X,\mathbb{R}/\mathbb{Z})$ under β. This shows $\ker(ch) \subseteq \text{im}(\beta)$. The opposite direction is straightforward. Let $\mathcal{V},\mathcal{W} \in K^1(X,\mathbb{R}/\mathbb{Z})$. The Chern character of the image of β, together with the exactness condition, gives $ch([V]-[W]) = [d(\omega_1-\omega_2)] = 0$. So, $[V]-[W]$ lies in the kernel of ch. This shows that $\text{im}(\beta) \subseteq \ker(ch)$ and hence the exactness at $K^0(X,\mathbb{Z})$. □

2.3 Mayer-Vietoris sequence in $\mathbb{R}/\mathbb{Z}$ K-theory

Let X be a smooth manifold. Suppose X is covered by two open subsets U and V. The sequence of inclusions

$$U \cap V \rightrightarrows U \sqcup V \to X$$

induces the Mayer-Vietoris sequence in K-theory with compact supports

$$\cdots \to K_c^*(U \cap V) \xrightarrow{\delta} K_c^*(U) \oplus K_c^*(V) \xrightarrow{s} K_c^*(X) \to \cdots \tag{2.3.1}$$

where δ is the signed inclusion $a \mapsto (a, -a)$ and s is the formal sum $(u, v) \mapsto u \oplus v$. Roughly speaking, elements of $K_c^0(-)$ are compactly supported vector bundles; whilst elements of $K_c^1(-)$ are compactly supported smooth invertible maps in the sense that for every point in the subset there exists a compact set K contained in that subset such that the maps are invertible inside K and are zero outside K.

Since K-theory is a generalised cohomology theory, it is a standard result that the sequence (2.3.1) is exact and the proof of this is similar to the case of compactly supported cohomology theory, cf. [18, Proposition 2.7]. By associating (2.3.1) with coefficients in $\mathbb{R}/\mathbb{Z}$, we get a long exact sequence in a covariant way

$$\cdots \to K_c^*(U \cap V, \mathbb{R}/\mathbb{Z}) \to K_c^*(U, \mathbb{R}/\mathbb{Z}) \oplus K_c^*(V, \mathbb{R}/\mathbb{Z}) \to K_c^*(X, \mathbb{R}/\mathbb{Z}) \to \cdots . \tag{2.3.2}$$

When X is compact, we write $K^*(X, \mathbb{R}/\mathbb{Z}) = K_c^*(X, \mathbb{R}/\mathbb{Z})$ for $* = 0, 1$. The description of elements of compactly supported $\mathbb{R}/\mathbb{Z}$ K-theory is quite straightforward. For instance, for $U \subset X$ an open subset, an element $\mathcal{V} \in K_c^1(U, \mathbb{R}/\mathbb{Z})$ is given by $((E, F, \alpha), (\nabla^E, \nabla^F), \omega_U)$, where (E, F, α) is a compactly supported triple that lies in $K_c^0(U)$, i.e. $\alpha : E \to F$ is a bundle homomorphism on U and is a bundle isomorphism outside a compact subset $K \subset U$; (∇^E, ∇^F) is a pair of Hermitian connections on E and F respectively; and ω_U is a compactly supported odd degree differential form on U such that $d\omega_U = ch(E) - ch(F)$.

Similarly, an element $\mathcal{W} \in K_c^0(U, \mathbb{R}/\mathbb{Z})$ is given by $(g_U, (d, g_U^{-1} dg_U), \mu_U)$, where $g_U : U \to U(N, \mathbb{C})$ (for large N) is a smooth map such that for all $x \in U$ there exists $K \subset U$ with $g_U(y) = 0$ for $y \in U \backslash K$; the second entry is a pair of flat connections associated to g_U; and μ_U is a compactly supported even degree differential form satisfying (2.1.2). The relation is the stabilisation under homotopy.

We will use employ this tool to show the main results in Section 5.1.2 and Section 5.2.2.

Chapter 3

Geometric K-homology

Informally, K-homology is the natural dual to Atiyah-Hirzebruch K-theory. There are three equivalent definitions of K-homology defined via homotopy, analytic and geometric approaches [12]. For our purposes, the model we will be using is the *geometric* K-homology, first introduced by Baum and Douglas [13]. In this chapter, we explain the terminology, discuss its relation with rational homology theory via the homological Chern character map and compute some basic examples. We will also establish the invariance of the Chern character under the K-homology relations.

3.1 The group $K_*(X)$

Definition 3.1.1. Let X be a smooth compact manifold. A K-cycle over X is a triple (M, E, f) where M is a closed Spin^c manifold over X, E is a complex vector bundle over M, and $f : M \to X$ is a continuous map.

A K-cycle (M, E, f) is odd (resp. even) if M is odd (resp. even) dimensional. In general, M is not necessarily connected and E is allowed to have different fibre dimensions on different connected components of M. For simplicity, we assume M to be connected throughout. The addition operation on K-cycles is given by the disjoint union $(M_1, E_1, f_1) \sqcup (M_2, E_2, f_2)$.

Definition 3.1.2. There are three relations imposed on the K-cycles:

1. **Direct sum-disjoint union**

   Let E_1 and E_2 be two complex vector bundles on M, then

   $$(M, E_1 \oplus E_2, f) \sim (M, E_1, f) \sqcup (M, E_2, f). \tag{3.1.1}$$

2. **Vector bundle modification**

   Let (M, E, f) be a K-cycle over X. Let H be a Spin^c vector bundle of even rank on M. Let $\underline{\mathbb{R}}$ be a trivial real line bundle over M. The direct sum $H \oplus \underline{\mathbb{R}}$ is a spin^c vector bundle. Denote by $\Sigma H = S(H \oplus \underline{\mathbb{R}})$ the unit sphere bundle over M, which inherits the Spin^c structure as the boundary of the unit disk bundle. Over ΣH there is a complex vector bundle β_H which is an associated principal Spin^c-bundle with the Bott generator vector bundle β. Each fibre of β_H is an even dimensional sphere S^{2p}. Let $\rho : \Sigma H \to M$ be a projection map. The composition $f \circ \rho$ is continuous. Then, the original K-cycle is related to the modified K-cycle

   $$(M, E, f) \sim (\Sigma H, \beta_H \otimes \rho^* E, f \circ \rho). \tag{3.1.2}$$

3. **Bordism**

   Let (M, E, f) and (M', E', f') be two K-cycles over X. Then, (M, E, f) is said to be bordant to (M', E', f') if there exists a triple (W, F, φ) where W is a compact Spin^c manifold with boundary ∂W such that

   $$(\partial W, F|_{\partial W}, \varphi|_{\partial W}) \sim (-M, E, f) \sqcup (M', E', f') \tag{3.1.3}$$

   where $-M$ denotes the same manifold M with the reverse Spin^c structure.

Isomorphism: Two K-cycles (M, E, f) and (M', E', f') are said to be isomorphic if and only if there exists a Spin^c structure preserving diffeomorphism $\psi : M \to M'$ such that $\psi^* E' \cong E$ and $f = f' \circ \psi$. Denote by $\{(M, E, f)\}$ the collection of all isomorphic K-cycles over X.

Remark 3.1.3. The bundle modification is sometimes viewed as the analog of Bott Periodicity in K-homology.

Definition 3.1.4. The geometric K-homology $K_*(X) = \{(M, E, f)\}/\sim$ is an abelian group. The addition of K-cycles is given by disjoint union. Such a group is $\mathbb{Z}_2$-graded, i.e. $K_*(X) = K_0(X) \oplus K_1(X)$, where the grading is given by the parity of M.

Example 3.1.1.

1. Let $X = \mathrm{pt}$. Then, its K-homology is $K_0(\mathrm{pt}) \cong \mathbb{Z}$ generated by $(\mathrm{pt}, \mathrm{pt} \times \mathbb{C}, \mathrm{Id})$. On the other hand, $K_1(\mathrm{pt}) = 0$.

2. Let $X = S^n$ be the n-sphere. When n is even, the K-homology of S^n is

$$K_0(S^n) \cong \mathbb{Z} \oplus \mathbb{Z} \text{ generated by } (\mathrm{pt}, \mathrm{pt} \times \mathbb{C}, \iota) \text{ and } (S^n, \beta, \mathrm{Id}), \tag{3.1.4}$$

where $\iota : \mathrm{pt} \to S^n$ is the inclusion map. Clearly, $K_1(S^n) = 0$ for even n. When n is odd, the only non-trivial group is

$$K_1(S^n) \cong \mathbb{Z} \text{ generated by } (S^n, \tau, \mathrm{Id}) \tag{3.1.5}$$

where τ is the trivial bundle on S^n.

3. Let $X = L_k = L(k; 1, ..., 1) = S^3/\mathbb{Z}_k$ be a three dimensional Lens space. The K-homology L_k is

$$K_0(L_k) \cong \mathbb{Z} \text{ generated by } (\mathrm{pt}, \mathrm{pt} \times \mathbb{C}, \iota)$$

where $\iota : \mathrm{pt} \to L_k$ is the inclusion map. The odd K-homology is

$$K_1(L_k) \cong \mathbb{Z} \oplus \mathbb{Z}_k$$

where the free and the torsion parts are generated by

$$(L_k, \tau_{L_k}, \mathrm{Id}_{L_k}) \text{ and } (L_k, \pi^*\beta_0^*, \mathrm{Id}_{L_k})$$

respectively. Here, τ_{L_k} is the trivial bundle over L_k and $\pi^*\beta_0^*$ is the pullback of the dual bundle of β_0 via the projection map $\pi : L_k \to S^2$. Here, we identify L_k with the sphere bundle $S(\beta_0^{\otimes k})$ of the k-tensor product of the canonical line bundle β_0 over S^2, cf. [31, 46].

4. Let $X = \mathbb{R}^n$. Consider the short exact sequence

$$0 \to \mathrm{pt} \to S^n \to \mathbb{R}^n \to 0 \tag{3.1.6}$$

where S^n the one-point compactification of $\mathbb{R}^n$. For a positive odd integer n, the induced sequence for odd K-homology is

$$0 \to K_1(\{\infty\}) \to K_1(S^n) \to K_1(\mathbb{R}^n) \to 0. \tag{3.1.7}$$

Note that this is covariant because K-homology is an extraordinary homology theory. It then follows that since $K_1(\{\infty\}) = K_1(\mathrm{pt}) = 0$, there is a natural isomorphism

$$K_1(\mathbb{R}^n) \cong K_1(S^n). \tag{3.1.8}$$

This means that a generator of $K_1(\mathbb{R}^n)$ can be viewed as a generator of $K_1(S^n)$ given by (3.1.5). Indeed, from the analytic side, the Dirac operators on $\mathbb{R}^n$ and S^n share a similar formula. On the other hand, for a positive even integer n, the induced sequence for even K-homology is

$$0 \to K_0(\{\infty\}) \to K_0(S^n) \to K_0(\mathbb{R}^n) \to 0. \tag{3.1.9}$$

By reducing the K_0-group, we obtain a natural isomorphism

$$\widetilde{K}_0(\mathbb{R}^n) \cong \widetilde{K}_0(S^n) \tag{3.1.10}$$

so a generator of $\widetilde{K}_0(\mathbb{R}^n)$ is given by the non-trivial generator in (3.1.4).

3.2 Homological Chern character

Definition 3.2.1. Let (M, E, f) be a K-cycle over a smooth compact manifold X. Define the homological Chern character map by

$$\begin{aligned} ch : K_{0/1}(X) &\longrightarrow H_{\mathrm{even/odd}}(X, \mathbb{Q}) \\ ch(M, E, f) &= f_*\mathrm{PD}(ch(E)\mathrm{Td}(M)) \end{aligned} \tag{3.2.1}$$

where $ch(E)$ is the topological Chern character (1.1.22) of E, $\mathrm{Td}(M)$ is the Todd class (1.1.18) of M, PD denotes the Poincaré duality isomorphism between homology and cohomology theories and $f_* : H_*(M,\mathbb{Q}) \to H_*(X,\mathbb{Q})$ is the induced map in rational homology.

Remark 3.2.2. Note that the degree of the characteristic class $ch(E)\mathrm{Td}(M)$ is always even for M of any dimension and for arbitrary complex vector bundles E. However, if M is odd dimensional, then by Poincaré duality and the de-Rham theorem, the induced homology class $f_*\mathrm{PD}(ch(E)\mathrm{Td}(M))$ is odd.

Example 3.2.1.

1. Let $X = S^n$ be n-dimensional sphere. For an odd positive integer n, the non-trivial Chern character map is

$$ch_{odd} : K_1(S^n) \to H_{odd}(S^n,\mathbb{Q})$$
$$ch(S^n,\tau,\mathrm{Id}) = \mathrm{Id}_*\mathrm{PD}(ch(\tau)\mathrm{Td}(S^n)) = \mathrm{rk}(\tau)[S^n] \tag{3.2.2}$$

where $\mathrm{Td}(S^n) \equiv 1$ and $[S^n]$ is the fundamental class of $H_n(S^n,\mathbb{Z})$. Thus, in this special case the image lies in the integral homology. For an even positive integer n, we have

$$ch_{even} : K_0(S^n) \to H_{even}(S^n,\mathbb{Q})$$
$$ch(S^n,\beta,\mathrm{Id}) = \mathrm{Id}_*\mathrm{PD}(ch(\beta)\mathrm{Td}(S^n)) = ch(\beta) \frown [S^n] = 1. \tag{3.2.3}$$

Note that for $n = 2r$, let $\beta_0 = 1 - L_0 \to S^2$ be the canonical line bundle, then $\beta = \beta_0 \boxtimes \cdots \boxtimes \beta_0$ is the generator of $K^0(S^{2r})$ and so

$$ch(\beta_0 \boxtimes \cdots \boxtimes \beta_0) = c_1(\beta_0)^r \tag{3.2.4}$$

showing that $ch(\beta)$ is integral and of top degree.

2. Let $X = L_k = S^3/\mathbb{Z}_k$ be a three dimensional Lens space. Then, the non-trivial Chern character map

$$ch_{odd} : K_1(L_k) \to H_{odd}(L_k,\mathbb{Q}) \tag{3.2.5}$$

admits an integral lift, cf.[46]. Since the odd homology of L_k is $H_1(L_k) \oplus$

$H_3(L_k) \cong \mathbb{Z}_k \oplus \mathbb{Z}$, the map (3.2.5) reduces to

$$ch_{1/3} : K_1(L_k) \to H_1(L_k, \mathbb{Z}) \oplus H_3(L_k, \mathbb{Z})$$

in which the map for the free and the torsion parts are respectively given by

$$ch_3(L_k, \tau_{L_k}, \mathrm{Id}_{L_k}) = \mathrm{PD}(ch(\tau_{L_k})\mathrm{Td}(L_k)) = \mathrm{rk}(L_k)[L_k] \in H_3(L_k, \mathbb{Z}) \cong \mathbb{Z}.$$

$$ch_1(L_k, \pi^*\beta_0^*, \mathrm{Id}_{L_k}) = \mathrm{PD}(ch(\pi^*\beta_0^*)\mathrm{Td}(L_k)) = -\pi^*(c_1(\beta_0)) \frown [L_k] \in H_1(L_k, \mathbb{Z}) \cong \mathbb{Z}_k.$$

Lemma 3.2.3. *The Chern character map* (3.2.1) *is well-defined under the K-homology relations.*

Proof. The approach of proof below follows from [15]. We show that the homological Chern character ch respects the three relations on K-cycles. The relation of direct sum-disjoint union is immediate:

$$\begin{aligned} ch(M, E \oplus E', f) &= f_*\mathrm{PD}(ch(E \oplus E')Td(M)) \\ &= f_*\mathrm{PD}((ch(E) + ch(E'))Td(M)) \\ &= ch(M, E, f) + ch(M, E', f). \end{aligned}$$

Hence, ch is group homomorphism. For bordism, let (W, F, φ) be a K-chain over X such that (3.1.3). Then, it is also immediate that

$$\begin{aligned} ch(\partial W, F|_{\partial W}, \varphi|_{\partial W}) &= ch((M, E, f) \sqcup (-M', E', f')) \\ &= ch(M, E, f) - ch(M', E', f'). \end{aligned}$$

In particular, the homological Chern character satisfies the relation

$$ch \circ \partial = \partial \circ ch \tag{3.2.6}$$

where the smooth 'boundary' map is given by

$$\partial : (W, F, \varphi) \mapsto (\partial W, F|_{\partial W}, \varphi|_{\partial W}). \tag{3.2.7}$$

For vector bundle modification, we need to establish the equality

$$ch(M, E, f) = ch(\Sigma H, \beta_H \otimes \rho^* E, f \circ \rho) \tag{3.2.8}$$

where $\rho : \Sigma H \to M$ is the projection map. By definition, the term on the RHS is

$$ch(\Sigma H, \beta_H \otimes \rho^* E, f \circ \rho) = (f \circ \rho)_* \mathrm{PD}_{\Sigma H}(ch(\beta_H \otimes \rho^* E)\mathrm{Td}(\Sigma H)). \tag{3.2.9}$$

Since the Chern character respects the tensor product and the pullback, we have

$$ch(\beta_H \otimes \rho^* E) = ch(\beta_H) \cdot \rho^* ch(E). \tag{3.2.10}$$

Note that on the level of singular chains ρ_* is the induced map between the chain groups of ΣH and M, which makes the following diagram

$$\begin{array}{ccc}
\Omega^\bullet(\Sigma H) & \xrightarrow{\rho_!} & \Omega^{\bullet-2p}(M) \\
\downarrow{\scriptstyle \mathrm{PD}_{\Sigma H}} & & \downarrow{\scriptstyle \mathrm{PD}_M} \\
C_{\dim \Sigma H - \bullet}(\Sigma H) & \xrightarrow{\rho_*} & C_{\dim M - (\bullet - 2p)}(M)
\end{array}$$

commute. Here, the upper horizontal map $\rho_!$ is the push-forward given by the integration along the fibre of $\rho : \Sigma H \to M$; the vertical PD_M and the vertical $\mathrm{PD}_{\Sigma H}$ are the Poincaré duality maps. Then, by commutativity we have

$$\rho_* \circ \mathrm{PD}_{\Sigma H} = \mathrm{PD}_M \circ \rho_!. \tag{3.2.11}$$

A connection on the tangent bundle $T\Sigma H$ amounts to the splitting

$$T\Sigma H = T^v \Sigma H \oplus T^h \Sigma H \cong TS^{2p} \oplus \rho^* TM$$

where $T^v \Sigma H$ and $T^h \Sigma H$ denote the vertical and the horizontal tangent spaces respectively. Then, we have

$$\mathrm{Td}(\Sigma H) = \mathrm{Td}(T\Sigma H) = \mathrm{Td}(S^{2p}) \cdot \rho^* \mathrm{Td}(M). \tag{3.2.12}$$

