Solutions for Cooperative Fuzzy Games and Their Application in Exchange Economies

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Solutions for Cooperative Fuzzy Games and Their Application in Exchange Economies

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Introduction

Game theory is a subject that uses mathematics to model and analyze the interaction between rational players. The main founders of game theory are Zermelo, Borel and von Neumann. Later, von Neumann and Morgenstern (1944, [81]) systematized and formalized game theory for the first time. Then, Nash (1950, [60]) proved the existence of equilibrium points using a fixed point theorem, which laid a solid foundation for the generalization of game theory. It is generally believed that game theory can be divided into cooperative games and non-cooperative games. The main difference between cooperative games and non-cooperative games is whether a binding agreement is signed among the players. The game is a cooperative game if such an agreement exists, otherwise it is a non-cooperative game. These two classes of games have attracted a large number of scholars. Non-cooperative games mainly study the players' choice of strategies when each player wants to maximize his utility, given each other's strategies. On the other hand, cooperative games mainly study how to find a fair and reasonable payoff vector to distribute the benefits generated by the players when they cooperate.

Fuzziness is everywhere in the real world. There are many fuzzy concepts in general language. For example, there is no absolute boundary between such opposing concepts as tall and short or clean and polluted. To reflect the fuzzy phenomenon scientifically, Zadeh (1965, [86]) published his paper *Fuzzy set*. Zadeh (1965, [86]) defined a fuzzy set as a class of objects with continuous membership grade, generalizing the notion of a set. A fuzzy number is a fuzzy set that satisfies specific conditions, generalizing the notion of a real number. In cooperative games, the uncertainty in the process of cooperation will lead to a fuzzy worth of a coalition. In economic models, preference reflects an agent's attitude to different consumption vectors. Here, the uncertainty of each agent's attitude to consumption vectors can be reflected in a fuzzy preference relation. Moreover, there can be uncertainty regarding the payoff allocation.

This thesis discusses solutions for cooperative games and exchange economies, giving special attention to fuzziness in these models. Starting with solutions for cooperative games with transferable utility (TU-games) (Chapters 1 and 2), we enrich the model by considering fuzzy payoffs in TU-games (Chapter 3), and finally consider fuzzy preferences in a model of an exchange economy (Chapter 5).

Overview of the thesis

TU-games describe situations in which players can form a coalition and obtain the worth of cooperation based on a binding agreement. Excess in a TU-game is usually regarded as a measure of the dissatisfaction of a coalition or a player with a given payoff vector, expressed by the excess of the coalition and the excess of the player, respectively. Roughly said, the excess of a coalition in a payoff vector is the difference between what this coalition can generate in cooperation and what the players in the coalition will receive. When considering a weight system that assigns a weight to each coalition, the weighted excess of a coalition is defined by multiplying the excess by the assignated weight in Derks and Haller (1999, [27]). When the weight system assigns to a coalition the inverse of the number of players in the coalition, the weighted excess of this coalition is called the per-capita excess. Sakawa and Nishizaki (1994, [70]) were the first to define the excess of a player by summing up the excesses of all coalitions to which he belongs. Later, Vanam and Hemachandra (2013, [80]) defined the per-capita excess of a player by summing up all the per-capita excesses of all coalitions to which he belongs. Moreover, Yanovskaya (2002, [85]) proposed a proportional excess, where the dissatisfaction of a coalition is measured as the ratio between the worth of a coalition and the assigned payoff.

A solution for TU-games assigns a payoff vector to every game, reflecting how the worth generated by cooperation is allocated over the players. Based on different excesses of coalitions or players as introduced above, several solutions for TU-games in the literature are defined by lexicographically minimizing excesses or balancing excesses over different players. For example, the nucleolus in Schmeidler (1969, [71]), the prenucleolus in Sobolev (1973, [77]), the prekernel in Maschler et al. (1972, [51]), the least square value in Ruiz et al. (1998, [69]), the lexicographical solution in Sakawa and Nishizaki (1994, [70]), the per-capita excess-sum allocation in Vanam and Hemachandra (2013, [80]), and the proportional prenucleolus in Yanovskaya (2002, [85]).

In TU-games, we need to find a reasonable method to distribute the worth of cooperation. This is also the main goal for cooperative games with fuzzy payoffs. An important way to evaluate a solution is to analyze its properties and characterize it. Shapley (1953, [72]) first characterized a new value (called Shapley value) using efficiency, symmetry, additivity and the null player property. The well-known solutions mentioned in the previous paragraph are axiomatized by different researchers. Peleg (1986, [65]) provided an axiomatization of the prekernel using nonemptiness, Pareto optimality, covariance under strategic equivalence, the equal treatment property, a reduced game property, and the converse reduced game property. Calvo and Gutiérrez (1996, [19]) characterized the prekernel by strong stability and balanced surplus properties. Ruiz et al. (1998, [69]) characterized the family of least square values by efficiency, linearity, symmetry, inessential game, and coalitional monotonicity.

Allocation in economic models depends on the preferences of the agents. In case of uncertainty about the preferences, we need a model of preferences that reflects the uncertainty of each agent's attitude to every pair of consumption vectors. The usual $0-1$ binary relation reflects that each agent's satisfaction degree of one consumption vector relative to another is either 0 or 1. Based on this, the preference relation can be defined by comparing the relative satisfaction degree of each pair of consumption vectors. To express the uncertainty of each agent's attitude to each pair of consumption vectors, Nakamura (1986, [59]) defined the $[0, 1]$ binary relation where each agent's satisfaction degree for either one of each pair of commodity vectors is a constant in the closed interval $[0, 1]$. Based on this, a new preference relation can be defined in the same way as the above mentioned preference relation. Researchers applied these preference relations to exchange economies, defined different economic models, and studied the existence of competitive equilibria.

The first part of this thesis (Chapters 2, 3 and 4) mainly studies solutions of cooperative games by defining different excesses. First, in Chapter 2, we consider the weighted excess of a player by summing up all the weighted excesses of all coalitions to which he belongs. Thus, we define a new measure of agent's dissatisfaction for a given allocation, and then we analyze corresponding solutions for cooperative games and their properties. Second, in Chapter 3, we consider affine (and convex) combinations of the (classical) excess and the proportional excess, which provides a new approach to measure the dissatisfaction for coalitions of players. We further study the associated solutions for cooperative games. Third, in Chapter 4, we define the fuzzy excesses of coalitions for cooperative games with fuzzy payoffs (cooperative fuzzy games) introduced by Mallozzi et al. (2011, [49]). Then, we define solutions for cooperative fuzzy games based on fuzzy excesses and study their properties.

Finally, in Chapter 5, we mainly focus on studying a fuzzy preference relation that better reflects the ambiguous attitude of agents towards consumption vectors in real life and apply it to exchange economies. Then, we use a fixed point theorem and quasi-variational inequalities to study the existence of fuzzy competitive equilibria in the new economic model with fuzzy preferences.

A more detailed organization of this work, which consists of 5 chapters, is as follows.

Chapter 1 contains known definitions and notations about cooperative games, fuzzy numbers, cooperative fuzzy games and pure exchange economies.

In Chapter 2, we deal with the weighted excesses of players in cooperative games which are obtained by summing up all the weighted excesses of all coalitions to which they belong. We first show that lexicographically minimizing the individual weighted excesses of players gives the same minimal weighted excess for every player. Moreover, we show that the associated payoff vector is the corresponding least square value. Second, we show that minimizing the variance of the players' weighted excesses on the preimputation set, again yields the corresponding least square value. Third, we show that these results give rise to lower and upper bounds for the core payoff vectors. Using these bounds, we define the weighted super core as a polyhedron that contains the core, which is one of the main set-valued solutions for both cooperative games as well as exchange economies. It turns out that the least square values can be seen as a center of this weighted super core, giving a third new characterization of the least square values. Finally, these lower and upper bounds for the core inspire us to introduce a new solution for cooperative TU games that has a strong similarity with the Shapley value.

In Chapter 3, we introduce a new approach to measure the dissatisfaction for coalitions of players in cooperative transferable utility games. This is done by considering affine (and convex) combinations of the classical excess and the proportional excess. Based on this so-called α-excess, we define new solution concepts for cooperative games, such as the α-prenucleolus and the α-prekernel. The classical prenucleolus and prekernel are special cases when $\alpha = 0$. We characterize the α-prekernel by strong stability and the α-balanced surplus property. Also, we show that the payoff vector generated by the α-prenucleolus belongs to the α-prekernel.

In Chapter 4, we propose a total order relation of fuzzy numbers based on the expected values of fuzzy numbers. We show that three concepts of the indifference fuzzy core, nucleolus and bargaining sets of cooperative games with fuzzy payoffs are well-defined using this total order relation. Moreover, we obtain that the indifference fuzzy bargaining sets coincide with the indifference fuzzy core for convex cooperative games with fuzzy payoffs. Moreover, we characterize the class of superadditive cooperative games with fuzzy payoffs for which these sets coincide.

In Chapter 5, we set up a new fuzzy binary relation on consumption sets to evaluate fuzzy preferences. Besides, we prove that there exists a continuous fuzzy order-preserving function on a reference set for a given fuzzy preference relation. Subsequently, we focus on a model of a pure exchange economy with fuzzy preferences. The existence of a fuzzy competitive equilibrium for a pure exchange economy with fuzzy preferences is shown by using a fixed point theorem. Finally, we show that fuzzy competitive equilibria can be characterized as a solution to an associated quasi-variational inequality, giving rise to an equilibrium solution.

Chapter 1

Preliminaries

1.1 Cooperative games

1.1.1 TU-games

A *characteristic function game with transferable utility* (TU-game for short) is a pair (N, v) consisting of a set $N = \{1, 2, \cdots, n\}$ of n players, and a *characteristic function* $v : 2^N \to \mathbb{R}$, such that $v(\emptyset) = 0$. The power set 2^N denotes the set of all subsets or *coalitions* of N. Since the set of players is fixed, we often shortly write v instead of (N, v). For each coalition $S \subseteq N$, $v(S)$ represents the *worth* that coalition S achieves when its members cooperate. The number of players in coalition $S \subseteq N$ is denoted by s. The set of all TU-games with player set N is denoted by G^N.

A game $v \in G^N$ is said to be *balanced* if for every map $\lambda : 2^N \to \mathbb{R}_+$ such that $\sum_{\substack{S \subseteq N \\ i \in S}} \lambda(S) = 1$ for every $i \in N$,

$$v(N) \geq \sum_{S \subseteq N} \lambda(S) v(S).$$

The class of all *balanced games* is denoted by G_B^N.

A game $v \in G^N$ is convex if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \text{ for every } S, T \in 2^N.$$

A game $v \in G^N$ is superadditive if

$$v(S) + v(T) \leq v(S \cup T)$$

for every $S, T \in 2^N$ with $S \cap T = \emptyset$.

It is obvious that a convex game must be superadditive.

A well-known basis of G^N is the collection of unanimity games $(N, u_T)_{T \in 2^N}$, where the *unanimity game* (N, u_T) is defined as

$$u_T(S) = \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

1.1.2 Solutions

A vector $x \in \mathbb{R}^n$ will be called a *payoff vector*, and $x(S) = \sum_{i \in S} x_i$ for every $S \in 2^N$. For a game v, we say that a payoff vector $x \in \mathbb{R}^n$ is

- *efficient* if $x(N) = v(N)$;
- *individually rational* if $x_i \geq v(\{i\})$ for all $i \in N$;
- *coalitionally rational* if $x(S) \geq v(S)$ for all $S \subseteq N$.

A solution is a function φ that assigns to every game $v \in G^N$ a set of n-dimensional payoff vectors. A solution φ is single-valued if $\varphi(N, v)$ consists of exactly one payoff vector for every game v. In that case, we usually write it as a function $\varphi : G^N \to \mathbb{R}^n$ with $\varphi(N, v) \in \mathbb{R}^n$ being the unique payoff vector assigned to the game. A single-valued solution is also called a *value*. A payoff vector x is said to be a *preimputation* if it is efficient. A preimputation is called an *imputation* if it is also individually rational. Let $\mathscr{I}^*(N, v)$ and $\mathscr{I}(N, v)$ be the preimputation set and the imputation set of game v, respectively. The *core* of game v consists of the set of efficient and coalitionally rational payoff vectors, and is denoted by $\mathscr{C}(N, v)$.

For every payoff vector $x \in \mathbb{R}^n$ and every nonempty coalition S, the *excess* of S at x is

$$e(S,x) = v(S) - x(S). \tag{1.2}$$

The excess, $e(S,x)$, can be viewed as a measure of the *dissatisfaction* of coalition S with respect to the payoff vector x. The core of a game $v \in G^N$ can be written as

$$\mathscr{C}(N,v) = \{x \in \mathbb{R}^n \mid x(N) = v(N) \text{ and } e(S,x) \leq 0 \ \forall\ S \subseteq N\}.$$

The core is stable in the sense that each of its elements cannot be blocked by any coalition.

For every $\varepsilon \in \mathbb{R}$, the *strong ε-core* (Shapley and Shubik 1963, 1966, [73, 74]) of a game $v \in G^N$ is given by

$$\mathscr{C}_\varepsilon(N,v) = \{x \in \mathscr{I}^*(N,v) \mid e(S,x) - \varepsilon \leq 0 \ \forall S \in 2^N \setminus \{\emptyset, N\}\},$$

and thus allows coalitions to get 'a bit' less than their worth. Using this, the *least core* of a game $v \in G^N$ is defined as $\mathscr{C}_\lambda(N,v)$, where $\lambda = \min\{\varepsilon \in \mathbb{R} \mid \mathscr{C}_\varepsilon(N,v) \neq \emptyset\}$. We denote the least core of game v by $\mathscr{LC}(N,v)$. Another well-known solution is the *prekernel* (Maschler et al. 1972, [51]), which tries to balance the payoffs of players in a pairwise comparison. We denote by $\Gamma_{ij}(N)$ the set of all coalitions containing player i, but not player j, that is, $\Gamma_{ij}(N) = \{S \subseteq N \mid i \in S, j \notin S\}$. If there is no confusion about the player set, we will shortly write Γ_{ij} instead of $\Gamma_{ij}(N)$. To formally define the prekernel of a game, we first need to calculate the maximal surplus of player i over another player j at $x \in \mathbb{R}^n$ in the game $v \in G^N$:

$$s^v_{ij}(x) = \max_{S \in \Gamma_{ij}} e(S,x) \tag{1.3}$$

is the maximal surplus (in terms of excess) that player i can obtain in a coalition without player j. The prekernel $\mathscr{PK}(N,v)$ of a game $v \in G^N$ balances, within the preimputation set, the surpluses by equalizing for every pair of players the maximal surplus of one player over the other. Formally,

$$\mathscr{P}\mathscr{K}(N,v) = \{x \in \mathscr{I}^*(N,v) \mid s^v_{ij}(x) = s^v_{ji}(x) \text{ for all } i,j \in N, i \neq j\}. \tag{1.4}$$

Next, we define the lexicographic order on $\mathbb{R}^m$, and the vector $\theta(x)$ which orders the coordinates of x in nonincreasing order. Let $m \in \mathbb{N}$ and $x, y \in \mathbb{R}^m$. The lexicographic order $\leq_L$ is defined as follows:

(i) $x <_L y$ if there exists an integer $k \in \mathbb{N}, 1 \leq k \leq m$ such that $x_i = y_i$ for $1 \leq i < k$ and $x_k < y_k$,

(ii) $x \leq_L y$ if $x <_L y$ or $x = y$.

Moreover, $\theta(x)$ is the vector where the coordinates of x are ordered in nonincreasing order: $\theta_1(x) \geq \theta_2(x) \geq \ldots \geq \theta_m(x)$.

The prenucleolus of game v (Sobolev 1973, [77]) is the unique preimputation $x \in \mathscr{I}^*(N,v)$ satisfying

$$\theta(e(S,x)_{S \in 2^N \setminus \{\emptyset\}}) \leq_L \theta(e(S,y)_{S \in 2^N \setminus \{\emptyset\}}) \ \forall y \in \mathscr{I}^*(N,v).$$

Similarly, for games with nonempty imputation set, the nucleolus (Schmeidler 1969, [71]) of game v is the unique imputation $x \in \mathscr{I}(N,v)$ satisfying

$$\theta(e(S,x)_{S \in 2^N \setminus \{\emptyset\}}) \leq_L \theta(e(S,y)_{S \in 2^N \setminus \{\emptyset\}}) \ \forall\, y \in \mathscr{I}(N,v).$$

For a game $v \in G^N$, the *minimal right vector* $a(N,v) \in \mathbb{R}^n$ is defined by

$$a_i(N,v) = \max_{S \ni i} \left[v(S) - \sum_{j \in S \setminus \{i\}} \mu_j(N,v) \right] \text{ for all } i \in N,$$

where $\mu(N,v) \in \mathbb{R}^n$ given by $\mu_j(N,v) = v(N) - v(N \setminus \{j\})$ is the *utopia vector* of game v. A game v is called *quasi-balanced* if $a_i(N,v) \leq \mu_i(N,v)$ for all $i \in N$ and $\sum_{i \in N} a_i(N,v) \leq v(N) \leq \sum_{i \in N} \mu_i(N,v)$. For a quasi-balanced game v, the *τ value, or compromise value,* $\tau(N,v)$ (Tijs 1981, [79], Borm et al. 1992, [16]), is the solution that assigns to every quasi-balanced game the linear combination of the utopia vector and the minimal right vector that is efficient, i.e.,

$$\tau_i(N,v) = \lambda\mu_i(N,v) + (1-\lambda)a_i(N,v),$$

with $\lambda \in [0,1]$ such that $\sum_{i\in N}\tau_i(N,v) = v(N)$.

For every game $v \in G^N$, every player $i \in N$, and every coalition $S \in 2^N\backslash\{\emptyset\}$, the *marginal contribution* of i to S in v, denoted by $m_i^S(N,v)$, is given by

$$m_i^S(N,v) = \begin{cases} v(S) - v(S\backslash\{i\}) & \text{if } i \in S, \\ v(S\cup\{i\}) - v(S) & \text{if } i \notin S. \end{cases} \tag{1.5}$$

The most well-known single-valued solution is the *Shapley value* (Shapley 1953, [72]). It assigns to every player in game v its expected marginal contribution, by assuming that all possible orders of entrance of the players to the grand coalition occur with equal probability,

$$\mathscr{S}\mathscr{H}_i(N,v) = \sum_{\substack{S\subseteq N\\ S\ni i}} \frac{(n-s)!(s-1)!}{n!}[v(S) - v(S\backslash\{i\})] \text{ for all } i \in N.$$

The *Banzhaf value* (Dubey and Shapley 1979, [32]), originally introduced in Banzhaf (1965, [12]) as a power index for voting games, assigns to every player in every game its expected marginal contribution, by assuming that every coalition not containing this player is equally likely to occur,

$$\mathscr{B}\mathscr{A}_i(N,v) = \frac{1}{2^{n-1}}\sum_{\substack{S\subseteq N\\ S\ni i}}[v(S) - v(S\backslash\{i\})] \text{ for all } i \in N.$$

Davis and Maschler (1963, [24]) introduced the Davis-Maschler bargaining set by considering *objections* and *counter-objections* made by single players.

For every $v \in G^N$ and $x \in \mathscr{I}(N,v)$, let $i,j \in N, i \neq j$. We say that an objection of i against j at the imputation x in the game v is a pair (S,y)

where $S \in \Gamma_{ij}$ and $y = (y_k)_{k \in S}$ satisfying

$$y(S) = v(S), \tag{1.6}$$

$$y_k > x_k \text{ for all } k \in S. \tag{1.7}$$

We further say that a counter-objection of j to the objection (S, y) of i at x is a pair (T, z) where $T \in \Gamma_{ji}$ and $z = (z_k)_{k \in T}$ satisfying

$$z(T) = v(T), \tag{1.8}$$

$$z_k \geq y_k \text{ for all } k \in T \cap S, \tag{1.9}$$

$$z_k \geq x_k \text{ for all } k \in T \setminus S. \tag{1.10}$$

The *Davis-Maschler bargaining set* $\mathscr{M}_1^{ind}(N, v)$ of v is a set of payoff vectors satisfying

$$\mathscr{M}_1^{ind}(N, v) = \{x \in \mathscr{I}(N, v) \mid \text{no player has a justified objection at } x\},$$

where the justified objection is an objection that has no counter-objection.

Moreover, Mas-Colell (1989, [52]) defined the Mas-Colell bargaining set by considering *objections* and *counter-objections* made by nonempty coalitions.

For every $v \in G^N$ and $x \in \mathscr{I}(N, v)$, we say that an objection of coalition S at x in the game v is a pair (S, y) where S is a nonempty coalition, $y = (y_k)_{k \in S}$ satisfying (1.6) with

$$y_k \geq x_k \text{ for all } k \in S, \tag{1.11}$$

and at least one of the inequalities in (1.11) is strict. We further say that a counter-objection of coalition T to the objection (S, y) at x is a pair (T, z) where T is a nonempty coalition, $z = (z_k)_{k \in T}$ satisfying (1.8),(1.9),(1.10) and at least one of the inequalities in (1.9) or (1.10) is strict.

The *Mas-Colell bargaining set* $\mathscr{MB}(N,v)$ of v is defined as

$$\mathscr{MB}(N,v) = \{x \in \mathscr{I}(N,v) \mid \text{no nonempty coalition} \\ \text{has a justified objection at } x\}.$$

Observe that for every $v \in G^N$, a pair (S,y) can be used as an objection by the players of S or coalition S at x if and only if $e(S,y) > 0$. Furthermore, a pair (T,z) is both types of counter-objections at y if and only if $e(T,z) \geq 0$. Consequently, it is true that $\mathscr{C}(N,v) \subseteq \mathscr{M}_1^{ind}(N,v)$ and $\mathscr{C}(N,v) \subseteq \mathscr{MB}(N,v)$.

1.2 Cooperative fuzzy games

First, we recall some definitions and notations about fuzzy sets and fuzzy numbers (see Zadeh (1965, [86]), Dubois (1980, [33])).

1.2.1 Fuzzy numbers

Denote the set of all real numbers by $\mathbb{R}$. A *fuzzy set* $\tilde{A}$ in $\mathbb{R}$ is characterized by a membership function $\mu_{\tilde{A}} : \mathbb{R} \to [0,1]$. The value of $\mu_{\tilde{A}}(x)$ can be interpreted as the degree of membership of x to $\tilde{A}$.

Let $\tilde{A}$ be a fuzzy set in $\mathbb{R}$ with membership function $\mu_{\tilde{A}}$. The fuzzy set is a *fuzzy number* if $\mu_{\tilde{A}} : \mathbb{R} \to [0,1]$ is a mapping with the following properties:

(i) $\mu_{\tilde{A}}$ is upper semi-continuous, i.e., for every $x_0 \in \mathbb{R}$ and every $\varepsilon > 0$, there exists a neighbourhood U of x_0 such that

$$\mu_{\tilde{A}}(x) \leq \mu_{\tilde{A}}(x_0) + \varepsilon$$

for all $x \in U$,

(ii) $\mu_{\tilde{A}}$ is convex, i.e.,

$$\mu_{\tilde{A}}(\lambda x + (1-\lambda)y) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$$

for all $x, y \in \mathbb{R}, \lambda \in [0,1]$,

(iii) $\mu_{\tilde{A}}$ is normal, i.e., $\exists\, x_0 \in \mathbb{R}$ for which $\mu_{\tilde{A}}(x_0) = 1$,

(iv) $\text{support}(\tilde{A}) = \{x \in \mathbb{R} \mid \mu_{\tilde{A}}(x) > 0\}$ is the support of $\tilde{A}$ and its closure $cl(\text{support}(\tilde{A}))$ is compact.

Let $\mathbb{FR}$ be the set of all fuzzy numbers in $\mathbb{R}$.

For every $\tilde{A} \in \mathbb{FR}$, there exist $a, b, c, d \in \mathbb{R}$, $L : [a,b] \to [0,1]$ nondecreasing and $R : [c,d] \to [0,1]$ nonincreasing such that the *membership function* $\mu_{\tilde{A}}(x)$ is given as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} L(x) & \text{if } a \leq x < b, \\ 1 & \text{if } b \leq x \leq c, \\ R(x) & \text{if } c < x \leq d, \\ 0 & \text{ otherwise.} \end{cases} \tag{1.12}$$

A fuzzy number is said to be *trapezoidal* and denoted by $\lfloor a, b, c, d \rfloor$ if the functions L and R are linear. We denote the set of all trapezoidal fuzzy numbers by $\mathbb{TR}$.

For every $A \in \mathbb{R}$, the corresponding fuzzy number $\tilde{A}$ is defined by

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 \text{ if } x = A, \\ 0 \text{ if } x \neq A. \end{cases} \tag{1.13}$$

That is the situation of $a = b = c = d = A$ in (1.12).

For every $J = [\underline{J}, \overline{J}] \in \mathbb{IR}$, where $\mathbb{IR}$ is the set of all closed intervals in $\mathbb{R}$, the corresponding fuzzy number $\tilde{J}$ is defined by

$$\mu_{\tilde{J}}(x) = \begin{cases} 1 \text{ if } \underline{J} \leq x \leq \overline{J}, \\ 0 \text{ otherwise.} \end{cases} \tag{1.14}$$

That is the situation of $a = b = \underline{J}, c = d = \overline{J}$ in (1.12).

The α-*level set* of a fuzzy number $\tilde{A} \in \mathbb{FR}, 0 \leq \alpha \leq 1$, denoted by $\tilde{A}[\alpha]$, is defined as

$$\tilde{A}[\alpha] = \begin{cases} \{x \in \mathbb{R} | \mu_{\tilde{A}}(x) \geq \alpha\} & \text{if } 0 < \alpha \leq 1, \\ cl(\text{support}(\tilde{A})) & \text{if } \alpha = 0. \end{cases}$$

It is clear that the α-level set of a fuzzy number is a closed bounded interval $[A_*(\alpha), A^*(\alpha)]$, where $A_*(\alpha)$ denotes the left-hand endpoint of $\tilde{A}[\alpha]$ and $A^*(\alpha)$ the right-hand endpoint of $\tilde{A}[\alpha]$.

Let $\tilde{A}$, $\tilde{B}$ be two fuzzy numbers and λ a real number. The arithmetic fuzzy addition $\tilde{A} \tilde{+} \tilde{B}$, subtraction $\tilde{A} \tilde{-} \tilde{B}$, and scalar multiplication $\lambda \tilde{B}$ are fuzzy numbers which have the membership functions $\mu_{(\tilde{A} \tilde{+} \tilde{B})}(z)$, $\mu_{(\tilde{A} \tilde{-} \tilde{B})}(z)$, and $\mu_{\lambda \tilde{A}}(z)$ defined, for every $z \in \mathbb{R}$, by

$$\mu_{(\tilde{A} \tilde{+} \tilde{B})}(z) = \sup_{y \in \mathbb{R}} \{\min\{\mu_{\tilde{A}}(y), \mu_{\tilde{B}}(z - y)\}\},$$

$$\mu_{(\tilde{A} \tilde{-} \tilde{B})}(z) = \sup_{y \in \mathbb{R}} \{\min\{\mu_{\tilde{A}}(y), \mu_{\tilde{B}}(y - z)\}\},$$

$$\mu_{\lambda \tilde{A}}(z) = \begin{cases} \mu_{\tilde{A}}(\frac{z}{\lambda}) & \text{if } \lambda \neq 0, \\ 0 & \text{if } \lambda = 0, \end{cases}$$

We denote fuzzy summation by $\tilde{\sum}$.