From (3.2.9), we compute

$$\begin{aligned}
&ch(\Sigma H, \beta_H \otimes \rho^* E, f \circ \rho)\\
&\quad = f_* \rho_* \mathrm{PD}_{\Sigma H}(ch(\beta_H \otimes \rho^* E) \cdot \mathrm{Td}(\Sigma H))\\
&\quad = f_* \rho_* \mathrm{PD}_{\Sigma H}(ch(\beta_H) \cdot \rho^* ch(E) \cdot \mathrm{Td}(\Sigma H)) \qquad \text{by (3.2.10)}\\
&\quad = f_* \mathrm{PD}_M \rho_!(ch(\beta_H) \cdot \rho^* ch(E) \cdot \mathrm{Td}(\Sigma H)) \qquad \text{by (3.2.11)}\\
&\quad = f_* \mathrm{PD}_M \rho_!(\rho^*(ch(E)\mathrm{Td}(M)) \cdot ch(\beta_H)\mathrm{Td}(S^{2p})) \qquad \text{by (3.2.12)}\\
&\quad = f_* \mathrm{PD}_M(ch(E)\mathrm{Td}(M)) \int_{S^{2p}} ch(\beta_H|_{S^{2p}})\mathrm{Td}(S^{2p})\\
&\quad = f_* \mathrm{PD}_M(ch(E)\mathrm{Td}(M))\\
&\quad = ch(M, E, f).
\end{aligned}$$

where the term $\int_{S^{2p}} ch(\beta_H|_{S^{2p}})\mathrm{Td}(S^{2p}) = 1$ by the Atiyah-Singer index theorem [9]. See also [14, Proposition 6]. This completes the proof. □

3.3 Mayer-Vietoris sequence in K-homology

Let X be a smooth compact manifold. Let $\{U, V\}$ be a good cover of X, i.e. U, V and their intersection $U \cap V$ are contractible. Consider the sequence

$$U \cap V \rightrightarrows U \sqcup V \to X = U \cup X.$$

Since K-homology is an extraordinary homology theory, there is an associated covariant exact sequence

$$\cdots \to K_*(U \cap V) \xrightarrow{i_* \oplus j_*} K_*(U) \oplus K_*(V) \xrightarrow{r_* \ominus s_*} K_*(X) \xrightarrow{\partial_*} \cdots$$

where i_*, j_*, r_*, s_* are induced inclusion maps. In particular,

$$\begin{aligned}
&i_* : K_1(U \cap V) \to K_1(U)\\
\text{(3.3.1)} \qquad &(N, \xi, \gamma) \mapsto i_*(N, \xi, \gamma) = (N, \xi, i \circ \gamma).
\end{aligned}$$

The induced K-cycles for j_*, r_* and s_* are defined similarly. Then, the first map $i_* \oplus j_*$ is given by the disjoint union $(N, \xi, i \circ \gamma) \sqcup (N, \xi, j \circ \gamma)$. The second map

$r_* \ominus s_*$ is given by $(N, \xi, r \circ \gamma) \sqcup -(N', \xi', s \circ \gamma')$. It is immediate that

$$(r_* \ominus s_*) \circ (i_* \oplus j_*) = 0.$$

Let (M, E, f) be a K-cycle over X. Assume that M^n can be partitioned into two compact manifolds with boundary M_1, M_2 of the same dimension by a hypersurface N^{n-1}, then the connecting map $\partial_* : K_*(X) \to K_{*-1}(U \cap V)$ is given by

$$\partial_*(M, E, f) = \partial(M_1, E_1, f_1) = (\partial M_1, E_1|_{\partial M_1}, f_1|_{\partial M_1})$$

where ∂ is the boundary map (3.2.7). An example of such M^n would be the n-sphere S^n, in which we take $M_1 = M_2 = D^n$ and $N^{n-1} = S^{n-1}$.

Since K-homology is $\mathbb{Z}_2$-graded, it reduces to a six-term exact sequence in K-homology

$$\begin{array}{ccccc} K_0(U \cap V) & \longrightarrow & K_0(U) \oplus K_0(V) & \longrightarrow & K_0(U \cup V) \\ \uparrow & & & & \downarrow \\ K_1(U \cup V) & \longleftarrow & K_1(U) \oplus K_1(V) & \longleftarrow & K_1(U \cap V). \end{array}$$

3.4 Perspective from physics

K-homology is deeply related to string theory in physics. In our case, we are only concern with Type IIA and IIB string theories since we are working with the complex K-theory of X. It was argued by Reis-Szabo [46, 50] that the fundamental objects in Type IIA and IIB string theories, known as *D-branes*, can be naturally described in terms of geometric K-cycles: for an oriented spin ten dimensional manifold X, a D-brane over X is a spin^c submanifold M of X, carrying the Chan-Paton bundle E, together with an embedding $f : M \hookrightarrow X$.

One of the justifications that K-homology is a more suitable arena for describing D-branes (suggested by Reis-Szabo [46]) than K-theory (suggested by Witten [52]) is that the former transforms covariantly under an induced "push-forward" map f_*, which is natural in terms of D-branes when f is an embedding. In contrast, K-theory behaves contravariantly with respect to f^*. Besides this, D-branes a priori carry stable vector bundles, rather than virtual bundles.

In [46], the relationship between K-homology relations on K-cycles and non-trivial dynamical aspects of D-branes was studied explicitly on the level of generators. In particular, the stability of D-branes corresponds to the decomposability of K-cycles. The decaying of unstable D-branes into stable bound states is given by a mechanism called *tachyon condensation* on their worldvolume; whilst the opposite is known as the *polarisation* of D-branes, in which D-branes expand or polarise into higher dimensional D-branes. It is also called the *dielectric effect*. Informally, this corresponds to the relation of vector bundle modification of K-cycles.

As an evident topological object, we can talk about the deformation of D-brane worldvolumes continuously together with the Chan-Paton gauge bundle over it. This corresponds to the bordism relation of K-homology. A qualitative meaning of this is that if a D-brane is bordant to a trivial cycle on X, then it carries the same charge as the vacuum. On the other hand, the direct sum of K-cycles corresponds to the gauge symmetry enhancement for coincident branes, which happens when M is wrapped by several D-branes.

Although we do not discuss anaytic K-homology in this thesis, there is also an explicit description of D-branes in terms of analytic K-cycles in [50]. In particular, Szabo linked operations of tachyons such as tachyon condensation and deformation of tachyons with the equivalence relation on analytic K-cycles, showing that D-branes *are* Fredholm modules.

We end by summarizing the discussion above in the following table.

K-cycles	D-branes
Direct sum-disjoint union $(M, E_1, f) + (M, E_2, f) = (M, E_1 \oplus E_2, f)$	Gauge symmetry enhancement for coincident brane
Bordism $(\partial W, F\|_{\partial W}, \phi\|_{\partial W}) \cong (M_1, E_1, f_1) + (-M_2, E_2, f_2)$	Continuous deformation of D-branes
Vector bundle modification $(M, E, f) \cong (\Sigma H, \beta_H \otimes \rho^* E, f \circ \rho)$	Dielectric effect, branes within branes

Chapter 4

The Dai-Zhang Toeplitz index theorem

In this chapter, we discuss a new and recent result by Dai and Zhang [21] on the generalisation of the classical Toeplitz index theorem to manifolds with boundary. In particular, they proved an index theorem for Toeplitz operators on *odd* dimensional compact spin manifolds with *even* dimensional boundary. This can be viewed as the *direct* odd dimensional analog of the celebrated Atiyah-Patodi-Singer index theorem [5–7]. By 'direct analog', we mean that the Dai-Zhang Toeplitz index theorem establishes an equality between the analytic index of a certain perturbed Toeplitz operator and the topological index of that manifold together with some boundary correction terms. One of the correction terms is a new eta-type spectral invariant — the Dai-Zhang eta-invariant. It fits into the role as a measure of spectral asymmetry in the even case. We will discuss the construction of the Dai-Zhang eta-invariant in Section 4.5.

For the reader's benefit, this chapter is accordingly devoted to the following reviews. Section 4.1: the Atiyah-Singer index theorem (even dimensional closed manifolds); Section 4.2: the Atiyah-Patodi-Singer index theorem (even dimensional manifolds with boundary); Section 4.3: the classical Toeplitz index theorem (odd dimensional closed manifolds); Section 4.4: the Dai-Zhang Toeplitz index theorem (odd dimensional manifolds with boundary).

No new result is contained in this chapter. Readers who are familiar with these notions can proceed to the next chapter and only come back for notation.

4.1 The Atiyah-Singer index theorem

Let X be a closed spin manifold of even dimension $2n$. Let $S = S(TX) = S^{\pm}$ be the spinor bundle over X, whose $\mathbb{Z}_2$-grading is given by (1.2.15). Let

$$\partial\!\!\!/ : \Gamma(S^{\pm}) \to \Gamma(S^{\mp}) \tag{4.1.1}$$

be the Dirac operator on X defined by (1.2.19), which is an elliptic self-adjoint first order differential operator. The Dirac operator $\partial\!\!\!/$ anti-commutes with $c(\omega_{\mathbb{C}})$, i.e. $\partial\!\!\!/ c(\omega_{\mathbb{C}}) = -c(\omega_C)\partial\!\!\!/$, since $c(\omega_{\mathbb{C}})$ anti-commutes with $c(e_i)$ and $\nabla^S \omega_{\mathbb{C}} = 0$. Thus, it can be written as

$$\partial\!\!\!/ = \begin{pmatrix} 0 & \partial\!\!\!/^- \\ \partial\!\!\!/^+ & 0 \end{pmatrix} \tag{4.1.2}$$

with respect to $S^{\pm}$. Since $\partial\!\!\!/$ is self-adjoint, the equality $(\partial\!\!\!/^{\pm})^* = \partial\!\!\!/^{\mp}$ holds. Let $L^2(S)$ be the space of L^2-sections of S, i.e. the L^2-completion of the space of smooth sections $\Gamma(S)$. By standard elliptic theory, the operator

$$\partial\!\!\!/ : L^2(S^{\pm}) \longrightarrow L^2(S^{\mp})$$

is a self-adjoint Fredholm operator (i.e. a self-adjoint bounded linear operator with closed range and with finite dimensional kernel and cokernel). Then, the analytic index of $\partial\!\!\!/^+$ is defined by

$$\mathrm{Ind}(\partial\!\!\!/^+) = \dim \ker(\partial\!\!\!/^+) - \dim \operatorname{coker}(\partial\!\!\!/^+) \in \mathbb{Z} \tag{4.1.3}$$

or equivalently $\mathrm{Ind}(\partial\!\!\!/^+) = \dim \ker(\partial\!\!\!/^+) - \dim \ker(\partial\!\!\!/^-)$. The analytic index Ind is a globally defined spectral invariant (under compact perturbations) of X.

Theorem 4.1.1 (Atiyah-Singer index theorem for spin Dirac operators [8–10])**.** *Let X be an even dimensional closed spin manifold. Let $\partial\!\!\!/$ be a Dirac operator on X. Then, its analytic index coincides with the topological index*

$$\mathit{Ind}(\partial\!\!\!/^+) = \int_X \hat{A}(TX, \nabla^{TX}) = \hat{A}(X) \tag{4.1.4}$$

where $\hat{A}(X)$ is the A-hat genus (1.1.15).

Remark 4.1.2. The non-triviality of the equality (4.1.4) is that whilst the LHS is an integer, the RHS is *not* in general. For instance, when X is a compact manifold of dimension 4, $\hat{A}(X) = -\frac{1}{24}p_1(X)$, where $p_1(X)$ is the rational (first) Pontryagin number of X. Hence, the Atiyah-Singer index theorem for Dirac operators asserts the integrality of $\hat{A}(X)$ when X is closed and spin.

Let E be a complex vector bundle over X. Let $S \otimes E$ be the tensor product bundle. Assume that $S \otimes E$ is equipped with a Hermitian metric and an induced unitary connection. Let $\partial\!\!\!/_{S\otimes E}$ be the twisted Dirac operator defined by (1.2.20). The $\mathbb{Z}_2$-grading (4.1.2) extends to

$$\partial\!\!\!/_{S\otimes E} = \begin{pmatrix} 0 & \partial\!\!\!/^-_{S\otimes E} \\ \partial\!\!\!/^+_{S\otimes E} & 0 \end{pmatrix}.$$

Theorem 4.1.3 (Atiyah-Singer index theorem for twisted spin Dirac operators [8–10])**.** *Let X be an even dimensional closed spin manifold. Let $\partial\!\!\!/_{S\otimes E}$ be the Dirac operator on a Dirac bundle $S \otimes E$. Then, we have*

$$Ind\, \partial\!\!\!/^+_{S\otimes E} = \int_X \hat{A}(TX, \nabla^{TX}) \wedge ch(E, \nabla^E), \tag{4.1.5}$$

where $ch(E, \nabla^E)$ is the Chern character form (1.1.22) *of E.*

Theorem 4.1.4 (Atiyah-Singer index theorem for twisted spin^c Dirac operators [8–10])**.** *Let X be an even dimensional closed spin^c manifold. Let S be a Spin^c spinor bundle on X as an associated bundle $P_{Spin^c} \times_{\rho^c} V$ for V a left $Cl(X)$-module. Let E be a complex vector bundle on X. Then,*

$$\begin{aligned} Ind(\partial\!\!\!/^+_{S\otimes E}) &= \int_X \hat{A}(TX, \nabla^{TX}) \wedge e^{\frac{c_1(L,\nabla^L)}{2}} \wedge ch(E, \nabla^E) \\ &= \int_X Td(TX, \nabla^{TX}) \wedge ch(E, \nabla^E) \end{aligned} \tag{4.1.6}$$

where L is the canonical line bundle of the spin^c structure and $Td(TX, \nabla^{TX})$ is the Todd form (1.1.17) *of X.*

4.2 The Atiyah-Patodi-Singer index theorem

Let X be a compact oriented even dimensional manifold with odd dimensional boundary $\partial X = Y$. Assume that X is isometric to $(0,1] \times Y$ in a collar neighbourhood of Y. Let $p : (0,1] \times Y \to Y$ be the projection map. Let $\partial\!\!\!/_X$ be the $\mathbb{Z}_2$-graded Dirac operator acting on sections of a $\mathbb{Z}_2$-graded Dirac bundle $S = S^{\pm}$ over X. Let $\partial\!\!\!/_Y$ be the Dirac operator acting on sections of an ungraded Dirac bundle S' over Y. Assume that there is an isomorphism $p^*(S' \oplus S') \cong S$ over the collar neighbourhood whose Clifford action of TX is given by

$$c(\partial_t) = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}; \quad c_X(\nu) = \begin{pmatrix} 0 & c_Y(\nu) \\ c_Y(\nu) & 0 \end{pmatrix}$$

for $t \in (0,1]$ and $\nu \in TY$. Then, near the boundary we have

$$\partial\!\!\!/_X = c(\partial_t)(\partial_t + \partial\!\!\!/_Y).$$

In matrix form, the Dirac operator can be written as

$$\partial\!\!\!/ = \begin{pmatrix} 0 & -\partial_t + \partial\!\!\!/_Y \\ \partial_t + \partial\!\!\!/_Y & 0 \end{pmatrix},$$

in which the off-diagonal terms are the formal adjoints of each other.

Let $L^2(S')$ be the L^2-completion of smooth sections of S', on which $\partial\!\!\!/_Y$ acts as an unbounded symmetric operator. In particular, $\partial\!\!\!/_Y$ is essentially self-adjoint and its closure has compact resolvent. Thus, $L^2(S')$ has spectral decomposition $\{\lambda_i, \phi_i\}$ where λ_i and ϕ_i are eigenvalues and eigenvectors respectively, i.e. it can be written as the direct sum of eigenspaces $\mathbb{E}_{\lambda_i}$ with respect to λ_i

$$L^2(S') = \bigoplus_{\lambda_i \in \mathrm{spec}(\partial\!\!\!/_Y)} \mathbb{E}_{\lambda_i}. \tag{4.2.1}$$

Definition 4.2.1. [5–7] The eta-function associated to $\partial\!\!\!/_Y$ is defined by the series

$$\eta(\partial\!\!\!/_Y, s) = \sum_{\lambda_i \neq 0} \frac{\mathrm{sgn}(\lambda_i)}{|\lambda_i|^s} \tag{4.2.2}$$

for $s \in \mathbb{C}$ and $\mathrm{Re}(s)$ sufficiently large. When $s = 0$, the term

$$\eta(\not{\partial}_Y) = \eta(\not{\partial}_Y, 0) \tag{4.2.3}$$

is called the Atiyah-Patodi-Singer eta-invariant.

Remark 4.2.2. In fact, the eta-invariant is defined for any self-adjoint elliptic differential operator acting on sections of a vector bundle over a closed manifold. The series (4.2.2) is absolutely convergent in the half plane $\mathrm{Re}(s) > \dim(Y)/m$, where m is the order of the operator $\not{\partial}_Y$. By Mellin transform, the eta-function (4.2.2) has an integral expression

$$\eta(\not{\partial}_Y, s) = \frac{1}{\Gamma(\frac{s+1}{2})} \int_0^\infty t^{\frac{s-1}{2}} \mathrm{Tr}(\not{\partial} e^{-t\not{\partial}^2}) dt. \tag{4.2.4}$$

When $s = 0$, the eta-invariant (4.2.3) can thus be written as

$$\eta(\not{\partial}_Y) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-\frac{1}{2}} \mathrm{Tr}(\not{\partial} e^{-t\not{\partial}^2}) dt. \tag{4.2.5}$$

The eta-invariant plays a crucial role as a boundary correction term in Theorem 4.2.3 below. Standard references on the eta-invariant are [16, 27, 44, 45].

Condition for $\not{\partial}_X$ being a Fredholm operator. Let

$$P_{\geq 0} : L^2(S') \to L^2_{\geq 0}(S') \tag{4.2.6}$$

be the orthogonal projection onto the non-negative eigenspaces of $\not{\partial}_Y$. Here, $L^2_{\geq 0}(S')$ denotes the direct sum of $\mathbb{E}'_\lambda$ with $\lambda \geq 0$. Such $P_{\geq 0}$ is often known as the *Atiyah-Patodi-Singer boundary projection*. Consider the restriction of $\not{\partial}_X$ to act on smooth sections $s|_Y$ subjected to the boundary problems

$$\begin{cases} \not{\partial}_X^+(s) = 0 \\ P_{\geq 0}(s|_Y) = 0 \end{cases} \quad \text{and} \quad \begin{cases} \not{\partial}_X^-(s) = 0 \\ P_{\geq 0}^\perp(s|_Y) = 0 \end{cases} \tag{4.2.7}$$

where $P_{\geq 0}^\perp = \mathrm{Id} - P_{\geq 0}$. Then, $\not{\partial}_X$ is a Fredholm operator. We denote the Fredholm index of $\not{\partial}_X$ by $\mathrm{Ind}(\not{\partial}_X^+)$. The celebrated Atiyah-Patodi-Singer index theorem asserts the following statement.

Theorem 4.2.3 (The Atiyah-Patodi-Singer index thereom for spin Dirac operators [5]). *Let X be an even dimensional compact spin manifold with odd dimensional closed boundary Y. In the above setting, the equality*

$$\text{(4.2.8)} \quad Ind(\partial\!\!\!/_X^+) = \int_X \hat{A}(TX, \nabla^{TX}) \wedge ch(E, \nabla^E) - \frac{\dim\ker(\partial\!\!\!/_Y) + \eta(\partial\!\!\!/_Y)}{2} \quad \in \mathbb{Z}$$

holds where $\eta(\partial\!\!\!/_Y)$ is the eta-invariant (4.2.3).

Remark 4.2.4. Theorem 4.2.3 extends to the case of twisted spin^c Dirac operators $\partial\!\!\!/_E^+$ twisted by a complex vector bundle E on an even dimensional spin^c compact manifold X. In particular, the integrand becomes $\mathrm{Td}(TX, \nabla^{TX}) \wedge ch(E, \nabla^E)$, similar to (4.1.6). In general, Theorem 4.2.3 holds for any elliptic first order differential operator Q on X. See [5–7, 17] for more details on this index theorem. An approach via b-calculus by Melrose is explained in [42].

4.3 The classical Toeplitz index theorem

Let X be an odd dimensional closed spin smooth manifold. Assume that X has a Riemannian structure. Let $\partial\!\!\!/_X$ be the (ungraded) self-adjoint Dirac operator acting on $L^2(S)$ for the (ungraded) spinor bundle $S \to X$. The Atiyah-Singer index theorem for Dirac operators does not extend non-trivially to this odd case because any elliptic differential operators on odd dimensional closed manifolds have vanishing index. Hence, in order to obtain a non-trivial index, *Toeplitz operators* on X are considered and will be explained below.

Let $g : X \to GL(N, \mathbb{C})$ be a smooth map into the general linear group for some large positive integer N. Without loss of generality, we assume the group $GL(N, \mathbb{C})$ is reduced to the unitary group $U(N) = U(N, \mathbb{C})$ via a given Riemannian metric on X. So, $g : X \to U(N)$ represents a class of $K^1(X)$ as in (1.1.32). Let τ be the complex trivial bundle upon which g acts on smoothly. Let $E \to X$ be a complex vector bundle. Let $L^2(S \otimes E \otimes \tau)$ be the space of L^2 sections, on which g extends to act as identity on $L^2(S \otimes E)$ and as an automorphism on τ. Then, the ungraded twisted Dirac operator

$$\partial\!\!\!/_X : L^2(S \otimes E \otimes \tau) \to L^2(S \otimes E \otimes \tau)$$

is self-adjoint. Now, we have an orthogonal decomposition

$$L^2(S \otimes E \otimes \tau) = \bigoplus_{\lambda \in \mathrm{spec}\, \partial\!\!\!/} \mathbb{E}_\lambda \tag{4.3.1}$$

where $\mathbb{E}_\lambda$ is the eigenspace associated to the eigenvalue λ. Define the orthogonal projection

$$P_{\geq 0} : L^2(S \otimes E \otimes \tau) \longrightarrow L^2_{\geq 0}(S \otimes E \otimes \tau)$$

where $L^2_{\geq 0}(S \otimes E \otimes \tau)$ corresponds to the eigenvalue $\lambda \geq 0$. Then, the Toeplitz operator T_g^E twisted by E is defined by

$$T_g^E := P_{\geq 0} g P_{\geq 0} : L^2_{\geq 0}(S \otimes E \otimes \tau) \longrightarrow L^2_{\geq 0}(S \otimes E \otimes \tau). \tag{4.3.2}$$

One can show that T_g^E is a Fredholm operator with parametrix $T^E_{g^{-1}}$. The odd analog of the Atiyah-Singer index theorem, which we call the classical Toeplitz index theorem, asserts the equality between the Fredholm index of the Toeplitz operator and the topological index of X.

Theorem 4.3.1 (The classical Toeplitz index theorem [13])**.** *Let X be an odd dimensional closed spin manifold. Let T_g^E be the Toeplitz operator defined by* (4.3.2)*. Then, the equality*

$$Ind\ T_g^E = -\int_X ch(g,d) \wedge ch(E, \nabla^E) \wedge \hat{A}(TX, \nabla^{TX}) \tag{4.3.3}$$

holds where $ch(g,d)$ is the odd Chern character given by (1.1.34) *and the other two terms are the standard characteristic forms as in* (4.1.5)*.*

Example 4.3.1. Let $X = S^1$. It was shown in [17] that

$$\mathrm{Ind}(T_g) = \mathrm{sf}(\partial\!\!\!/, g^{-1}\partial\!\!\!/ g) = -\int_X ch(g), \tag{4.3.4}$$

where $\partial\!\!\!/ = -id/d\theta$ is the usual Dirac operator on S^1 and sf is the spectral flow of the one parameter family

$$\partial\!\!\!/(s) = s\partial\!\!\!/ + (1-s) g^{-1} \partial\!\!\!/ g$$

for $s \in [0,1]$. The first equality of (4.3.4) follows from [17, Theorem 17.17]; the second equality follows from [26, Theorem 2.8]. In particular, when $g(\theta) = \exp(i\theta)$ is a smooth map of degree 1, then (4.3.4) is the K-theory Poincaré duality pairing

$$K^1(S^1) \times K_1(S^1) \to \mathbb{Z}; \quad (g, \partial\!\!\!/_{S^1}) \mapsto \mathrm{Ind}(T_g) = -\int_X ch(g) = \deg(g) = 1, \tag{4.3.5}$$

where $T_g = P_{\geq 0} g P_{\geq 0}$ and $P_{\geq 0}$ is the orthogonal projection onto the non-negative eigenspaces of $\partial\!\!\!/_{S^1}$. Here, we use the fact that $\mathrm{Ind}(T_g) = \mathrm{Ind}(\tilde{T}_g)$ for $\tilde{T}_g = P_{\geq 0} - gP_{<0}$, cf. [16, 17].

4.4 The Dai-Zhang Toeplitz index theorem

In [21], Dai and Zhang have extended the classical Toeplitz index theorem to the case of manifolds with boundary. It is indeed the *direct* odd analog to the Atiyah-Patodi-Singer index theorem (4.2.3). In particular, there are contributions from the boundary terms associated to an elliptic differential operator on an even dimensional closed manifold Y. In the following, we sketch the idea and the setup behind the formulation of the Dai-Zhang index theorem.

Let X be an odd dimensional compact spin Riemannian manifold with boundary Y. Assume that Y has a Riemannian structure induced from that of X. Denote the Riemannian metrics on X and Y as g^X and g^Y respectively. Assume that X is isometric to $(0,1] \times Y$ in a collar neighborhood of Y. Hence, we may assume that the metric is of product type over the cylinder

$$g^X|_{(0,1]\times Y} = dx^2 \oplus g^Y.$$

Let $E \to X$ be a complex vector bundle and $S \to X$ the canonical spinor bundle. Over the cylinder $(0,1] \times Y$, the Dirac operator $\partial\!\!\!/^E = \sum c \circ \nabla^{S\otimes E}$ takes the form

$$\partial\!\!\!/^E_X = c(\partial_x)(\partial_x + \partial\!\!\!/^E_Y).$$

The Dirac operator $\partial\!\!\!/^E_Y$ is elliptic and self-adjoint. Here, we abuse notation by writing $\partial\!\!\!/^E_Y$ for the Dirac operator twisted by $E|_Y$, and it is assumed to be independent of the parameter in $(0,1]$. As in [5], we obtain a *global* elliptic boundary condition for $\not{\partial}^E$ by considering the orthogonal projection

$$P_Y : L^2(S \otimes E \otimes \tau|_Y) \to L^2_{>0}(S \otimes E \otimes \tau|_Y). \tag{4.4.1}$$

Since Y is even dimensional, we need a choice of 'half-space' in the kernel of $\not{\partial}_Y$ to obtain an elliptic self-adjoint boundary condition. This is a Lagrangian subspace $L \subset \ker(\not{\partial}^E_Y)$ such that $c(\partial_x)L = L^\perp \cap \ker(\not{\partial}^E_Y)$ and $\dim(L) = \dim(L^\perp)$. Let P_L be the orthogonal projection onto L

$$P_L : L^2(S \otimes E \otimes \tau|_Y) \to L. \tag{4.4.2}$$

Consider the modified Atiyah-Patodi-Singer projection

$$P^\partial = P_Y + P_L \tag{4.4.3}$$

from (4.4.1) and (4.4.2). By abuse of notation, *a choice of L is assumed* when we write P^∂. Thus, for $\nu \in L^2(S \otimes E \otimes \tau)$,

$$\begin{cases} \not{\partial}^E_X(\nu) = 0 \\ P^\partial(\nu|_Y) = 0 \end{cases}$$

is a well-defined elliptic self-adjoint boundary problem $(\not{\partial}^E_X, P^\partial)$. Let $\not{\partial}^E_{P^\partial}$ be the corresponding elliptic self-adjoint operator which acts on $L^2_{\geq 0, P^\partial}(S \otimes E)$. Let $g : X \to U(N)$ be a K^1-representative on X, which acts as an automorphism on the trivial bundle τ and as the identity on $L^2_{P^\partial}(S \otimes E)$. This extends to an action on $L^2_{P^\partial}(S \otimes E \otimes \tau)$. Denote by

$$P_{P^\partial} : L^2(S \otimes E \otimes \tau) \to L^2_{\geq 0, P^\partial}(S \otimes E \otimes \tau)$$

the orthogonal projection onto the non-negative eigenspaces

$$L^2_{\geq 0, P^\partial}(S \otimes E \otimes \tau) = \bigoplus_{\lambda \in \mathrm{spec}(\not{\partial}_{P^\partial}), \lambda \geq 0} \mathbb{E}_\lambda.$$

In order to define generalised Toeplitz operators, we consider $gP^\partial g^{-1}$ the orthogonal projection onto $L^2(S \otimes E \otimes \tau|_Y)$. Let $(\partial\!\!\!/^E, gP^\partial g^{-1})$ be another associated elliptic boundary problem. Let $\partial\!\!\!/^E_{gP^\partial g^{-1}}$ be the corresponding Dirac operator and let $P_{gP^\partial g^{-1}}$ be the orthogonal projection

$$P_{gP^\partial g^{-1}} : L^2(S \otimes E \otimes \tau) \to L^2_{\geq 0, gP^\partial g^{-1}}(S \otimes E \otimes \tau).$$

Definition 4.4.1. In the above setting, the generalised Toeplitz operator $T^E_{P^\partial}$ is defined as the composition

$$T^E_{g,P^\partial} = P_{gP^\partial g^{-1}} \circ g \circ P_{P^\partial} : L^2_{\geq 0, P^\partial}(S \otimes E \otimes \tau) \to L^2_{\geq 0, gP^\partial g^{-1}}(S \otimes E \otimes \tau). \tag{4.4.4}$$

To ensure the existence of such a Lagrangian subspace L, we have to make the following assumption.

Assumption 1. *The Dirac operator $\partial\!\!\!/^{E,+}_Y$ on an even dimensional boundary manifold Y has vanishing index , i.e. $Ind(\partial\!\!\!/^{E,+}_Y) = 0$.*

Let X be an odd dimensional spin manifold with boundary Y. Let g^{TX}, ∇^{TX} and R^{TX} be respectively the Riemannian metric, its associated Levi-Civita connection and its curvature on X. Let $E \to X$ be a complex vector bundle equipped with a Hermitian connection ∇^E. Let $\partial\!\!\!/^{E\otimes\tau}$ be the Dirac operator acting on $L^2(S \otimes E \otimes \tau)$.

Let $\mathcal{P}_X$ be the Calderón projection associated to $\partial\!\!\!/^{E\otimes\tau}$ on X, which is an orthogonal projection onto $L^2(S\otimes E\otimes\tau|_Y)$. Then, $\mathcal{P}_X - P^\partial$ is a pseudodifferential operator of order less than zero, c.f. [17]. Let $\mathrm{Mas}(gP^\partial g^{-1}, P^\partial, \mathcal{P}_X)$ be the Maslov triple index in the sense of Kirk-Lesch [32].

Theorem 4.4.2 (The Dai-Zhang Toeplitz index theorem [21])**.** *In the above setting, together with Assumption 1, we have the following identity*

$$\begin{aligned} Ind(T^E_{g,P^\partial}) = -\varphi \int_X & \widehat{A}(TX, \nabla^{TX}) \wedge ch(E, \nabla^E) \wedge ch(g, d) \\ & - \bar{\eta}(Y, g) + Mas(gP^\partial g^{-1}, P^\partial, \mathcal{P}_X) \quad \in \mathbb{Z}. \end{aligned} \tag{4.4.5}$$

where φ is the constant term $\left(2\pi\sqrt{-1}\right)^{-(dim\ M-1)/2}$ and $\bar{\eta}(Y,g)$ is the Dai-Zhang eta-invariant as explained below.

4.5 The Dai-Zhang eta-invariant

The Dai-Zhang eta-invariant $\bar{\eta}(Y,g)$ arises as (one of) the boundary correction terms in (4.4.5), which fits in the role of the Atiyah-Patodi-Singer eta-invariant (4.2.3) in this even case. The remainder of this section is devoted to the construction of such an *even* eta-invariant and to see how a K^1-representative is incorporated in a fundamental way. This is particularly important in the formulation of the analytic pairing (5.1.2) when such an element lifts to an element of $K^0(X,\mathbb{R}/\mathbb{Z})$. Full details on the analysis of this even eta-invariant can be found in [20–22].

Let X be an even dimensional closed spin^c manifold and E be a complex vector bundle over X. Let $\partial\!\!\!/_{E,X}$ be the Dirac operator on X twisted by E. Consider the cylinder $[0,1]\times X$ with a product metric near the boundary.

- Twist $\partial\!\!\!/_{E,X}$ by $g: X\to U(N)$, acting on $L^2(S\otimes E\otimes\tau)$. The smooth map g acts as the identity on $L^2(S\otimes E)$ and as an automorphism on τ. Denote this by $\partial\!\!\!/^g_{E\otimes\tau,X}$;

- Extend $S\otimes E\otimes\tau$ trivially to the cylinder $[0,1]\times X$, i.e. over each $t\in[0,1]$ there is a copy of E. Let $\psi=\psi(t)$ be a cut-off function on $[0,1]$ which is identically 1 in a ϵ-neighborhood of X for small $\epsilon>0$ and 0 outside of a 2ϵ-neighborhood of X. Consider the Dirac-type operator

  $$\partial\!\!\!/^{\psi}_{E\otimes\tau,X\times[0,1]}=(1-\psi)\partial\!\!\!/_{E\otimes\tau}+\psi g\partial\!\!\!/_{E\otimes\tau}g^{-1}$$

  and its conjugation

  $$\partial\!\!\!/^{\psi,g}_{E\otimes\tau,X\times[0,1]}=g^{-1}\partial\!\!\!/^{\psi}_{E}g=\partial\!\!\!/_{E\otimes\tau}+(1-\psi)g^{-1}[\partial\!\!\!/_{E\otimes\tau},g]; \tag{4.5.1}$$

- Assume that the Lagrangian $L\subset\ker(\partial\!\!\!/^g_{E\otimes\tau,X})$ exists and fix a choice of L. Equip (4.5.1) on one end $X\times\{0\}\cong X$ with the modified Atiyah-Patodi-

Singer boundary conditions (4.4.3)

$$P^{\partial} = P_{\geq 0} + P_L : L^2_{\geq 0}(S \otimes E \otimes \tau) \to L^2_{\geq 0}(S \otimes E \otimes \tau|_X) \oplus L. \tag{4.5.2}$$

Equip (4.5.1) on the other end $X \times \{1\}$ with the complementary conjugated orthogonal projection $\mathrm{Id} - g^{-1}P^{\partial}g$.

Then, $(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}, P^{\partial}, \mathrm{Id} - g^{-1}P^{\partial}g)$ is a self-adjoint elliptic boundary problem. For simplicity, we denote the boundary problem by $\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}$, i.e. with the boundary conditions implicitly implied. Let the eta-function of $\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}$ be given by the usual formula

$$\eta(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}, s) = \sum_{\lambda\neq 0} \frac{sgn(\lambda)}{|\lambda|^s} \tag{4.5.3}$$

for $\mathrm{Re}(s)$ sufficiently large and with the sum running through all non-zero eigenvalues λ of $\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}$. Take $\eta(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}) := \eta(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}, 0)$. Let $\hat{\eta}(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]})$ be the full eta-invariant defined by

$$\widehat{\eta}(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}) = \frac{\eta(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}) + h(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]})}{2} \tag{4.5.4}$$

where $h(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}) = \dim\ker(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]})$.

Definition 4.5.1. [21] With the construction above, an eta-type invariant on an even dimensional closed manifold is defined by

$$\bar{\eta}(X, E, g) = \widehat{\eta}(\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}) - \mathrm{sf}\big\{\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}(s); s \in [0,1]\big\} \tag{4.5.5}$$

where the second term is the spectral flow of $\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}(s)$, defined by

$$\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}(s) = D_{E\otimes\tau} + (1 - s\psi)g^{-1}\not{\partial}_{E\otimes\tau}g \tag{4.5.6}$$

on $X \times [0,1]$, with boundary projections P^{∂} on $X \times \{0\}$ and $\mathrm{Id} - g^{-1}P^{\partial}g$ on $X \times \{1\}$. That is, $\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}(s)$ is a path connecting $g^{-1}\not{\partial}_{E\otimes\tau}g$ and $\not{\partial}^{\psi,g}_{E\otimes\tau,X\times[0,1]}$. We call (4.5.5) the Dai-Zhang eta-invariant.

Remark 4.5.2. A priori, the spectral flow in (4.5.5) is an integer which measures the net change of positive crossing (from negative to positive eigenvalues across 0) and negative crossing (from positive to negative eigenvalues across 0). Thus, upon reduction modulo $\mathbb{Z}$, we obtain an $\mathbb{R}/\mathbb{Z}$-valued invariant in the sense of Atiyah-Patodi-Singer [7]:

$$\bar{\eta}(\eth^{\psi,g}_{E\otimes\tau,X\times[0,1]}) := \bar{\eta}(X,E,g) \equiv \widehat{\eta}(\eth^{\psi,g}_{E\otimes\tau,X\times[0,1]}) \bmod \mathbb{Z}.$$

4.5.1 Properties of the Dai-Zhang eta-invariant

One observes that the RHS of (4.5.5) involves much more data than the LHS. It is natural to ask whether this invariant is well-defined under the change of this data. In fact, Dai and Zhang [20, 21] have shown that the eta-invariant only depends on the triple (X,E,g), as summarised in the following.