For $\alpha \in [0, 1]$, the α-level sets of the fuzzy addition and the scalar multiplication have the following properties:

$$(\tilde{A} \tilde{+} \tilde{B})[\alpha] = [A_*(\alpha) + B_*(\alpha), A^*(\alpha) + B^*(\alpha)],$$

$$(\lambda \tilde{A})[\alpha] = [\lambda A_*(\alpha), \lambda A^*(\alpha)] \text{ if } \lambda > 0,$$

$$(\lambda \tilde{A})[\alpha] = [\lambda A^*(\alpha), \lambda A_*(\alpha)] \text{ if } \lambda < 0.$$

Let $\tilde{A} = \lfloor a^1, a^2, a^3, a^4 \rfloor$, $\tilde{B} = \lfloor b^1, b^2, b^3, b^4 \rfloor$ be two trapezoidal fuzzy numbers and λ a real number. The fuzzy addition $\tilde{A} \tilde{+} \tilde{B}$ and the scalar multiplication $\lambda \tilde{B}$ are trapezoidal fuzzy numbers, defined as follows:

$$\tilde{A} \tilde{+} \tilde{B} = \lfloor a^1 + b^1, a^2 + b^2, a^3 + b^3, a^4 + b^4 \rfloor,$$

$$\lambda\tilde{A} = \lfloor \lambda a^1, \lambda a^2, \lambda a^3, \lambda a^4 \rfloor \text{ if } \lambda > 0,$$

$$\lambda\tilde{A} = \lfloor \lambda a^4, \lambda a^3, \lambda a^2, \lambda a^1 \rfloor \text{ if } \lambda < 0.$$

The *expected value* $E(\tilde{A})$ of a fuzzy number $\tilde{A}$ is defined by Heilpern (1992, [39]) as

$$E(\tilde{A}) = \tfrac{1}{2}\int_0^1 (A_*(\alpha) + A^*(\alpha))d\alpha.$$

For every $\tilde{A}, \tilde{B} \in \mathbb{FR}$, the expected values of fuzzy numbers satisfy the following properties:

$$E(\tilde{A}\tilde{+}\tilde{B}) = E(\tilde{A}) + E(\tilde{B}), E(\tilde{A}\tilde{-}\tilde{B}) = E(\tilde{A}) - E(\tilde{B}).$$

A *partial order relation of intervals* was defined in Branzei et al. (2010, [18]). Let $J, K \in \mathbb{IR}$ with $J = [\underline{J}, \overline{J}], K = [\underline{K}, \overline{K}]$. We say $J \trianglerighteq K$ if and only if $\underline{J} \geq \underline{K}$ and $\overline{J} \geq \overline{K}$; $J \triangleright K$ if and only if $J \trianglerighteq K$ and $J \neq K$.

A *total order relation of intervals* was proposed by Han et al. (2012, [37]). Let $J, K \in \mathbb{IR}$ with $J = [\underline{J}, \overline{J}], K = [\underline{K}, \overline{K}]$. We say $J \gtrsim K$ if and only if $\frac{1}{2}(\underline{J} + \overline{J}) \geq \frac{1}{2}(\underline{K} + \overline{K})$; $J \sim K$ if and only if $\frac{1}{2}(\underline{J} + \overline{J}) = \frac{1}{2}(\underline{K} + \overline{K})$; $J > K$ if and only if $\frac{1}{2}(\underline{J} + \overline{J}) > \frac{1}{2}(\underline{K} + \overline{K})$.

A *partial order relation of fuzzy numbers* was defined in Mallozzi et al. (2011, [49]). Let $\tilde{A}, \tilde{B} \in \mathbb{FR}$. $\tilde{A} \succeq \tilde{B}$ if and only if $\tilde{A}[\alpha] \trianglerighteq \tilde{B}[\alpha]$ for every $\alpha \in [0,1]$, where $\tilde{A}[\alpha] \trianglerighteq \tilde{B}[\alpha]$ if and only if $A_*(\alpha) \geq B_*(\alpha)$ and $A^*(\alpha) \geq B^*(\alpha)$.

Let X be a subset of $\mathbb{R}^l$ and $\tilde{f} : X \to \mathbb{FR}$ a fuzzy mapping parameterized by $\tilde{f}(x) = \{(f(x)_*(\alpha), f(x)^*(\alpha), \alpha) : \alpha \in [0,1]\}$ for each $x \in X$. The expected mapping $f_E(x)$ for every $x \in X$ defined as

$$f_E(x) = \frac{1}{2}\int_0^1 [f(x)_*(\alpha) + f(x)^*(\alpha)]d\alpha,$$

is a real-valued function. The expected values of fuzzy numbers may be used as a ranking function. For example, Chanas and Kasperski (2003, [20]) compared every pair of fuzzy numbers and said that $\tilde{A}$ is not less than $\tilde{B}$ if and only if $E(\tilde{A}) \geq E(\tilde{B})$. However, there exist distinct fuzzy numbers $\tilde{A}$ and $\tilde{B}$ such that $E(\tilde{A}) = E(\tilde{B})$, which means that the definition of this ranking criterion is a partial order relation.

1.2.2 Solutions of cooperative fuzzy games

TU-games only consider situations where coalitional values are real numbers. However, there exist many uncertain factors during the process of negotiation and coalition forming. As a result, the players often only know imprecise information about the outcome of cooperation. Intervals can express the lower and upper bounds of the coalitional values, and hence researchers proposed interval games, such as Alparslan Gök et al. (2008, 2010, [4, 5]). To generalize interval games, Mareš (1999, [50]) and Mallozzi et al. (2011, [49]) formulated the vagueness of the coalitional values by fuzzy numbers and introduced cooperative games with fuzzy payoffs (cooperative fuzzy games for short).

An *interval game* introduced by Alparslan Gök et al. (2008, [4]) is a pair (N, ν) where $N = \{1, 2, \cdots, n\}$ is the set of players, and $\nu : 2^N \to \mathbb{IR}$ is the characteristic function such that $\nu(\emptyset) = 0$. We often shortly write ν instead of (N, ν). For each coalition $S \subseteq N$, $\nu(S)$ represents the worth that coaliton S achieves when its members cooperate. In other words, the coalitional values are intervals. Denote the set of all interval games with player set N by $\mathcal{I}G^N$. An interval game $\nu \in \mathcal{I}G^N$ is said to be *$\mathcal{I}$-balanced* if for every map $\lambda : 2^N \to \mathbb{R}_+$ such that $\sum_{\substack{S \subseteq N \\ i \in S}} \lambda(S) = 1$ for every $i \in N$, we have $\nu(N) \trianglerighteq \sum_{S \subseteq N} \lambda(S)\nu(S)$. The *interval imputation set* $\mathscr{I}(N, \nu)$ and the *interval core* $\mathscr{C}(N, \nu)$ of an interval game ν are defined by the partial order relation $\trianglerighteq$ of intervals, which is defined by comparing the endpoints of the intervals, as follows:

$$\mathscr{I}(N, \nu) = \{(J_1, \cdots, J_n) \in \mathbb{IR}^n \mid \sum_{i \in N} J_i = \nu(N) \text{ and } J_i \trianglerighteq \nu(i)\ \forall i \in N\},$$

$$\mathscr{C}(N, \nu) = \{(J_1, \cdots, J_n) \in \mathbb{IR}^n \mid \sum_{i \in N} J_i = \nu(N) \text{ and } \sum_{i \in S} J_i \trianglerighteq \nu(S)\ \forall S \subseteq N\},$$

where $\sum_{i \in S} J_i = [\sum_{i \in S} \underline{J_i}, \sum_{i \in S} \overline{J_i}]$ for every $S \subseteq N$.

Han et al. (2012, [37]) introduced the total order relation $\gtrsim$ of intervals, which is defined by comparing the midpoints of the intervals. They defined an alternative imputation set and interval core of an interval game ν. The indifference interval imputation set $\mathscr{I}'(N,\nu)$ is

$$\mathscr{I}'(N,\nu) = \{(J_1,\cdots,J_n) \in \mathbb{IR}^n \mid \sum_{i\in N} J_i \sim \nu(N) \text{ and } J_i \gtrsim \nu(i)\ \forall i \in N\},$$

and the indifference interval core $\mathscr{C}'(N,\nu)$ is

$$\mathscr{C}'(N,\nu) = \{(J_1,\cdots,J_n) \in \mathbb{IR}^n \mid \sum_{i\in N} J_i \sim \nu(N) \text{ and } \sum_{i\in S} J_i \gtrsim \nu(S)\ \forall S \subseteq N\},$$

where $\sum_{i\in S} J_i = [\sum_{i\in S} \underline{J_i}, \sum_{i\in S} \overline{J_i}]$ for every $S \subseteq N$.

A *cooperative fuzzy game* introduced in Mallozzi et al. [49] is a pair $(N,\tilde{v})$, where $N = \{1,2,\cdots,n\}$ is the set of players and $\tilde{v} : 2^N \to \mathbb{FR}$ is a mapping which assigns to every coalition $S \in 2^N$ a fuzzy number with $\tilde{v}(\emptyset) = 0$. We often shortly write $\tilde{v}$ instead of $(N,\tilde{v})$. For every coalition $S \subseteq N$, $\tilde{v}(S)$ denotes the fuzzy worth that coalition S achieves when its members cooperate. The class of all cooperative fuzzy games with player set N is denoted by $\mathcal{F}G^N$. A cooperative fuzzy game $\tilde{v} \in \mathcal{F}G^N$ is said to be *$\mathcal{F}$-balanced* if for every map $\lambda : 2^N \to \mathbb{R}_+$ such that $\sum_{\substack{S\subseteq N\\ i\in S}} \lambda(S) = 1$ for every $i \in N$, we have $\tilde{v}(N) \succeq \tilde{\sum}_{S\subseteq N}\lambda(S)\tilde{v}(S)$. For simplicity, we write $\tilde{v}(i)$ instead of $\tilde{v}(\{i\})$ for every $i \in N$, and use the standard notation $\tilde{x}(S) := \tilde{\sum}_{i\in S}\tilde{x}_i$ for every $\emptyset \neq S \subseteq N$ and $\tilde{x}(\emptyset) = 0$. A solution of $\tilde{v} \in \mathcal{F}G^N$ is a function $\Psi : \mathcal{F}G^N \to \mathbb{FR}^n$ assigning to every cooperative fuzzy game $\tilde{v}$ a set of n-dimensional payoff vectors with fuzzy numbers. Mallozzi et al. (2011, [49]) defined the *fuzzy core* ($\mathcal{F}$-core for short) of a cooperative fuzzy game $\tilde{v}$ using the partial order relation $\succeq$ of fuzzy numbers as follows:

$$\mathscr{C}^{\mathcal{F}}(N,\tilde{v}) = \{(\tilde{x}_1,\cdots,\tilde{x}_n) \in \mathbb{FR}^n \mid \tilde{x}(N) = \tilde{v}(N) \text{ and } \tilde{x}(S) \succeq \tilde{v}(S)\ \forall S \subseteq N\}.$$

1.3 Pure exchange economies

First, we recall different definitions of binary relations and the corresponding preference relations.

1.3.1 Preference relations

Preference reflects an individual's attitude toward a collection of objects. Next, we recall the $0-1$ binary relation.

Let X be a reference set. A $0-1$ binary relation $\mathfrak{R}_1$ in $X \times X$ is a pair $(X, \mu_{\mathfrak{R}_1})$, where $\mu_{\mathfrak{R}_1} : X \times X \to \{0, 1\}$ is the satisfaction function of $\mathfrak{R}_1$, and for $x, y \in X$, $\mu_{\mathfrak{R}_1}(x, y)$ represents the satisfaction degree of x relative to y; $\mu_{\mathfrak{R}_1}(y, x)$ represents the satisfaction degree of y relative to x; $\mu_{\mathfrak{R}_1}(x, y)$ and $\mu_{\mathfrak{R}_1}(y, x)$ are the relative satisfaction degree of x and y.

Based on the $0-1$ binary relation $\mathfrak{R}_1$, the preference relation $\succsim_{\mathfrak{R}_1}$ on X can be defined as follows:

For any $x, y \in X$, if $\mu_{\mathfrak{R}_1}(x, y) \geq \mu_{\mathfrak{R}_1}(y, x)$, we say that x is weakly preferred to y, denoted by $x \succsim_{\mathfrak{R}_1} y$; if $\mu_{\mathfrak{R}_1}(x, y) = \mu_{\mathfrak{R}_1}(y, x) = 1$, we say that x is indifferent to y, denoted by $x \sim_{\mathfrak{R}_1} y$; if $\mu_{\mathfrak{R}_1}(x, y) > \mu_{\mathfrak{R}_1}(y, x)$, we say that x is strongly preferred to y, denoted by $x \succ_{\mathfrak{R}_1} y$.

Since the $0-1$ binary relation is complete, any two elements $x, y \in X$ are comparable.

An agent i's preference relation on a collection of objects is denoted by $\succsim^i_{\mathfrak{R}_1}$. According to a conclusion in Debreu (1954, [26]), under certain conditions, there exists a utility function: $u_i : \mathbb{R}^l_+ \to \mathbb{R}$ such that $u_i(x_i) \geq u_i(x'_i)$ if and only if x_i is preferred or indifferent to x'_i for agent i.

Often an agent is not that definite about its preference with respect to two alternatives as reflected in $\mu_{\mathfrak{R}_1}$. Therefore, Nakamura (1986, [59]) introduced the following fuzzy binary relation.

Let X be a reference set. A fuzzy binary relation $\mathfrak{R}_2$ in $X \times X$ is a pair $(X, \mu_{\mathfrak{R}_2})$, where $\mu_{\mathfrak{R}_2} : X \times X \to [0, 1]$ is the satisfaction function of $\mathfrak{R}_2$, and for $x, y \in X$, $\mu_{\mathfrak{R}_2}(x, y)$ represents the satisfaction degree of x relative to y.

Based on the fuzzy binary relation $\mathfrak{R}_2$, the fuzzy preference relation $\succsim_{\mathfrak{R}_2}$ on X can be defined as follows:

For any $x, y \in X$, if $\mu_{\mathfrak{R}_2}(x, y) \geq \mu_{\mathfrak{R}_2}(y, x)$, we say that x is weakly preferred to y, denoted by $x \succsim_{\mathfrak{R}_2} y$; if $\mu_{\mathfrak{R}_2}(x, y) = \mu_{\mathfrak{R}_2}(y, x) \in (0, 1]$, we say that x is indifferent to y, denoted by $x \sim_{\mathfrak{R}_2} y$; if $\mu_{\mathfrak{R}_2}(x, y) > \mu_{\mathfrak{R}_2}(y, x)$, we say that x is strongly preferred to y, denoted by $x \succ_{\mathfrak{R}_2} y$; if $\mu_{\mathfrak{R}_2}(x, y) = \mu_{\mathfrak{R}_2}(y, x) = 0$, we say that x is uncomparable to y.

Under certain conditions, there exists a fuzzy utility function: $\hat{u}_i : \mathbb{R}^l_+ \to [0, 1]$ such that $\hat{u}_i(x_i) \geq \hat{u}_i(x'_i)$ if and only if x_i is preferred or indifferent to x'_i for agent i.

1.3.2 Existence of a competitive equilibrium

In this thesis, all the vectors in exchange economy models are in bold to make them more transparent. For every pair of vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$, $\mathbf{x} > \mathbf{y}$ means $x_i > y_i$ for all i; $\mathbf{x} \geqq \mathbf{y}$ means $x_i \geq y_i$ for all i; and $\mathbf{x} \geqslant \mathbf{y}$ means $\mathbf{x} \geqq \mathbf{y}$ but not $\mathbf{x} = \mathbf{y}$. The scalar product $\sum_{i=1}^m x_i y_i$ of two members $\mathbf{x}$ and $\mathbf{y}$ of $\mathbb{R}^m$ is denoted by $\langle \mathbf{x}, \mathbf{y} \rangle$.

In a pure exchange economy, we consider a marketplace consisting of l different goods, indexed by $h \in H = \{1, \cdots, l\}$, and n agents, denoted by $i \in N = \{1, \cdots, n\}$. Every agent i has an initial endowment vector: $\mathbf{w}_i = (w_{i1}, \cdots, w_{il}) \in \mathbb{R}^l_+$. The consumption vector of agent i is $\mathbf{x}_i = (x_{i1}, \cdots, x_{il}) \in X_i \subseteq \mathbb{R}^l_+$, where x_{ih} is consumption of agent i of commodity h. X_i is interpreted as the consumption set of agent i. We denote a price vector by $\mathbf{p} = (p_1, \cdots, p_l) \in \mathbb{R}^l_+$, where p_h is the price of commodity h. As it is standard in economic theory, an agent's choice from a given set of alternative consumption vectors is assumed to be made in accordance with his preference $\succsim^i_{\mathfrak{R}_1}$ (see page 19), which is represented by the utility function u_i explained in Subsection 1.3.1. We denote a pure exchange economy by $\mathcal{E} = (H, N, (X_i, \succsim^i_{\mathfrak{R}_1}, \mathbf{w}_i)_{i \in N})$.

An agent wants to maximize his utility among all consumption vectors that belong to his budget set. The budget set $B_i(\mathbf{p})$ of agent i is the set of admissible consumption vectors that are affordable for the agent at price vector $\mathbf{p} = (p_1, \cdots, p_l)$ with the value generated by his initial endowment vector $\mathbf{w}_i$, i.e.,

$$B_i(\mathbf{p}) = \{\mathbf{x}_i \in X_i \mid \langle \mathbf{p}, \mathbf{x}_i \rangle \leq \langle \mathbf{p}, \mathbf{w}_i \rangle\}.$$

This leads to the following optimization problem, for all $i \in N$ and $\mathbf{p} \in P$,

$$\max_{\mathbf{x}_i \in B_i(\mathbf{p})} u_i(\mathbf{x}_i). \tag{1.15}$$

In turn, the agent's income can be regarded as the receipts from sales of the initial endowments. The market for every good is usually considered to be in equilibrium if the supply of the good equals the demand for it. However, the price of some good may be zero, which means supply will exceed demand. The aggregate excess demand is $\mathbf{z} = (z_1, \cdots, z_l) \in \mathbb{R}^l$, where $z_h = \sum\limits_{i \in N} (x_{ih} - w_{ih})$ and $x_{ih} - w_{ih}$ is agent i's excess demand of good $h \in H$.

Definition 1.1. For the pure exchange economy $\mathcal{E}$, a pair $(\bar{\mathbf{p}}, \bar{\mathbf{x}})$ is said to be a *competitive equilibrium* of $\mathcal{E}$ if it satisfies the following conditions:

(1) $\bar{\mathbf{x}}_i \in \arg \max\limits_{\mathbf{x}_i \in B_i(\bar{\mathbf{p}})} u_i(\mathbf{x}_i)$.

(2) $\bar{\mathbf{p}} \in P = \{\mathbf{p} \mid \mathbf{p} \in \mathbb{R}^l, \mathbf{p} \geqq \mathbf{0}, \sum\limits_{h \in H} p_h = 1\}$.

(3) $\bar{\mathbf{z}} \leqq \mathbf{0}, \langle \bar{\mathbf{p}}, \bar{\mathbf{z}} \rangle = 0$, where $\bar{z}_h = \sum\limits_{i \in N} (\bar{x}_{ih} - w_{ih})$

Notice that in Condition (1), $u_i(\bar{\mathbf{x}}_i)$ depends on the value of $\bar{\mathbf{p}}$ and represents the utility function of agent i. Condition (2) implies that prices should be nonnegative and not all zero. Without any loss of generality, we can normalize the vector $\bar{\mathbf{p}}$ by restricting the sum of its coordinates to be 1. The first part of Condition (3), i.e., $\bar{\mathbf{z}} \leqq \mathbf{0}$, indicates that agents in the economy cannot consume more than their initial endowments. The second part of Condition (3), i.e., $\langle \bar{\mathbf{p}}, \bar{\mathbf{z}} \rangle = 0$, implies that the net value of trade is zero. All money that is paid for demanded goods by consumers, is received by consumers who have the initial endowments.

We introduce an abstract economy and define an equilibrium of an abstract economy.

An *abstract economy* consists of n agents $N = \{1, \cdots, n\}$, each of whom has an action set $\mathcal{H}_i \subseteq \mathbb{R}^l$ and a payoff function f_i defined over $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n$. The choice of agent i is restricted to the set $A_i(\mathbf{a}_{-i}) \subseteq \mathcal{H}_i$, which is a set-valued function defined for each point $\mathbf{a}_{-i} \in \mathcal{H}_{-i} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_{i-1} \times \mathcal{H}_{i+1} \times \cdots \times \mathcal{H}_n$. Formally, we denote an abstract economy as $(N, (\mathcal{H}_i, f_i, A_i(\mathbf{a}_{-i}))_{i \in N})$.

Definition 1.2. Let $(N, (\mathcal{H}_i, f_i, A_i(\mathbf{a}_{-i}))_{i \in N})$ be an abstract economy. $\bar{\mathbf{a}}$ is an *equilibrium* point if

$$f_i(\bar{\mathbf{a}}_{-i}, \bar{\mathbf{a}}_i) = \max_{\mathbf{a}_i \in A_i(\bar{\mathbf{a}}_{-i})} f_i(\bar{\mathbf{a}}_{-i}, \mathbf{a}_i) \text{ for all } \bar{\mathbf{a}}_i \in A_i(\bar{\mathbf{a}}_{-i}), i \in N.$$

We recall some definitions in Debreu (1952, [25]). The *graph* of $A_i(\mathbf{a}_{-i})$ is the set $\{\mathbf{a} \mid \mathbf{a}_i \in A_i(\mathbf{a}_{-i})\}$. The set-valued function $A_i(\mathbf{a}_{-i})$ is said to be *continuous* at $\mathbf{a}^0_{-i}$ if for every sequence $\{\mathbf{a}^{(k)}_{-i}\}$ converging to $\mathbf{a}^0_{-i}$, there exists a sequence $\{\mathbf{a}^{(k)}_i\}$ converging to $\mathbf{a}^0_i$ such that $\mathbf{a}^{(k)}_i \in A_i(\mathbf{a}^{(k)}_{-i})$ for all k.

Lemma 1.3. *An abstract economy $(N, (\mathcal{H}_i, f_i, A_i(\mathbf{a}_{-i}))_{i \in N})$ has an equilibrium point if*

(i) for each $i, \mathcal{H}_i$ is compact and convex, $f_i(\mathbf{a}_{-i}, \mathbf{a}_i)$ is continuous on $\mathcal{H}$ and quasi-concave in $\mathbf{a}_i$;

(ii) for every $\mathbf{a}_{-i}$, $A_i(\mathbf{a}_{-i})$ is a continuous function whose graph is a closed set; and

(iii) for every $\mathbf{a}_{-i}$, the set $A_i(\mathbf{a}_{-i})$ is convex and nonempty.

This lemma gives conditions for the existence of an equilibrium of an abstract economy. Under the following assumptions, Arrow and Debreu (1954, [8]) prove the existence of a competitive equilibrium of pure exchange economy $\mathcal{E}$ by defining an abstract economy whose equilibrium points have all the properties of a competitive equilibrium.

What follows are certain assumptions about the consumption units in a pure exchange economy.

For every good $h = 1, \cdots, l$, the rate of consumption of agent $i = 1, \cdots, m$ is necessarily non-negative, i.e., $x_{ih} \geq 0$.

Assumption I The set of consumption vectors X_i available to an individual $i = 1, 2, \cdots, m$ is a closed convex subset of $\mathbb{R}^l_+$.

Assumption II For all $\mathbf{x}'_i \in X_i$, the sets $\{\mathbf{x}_i \in X_i \mid \mathbf{x}_i \precsim^i \mathbf{x}'_i\}$ and $\{\mathbf{x}_i \in X_i \mid \mathbf{x}'_i \precsim^i \mathbf{x}_i\}$ are closed.

Assumption II ensures the continuity of $u_i(\mathbf{x}_i)$.

Assumption III For every $\mathbf{x}_i \in X_i$, there is $\mathbf{x}'_i \in X_i$ such that $u_i(\mathbf{x}'_i) > u_i(\mathbf{x}_i)$.

Assumption III assumes that there is no saturation point, no consumption vector that an individual would prefer to all others.

Assumption IV If $u_i(\mathbf{x}_i) > u_i(\mathbf{x}'_i)$ and $0 < \lambda < 1$, then $u_i[\lambda \mathbf{x}_i + (1 - \lambda)\mathbf{x}'_i] > u_i(\mathbf{x}'_i)$.

Assumption IV corresponds to the usual assumption that the indifference surfaces are convex in the sense that the set $\{\mathbf{x}_i \in X_i \mid u_i(\mathbf{x}_i) \geq a\}$ is a convex set for every fixed real number a.

We also suppose that agent i possesses an initial endowment vector $\mathbf{w}_i$ of different goods available.

Assumption V For some $\mathbf{x}_i \in X_i$, $\mathbf{x}_i < \mathbf{w}_i$.

Assumption V ensures that every agent could exhaust his initial endowments in some feasible way and still have a positive amount of each good available for trading in the pure exchange economy.

Theorem 1.4. *For a pure exchange economy $\mathcal{E}$, if $\mathcal{E}$ satisfies Assumptions I-V, then there is a competitive equilibrium of $\mathcal{E}$.*

Next, we recall the assumptions needed to prove the existence of a competitive equilibrium of a pure exchange economy by using the quasi-variational inequality .

We assume for $i \in N$:

(i) u_i is continuous and strictly concave on X_i.

(ii) For every $\mathbf{p} \in P$ and $\mathbf{x}_i \in B_i(\mathbf{p})$, $\nabla u_i(\mathbf{x}_i) \neq 0$.

(iii) For each $\mathbf{p} \in P$ and $\mathbf{x}_i \in \partial B_i(\mathbf{p})$, $\frac{\partial u_i(\mathbf{x}_i)}{\partial x_{ih}} > 0$ when $x_{ih} = 0$, $h \in H$.

(iv) $\lim\limits_{\substack{\|\mathbf{x}_i\| \to +\infty \\ \mathbf{x}_i \in B_i(\mathbf{p})}} u_i(\mathbf{x}_i) = -\infty$.

(v) Every agent is endowed with a positive quantity of at least one good, i.e.,

$$\forall i \in N, \exists h : w_{ih} > 0.$$

Under Assumptions (i)–(v), for all $i \in N$, the maximization problem (1.15) has a unique solution $\bar{\mathbf{x}}_i(\mathbf{p})$ for each $\mathbf{p} \in P$, denoted by $\bar{\mathbf{x}}_i$.

The competitive equilibrium of Definition 1.1 is equivalent to the following statement:

Proposition 1.5. *For the pure exchange economy $\mathcal{E}$, let $\bar{\mathbf{p}} \in P$ and $\bar{\mathbf{x}} \in B(\bar{\mathbf{p}}) = \prod\limits_{i \in N} B_i(\bar{\mathbf{p}})$. The pair $(\bar{\mathbf{p}}, \bar{\mathbf{x}}) \in P \times B(\bar{\mathbf{p}})$ is a competitive equilibrium if and only if*

$$u_i(\bar{\mathbf{x}}_i) = \max_{\mathbf{x}_i \in B_i(\bar{\mathbf{p}})} u_i(\mathbf{x}_i) \textit{ for all } i \in N,$$

and

$$z_h = \sum_{i \in N} (\bar{x}_{ih} - w_{ih}) \leq 0 \textit{ for all } h \in H.$$

In order to characterize a competitive equilibrium of a pure exchange economy as a solution to a related quasi-variational inequality, Anello et al. (2010, [6]) provide the following theorem.

Theorem 1.6. *Let $\mathcal{E} = (H, N, (X_i, \succsim^i_{\mathfrak{R}_1}, \mathbf{w}_i)_{i \in N})$ be a pure exchange economy. The pair $(\bar{\mathbf{p}}, \bar{\mathbf{x}}) \in P \times B(\bar{\mathbf{p}})$ is a competitive equilibrium of a pure exchange economy if and only if $(\bar{\mathbf{p}}, \bar{\mathbf{x}})$ is a solution to the following quasi-variational inequality*

$$\sum_{i \in N} \langle -\nabla u_i(\bar{\mathbf{x}}_i), (\mathbf{x}_i - \bar{\mathbf{x}}_i) \rangle - \left\langle \sum_{i \in N} (\bar{\mathbf{x}}_i - \mathbf{w}_i), (\mathbf{p} - \bar{\mathbf{p}}) \right\rangle \geq 0. \tag{1.16}$$

Chapter 2

Individual weighted excess and least square values

2.1 Introduction

In cooperative games with transferable utility, the lexicographical framework can provide a wide variety of objectives to be minimized, and it has given rise to an entire class of solution concepts. Two of the most popular solutions, the nucleolus defined by Schmeidler (1969, [71]) and the prenucleolus proposed by Sobolev (1973, [77]), are the outcome of a lexicographic minimization procedure over the excess vector that can be connected with every coalition. For every payoff vector, the excess of a coalition is the difference between the coalitional value and the total payoff allocated to the members of the coalition, and thus can be seen as a measure of dissatisfaction for the coalition. Since the sum of all excesses is constant over the preimputation set, a decrease of the highest excess will definitely result in the increase of at least one other excess. Based on the above fact, Ruiz et al. (1996, [68]) introduced the least square prenucleolus which minimizes the variance of the excesses of the coalitions under the assumption that all coalitions are equally important, i.e., all the excesses are given the same weight. Later, Ruiz et al. (1998, [69]) relaxed this assumption by allowing different weights for different coalitions. Hence, they introduced a function of coalitional weights and studied a family of symmetric values, called the LS family, obtained by minimizing the weighted variance of the excesses of all coalitions. Successively, Derks and Haller (1999, [27]) considered the weighted excess obtained by multiplying the ordinary excess of each coalition with a coalition specific positive coefficient or weight, and presented the weighted nucleolus.

The solutions mentioned above are based on the excesses of all coalitions, reflecting the dissatisfaction of the coalitions. Aiming to evaluate a payoff vector by means of the dissatisfaction of every player, Sakawa and Nishizaki (1994, [70]) presented the excess of a player by summing up all the excesses of all coalitions which he belongs to, and defined the lexicographical solution in view of the players' excesses. Vanam and Hemachandra (2013, [80]) took into account the per-capita excess-sum of every player, and proposed the per-capita excess-sum allocation in a TU cost game.

The goal of the current chapter is to explore the effect of allowing different weights for different coalitions, but considering individual excesses. We consider the *weighted excess of a player* by summing up all the weighted excesses of coalitions to which he belongs. It can be interpreted as the weighted dissatisfaction of a player with respect to the proposed payoff. Firstly, we show that lexicographically minimizing the weighted excesses of players yields the same weighted excess for every player. Moreover, taking the same weighted excess for all players as in Sakawa and Nishizaki (1994, [70]) and Molina and Tejada (2002, [56]), it turns out that the corresponding solution is a least square value as proposed by Ruiz et al. (1998, [69]). Second, by minimizing the variance of the weighted excesses of all players, we again obtain the corresponding least square value. This insight leads us to obtain an alternative axiomatic characterization of the least square (LS) family by efficiency and an equal weighted dissatisfaction property. Third, the results above give rise to an upper bound and a lower bound for the core and, using these bounds, we define the *weighted super core*. It is further shown that every least square value is obtained as some center of the corresponding weighted super core for every weight system.

Inspired by the midpoint of these two bounds, a Shapley-like value is proposed by assigning to every player in every game its expected weighted marginal contribution. Moreover, this value can be characterized similarly to the Shapley value by weighted efficiency, weighted dummy player property, additivity and symmetry. However, it is not efficient. To obtain an efficient solution, we consider two different methods of normalization, an additive and a multiplicative, respectively introduced by Hammer and Holzman (1987, [36]) and Dubey and Shapley (1979, [32]). It turns out that the additive normalization coincides with the corresponding ESL-value defined by Ruiz et al. (1998, [69]).

The chapter, which is based on Zhang et al. (2021, [89]), is organized as follows. Section 2.2 recalls some definitions for cooperative games. Section 2.3 introduces the individual weighted excess of a player and shows that lexicographically minimizing these excesses yields the corresponding least square value with equal excess for every player. In Section 2.4, we show that minimizing the variance of the weighted excesses also yields the least square values. In Section 2.5, we introduce the weighted super core as a polyhedron using core lower and upper bounds that are determined using the insights from the previous sections. We show that the least square value is the center of the corresponding weighted super core. Section 2.6 introduces a Shapley-like value based on the core bounds determined before. Section 2.7 concludes with a brief summary. In the Appendix, we give an axiomatization of the p-weighted Shapley value.

2.2 Definitions and Notations

In this section, we discuss some definitions that are variations of concepts discussed in Chapter 1, specifically on concepts related to the excesses of coalitions and the pre(nucleolus).

Lexicographically minimizing the excess over the set of imputations (respectively preimputations) gives the so-called nucleolus defined by Schmeidler (1969, [71]) (respectively prenucleolus proposed by Sobolev (1973,

[77])) as solution. Instead of minimizing the (coalitional) excesses, in order to better reflect the dissatisfaction of the players themselves, Sakawa and Nishizaki (1994, [70]) proposed the *excess of a player* at a payoff vector x by summing up all the excesses of coalitions (see Eq. (1.2)) to which he belongs,

$$w(i,x) = \sum_{\substack{S \subseteq N \\ S \ni i}} e(S,x). \tag{2.1}$$

On the basis of the lexicographical order $\leq_L$ (see page 10), Sakawa and Nishizaki (1994, [70]) defined the *lexicographical solution*, which minimizes the excesses of all players.

Vanam and Hemachandra (2013, [80]) took into account the *per-capita excess-sum of player i* at an imputation x, i.e.,

$$pce_i(x) = \sum_{\substack{S \subseteq N \\ S \ni i}} \frac{1}{s} e(S,x), \tag{2.2}$$

and proposed the *per-capita excess-sum allocation* of a TU cost game[1], which minimizes the per-capita excess-sum of every player in the lexicographical order $\leq_L$.

Given *weights* $p_S^N > 0$ for $\emptyset \subset S \subset N$ for every game $v \in G^N$ and every payoff vector $x \in \mathscr{I}(N,v)$, Derks and Haller (1999, [27]) defined the *weighted excess*

$$e^p(S,x) = p_S^N e(S,x) = p_S^N (v(S) - x(S)), \tag{2.3}$$

and the corresponding weighted nucleolus by lexocographically minimizing the weighted excess.

Since in this thesis we take the player set N to be fixed, from now on we will suppress the superindex N and write the weight of coalition S simply as p_S.

[1]A cost game is defined similar as a (profit) game, except that the interpretation of the worth of a coalition is the total cost that a coalition of players has to face jointly. Some solutions need to be redefined accordingly, for example the (anti-)core of a cost game is the set of efficient payoff vectors such that no coalition pays more than its own cost.

Ruiz et al. (1998, [69]) also regarded games with coalitional weights and restricted their attention to *symmetric* weight systems, which assign the same weight to coalitions of the same size. In that case, a *weight system* $p = (p_S)_{S\subseteq N}$ can be written as $p = (p_s)_{1\leq s\leq n}$, where $p_S = p_s$ for every $S \subseteq N$ with $|S| = s$. They considered the following minimization problem,

$$\min_{x\in\mathbb{R}^n} \quad \sum_{S\subseteq N} p_s(e(S,x) - \bar{e}(v))^2 \quad \text{s.t.} \quad \sum_{i\in N} x_i = v(N), \tag{2.4}$$

where

$$\bar{e}(v) = \bar{e}(v,x) = \frac{1}{2^n - 1} \sum_{S\subseteq N} e(S,x)$$

is the average excess for x, which is constant for every efficient payoff vector.

Given a weight system $p = (p_s)_{1\leq s\leq n}$, the corresponding p-*least square value* (p-LS value for short), is the value that assigns to every game v the solution of the minimization problem (2.4), and is given by

$$\mathrm{LS}_i^p(N,v) = \frac{v(N)}{n} + \frac{1}{n\alpha}\left(na_i^p(v) - \sum_{j\in N} a_j^p(v)\right) \text{ for every } i \in N, \tag{2.5}$$

where

$$\alpha = \sum_{s=1}^{n-1} p_s \binom{n-2}{s-1} \text{ and } a_i^p(v) = \sum_{\substack{S\subseteq N\\ S\ni i}} p_s v(S). \tag{2.6}$$

A value $\varphi : G^N \to \mathbb{R}^n$ belongs to the *least square family* (LS family for short) if there exists a weight system p such that $\varphi(N,v) = LS^p(N,v)$ for all $v \in G^N$.

In order to establish some basic properties of the solutions of the LS family, Ruiz et al. (1998, [69]) restated (2.5) as

$$\mathrm{LS}_i^p(N,v) = \frac{v(N)}{n} + \sum_{\substack{S\subset N\\ S\ni i}} \varrho_s \frac{v(S)}{s} - \sum_{\substack{S\subset N\\ S\not\ni i}} \varrho_s \frac{v(S)}{n-s}, \tag{2.7}$$

where $\varrho_s = \frac{s(n-s)}{n}\frac{p_s}{\alpha}$.

We recall the following well-known axioms for a solution φ,

- *Efficiency*: For each game $v \in G^N$, $\sum_{i \in N} \varphi_i(N,v) = v(N)$.
- *Symmetry*: For each game $v \in G^N$ and each permutation $\sigma : N \to N$, let $\sigma v \in G^N$ be given by $\sigma v(S) = v(\sigma(S))$, $S \subseteq N$. Then, $\varphi_{\sigma(i)}(N, \sigma v) = \varphi_i(N,v)$ for all $i \in N$.
- *Linearity*: For every two games $v, w \in G^N$ and $a, b \in \mathbb{R}$, $\varphi(N, av + bw) = a\varphi(N,v) + b\varphi(N,w)$, where $(av+bw)(S) = a \cdot v(S) + b \cdot w(S)$ for all $S \subseteq N$. Particularly, this property is called *additivity* when $a = b = 1$.

Ruiz et al. (1998, [69]) further showed that a value φ on G^N is efficient, linear and symmetric if, and only if, there exists a unique collection of real constants $\{\varrho_s\}_{s=1,\cdots,n-1}$ such that for every game $v \in G^N$, the payoff vector $(\varphi_i(N,v))_{i \in N}$ is given by formula (2.7). These values are called ESL-values.

Consider the grand coalition N in the game v. For given constants d_{ij} for all $i, j \in N$, we say that they are compatible if $d_{ii} = 0$, $d_{ij} = -d_{ji}$ and $d_{ij} + d_{jk} = d_{ik}$ for all $i, j, k \in N$. We say that a payoff vector $(x_i)_{i \in N}$ preserves differences according to d_{ij} if

$$x_i - x_j = d_{ij} \text{ for all } i, j \in N.$$

Given compatible constants $\{d_{ij}\}_{i,j \in N}$, Hart and Mas-Colell (1989, [38]) verified that there exists a single efficient payoff vector x that preserves differences:

$$x_i = \frac{1}{n}\left(v(N) + \sum_{j \in N} d_{ij}\right),\ x_j = x_i - d_{ij}.$$

2.3 p-least square values and the p-weighted excess-sum prenucleolus

In the remaining of this chapter, we use a system of weights $p = (p_S)_{S \subseteq N}$ satisfying

$$p_S \geq 0 \text{ for all } S \in 2^N \setminus \{\emptyset\} \text{ and } p_S > 0 \text{ for some coalition } S \neq N.$$

Inspired by the ideas of Ruiz et al. (1998, [69]), but considering the individual excess as in Sakawa and Nishizaki (1994, [70]) (see Eq. (2.1)) and using weights as in Derks and Haller (1999, [27]) (see Eq. (2.3)), we represent the *weighted dissatisfaction* of the players by defining the *weighted excess of every player*.

Definition 2.1. Given a system of weights p, a game $v \in G^N$, a preimputation $x \in \mathscr{I}^*(N, v)$ and a coalition $S \subseteq N$,

$$w^p(i, x) = \sum_{\substack{S \subseteq N \\ S \ni i}} p_S e(S, x) \tag{2.8}$$

is called the weighted excess of player i with respect to preimputation x.

The weighted excess of a player is the sum of all the weighted excesses of the coalitions to which he belongs, and as such it may be interpreted as the weighted dissatisfaction of the player towards the proposed payoff. Notice that the weight system only depends on coalition S and the player set N, and not on the worth of the coalition or the preimputation.

Consider the n-dimensional vector $\theta(w^p(i, x)_{i \in N})$, whose components are arranged in nonincreasing order. Just like the prenucleolus is obtained by lexicographically minimizing the coalitional excesses over all preimputations, we lexicographically minimize the individual weighted excesses over all preimputations (see page 10).

Definition 2.2. For every weight system p and every game $v \in G^N$, the

p-weighted excess-sum prenucleolus is the set of payoff vectors that lexicographically minimizes the vector of excesses $(w^p(i,x))_{i\in N}$ over the preimputation set

$$\begin{aligned}\mathscr{PN}^p(N,v) &= \{x \in \mathscr{I}^*(N,v) | \theta(w^p(i,x)_{i\in N}) \leq_L \theta(w^p(i,y)_{i\in N}) \\ &\qquad \forall y \in \mathscr{I}^*(N,v)\}.\end{aligned}$$

If the weight system $p_S = 1$ for all $S \subseteq N$, then the p-weighted excess-sum prenucleolus is the lexicographical solution defined by Sakawa and Nishizaki (1994, [70]) and lexicographically minimizes (2.1). The p-weighted excess-sum prenucleolus becomes the per-capita excess-sum allocation of a cost game as defined by Vanam and Hemachandra (2013, [80]) if the weight system is given by $p_S = \frac{1}{|S|}$, see Eq. (2.2).

Remark 2.1. According to the results in Justman (1977, [43]), we have the following statements.

(i) If $\mathscr{I}^*(N,v)$ is nonempty and compact, and if all $w^p(i,x)$, $i \in N$, are continuous, then $\mathscr{PN}^p(N,v) \neq \emptyset$.

(ii) If $\mathscr{I}^*(N,v)$ is convex and all $w^p(i,x)$, $i \in N$, are convex, then $\mathscr{PN}^p(N,v)$ is convex and $w^p(i,x) = w^p(i,y)$ for all $i \in N$ and all $x, y \in \mathscr{PN}^p(N,v)$.

Inspired by the method provided by Peleg and Sudhölter (2007, [66]), let $y \in \mathscr{I}^*(N,v)$ and define

$$\mathscr{I}'(N,v) = \{x \in \mathscr{I}^*(N,v) \mid e(S,x) \leq \max_{S\subseteq N} e(S,y) \ \forall S \subseteq N\}.$$

Since $\mathscr{I}'(N,v)$ is nonempty, convex, and compact, from Remark 2.1, we obtain that the p-weighted excess-sum prenucleolus is a singleton.

Theorem 2.3. *Given every weight system p, the p-weighted excess-sum prenucleolus is a singleton for every game.*

The weight system p_S has several interpretations: the probability of coalition S to form; the power of coalition S in the bargaining process; the stability degree of coalition S. Next, as in Ruiz et al. (1998, [69]), we consider symmetric weight systems $p = (p_s)_{1 \leq s \leq n}$ where coalitions of the same size have the same weight. It turns out that in the p-weighted excess-sum prenucleolus, the individual weighted excesses are the same for every player.

Theorem 2.4. *Let $p = (p_s)_{1 \leq s \leq n}$ be a symmetric weight system and let $v \in G^N$. For each $x \in \mathscr{P}\mathscr{N}^p(N, v)$ and $i, j \in N$, it holds that*

$$w^p(i, x) = w^p(j, x) = \frac{1}{n}\left(\sum_{k \in N} a_k^p(v) - (\alpha + n\beta)v(N)\right), \tag{2.9}$$

where $\beta = \sum_{s=2}^{n} p_s \binom{n-2}{s-2}$ and α and $a_i^p(v)$ are given by (2.6).

Proof. We prove the theorem in four steps.

(i) Recall that $a_i^p(v) = \sum_{\substack{S \subseteq N \\ S \ni i}} p_s v(S)$, $i \in N$. For every $x \in \mathscr{I}^*(N, v)$ and $i \in N$, we have

$$\begin{aligned}
w^p(i, x) &= \sum_{\substack{S \subseteq N \\ S \ni i}} p_s(v(S) - x(S)) \\
&= \sum_{\substack{S \subseteq N \\ S \ni i}} p_s v(S) - \sum_{\substack{S \subseteq N \\ S \ni i}} p_s x(S) \\
&= a_i^p(v) - \left(\sum_{\substack{S \subseteq N \\ S \ni i}} p_s x_i + \sum_{j \in N \setminus \{i\}} \sum_{\substack{S \subseteq N \\ S \ni i,j}} p_s x_j\right) \\
&= a_i^p(v) - \left(\sum_{s=0}^{n-1} \binom{n-1}{s} p_{s+1} x_i + \sum_{j \in N \setminus \{i\}} \sum_{s=0}^{n-2} \binom{n-2}{s} p_{s+2} x_j\right) \\
&= a_i^p(v) - \sum_{s=1}^{n} \binom{n-1}{s-1} p_s x_i - \sum_{j \in N \setminus \{i\}} \left(\sum_{s=2}^{n} \binom{n-2}{s-2} p_s x_j\right) \\
&= a_i^p(v) - \sum_{s=2}^{n-1} \binom{n-1}{s-1} p_s x_i - \binom{n-1}{0} p_1 x_i - \binom{n-1}{n-1} p_n x_i
\end{aligned}$$

$$
\begin{aligned}
& - \sum_{j \in N \setminus \{i\}} \sum_{s=2}^{n} \binom{n-2}{s-2} p_s x_j \\
= \; & a_i^p(v) - \left(\sum_{s=2}^{n-1} \binom{n-1}{s-1} p_s + p_1 + p_n \right) x_i \\
& - \left(\sum_{s=2}^{n} \binom{n-2}{s-2} p_s \right) \sum_{j \in N \setminus \{i\}} x_j \\
= \; & a_i^p(v) - \left(\sum_{s=2}^{n-1} \left(\binom{n-2}{s-1} + \binom{n-2}{s-2} \right) p_s + p_1 + p_n \right) x_i \\
& - \left(\sum_{s=2}^{n} \binom{n-2}{s-2} p_s \right) (v(N) - x_i) \\
= \; & a_i^p(v) - \left(\sum_{s=2}^{n-1} \binom{n-2}{s-1} p_s + p_1 + p_n - \binom{n-2}{n-2} p_n \right) x_i \\
& - \left(\sum_{s=2}^{n} \binom{n-2}{s-2} p_s \right) v(N) \\
= \; & a_i^p(v) - \left(\sum_{s=1}^{n-1} \binom{n-2}{s-1} p_s \right) x_i - \left(\sum_{s=2}^{n} \binom{n-2}{s-2} p_s \right) v(N) \\
= \; & a_i^p(v) - \alpha x_i - \beta v(N)
\end{aligned}
\tag{2.10}
$$

(ii) Adding up the individual excesses over all individual players gives

$$
\begin{aligned}
\sum_{i \in N} w^p(i,x) &= \sum_{i \in N} (a_i^p(v) - \alpha x_i - \beta v(N)) \\
&= \sum_{i \in N} a_i^p(v) - \alpha \sum_{i \in N} x_i - \beta \sum_{i \in N} v(N) \\
&= \sum_{i \in N} a_i^p(v) - (\alpha + n\beta) v(N).
\end{aligned}
\tag{2.11}
$$

(iii) Next, we show that the individual weighted excess is the same for every player. On the contrary, assume that $x \in \mathscr{P}\mathscr{N}^p(N,v)$ such that there are $i, j \in N$, $i \neq j$, with $w^p(i,x) \neq w^p(j,x)$. Without loss of generality, fix

$i, j \in N$ with $w^p(i,x) > w^p(j,x)$ and $w^p(i,x) = \max_{k\in N} w^p(k,x)$. Define $c = \frac{w^p(i,x)-w^p(j,x)}{2}$ and then construct a payoff vector x' meeting

$$w^p(k,x') = \begin{cases} w^p(i,x) - c \text{ if } k = i, \\ w^p(j,x) + c, \text{ if } k = j, \\ w^p(k,x), \text{ if } k \neq i,j. \end{cases} \tag{2.12}$$

Obviously, $(w^p(i,x') - w^p(i,x)) + (w^p(j,x') - w^p(j,x)) = 0$ and, by Eq. (2.11)

$$\sum_{k\in N} w^p(k,x') = \sum_{k\in N} w^p(k,x) = \sum_{i\in N} a_i^p(v) - (\alpha + n\beta)v(N).$$

By Eq. (2.10), we have

$$w^p(i,x') - w^p(i,x) = \alpha(x_i - x_i') \text{ and } w^p(j,x') - w^p(j,x) = \alpha(x_j - x_j').$$

Thus, we have $(x_i - x_i') + (x_j - x_j') = 0$. Therefore, $x' \in \mathscr{I}^*(N,v)$. However, from the construction of the payoff vector x',

$$w^p(j,x') = w^p(i,x') < w^p(i,x)$$

and, for $k \in N \setminus \{i,j\}$,

$$w^p(k,x') = w^p(k,x) \leq w^p(i,x)$$

and $\theta(w^p(k,x')_{k\in N}) <_L \theta(w^p(k,x)_{k\in N})$. This establishes a contradiction to our premise $x \in \mathscr{PN}^p(N,v)$ and, therefore, $w^p(i,x) = w^p(j,x)$ for all $i, j \in N$.

(iv) From (ii) and (iii) above, we can directly derive that the individual weighted excesses for every $x \in \mathscr{I}^*(N,v)$, $i \in N$, are given by

$$w^p(i,x) = \frac{1}{n}\sum_{j\in N} w^p(j,x) = \frac{1}{n}\sum_{j\in N} a_j^p(v) - (\alpha + n\beta)v(N). \qquad \square$$

It turns out that, for every symmetric weight system, the p-weighted excess-sum prenucleolus coincides with the corresponding least square value given by Eq. (2.5).

Proposition 2.5. *Let p be a symmetric weight system and $v \in G^N$. Then,*

$$\mathscr{P}\mathscr{N}^p(N,v) = LS^p(N,v).$$

In the proof of Proposition 2.5, we use the following lemma.

Lemma 2.6. *For every $v \in G^N$ and $i, j \in N$, we have*

(i) $a_i^p(v) - a_j^p(v) = \sum_{S \subseteq N \setminus \{i,j\}} p_{s+1}[v(S \cup \{i\}) - v(S \cup \{j\})].$

(ii) $na_i^p(v) - \sum_{j \in N} a_j^p(v) = \sum_{j \in N \setminus \{i\}} \sum_{S \subseteq N \setminus \{i,j\}} p_{s+1}[v(S \cup \{i\}) - v(S \cup \{j\})].$

(iii) $a_{-i}^p(v) - a_{-j}^p(v) = \sum_{S \subseteq N \setminus \{i,j\}} p_{s+1}[v(S \cup \{j\}) - v(S \cup \{i\})].$

Proof.

(i)
$$\begin{aligned}
a_i^p(v) - a_j^p(v) &= \sum_{\substack{S \subseteq N \\ S \ni i}} p_s v(S) - \sum_{\substack{S \subseteq N \\ S \ni j}} p_s v(S) \\
&= \sum_{S \subseteq N \setminus \{i,j\}} [p_{s+1} v(S \cup \{i\}) + p_{s+2} v(S \cup \{i,j\})] \\
&\quad - \sum_{S \subseteq N \setminus \{i,j\}} [p_{s+1} v(S \cup \{j\}) + p_{s+2} v(S \cup \{i,j\})] \\
&= \sum_{S \subseteq N \setminus \{i,j\}} p_{s+1}[v(S \cup \{i\}) - v(S \cup \{j\})].
\end{aligned}$$

(ii)
$$\begin{aligned}
na_i^p(v) - \sum_{j \in N} a_j^p(v) &= \sum_{j \in N} [a_i^p(v) - a_j^p(v)] \\
&= \sum_{j \in N \setminus \{i\}} [a_i^p(v) - a_j^p(v)] \\
&= \sum_{j \in N \setminus \{i\}} \sum_{S \subseteq N \setminus \{i,j\}} p_{s+1}[v(S \cup \{i\}) - v(S \cup \{j\})],
\end{aligned}$$

where the last equality follows from part (i).

(iii) $$a^p_{-i}(v) - a^p_{-j}(v) = \sum_{k\in N\setminus\{i\}} a^p_k(v) - \sum_{k\in N\setminus\{j\}} a^p_k(v)$$
$$= a^p_j(v) - a^p_i(v)$$
$$= \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{j\}) - v(S\cup\{i\})],$$

where the last equality follows from part (i). □

Proof of Proposition 2.5. For every $x \in \mathscr{I}^*(N,v)$, by Eq. (2.10), the weighted excess of player i with respect to x,

$$w^p(i,x) = a^p_i(v) - \alpha x_i - \beta v(N), \tag{2.13}$$

is a constant. Let x be a preimputation meeting Eq. (2.9). Then,

$$x_i - x_j = \frac{1}{\alpha}[a^p_i(v) - a^p_j(v)] = \frac{1}{\alpha}\sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{i\}) - v(S\cup\{j\})]$$

for all $i,j\in N$, where the first equality follows from Eq. (2.10) and the second equality follows from Lemma 2.6 (i).

Let us consider the constants $d^p_{ij} = \frac{1}{\alpha}[a^p_i(v) - a^p_j(v)]$, for all $i,j\in N$. It is easily seen that the system $\{d^p_{ij}\}_{i,j\in N}$ satisfies $d^p_{ii} = 0$, $d^p_{ij} = -d^p_{ji}$ and $d^p_{ij} + d^p_{jk} = d^p_{ik}$, for all $i,j,k\in N$. Furthermore, x preserves differences according to $\{d^p_{ij}\}_{i,j\in N}$, i.e., $x^i - x^j = d^p_{ij}$ for all $i,j\in N$. Thus, by Hart and Mas-Colell (1989, [38]), there exists a unique efficient payoff vector x that preserves $\{d^p_{ij}\}_{i,j\in N}$ and it is given by

$$x_i = \frac{1}{n}\left(v(N) + \sum_{j\in N} d^p_{ij}\right) \text{ and } x_j = x_i - d^p_{ij}.$$

That is, for every $i \in N$,

$$x_i = \frac{v(N)}{n} + \frac{1}{n\alpha}\left(na_i^p(v) - \sum_{j \in N} a_j^p(v)\right). \qquad \square$$

As a direct consequence of Lemma 2.6 (ii), the p-LS value can be written as

$$\begin{aligned}\mathrm{LS}_i^p(N,v) \quad = \quad & \frac{v(N)}{n} + \frac{1}{n\sum\limits_{s=0}^{n-2} p_{s+1}\binom{n-2}{s}} \sum_{j \in N\setminus\{i\}} \sum_{S \subseteq N\setminus\{i,j\}} p_{s+1}[v(S \cup \{i\}) \\ & -v(S \cup \{j\})]\end{aligned} \tag{2.14}$$

This expression makes clear that a least square value assigns to every player i an equal share in the worth of the grand coalition, but corrects this by the average weighted difference in contributions of player i and every other player j to coalitions they do not belong to. (Here, $\sum_{s=0}^{n-2} p_{s+1}\binom{n-2}{s}$ is the sum of the weights put on all coalitions containing i and every other player $j \neq i$.)