For the choice of cut-off functions:

Proposition 4.5.3. *[21, Proposition 5.1] The invariant $\bar{\eta}(X,E,g,\psi)$ is independent of the cut-off function ψ.*

For the choice of length of radial interval of cylinder $[0,a]\times X$:

Proposition 4.5.4. *[20, Lemma 3.1] Upon reducing modulo $\mathbb{Z}$, the invariant $\bar{\eta}(X,E,g)$ is independent of a.*

Corollary 4.5.5. *[20] The adiabatic limit of this invariant is*

$$\bar{\eta}(X,E,g) = \lim_{a\to\infty} \bar{\eta}(\eth^{\psi,g}_{E,[0,a]\times X}).$$

For the choice of boundary projections:

Let P be a Cl(1)-spectral section with respect to P^∂, introduced by Melrose-Piazza [43] as a generalisation of the modified Atiyah-Patodi-Singer boundary projection P^∂ associated to L. In particular, P differs from P^∂ by a finite dimensional subspace. Both the spectral section P and its conjugation $g^{-1}Pg$ remain self-adjoint elliptic boundary projections for $\eth^E_X$. The Dai-Zhang eta-invariant is defined in the same way.

Proposition 4.5.6. *[21, Proposition 5.6] Let P and Q be any two Cl(1)-spectral sections with respect to P^{∂}. Then,*

$$\bar{\eta}(X, E, P) \equiv \bar{\eta}(X, E, Q) \ mod\ \mathbb{Z}.$$

For the change of g under continuous deformation:

Proposition 4.5.7. *[21] Let $\{g_t\}_{t\in[0,1]}$ be a smooth family of maps from $X \to U(N)$ connecting g_0 and g_1. Then, the equality*

(4.5.7)
$$\bar{\eta}(X, E, g_1) - \bar{\eta}(X, E, g_0) = \varphi \int_X \hat{A}(TX, \nabla^{TX}) \wedge ch(E, \nabla^E) \wedge Tch(g_t, d) \ mod\ \mathbb{Z}$$

holds where φ is the constant as in (4.4.5) and $Tch(g_t, d)$ is the transgression form (1.1.35) of the odd Chern character $ch(g_t)$.

Chapter 5

Analytic Pontryagin duality in K-theory

5.1 The even case: $K^0(X, \mathbb{R}/\mathbb{Z}) \times K_0(X)$

This section is dedicated to the proof of the following theorem.

Theorem 5.1.1. *Let M be an even dimensional closed spin^c manifold and let E be a complex vector bundle over M. Let X be a smooth compact manifold, together with a smooth map $f : M \to X$. Let $h = g \circ f : M \to U(N)$ be a K^1-representative of M and let τ be the trivial bundle in which h acts as an automorphism. Let $\partial\!\!\!/^{\psi,h}_{E\otimes\tau, M\times[0,1]}$ be the Dirac operator twisted by E and τ on the cylinder $M \times [0,1]$*

$$\partial\!\!\!/^{\psi,h}_{E\otimes\tau, M\times[0,1]} = \partial\!\!\!/_{E\otimes\tau} + (1-\psi)h^{-1}[\partial\!\!\!/_{E\otimes\tau}, h]. \tag{5.1.1}$$

Let $\bar{\eta}(\partial\!\!\!/^{\psi,h}_{E\otimes\tau, M\times[0,1]})$ be its reduced eta-invariant (4.5.5). Then, the analytic pairing

$$\begin{aligned} &K^0(X, \mathbb{R}/\mathbb{Z}) \times K_0(X) \longrightarrow \mathbb{R}/\mathbb{Z} \\ &\langle (g, (d, g^{-1}dg), \mu), (M, E, f) \rangle \\ &= \bar{\eta}\big(\partial\!\!\!/^{\psi,h}_{E\otimes\tau, M\times[0,1]}\big) - \int_M f^*\mu \wedge ch(E) \wedge Td(M) \ mod\ \mathbb{Z} \end{aligned} \tag{5.1.2}$$

is well-defined and non-degenerate.

5.1.1 Well-definedness of K^0 pairing

Well-defined on the level of cycle:

Proposition 5.1.2. *The analytic pairing* (5.1.2) *is independent of the Riemannian metric of the manifold M, the Hermitian metric and the connection on the complex vector bundle E.*

Proof. Fix a $\mathbb{R}/\mathbb{Z}$ K^0-cocycle $(r,(d,r^{-1}dr),\mu)$. For $i=1,2$, let $M_i=(M,g_i)$ be the same even dimensional manifold with different Riemannian metrics g_i, which is the boundary of a cylinder $N=M\times[0,1]$, i.e. $\partial N\cong M_1\sqcup -M_2$. Let

$$g_\gamma=\gamma(t)+(dt)^2 \tag{5.1.3}$$

be the extended metric on N, where $\gamma(t)$ is a path in the space of Riemannian metrics on M. Let g^{E_i},∇^{E_i} be the metric and Hermitian connection on $E_i=E|_{M_i}$ respectively. Let τ be the trivial bundle upon which $h=r\circ f:M\to U(N)$ acts as an automorphism. Let $\nabla^{E\otimes\tau}$ be the Hermitian tensor product connection on $E\otimes\tau$. Set

$$\nabla_p^{E\otimes\tau}=\partial_t\wedge dt+p(t)$$

to be a path of connections on $E\otimes\tau$ extended to N, where $p(t)$ is a path of connections on $E\otimes\tau$ over M. Let $\partial\!\!\!/^{\psi,h}_{E_i\otimes\tau,M\times[0,1]}$ be the corresponding Dirac operators at the two ends $M\times\{i\}$. Let $\bar\eta(M_i,E_i,h)=\bar\eta(\partial\!\!\!/^{\psi,h}_{E_i\otimes\tau,M\times[0,1]})$. Then, we only need to compute

$$\begin{aligned}&\bar\eta(M_1,E_1,h)-\bar\eta(M_2,E_2,h)\\ &-\left(\int_{M_1}\mathrm{Td}(\Omega_{M_1})\wedge ch(\nabla^{E_1})\wedge f^*\mu-\int_{M_2}\mathrm{Td}(\Omega_{M_2})\wedge ch(\nabla^{E_2})\wedge f^*\mu\right)\bmod\mathbb{Z}\end{aligned} \tag{5.1.4}$$

where Ω_{M_i} is the respective Riemannian curvature of M_i for $i=1,2$.

Let θ be the transgression form of $\mathrm{Td}\wedge ch$ on N such that

$$d\theta=\mathrm{Td}(\Omega_{M_1})\wedge ch(\nabla^{E_1})-\mathrm{Td}(\Omega_{M_2})\wedge ch(\nabla^{E_2}).$$

The integral part of (5.1.4) is immediate:

(5.1.5)
$$\int_{M_1} \mathrm{Td}(\Omega_{M_1}) \wedge ch(\nabla^{E_1}) \wedge f^*\mu - \int_{M_2} \mathrm{Td}(\Omega_{M_2}) \wedge ch(\nabla^{E_2}) \wedge f^*\mu = \int_{\partial N} d\theta \wedge f^*\mu \bmod \mathbb{Z}.$$

By the Dai-Zhang Toeplitz index formula (4.4.5), upon reducing modulo $\mathbb{Z}$,

$$\bar{\eta}(M_1, E_1, h) - \bar{\eta}(M_2, E_2, h) = \int_N \mathrm{Td}(\Omega_{g_\gamma}) \wedge ch(\nabla_p^E) \wedge ch(h, d) \bmod \mathbb{Z} \tag{5.1.6}$$

where Ω_{g_γ} is the Riemannian curvature of N and $ch(h, d)$ is the odd Chern character of h. By Stokes theorem, the left hand side of (5.1.6) is $\int_{\partial N} \theta \wedge ch(h, d)$. By the exactness condition (2.1.2) and Stokes theorem again, the difference (5.1.4) is zero. □

Well-defined under the $\mathbb{R}/\mathbb{Z}$ K^0-theory relation:

Proposition 5.1.3. *The analytic pairing* (5.1.2) *respects the* $\mathbb{R}/\mathbb{Z}$ K^0*-relation* (2.1.4).

Proof. Fix a K_0-cycle (M, E, f). For $i = 1, 2, 3$, consider $\mathbb{R}/\mathbb{Z}$ K^0-cocycles $\mathcal{E}_i = (r_i, (d, r_i^{-1} dr_i), \mu_i)$ such that $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$, i.e. $r_2 \simeq r_1 \oplus r_3$ and satisfying (2.1.6):

$$\mu_2 - \mu_1 - \mu_3 = Tch(r_1, r_2, r_3).$$

Let $h_i = r_i \circ f : M \to U(N_i)$. For simplicity, we denote (5.1.2) by $\bar{\eta}(\mathcal{E}_i^M)$ for each i. Assume there is a smooth path h_t connecting h_2 and $h_1 \oplus h_3$, both of which sit at each end of the cylinder $M \times [0, 1]$ respectively. Moreover, assume that the extension to the cylinder is compatible with all of the relevant data associated to each end, for instance there is a path $\tilde{\mu}_t = f^*\mu_t$ connecting $\tilde{\mu}_2$ at $M \times \{0\}$ and $\tilde{\mu}_1 + \tilde{\mu}_3$ at $M \times \{1\}$ by some suitable cut-off function. Then, by applying the Dai-Zhang Toeplitz index theorem for $M \times [0, 1]$ using (4.4.5) and Stokes theorem, we compute

$$\begin{aligned}
&\bar{\eta}(\mathcal{E}_2^M) - \bar{\eta}(\mathcal{E}_1^M) - \bar{\eta}(\mathcal{E}_3^M) \\
&\qquad = \int_{[0,1]\times M} \mathrm{Td}([0, 1] \times M) \wedge ch([0, 1] \times E) \wedge \big(ch(h_t; t \in [0, 1]) - d\tilde{\mu}_t\big) \bmod \mathbb{Z}
\end{aligned}$$

$$= \int_M \int_0^1 \mathrm{Td}(M) \wedge ch(E) \wedge \big(ch(h_t; t \in [0,1]) - d\tilde{\mu}_t\big) \bmod \mathbb{Z}$$
$$= \int_M \mathrm{Td}(M) \wedge ch(E) \wedge \big(\mathrm{T}ch(h_1,h_2,h_3) - (\tilde{\mu}_2 - \tilde{\mu}_1 - \tilde{\mu}_3)\big) \bmod \mathbb{Z} = 0.$$

This shows that $\bar{\eta}(\mathcal{E}_2^M) = \bar{\eta}(\mathcal{E}_1^M) + \bar{\eta}(\mathcal{E}_3^M)$ whenever $r_2 \simeq r_1 \oplus r_3$. □

Well-defined under the K-homology relation:

Lemma 5.1.4. *The analytic term $\bar{\eta}\big(\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}\big)$ respects the K-homology relation [13, §11].*

Proof. The following approach is inspired by [15]. For simplicity, we denote $\bar{\eta}\big(\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]}\big)$ by $\bar{\eta}(M,E,h)$. It is straightforward for the case of direct sum-disjoint union, i.e.

$$\bar{\eta}\big((M,E_1,h)\sqcup(M,E_2,h)\big) = \bar{\eta}(M,E_1\oplus E_2,h) = \bar{\eta}(M,E_1,h)+\bar{\eta}(M,E_2,h) \tag{5.1.7}$$

where the Dirac operator splits into $\partial\!\!\!/^{\psi,h}_{E_1\otimes\tau,M\times[0,1]} \oplus \partial\!\!\!/^{\psi,h}_{E_2\otimes\tau,M\times[0,1]}$. For bordism, let (W,F,φ) be a K-chain such that $(\partial W, F|_{\partial W}, \varphi|_{\partial W}) \cong (M,E,f) \sqcup (-M',E',f')$. Then,

$$\bar{\eta}(\partial W, F|_{\partial W}, \varphi|_{\partial W}) = \bar{\eta}\big((M,E,f) \sqcup (-M',E',f')\big) = \bar{\eta}(M,E,f) + \bar{\eta}(-M',E',f)$$

where the Dirac operator is given by $\partial\!\!\!/^{\psi,h}_{E\otimes\tau,M\times[0,1]} \oplus \partial\!\!\!/^{\psi,h'}_{E'\otimes\tau',M'\times[0,1]}$.

The relation of vector bundle modification is given by

$$(M,E,f) \sim (\Sigma H, \beta \otimes \rho^* E, f \circ \rho), \tag{5.1.8}$$

where H is a spin^c vector bundle over M, $\underline{\mathbb{R}}$ is the trivial real line bundle, $\Sigma H = S(H \oplus \underline{\mathbb{R}})$ is the sphere bundle, $\rho : \Sigma H \to M$ is the projection and β is the Bott bundle over ΣH. Since M is an even dimensional spin^c manifold, so is ΣH. Thus, the consideration of the Dai-Zhang eta-invariant $\eta(\Sigma H, \beta \otimes \rho^* E, f \circ \rho)$ is valid. Via $r : X \to U(N)$, the composition $g = r \circ f$ represents an element of $K^1(M)$ and the composition $h = g \circ \rho : S(H \oplus \underline{\mathbb{R}}) \to U(N)$ represents an element of $K^1(S(H \oplus \underline{R}))$. Let τ be the trivial bundle upon which g acts as an automorphism and S_M be the spinor bundle on M. Now, we extend the tensor product bundle $S_M \otimes E \otimes \tau$ on M trivially to the cylinder $M \times [0,1]$, denoted by $S_{M\times[0,1]} \otimes F$ for $F = E \otimes \tau$. By the Dai-Zhang construction, we obtain the associated Dirac operator $\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}$. Let $\widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}}$ be its lift to $\Sigma H \times [0,1]$. This requires some explanation. Note that, there is a lift $\widetilde{S_{M\times[0,1]}}$ of $S_{M\times[0,1]} \otimes F$ to $\Sigma H \times [0,1]$ via $\rho' = \rho \times t$ where $t \in [0,1]$. Let $S_{S^{2p}}$ be the spinor bundle on the even spheres S^{2p}. Denote its lift to $\Sigma H \times [0,1]$ by $\widetilde{S_{S^{2p}}}$. Then, by [9] there is an isomorphism of the tensor product

$$S_{\Sigma H\times[0,1]} \cong \widetilde{S_{M\times[0,1]}} \hat{\otimes} \widetilde{S_{S^{2p}}}$$

where $S_{\Sigma H\times[0,1]}$ is the primitive spinor bundle associated to $T(\Sigma H \times [0,1])$ of the spin^c manifold $\Sigma H \times [0,1]$. The 'full' bundle data on $\Sigma H \times [0,1]$ is now

$$S_{\Sigma H\times[0,1]} \otimes \widetilde{\beta} \otimes (\rho')^* F. \tag{5.1.9}$$

Let $\partial\!\!\!/_{\beta,S^{2p}}$ be the Dirac operator on S^{2p} twisted by the Bott bundle β, with $\widetilde{\partial\!\!\!/_{\beta,S^{2p}}}$ its lift to $\Sigma H \times [0,1]$, acting on (5.1.9) via $\widetilde{S_{S^{2p}}}$ and $\widetilde{\beta}$ and by the identity on others. On the other hand, the lift $\widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}}$ acts on (5.1.9) via $\widetilde{S_{M\times[0,1]}}$ and $\widetilde{(\rho')^*F}$ and by the identity on others. That is, both of the lifted Dirac operators $\widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}}$ and $\widetilde{\partial\!\!\!/_{\beta,S^{2p}}}$ act on the bundle (5.1.9), as well as the primitive spin^c Dirac operator $\partial\!\!\!/^{\psi',h}_{\widetilde{\beta}\otimes(\rho')^*F,\Sigma H\times[0,1]}$. Let P be the sharp product of the two operators

$$P = \widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}} \# \widetilde{\partial\!\!\!/_{\beta,S^{2p}}} = \begin{pmatrix} \widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}} \otimes 1 & 1 \otimes \widetilde{\partial\!\!\!/_{\beta,S^{2p}}}^{-} \\ 1 \otimes \widetilde{\partial\!\!\!/_{\beta,S^{2p}}}^{+} & -\widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}} \otimes 1 \end{pmatrix}. \tag{5.1.10}$$

It is an elliptic operator on $\Sigma H \times [0,1]$ acting on the bundle (5.1.9). Moreover, P can be identified with the primitive Dirac operator $\partial\!\!\!/^{\psi',h}_{\widetilde{\beta}\otimes(\rho')^*F,\Sigma H\times[0,1]}$ on $\Sigma H \times [0,1]$ by the local triviality of the fibration $\Sigma H \times [0,1] \to M \times [0,1]$. One can alternatively view P as the sharp product

$$\widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}} \# \mathcal{D}_\beta$$

where $\mathcal{D}_\beta$ is a family of elliptic operators $\mathcal{D}$ given by the Dirac operator on $S(H_m \oplus \underline{\mathbb{R}})$ for $m \in M \times [0,1]$, and $(\mathcal{D}_\beta)_m$ is identified with $\widetilde{\partial\!\!\!/_{\beta,S^{2p}}}$ by [14, Proposition 7].