We conclude this section with the following remark. Given every weight system p, the solution satisfying that all individual weighted excesses are equal is defined as follows

$$\mathscr{E}\mathscr{S}^p(N,v) = \{x \in \mathscr{I}^*(N,v) | w^p(1,x) = \cdots = w^p(n,x)\}. \tag{2.15}$$

If the weight system $p_S = 1$ for all $S \subseteq N$, this solution becomes the equalizer solution of a crisp game defined by Molina and Tejada (2002, [56]).

2.4 Minimizing the variance of individual weighted excesses

In the previous section, we gave a characterization of the least square values by lexicographically minimizing the (weighted) individual player excesses. In this section, we consider a least square method, but using the individual weighted excesses as considered in Section 2.3.

Ruiz et al. (1998, [69]) minimize the sum of squared differences from the coalitional excesses and average excess to obtain the least square values. We now minimize the sum of squared diffferences of the individual weighted excesses and the per capita weighted excess

$$\bar{w}(v) = \frac{1}{n}\sum_{i\in N} w^p(i,x) = \frac{1}{n}\left(\sum_{i\in N} a_i^p(v) - (\alpha + n\beta)v(N)\right).$$

Given a symmetric weight system p, we consider the following problem for a game $v \in G^N$:

Problem 1:

$$\min_{x\in\mathbb{R}^n} \quad \sum_{i\in N}\left(\sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x) - \bar{w}(v)\right)^2$$
$$\text{s.t.} \quad \sum_{i\in N} x_i = v(N).$$

Notice that for $c \in \mathbb{R}$,

$$\sum_{i\in N}\left(\sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x) - c\right)^2 = \sum_{i\in N}\left(\sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x)\right)^2 + nc^2 - 2c\sum_{i\in N}\sum_{\substack{S\subseteq N\\ S\ni i}} p_s e(S,x),$$

where the last summation is constant over the preimputation set since

$$\sum_{i \in N} \sum_{\substack{S \subseteq N \\ S \ni i}} p_s e(S, x) = \sum_{i \in N} w^p(i, x) = \sum_{i \in N} a_i^p(v) - (\alpha + n\beta) v(N).$$

As a consequence, substituting $\bar{w}(v)$ in the objective function in Problem 1 by a constant c, the resulting objective function differs only in a constant, and the optimal solution remains unchanged. Particularly, for $c = 0$, the optimal solution of Problem 1 is that of the following problem.

Problem 2:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \sum_{i \in N} \left(\sum_{\substack{S \subseteq N \\ S \ni i}} p_s e(S, x) \right)^2 \\ \text{s.t.} \quad & \sum_{i \in N} x_i = v(N). \end{aligned}$$

This gives another characterization of the least square values.

Theorem 2.7. *For each symmetric weight system p and for each game $v \in G^N$, the unique solution of Problem 1 is $LS^p(N, v)$.*

Proof. By working out the Hessian matrix, it can easily be checked that the objective function in Problem 2 is strictly convex in $\mathbb{R}^n$. Moreover, it is obvious that the objective function is continuous. Since the feasible set is convex and determined by an equality constraint, there is at most one optimal solution, and the Lagrange conditions are necessary and sufficient for a point to be the optimal solution. The Lagrangian of Problem 2 is

$$L(x, \lambda) = \sum_{i \in N} \left(\sum_{\substack{S \subseteq N \\ S \ni i}} p_s e(S, x) \right)^2 + \lambda \Big(\sum_{i \in N} x_i - v(N) \Big).$$

Then, the partial derivative with respect to x_i, $i \in N$, of $L(x, \lambda)$ is given by $L_{x_i}(x, \lambda)$

$$\begin{aligned}
&= -2\sum_{s=1}^{n} p_s \binom{n-1}{s-1} \sum_{\substack{S\subseteq N\\ S\ni i}} p_s(v(S)-x(S)) \\
&\quad -2\sum_{s=2}^{n} p_s \binom{n-2}{s-2} \sum_{j\in N\setminus\{i\}} \sum_{\substack{S\subseteq N\\ S\ni j}} p_s(v(S)-x(S)) + \lambda \\
&= -2\sum_{s=1}^{n} p_s \binom{n-1}{s-1} (a_i^p(v) - \alpha x_i - \beta v(N)) \\
&\quad -2\beta \sum_{j\in N\setminus\{i\}} (a_j^p(v) - \alpha x_j - \beta v(N)) + \lambda \\
&= -2\sum_{s=1}^{n} p_s \binom{n-1}{s-1} a_i^p(v) + 2\sum_{s=1}^{n} p_s \binom{n-1}{s-1} \alpha x_i + 2\sum_{s=1}^{n} p_s \binom{n-1}{s-1} \beta v(N) \\
&\quad -2\beta \left(\sum_{j\in N\setminus\{i\}} a_j^p(v) + \alpha x_i - (\alpha + (n-1)\beta) v(N) \right) + \lambda \\
&= -2\sum_{s=1}^{n} p_s \binom{n-1}{s-1} a_i^p(v) - 2\beta \sum_{j\in N\setminus\{i\}} a_j^p(v) + 2\alpha \left(\sum_{s=1}^{n} p_s \binom{n-1}{s-1} - \beta \right) x_i \\
&\quad +2\beta \left(\sum_{s=1}^{n} p_s \binom{n-1}{s-1} - \beta + \alpha + n\beta \right) v(N) + \lambda \\
&= -2\gamma a_i^p(v) - 2\beta a_{-i}^p(v) + 2\alpha^2 x_i + 2\beta(2\alpha + n\beta) v(N) + \lambda = 0
\end{aligned}$$

where

$$\gamma = \sum_{s=1}^{n} p_s \binom{n-1}{s-1} \text{ and } a_{-i}^p(v) = \sum_{j\in N\setminus\{i\}} a_j^p(v), \tag{2.16}$$

the second equation follows from Eq. (2.10) and the last equation holds for $\gamma - \beta = \alpha$.

Evidently, the derivative about λ gives the efficiency constraint

$$L_\lambda(x,\lambda) = \sum_{i\in N} x_i - v(N) = 0.$$

A simple calculation solves this linear system and shows that the unique point x satisfying these conditions is given by

$$x_i^p = \frac{v(N)}{n} + \frac{\gamma}{n\alpha^2}\left(na_i^p(v) - \sum_{j\in N} a_j^p(v)\right) + \frac{\beta}{n\alpha^2}\left(na_{-i}^p(v) - \sum_{j\in N} a_{-j}^p(v)\right) \tag{2.17}$$

for every $i \in N$, $\alpha = \sum_{s=1}^{n-1} p_s\binom{n-2}{s-1}$, $a_i^p(v) = \sum_{\substack{S\subseteq N\\ S\ni i}} p_s v(S)$, $\beta = \sum_{s=2}^{n} p_s\binom{n-2}{s-2}$, and $a_{-i}^p(v) = \sum_{j\in N\setminus\{i\}} a_j^p(v)$, as defined in (2.6), (2.9) and (2.16). From Lemma 2.6 (iii) and $\gamma - \beta = \alpha$, Eq. (2.17) becomes

$$\begin{aligned} x_i^p &= \frac{v(N)}{n} + \frac{1}{n\alpha}\left(na_i^p(v) - \sum_{j\in N} a_j^p(v)\right) \\ &= \frac{v(N)}{n} + \frac{1}{n\sum_{s=0}^{n-2} p_{s+1}\binom{n-2}{s}} \sum_{j\in N\setminus\{i\}} \sum_{S\subseteq N\setminus\{i,j\}} p_{s+1}[v(S\cup\{i\}) \\ &\quad - v(S\cup\{j\})]. \end{aligned}$$

This coincides with Eq. (2.5) of the p-LS value. □

So far, we have seen that p-least square values can be obtained both as the allocation that lexicographically minimizes the individual weighted excesses and as the allocation that minimizes the variance of the individual weighted excesses. Using this, we propose a new axiomatic characterization of the p-LS values which requires equal individual weighted excesses for each player.

- *Equal p-weighted dissatisfaction property*: Let p be a symmetric weight system. The solution φ satisfies equal p-weighted dissatisfaction if for every game $v \in G^N$, $w^p(i,\varphi(v)) = w^p(j,\varphi(v))$ for every $i,j \in N$ with

$i \neq j$.

Together with efficiency, this property characterizes the corresponding p-least square value.

Theorem 2.8. *Let p be a symmetric weight system. A value $\varphi : G^N \to \mathbb{R}^n$ satisfies efficiency and the equal p-weighted dissatisfaction property if, and only if, φ is the p-LS value.*

Proof. It can be easily checked that each value defined by (2.5) satisfies the two axioms with the corresponding weight system p. To see the converse, let φ be a value satisfying the two axioms for some symmetric weight system p. On the contrary, suppose that there are two different values $\varphi^1(v), \varphi^2(v) \in \mathbb{R}^n$ that verify the two properties. On account of the equal p-weighted dissatisfaction property, it is true that

$w^p(i, \varphi^1(v)) = w^p(j, \varphi^1(v))$ and $w^p(i, \varphi^2(v)) = w^p(j, \varphi^2(v))$ for every $i, j \in N$.

Since the sum of the weighted excesses of all players is constant, (see Theorem 2.4), it holds that

$$w^p(i, \varphi^1(v)) = \frac{1}{n}\left(\sum_{k \in N} a_k^p(v) - (\alpha + n\beta)v(N)\right) = w^p(i, \varphi^2(v)).$$

Moreover,

$$w^p(i, \varphi^1(v)) = \sum_{\substack{S \subseteq N \\ S \ni i}} p_s(v(S) - \varphi^1(S))$$

and

$$w^p(i, \varphi^2(v)) = \sum_{\substack{S \subseteq N \\ S \ni i}} p_s(v(S) - \varphi^2(S))$$

imply $\sum_{\substack{S \subseteq N \\ S \ni i}} p_s \varphi^1(S) = \sum_{\substack{S \subseteq N \\ S \ni i}} p_s \varphi^2(S)$. Since $p_s \geq 0$ for all $1 \leq s \leq n$, and $p_s > 0$ for at least one $1 \leq s < n$, and the equality should hold for every game, it must be $\varphi_i^1(v) = \varphi_i^2(v)$ for every $i \in N$. □

2.5 The p-LS value as center of the weighted super core

In this section, we consider balanced games, i.e. games $v \in G^N$ with a nonempty core. We denote by G_B^N the class of balanced games on player set N. Let $v \in G_B^N$ and $x \in \mathscr{C}(N,v)$. It is obvious that $e(S,x) \leq 0$ for every $S \subseteq N$. Consequently,

$$w^p(i,x) = \sum_{\substack{S \subseteq N \\ S \ni i}} p_s e(S,x) \leq 0,\, i \in N,$$

which allows us to define lower bounds $lo_i^p(v)$, $i \in N$, for core elements. From (2.13) in the proof of Proposition 2.5, it follows that

$$x_i \geq \frac{a_i^p(v) - \beta v(N)}{\alpha} = lo_i^p(v).$$

Besides, summing over all core constraints with $i \notin S$, it holds that

$$\sum_{S \subseteq N \setminus \{i\}} p_s x(S) \geq \sum_{S \subseteq N \setminus \{i\}} p_s v(S). \tag{2.18}$$

Since

$$\begin{aligned} \sum_{S \subseteq N \setminus \{i\}} p_s x(S) &= \sum_{j \in N \setminus \{i\}} \sum_{\substack{S \subseteq N \setminus \{i\} \\ S \ni j}} p_s x_j = \sum_{j \in N \setminus \{i\}} \left(\sum_{s=1}^{n-1} p_s \sum_{\substack{S \subseteq N \setminus \{i\} \\ S \ni j,\ |S|=s}} x_j \right) \\ &= \sum_{j \in N \setminus \{i\}} \sum_{s=1}^{n-1} \binom{n-2}{s-1} p_s x_j = \sum_{s=1}^{n-1} \binom{n-2}{s-1} p_s \sum_{j \in N \setminus \{i\}} x_j \\ &= \alpha \sum_{j \in N \setminus \{i\}} x_j = \alpha(v(N) - x_i), \end{aligned}$$

by (2.18), we obtain upper bounds $up_i^p(v)$, $i \in N$, given by

$$x_i \leq \frac{\alpha v(N) - \sum_{S \subseteq N \setminus \{i\}} p_s v(S)}{\alpha} = up_i^p(v).$$

This inspires us to define the following set valued solution that contains the core.

Definition 2.9. For every game $v \in G^N$ and every weight system p, the p-weighted super core of a game is given by

$$\mathscr{S}\mathscr{C}^p(N,v) = \{x \in \mathbb{R}^n \mid lo_i^p(v) \leq x_i \leq up_i^p(v)\} \ \forall i \in N$$

Observe that these bounds of the weighted super core have the following properties:

(i) The midpoint of each of these bounds of the p-weighted super core for $v \in G^N$ is[2]

$$\frac{lo_i^p(v) + up_i^p(v)}{2} = \frac{\sum_{S \subseteq N \setminus \{i\}} (p_{s+1} v(S \cup \{i\}) - p_s v(S)) + (\alpha - \beta) v(N)}{2\alpha}$$

for every $i \in N$.

(ii) The difference between these bounds of the p-weighted super core of $v \in G^N$ is the same for every player, and is given by[3]

$$lo_i^p(v) - up_i^p(v) = \frac{\sum_{S \subseteq N} p_s v(S) - (\alpha + \beta) v(N)}{\alpha} \text{ for every } i \in N.$$

For the symmetric weight system $p_s = 1$, that is, $\alpha = \beta = 2^{n-2}$, these bounds coincide with those in Vanam and Hemachandra (2013, [80]). In this case, the midpoint of each of these bounds gives rise to the Banzhaf value defined by Banzhaf (1965, [12]).

It is easily seen that the core is contained in the weighted super core of a game v. It turns out that, for every weight system, the vector that

[2]This follows from substituting $a_i^p(v)$ in $lo_i^p(v) + up_i^p(v) = \frac{a_i^p(v) - \beta v(N)}{\alpha} + \frac{\alpha v(N) - \sum_{S \subseteq N \setminus \{i\}} p_s v(S)}{\alpha} = \frac{\sum_{S \subseteq N \setminus \{i\}} p_s v(S) + (\alpha - \beta) v(N) - \sum_{S \subseteq N, S \ni i} p_s v(S)}{\alpha} = \frac{\sum_{S \subseteq N \setminus \{i\}} (p_{s+1} v(S \cup \{i\}) - p_s v(S)) + (\alpha - \beta) v(N)}{\alpha}$.

[3]This follows from substituting $a_i^p(v)$ in $lo_i^p(v) - up_i^p(v) = \frac{a_i^p(v) - \beta v(N)}{\alpha} - \frac{\alpha v(N) - \sum_{S \subseteq N \setminus \{i\}} p_s v(S)}{\alpha} = \frac{\sum_{S \subseteq N \setminus \{i\}} p_s v(S) - (\alpha + \beta) v(N) + \sum_{S \subseteq N, S \ni i} p_s v(S)}{\alpha} = \frac{\sum_{S \subseteq N} p_s v(S) - (\alpha + \beta) v(N)}{\alpha}$.

equalizes the difference between the realized payoff and the lower bound payoff over all players is the payoff vector assigned by the corresponding least square value. In this sense, the p-LS value can be seen as the center of the weighted super core.

Theorem 2.10. *Let $v \in G^N$ and p a weight system. If $x \in \mathbb{R}^n$ with $x_i - lo_i^p(v) = x_j - lo_j^p(v)$ for each $i, j \in N$, then $x = LS^p(N, v)$.*

Proof. Let $i \in N$. Obviously, for $j \in N \setminus \{i\}$, $x_i - lo_i^p(v) = x_j - lo_j^p(v)$ implies

$$x_i - x_j = lo_i^p(v) - lo_j(v) = \frac{a_i^p(v) - \beta v(N)}{\alpha} - \frac{a_j^p(v) - \beta v(N)}{\alpha} = \frac{a_i^p(v) - a_j^p(v)}{\alpha}.$$

Adding over all $j \in N$ and by efficiency of x, $nx_i - v(N) = \frac{na_i^p(v) - \sum_{j \in N} a_j^p(v)}{\alpha}$. Then,

$$x_i = \frac{v(N)}{n} + \frac{1}{n\alpha}\left(na_i^p(v) - \sum_{j \in N} a_j^p(v)\right).$$

From (2.5), we conclude that $x = \text{LS}^p(N, v)$. □

2.6 A weighted Shapley-like value

Inspired by the midpoint of the two bounds of the payoff in the p-*weighted super core*, we can define the p-*weighted Shapley value* for a weight system p as

$$\mathscr{SH}_i^p(N, v) = \sum_{S \subseteq N \setminus \{i\}} \frac{s!(n-s-1)!}{n!}(p_{s+1}v(S \cup \{i\}) - p_s v(S)),\ i \in N.$$

Here, $\frac{s!(n-s-1)!}{n!}$ is the probability that player i joins coalition S and that the *weighted marginal contribution* $p_{s+1}v(S \cup \{i\}) - p_s v(S)$ is paid to player i for joining coalition S. Hence, the p-weighted Shapley value is the expected weighted contribution of player i in the game $v \in G^N$. Unfortunately, the p-weighted Shapley value need not be efficient. Next, we characterize the

p-weighted Shapley value for a symmetric weight system. For this, we need to introduce new properties.

Let p be a symmetric weight system. Player $i \in N$ is called a *p-weighted dummy* in the game $v \in G^N$ if

$$p_{s+1}v(S \cup \{i\}) - p_s v(S) = p_1 v(i).$$

- *p-weighted dummy player property*: For every $v \in G^N$ and every p-weighted dummy player $i \in N$, it holds that $\varphi_i(v) = p_1 v(i)$.
- *p-weighted efficiency*: For every game $v \in G^N$, $\sum_{j \in N} \varphi_j(N, v) = p_n v(N)$.

The following result follows straightforward from the definition of the weighted Shapley value. The proof is therefore omitted.

Proposition 2.11. $\sum_{j \in N} \mathscr{S}\mathscr{H}^p_j(N, v) = p_n v(N)$ *for every* $v \in G^N$ *and every symmetric weight system* $p = (p_s)_{1 \leq s \leq n}$.

Similar as the axiomatization of the Shapley value by efficiency, symmetry, the dummy player property and additivity, we can prove the following.

Theorem 2.12. *Let p be a symmetric weight system. The p-weighted Shapley value $\mathscr{S}\mathscr{H}^p : G^N \to \mathbb{R}^n$ is the unique value on G^N with the following four properties: symmetry, p-weighted dummy player property, additivity and p-weighted efficiency.*

The proof follows the same lines as the original proof in Shapley (1953, [72]). The only difference is that, instead of using unanimity games as a basis for the class of games with player set N, we need to use p-weighted unanimity games, u^p_T, $\emptyset \neq T \subseteq N$, defined as $u^p_T(S) = \frac{1}{p_s}$ if $T \subseteq S$ and $u^p_T(S) = 0$ otherwise. The proof is, therefore, in Appendix.

Efficiency is a crucial requirement if one is looking for a solution that can be accepted by all the players. This leads us to consider an "efficient normalization" of the p-weighted Shapley value. One can obtain an efficient normalization by adding the same constant to all its components as in Hammer and Holzman (1987, [36]). Consequently, the *additive normalized weighted Shapley value* $\widehat{\mathscr{SH}}_i^p(N,v)$ in G^N is given by

$$\widehat{\mathscr{SH}}_i^p(N,v) = \mathscr{SH}_i^p(N,v) + \frac{1}{n}\left(v(N) - \sum_{j\in N}\mathscr{SH}_j^p(N,v)\right)$$

for every $i \in N$. Actually, this normalized p-weighted Shapley value is an ESL-value proposed by Ruiz et al. (1998, [69]).

Another possible normalization is to multiply all components by the same constant as in Dubey and Shapley (1979, [32]), and obtain the multiplicative normalized p-weighted Shapley value

$$\overline{\mathscr{SH}}_i^p(N,v) = \frac{\mathscr{SH}_i^p(N,v)}{\sum_{i\in N}\mathscr{SH}_i^p(N,v)}v(N) = \frac{1}{p_n}\mathscr{SH}_i^p(N,v),\ i \in N,$$

the second equality is only possible when $p_n > 0$.

We illustrate these solutions with two well-known examples. First, we study a bankruptcy game as introduced in O'Neill (1982, [63]).

Example 2.1. A bankruptcy problem is described by a tuple (N, E, d) where a set N of agents have rightful demands, given by $d \in \mathbb{R}_+^N$, over the scarce estate E, that is, $E \leq d(N)$. The associated bankruptcy game, $(N, v_{E,d})$, is defined, for $S \subseteq N$, as $v_{E,d}(S) = \max\{0, E - d(N \setminus S)\}$.

Let (N, E, d) be a bankruptcy problem with $N = \{1, 2, 3\}$, estate $E = 80$, and three claims $d_1 = 30$, $d_2 = 40$, $d_3 = 60$, and consider the associated bankruptcy game $(N, v_{E,d})$ given by $v_{E,d}(\{1\}) = v_{E,d}(\{2\}) = 0$, $v_{E,d}(\{3\}) = 10$, $v_{E,d}(\{1, 2\}) = 20$, $v_{E,d}(\{1, 3\}) = 40$, $v_{E,d}(\{2, 3\}) = 50$, $v_{E,d}(\{1, 2, 3\}) = 80$. Let $p = (p_1, p_2, p_3)$ be a symmetric weight system. Then,

$$\widehat{\mathscr{SH}}_i^p(N,v) = \mathrm{LS}^p(N,v) = \left(\frac{80}{3} - \frac{10}{3}\frac{p_1+4p_2}{p_1+p_2}, \frac{70}{3}, \frac{80}{3} + \frac{10}{3}\frac{2p_1+5p_2}{p_1+p_2}\right)$$

and

$$\overline{\mathscr{SH}}_i^p(N,v) = \left(\frac{80}{3} - \frac{5}{3}\frac{p_1+4p_2}{p_3}, \frac{80}{3} - \frac{5}{3}\frac{p_1+p_2}{p_3}, \frac{80}{3} + \frac{5}{3}\frac{2p_1+5p_2}{p_3}\right).$$

Next, we compare the outcomes of these solutions with the allocations proposed by some well-known bankruptcy rules: the constrained equal awards (CEA), constrained equal losses (CEL), and Talmud (Tal) rules (cf. Auman and Maschler (1985, [11]), for Talmud rule see contested garment consistent rule); the random arrival (RA) rule (cf. O'Neill (1982, [63]) as recursive completion); and the adjusted proportional (AP) rule (cf. Curiel et al. (1987, [21]).[4]

$$\text{CEA}(N,E,d)=\left(\frac{80}{3},\frac{80}{3},\frac{80}{3}\right),\ \text{CEL}(N,E,d)=\left(\frac{40}{3},\frac{70}{3},\frac{130}{3}\right),$$

$$\text{Tal}(N,E,d)=\left(15,\frac{45}{2},\frac{85}{2}\right),\ \text{RA}(N,E,d)=\left(\frac{55}{3},\frac{70}{3},\frac{115}{3}\right),$$

$$\text{and AP}(N,E,d)=\left(\frac{35}{2},\frac{70}{3},\frac{235}{6}\right).$$

Moreover, for the 3-person bankruptcy game with weight system p, it holds that

$$\text{LS}^p(N,v)=\begin{cases}\text{CEL}(N,E,d) & \text{if } p_1=0,\\ \text{RA}(N,E,d) & \text{if } p_1=p_2,\ p_1\neq 0,\\ \text{AP}(N,E,d) & \text{if } p_2=\frac{7}{5}p_1,\ p_1\neq 0,\end{cases}$$

and

$$\overline{\mathscr{SH}}_i^p(N,v)=\begin{cases}\text{CEA}(N,E,d) & \text{if } p_1=p_2=0,\\ \text{CEL}(N,E,d) & \text{if } p_3=\frac{1}{2}p_2,\ p_2\neq 0,\ p_1=0,\\ \text{Tal}(N,E,d) & \text{if } p_3=p_1=\frac{2}{3}p_2,\ p_2\neq 0,\\ \text{RA}(N,E,d) & \text{if } p_3=p_1=p_2,\ p_2\neq 0,\\ \text{AP}(N,E,d) & \text{if } p_3=\frac{6}{5}p_1,\ p_2=\frac{7}{5}p_1,\ p_1\neq 0.\end{cases}$$

[4]Let (N,E,d) be a bankruptcy problem. The constrained equal awards rule is defined by $\text{CEA}_i(N,E,d)=\min\{\alpha,d_i\}$ for each $i\in N$ with α such that $\sum_{i\in N}\text{CEA}_i(N,E,d)=E$; the constrained equal losses rule is defined by $\text{CEL}_i(N,E,d)=\max\{0,d_i-\beta\}$ for each $i\in N$ with β such that $\sum_{i\in N}\text{CEL}_i(N,E,d)=E$; the Talmud rule is defined by $\text{Tal}(N,E,d)=\text{CEA}(N,E,\frac{1}{2}d)$ if $\sum_{i\in N}\frac{d_i}{2}\geq E$ and $\text{Tal}(N,E,d)=\frac{1}{2}d+\text{CEL}(N,E-\frac{1}{2}d(N),\frac{1}{2}d)$ otherwise; the random arrival rule is defined by $\text{RA}_i(N,E,d)=\sum_{S\subseteq N\setminus\{i\}}\frac{s!(n-s-1)!}{n!}\min\{d_i,\max\{0,E-d(S)\}\}$ for each $i\in N$.

Next, we consider an airport game as introduced in Littlechild and Owen (1973, [46]).

Example 2.2. In an airport problem, a group of aircrafts need different landing lengths which have different associated costs. Smaller aircrafts can use the same runway as bigger aircrafts, but not the other way around. Let $C_1, \ldots, C_n$ represent the costs associated to the different types of aircrafts, with $C_1 \leq C_2 \leq \ldots \leq C_n$. Littlechild and Owen (1973, [46]) modelled the corresponding allocation cost problem using an associated cost game defined by $c(S) = \max\{C_i | i \in S\}$ for each $S \subseteq N$.

Let (N, C) be an airport problem with $N = \{1, 2, 3\}$, three different needs on runways, and three costs C_1, C_2, C_3, $C_1 \leq C_2 \leq C_3$, and consider the associated airport game (N, c) given by $c(\{1\}) = C_1$, $c(\{2\}) = c(\{1,2\}) = C_2$, $c(\{3\}) = c(\{1,3\}) = c(\{2,3\}) = c(\{1,2,3\}) = C_3$. Let $p = (p_1, p_2, p_3)$ be a symmetric weight system. Then, the components of the p-LS value are

$$x_1 = \frac{C_3}{3} + \frac{1}{3(p_1 + p_2)} \left[p_1(2C_1 - C_3 - C_2) + p_2(C_2 - C_3)\right],$$

$$x_2 = \frac{C_3}{3} + \frac{1}{3(p_1 + p_2)} \left[p_1(2C_2 - C_3 - C_1) + p_2(C_2 - C_3)\right],$$

$$x_3 = \frac{C_3}{3} + \frac{1}{3(p_1 + p_2)} \left[p_1(2C_3 - C_2 - C_1) + 2p_2(C_3 - C_2)\right],$$

and the components of the p-weighted Shapley value are

$$x_1 = \frac{C_3}{3} + \frac{p_1(2C_1 - C_2 - C_3) + p_2(C_2 - C_3)}{6p_3},$$

$$x_2 = \frac{C_3}{3} + \frac{p_1(2C_2 - C_1 - C_3) + p_2(C_2 - C_3)}{6p_3},$$

$$x_3 = \frac{C_3}{3} + \frac{p_1(2C_3 - C_1 - C_2) + p_2(2C_3 - 2C_2)}{6p_3}.$$

Next, we compare the outcomes of these solutions with the allocation proposed by some well-known rules for airport problems: the sequential equal contributions (SEC) rule (cf. Littlechild and Owen (1973, [46]), for the sequential equal contributions rule see the Shapley value of the airport problem), the slack maximizer (SM) rule (cf. Littlechild (1974, [47]), for slack maximizer rule see the nucleolus of the airport problem proposed by Albizuri et al. (2018, [2])), and the constrained equal benefits (CEB) rule[5] (cf. Potters and Sudhölter (1999, [67])).

$$\mathrm{SEC}(N,C)=\left(\frac{C_1}{3},\frac{C_1}{3}+\frac{C_2-C_1}{2},\frac{C_1}{3}+\frac{C_2-C_1}{2}+C_3-C_2\right),$$

$$\mathrm{SM}(N,C)=\left(\min\{\frac{C_1}{2},\frac{C_2}{3}\},\frac{C_2}{2}-\min\{\frac{C_1}{4},\frac{C_2}{6}\},C_3-\frac{C_2}{2}-\min\{\frac{C_1}{4},\frac{C_2}{6}\}\right),$$

and

$$\mathrm{CEB}(N,C)=\left(\frac{2C_1-C_2}{3},\frac{2C_2-C_1}{3},\frac{3C_3-C_2-C_1}{3}\right).$$

Furthermore, for the 3-person airport game with weight system p, it holds that

$$\mathrm{LS}^p(N,v)=\begin{cases}\mathrm{SEC}(N,C) & \text{if } p_1=p_2\neq 0,\\ \mathrm{CEB}(N,C) & \text{if } p_2=0,\ p_1\neq 0;\end{cases}$$

$$\mathrm{LS}^p(N,v)=\begin{cases}\mathrm{SM}(N,C) & \text{if } p_1=\frac{2C_2-3C_1}{2C_2-C_1}p_2\neq 0,\ \text{and } C_1\leq\frac{2}{3}C_2,\\ \mathrm{SM}(N,C) & \text{if } p_1p_2\neq 0,\ \text{and } C_1=C_2;\end{cases}$$

and

$$\overline{\mathscr{SH}}^p(N,v)=\begin{cases}\mathrm{SEC}(N,C) & \text{if } p_1=p_2=p_3\neq 0,\\ \mathrm{CEB}(N,C) & \text{if } p_1=2p_3\neq 0,\ p_2=0;\end{cases}$$

$$\overline{\mathscr{SH}}^p(N,v)=\begin{cases}\mathrm{SM}(N,C) & \text{if } p_1=\frac{2C_2-3C_1}{2C_2-2C_1}p_3,\ p_2=\frac{2C_2-C_1}{2C_2-2C_1}p_3,\ p_3\neq 0,\\ & \qquad\text{and } C_1\leq\frac{2}{3}C_2,\\ \mathrm{SM}(N,C) & \text{if } p_1p_2\neq 0,\ \text{and } C_1=C_2.\end{cases}$$

[5]Let (N,C) be an airport problem. The sequential equal contributions rule is defined by $\mathrm{SEC}_i(N,C)=\sum_{k=1}^{i}\frac{C_k-C_{k-1}}{n+1-k}$, for each $i\in N$; the slack maximizer rule with $n\geq 2$ is given inductively by $\mathrm{SM}_i(N,C)=\min_{l=i}^{n-1}\frac{C_l-\sum_{k=0}^{i-1}\mathrm{SM}_k(N,C)}{l-i+2}$, for each $i=1,\cdots,n-1$ and $\mathrm{SM}_n(N,C)=\mathrm{SM}_{n-1}(N,C)+C_n-C_{n-1}$ beginning with $\mathrm{SM}_0(N,C)=C_0=0$; the constrained equal benefits rule is defined by $\mathrm{CEB}_i(N,C)=\max\{c_i-\beta,0\}$ for each $i\in N$ with $\beta\in\mathbb{R}_+$ such that $\sum_{i\in N}\mathrm{CEB}_i(N,C)=C_n$.

2.7 Conclusions

In this chapter, we give three characterizations of the least square values for cooperative TU-games: (i) by lexicographically minimizing the individual weighted excesses of players, (ii) by minimizing the variance of the players' weighted excesses on the preimputation set, and (iii) by showing that they are the center of the weighted super core defined by certain lower and upper bounds for the core payoff vectors. Based on these lower and upper bounds, we present a new solution similar to the Shapley value for cooperative TU-games. Finally, we illustrate these solutions in two well-known examples that are studied in the literature: bankruptcy games and airport games.

These results not only give more insight in the least square values, specifically regarding the effect of weights assigned to individuals instead of coalitional weights, but also provide inspiration for new solutions such as the p-weighted super core and the p-weighted Shapley value. Some ideas for further investigation are the following. First, it would be interesting to consider other solutions by dividing the weighted marginal contribution $(p_{s+1}v(S \cup i) - p_s v(S))$ according to other different ratios, such as dividing them equally. Second, since the p-weighted Shapley value is not efficient, it is relevant to characterize the (additive or multiplicative) normalized p-weighted Shapley value. Third, it is of interest to study the efficient point on the segment between the lower and upper bounds, similar as the τ-value proposed by Tijs (1981, [79]), which is defined as the efficient point between the minimal right vector and the utopia vector.

Appendix: the proof of Theorem 2.12

It is obvious that the p-weighted Shapley value satisfies symmetry, p-weighted dummy player property, additivity and p-weighted efficiency. It remains to show that there exists only one solution satisfying these properties. Let $\varphi : G^N \to \mathbb{R}^n$ be a value with the four mentioned properties. For every $\emptyset \neq T \subseteq N$ and $p = (p_s)_{1 \leq s \leq n}$, consider the weighted unanimity n-person game u_T^p as

$$u_T^p(S) = \begin{cases} \frac{1}{p_s}, & \text{if } T \subseteq S, \\ 0, & \text{otherwise.} \end{cases} \tag{2.19}$$

Let $\emptyset \neq T \subseteq N$ and $\alpha \in \mathbb{R}$. Every player $i \in N \setminus T$ is a weighted dummy player in the n-person game αu_T^p, and hence, by the weighted dummy player property for φ, it is true that

$$\varphi_i(\alpha u_T^p) = \alpha p_1 u_T^p(i) = 0 \text{ for each } i \in N \setminus T.$$

Let $j, k \in T$. There is a permutation $\sigma : N \to N$ satisfying $\sigma(j) = k$ and $\sigma(i) \in T$ for all $i \in T$. Then $\sigma(\alpha u_T^p) = \alpha u_T^p$, and hence, by the symmetry property for φ, we get

$$\varphi_k(\alpha u_T^p) = \varphi_{\sigma(j)}(\sigma(\alpha u_T^p)) = \varphi_j(\alpha u_T^p), \text{ for each } j, k \in T.$$

Moreover, $\sum\limits_{r \in N} \varphi_r(\alpha u_N^p) = \alpha p_n u_T^p(N) = \alpha$ by the weighted efficiency property for φ. Thus, it yields that

$$\varphi_i(\alpha u_T^p) = \begin{cases} \frac{\alpha}{|T|}, & \text{if } i \in T, \\ 0, & \text{if } i \in N \setminus T. \end{cases}$$

It is well known that the set $\{u_T \in G^N \mid \emptyset \neq T \subseteq N\}$ of all unanimity games forms a basis of the linear space G^N, where

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S, \\ 0, & \text{otherwise.} \end{cases}$$

For every $v \in G^N$, it holds that

$$v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \triangle_v(T) u_T, \tag{2.20}$$

where $\triangle_v(T) = \sum\limits_{S \subseteq T} (-1)^{(|T|-|S|)} v(S)$. Taking $\triangle_v^p(T) = \sum\limits_{S \subseteq T} (-1)^{(|T|-|S|)} p_s v(S)$, (2.20) is equivalent to

$$v = \sum_{\substack{T \subseteq N \\ T \neq \emptyset}} \triangle_v^p(T) u_T^p, \text{ for every } v \in G^N.$$

Together with the additivity property, we have

$$\varphi_i(v) = \sum_{\substack{T \subseteq N \\ T \ni i}} \frac{\triangle_v^p(T)}{|T|}.$$

$\square$

Chapter 3

The prekernel of cooperative games with α-excess

3.1 Introduction

The *excess* of a coalition at a given payoff vector in transferable utility (TU) games represents the gain or loss of the coalition if its members withdraw from the game in order to form their own coalition. Usually, the excess of a coalition can be viewed as the dissatisfaction of the coalition at the proposed payoff vector. The classical excess is defined by the difference between the worth of a coalition and the payoff assigned to the coalition members. The most popular solutions such as the core [35], the Shapley value [72], the nucleolus [71], the prenucleolus [77] and the (pre)kernel [51] can be characterized on the basis of this classical excess. Especially, Peleg and Sudhölter [66] provided an axiomatization of the prekernel, which avoids any reference to interpersonal comparison of utilities. They verified that there is a unique solution on the set of all TU-games that satisfies *nonemptiness, Pareto optimality, covariance under strategic equivalence, the equal treatment property, a reduced game property*, and *the converse reduced game property*. In view of the stability of a preimputation, which means that no player has incentives to move from the preimputation, Calvo and

Gutiérrez (1996, [19]) first defined the *strong stability property*. The least core of a TU-game is characterized using this property. They also proposed the *balanced surplus property* similar to the balanced contributions property of Myerson (1980, [58]). By means of these two properties, they gave a new characterization of the prekernel of a TU-game.

Thereafter, Lemaire (1991, [45]) presented the *relative excess* to measure the dissatisfaction of every coalition as the quotient of the usual excess and the coalitional value, and defined the proportional nucleolus. In addition, Yanovskaya (2002, [85]) defined proportional solutions for the class of positive TU-games with all nonempty coalitional values strictly positive, depending only on the *proportional excess*, which is defined as the quotient of the coalitional value and the coalitional payoff. Actually, the relative excess is ordinally equivalent to the proportional excess. Successively, the proportional prenucleolus and nucleolus were characterized by Naumova (2014, [61]).

Which definition of excess is most appropriate, depends on the application one has in mind. To avoid ignoring some player's benefit for the general case, our aim in this chapter is to define a more general excess (called *α-excess*) by considering affine combinations of the classical excess and the proportional excess for the class of positive TU-games. Based on this α-excess, we modify solutions like the core, ε-core, least core, (pre)nucleolus and prekernel for positive TU-games. In this way, corresponding α-solutions for positive TU-games are obtained. First, we show that the core and the α-core coincide for positive TU-games. However, we will see that this is not the case for the modifications of the prekernel, the least core, and the prenucleolus. Second, we prove that the α-prenucleolus is always contained in the α-prekernel. Third, we characterize the α-prekernel by strong stability and an α-balanced surplus property.

This chapter, which is based on Zhang et al. (2021, [90]), is organized as follows. In Section 3.2, we recall some related preliminaries about cooperative game theory. Section 3.3 introduces the α-excess of a coalition, defines modifications of solutions using this modified excess, and characterizes the α-least core and α-prekernel by strong stability and α-balanced surplus properties. In Section 3.4, we define the α-prenucleolus and α-nucleolus. Also, we verify that the α-prenucleolus is contained in the α-prekernel for all α. Section 3.5 concludes with a brief summary.

3.2 Definitions and Notations

In this section, we add some definitions and notations that are specific for this chapter, but are not mentioned in Chapter 1.

The class G^{N_+} of positive TU-games where all worths of nonempty coalitions are positive is given by

$$G^{N_+} = \{(N, v) \mid v(S) > 0 \; \forall S \subseteq N, S \neq \emptyset\}.$$

Positive versions of solutions are defined in such a way that they only consider positive payoff vectors. Specifically, the positive *preimputation set* of a game $v \in G^N$ is given by

$$\mathscr{I}^*_{++}(N, v) = \{x \in \mathbb{R}^n_{++} \mid x(N) = v(N)\}.$$

The positive *imputation set* of a game $v \in G^N$ is given by

$$\mathscr{I}_{++}(N, v) = \{x \in \mathscr{I}^*_{++}(N, v) \mid x_i \geq v(\{i\}) \; \forall i \in N\}.$$

The positive *core* of a game $v \in G^N$ is given by

$$\mathscr{C}_{++}(N, v) = \{x \in \mathscr{I}^*_{++}(N, v) \mid x(S) \geq v(S) \; \forall S \subseteq N\}.$$

Yanovskaya (2002, [85]) considers a proportional excess function, where the dissatisfaction of a coalition is measured as the ratio between the worth of a coalition and the assigned payoff. Formally, for $v \in G^{N_+}$, $x \in \mathbb{R}^n_{++}$ and $S \subseteq N$, the *proportional excess* of S at x is

$$\overline{e}(S, x) = \frac{v(S)}{x(S)}.$$

Whereas the classical excess (see (1.2), page 9) of a coalition that exactly gets its worth is equal to 0, for such a coalition the proportional excess is equal to 1.

3.3 The α-prekernel in TU-games

We begin with an example that illustrates the difference between the classical and proportional excesses introduced before.

Example 3.1. Let $N = \{1, 2, 3\}$ be three companies. Assume that these three companies lost money in cooperation. Let v be defined by $v(\{1, 2, 3\}) = 1194$, $v(\{1\}) = 10$, $v(\{2\}) = v(\{3\}) = 1000$, and $v(S) = \rho$ otherwise, ρ being a sufficiently small positive number. For the given payoff vector $(2, 992, 200)$, it holds that $e(\{1\}, x) = e(\{2\}, x) = 8$, $e(\{3\}, x) = 800$, however, $\bar{e}(\{1\}, x) = \bar{e}(\{3\}, x) = 5$, $\bar{e}(\{2\}, x) = \frac{125}{124}$.

Considering Example 3.1, now comes the question, which excess is better to measure the dissatisfaction of the companies at the payoff vector $(2, 992, 200)$? The rich company can "tolerate" a moderate or small loss more than the poor company. However, it does not "tolerate" a very large loss either. From our perspective, it is not obvious that one should consider either the classical excess or the proportional excess. In such cases, an affine (or convex) combination of these two excesses might be more reasonable. Consequently, also variations of solutions, such as the prekernel and the prenucleolus, based on an affine or convex combination of these two excesses, might be reasonable solution concepts.

Definition 3.1. Given $\alpha \in \mathbb{R}$, a game $v \in G^{N_+}$, a positive payoff vector $x \in \mathbb{R}^n_{++}$ and a coalition $S \subseteq N$, $S \neq \emptyset$, the α-excess of coalition S with respect to x is given by

$$e^\alpha_v(S, x) = \alpha \frac{v(S)}{x(S)} + (1 - \alpha)(v(S) - x(S)). \tag{3.1}$$

For $S = \emptyset$, we define $e^\alpha_v(\emptyset, x) = 0$ for all $v \in G^{N_+}$ and $x \in \mathbb{R}^n_{++}$.

If there is no confusion about the game $v \in G^{N_+}$, we will shortly write $e^\alpha(S, x)$ instead of $e^\alpha_v(S, x)$. Specifically, when $\alpha \in [0, 1]$, we speak about a convex combination of the classical and proportional excess. Observe that we obtain the classical excess as a special case of α-excess by taking $\alpha = 0$, and the proportional excess as special case when taking $\alpha = 1$. Similar as the classical and proportional excess, the α-excess represents the gain (or loss, if it is less than 1) to the coalition S if its members depart from an agreement that yields x in order to form their own coalition, but allow a trade-off between the classical and proportional excess.

In view of the concept of α-excess, the definition of the core of a positive game $v \in G^{N_+}$ could be modified by considering those imputations which α-excess is at most equal to α, i.e. one could consider

$$\mathscr{C}^\alpha(N,v) = \{x \in \mathscr{I}^*_{++}(N,v) \mid e^\alpha(S,x) \leq \alpha \;\forall S \subseteq N\}.$$

However, it turns out that for every $\alpha \in [0,1]$ this coincides with the classical core ($\alpha = 0$) as long as we consider only positive payoff vectors.

Proposition 3.2. *For every $\alpha \in [0,1]$ and $v \in G^{N_+}$, we have $\mathscr{C}^\alpha(N,v) = \mathscr{C}(N,v)$.*

Proof. Notice that for $v \in G^{N_+}$, $\mathscr{C}(N,v) = \mathscr{C}_{++}(N,v)$ since $v(\{i\}) > 0$ $\forall i \in N$. For every $\alpha \in [0,1]$, $v \in G^{N_+}$, and $x \in \mathbb{R}^n_{++}$, we have

$$\begin{aligned}
e^\alpha(S,x) \leq \alpha \quad &\Leftrightarrow \quad \alpha\frac{v(S)}{x(S)} + (1-\alpha)(v(S)-x(S)) \leq \alpha \\
&\Leftrightarrow \quad \frac{\alpha v(S) + (1-\alpha)(v(S)-x(S))x(S)}{x(S)} \leq \alpha \\
&\Leftrightarrow \quad \alpha v(S) + (1-\alpha)(v(S)-x(S))x(S) \leq \alpha x(S) \\
&\Leftrightarrow \quad \alpha(v(S)-x(S)) + (1-\alpha)(v(S)-x(S))x(S) \leq 0 \\
&\Leftrightarrow \quad (v(S)-x(S))(\alpha + (1-\alpha)x(S)) \leq 0 \\
&\Leftrightarrow \quad v(S)-x(S) \leq 0 \Leftrightarrow e(S,x) \leq 0
\end{aligned}$$

where the last but one equivalence follows since $\alpha + (1-\alpha)x(S) > 0$ for all $x \in \mathbb{R}^n_{++}$. □

From this proposition, we can conclude that considering different α-excesses from our class to measure the dissatisfaction of coalitions, has no effect on the definition of the core. However, we will see that it does affect the definition of the prekernel, the least core, and the prenucleolus.

First, modifying the ε-core of a game $v \in G^{N_+}$, we obtain the *$\alpha\varepsilon$-core* given by

$$\mathscr{C}_\varepsilon^\alpha(N,v) = \{x \in \mathscr{I}_{++}^*(N,v) \mid e^\alpha(S,x) - \varepsilon \leq \alpha \;\forall S \subseteq N, S \neq \emptyset\},$$

and the *α-least core* of a game $v \in G^{N_+}$ being $\mathscr{C}_\lambda^\alpha(N,v)$, where $\lambda = \lambda^{v,\alpha} = \min\{\varepsilon \in \mathbb{R} \mid \mathscr{C}_\varepsilon^\alpha(N,v) \neq \emptyset\}$. We denote the α-least core of game v by $\mathscr{L}\mathscr{C}^\alpha(N,v)$. If there does not exist a minimal $\varepsilon \in \mathbb{R}$ such that $\mathscr{C}_\varepsilon^\alpha(N,v) \neq \emptyset$, then $\mathscr{L}\mathscr{C}^\alpha(N,v) = \emptyset$.

Example 3.2. Consider the 3-person game v defined as $v(\{1,2,3\}) = 1$, $v(\{1,2\}) = v(\{1,3\}) = \frac{1}{2}$, and $v(S) = \rho$ otherwise, ρ being a sufficiently small positive number. The symbol $\underline{\text{conv}}$ indicates the convex hull excluding the boundary points with a component of 0. We find out that

$$\begin{aligned}\mathscr{C}_{++}(N,v) &= \left\{x \in \mathbb{R}_{++}^3 \mid x_1+x_2+x_3 = 1, x_2 \leq \frac{1}{2}, x_3 \leq \frac{1}{2}\right\}\\ &= \underline{\text{conv}}\left\{(1,0,0),(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},0,\frac{1}{2}),(0,\frac{1}{2},\frac{1}{2})\right\},\end{aligned}$$

see the area ABCD in Fig.3.1. We show that the $\alpha\varepsilon$-cores are different for the classical and proportional excess. For $\alpha = 0$, we have

$$\begin{aligned}\mathscr{C}_\varepsilon^0(N,v) &= \Big\{x \in \mathbb{R}_{++}^3 \mid x_1+x_2+x_3 = 1, x_1 \geq -\varepsilon, -\varepsilon \leq x_2 \leq \frac{1}{2}+\varepsilon\,,\\ &\qquad -\varepsilon \leq x_3 \leq \frac{1}{2}+\varepsilon\Big\}.\end{aligned}$$

Then, for $\varepsilon \geq 0$, we have

$$\begin{aligned}\mathscr{C}_\varepsilon^0(N,v) &= \underline{\text{conv}}\Big\{(1,0,0),(\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon,0),(\frac{1}{2}-\varepsilon,0,\frac{1}{2}+\varepsilon)\,,\\ &\qquad (0,\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon),(0,\frac{1}{2}+\varepsilon,\frac{1}{2}-\varepsilon)\Big\},\end{aligned}$$

see the area ABCDE in Fig.3.2. If $\varepsilon < 0$,

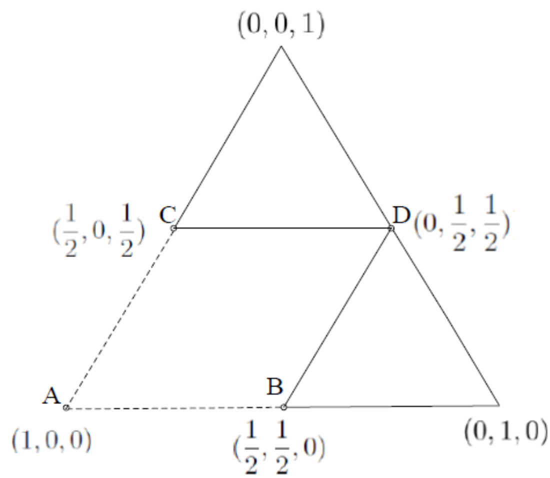

FIGURE 3.1: The positive core $\mathscr{C}_+(N, v)$

$$\begin{aligned}\mathscr{C}^0_\varepsilon(N,v) &= \underline{\text{conv}}\Big\{(1+2\varepsilon,-\varepsilon,-\varepsilon),(\frac{1}{2},\frac{1}{2}+\varepsilon,-\varepsilon),(\frac{1}{2},-\varepsilon,\frac{1}{2}+\varepsilon)\,, \\ &\quad (-2\varepsilon,\frac{1}{2}+\varepsilon,\frac{1}{2}+\varepsilon)\Big\},\end{aligned}$$

see the area ABCD in Fig.3.3. Moreover, $\mathscr{C}^0_\varepsilon(N,v) \neq \emptyset$ iff $\varepsilon \geq -\frac{1}{4}$. Thus, $\mathscr{L}\mathscr{C}^0(N,v) = \{(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\}$.

Next, consider the case that $\alpha = 1$.

$$\begin{aligned}\mathscr{C}^1_\varepsilon(N,v) &= \Big\{x \in \mathbb{R}^3_{++} \mid x_1+x_2+x_3 = 1, x_1 \geq \frac{-\varepsilon}{\varepsilon+1}, x_2 \leq \frac{2\varepsilon+1}{2(\varepsilon+1)}\,, \\ &\quad x_3 \leq \frac{2\varepsilon+1}{2(\varepsilon+1)}\Big\}.\end{aligned}$$

If $\varepsilon \geq 0$,

$$\begin{aligned}\mathscr{C}^1_\varepsilon(N,v) &= \underline{\text{conv}}\Big\{(1,0,0),(\frac{1}{2(\varepsilon+1)},\frac{2\varepsilon+1}{2(\varepsilon+1)},0),(\frac{1}{2(\varepsilon+1)},0,\frac{2\varepsilon+1}{2(\varepsilon+1)})\,, \\ &\quad (0,\frac{1}{2(\varepsilon+1)},\frac{2\varepsilon+1}{2(\varepsilon+1)}),(0,\frac{2\varepsilon+1}{2(\varepsilon+1)},\frac{1}{2(\varepsilon+1)})\Big\}.\end{aligned}$$

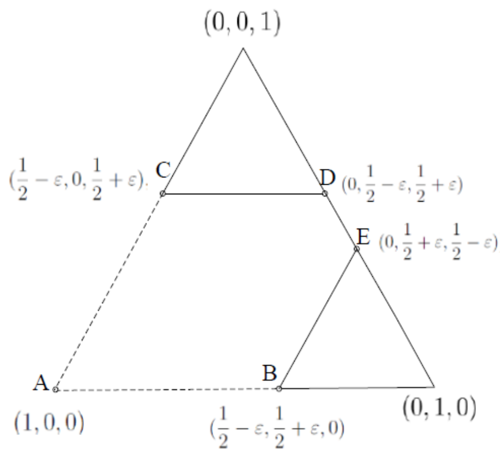

FIGURE 3.2: The 0ε-core, $\mathscr{C}^0_\varepsilon(N,v)$, when $\varepsilon \geq 0$

Now, the shape of $\mathscr{C}^1_\varepsilon(N,v)$ is the same as that of $\mathscr{C}^0_\varepsilon(N,v)$ when $\varepsilon \geq 0$ in Fig.3.2, but it is determined by different extreme points. If $-1 < \varepsilon < 0$,

$$\begin{aligned}\mathscr{C}^1_\varepsilon(N,v) \quad = \quad \underline{\text{conv}}\Big\{&(1,0,0), (\frac{1}{2(\varepsilon+1)}, \frac{2\varepsilon+1}{2(\varepsilon+1)}, 0), (\frac{1}{2(\varepsilon+1)}, 0, \frac{2\varepsilon+1}{2(\varepsilon+1)}),\\ &(\frac{-2\varepsilon}{2(\varepsilon+1)}, \frac{2\varepsilon+1}{2(\varepsilon+1)}, \frac{2\varepsilon+1}{2(\varepsilon+1)})\Big\},\end{aligned}$$

see the area ABCD in Fig.3.4. Hence, $\mathscr{C}^1_\varepsilon(N,v) \neq \mathscr{C}^0_\varepsilon(N,v)$ as long as $\varepsilon \neq 0$ or $\varepsilon \neq -\frac{1}{2}$. Also, $\mathscr{C}^1_\varepsilon(N,v) \neq \emptyset$ iff $\varepsilon > -\frac{1}{2}$. Thus, $\mathscr{L}\mathscr{C}^1(N,v) = \emptyset \neq \mathscr{L}\mathscr{C}^0(N,v)$. □

Example 3.2 indicates that the $\alpha\varepsilon$-core and the α-least core are different for different α. To define the α-prekernel, we first adapt the definition of maximal surplus.

Definition 3.3. Given $\alpha \in [0,1]$, $v \in G^{N_+}$, and $x \in \mathbb{R}^n_{++}$, the maximal α-surplus of player i over another player j at x in the game v is given by

$$s^{v,\alpha}_{ij}(x) = \max_{S\in\Gamma_{ij}} e^\alpha(S,x). \tag{3.2}$$

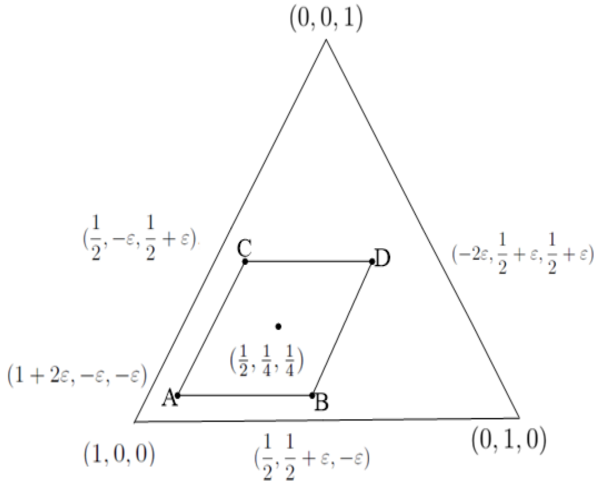

FIGURE 3.3: The 1ε-core, $\mathscr{C}^0_\varepsilon(N, v)$, when $\varepsilon < 0$

Notice that a maximal surplus not less than (or not greater than) 1 of i over j at a payoff vector x can be interpreted as the maximal (or minimal) amount that player i can gain (or lose) without cooperation with j. Consequently, the maximum α-surplus can be regarded as another measure of the power of player i to threaten player j at the preimputation x.