By [7], the eta-invariant of the sharp product operator P can be calculated by

$$\eta(P)=\operatorname{Ind}\Big(\widetilde{\partial\!\!\!/_{\beta,S^{2p}}}^{+}\Big)\cdot\eta\Big(\widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}}\Big)=\eta\Big(\widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}}\Big)$$

since $\operatorname{Ind}\Big(\widetilde{\partial\!\!\!/_{\beta,S^{2p}}}^{+}\Big)=1$ by the Atiyah-Singer index theorem (4.1.3). This shows that

$$\eta\Big(\partial\!\!\!/^{\psi',h}_{\widetilde{\beta}\otimes(\rho')^*F,\Sigma H\times[0,1]}\Big)=\eta\Big(\widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}}\Big).$$

The rest of the proof involves the argument of the dimension of the kernel of the Dirac operator, which is standard. In particular, the kernel of P or equivalently $\partial\!\!\!/^{\psi',h}_{\widetilde{\beta}\otimes(\rho')^*F,\Sigma H\times[0,1]}$ coincides with the kernel of $\widetilde{\partial\!\!\!/^{\psi,g}_{F,M\times[0,1]}}$. Thus, the reduced eta-invariant is invariant under vector bundle modification. □

Lemma 5.1.5. *The integral term $\int_M f^*\mu\wedge ch(E)\wedge Td(M)$ mod $\mathbb{Z}$ respects the K-homology relation [13, §11].*

Proof. Fix a $\mathbb{R}/\mathbb{Z}$ K^0-cocycle $\mathcal{V}=(g,(d,g^{-1}dg),\mu)$. Let $\mathcal{E}=(M,E,f)$ be a K_0-cycle. For direct sum-disjoint union, it is straightforward to see that the integral of the sum splits into the sum of the integral. For bordism, consider a K-chain (W,F,g) and by pairing $\mathcal{V}$ with each term in $(\partial W,F|_{\partial W},g|_{\partial W})\cong(M,E,f)\sqcup(-M',E',f')$, it is immediate that

$$\begin{aligned}&\int_W (g|_{\partial W})^*\mu\wedge ch(F|_{\partial W})\wedge \mathrm{Td}(\partial W)\mathrm{mod}\ \mathbb{Z}\\&\qquad=\int_M f^*\mu\wedge ch(E)\wedge\mathrm{Td}(M)\mathrm{mod}\ \mathbb{Z}+\int_{M'} f'^*\mu\wedge ch(E')\wedge\mathrm{Td}(M')\mathrm{mod}\ \mathbb{Z}.\end{aligned}$$

For vector bundle modification, $(M,E,f)\sim(\Sigma H,\beta\otimes\rho^*E,f\circ\rho)$, we compute

$$\begin{aligned}&\int_{\Sigma H}(f\circ\rho)^*\omega\wedge ch(\beta\otimes\rho^*F)\wedge\mathrm{Td}(\Sigma H)\ \mathrm{mod}\ \mathbb{Z}\\&\qquad=\sum_{U_\alpha}\varphi_\alpha\int_{U_\alpha\times S^{2p}} f^*(\omega|_{U\alpha})\wedge ch(\beta)\otimes ch(E|_{U_\alpha})\wedge\mathrm{Td}(U_\alpha\times S^{2p})\ \mathrm{mod}\ \mathbb{Z}\\&\qquad=\sum_{U_\alpha}\varphi_\alpha\int_{U_\alpha} f^*(\omega|_{U\alpha})\wedge ch(E|_{U_\alpha})\wedge\mathrm{Td}(U_\alpha)\int_{S^{2p}} ch(\beta)\wedge\mathrm{Td}(S^{2p})\ \mathrm{mod}\ \mathbb{Z}\\&\qquad=\sum_{U_\alpha}\varphi_\alpha\int_{U_\alpha} f^*(\omega|_{U\alpha})\wedge ch(E|_{U_\alpha})\wedge\mathrm{Td}(U_\alpha)\ \mathrm{mod}\ \mathbb{Z}\end{aligned}$$

$$= \int_M f^*\omega \wedge ch(E) \wedge \mathrm{Td}(M) \bmod \mathbb{Z}.$$

Here, φ_α is a partition of unity subordinate to an open cover $\{U_\alpha\}$ of M and the second integral (over S^{2p}) on the second line is known to have index 1 by the Atiyah-Singer index theorem. This completes the proof. □

Proposition 5.1.6. *The analytic pairing* (5.1.2) *respects the K-homology relation [13, §11].*

Proof. The link between these two terms (via the exactness condition (2.1.4)) does not play a role here, so the claim follows from Lemma 5.1.4 and Lemma 5.1.5. □

5.1.2 Non-degeneracy of K^0 pairing

We show the non-degeneracy by an argument of Mayer-Vietoris sequence (cf. Section 2.3) for the K^0 pairing. The approach adapted here is inspired by Savin-Sternin [48], in which their argument works for the duality pairing on abstract cycles. In contrast, the following proof is much more delicate as explicit (co)cycles are involved. First, we show that (5.1.2) is an isomorphism for a contractible open set $U \cong \mathbb{R}^n$. Then, by the assumption that the isomorphism holds for contractible U, V and intersection $U \cap V$, it holds for $X = U \cup V$. Lastly, we apply an induction on the size of the open cover.

To do this, we need a description of $K_0(U) \cong K_0(\mathbb{R}^n)$ for positive even n. Consider the short exact sequence of the induced K_0 groups associated to the one-point compactification of the Euclidean space $\mathbb{R}^n$

$$0 \longrightarrow K_0(\{\infty\}) \longrightarrow K_0(S^n) \longrightarrow K_0(\mathbb{R}^n) \longrightarrow 0. \tag{5.1.11}$$

Recall that for even n, the geometric K-homology of even sphere $K_0(S^n)$ is

$$K_0(S^n) \cong \mathbb{Z}\langle(\mathrm{pt}, \mathrm{pt} \times \mathbb{C}, i)\rangle \oplus \mathbb{Z}\langle(S^n, \beta, \mathrm{Id})\rangle. \tag{5.1.12}$$

Here $i : \{\infty\} \to S^n$ is the inclusion map and β is the non-trivial Bott bundle over S^n. Since $K_0(\{\infty\}) \cong \mathbb{Z}$ is generated by $(\mathrm{pt}, \mathrm{pt} \times \mathbb{C}, \mathrm{Id})$ and coincides with

$\ker[K_0(S^n) \to K_0(\mathbb{R}^n)]$, we obtain

$$K_0(\mathbb{R}^n) \cong \widetilde{K}_0(S^n)$$

where $\widetilde{K}_0(S^n)$ denotes the reduced K-homology of S^n, generated by the non-trivial cycle. Thus, it suffices to consider the pairing in $\widetilde{K}_0(S^n)$.

Remark 5.1.7. For $n = 2$, recall that from Example 1.1.5 the Bott bundle is $\beta_0 = L_0 - 1 \in \widetilde{K}^0(S^2)$. To see the Bott bundle over n-spheres for $n > 2$, we observe that by the multiplicative property of reduced K-theory of S^2

$$\widetilde{K}^0(S^2) \times \cdots \times \widetilde{K}^0(S^2) \to \widetilde{K}^0(S^2 \wedge \cdots \wedge S^2) = \widetilde{K}^0(S^n)$$

where $n = 2r$ for r times the wedge of 2-spheres. Then, the Bott bundle $\beta \in \widetilde{K}^0(S^n)$ is $\beta = \beta_0 \boxtimes \cdots \boxtimes \beta_0 = (L_0 - 1)^r$. Its Chern character given by (3.2.4) is thus integral.

Consider the following exact sequence

$$\cdots \to \widetilde{K}^0(S^n) \to \widetilde{K}^0(S^n,\mathbb{R}) \to \widetilde{K}^0(S^n,\mathbb{R}/\mathbb{Z}) \to \widetilde{K}^1(S^n) \to \cdots$$

Recall that the odd K-theory $K^1(S^n)$ can be regarded as the set of homotopy classes $[S^n, U(\infty)]$ of continuous maps from S^n to the stablised unitary group $U(\infty)$, which is by definition the n-th homotopy group $\pi_n(U(\infty))$. By Bott Periodicity, $K^1(S^n) \cong \pi_n(U(\infty))$ is trivial when n is even. Hence we have

$$\cdots \to \widetilde{K}^0(S^n) \xrightarrow{ch} \widetilde{K}^0(S^n,\mathbb{R}) \to \widetilde{K}^0(S^n,\mathbb{R}/\mathbb{Z}) \to 0.$$

By viewing $\widetilde{K}^0(S^n,\mathbb{R}/\mathbb{Z})$ as the cokernel of ch, its generator is represented by

$$(0, 0, \mu - ch(\beta)), \tag{5.1.13}$$

where $\mu \in \Omega^{\text{even}}(S^n)/d\Omega$ such that $d\mu = 0$ and $\beta \in \widetilde{K}^0(S^n)$.

Now, we are ready to show that the Pontryagin duality K^0 pairing implemented by (5.1.2) is non-degenerate.

The case of the K^0 pairing for $\mathbb{R}^n$ for positive even n reduces to the $\widetilde{K}^0$ pairing for S^n. In particular, it suffices to show that the map

$$\widetilde{K}^0(S^n, \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(\widetilde{K}_0(S^n), \mathbb{R}/\mathbb{Z})$$

implemented by

$$\bar{\eta}\big(\not{\partial}^{\beta}_{S^n\times[0,1]}\big) - \int_{S^n} \big(\mu - ch(\beta)\big) \wedge ch(\beta) \wedge \mathrm{Td}(S^n) \bmod \mathbb{Z} \tag{5.1.14}$$

is an isomorphism, which then implies the non-degeneracy of the pairing. Since we are working on generators, the injectivity is implied and we only need to show the surjectivity, i.e. it suffices to show that the image of the pairing is not identically zero in $\mathbb{R}/\mathbb{Z}$. Since TS^n is stably trivial, the Todd form $\mathrm{Td}(S^n) = 1$. The integrand then consists of two parts:

$$\mu \wedge ch(\beta) \text{ and } ch(\beta) \wedge ch(\beta).$$

It is clear that the integration of $ch(\beta)^2$ over S^n is zero modulo $\mathbb{Z}$ since $ch(\beta)$ is just the wedge product of $c_1(L)$ and is already the top degree form on S^n. For $\mu \wedge ch(\beta)$, since $ch(\beta)$ is already the top degree form on S^n, only the lowest term (the 0-form of μ) survives in the integration. The 0-form is in general an $\mathbb{R}$-valued function on S^n. Hence, we conclude that

$$\int_{S^n} \mu \wedge ch(\beta) \wedge \mathrm{Td}(S^n) \bmod \mathbb{Z} \neq 0. \tag{5.1.15}$$

To consider the reduced Dai-Zhang eta-invariant $\bar{\eta}$ of the even sphere S^n, we need to compute $\eta\big(\not{\partial}^{\beta}_{S^n\times[0,1]}\big)$. Let $\not{\partial}_{\beta,S^n} = \not{\partial}^{\pm}_{\beta,S^n}$ be the $\mathbb{Z}_2$-graded Dirac operator on S^n twisted by β. By the Atiyah-Singer index theorem,

$$\mathrm{Ind}(\not{\partial}^{+}_{\beta,S^n}) = \int_{S^n} ch(\beta) \wedge \mathrm{Td}(S^n) = 1 \neq 0. \tag{5.1.16}$$

Hence, we cannot directly apply the method of Dai-Zhang to compute the eta-invariant, as it requires the vanishing of $\mathrm{Ind}(\not{\partial}^{+}_{\beta,S^n})$ to ensure the existence of Lagrangian subspaces in $\ker(\not{\partial}_{\beta,S^n})$ for the modified boundary conditions. To circumvent this, we adopt a method suggested in [20].

First, we extend β over S^n trivially to the cylinder $S^n \times [0,1]$. Then, the two ends of the interval $[0,1]$ is identified into a circle, and glue the bundle over $S^n \times \{0\}$ and $S^n \times \{1\}$ using a smooth automorphism in $K^1(S^n)$. Since $K^1(S^n)$ is trivial for even n, the gluing map is just the identity (infinite) matrix in $U(\infty)$ and the bundles at the two ends are identified trivially, giving a well-defined bundle $\beta' \to S^n \times S^1$. Moreover, $S^n \times S^1$ is closed and therefore no boundary conditions are required. Let $\not{\partial}_{\beta', S^n \times S^1}$ be the resulting twisted Dirac operator. It can be rewritten as the sharp product

$$\not{\partial}_{\beta', S^n \times S^1} = \not{\partial}_{\beta, S^n} \# \not{\partial}_{S^1} = \begin{pmatrix} \not{\partial}_{S^1} \otimes 1 & 1 \otimes \not{\partial}^-_{\beta, S^n} \\ 1 \otimes \not{\partial}^+_{\beta, S^n} & -\not{\partial}_{S^1} \otimes 1 \end{pmatrix}. \tag{5.1.17}$$

Then, its Atiyah-Patodi-Singer eta-invariant is

$$\eta_{\mathrm{APS}}(\not{\partial}_{\beta', S^n \times S^1}) = \mathrm{Ind}(\not{\partial}^+_{\beta, S^n}) \cdot \eta_{\mathrm{APS}}(\not{\partial}_{S^1}) = 1 \cdot 0 = 0.$$

To determine the kernel of (5.1.17), let $\begin{pmatrix} x_1 \otimes y_1 \\ x_2 \otimes y_2 \end{pmatrix}$ be the spinors. Then, the calculation reduces to

$$\begin{cases} \not{\partial}_{S^1}(x_1) \otimes y_1 = -x_2 \otimes \not{\partial}^-_{\beta, S^n}(y_2) \\ \not{\partial}_{S^1}(x_2) \otimes y_2 = x_1 \otimes \not{\partial}^{\beta,+}_{S^n}(y_1). \end{cases} \tag{5.1.18}$$

For instance, if $(x_1, x_2) \in \ker(\not{\partial}_{S^1})$, then $(y_1, y_2) \in \ker(\not{\partial}^{\beta,+}_{S^n} \oplus \not{\partial}^{\beta,-}_{S^n})$. In particular, we have

$$\ker(\not{\partial}^{\beta'}_{S^n \times S^1}) \cong \ker(\not{\partial}_{S^1}) \dot{\cap} \big(\ker(\not{\partial}^{\beta,+}_{S^n}) \oplus \ker(\not{\partial}^{\beta,-}_{S^n}) \big) \tag{5.1.19}$$

where $\dot{\cap}$ means the 'intersection' of elements as in spinors, not as the intersection of spaces. Since the Dirac operator $\not{\partial}^+_{\beta, S^n}$ has one dimensional kernel and zero dimensional cokernel, we conclude that

$$\dim \ker(\not{\partial}_{\beta', S^n \times S^1}) \equiv \dim \ker(\not{\partial}_{S^1}) = 1. \tag{5.1.20}$$

Hence, for all positive even n, the reduced Dai-Zhang eta invariant is

$$\bar{\eta}(\partial\!\!\!/_{\beta', S^n \times S^1}) \equiv \frac{1}{2} \bmod \mathbb{Z}. \tag{5.1.21}$$

By (5.1.15) and (5.1.21), we conclude the following lemma.

Lemma 5.1.8. *For even $n \in \mathbb{Z}_+$, the map $\widetilde{K}^0(S^n, \mathbb{R}/\mathbb{Z}) \to Hom(\widetilde{K}_0(S^n), \mathbb{R}/\mathbb{Z})$ implemented by* (5.1.14) *is an isomorphism, and thus so is the case of $U \cong \mathbb{R}^n$.*

Since $K_1(\mathbb{R}^n) \cong K_1(S^n) \cong 0$ for even n, the relevant part of the Mayer-Vietoris sequence in the analytic K^0 pairing is

$$\begin{array}{ccccccc}
\longrightarrow K_c^0(U \cap V, \mathbb{R}/\mathbb{Z}) & \longrightarrow & K_c^0(U, \mathbb{R}/\mathbb{Z}) \oplus K_c^0(V, \mathbb{R}/\mathbb{Z}) & \longrightarrow & K^0(U \cup V, \mathbb{R}/\mathbb{Z}) \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
\to \mathrm{Hom}(K_0(U \cap V), \mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}(K_0(U), \mathbb{R}/\mathbb{Z}) \oplus \mathrm{Hom}(K_0(V), \mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}(K_0(U \cup V), \mathbb{R}/\mathbb{Z}) \to
\end{array}$$

Lemma 5.1.9. *Assume the isomorphism holds for contractible open sets U, V and the intersection $U \cap V$. Then, it holds for $X = U \cup V$, i.e. the map $K^0(X, \mathbb{R}/\mathbb{Z}) \to Hom(K_0(X), \mathbb{R}/\mathbb{Z})$ implemented by* (5.1.2) *is an isomorphism.*

Proof. The result follows by Lemma 5.1.8 and by the Five lemma. □

Proof. (of Theorem 5.1.1) The non-degeneracy of the pairing is implied by the isomorphism as in Lemma 5.1.9. The last step is to induct on the size of open cover of X. The base case is that we have shown $K^i(X, \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(K_i(X), \mathbb{R}/\mathbb{Z})$ implemented by the analytic pairing (5.1.2) is an isomorphism for U, V and $U \cap V$, where $\{U, V\}$ is a good cover of X.

Let $\{U_0, \dots, U_{p-1}\}$ be any open cover of X of size p. Let $V = U_0 \cup \cdots \cup U_{p-2}$. The induction hypothesis is the following: assume that the isomorphism holds for V, U_{p-1} and the non-empty intersection $V \cap U_{p-1}$, which then holds for $V \cup U_{p-1}$. Now, consider a good cover $\{U_0', \dots, U_p'\}$ of X of size $p+1$. Let

$$V' = U_0' \cup \cdots U_{p-1}'.$$

By the induction hypothesis, the isomorphism holds for V' (since V' is of size p union) and U'_p. The Mayer-Vietoris sequence for V', U'_p and $V' \cap U'_p$ is

$$\cdots \to K^i(V' \cap U'_p, \mathbb{R}/\mathbb{Z}) \to K^i(V', \mathbb{R}/\mathbb{Z}) \oplus K^i(U'_p, \mathbb{R}/\mathbb{Z}) \to K^i(V' \cup U'_p, \mathbb{R}/\mathbb{Z}) \to \cdots .$$

To claim the isomorphism for the union $V' \cup U'_p$, we only need to consider the intersection. Note that

$$V' \cap U'_p = (U'_0 \cap U'_p) \cup \cdots \cup (U'_{p-1} \cap U'_p).$$

It is the p-union of contractible sets. By the induction hypothesis, the isomorphism holds for $V' \cap U'_p$. By the Mayer-Vietoris sequence and the Five lemma, we conclude that the isomorphism holds for $V' \cup U'_p$. This completes the proof of the non-degeneracy of the analytic K^0 pairing. □

5.2 The odd case: $K^1(X, \mathbb{R}/\mathbb{Z}) \times K_1(X)$

The analytic pairing in the odd case, or what we refer to as the *analytic K^1-pairing*, was first introduced by Lott [36]. We stress that in this section we *do not re-prove* what Lott had already proved, but show the well-definedness and the non-degeneracy of the analytic pairing (5.2.3), which is not immediately clear in his paper. Another upshot of providing a full proof of this observation is that we provide evidence of the non-triviality of such a pairing. In particular, we will consider an example which forms a crucial intermediate step in the proof.

Setup. Let X be a smooth compact manifold. Let M be an odd dimensional closed spin^c manifold and E be a complex vector bundle over M equipped with a Hermitian connection ∇^E. Let $f : M \to X$ be a smooth map. This defines a triple $(M, E, f) \in K_1(X)$. Let S_M be the spinor bundle associated to TM. Let $\partial\!\!\!/_{E\otimes f^*V,M}$ be the twisted Dirac operator acting on $L^2(S_M \otimes E \otimes f^*V)$ defined via the tensor product connection $\nabla^{E\otimes f^*V}$, which locally takes the form

$$\partial\!\!\!/_{E\otimes f^*V,M} := \sum_i c(e_i)\nabla^{E\otimes f^*V}_{e_i} \tag{5.2.1}$$

where $\{e_i\}$ is an orthonormal frame of the tangent bundle TM. The Dirac operator $\not{\partial}_{E\otimes f^*V,M}$ is an elliptic differential operator on an odd dimensional closed manifold M. Let $\eta(\not{\partial}_{E\otimes f^*V,M}) := \eta(\not{\partial}_{E\otimes f^*V,M}, 0)$ be the Atiyah-Patodi-Singer eta-invariant associated to $\not{\partial}_{E\otimes f^*V,M}$, given by (4.2.3). In our case, we consider the full eta-invariant (the boundary correction terms of (4.2.8)) and reduce modulo the integers

$$\bar{\eta}(\not{\partial}_{E\otimes f^*V,M}) = \frac{\eta(\not{\partial}_{E\otimes f^*V,M}) + h(\not{\partial}_{E\otimes f^*V,M})}{2} \mod \mathbb{Z} \tag{5.2.2}$$

where $h(\not{\partial}_{E\otimes f^*V,M})$ is the dimension of the kernel of $\not{\partial}_{E\otimes f^*V,M}$. For simplicity, we call (5.2.2) the *reduced* eta-invariant. Observe that the reduced eta-invariant is dependent on the geometry of the manifold and the bundle, e.g. the Riemannian metric on M and the Hermitian metric and connection on both E and f^*V.

Theorem 5.2.1 (Lott's analytic K^1-pairing [36])**.** *In the above setting, the analytic K^1-pairing is given by*

$$\begin{aligned} &K^1(X, \mathbb{R}/\mathbb{Z}) \times K_1(X) \to \mathbb{R}/\mathbb{Z} \\ &\langle (V, \nabla^V, \omega), (M, E, f)\rangle \\ &= \bar{\eta}(\not{\partial}_{E\otimes f^*V,M}) - \int_M f^*\omega \wedge ch(E, \nabla^V) \wedge Td(M) \ mod\ \mathbb{Z}. \end{aligned} \tag{5.2.3}$$

Note that (5.2.3) is the odd analog of (5.1.2). As mentioned before, it is not immediately clear that the pairing (5.2.3) is well-defined since both of the terms in (5.2.3) depend on the geometry of the manifold and the bundle. Thus, the rest of this section is devoted to the proof of the following claim.

Theorem 5.2.2. *The analytic K^1-pairing* (5.2.3) *is well-defined and non-degenerate.*

5.2.1 Well-definedness of K^1 pairing

Well-defined on the level of cycle:

The idea is very similar to the even case. Fix a $\mathbb{R}/\mathbb{Z}$ K^1-cocycle (V, ∇^V, ω). Let $M_1 = (M, g_1)$ and $M_2 = (M, g_2)$ be the same manifold equipped with two Riemannian metrics g_1 and g_2 respectively. Let $N = M \times [0, 1]$ be a cylinder with boundary $\partial N = M_1 \sqcup -M_2$, i.e. N is equipped with a suitable metric such that g_1 can be deformed continuously into g_2. To be more precise, let $\gamma : [0, 1] \to \mathrm{Met}(M)$

be a path in the space of Riemannian metrics on M. Then,

$$g_\gamma = \gamma(t) + (dt)^2$$

is the extended metric on the cylinder N. The complex vector bundle E extends trivially to the whole of N, and we still denoted this extension by E. Let g^{E_1}, g^{E_2} and $\nabla^{E_1}, \nabla^{E_2}$ be the metric and Hermitian connection on $E_1 = E|_{M_1}$ and $E_2 = E|_{M_2}$ respectively. Let $p : [0,1] \to \mathrm{Conn}(E)$ be a path in the space of connections on E. Its extension to N is

$$\nabla_p^E = \frac{\partial}{\partial t} \wedge dt.\mathrm{Id}_E + p(t).$$

Let (f^*V, ∇^{f^*V}) be the pullback of V via f. For $i = 1, 2$ let $E_i \otimes f^*V$ be the tensor product bundles with connections $\nabla_p^{E_i\otimes f^*V}$. Let $\not{\partial}_{M_1}^{E_1\otimes f^*V}$ and $\not{\partial}_{M_2}^{E_2\otimes f^*V}$ be the corresponding Dirac operators twisted by $E_1 \otimes f^*V$ and $E_2 \otimes f^*V$ at the two ends. The well-definedness of the pairing boils down to the equation

$$\begin{aligned} &\bar\eta(\not{\partial}_{M_1}^{E_1\otimes f^*V}) - \bar\eta(\not{\partial}_{M_2}^{E_2\otimes f^*V}) \\ &-\left(\int_{M_1} \mathrm{Td}(\Omega_{M_1}) \wedge ch(\nabla^{E_1}) \wedge f^*\omega - \int_{M_2} \mathrm{Td}(\Omega_{M_2}) \wedge ch(\nabla^{E_2}) \wedge f^*\omega\right) \bmod \mathbb{Z} \end{aligned} \tag{5.2.4}$$

where Ω_{M_i} are the respective Riemannian curvatures of M_i, for $i = 1, 2$.

Let θ be the transgression form of the characteristic form $\mathrm{Td}\wedge ch$ on N such that

$$d\theta = \mathrm{Td}(\Omega_{M_1}) \wedge ch(\nabla^{E_1}) - \mathrm{Td}(\Omega_{M_2}) \wedge ch(\nabla^{E_2}).$$

The integral part of (5.2.4) is immediate:

$$\int_{M_1} \mathrm{Td}(\Omega_{M_1})\wedge ch(\nabla^{E_1})\wedge f^*\omega - \int_{M_2} \mathrm{Td}(\Omega_{M_2})\wedge ch(\nabla^{E_2})\wedge f^*\omega = \int_{\partial N} d\theta\wedge f^*\omega \bmod \mathbb{Z}. \tag{5.2.5}$$

By reducing the Atiyah-Patodi-Singer index formula (4.2.8) modulo $\mathbb{Z}$, we have

$$0 = \int_N \mathrm{Td}(\Omega_{g_\gamma}) \wedge ch(\nabla_p^E) \wedge ch(f^*\nabla^V) - (\bar\eta(\not{\partial}_{M_1}^{E_1\otimes f^*V}) - \bar\eta(\not{\partial}_{M_2}^{E_2\otimes f^*V})) \bmod \mathbb{Z}$$

where Ω_{g_γ} is the Riemannian curvature of N. By Stokes theorem, we obtain

$$\bar{\eta}(\not{\partial}_{M_1}^{E_1\otimes f^*V}) - \bar{\eta}(\not{\partial}_{M_2}^{E_2\otimes f^*V}) = \int_{\partial N} \theta \wedge f^*ch(\nabla^V). \tag{5.2.6}$$

Since $d\theta$ and $f^*ch(\nabla^V)$ are both characteristic forms and are hence closed, by Stokes theorem again and by (2.2.2), the difference (5.2.4) is zero.

Proposition 5.2.3. *The analytic K^1-pairing is independent of the Riemannian metric of the manifold M and the choice of Hermitian connection on E. A similar approach shows that it is also independent of the choice of connection on f^*V.*

Well-defined under the $\mathbb{R}/\mathbb{Z}$ K^1-theory relation

Fix an odd K-cycle (M, E, f). For $i = 1, 2, 3$, consider $\mathcal{V}_i = (V_i, \nabla^{V_i}, \omega_i) \in K^1(X, \mathbb{R}/\mathbb{Z})$ such that $\mathcal{V}_2 = \mathcal{V}_1 + \mathcal{V}_3$, i.e. $V_2 \cong V_1 \oplus V_3$ and $\omega_2 - \omega_1 - \omega_3 = CS(\nabla^{V_1}, \nabla^{V_2}, \nabla^{V_3})$. For simplicity, denote by $\bar{\eta}(f^*\mathcal{V}_i)$ the K^1 pairing between a fixed (M, E, f) and each $\mathcal{V}_i$, whose pairing formula is given by (5.2.3). Now, let $p : [0,1] \times M \to M$ be the obvious projection. Let $\widetilde{V} \to [0,1] \times M$ be a complex vector bundle which restricts to p^*V_2 near $\{1\} \times M$ and to $p^*(V_1 \oplus V_3)$ near $\{0\} \times M$. Let $\nabla^{\widetilde{V}}$ be a unitary connection on $\widetilde{V}$. Let $\widetilde{\omega} \in \Omega^{\mathrm{odd}}([0,1] \times M)/d\Omega$ such that $\widetilde{\omega}$ restricts to $p^*\omega_2$ near $\{1\} \times M$ and to $p^*(\omega_1 + \omega_3)$ near $\{0\} \times M$.

By the Atiyah-Patodi-Singer index theorem and Stokes theorem, we compute

$$\begin{aligned}
&\bar{\eta}(f^*\mathcal{V}_2) - \bar{\eta}(f^*\mathcal{V}_1) - \bar{\eta}(f^*\mathcal{V}_3)\\
&\qquad = \int_{[0,1]\times M} \mathrm{Td}([0,1] \times M) \wedge ch([0,1] \times E) \wedge f^*\big(ch(\nabla^{\widetilde{V}}) - d\widetilde{\omega}\big) \bmod \mathbb{Z}\\
&\qquad = \int_M \int_0^1 \mathrm{Td}(M) \wedge ch(E) \wedge f^*\big(ch(\nabla^{\widetilde{V}}) - d\widetilde{\omega}\big) \bmod \mathbb{Z}\\
&\qquad = \int_M \mathrm{Td}(M) \wedge ch(E) \wedge f^*\big(CS(\nabla^{V_1}, \nabla^{V_2}, \nabla^{V_3}) - (\omega_2 - \omega_1 - \omega_3)\big) \bmod \mathbb{Z} = 0.
\end{aligned}$$

This shows that $\bar{\eta}(f^*\mathcal{V}_2) = \bar{\eta}(f^*\mathcal{V}_1) + \bar{\eta}(f^*\mathcal{V}_3)$ whenever $\mathcal{V}_2 = \mathcal{V}_1 + \mathcal{V}_3$. □

Well-defined under the K-homology relation:

Let $\mathcal{V} \in K^1(X, \mathbb{R}/\mathbb{Z})$. We show that the reduced eta invariant $\bar{\eta}(f^*\mathcal{V})$ respects the K-homology relation (3.1.2). For simplicity, we will show this separately for the analytic part and the integral part. Unlike the previous proof, this is valid because the exactness condition (2.2.2) does not play a role in the following proof.

Lemma 5.2.4. *Let (M,E,f) be an odd K-cycle over X. Let $\not{\partial}_F^M$ be the Dirac operator on M twisted by $F=E\otimes f^*V$. Then, the reduced eta-invariant $\bar{\eta}(\not{\partial}_F^M)$ respects the K-homology relation* (3.1.2).

Proof. For simplicity, we write $\bar{\eta}(M,F,f)$ as the reduced eta-invariant. Let $F'=E'\otimes f^*V$. By the distributivity of the tensor product of vector bundles, we have $S\otimes(F\oplus F')=(S\otimes F)\oplus(S\otimes F')$. Then, the corresponding Dirac operator is $\not{\partial}_F^M\oplus\not{\partial}_{F'}^M$. It follows that

$$\bar{\eta}\big((M,F,f)\sqcup(M,F',f)\big)=\bar{\eta}(M,F\oplus F',f)=\bar{\eta}(M,F,f)+\bar{\eta}(M,F',f). \tag{5.2.7}$$

For bordism, let $(W,\widetilde{F},\varphi)$ be a K-chain such that $(\partial W,\widetilde{F}|_{\partial W},\varphi|_{\partial W})\cong(M,F,f)\sqcup(-M',F',f')$. Since the Dirac operator is given by $\not{\partial}_F^M\oplus\not{\partial}_{F'}^{-M}$, we have

$$\bar{\eta}(\partial W,\widetilde{F}|_{\partial W},\varphi|_{\partial W})=\bar{\eta}\big((M,F,f)\sqcup(-M',F',f')\big)=\bar{\eta}(M,F,f)+\bar{\eta}(-M',F',f).$$

For vector bundle modification, consider the unit sphere bundle $\rho:\Sigma H=S(H\oplus\underline{\mathbb{R}})\to M$, where H is an even dimensional spin^c vector bundle and $\underline{\mathbb{R}}$ is a trivial real line bundle over M. Note that ΣH is a spin^c vector bundle and is itself an odd dimensional closed manifold. Let β_H be the Bott bundle over ΣH. Let S_M (resp. $S_{S^{2p}}$) be the spinor bundle on M (resp. S^{2p}) and denote by $\widetilde{S_M}$ (resp. $\widetilde{S_{S^{2p}}}$) its lift to ΣH. Let $S_{\Sigma H}$ be the primitive spinor bundle over ΣH. By [9], there is an isomorphism of the tensor product

$$\widetilde{S_M}\otimes\widetilde{S_{S^{2p}}}\cong S_{\Sigma H}. \tag{5.2.8}$$

In particular, the "full" bundle data on ΣH is now $S_{\Sigma H}\otimes\beta_H\otimes\rho^*F$. Let $\not{\partial}_F^M$ be the twisted Dirac operator on M and let $\widetilde{\not{\partial}_F^M}$ be its lift to act on $L^2(S_{\Sigma H}\otimes\beta_H\otimes\rho^*F)$.

On the other hand, let $\not{\partial}_\beta^{S^{2p}}$ be the self-adjoint twisted Dirac operator on S^{2p}, whose positive part $\not{\partial}_\beta^{S^{2p},+}$ has index 1. Denote by $\widetilde{\not{\partial}_\beta^{S^{2p}}}$ its lift to ΣH, which also acts on $L^2(S_{\Sigma H}\otimes\beta_H\otimes\rho^*F)$. Then, by [7] their sharp product is given by

$$P = \widetilde{\partial\!\!\!/_F^M} \,\#\, \widetilde{\partial\!\!\!/_\beta^{S^{2p}}} = \begin{pmatrix} \widetilde{\partial\!\!\!/_F^M} \otimes 1 & 1 \otimes \widetilde{\partial\!\!\!/_\beta^{S^{2p}}}^- \\ 1 \otimes \widetilde{\partial\!\!\!/_\beta^{S^{2p}}}^+ & -\widetilde{\partial\!\!\!/_F^M} \otimes 1 \end{pmatrix}.$$

It is readily verified that P is an elliptic operator on ΣH. Moreover, P can be identified with the primitive spin^c Dirac operator $\partial\!\!\!/^{\Sigma H}_{\beta_H \otimes \rho^* F}$ on ΣH, by the local triviality of the fibration $\Sigma H \to M$. Strictly speaking, P is the sharp product $\widetilde{\partial\!\!\!/_F^M} \,\#\, \mathcal{D}_\beta$ where $\mathcal{D}_\beta$ is a family of elliptic operators $\mathcal{D}$ given by the Dirac operator on $S(H_m \oplus \underline{\mathbb{R}})$ for each $m \in M$, and $(\mathcal{D}_\beta)_m$ can be identified with $\widetilde{\partial\!\!\!/_\beta^{S^{2p}}}$, cf. [14, Proposition 7 & 16].

By [7], the eta-invariant of the sharp product operator P is given by

$$\eta(P) = \mathrm{Ind}\Big(\partial\!\!\!/_\beta^{S^{2p},+}\Big) \cdot \eta(\partial\!\!\!/_F^M) = \eta(\partial\!\!\!/_F^M), \tag{5.2.9}$$

since $\mathrm{Ind}(\partial\!\!\!/_\beta^{S^{2p},+}) = 1$ by the Atiyah-Singer index theorem (4.1.3) . Hence,

$$\eta\Big(\partial\!\!\!/^{\Sigma H}_{\beta_H \otimes \rho^* F}\Big) = \eta(\partial\!\!\!/_F^M). \tag{5.2.10}$$

It is readily verified that the kernel of P and the kernel of $\partial\!\!\!/_F^M$ coincide. This shows that the reduced eta-invariant $\bar{\eta}(\partial\!\!\!/_F^M)$ is invariant under vector bundle modification. □

Lemma 5.2.5. *The integral part of the pairing* (5.2.3) *respects the K-homology relation* (3.1.2)*.*

Proof. The proof is almost the same (except on the (co)cycles) as that of Lemma 5.1.5 since the integral term of (5.2.3) does not involve the pullback bundle f^*V. □

By Lemma 5.2.4 and Lemma 5.2.5, we conclude the following.

Proposition 5.2.6. *The analytic K^1-pairing given by* (5.2.3) *is well-defined under the K-homology relation* (3.1.2)*.*

5.2.2 Non-degeneracy of K^1-pairing

This section is devoted to the full proof of the non-degeneracy of Theorem 5.2.2. The main strategy is the same — we approach by applying the Mayer-Vietoris sequence of $\mathbb{R}/\mathbb{Z}$ K-theory (Section 2.3) and of K-homology (Section

3.3) and the Five lemma. However, there are some subtleties which require careful arguments.

Let $\{U, V\}$ be a good open cover of size two of X with $X = U \cup V$. There are two things to show:

1. the map $K^1(X,\mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(K_1(X),\mathbb{R}/\mathbb{Z})$ implemented by the pairing (5.2.3) is an isomorphism for contractible U,

2. if the pairing is an isomorphism for contractible U, V and the intersection $U \cap V$, then it is an isomorphism for $X = U \cup V$.

To show the first point, it suffices to consider $K^1(\mathbb{R}^n,\mathbb{R}/\mathbb{Z})$ for odd $n \in \mathbb{Z}_+$. The corresponding K-homology is $K_1(\mathbb{R}^n)$. Let

$$S^n = \mathbb{R}^n \cup \{\infty\} \tag{5.2.11}$$

be the one-point compactification of Euclidean space $\mathbb{R}^n$. Then, we have the associated short exact sequence of K-homology

$$0 \longrightarrow K_1(\{\infty\}) \longrightarrow K_1(S^n) \longrightarrow K_1(\mathbb{R}^n) \longrightarrow 0.$$

Since $K_1(\{\infty\}) = K_1(\mathrm{pt}) = 0$, there is a natural isomorphism

$$K_1(\mathbb{R}^n) \cong K_1(S^n). \tag{5.2.12}$$

Thus, the analytic K^1-pairing on $\mathbb{R}^n$ reduces to the case of S^n. Recall from Example 3.1.1 that when n is even $K_1(S^n) = 0$, and when n is odd $K_1(S^n) = \mathbb{Z}$ which is explicitly generated by (S^n, τ, Id). Here, τ is the trivial bundle over odd dimensional sphere S^n and $\mathrm{Id} : S^n \to S^n$ is the identity map. The analytic pairing is thus given by

$$\begin{gathered} K_1(S^n) \times K^1(S^n,\mathbb{R}/\mathbb{Z}) \longrightarrow \mathbb{R}/\mathbb{Z} \\ \langle (S^n, \tau, \mathrm{Id}), (\tau, d, \omega) \rangle = \bar{\eta}(\partial\!\!\!/_{S^n}) - \int_{S^n} \omega \wedge ch(\tau) \wedge \mathrm{Td}(S^n) \bmod \mathbb{Z}. \end{gathered} \tag{5.2.13}$$

The analytic term of (5.2.13) is the reduced eta-invariant $\bar{\eta}$ of the Dirac operator twisted by the trivial bundle on S^n and ω is an odd degree closed form since

$d\omega = ch(\tau) - rk(\tau) = 0$.

Since we are working on generators, the injectivity of (5.2.13) is implied. To show surjectivity, we only need to show that the RHS of (5.2.13) is non-zero. First, note that $\mathrm{Td}(S^n) \equiv 1$ for all positive odd n. Since $ch(\tau) \equiv rk(\tau)$, the integral term is dominated by

$$-rk(\tau) \int_{S^n} \omega \bmod \mathbb{Z}.$$

In general, the integral of ω over S^n is $\mathbb{R}$-valued and this yields an element in $\mathbb{R}/\mathbb{Z}$ upon reducing modulo $\mathbb{Z}$. For the analytic term, we consider two cases separately and summarise in the table below.

n	$\eta(\not{\partial}_{S^n})$	$\dim\ker(\not{\partial}_{S^n})$	$\bar{\eta}(\not{\partial}_{S^n})$
1	0	1	$\frac{1}{2}$
≥ 3	0	0	0

When $n = 1$, the eta-invariant of the Dirac operator $\not{\partial}_{S^1} = -id/d\theta$ (with respect to the disconnected-cover-spin-structure) is known to be 0 and has kernel of dimension 1. For $n \geq 3$, the eta-invariant remains zero since there are orientation reversing diffeomorphisms on odd dimensional spheres which essentially flip the signs of the eigenvalues and this yields a symmetric spectrum. Since S^n is a compact spin manifold which admits positive scalar curvature, by Lichnerowicz's theorem (cf. [33, Section II, Corollary 8.9]), the Dirac operator has no harmonic spinors, so the dimensional of the kernel is zero. The resulting reduced eta invariant $\bar{\eta}$ is zero. Hence, for odd $n \in \mathbb{Z}_+$ the pairing (5.2.13) has image not identically zero.