Definition 3.4. Given $\alpha \in [0, 1]$, and $x \in \mathscr{I}^*_{++}(N, v)$, the α-prekernel $\mathscr{PK}^\alpha(N, v)$ of the game $v \in G^{N+}$ is the set of preimputations x given by

$$\mathscr{PK}^\alpha(N, v) = \{x \in \mathscr{I}^*_{++}(N, v) \mid s^{v,\alpha}_{ij}(x) = s^{v,\alpha}_{ji}(x) \text{ for all } i, j \in N, i \neq j\}. \tag{3.3}$$

Obviously, for $\alpha = 0$, the closure of the α-prekernel coincides with the traditional prekernel. Similar as the prekernel, for every $\alpha \in [0, 1]$, the corresponding α-prekernel balances the surpluses pairwise, but using the modified α-excess where dissatisfaction is measured by a mix of the difference and the ratio of potential and realized payoffs $v(S)$, respectively $x(S)$.

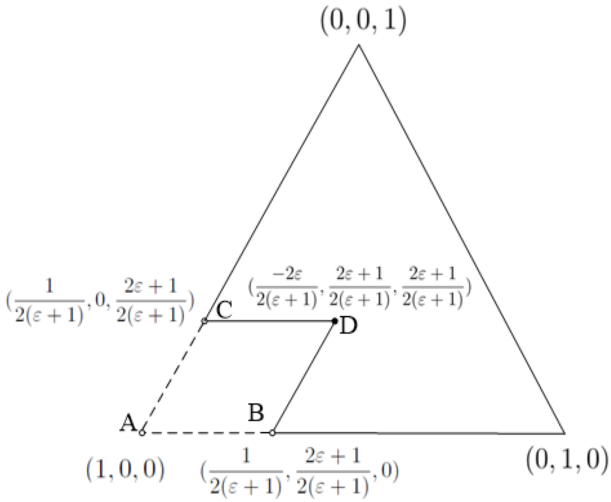

FIGURE 3.4: The 1ε-core, $\mathscr{C}^1_\varepsilon(N,v)$, when $-1<\varepsilon<0$

We illustrate that the α-prekernel is different for different α with the following example.

Example 3.3. Let $N=\{1,2,3,4\}$ and let v be defined by $v(\{1,2,3,4\}) = v(\{1,2\}) = v(\{3,4\}) = 1$, $v(\{2\}) = v(4) = 2\rho$ and $v(S)=\rho$ otherwise, ρ being a sufficiently small positive number. For $\alpha=0$, we find that $e(\{12\},x) = 1-x_1-x_2$, $e(\{34\},x)=1-x_3-x_4$, $e(\{2\},x)=2\rho-x_2$, $e(\{4\},x)=2\rho-x_4$, $e(S,x)=v(S)-x(S)$ otherwise. Thus,

$$s^{v,0}_{13}(x) = s^{v,0}_{14}(x) = \max\{\rho - x_1, 1-x_1-x_2\},$$

$$s^{v,0}_{23}(x) = s^{v,0}_{24}(x) = \max\{2\rho - x_2, 1-x_1-x_2\},$$

$$s^{v,0}_{31}(x) = s^{v,0}_{32}(x) = \max\{\rho - x_3, 1-x_3-x_4\},$$

$$s^{v,0}_{41}(x) = s^{v,0}_{42}(x) = \max\{2\rho - x_4, 1-x_3-x_4\},$$

$$s^{v,0}_{34}(x) = \rho - x_3, s^{v,0}_{43}(x) = 2\rho - x_4,$$

$$s^{v,0}_{12}(x) = \rho - x_1, s^{v,0}_{21}(x) = 2\rho - x_2.$$

Following from the definition of the 0-prekernel, i.e., $s_{ij}^{v,0}(x) = s_{ji}^{v,0}(x)$ for all $i, j \in N, i \neq j$, and ρ being a sufficiently small positive number, we know that the 0-prekernel is the set

$$\{x \in \mathscr{I}_{++}(N, v) \mid x_1 = x_3 = \frac{1}{4} - \frac{1}{2}\rho \text{ and } x_2 = x_4 = \frac{1}{4} + \frac{1}{2}\rho\}.$$

For $\alpha = 1$, it is found that $\bar{e}(\{12\}, x) = \frac{1}{x_1+x_2}$, $\bar{e}(\{34\}, x) = \frac{1}{x_3+x_4}$, $\bar{e}(\{2\}, x) = \frac{2\rho}{x_2}$, $\bar{e}(\{4\}, x) = \frac{2\rho}{x_4}$, $\bar{e}(S, x) = \frac{\rho}{x(S)}$ otherwise. Thus,

$$s_{13}^{v,1}(x) = s_{14}^{v,1}(x) = \max\{\frac{\rho}{x_1}, \frac{1}{x_1 + x_2}\},$$

$$s_{23}^{v,1}(x) = s_{24}^{v,1}(x) = \max\{\frac{2\rho}{x_2}, \frac{1}{x_1 + x_2}\},$$

$$s_{31}^{v,1}(x) = s_{32}^{v,1}(x) = \max\{\frac{\rho}{x_3}, \frac{1}{x_3 + x_4}\},$$

$$s_{41}^{v,1}(x) = s_{42}^{v,1}(x) = \max\{\frac{2\rho}{x_4}, \frac{1}{x_3 + x_4}\},$$

$$s_{34}^{v,1}(x) = \frac{\rho}{x_3}, s_{43}^{v,1}(x) = \frac{2\rho}{x_4},$$

$$s_{12}^{v,1}(x) = \frac{\rho}{x_1}, s_{21}^{v,1}(x) = \frac{2\rho}{x_2}.$$

By the definition of the 1-prekernel, $s_{ij}^{v,1}(x) = s_{ji}^{v,1}(x)$ for all $i, j \in N, i \neq j$, and ρ being a sufficiently small positive number, we get that $x_4 = 2x_3$, $x_2 = 2x_1$ and $x_1 = x_3$. Therefore, the 1-prekernel is $(\frac{1}{6}, \frac{1}{3}, \frac{1}{6}, \frac{1}{3})$, which is different from the 0-prekernel as long as $\rho \neq \frac{1}{6}$. □

Now, given $v \in G^{N_+}$ and $\alpha \in \mathbb{R}$, we define the surplus function as the map $s^{v,\alpha} : \mathscr{I}^*_{++}(N, v) \to \mathbb{R}^n$ with

$$s_i^{v,\alpha}(x) = \max\{e^\alpha(S, x) \mid S \subset N, i \in S\} \text{ for all } i \in N, x \in \mathscr{I}^*_{++}(N, v),$$

i.e., $s_i^{v,\alpha}(x)$ is the maximum α-surplus that player i can obtain by cooperation given payoff vector x. Recall that $s_{ij}^{v,\alpha}(x)$ gives the potential gain of player i with respect to player j (see Definition 3.3). Thus, $s_{ij}^{v,\alpha}(x)$ is a relational surplus comparing the positions of two players in a game, whereas the α-surplus of player i, $s_i^{v,\alpha}(x)$, is an individual measure for player i's position in the game. We call a preimputation α-strongly stable if the individual α-surplusses are equal for all players.

Definition 3.5. Given $\alpha \in [0,1]$, the preimputation $x \in \mathscr{I}^*_{++}(N,v)$ is said to be *α-strongly stable* for game $v \in G^{N_+}$ if $s_i^{v,\alpha}(x) = s_j^{v,\alpha}(x)$ for all $i,j \in N$.

We now provide another characterization of the α-least core $\mathscr{LC}^\alpha(N,v)$ for a positive game, by showing that it consists of all α-strongly stable payoff vectors for every $\alpha \in [0,1]$. For notational convenience, we often write λ instead of $\lambda^{v,\alpha}$ if there is no confusion about v and α.

Theorem 3.6. *For every $\alpha \in [0,1]$ and $v \in G^{N_+}$, $x \in \mathscr{LC}^\alpha(N,v)$ if and only if $x \in \mathscr{I}^*_{++}(N,v)$ and $\lambda^{v,\alpha} = s_i^{v,\alpha}(x) - \alpha, \forall i \in N$.*

Proof. Take $\alpha \in [0,1]$ and $v \in G^{N_+}$.

'Only if': Assume that $x \in \mathscr{LC}^\alpha(N,v)$. Then, by definition $x(N) = v(N)$ and $e^\alpha(S,x) \leq \alpha + \lambda$, for every $S \in 2^N \setminus \{\emptyset, N\}$. In addition, owing to the definition of $\lambda = \lambda^{v,\alpha}$, there is a coalition $T \in 2^N \setminus \{\emptyset, N\}$ such that $e^\alpha(T,x) = \alpha + \lambda$. Denote $\mathcal{T} = \{T \in 2^N \setminus \{\emptyset, N\} \mid e^\alpha(T,x) = \alpha + \lambda\}$. We assert that for every $i \in N$, there exists $T \in \mathcal{T}$, such that $i \in T$. On the contrary, assume that $\exists i \in N$ such that $e^\alpha(S,x) < \alpha + \lambda$, for every $S \subset N,\ S \neq N$ with $i \in S$. Let $\beta_1 = \max\{e^\alpha(S,x) \mid i \in S, S \neq N\} < \alpha + \lambda$, and let $y \in \mathbb{R}^n_{++}$ be defined by

$$y_k = \begin{cases} x_k - \beta_2, \text{ if } k = i, \\ x_k + \frac{\beta_2}{n-1} \text{ if } k \neq i, \end{cases} \tag{3.4}$$

where $0 < \beta_2 < \min\limits_{S \in 2^N \setminus \{\emptyset,N\}} \{\frac{\lambda - \beta_1}{1-\alpha}, \frac{(n-1)\underline{x}x(S)}{(n-s)(\overline{v}+\underline{x})}\}$, $\underline{x} = \min\limits_{S \in 2^N \setminus \{\emptyset,N\}} x(S)$, and $\overline{v} = \max\limits_{S \in 2^N \setminus \{\emptyset,N\}} v(S)$.

If $S = \{i\}$, then $0 < \beta_2 < \min\limits_{S \in 2^N \setminus \{\emptyset,N\}} \{\frac{\lambda - \beta_1}{1-\alpha}, \frac{\underline{x}x_i}{(\overline{v}+\underline{x})}\}$. We obtain that $x_i > \beta_2$ since $x_i > \frac{\underline{x}}{\overline{v}+\underline{x}} x_i$ and $\beta_2 < \frac{\underline{x}}{\overline{v}+\underline{x}} x_i$. Thus, $y \in \mathscr{I}^*_{++}(N,v)$. We show that $e^\alpha(S,y) < \alpha+\lambda$ for every $S \in 2^N \setminus \{\emptyset, N\}$ establishing a contradiction to the definition of λ.

In the case that $i \notin S \neq \emptyset$, it holds that

$$\begin{aligned}
e^\alpha(S,y) =& \alpha\frac{v(S)}{x(S)+\frac{s\beta_2}{n-1}} + (1-\alpha)(v(S)-x(S)-\frac{s\beta_2}{n-1})\\
<& \alpha\frac{v(S)}{x(S)} + (1-\alpha)(v(S)-x(S)) - (1-\alpha)\frac{s\beta_2}{n-1}\\
=& e^\alpha(S,x) - (1-\alpha)\frac{s\beta_2}{n-1}\\
<& e^\alpha(S,x) \leq \alpha+\lambda.
\end{aligned}$$

In the case that $i \in S \neq N$, we obtain that

$$\begin{aligned}
e^\alpha(S,y) &= \alpha\frac{v(S)}{x(S)-\beta_2+\frac{(s-1)\beta_2}{n-1}} + (1-\alpha)\left(v(S)-x(S)+\beta_2-\frac{(s-1)\beta_2}{n-1}\right)\\
&= \alpha\left(1+\frac{\frac{n-s}{n-1}\beta_2}{x(S)-\frac{n-s}{n-1}\beta_2}\right)\frac{v(S)}{x(S)}\\
&\quad +(1-\alpha)\left(v(S)-x(S)+\beta_2-\frac{(s-1)\beta_2}{n-1}\right)\\
&= \alpha\frac{v(S)}{x(S)} + \alpha\left(\frac{\frac{n-s}{n-1}\beta_2}{x(S)-\frac{n-s}{n-1}\beta_2}\right)\frac{v(S)}{x(S)}\\
&\quad +(1-\alpha)(v(S)-x(S)) + (1-\alpha)\left(\beta_2-\frac{(s-1)\beta_2}{n-1}\right)\\
&< e^\alpha(S,x) + \alpha\frac{\frac{n-s}{n-1}\beta_2}{x(S)-\frac{n-s}{n-1}\beta_2}\frac{\overline{v}}{\underline{x}} + (1-\alpha)\beta_2\\
&\leq \beta_1+\alpha+(1-\alpha)\beta_2 < \alpha+\lambda,
\end{aligned}$$

where the second inequality follows from $0 < \alpha\frac{\frac{n-s}{n-1}\beta_2}{x(S)-\frac{n-s}{n-1}\beta_2}\frac{\overline{v}}{\underline{x}} < 1$ (which follows since, by definition of β_2, $\left(x(S)-\frac{n-s}{n-1}\beta_2\right)\underline{x} > (x(S)-\frac{\underline{x}x(S)}{\overline{v}+\underline{x}})\underline{x} = \frac{\underline{x}\overline{v}x(S)}{\overline{v}+\underline{x}} > \frac{n-s}{n-1}\beta_2\overline{v}$), and the last inequality follows from $(1-\alpha)\beta_2 < \lambda-\beta_1$.

Hence, $e^\alpha(S,y) < \alpha+\lambda$ for every $S \in 2^N \setminus \{\emptyset, N\}$, which contradicts with the definition of λ. Therefore, $s_i^{v,\alpha}(x) = \alpha+\lambda$ for every $i \in N$.

'If': Assume that $x \in \mathscr{I}^*_{++}(N,v)$ and $s_i^{v,\alpha}(x) = \alpha + \lambda$ for every $i \in N$. Then, for every $S \in 2^N \setminus \{\emptyset, N\}$, there exists $i \in S$ such that $e^\alpha(S,x) - \lambda \leq s_i^{v,\alpha}(x) - \lambda = \alpha$. Therefore, $e^\alpha(S,x) \leq \lambda + \alpha$ for every $S \in 2^N \setminus \{\emptyset, N\}$. That is to say, $x \in \mathscr{LC}^\alpha(N,v)$. □

Above, we considered two ways to evaluate the position of a player i in a game v. First, with respect to every other player $j \neq i$, the surplus $s_{ij}^{v,\alpha}(x)$ compares the relative position of i with respect to every other player j. In the prekernel, these surpluses are α-balanced for every pair of players. Second, the α-surplus $s_i^{v,\alpha}(x)$ is a measure of the overall position of player i in the game. Instead of comparing payoff vectors by only the individual or pairwise surpluses separately, we combine these surpluses, and compare payoff vectors by balancing the differences between the pairwise and individual surpluses.

Definition 3.7. For a given $\alpha \in [0,1]$, $x \in \mathscr{I}^*_{++}(N,v)$ satisfies the α-balanced surplus property if

$$s_i^{v,\alpha}(x) - s_{ij}^{v,\alpha}(x) = s_j^{v,\alpha}(x) - s_{ji}^{v,\alpha}(x) \text{ for every } i,j \in N.$$

We can characterize the α-prekernel using α-strong stability and this α-balanced surplus property.

Theorem 3.8. *Given $\alpha \in [0,1]$, $x \in \mathscr{PK}^\alpha(N,v)$ if and only if $x \in \mathscr{I}^*_{++}(N,v)$ is α-strongly stable and satisfies the α-balanced surplus property.*

Proof. 'Only If:' Let $x \in \mathscr{PK}^\alpha(N,v)$ and take every $i,j \in N,\ i \neq j$. By the definition of the α-prekernel, it holds that $s_{ij}^{v,\alpha}(x) = s_{ji}^{v,\alpha}(x)$. Hence, we get that

$$\begin{aligned} s_i^{v,\alpha}(x) &= \max\{s_{ij}^{v,\alpha}(x), \max\{e^\alpha(T,x) \mid \{i,j\} \subseteq T \neq N\}\} \\ &= \max\{s_{ji}^{v,\alpha}(x), \max\{e^\alpha(T,x) \mid \{j,i\} \subseteq T \neq N\}\} \\ &= s_j^{v,\alpha}(x), \end{aligned}$$

which implies that x is α-strongly stable. Since, additionally $x \in \mathscr{PK}^\alpha(N,v)$, and thus $s_{ij}^{v,\alpha}(x) = s_{ji}^{v,\alpha}(x)$, x satisfies the α-balanced surplus property.

'If:' Let x be α-strongly stable and satisfy the α-balanced surplus property. By x being α-strongly stable, $s_i^{v,\alpha}(x) = s_j^{v,\alpha}(x)$ for every $i, j \in N$, $i \neq j$. Then, since x verifies the α-balanced surplus property, it holds that $s_{ij}^{v,\alpha}(x) = s_{ji}^{v,\alpha}(x)$ for every $i \neq j$, and therefore $x \in \mathscr{P}\mathscr{K}^\alpha(N,v)$. $\square$

3.4 The α-prenucleolus and the α-nucleolus

In this section, based on the lexicographical order $\leq_L$ (see page 10), considering the 2^n-dimensional vector $\theta(e^\alpha(S,x)_{S\subseteq N})$, whose components are arranged in nonincreasing order, we propose the α-prenucleolus and α-nucleolus of a cooperative game as follows.

Definition 3.9. Let $\alpha \in [0,1]$. For every game $v \in G^{N_+}$, the α-prenucleolus $\mathscr{P}\mathscr{N}^\alpha(N,v)$ and the α-nucleolus $\mathscr{N}^\alpha(N,v)$ which minimize the excess $e^\alpha(S,x)$ of every coalition over the preimputation set, respectively, the imputation set, are defined as follows

$$\begin{aligned}\mathscr{P}\mathscr{N}^\alpha(N,v) &= \{x \in \mathscr{I}^*_{++}(N,v) | \theta(e^\alpha(S,x)_{S\subseteq N}) \leq_L \theta(e^\alpha(S,y)_{S\subseteq N}) \\ &\qquad \forall y \in \mathscr{I}^*_{++}(N,v)\}\end{aligned}$$

and

$$\begin{aligned}\mathscr{N}^\alpha(N,v) &= \{x \in \mathscr{I}_{++}(N,v) | \theta(e^\alpha(S,x)_{S\subseteq N}) \leq_L \theta(e^\alpha(S,y)_{S\subseteq N}) \\ &\qquad \forall y \in \mathscr{I}_{++}(N,v)\}.\end{aligned}$$

Remark 3.1. Owing to the results obtained by Justman (1977, [43]), we have the following statements. For every given $\alpha \in [0,1]$,

(i) If $\mathscr{I}_{++}(N,v)$ is nonempty and compact and if all $e^\alpha(S,x)$, $S \subseteq N$, are continuous with respect to the second variable, then $\mathscr{N}^\alpha(N,v) \neq \emptyset$.

(ii) If $\mathscr{I}_{++}(N,v)$ is convex and all $e^\alpha(S,x)$, $S \subseteq N$, are convex with respect to the second variable, then $\mathscr{N}^\alpha(N,v)$ is convex and $e^\alpha(S,x) = e^\alpha(S,y)$ for all $S \subseteq N$ and all $x, y \in \mathscr{N}^\alpha(N,v)$.

Inspired by the method provided by Peleg and Sudhölter (2007, [66]), define

$$\begin{aligned}\mathscr{I}'_{++}(N,v) &= \{x \in \mathscr{I}^*_{++}(N,v) | \max_{S \subseteq N} e^{\alpha}(S,x) \leq \max_{S \subseteq N} e^{\alpha}(S,y) \\ &\qquad \forall y \in \mathscr{I}^*_{++}(N,v)\}.\end{aligned}$$

Following from Remark 3.1, we obtain that the α-nucleolus is a singleton. Also, for every $x \in \mathscr{I}^*_{++}(N,v)$, in the definition of $\mathscr{P}\mathscr{N}^{\alpha}(N,v)$, we may replace $\mathscr{I}^*_{++}(N,v)$ by the compact, nonempty, and convex set $\mathscr{I}'_{++}(N,v)$. Thus, the α-prenucleolus is also a singleton.

Theorem 3.10. *Given $\alpha \in [0,1]$, for every game $v \in G^{N_+}$: (i) the α-nucleolus is a singleton, and (ii) the α-prenucleolus is a singleton.*

From now on, we often write the α-prenucleolus of game v just as its unique element, and denote it by $\nu^{\alpha}(N,v) \in \mathbb{R}^n_{++}$. Next, we show that the α-prenucleolus is an element of the α-prekernel.

Theorem 3.11. *For every game $v \in G^{N_+}$ and for all $\alpha \in [0,1]$, $\nu^{\alpha}(N,v) \in \mathscr{P}\mathscr{K}^{\alpha}(N,v)$.*

Proof. Let $\alpha \in [0,1]$ and $x^{\alpha} = \nu^{\alpha}(N,v)$. We show that $x^{\alpha} \in \mathscr{P}\mathscr{K}^{\alpha}(N,v)$. On the contrary, assume that there exists $\bar{\alpha} \in [0,1]$ such that $x^{\bar{\alpha}} \notin \mathscr{P}\mathscr{K}^{\bar{\alpha}}(N,v)$. For easiness of notation, let $x = x^{\bar{\alpha}}$. Since $x \in \mathscr{I}^*_{++}(N,v)$, there exist two distinct players $i,j \in N$ with $s^{v,\bar{\alpha}}_{ij}(x) > s^{v,\bar{\alpha}}_{ji}(x)$.

First, we show that there exists δ with $0 < \delta < \hat{x} = \min_{S \in \Gamma_{ji}(N)} x(S)$ such that

$$s^{v,\bar{\alpha}}_{ji}(x) = s^{v,\bar{\alpha}}_{ij}(x) - \delta - \frac{\bar{\alpha}\delta\hat{v}}{(\hat{x}-\delta)\hat{x}}, \tag{3.5}$$

where $\hat{v} = \max_{S \in \Gamma_{ji}(N)} v(S)$. This is equivalent to showing that the second degree equation on δ

$$\hat{x}\delta^2 - [(s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))\hat{x} + \hat{x}^2 + \bar{\alpha}\hat{v}]\delta + (s^{v,\bar{\alpha}}_{ij}(x) - s^{v,\bar{\alpha}}_{ji}(x))\hat{x}^2 = 0 \tag{3.6}$$

has at least one real solution. This is true when the discriminant of the equation is non-negative, i.e.,

$$[(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x))\hat{x} + \hat{x}^2 + \bar{\alpha}\hat{v}]^2 - 4\hat{x}^3(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x)) \geq 0,$$

or, equivalently after some algebra[1],

$$[(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x))\hat{x} - \hat{x}^2]^2 + \bar{\alpha}^2\hat{v}^2 + 2[(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x))]\hat{x}\hat{v}\bar{\alpha} \geq 0. \quad (3.7)$$

The formula (3.7) holds since every term in the inequality is non-negative. Therefore, the quadratic equation (3.6) has at least one solution, and therefore there exists $\delta \in (0, \hat{x})$ for which (8) holds.

Second, we define $y \in \mathbb{R}^n_{++}$ as

$$y_k = \begin{cases} x_k + \delta, \text{ if } k = i, \\ x_k - \delta, \text{ if } k = j, \\ x_k, \text{ otherwise}, \end{cases} \quad (3.8)$$

and show that $y \in \mathscr{I}^*_{++}(N, v)$ with $\theta((e^\alpha(S, y))_{S\subseteq N}) <_L \theta((e^\alpha(S, x))_{S\subseteq N})$.

On the one hand, $x \in \mathscr{I}^*_{++}(N, v)$ implies

$$\sum_{k\in N} y_k = \sum_{k\in N} x_k = v(N),$$

while $x \in \mathscr{I}^*_{++}(N, v)$ and $0 < \delta < \hat{x} = \min_{S\in\Gamma_{ji}(N)} x(S) \leq x_j$ imply $y_k > 0$ for all $k \in N$. Thus, $y \in \mathscr{I}^*_{++}(N, v)$.

On the other hand, to show $\theta((e^\alpha(S, y))_{S\subseteq N}) <_L \theta((e^\alpha(S, x))_{S\subseteq N})$, we consider the following three cases. Denote

$$\mathcal{S} = \{S \in 2^N \setminus \Gamma_{ij}(N) \mid e^\alpha(S, x) \geq s_{ij}^{v,\alpha}(x)\} \text{ and } \tilde{s} = |\mathcal{S}|.$$

[1] This follows since $[(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x))\hat{x} + \hat{x}^2 + \bar{\alpha}\hat{v}]^2 - 4\hat{x}^3(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x)) = (s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x))^2\hat{x}^2 + \hat{x}^4 + \bar{\alpha}^2\hat{v}^2 + 2\hat{x}^3(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x)) + 2(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x))\hat{x}\hat{v}\bar{\alpha} + 2\hat{x}^2\hat{v}\bar{\alpha} - 4\hat{x}^3(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x)) = (s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x))^2\hat{x}^2 - 2\hat{x}^3(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x)) + \hat{x}^4 + \bar{\alpha}^2\hat{v}^2 + 2(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x))\hat{x}\hat{v}\bar{\alpha} + 2\hat{x}^2\hat{v}\bar{\alpha} = [(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x))\hat{x} - \hat{x}^2]^2 + \bar{\alpha}^2\hat{v}^2 + 2[(s_{ij}^{v,\bar{\alpha}}(x) - s_{ji}^{v,\bar{\alpha}}(x))]\hat{x}\hat{v}\bar{\alpha}$.

(i) First, if $S \in 2^N \setminus (\Gamma_{ij}(N) \cup \Gamma_{ji}(N))$ then $e^\alpha(S,y) = e^\alpha(S,x)$ in view of the form of y as in (3.8).

(ii) Second, if $S \in \Gamma_{ij}(N)$,

$$e^\alpha(S,y) = \alpha \frac{v(S)}{x(S)+\delta} + (1-\alpha)(v(S) - x(S) - \delta) < e^\alpha(S,x).$$

(iii) Third, if $S \in \Gamma_{ji}(N)$, then

$$\begin{aligned}
e^\alpha(S,y) =& \alpha \frac{v(S)}{x(S)-\delta} + (1-\alpha)(v(S) - x(S) + \delta) \\
=& \alpha \frac{x(S)}{x(S)-\delta} \frac{v(S)}{x(S)} + (1-\alpha)(v(S) - x(S)) + (1-\alpha)\delta \\
=& \alpha \left(1 + \frac{\delta}{x(S)-\delta}\right) \frac{v(S)}{x(S)} + (1-\alpha)(v(S) - x(S)) + (1-\alpha)\delta \\
=& e^\alpha(S,x) + \alpha \frac{\delta}{x(S)-\delta} \frac{v(S)}{x(S)} + (1-\alpha)\delta \\
\leq& s_{ji}^{v,\alpha}(x) + \left(\frac{\alpha}{x(S)-\delta} \frac{v(S)}{x(S)} + 1\right) \delta \\
\leq& s_{ij}^{v,\alpha}(x) - \frac{\alpha(v_{max} - v(S))}{(x_{min} - \delta) x_{min}} \delta \\
<& e^\alpha(S,x)
\end{aligned}$$

where the first inequality follows by definition of $s_{ji}^{v,\alpha}(x)$, and the second from (3.5) and the definition of x_{min}.

Let $S^1, S^2 \in 2^N$ be such that $e^\alpha(S^1,x) \geq e^\alpha(S,x)$ for every $S \in 2^N$ and $e^\alpha(S^2,y) \geq e^\alpha(S,y)$ for every $S \in 2^N$. From the above three cases, it holds that $e^\alpha(S,y) \leq e^\alpha(S,x)$ for every $S \in 2^N$. Thus, $e^\alpha(S^2,y) \leq e^\alpha(S^2,x) \leq e^\alpha(S^1,x)$. Therefore, it holds that[2] $\theta_t(y) \leq \theta_t(x)$ for all $t \leq \tilde{s}$, and $\theta_{\tilde{s}+1}(y) < s_{ij}^{v,\alpha}(x) = \theta_{\tilde{s}+1}(x)$ if $\theta_t(y) = \theta_t(x)$ for all $t \leq \tilde{s}$, for every $\alpha \in [0,1]$. That is to say, $\theta(y) <_L \theta(x)$, and thus $x \neq \nu^\alpha(N,v)$ and the desired contradiction has been obtained. □

[2]If there is no confusion, we sometimes shortly write $\theta(x)$ and $\theta(y)$ instead of $\theta(e^\alpha(S,x)_{S\subseteq N})$, respectively, $\theta(e^\alpha(S,y)_{S\subseteq N})$

From Theorems 3.10 and 3.11, we obtain the following corollary.

Corollary 3.12. *For every game* $v \in G^{N_+}$ *and every* $\alpha \in [0,1]$, $\mathscr{PK}^{\alpha}(N,v) \neq \emptyset$.

3.5 Conclusions

In this chapter, we propose a family of excesses (α-excess) for positive TU-games that measure the dissatisfaction of every coalition and generalizes the classical and proportional excesses. Then, the corresponding solutions, such as the α-least core, the α-(pre)nucleolus, and the α-(prekernel) are defined based on the α-excess. We give a characterization of the α-prekernel by strong stability and the α-balanced surplus property. Meanwhile, the α-least core of a positive TU-game can be characterized in terms of strong stability. Finally, we introduce the α-prenucleolus and α-nucleolus, and showed that, for every game, these are singletons and the unique α-prenucleolus element belongs to the corresponding α-prekernel.

For future research, we intend to modify the famous Davis and Maschler reduced game (Davis and Maschler (1965)), taking account of the modified excess. There is a large literature on reduced game consistency. Reduced game consistency requires that, after some players leave the game with the payoffs assigned to them by a solution, applying the same solution on the reduced game on the remaining players gives these remaining players the same payoff as in the original game. Different solutions can be characterized by different reduced game properties, where the difference is in the way the reduced game is defined. For the α-prekernel, we might consider the following reduced game. Let $\alpha \in [0,1]$. Given a game $v \in G^{N_+}$, a nonempty coalition S, and a positive payoff vector x, the α-reduced game on S at x, denoted $(S, v^{\alpha}_{x,S})$, is the game defined by

$$v^{\alpha}_{x,S}(T) = \begin{cases} 0, \text{ if } T = \emptyset, \\ v(N) - x(N \setminus T), \text{ if } T = S, \\ \frac{\alpha}{\alpha+(1-\alpha)x(T)} \max\limits_{Q \subseteq N \setminus S} \left\{ \frac{v(T \cup Q)x(T)}{x(T \cup Q)} \right\} + \\ \qquad \frac{(1-\alpha)x(T)}{\alpha+(1-\alpha)x(T)} \max\limits_{Q \subseteq N \setminus S} \{v(T \cup Q) - x(Q)\}, \text{ if } T \subsetneq S. \end{cases}$$

It can be shown that the α-prekernel satisfies the corresponding reduced game property. However, a characterization of the α-prekernel using the α-reduced game property is still an open problem.

Chapter 4

On the core, nucleolus and bargaining sets of cooperative games with fuzzy payoffs

4.1 Introduction

Cooperative games with fuzzy payoffs (cooperative fuzzy game in short) were introduced by Mareš (1999, [50]) to model cooperative situations where the worth of every coalition is a fuzzy number. They are useful to analyze and resolve cooperative situations under fuzzy environments. In Mallozzi et al. (2011, [49]), the authors motivate the analysis of cooperative fuzzy games by relating them to bankruptcy problems.

For a bankruptcy problem, an estate has to be divided among some creditors. In the classical model, every creditor has an individual claim on a certain amount of the estate, and the total claim is weakly larger than the estate. The individual claim is the best value $\overline{c}$ of a creditor, while nothing is said about the minimum value $\underline{c}$ that a creditor is willing to accept. It is not surprising that every creditor can try to improve his situation by giving bounds $\underline{c}$ and $\overline{c}$ for what he is willing to accept. In this way, it becomes an interval bankruptcy problem introduced by Branzei et al. (2003, [17]).

Actually, if a creditor claims with his minimum and best value bounds, it is natural to assume that the creditor prefers the amounts in the interval $[\underline{c}, \overline{c}]$ according to some increasing utility function. In addition, the result of asset evaluation may vary for different assessment methods. Accordingly, the estate and the claims of creditors become fuzzy numbers, resulting in a fuzzy bankruptcy problem. Obviously, the classical bankruptcy problem and the interval bankruptcy problem are special cases of the fuzzy bankruptcy problem. The main question is how to share the estate among the creditors. To answer this question, we can make use of fuzzy bankruptcy games. Important for this aim are the core and bargaining set for cooperative fuzzy games, which are the focus of this chapter.

Based on the partial order relation defined by the α-level sets of fuzzy numbers, the $\mathcal{F}$-core proposed by Mallozzi et al. (2011, [49]) is a set-valued solution that distributes the worth of the grand coalition and guarantees that the players of every coalition gain at least what they could obtain by themselves. In other words, the $\mathcal{F}$-core consists of feasible outcomes which cannot be improved upon by every coalition. Depending on the partial order relation of fuzzy numbers, Mallozzi et al. (2011, [49]) introduced $\mathcal{F}$-balancedness for fuzzy interval cooperative games, which requires (in the spirit of Bondareva (1962, [15]) and Shapley (1967, [75])) that allowing players to work part time in different coalitions is less profitable than working full time in the grand coalition. They showed that if the $\mathcal{F}$-core is not empty, then the cooperative fuzzy game is $\mathcal{F}$-balanced, but the opposite need not be true as opposed to balancedness for cooperative games. Wang and Zhang (2016, [84]) gave three sufficient conditions for non-emptiness of the $\mathcal{F}$-core.

Mallozzi et al. (2011, [49]) used a partial order relation to compare a pair of fuzzy numbers. This partial order relation can only compare every pair of fuzzy numbers with one's left-hand endpoint being greater than or equal to the other's left-hand endpoint and one's right-hand endpoint being greater than or equal to the other's right-hand endpoint of the α-level set. There exist many pairs of fuzzy numbers with one's left-hand endpoint being greater than the other's left-hand endpoint, but one's right-hand endpoint being less than the other's right-hand endpoint of the α-level set. In this case, we don't know which fuzzy number is greater by the partial order relation used in Mallozzi et al. (2011, [49]). The $\mathcal{F}$-core provides a possible way out in decision making processes where the vagueness over the worth of every coalition is given by means of a fuzzy number. Due to the partial order relation, payoffs that are uncomparable with coalitional values do not belong to the $\mathcal{F}$-core. What's worse, it may be empty in many cases for the same reason. As a consequence, we may hardly select some reasonable allocations to keep the grand coalition stable. Another issue is that given an allocation in the $\mathcal{F}$-core, players may hesitate on whether to depart from the grand coalition or not, since they are not sure that their allocated payoff is greater than the worth they may obtain on their own. The common root of these issues lies in the ranking criterion of fuzzy numbers.

In this current chapter, which is based on Zhang et al. (2019, [87]), we settle these issues by defining a total order relation based on the expected values of fuzzy numbers, where the expected value of a fuzzy number is the mean of the midpoints of the α-level sets of fuzzy numbers with a uniform distribution in the interval $[0, 1]$. By defining a total order relation based on the expected values of fuzzy numbers, we can now easily compare each pair of fuzzy numbers. This allows to give better definitions for the core and bargaining set of cooperative fuzzy games.

The indifference fuzzy core defined in this chapter contains all indifferently efficient payoff vectors for which the players of every coalition receive at least what they could gain by themselves in view of the total order relation of fuzzy numbers. In this case, the above-mentioned drawback of the $\mathcal{F}$-core is overcome. Simultaneously, the players are determined to join the grand coalition so long as they receive in the allocation no less than the worth obtained by themselves. Moreover, we conclude that the indifference fuzzy core is nonempty for convex cooperative fuzzy games. However, the indifference fuzzy core may be empty. We introduce a necessary and sufficient condition for non-emptiness of the indifference fuzzy core called balancedness. This improves upon $\mathcal{F}$-balancedness defined by the partial order relation of fuzzy numbers, which is a necessary but not sufficient condition for non-emptiness of the $\mathcal{F}$-core.

Each player aspires to obtain his maximum payoff in the indifference fuzzy core. This creates tension among the players since not all of them can achieve their maximum payoffs in an element of the indifference fuzzy core. It is, then, hard to say which of the payoff vectors is better in the indifference fuzzy core. Here, we introduce the indifference fuzzy nucleolus by lexicographically minimizing the excesses of coalitions over the nonempty compact convex indifference fuzzy imputation set, which is based on the total order relation. We show that there exists at least one fuzzy payoff vector in the indifference fuzzy nucleolus. It is also proved that the indifference fuzzy nucleolus is a subset of the indifference fuzzy core when the indifference fuzzy core is nonempty.

The indifference fuzzy core of the cooperative fuzzy game may be empty, which makes us to consider other solution concepts. Additionally, the concepts which have been introduced in the previous paragraphs neglect the bargaining process that may actually take place during the cooperative process. No agreement will be reached if every player demands the maximum he can get in the coalition. In this chapter, we assume that all players can bargain together, and settle at a unanimous outcome which is based on the threats and counterthreats that they possess. Following this idea, we define and analyze the indifference fuzzy bargaining set as the set of all the unanimous outcomes for cooperative fuzzy games. Based on the allowed threats and counterthreats, we introduce two indifference fuzzy bargaining sets: the Davis-Maschler indifference fuzzy bargaining set in which no player has a justified objection at an indifference fuzzy imputation against every other player; and the Mas-Colell indifference fuzzy bargaining set in which no coalition has a justified objection at an indifference fuzzy imputation against every other coalition. Only objections without counter-objections are considered as justified. Consequently, the indifference fuzzy core is a subset of the indifference fuzzy bargaining set since blocking a fuzzy allocation becomes more difficult. Moreover, we characterize the class of superadditive cooperative fuzzy games for which the indifference fuzzy bargaining set and the indifference fuzzy core coincide. We further conclude that the Davis-Maschler indifference fuzzy bargaining set of a superadditive cooperative fuzzy game is a subset of the Mas-Colell indifference fuzzy bargaining set. Similarly as for TU-games, the Davis-Maschler indifference fuzzy bargaining set, the Mas-Colell indifference fuzzy bargaining set, and the indifference fuzzy core of convex cooperative fuzzy games coincide.

The chapter is organized as follows. Section 4.2 introduces the indifference fuzzy core and the balanced condition for cooperative fuzzy games. The indifference fuzzy nucleolus is defined in Section 4.3. Section 4.4 proposes two indifference fuzzy bargaining sets, both of which are equal to the indifference fuzzy core for convex cooperative fuzzy games. In Section 4.5, we provide the proof of the equality between the two indifference fuzzy baragining sets and the indifference fuzzy core for superadditive cooperative fuzzy games. The last section concludes with a brief summary.

4.2 Indifference fuzzy core

We start this section with an example to explain a problem caused by the partial order relation $\succeq$ in Mallozzi et al. (2011, [49]).

Example 4.1. Let's consider a fuzzy bankruptcy problem with an estate $\tilde{e}$ divided among n creditors. Let the estate $\tilde{e} = \lfloor 1, 3, 5, 8 \rfloor$ and the first creditor's claim $\tilde{c}_1 = \lfloor 2, \frac{8}{3}, \frac{11}{2}, \frac{15}{2} \rfloor$. We work out $\tilde{e}[\alpha] = [1 + 2\alpha, 8 - 3\alpha], \tilde{c}_1[\alpha] = [2 + \frac{2}{3}\alpha, -2\alpha + \frac{15}{2}], \alpha \in [0, 1]$. Obviously, $\tilde{e}$ and $\tilde{c}_1$ are non-comparable by the partial order relation $\succeq$ used by Mallozzi et al. (2011, [49]). Hence, it is impossible to give an allocation of this fuzzy bankruptcy problem.

As illustrated in Example 4.1, the partial order relation in Mallozzi et al. (2011, [49]) may bring problems to provide a "fair" allocation. The difficulty raises when we have two fuzzy numbers $\tilde{A}, \tilde{B} \in \mathbb{FR}$ with $A_*(\alpha) < B_*(\alpha)$ but $A^*(\alpha) > B^*(\alpha)$ for every $\alpha \in [0, 1]$. In this case, $\tilde{A}$ and $\tilde{B}$ cannot be compared with the partial order relation $\succeq$. A total order relation of fuzzy numbers should be defined to solve such difficult situations. Following from the total order criterion for intervals presented by Han et al. (2012, [37]), we define the following order relation of fuzzy numbers.

Definition 4.1. Let $\tilde{A}, \tilde{B} \in \mathbb{FR}$.

$\tilde{A} \gtrapprox \tilde{B}$ if $\tilde{A}[\alpha] \gtrsim \tilde{B}[\alpha]$ for every $\alpha \in [0,1]$, where $\tilde{A}[\alpha] \gtrsim \tilde{B}[\alpha]$ if $\frac{A_*(\alpha)+A^*(\alpha)}{2} \geq \frac{B_*(\alpha)+B^*(\alpha)}{2}$.

$\tilde{A} \backsimeq \tilde{B}$ if $\tilde{A}[\alpha] \sim \tilde{B}[\alpha]$ for every $\alpha \in [0,1]$, where $\tilde{A}[\alpha] \sim \tilde{B}[\alpha]$ if $\frac{A_*(\alpha)+A^*(\alpha)}{2} = \frac{B_*(\alpha)+B^*(\alpha)}{2}$.

For Example 4.1, $\frac{1}{2}(1 + 2\alpha + 8 - 3\alpha) = \frac{9-\alpha}{2}, \frac{1}{2}(2 + \frac{2}{3}\alpha - 2\alpha + \frac{15}{2}) = -\frac{2}{3}\alpha + \frac{19}{4}, \alpha \in [0,1]$. It is obvious that $\tilde{e}$ and $\tilde{c}_1$ are non-comparable yet by the order relation $\gtrapprox$ introduced in the above paragraph. Thus, the order relation $\gtrapprox$ is a partial order relation for fuzzy numbers.

Definition 4.2. Let $\tilde{A}, \tilde{B} \in \mathbb{FR}$.

$\tilde{A}$ is *weakly superior* to $\tilde{B}$, denoted by $\tilde{A} \succcurlyeq \tilde{B}$, if $E(\tilde{A}) \geq E(\tilde{B})$.

$\tilde{A}$ and $\tilde{B}$ are indifferent, denoted by $\tilde{A} \approx \tilde{B}$, if $E(\tilde{A}) = E(\tilde{B})$.

$\tilde{A}$ is *superior* to $\tilde{B}$, denoted by $\tilde{A} \succ \tilde{B}$, if $E(\tilde{A}) > E(\tilde{B})$.

For Example 4.1, $E(\tilde{e}) = \frac{17}{4}, E(\tilde{c}_1) = \frac{53}{12}$, we know $\tilde{e} \prec \tilde{c}_1$ by Definition 4.2. Thus, the first creditor's claim $\tilde{c}_1$ is greater than the estate $\tilde{e}$.

Particularly, for every symmetrical trapezoidal fuzzy number $\lfloor a, b, c, d \rfloor$, i.e., $b - a = d - c, L(x)$ and $R(x)$ are linear, the α-level set is $[(b-a)\alpha + a, (c-d)\alpha + d]$. Thus, the sum $a + d$ of the endpoints of the α-level set does not depend on α. In this case, the midpoint $\frac{a+d}{2}$ of the α-level set is equal to the expected value $E(\lfloor a, b, c, d \rfloor) = \frac{1}{2}\int_0^1 (a+d)d\alpha$ of the symmetrical trapezoidal fuzzy number. Thus, for symmetrical trapezoidal fuzzy numbers, the partial order relation $\gtrapprox$ in Definition 4.1 by comparing the midpoints of the α-level sets of fuzzy numbers is the total order relation $\succcurlyeq$ in Definition 4.2 by comparing the expected values of fuzzy numbers.

Remark 4.1. (i) If $\tilde{A} \succeq \tilde{B}$ as defined in Mallozzi et al. (2011, [49]), then $\tilde{A} \succcurlyeq \tilde{B}$.

(ii) For every $A, B \in \mathbb{R}$, if $A \geq B$, then $A \succcurlyeq B$.

(iii) For every $A, B \in \mathbb{IR}$, if $A \gtrsim B$ as defined in Han et al. [37], then $A \succcurlyeq B$.

From the definition of the total order relation of fuzzy numbers, it is easily seen that for every $\tilde{A}, \tilde{B} \in \mathbb{FR}$, the maximum and minimum values are defined as

$$\tilde{C} \approx \max\{\tilde{A}, \tilde{B}\} \Leftrightarrow E(\tilde{C}) = \max\{E(\tilde{A}), E(\tilde{B})\},$$

$$\tilde{C} \approx \min\{\tilde{A}, \tilde{B}\} \Leftrightarrow E(\tilde{C}) = \min\{E(\tilde{A}), E(\tilde{B})\}.$$

The $\mathcal{F}$*-core* may contain few payoff vectors, or even no one, just because of the partial order relation. Even worse, players may not know whether to take part in the grand coalition or not, since they may not be sure whether their payoff is greater than their coalitional value. The key to solve such problems is the total order relation of fuzzy numbers. Here, we propose the indifference fuzzy core of a cooperative fuzzy game based on the total order relation $\succcurlyeq$ of fuzzy numbers. In a cooperative fuzzy game $\tilde{v}$ (For details we refer to page 18), we say that a payoff vector $\tilde{x} \in \mathbb{FR}^n$ is *indifferently efficient* if $\tilde{x}(N) \approx \tilde{v}(N)$; *individually rational* if $\tilde{x}_i \succcurlyeq \tilde{v}(i)$ for all $i \in N$; and *coalitionally rational* if $\tilde{x}(S) \succcurlyeq \tilde{v}(S)$ for all $S \subseteq N$. A payoff vector $\tilde{x}$ is called an *indifference fuzzy imputation* if it is indifferently efficient and individually rational. $\mathscr{I}(N, \tilde{v})$ denotes the indifference fuzzy imputation set being the set of indifference fuzzy imputations.

Definition 4.3. The *indifference fuzzy core* $\mathscr{C}(N, \tilde{v})$ of a cooperative fuzzy game $\tilde{v} \in \mathcal{F}G^N$ (see page 18) is defined as

$$\mathscr{C}(N, \tilde{v}) = \{(\tilde{x}_1, \cdots, \tilde{x}_n) \in \mathbb{FR}^n \mid \tilde{x}(N) \approx \tilde{v}(N) \text{ and } \tilde{x}(S) \succcurlyeq \tilde{v}(S)\ \forall S \subseteq N\}.$$

Remark 4.2. (i) Obviously, it is true that $\mathscr{C}^{\mathcal{F}}(N, \tilde{v}) \subseteq \mathscr{C}(N, \tilde{v})$ (see page 18).
(ii) If the cooperative fuzzy game $\tilde{v} \in \mathcal{F}G^N$ degenerates into the classical game $v \in G^N$, $\mathscr{C}(N, \tilde{v}) = \mathscr{C}(N, v)$. If the cooperative fuzzy game $\tilde{v} \in \mathcal{F}G^N$ degenerates into the interval game $\nu \in \mathcal{I}G^N$, $\mathscr{C}(N, \tilde{v}) = \mathscr{C}'(N, \nu)$, where $\mathscr{C}'(N, \nu)$ is the indifference interval core of an interval game defined by Han et al. (2012, [37]) (see pages 17-18).

Example 4.2. Let the cooperative fuzzy game $\tilde{v} \in \mathcal{F}G^N$ with $N = \{1, 2, 3\}$ and $\tilde{v}(i) = \lfloor 0, 0, 0, 0 \rfloor$, $i = 1, 2, 3$, $\tilde{v}(12) = \lfloor 1, 2, 2, 3 \rfloor, \tilde{v}(23) = \lfloor 1, 1, 2, 2 \rfloor$, $\tilde{v}(13) = \lfloor 4, 5, 6, 7 \rfloor$, $\tilde{v}(123) = \lfloor 4, 5, 7, 8 \rfloor$. On Table 4.1, we give the coali-

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{12\}$	$\{13\}$	$\{23\}$	N
$E(\tilde{v}(S))$	0	0	0	0	2	$\frac{11}{2}$	$\frac{3}{2}$	6

TABLE 4.1 Coalitional expected values in Example 4.2

tional expected values of $\tilde{v}$.

We may check the fuzzy payoff vector $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \in \mathscr{C}(N, \tilde{v})$, where $\tilde{x}_1 = \lfloor 0, 1, 3, 6 \rfloor$, $\tilde{x}_2 = \lfloor 0, 0, 1, 1 \rfloor$ and $\tilde{x}_3 = \lfloor 2, 3, 3, 4 \rfloor$. On Table 4.2, we give

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{12\}$	$\{13\}$	$\{23\}$	N
$E(\tilde{x}(S))$	0	$\frac{5}{2}$	$\frac{1}{2}$	3	3	$\frac{11}{2}$	$\frac{7}{2}$	6

TABLE 4.2 Coalitional expected payoffs given by $\tilde{x}$ in Example 4.2

the coalitional expected payoffs given by $\tilde{x}$. By Tables 4.1 and 4.2, it readily follows that $\tilde{x}_1 \tilde{+} \tilde{x}_2 \tilde{+} \tilde{x}_3 \approx \tilde{v}(123), \tilde{x}_i \succ \tilde{v}(i), \tilde{x}_1 \tilde{+} \tilde{x}_2 \succ \tilde{v}(12), \tilde{x}_1 \tilde{+} \tilde{x}_3 \approx \tilde{v}(13)$ and $\tilde{x}_2 \tilde{+} \tilde{x}_3 \succ \tilde{v}(23)$. However, $\tilde{x}_1 \tilde{+} \tilde{x}_2$ and $\tilde{v}(12)$ are non-comparable by the partial order relation $\succeq$ of fuzzy numbers since $(\tilde{x}_1 \tilde{+} \tilde{x}_2)[\alpha] = [\alpha, 7 - 3\alpha]$ and $\tilde{v}(12)[\alpha] = [\alpha + 1, 3 - \alpha]$. Thus, $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) \notin \mathscr{C}^{\mathcal{F}}(N, \tilde{v})$.

From Example 4.2, we see that $\mathscr{C}(N, \tilde{v}) \subseteq \mathscr{C}^{\mathcal{F}}(N, \tilde{v})$ need not be true for every $\tilde{v} \in \mathcal{F}G^N$, meaning that there exist payoff vectors in the indifference fuzzy core that do not belong to the $\mathcal{F}$-core defined by Mallozzi et al. (2011, [49]). In other words, players determine to join in the grand coalition, since their payoffs from those allocations in the indifference fuzzy core are greater than the worths acquired on their own by the total order relation $\succcurlyeq$ of fuzzy numbers.

The *expected game* (N, v_E) of $\tilde{v} \in \mathcal{F}G^N$, defined by $v_E(S) = E(\tilde{v}(S))$ for every $S \in 2^N$, is a TU-game.

For Example 4.2, we consider the expected game (N, v_E). Obviously, the expected vector $(E(\tilde{x}_1), E(\tilde{x}_2), E(\tilde{x}_3)) = (\frac{5}{2}, \frac{1}{2}, 3) \in \mathscr{C}(N, v_E)$. Conversely, suppose that there is a fuzzy payoff vector $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ such that $(E(\tilde{y}_1), E(\tilde{y}_2), E(\tilde{y}_3)) = (\frac{5}{2}, \frac{1}{2}, 3) \in \mathscr{C}(N, v_E)$, then $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) \in \mathscr{C}(N, \tilde{v})$. It turns out that such relation holds for every cooperative fuzzy game. Therefore, the indifference fuzzy core is nonempty if and only if the associated expected game is balanced.

Theorem 4.4. *For every $\tilde{v} \in \mathcal{F}G^N$, $\mathscr{C}(N, \tilde{v}) \neq \emptyset$ if and only if $\mathscr{C}(N, v_E) \neq \emptyset$.*

Proof. Let $\tilde{v} \in \mathcal{F}G^N$.

'Only if': assume that $\mathscr{C}(N, \tilde{v}) \neq \emptyset$. There exists a fuzzy payoff vector $(\tilde{x}_1, \cdots, \tilde{x}_n)$ such that

$$\tilde{x}(N) \approx \tilde{v}(N) \text{ and } \tilde{x}(S) \succcurlyeq \tilde{v}(S) \text{ for every } S \subseteq N.$$

Let $(y_1, \cdots, y_n) = (E(\tilde{x}_1), \cdots, E(\tilde{x}_n))$. We have

$$\sum_{i \in N} y_i = \sum_{i \in N} E(\tilde{x}_i) = E(\tilde{\sum_{i \in N}} \tilde{x}_i) = E(\tilde{v}(N)) = v_E(N),$$

and

$$\sum_{i \in S} y_i = \sum_{i \in S} E(\tilde{x}_i) = E(\tilde{\sum_{i \in S}} \tilde{x}_i) \geq E(\tilde{v}(S)) = v_E(S),$$

for every $S \subseteq N$. Hence, $\mathscr{C}(N, v_E) \neq \emptyset$.

'If': let $\mathscr{C}(N, v_E) \neq \emptyset$. Then, there exists a real payoff vector $(x_1, \cdots, x_n)$ such that

$$\sum_{i \in N} x_i = v_E(N) \text{ and } \sum_{i \in S} x_i \geq v_E(S) \text{ for every } S \subseteq N.$$

Let $(\tilde{z}_1, \cdots, \tilde{z}_n)$ with $\tilde{z}_i = \lfloor \frac{x_i}{2}, x_i, x_i, \frac{3x_i}{2} \rfloor$ for every $i \in N$. It is easily seen that $E(\tilde{z}_i) = x_i$ for each $i \in N$. Then,

$$E(\tilde{\sum_{i \in N}} \tilde{z}_i) = \sum_{i \in N} E(\tilde{z}_i) = \sum_{i \in N} x_i = v_E(N) = E(\tilde{v}(N)),$$

and

$$E(\tilde{\sum_{i \in S}} \tilde{z}_i) = \sum_{i \in S} E(\tilde{z}_i) = \sum_{i \in S} x_i \geq v_E(S) = E(\tilde{v}(S)),$$

which implies

$$\tilde{\sum_{i\in N}}\tilde{z}_i \approx \tilde{v}(N) \text{ and } \tilde{\sum_{i\in S}}\tilde{z}_i \succcurlyeq \tilde{v}(S).$$

Therefore, $\mathscr{C}(N,\tilde{v}) \neq \emptyset$. □

We say that $\tilde{v} \in \mathcal{F}G^N$ is convex if

$$\tilde{v}(S)\tilde{+}\tilde{v}(T) \preccurlyeq \tilde{v}(S\cup T)\tilde{+}\tilde{v}(S\cap T) \text{ for every } S,T \in 2^N.$$

The class of all *convex cooperative fuzzy games* is denoted by $\mathcal{C}^N$.

Shapley (1971, [76]) established that the core of a convex TU-game is nonempty. By Theorem 4.4, we know that the indifference fuzzy core is nonempty if the expected game is convex. It turns out that if the cooperative fuzzy game is convex, then the expected game is convex.