Lemma 5.2.7. *For positive odd n, the map $K^1(S^n, \mathbb{R}/\mathbb{Z}) \to Hom(K_1(S^n), \mathbb{R}/\mathbb{Z})$ given by* (5.2.13) *is an isomorphism, and so is the K^1 pairing for $\mathbb{R}^n \cong U$.*

It follows immediately that since n is odd, the K_0-homology group of odd spheres (and hence $\mathbb{R}^n$), as well as $K^0(\mathbb{R}^n, \mathbb{R}/\mathbb{Z})$ are all trivial. So, for odd n the map between $K^0(\mathbb{R}^n, \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(K_0(\mathbb{R}^n), \mathbb{R}/\mathbb{Z})$ is trivially an isomorphism. The only non-trivial and relevant part of the Mayer-Vietoris sequence is

$$\begin{array}{ccccccc} \longrightarrow K_c^1(U\cap V,\mathbb{R}/\mathbb{Z}) & \longrightarrow & K_c^1(U,\mathbb{R}/\mathbb{Z})\oplus K_c^1(V,\mathbb{R}/\mathbb{Z}) & \longrightarrow & K^1(U\cup V,\mathbb{R}/\mathbb{Z}) \longrightarrow \\ \downarrow & & \downarrow & & \downarrow \\ \to \mathrm{Hom}(K_1(U\cap V),\mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}(K_1(U),\mathbb{R}/\mathbb{Z})\oplus \mathrm{Hom}(K_1(V),\mathbb{R}/\mathbb{Z}) & \to & \mathrm{Hom}(K_1(U\cup V),\mathbb{R}/\mathbb{Z}) \to \end{array}$$

where the upper horizontal row is the Mayer-Vietoris sequence for $\mathbb{R}/\mathbb{Z}$ K^1-theory, the bottom row is the Mayer-Vietoris sequence for the Hom functor and each vertical line is the map implemented by the analytic pairing. We can now state the following result.

Lemma 5.2.8. *Suppose the isomorphism holds for contractible U, V and the intersection $U \cap V$. Then, for $X = U \cup V$, the map $K^1(X, \mathbb{R}/\mathbb{Z}) \to Hom(K_1(X), \mathbb{R}/\mathbb{Z})$ implemented by the analytic pairing* (5.2.3) *is an isomorphism.*

Proof. The result follows from Lemma 5.2.7 and the Five lemma. □

Proof. (of Theorem 5.2.2) The non-degeneracy of the pairing is implied by the isomorphism as in Lemma 5.2.8. For the general case, we apply induction on the size of the open cover, as in the proof of Lemma 5.1.9. This completes the proof for the analytic K^1-pairing. □

5.3 Perspective from physics

As we have explained in Chapter 3, geometric K-homology is a natural platform in describing D-branes on X. Both of the analytic K^1 and K^0 pairings have the following qualitative meaning: they measure the Aharonov-Bohm phase at infinity of Type IIA and Type IIB string theories respectively. We recall that Type IIA (resp. IIB) string theory is classified by $K_1(X)$ (resp. $K_0(X)$) whose elements are stable supersymmetric Dp-branes for all even $0 \leq p \leq 8$ (resp. for all odd $-1 \leq p \leq 9$), cf. [46, Section 2.2].

The Aharonov-Bohm effect in D-branes can be informally described as follows. Let X_9 be a smooth compact spin^c 9-manifold with boundary X. In the odd case, we consider the Type IIA String theory on $X_9 \times \mathbb{R}_t$. Assume that a brane produces a torsion flux. This flux defines an element of $K^0_{\mathrm{Tors}}(X)$. From the short exact sequence (2.2.11)

$$K^1(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} K^0_{\mathrm{Tors}}(X, \mathbb{Z}) \to 0,$$

this flux lifts to an element of $K^1(X, \mathbb{R}/\mathbb{Z})$. On the other hand, a test brane defines an element of $K^0(X)$. By the module multiplication (2.2.6), a pair consisting of a torsion flux and a test brane defines another element $\mathcal{V}$ of $K^1(X, \mathbb{R}/\mathbb{Z})$. Then, the Aharonov-Bohm phase is obtained by "transporting" $\mathcal{V}$ along an element of $K_1(X)$, which is a lift of an element of $H_1(X)$. This happens topologically. The pairing (5.2.3) provides an analytical description of such a transportation in terms of the eta-invariant of Dirac operators on X, cf. [37, 51]. The reason as to why the resulting $\mathbb{R}/\mathbb{Z}$-valued invariant of the analytic K^1-pairing (5.2.3) *is* the Aharonov-Bohm phase is explained in [51].

Analogously, the even case of the analytic pairing corresponds to the Aharonov-Bohm effect in Type IIB String theory on X. In this case, the torsion flux defines an element of $K^1_{\mathrm{Tors}}(X)$, which lifts to $K^0(X, \mathbb{R}/\mathbb{Z})$ from the short exact sequence (2.1.20)

$$K^0(X, \mathbb{R}/\mathbb{Z}) \xrightarrow{\beta} K^1_{\mathrm{Tors}}(X) \to 0.$$

Then, by module multiplication a pair consisting of a torsion flux and test brane define another element $\mathcal{E}$ of $K^0(X, \mathbb{R}/\mathbb{Z})$. Analytically, the Aharonov-Bohm phase at infinity is then measured by evaluating $\mathcal{E}$ over some cycles in $K_0(X)$ by the pairing formula (5.1.2) in terms of the Dai-Zhang eta-invariant of Dirac operators (5.1.1) on X.

Chapter 6

Analytic Pontryagin duality in cohomology theory

6.1 Degree 1: $H^1(X,\mathbb{R}/\mathbb{Z}) \times H_1(X,\mathbb{Z})$

In this section, we study the analytic Pontryagin duality pairing in the cohomology of degree one, which is another phase calculation of the Aharonov-Bohm effect in Quantum Mechanics, c.f. [25]. Let X be a smooth compact manifold. By Fact 1, the group $H_1(X)$ is identified with the first oriented bordism group $\Omega_1^{or}(X)$, whose element is given by $[S^1 \xrightarrow{\gamma} X]$. Then, the (classical) topological pairing

$$H^1(X,\mathbb{R}/\mathbb{Z}) \times H_1(X) \to \mathbb{R}/\mathbb{Z} \tag{6.1.1}$$

given by

$$\big(A, [S^1 \xrightarrow{\gamma} X]\big) \mapsto \int_{S^1} \gamma^* A \text{ mod } \mathbb{Z} \tag{6.1.2}$$

is the holonomy of a (pullback) flat connection A over a closed curve. Apart from the classical pairing, there is also an analytic aspect. Let $\not{\partial}_{S^1} = -id/d\theta$ be the usual Dirac operator on S^1, with respect to the disconnected-cover spin structure, given by

$$\tau = S^1 \times \mathbb{C} = \mathbb{R} \times \mathbb{C}/\sim, \tag{6.1.3}$$

where $(t,z) \sim (t',z')$ if and only if $t - t' \in \mathbb{Z}, z = z'$. In other words, this is the 'bad' spin structure of S^1 that does not extend to the disc $\mathbb{D}$. The group $H^1(X,\mathbb{R}/\mathbb{Z})$ is usually interpreted as the set that classifies all of the isomorphic flat complex line bundles with connections over X whose first Chern classes are torsion in $H^2(X,\mathbb{Z})$. The pullback, via γ, defines a flat complex line bundle over S^1, which is necessarily trivial by a torsionality argument. More precisely, let

$$L_\rho = \widetilde{X} \times_\rho U(1) \tag{6.1.4}$$

be the associated line bundle defined by an unitary representation $\rho : \pi_1(X) \to U(1)$. This bundle is flat and has the first Chern class $c_1(L_\alpha) \in H^2_{\mathrm{Tors}}(X,\mathbb{Z})$. Via $\gamma : S^1 \to X$, we obtain the unitary representation ρ' through the composition

$$\rho' = \rho \circ \gamma_* : \pi_1(S^1) \to U(1),$$

which defines the flat line bundle

$$\tilde{L} := L_{\rho'} = \mathbb{R} \times_{\rho'} U(1) \tag{6.1.5}$$

over S^1. A section of $\tilde{L}$ takes the form $f(\theta)v_{\rho'}(\theta)$, where f is a function on S^1 and $\nu_{\rho'}$ is a generating section given by

$$\nu_{\rho'}(\theta) = \exp(2\pi i a\theta), \quad a \in (0,1). \tag{6.1.6}$$

Let $\partial\!\!\!/^{\tilde{L}}_{S^1}$ be the twisted-by-$\tilde{L}$ Dirac operator on S^1. It is an ordinary self-adjoint elliptic differential operator. According to [6, 27], its eigenvalues are $\lambda_n = n + a$, where n is an integer obtained by differentiating f. Then, its Atiyah-Patodi-Singer eta-invariant is

$$\eta_{APS}(\partial\!\!\!/^{\tilde{L}}_{S^1}) = 1 - 2a. \tag{6.1.7}$$

which is non-zero in general, yielding the non-triviality of the eta-invariant.

Moreover, since $\dim\ker(\not{\partial}^{\tilde{L}}_{S^1}) = 1$, the reduced eta-invariant is

$$\bar{\eta}_{APS}(\not{\partial}^{\tilde{L}}_{S^1}) = \frac{\eta_{APS}(\not{\partial}^{\tilde{L}}_{S^1}) + \dim\ker(\not{\partial}^{\tilde{L}}_{S^1})}{2} = 1 - a \bmod \mathbb{Z} \tag{6.1.8}$$

which is again non-vanishing.

Theorem 6.1.1. *Let X be a smooth compact manifold and let $\gamma : S^1 \to X$ be a loop. Let $\tilde{L}$ be the associated flat line bundle over S^1 defined by* (6.1.5) *via γ. Let $\not{\partial}^{\tilde{L}}_{S^1}$ be the corresponding twisted Dirac operator. Then, the analytic pairing $H^1(X, \mathbb{R}/\mathbb{Z}) \times H_1(X) \to \mathbb{R}/\mathbb{Z}$ is given by*

$$\langle A, [S^1 \xrightarrow{\gamma} X]\rangle = \bar{\eta}_{APS}(\not{\partial}^{\tilde{L}}_{S^1}) - \int_{S^1} \gamma^* A \ \mathit{mod}\ \mathbb{Z}. \tag{6.1.9}$$

This pairing is well-defined and non-degenerate.

Proof. The validity and non-triviality of the analytic part of (6.1.8) are discussed above. The topological term is the (reduced modulo $\mathbb{Z}$) holonomy of a flat connection A over a closed curve. This pairing formula is a special case of the analytic K^1-pairing in Section 5.2. In particular, the well-definedness and the non-degeneracy follow from the general case. For instance, the pullback data defines a triple $(\tilde{L}, \nabla^{\tilde{L}}, \omega)$ over S^1, where ω is a 1-form satisfying

$$d\omega = c_1(\tilde{L}, \nabla^{\tilde{L}}) = 0. \tag{6.1.10}$$

From a standard calculation of the curvature $F_{\nabla^{\tilde{L}}} = \nabla^{\tilde{L}} \circ \nabla^{\tilde{L}} = dA'$, where $A' = \gamma^* A$, we see that ω is cohomologous to A'. By Stokes theorem, the topological integration is independent of the choice of 1-form. The others are routine work and are left to the reader. □

6.2 Degree 2: $H^2(X, \mathbb{R}/\mathbb{Z}) \times H_2(X, \mathbb{Z})$

In this section, we study the analytic Pontryagin duality pairing in the $\mathbb{R}/\mathbb{Z}$-cohomology of degree two. Let X be a smooth compact manifold. The classical (topological) pairing

$$H^2(X,\mathbb{R}/\mathbb{Z}) \times H_2(X,\mathbb{Z}) \to \mathbb{R}/\mathbb{Z} \tag{6.2.1}$$

is given by the holonomy of the pullback of a representative ω in $H^2(X,\mathbb{R}/\mathbb{Z})$ over some singular cycle c in $H_2(X,\mathbb{Z})$ via a continuous map $f : c \to X$.

Fact 1 ([47, IV. 7.35])**.** *Every homology class $z \in H_i(X,\mathbb{Z})$ with $i \leq 6$ can be represented by a smooth manifold. Let $\Omega_i^{or}(X)$ be the i-th oriented bordism group of X. The map*

$$\Omega_i^{or}(X) \to H_i(X,\mathbb{Z}); \quad (S,f) \mapsto f_*[S]$$

is an isomorphism for $i \leq 3$.

Without loss of generality, we replace c by an oriented, connected, closed Riemannian surface Σ. Let $[f : \Sigma \to X] \in H_2(X)$, with the equivalence relation given by thin bordism, c.f. [47]. Then, the pairing (6.2.1) can be expressed as

$$\big(\omega, [\Sigma \xrightarrow{f} X]\big) \mapsto \int_\Sigma f^*\omega \bmod \mathbb{Z}. \tag{6.2.2}$$

By classification results, there are 3 cases: $\Sigma_0 = S^2$ (of genus zero), $\Sigma_1 = T^2 = S^1 \times S^1$ (of genus 1), and in general $\Sigma_{2g} = T^{\#g}$ (of genus $2g$ for $g > 1$). Since genus$(T^{\#g}) >$ genus(S^2), there exists a degree 1 map $\phi : T^{\#g} \to S^2$, see [28]. Hence, it suffices to consider $\Sigma = S^2$, and the other cases follow by the composition

$$\begin{array}{ccc} \Sigma_{2g} & \xrightarrow{f'} & X. \\ {\scriptstyle \phi}\big\uparrow & \nearrow_{f} & \\ S^2 & & \end{array}$$

From (6.2.2), this reduces to the analytic pairing on S^2 by pullback. In the literature, the geometric object associated to $f^*\omega$ is often known as a *flat gerbe with connection* over S^2, c.f. [30]. However, it is not clear how to 'twist' a Dirac operator on S^2 with a gerbe. To circumvent this, we use the Hermitian local bundles of Melrose [41].

6.2.1 Representative of $H^2(S^2, \mathbb{R}/\mathbb{Z})$ as projective line bundles

For $i = 2, 3, 4$, let $\mathrm{Diag}_i = \{(x, \ldots, x) \in S^2 \times \cdots \times S^2\}$ be the diagonal of S^2.

Definition 6.2.1 ([41])**.** A Hermitian local line bundle L over S^2 is a complex line bundle over a neighbourhood V_2 of the diagonal Diag_2, together with a tensor product isomorphism of smooth bundles

$$\pi_3^* L \otimes \pi_1^* L \xrightarrow{\cong} \pi_2^* L \tag{6.2.3}$$

over a neighbourhood V_3 of the diagonal Diag_3, where $\pi_i : S^2 \times S^2 \times S^2 \to S^2 \times S^2$ is the projection omitting the i-th element $\pi_i(x_1, x_2, x_3) = (\widehat{x_i})$, and satisfying the associativity condition $L_{(x,y)} \otimes L_{(y,z)} \otimes L_{(z,t)} \to L_{(x,t)}$ on a sufficiently small neighbourhood of the diagonal Diag_4.

Strictly speaking, L is *not* a genuine line bundle but is only projective in the sense of [40]. It is only defined locally over some neighbourhood of the diagonal. More precisely, choose a good cover $\{U_i\}$ of S^2, then the product $U_i \times U_i$ defines an open cover of Diag_2, which is contained in small neighbourhood V_2, i.e. $\mathrm{Diag}_2 \subset U_i \times U_i \subset V_2 \subset S^2 \times S^2$. Choose $p_i \in U_i$ and consider the 'left' and 'right' bundles

$$\mathcal{L}_{i,p_i} = L|_{U_i \times \{p_i\}}, \quad \mathcal{R}_{p_i,i} = L|_{\{p_i\} \times U_i}. \tag{6.2.4}$$

Then, by the composition law (6.2.3), a line bundle

$$L = \mathcal{L}_{i,p_i} \otimes \mathcal{R}_{p_i,i} \tag{6.2.5}$$

is defined over $U_i \times U_i$. Moreover, there is a dual bundle identification $\mathcal{R}_{p_i,i} \cong \mathcal{L}_{i,p_i}^{-1}$ over U_i. By [41, Lemma 1], a local line bundle L on S^2 can be equipped with a multiplicative unitary structure and a multiplicative Hermitian connection. A connection ∇ is multiplicative if for a local section u of L near $(x, y) \in U_i \times U_i$ with $\nabla u = 0$ at (x, y), and for a local section v of L near $(y, z) \in U_i \times U_i$ with $\nabla v = 0$ at (y, z), the composition $C(u, v)$ of (6.2.3) is locally constant at (x, y, z).

The multiplicative Hermitian structure is taken as a Hermitian structure $g(\cdot, \cdot)_i$ on each $\mathcal{L}_{i,p_i}$. Using the dual identification on $\mathcal{R}_{p_i,i}$, this defines a Hermitian structure on it over U_i and thus L over $U_i \times U_i$ via (6.2.5). Using a partition of unity ρ_i subordinate to $U_i \times U_i$, the Hermitian structure $g(u,v) = \sum_i (\rho_i \times \rho_i) g(u,v)_i$ is well-defined.

By [41, Proposition 2], there is a one-to-one correspondence between the group $H^2(S^2,\mathbb{R})$ and the set of Hermitian local line bundles modulo unitary multiplicative isomorphisms in some neighbourhood of the diagonal. In particular, the representative closed 2-forms are identified with the curvature of the Hermitian local line bundle, i.e. $[B/2\pi] \in H^2(S^2,\mathbb{R})$ and $B = \nabla \circ \nabla$ for (L,∇). It is the first Chern class of L. In this way, we have obtained another geometric interpretation of $H^2(S^2,\mathbb{R}/\mathbb{Z})$.

Lemma 6.2.2. *The group $H^2(S^2,\mathbb{R}/\mathbb{Z})$ is isomorphic to the quotient of $H^2(S^2,\mathbb{R})$ by the reduced cohomology $\tilde{H}^2(S^2,\mathbb{Z})$.*

Proof. Consider the long exact sequence

(6.2.6)
$$\cdots \to H^1(S^2,\mathbb{R}/\mathbb{Z}) \xrightarrow{c_1} H^2(S^2,\mathbb{Z}) \to H^2(S^2,\mathbb{R}) \to H^2(S^2,\mathbb{R}/\mathbb{Z}) \to H^3(S^2,\mathbb{Z}) \to \cdots$$

where the first[1] map $c_1 : H^1(S^2,\mathbb{R}/\mathbb{Z}) \to H^2(S^2,\mathbb{Z})$ is the first Chern class. Let L_0 be a flat line bundle over S^2. Then, there are two cases:

$$c_1(L_0) = 0 \quad \text{or} \quad c_1(L_0) \in H^2_{\text{Tors}}(S^2,\mathbb{Z}).$$

Since $H^2(S^2,\mathbb{Z}) \cong \mathbb{Z}$ is non-torsion, we have $c_1(L_0) \equiv 0$. So, all such flat line bundles are necessarily trivial. They are labelled by the integer 0 in $\mathbb{Z}$. Let $\tilde{H}^2(S^2,\mathbb{Z})$ be the group generated by the Bott bundle $\beta = L - 1$ which corresponds to the generator $1 \in \mathbb{Z}$. Since $H^3(S^2,\mathbb{Z}) = 0$, from (6.2.6) we get

$$0 \to H^2(S^2,\mathbb{Z})/\mathrm{im}(c_1) \to H^2(S^2,\mathbb{R}) \to H^2(S^2,\mathbb{R}/\mathbb{Z}) \to 0 \tag{6.2.7}$$

and thus $H^2(S^2,\mathbb{R}/\mathbb{Z}) \cong H^2(S^2,\mathbb{R})/\tilde{H}^2(S^2,\mathbb{Z})$. □

[1] For clarity, the notation $H^1(S^1,\mathbb{R}/\mathbb{Z})$ denotes the circle group $\mathbb{R}/\mathbb{Z} \cong U(1)$ equipped with the discrete topology. This should not be confused with the notation $H^1(S^2,\underline{U(1)}) \cong H^2(S^2,\mathbb{Z})$ where $\underline{U(1)}$ denotes the sheaf of germs of $U(1)$-valued functions on X. In particular, by standard bundle theory the latter classifies all principal $U(1)$-bundles, in which $U(1)$ is the circle group equipped with the usual topology.

Remark 6.2.3. In other words, we interpret a representative of $H^2(S^2, \mathbb{R}/\mathbb{Z})$ as a *pure* Hermitian local line bundle L over S^2, in the sense that it is 'trivial' when it descends to an ordinary non-trivial line bundle on S^2.

6.2.2 Projective Dirac operator on S^2 twisted by L

Let L be a pure Hermitian local line bundle with connection over S^2 defined above, whose (normalized) curvature is a representative in $H^2(S^2, \mathbb{R})$. The appropriate notion of the twisted Dirac operator is the *projective Dirac operator* $\partial\!\!\!/^L_{S^2,\mathrm{proj}}$ introduced by Mathai, Melrose and Singer, c.f. [39–41]. See also [38] for its relation with transversally elliptic operators. Such an operator is a projective elliptic differential operator of order one defined on some neighbourhood of the diagonal Diag_2, with its kernel supported on the intersection of that neighbourhood and where L exists. From [39], there is a projective spinor bundle S over S^2 associated to the Azumaya bundle $\mathbb{C}l(TS^2)$. Since S^2 is spin^c, it can be viewed as the lift of the ordinary spinor bundle, also denoted as S, trivially to some ϵ-neighbourhood N_ϵ of the diagonal. Then, the projective bundle $S \otimes L$ is defined over

$$N'_\epsilon := N_\epsilon \cap U_i \times U_i \supset \mathrm{Diag}_2.$$

Let $\nabla^{S\otimes L}$ be the tensor product connection, defined by taking an appropriate partition of unity subordinate to N'_ϵ. Such a tensor product connection always exists, by the existence of the multiplicative Hermitian connection of L defined above, and the restriction to N'_ϵ of a global spin connection on S. The projective Dirac operator is given in term of distributions

$$\partial\!\!\!/^L_{S^2,\mathrm{proj}} := cl \cdot \nabla^{S\otimes L}_{\mathrm{left}}(\kappa_{\mathrm{Id}}) \tag{6.2.8}$$

where $\kappa_{\mathrm{Id}} = \delta(z-z')\mathrm{Id}_{S\otimes L}$ is the kernel of the identity operator in $\mathrm{Diff}^1(S^2, S\otimes L)$; $\nabla^{S\otimes L}_{\mathrm{left}}$ is the connection restricted to the left variables and cl denotes the Clifford action of T^*S^2 on the left. The projective Dirac operator $\partial\!\!\!/^L_{S^2,\mathrm{proj}}$ is elliptic and is odd with respect to the $\mathbb{Z}_2$-grading

$$\partial\!\!\!/^{L,\pm}_{S^2,\mathrm{proj}} \in \mathrm{Diff}^1(S^2; S^\pm \otimes L, S^\mp \otimes L).$$

By [39, Theorem 1], the projective analytical index of the positive part $\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}}$ is given by

$$\mathrm{Ind}(\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}}) = \mathrm{Tr}(\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}}Q - 1_{S^-\otimes L}) - \mathrm{Tr}(Q\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}} - 1_{S^+\otimes L}) \tag{6.2.9}$$

for *any* parametrix Q of $\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}}$. By [41, Theorem 2] (and also [39, Theorem 2]), the projective analog of the Atiyah-Singer index formula of the positive part is given by

$$\mathrm{Ind}\big(\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}}\big) = \int_{S^2} \mathrm{Td}(S^2)\wedge\exp(B/2\pi) \quad \in \mathbb{R} \tag{6.2.10}$$

where $\exp(B/2\pi)$ denotes the first Chern class of the local line bundle L.

6.2.3 Analytic pairing formula in $H^2(S^2,\mathbb{R}/\mathbb{Z})$

To formulate the analytic pairing in the case of $H^2(S^2,\mathbb{R}/\mathbb{Z})$, we need to consider the eta-invariant for projective Dirac operators. There are two subtleties here. Firstly, for parity reasons we need an even analog of the eta-invariant. Secondly, the operator involved is projective and does not have a spectrum. Thus, there is no well-defined notion of spectral asymmetry yet.

To tackle the first point, we adopt the Dai-Zhang eta-invariant (4.5.5) of an elliptic operator on S^2. To incorporate the even eta-invariant in this projective case, we have to make several assumptions.

Definition 6.2.4. Define

$$\eta_{DZ}\big(\partial\!\!\!/^{L}_{S^2,\mathrm{proj}}\big) := \eta_{APS}\big(\partial\!\!\!/^{L}_{S^2\times S^1,\mathrm{proj}}\big) \tag{6.2.11}$$

where η_{DZ} (resp. η_{APS}) denote the (unreduced) eta-invariant of the projective Dirac operator on S^2 of Dai-Zhang (resp. on $S^2\times S^1$ of Atiyah-Patodi-Singer).

Both of the LHS and RHS of (6.2.11) are not well-defined, since these operators are projective. However, we can still work on the RHS. In particular, this definition is consistent with the construction of the Dai-Zhang eta-invariant, which is done on the extension of S^2 to the cylinder $S^2\times[0,1]$, cf. Section 4.5. Moreover, we use the fact that the projective analytical index $\mathrm{Ind}(\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}})$ given by (6.2.9) is non-zero. Then, we circumvent the technical assumption in the Dai-Zhang construction (requiring vanishing index) by considering the gluing of the bundle data on both ends $S^2 \times \{0\}$ and $S^2 \times \{1\}$ by a K^1-representative g. Since $K^1(S^2) \equiv 0$ by Bott Periodicity, the bundle $S \otimes L$, extended trivially over to $S^2 \times [0,1]$, is glued trivially without any twisting at either end. This justifies the notation $\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}}$.

Next, to calculate the RHS of (6.2.11) , we rewrite the operator $\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}}$ as the *sharp product* of elliptic operators on the product manifold

$$R := \partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}} = \partial\!\!\!/^L_{S^2,\mathrm{proj}} \# \partial\!\!\!/_{S^1} = \begin{pmatrix} \partial\!\!\!/_{S^1} \otimes 1 & 1 \otimes \partial\!\!\!/^{L,-}_{S^2,\mathrm{proj}} \\ 1 \otimes \partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}} & -\partial\!\!\!/_{S^1} \otimes 1 \end{pmatrix}. \tag{6.2.12}$$

Here, $\partial\!\!\!/_{S^1}$ is the ordinary Dirac operator on S^1 given by $\partial\!\!\!/_{S^1} = -id/d\theta$. Note that both of the usual Dirac operator S^1 and the projective Dirac operator $\partial\!\!\!/^L_{S^2,\mathrm{proj}}$ are elliptic, and so is $\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}}$. This can be seen from the square of (6.2.12)

$$R^2 = \begin{pmatrix} \partial\!\!\!/^2_{S^1} \otimes 1 + 1 \otimes \partial\!\!\!/^{L,-}_{S^2,\mathrm{proj}} \partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}} & 0 \\ 0 & \partial\!\!\!/^2_{S^1} \otimes 1 + 1 \otimes \partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}} \partial\!\!\!/^{L,-}_{S^2,\mathrm{proj}} \end{pmatrix}. \tag{6.2.13}$$

Moreover, it is readily verified that R is self-adjoint. Nevertheless, R is still projective and does not have a spectrum. To interpret the RHS term of (6.2.11), we define a projective analog of the usual relation of the eta-invariant of the sharp product [9, 27] .

Definition 6.2.5. Let $P = P^{\pm}$ be a projective Dirac operator (with $P^+ = (P^-)^*$) on an even dimensional closed manifold M_1 and let A be an ordinary self-adjoint Dirac operator on an odd dimensional closed manifold M_2. Let R' be the sharp product of P and A, as an elliptic differential operator on the product manifold $M_1 \times M_2$, given by the following formula similar to (6.2.12)

$$R' := P \# A = \begin{pmatrix} A \otimes 1 & 1 \otimes P^- \\ 1 \otimes P^+ & -A \otimes 1 \end{pmatrix}. \tag{6.2.14}$$

Define its projective Atiyah-Patodi-Singer eta-invariant as

$$\eta_{APS}(R') := \mathrm{Ind}(P^+) \cdot \eta_{APS}(A) \tag{6.2.15}$$

where $\mathrm{Ind}(P^+)$ is the projective analytic index given by the similar formula (6.2.9) and $\eta_{APS}(A)$ denotes the usual eta-invariant of A.

Remark 6.2.6. Note that it might be misleading to write $\eta_{APS}(R')$, since R' has no spectrum. The point here is that we view the LHS of (6.2.15) as the projective analog of the measure of the spectral asymmetry, given by the product of the two terms on the RHS of (6.2.15). This is valid because the projective analytical index is indepedent of the choice of parametrix Q of P^+ and the other term is just the usual Atiyah-Patodi-Singer eta-invariant.

Remark 6.2.7. Moreover, Definition 2.5 holds for the ordinary case: when both P and A are ordinary Dirac operators, or more generally elliptic differential [2] operators. For the benefit of the reader, we illustrate an argument in [27] on the equality of (6.2.15) when P and A and thus R' are elliptic differential. Let $\Delta^+ = P^*P$ and $\Delta^- = PP^*$ be the associated Laplacians. Let $\{\lambda_i, \nu_i\}$ be a spectral resolution of Δ^+ on

$$\ker(\Delta^+)^\perp = \mathrm{Range}(P^-). \tag{6.2.16}$$

Then, $\{\lambda_i, P\nu_i/\sqrt{\lambda_i}\}$ is a spectral resolution of Δ^- on

$$\ker(\Delta^-)^\perp = \mathrm{Range}(P^+). \tag{6.2.17}$$

Observe that on the space $V_i = \mathrm{span}\{(\nu_i \oplus 0), (0 \oplus P\nu_i/\sqrt{\lambda_i})\}$, the operator R' is given by

$$R'_i = \begin{pmatrix} k & \sqrt{\lambda_i} \\ \sqrt{\lambda_i} & -k \end{pmatrix}, \tag{6.2.18}$$

which has eigenvalues

$$\pm\sqrt{k^2 + \lambda_i}.$$

[2] See [7, Pg 85] for this statement on the eta-invariant of the sharp product of two elliptic *differential* operators. This is not true if either one is *pseudodifferential*. One cannot apply the approximating-R'-by-pseudodifferential-operator argument under the natural Fredholm topology (c.f. [27, Sec. 3.7]) since it is not clear that the eta-invariant is continuous in the Fredholm topology. However, by some perturbation method, Gilkey [27, Sec 3.8.4] shows that it still holds when P or A is pseudodifferential.

Since $\lambda_i > 0$, the eigenvalues are non-zero and taking the eta-invariant is equivalent to taking the summation of these eigenvalues, which is zero. So, on V_i it does not contribute to the eta. On the other hand, the complement of $\oplus_i V_i$ is

$$W = (\ker(\Delta^+) \oplus 0) \oplus (0 \oplus \ker(\Delta^-)). \tag{6.2.19}$$

On W, the operator R' is given by

$$R' = \begin{pmatrix} k \cdot \pi_{\ker(\Delta^+)} & 0 \\ 0 & -k \cdot \pi_{\ker(\Delta^-)} \end{pmatrix}. \tag{6.2.20}$$

Then, taking the eta is equivalent to taking the normalised trace of (6.2.20), which gives

$$\eta(R') = \sum \operatorname{sgn}(k) \cdot [\operatorname{Tr}(\pi_{\ker(\Delta^+)}) - \operatorname{Tr}(\pi_{\ker(\Delta^-)})] = \eta(A) \cdot \operatorname{Ind}(P^+). \tag{6.2.21}$$

Unfortunately, this does not extend to the projective case. In particular, the equality of $PQ - 1 = \pi_{\ker(\Delta^+)}$ and $QP - 1 = \pi_{\ker(\Delta^-)}$ does not hold because the projective operators P and Q and thus PQ and QP are supported on some neighourhood of the diagonal, but the orthogonal projections $\pi_{\ker(\Delta^\pm)}$ are by no means only supported on a small neighbourhood of the diagonal. This should justify the ad hoc definition of (6.2.15), although at the current stage it is not clear how to show such a relation in the projective case.

Let $\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}}$ be the projective Dirac operator on $S^2 \times S^1$ given by (6.2.12). By Definition 2.5, its projective Atiyah-Patodi-Singer eta-invariant is

$$\eta_{APS}\big(\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}}\big) := \operatorname{Ind}(\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}}) \cdot \eta_{APS}(\partial\!\!\!/_{S^1}) \tag{6.2.22}$$

where $\operatorname{Ind}(\partial\!\!\!/^{L,+}_{S^2,\mathrm{proj}})$ is the projective analytical index in (6.2.9) and $\eta_{APS}(\partial\!\!\!/_{S^1})$ denotes the usual eta-invariant of the ordinary Dirac operator on S^1.

Corollary 6.2.8. $\eta_{APS}\big(\partial\!\!\!/^L_{S^2\times S^1,proj}\big) = 0$.

Proof. This follows from the fact that $\eta_{APS}(\partial\!\!\!/_{S^1}) = 0$. □

On the other hand, due to projectiveness, the kernel of $\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}}$ is not well-defined.

Assumption 2. *Define* $h(P\#A) := \dim\ker(A)$.

Definition 6.2.9. Let P, A and R' as in Definition 6.2.5. Define the reduced eta-invariant of the projective Dirac operator R' by

$$\bar{\eta}_{APS}(R') = \frac{\eta(R') + h(R')}{2} \bmod \mathbb{Z}. \tag{6.2.23}$$

Corollary 6.2.10. *Let* $M_2 = S^1$. *Take* $P = \partial\!\!\!/^L_{S^2,proj}$ *and* $A = \partial\!\!\!/_{S^1}$. *By Assumption 2, we have*

$$h(\partial\!\!\!/^L_{S^2\times S^1,proj}) = \dim\ker(\partial\!\!\!/_{S^1}) = 1 \tag{6.2.24}$$

$$\bar{\eta}_{APS}(\partial\!\!\!/^L_{S^2\times S^1,proj}) = \frac{\eta(\partial\!\!\!/^L_{S^2\times S^1,proj}) + h(\partial\!\!\!/^L_{S^2\times S^1,proj})}{2} \ mod\ \mathbb{Z} = \frac{1}{2} \ mod\ \mathbb{Z}. \tag{6.2.25}$$

Combining the discussions above, we are now ready to state the result of this section.

Theorem 6.2.11. *Let X be a smooth compact manifold. Let S^2 be the Riemannian 2-sphere, together with a smooth map $f : S^2 \to X$. Let L be the Hermitian local line bundle whose normalised curvature is $B/2\pi$, defined by the pullback of a representative $\omega/2\pi$ in $H^2(X,\mathbb{R}/\mathbb{Z})$ via f. Let $\partial\!\!\!/^L_{S^2,proj}$ be the projective Dirac operator twisted by L on S^2. Then, the analytic pairing $H^2(X,\mathbb{R}/\mathbb{Z}) \times H_2(X) \to \mathbb{R}/\mathbb{Z}$ is given by*

$$\left\langle \frac{\omega}{2\pi}, [\Sigma \xrightarrow{f} X] \right\rangle = \bar{\eta}_{DZ}(\partial\!\!\!/^L_{S^2,proj}) - \int_{S^2} \frac{B}{2\pi} \ mod\ \mathbb{Z}. \tag{6.2.26}$$

Moreover, it is non-degenerate and well-defined.

Proof. From (6.2.11), we consider the reduced Dai-Zhang eta invariant $\bar{\eta}_{DZ}(\partial\!\!\!/^L_{S^2,\mathrm{proj}})$ as the invariant $\bar{\eta}_{APS}(\partial\!\!\!/^L_{S^2\times S^1,\mathrm{proj}})$ defined by Corollary 6.2.8, Assumption 2, and (6.2.25). Its justification has been given above, which follows from the extension to the cylinder and trivial gluing at both ends. The topological part (the second term) is the (modulo $\mathbb{Z}$) holonomy of the (normalised) curvature 2-form associated to the representative L over S^2. To show non-degeneracy, it suffices to show that

$$H^2(S^2, \mathbb{R}/\mathbb{Z}) \to \mathrm{Hom}(H_2(S^2, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \tag{6.2.27}$$

implemented by the formula (6.2.26) is an isomorphism. Notice that we are actually working on generators on both groups, i.e. the generator L in $H^2(S^2, \mathbb{R}/\mathbb{Z})$ and the fundamental class $[S^2]$ in $H_2(S^2)$. So, the injectivity is implied. For surjectivity, it suffices to show that the map is non-zero, and thus sending generator to generator in $\mathbb{R}/\mathbb{Z}$. Let $k \in \mathbb{R}$ be the integration of the topological term. Together with (6.2.25), the pairing (6.2.26) reduces to $1/2 - k$ modulo $\mathbb{Z}$, which is non-zero in general. The isomorphism implies that the analytic pairing is non-degenerate. The well-definedness follows as a special case of the analytic pairing in $\mathbb{R}/\mathbb{Z}$ K^0-theory with torsion twists, which will be proven elsewhere [34]. $\square$

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