Theorem 4.5. *If $\tilde{v} \in \mathcal{C}^N$, then the expected game (N, v_E) of $\tilde{v}$ is a convex game.*

Proof. Since $\tilde{v} \in \mathcal{C}^N$, it holds that for all $S,T \in 2^N$,

$$\tilde{v}(S)\tilde{+}\tilde{v}(T) \preccurlyeq \tilde{v}(S\cup T)\tilde{+}\tilde{v}(S\cap T).$$

By the properties of the expected value and the definition of $\preccurlyeq$, for all $S,T \subseteq N$,

$$v_E(S) + v_E(T) \leq v_E(S\cup T) + v_E(S\cap T),$$

which means that (N, v_E) is convex. Thus, $\mathscr{C}(N, v_E) \neq \emptyset$. □

From Theorems 4.4 and 4.5, we obtain the following immediate result.

Corollary 4.6. *$\mathscr{C}(N,\tilde{v}) \neq \emptyset$ for every $\tilde{v} \in \mathcal{C}^N$.*

Definition 4.7. For every $\tilde{v} \in \mathcal{F}G^N$, we say that $\tilde{v}$ is *balanced* if

$$\tilde{v}(N) \succcurlyeq \tilde{\sum_{S\subseteq N}}\lambda(S)\tilde{v}(S)$$

for every map $\lambda : 2^N \to \mathbb{R}_+$ with $\sum_{\substack{S\subseteq N\\ i\in S}} \lambda(S) = 1$ for all $i \in N$.

Remark 4.3. Definition 4.7 is in line with $\mathcal{F}$-balancedness introduced by Mallozzi et al. (2011, [49]), $\mathcal{I}$-balancedness defined by Alparslan Gök et al. (2008, [4]), and balancedness presented by Bondareva (1962, [15]) and Shapley (1967, [75]).

Example 4.3. Let $\tilde{v} \in \mathcal{F}G^N$ with $N = \{1,2,3\}$ and $\tilde{v}(i) = \lfloor 0,0,0,0 \rfloor, i = 1,2,3$, $\tilde{v}(12) = \lfloor 8,8,9,9 \rfloor$, $\tilde{v}(23) = \lfloor 7,8,9,10 \rfloor$, $\tilde{v}(13) = \lfloor 4,6,10,14 \rfloor$, $\tilde{v}(123) = \lfloor 10,12,14,16 \rfloor$. We check all the minimal balanced collections[1]:

$$1 \cdot \tilde{v}(1) \tilde{+} 1 \cdot \tilde{v}(2) \tilde{+} 1 \cdot \tilde{v}(3) = \lfloor 0,0,0,0 \rfloor \preccurlyeq \tilde{v}(123) = \lfloor 10,12,14,16 \rfloor;$$
$$1 \cdot \tilde{v}(1) \tilde{+} 1 \cdot \tilde{v}(23) = \lfloor 7,8,9,10 \rfloor \preccurlyeq \tilde{v}(123) = \lfloor 10,12,14,16 \rfloor;$$
$$1 \cdot \tilde{v}(3) \tilde{+} 1 \cdot \tilde{v}(12) = \lfloor 8,8,9,9 \rfloor \preccurlyeq \tilde{v}(123) = \lfloor 10,12,14,16 \rfloor;$$
$$1 \cdot \tilde{v}(2) \tilde{+} 1 \cdot \tilde{v}(13) = \lfloor 4,6,10,14 \rfloor \preccurlyeq \tilde{v}(123) = \lfloor 10,12,14,16 \rfloor;$$
$$\frac{1}{2} \cdot \tilde{v}(12) \tilde{+} \frac{1}{2} \cdot \tilde{v}(23) \tilde{+} \frac{1}{2} \cdot \tilde{v}(13) = \lfloor \frac{19}{2}, 11, 14, \frac{33}{2} \rfloor \preccurlyeq \tilde{v}(123) = \lfloor 10,12,14,16 \rfloor.$$

So $\tilde{v}$ is balanced. However, $(\frac{1}{2} \cdot \tilde{v}(12) \tilde{+} \frac{1}{2} \cdot \tilde{v}(23) \tilde{+} \frac{1}{2} \cdot \tilde{v}(13))[\alpha] = \lfloor \frac{19}{2}, 11, 14, \frac{33}{2} \rfloor[\alpha] = [\frac{3}{2}\alpha + \frac{19}{2}, \frac{33}{2} - \frac{5}{2}\alpha], \tilde{v}(123)[\alpha] = [10 + 2\alpha, 16 - 2\alpha]$. Obviously, $\lfloor \frac{19}{2}, 11, 14, \frac{33}{2} \rfloor$ and $\tilde{v}(123)$ are non-comparable by the partial order relation $\succeq$ of fuzzy numbers. Thus, $\tilde{v}$ is not $\mathcal{F}$-balanced.

Furthermore, we consider the expected game (N, v_E) of the game $\tilde{v}$, which coalitional values are given in Table 4.3. It can be verified that

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{12\}$	$\{13\}$	$\{23\}$	N
$v_E(S)$	0	0	0	0	8.5	8.5	8.5	13

TABLE 4.3 Expected game in Example 4.3

$(\frac{13}{3}, \frac{13}{3}, \frac{13}{3}) \in \mathscr{C}(N, v_E)$. By Theorem 4.4, it directly follows that $\mathscr{C}(N, \tilde{v}) \neq \emptyset$. We find that the conclusion is true for every $\tilde{v} \in \mathcal{F}G^N$.

Theorem 4.8. *For every $\tilde{v} \in \mathcal{F}G^N$, $\mathscr{C}(N, \tilde{v}) \neq \emptyset$ if and only if $\tilde{v}$ is balanced.*

[1]A coalition $\{S\} \subseteq 2^N$ is minimally balanced if it is balanced and does not contain a proper balanced subcollection. It is known that a TU-game is balanced if and only if all minimally balanced inequalities are satisfied (c.f. Shapley (1967, [75])). Following the same line as in the proof for TU-games, it is easily seen that a cooperative fuzzy game is balanced if and only if all minimally balanced inequalities are satisfied.

Proof. Let $\tilde{v} \in \mathcal{F}G^N$. From Theorem 4.4, $\mathscr{C}(N, \tilde{v}) \neq \emptyset$ if and only if $C(N, v_E) \neq \emptyset$, which is equivalent to (N, v_E) being balanced, i.e.,

$$v_E(N) \geq \sum_{S \subseteq N} \lambda(S) v_E(S)$$

for each map $\lambda : 2^N \to \mathbb{R}_+$ with $\sum_{\substack{S \subseteq N \\ i \in S}} \lambda(S) = 1$ for all $i \in N$. This is equivalent to

$$\tilde{v}(N) \succcurlyeq \tilde{\sum_{S \subseteq N}} \lambda(S) \tilde{v}(S)$$

for each map $\lambda : 2^N \to \mathbb{R}_+$ with $\sum_{\substack{S \subseteq N \\ i \in S}} \lambda(S) = 1$ for all $i \in N$. Thus, $\mathscr{C}(N, \tilde{v}) \neq \emptyset$ if and only if $\tilde{v}$ is balanced. □

Remark 4.4. Balancedness is a necessary and sufficient condition for non-emptiness of the indifference fuzzy core, but $\mathcal{F}$-balancedness is only a necessary condition for non-emptiness of the $\mathcal{F}$-core.

4.3 Indifference fuzzy nucleolus

Given a cooperative fuzzy game $\tilde{v}$, for every $\tilde{x} \in \mathscr{I}(N, \tilde{v})$ and every coalition $S \in 2^N$, the *excess* $\tilde{e}(S, \tilde{x})$ of S at $\tilde{x}$ is defined to be

$$\tilde{e}(S, \tilde{x}) = \tilde{v}(S) \tilde{-} \tilde{x}(S).$$

It can be easily checked that for every payoff vector $\tilde{x} \in \mathbb{FR}^n$, if $\tilde{x} \in \mathscr{C}(N, \tilde{v})$, then $\tilde{e}(S, \tilde{x}) \preccurlyeq \tilde{0}$ for all $S \subseteq N$, where $\tilde{e}(S, \tilde{x}) \approx \tilde{0}$ implies that $E(\tilde{e}(S, \tilde{x})) = 0$, i.e., $E(\tilde{v}(S)) = E(\tilde{x}(S))$, which means that $\tilde{v}(S) \approx \tilde{x}(S)$.

The nucleolus of TU-games defined by Schmeidler (1969, [71]) consists of payoff vectors which minimize the excesses of coalitions in the lexicographic order over the compact convex imputation set. We define the indifference fuzzy nucleolus for cooperative fuzzy games. First, we define the fuzzy lexicographic order $\preccurlyeq_L$ by the total order relation in Definition 4.2.

Let $m \in \mathbb{N}$ and $\tilde{x}, \tilde{y} \in \mathbb{FR}^m$. The fuzzy lexicographic order $\preccurlyeq_L$ is defined as follows:

(i) $\tilde{x} \prec_L \tilde{y}$ if there exists an integer $k \in \mathbb{N}, 1 \leq k \leq m$ such that $\tilde{x}_i \approx \tilde{y}_i$ for $1 \leq i < k$ and $\tilde{x}_k \prec \tilde{y}_k$.

(ii) $\tilde{x} \preccurlyeq_L \tilde{y}$ if $\tilde{x} \prec_L \tilde{y}$ or $\tilde{x} \approx \tilde{y}$.

Moreover, $\tilde{\theta}(\tilde{x})$ is the vector where the coordinates of $\tilde{x}$ are ordered in nonincreasing order: $\tilde{\theta}_1(\tilde{x}) \succcurlyeq \tilde{\theta}_2(\tilde{x}) \succcurlyeq \cdots \succcurlyeq \tilde{\theta}_m(\tilde{x})$.

Definition 4.9. For every cooperative fuzzy game $\tilde{v} \in \mathcal{FG}^N$, the *indifference fuzzy nucleolus* is the set of fuzzy payoff vectors that lexicographically minimizes the vector of fuzzy excesses $(\tilde{e}(S, \tilde{x}))_{S \subseteq N}$ over the imputation set

$$\begin{aligned} \mathscr{N}(N, \tilde{v}) \quad = \quad & \{\tilde{x} \in \mathscr{I}(N, \tilde{v}) | \tilde{\theta}(\tilde{e}(S, \tilde{x}))_{S \subseteq N} \preccurlyeq_L \tilde{\theta}(\tilde{e}(S, \tilde{y}))_{S \subseteq N} \\ & \forall \tilde{y} \in \mathscr{I}(N, \tilde{v})\}. \end{aligned}$$

Remark 4.5. The indifference fuzzy nucleolus of a cooperative fuzzy game that degenerates into a TU-game coincides with the nucleolus of that TU-game.

Theorem 4.10. *For every $\tilde{v} \in \mathcal{FG}^N$, $\mathscr{N}(N, \tilde{v}) \neq \emptyset$.*

Proof. Let $X_0 = \mathscr{I}(N, \tilde{v})$. Denote

$$S_1 \in \arg \min_{\tilde{x} \in X_0} \max_{S \in 2^N} \tilde{e}(S, \tilde{x}) \text{ and } \tilde{\theta}_1(\tilde{x}) \approx \tilde{e}(S_1, \tilde{x}),$$

$$X_1 = \{\tilde{x} \in X_0 \mid \max_{S \in 2^N} \tilde{e}(S, \tilde{x}) \approx \tilde{\theta}_1(\tilde{x})\},$$

$$S_2 \in \arg \min_{\tilde{x} \in X_1} \max_{S \in 2^N \setminus S_1} \tilde{e}(S, \tilde{x}) \text{ and } \tilde{\theta}_2(\tilde{x}) \approx \tilde{e}(S_2, \tilde{x}),$$

$$X_2 = \{\tilde{x} \in X_1 \mid \max_{S \in 2^N \setminus S_1} \tilde{e}(S, \tilde{x}) \approx \tilde{\theta}_2(\tilde{x})\},$$

$$\cdots$$

$$S_i \in \arg \min_{\tilde{x} \in X_{i-1}} \max_{S \in 2^N \setminus \{S_1, S_2, \cdots, S_{i-1}\}} \tilde{e}(S, \tilde{x}) \text{ and } \tilde{\theta}_i(\tilde{x}) \approx \tilde{e}(S_i, \tilde{x}),$$

$$X_i = \{\tilde{x} \in X_{i-1} \mid \max_{S \in 2^N \setminus \{S_1, S_2, \cdots, S_{i-1}\}} \tilde{e}(S, \tilde{x}) \approx \tilde{\theta}_i(\tilde{x})\}.$$

Notice that for any $i \in N$, X_i, S_i are well-defined, since X_0 is a compact set and $\tilde{e}(S, \tilde{x})$ is continuous on $\tilde{x}$ for each $S \in 2^N \setminus \{\emptyset\}$. Since n is a finite number, it holds that

$$S_{2^n} = \arg \min_{\tilde{x} \in X_{2^n-1}} \max_{S \in 2^N \setminus \{S_1, S_2, \cdots, S_{2^n-1}\}} \tilde{e}(S, \tilde{x}),$$

and

$$\tilde{\theta}_{2^n}(\tilde{x}) \approx \tilde{e}(S_{2^n}, \tilde{x}),$$

$$X_{2^n} = \{\tilde{x} \in X_{2^n-1} \mid \max_{S \in 2^N \setminus \{S_1, S_2, \cdots, S_{2^n-1}\}} \tilde{e}(S, \tilde{x}) \approx \tilde{\theta}_{2^n}(\tilde{x})\}.$$

Therefore, X_{2^n} is the indifference fuzzy nucleolus of the cooperative fuzzy game $\tilde{v}$. □

From the proof of Theorem 4.10, we deduce the following corollary immediately.

Corollary 4.11. *For every $\tilde{v} \in \mathcal{F}G^N$, if $\tilde{x}, \tilde{y} \in \mathscr{N}(N, \tilde{v})$, then $\tilde{x} \approx \tilde{y}$.*

Remark 4.6. The indifference fuzzy nucleolus of a cooperative fuzzy game is nonempty. Unlike for TU-games, it may consist more than one point, since two indifferent fuzzy payoff vectors may belong to the indifference fuzzy nucleolus.

Theorem 4.12. *For every $\tilde{v} \in \mathcal{F}G^N$ with $\mathscr{C}(N, \tilde{v}) \neq \emptyset$, $\mathscr{N}(N, \tilde{v}) \subseteq \mathscr{C}(N, \tilde{v})$.*

Proof. Suppose that there exists an indifference fuzzy imputation satisfying $\tilde{x}^* \in \mathscr{N}(N, \tilde{v})$ with $\tilde{x}^* \notin \mathscr{C}(N, \tilde{v})$. Let $\tilde{\theta}(\tilde{x}^*) \approx (\tilde{\theta}_1(\tilde{x}^*), \tilde{\theta}_2(\tilde{x}^*), \cdots, \tilde{\theta}_{2^n}(\tilde{x}^*))$, where $\tilde{\theta}_i(\tilde{x}^*) := \tilde{e}(S_i, \tilde{x}^*)$, $S_i \in 2^N$ defined as in the proof of Theorem 4.10. Since $\tilde{x}^* \notin \mathscr{C}(N, \tilde{v})$, there exists a coalition $S \in 2^N$ satisfying $\tilde{e}(S, \tilde{x}^*) \succ \tilde{0}$. Moreover, there exists an indifference fuzzy imputation $\tilde{y} \in \mathscr{C}(N, \tilde{v})$ such that for every $T_i \in 2^N$, $\tilde{e}(T_i, \tilde{y}) \preccurlyeq \tilde{0}$ and $\tilde{\theta}(\tilde{y}) \approx (\tilde{\theta}_1(\tilde{y}), \tilde{\theta}_2(\tilde{y}), \cdots, \tilde{\theta}_{2^n}(\tilde{y}))$, where $\tilde{\theta}_i(\tilde{y}) := \tilde{e}(T_i, \tilde{y})$. It is concluded that $\tilde{e}(S, \tilde{x}^*) \preccurlyeq \tilde{\theta}_1(\tilde{x}^*) \preccurlyeq \tilde{\theta}_1(\tilde{y}) \preccurlyeq \tilde{0}$ which establishes a contradiction to our premise $\tilde{x}^* \in \mathscr{N}(N, \tilde{v})$. Hence, $\tilde{x}^* \in \mathscr{C}(N, \tilde{v})$. □

4.4 Indifference fuzzy bargaining sets

4.4.1 Two types of indifference fuzzy bargaining sets

In this section, we pay attention to two indifference fuzzy bargaining sets based on *objections* and *counter-objections*. Let $i, j \in N$ be such that $i \neq j$. The set of all coalitions containing player i, but not player j, is denoted by $\Gamma_{ij} = \{S \subseteq N \mid i \in S, j \notin S\}$.

For every $\tilde{v} \in \mathcal{F}G^N$ and $\tilde{x} \in \mathscr{I}(N, \tilde{v})$, we say that an objection of i against j at the indifference fuzzy imputation $\tilde{x}$ in the cooperative fuzzy game $\tilde{v}$ is a pair $(S, \tilde{y})$ where $S \in \Gamma_{ij}$ and $\tilde{y} = (\tilde{y}_k)_{k\in S}$ satisfying

$$\tilde{y}(S) \approx \tilde{v}(S), \tag{4.1}$$

$$\tilde{y}_k \succ \tilde{x}_k \text{ for all } k \in S. \tag{4.2}$$

We further say that a counter-objection of j to the objection $(S, \tilde{y})$ of i at $\tilde{x}$ is a pair $(T, \tilde{z})$ where $T \in \Gamma_{ji}$ and $\tilde{z} = (\tilde{z}_k)_{k\in T}$ satisfying

$$\tilde{z}(T) \approx \tilde{v}(T), \tag{4.3}$$

$$\tilde{z}_k \succcurlyeq \tilde{y}_k \text{ for all } k \in T \cap S, \tag{4.4}$$

$$\tilde{z}_k \succcurlyeq \tilde{x}_k \text{ for all } k \in T \setminus S. \tag{4.5}$$

Definition 4.13. The *Davis-Maschler indifference fuzzy bargaining set* $\mathscr{M}_1^{ind}(N, \tilde{v})$ of the game $\tilde{v}$ is defined by

$$\mathscr{M}_1^{ind}(N, \tilde{v}) = \{\tilde{x} \in \mathscr{I}(N, \tilde{v}) \mid \text{no individual player has a justified objection at } \tilde{x}\},$$

where a justified objection is an objection that has no counter-objection.

For every $\tilde{v} \in \mathcal{F}G^N$ and $\tilde{x} \in \mathscr{I}(N, \tilde{v})$, we say that an objection of coalition S at $\tilde{x}$ in the cooperative fuzzy game $\tilde{v}$ is a pair $(S, \tilde{y})$ where S is a nonempty coalition, $\tilde{y} = (\tilde{y}_k)_{k\in S}$ satisfying (4.1) with

$$\tilde{y}_k \succcurlyeq \tilde{x}_k \text{ for all } k \in S, \tag{4.6}$$

and at least one of the inequalities in (4.6) is strict. We further say that a counter-objection of coalition T to the objection $(S, \tilde{y})$ at $\tilde{x}$ is a pair $(T, \tilde{z})$ where T is a nonempty coalition, $\tilde{z} = (\tilde{z}_k)_{k \in T}$ satisfying (4.3),(4.4),(4.5) and at least one of the inequalities in (4.4) or (4.6) is strict.

Definition 4.14. The *Mas-Colell indifference fuzzy bargaining set* $\mathscr{M}\mathscr{B}(N, \tilde{v})$ of the game $\tilde{v}$ is defined by

$$\mathscr{M}\mathscr{B}(N, \tilde{v}) = \{\tilde{x} \in \mathscr{I}(N, \tilde{v}) \mid \text{no nonempty coalition} \\ \text{has a justified objection at } \tilde{x}\},$$

where a justified objection is an objection that has no counter-objection.

Remark 4.7. Observe that for every $\tilde{v} \in \mathcal{F}G^N$, a pair $(S, \tilde{y})$ can be used as an objection by the players of S or coalition S at $\tilde{x}$ if and only if $\tilde{e}(S, \tilde{y}) \succ \tilde{0}$. Furthermore, a pair (T, z) is both types of counter-objections at $\tilde{y}$ if and only if $\tilde{e}(T, \tilde{z}) \not\succcurlyeq \tilde{0}$. Thus, $\mathscr{C}(N, \tilde{v}) \subseteq \mathscr{M}_1^{ind}(N, \tilde{v})$ and $\mathscr{C}(N, \tilde{v}) \subseteq \mathscr{M}\mathscr{B}(N, \tilde{v})$.

These two indifference fuzzy bargaining sets of a cooperative fuzzy game that degenerates into a TU-game conincide with the corresponding bargaining sets of that TU-game defined by Davis and Maschler (1963, [24]) and Mas-Colell (1989, [52]) (see pages 12-13).

4.4.2 Equality between solutions on convex cooperative fuzzy games

Maschler et al. (1972, [51]) proved that the Davis-Maschler bargaining set (see page 12) of a convex TU-game equals the core of the game. In this section, we prove that both the Davis-Maschler indifference fuzzy bargaining set and the Mas-Colell indifference fuzzy bargaining set are equal to the indifference fuzzy core for convex cooperative fuzzy games. The proofs are based on an excess fuzzy game and a monotonic fuzzy cover.

Given a payoff vector $\tilde{x} \in \mathbb{FR}^n$, the *excess fuzzy game* $(N, \tilde{\omega}_{\tilde{x}})$ of $\tilde{v} \in \mathcal{F}G^N$ is defined by

$$\tilde{\omega}_{\tilde{x}}(S) = \tilde{e}(S, \tilde{x}) \text{ for all } S \subseteq N.$$

Notice that, if $\tilde{v}$ is convex, then $(N, \tilde{\omega}_{\tilde{x}})$ is convex for every $\tilde{x} \in \mathbb{FR}^n$ as well.

The *monotonic fuzzy cover* $(N, \tilde{v}_m)$ of $\tilde{v} \in \mathcal{F}G^N$ is given by

$$\tilde{v}_m(S) :\approx \max_{T \subseteq S} \tilde{v}(T) \text{ for all } S \subseteq N.$$

Proposition 4.15. *For every $\tilde{v} \in \mathcal{F}G^N$ and its associated monotonic fuzzy cover $(N, \tilde{v}_m)$, the following statements hold,*

(i) $\tilde{v}_m(S) \succcurlyeq \tilde{v}(S)$ for all $S \subseteq N$.

(ii) $\tilde{v}_m(S) \preccurlyeq \tilde{v}_m(T)$ for all $S \subseteq T \subseteq N$.

(iii) If $\tilde{v}$ is convex, then $(N, \tilde{v}_m)$ is convex.

Proof. The statements (i) and (ii) follow immediately from the definition of monotonic fuzzy cover. It remains to prove statement (iii). Let $\tilde{v}$ be convex. For every $S \subseteq N$ and $T \subseteq N$, there exists $S_1 \subseteq S, T_1 \subseteq T$ such that $\tilde{v}_m(S) \approx \tilde{v}(S_1)$ and $\tilde{v}_m(T) \approx \tilde{v}(T_1)$. It follows that

$$\tilde{v}(S_1) \tilde{+} \tilde{v}(T_1) \preccurlyeq \tilde{v}(S_1 \cup T_1) \tilde{+} \tilde{v}(S_1 \cap T_1).$$

Then,

$$\begin{array}{rcl} \tilde{v}_m(S) \tilde{+} \tilde{v}_m(T) & \approx & \tilde{v}(S_1) \tilde{+} \tilde{v}(T_1) \\ & \preccurlyeq & \tilde{v}(S_1 \cup T_1) \tilde{+} \tilde{v}(S_1 \cap T_1) \\ & \preccurlyeq & \tilde{v}_m(S_1 \cup T_1) \tilde{+} \tilde{v}_m(S_1 \cap T_1) \\ & \preccurlyeq & \tilde{v}_m(S \cup T) \tilde{+} \tilde{v}_m(S \cap T). \end{array}$$

Hence, $(N, \tilde{v}_m)$ is convex. □

Given a cooperative fuzzy game $\tilde{v}$, for every payoff vector $\tilde{x} \in \mathbb{FR}^n$, we define the maximal excess fuzzy game $(N, \tilde{\omega}_{\tilde{x},m})$ on the player set N by

$$\tilde{\omega}_{\tilde{x},m}(S) :\approx \max_{T \subseteq S} \tilde{e}(T, \tilde{x}) \text{ for all } S \subseteq N.$$

Obviously, for every payoff vector $\tilde{x} \in \mathbb{FR}^n$,

(i) $\tilde{\omega}_{\tilde{x},m}(\emptyset) = 0$;

(ii) $\tilde{\omega}_{\tilde{x},m}(S) \succcurlyeq \max\{\tilde{e}(S,\tilde{x}), \tilde{0}\}$ for all $S \subseteq N$;

(iii) $S \subseteq T$ implies $\tilde{\omega}_{\tilde{x},m}(S) \preccurlyeq \tilde{\omega}_{\tilde{x},m}(T)$, i.e., $(N, \tilde{\omega}_{\tilde{x},m})$ is monotonic;

(iv) $\tilde{x}$ is individually rational if and only if $\tilde{\omega}_{\tilde{x},m}(i) \approx \tilde{0}$ for all $i \in N$;

(v) $\tilde{x}$ is coalitionally rational if and only if $\tilde{\omega}_{\tilde{x},m}(N) \approx \tilde{0}$.

Notice that $(N, \tilde{\omega}_{\tilde{x},m})$ is the monotonic fuzzy cover of the excess fuzzy game $(N, \tilde{\omega}_{\tilde{x}})$.

The restricted game $(T, \tilde{v}_T)$ of $\tilde{v} \in \mathcal{F}G^N$ is given by $\tilde{v}_T(S) := \tilde{v}(S)$ for every $S \subseteq T \subseteq N$.

Theorem 4.16. $\mathscr{C}(N,\tilde{v}) = \mathscr{M}_1^{ind}(N,\tilde{v})$ *for every* $\tilde{v} \in \mathcal{C}^N$.

Proof. For every $\tilde{v} \in \mathcal{C}^N$, from Therorem 4.5 and Remark 4.7, it follows that $\mathscr{C}(N,\tilde{v}) \subseteq \mathscr{M}_1^{ind}(N,\tilde{v})$. Next, we need to show that $\mathscr{M}_1^{ind}(N,\tilde{v}) \subseteq \mathscr{C}(N,\tilde{v})$. It is sufficient to prove that $\mathscr{I}(N,\tilde{v}) \setminus \mathscr{C}(N,\tilde{v}) \subseteq \mathscr{I}(N,\tilde{v}) \setminus \mathscr{M}_1^{ind}(N,\tilde{v})$. Let $\tilde{x} \in \mathscr{I}(N,\tilde{v}) \setminus \mathscr{C}(N,\tilde{v})$. Choose $T \subseteq N$ as a maximal coalition of the largest fuzzy excess at the indifference fuzzy imputation $\tilde{x}$, i.e., a coalition $T \subseteq N$ satisfying that

$$\tilde{e}(T,\tilde{x}) \succcurlyeq \tilde{e}(S,\tilde{x}) \text{ for all } S \subseteq N,$$
$$\tilde{e}(T,\tilde{x}) \succ \tilde{e}(S,\tilde{x}) \text{ for all } S \subseteq N \text{ with } T \subseteq S, S \neq T.$$

Now $\tilde{x} \in \mathscr{I}(N,\tilde{v}) \setminus \mathscr{C}(N,\tilde{v})$ implies $\tilde{e}(T,\tilde{x}) \succ \tilde{0}$ and $2 \le |T| \le n-1$ for the specific coalition T. Thus, for its maximal excess fuzzy game $(T, \tilde{\omega}_{\tilde{x},m})$, it should be $\tilde{\omega}_{\tilde{x},m}(T) \succ \tilde{0}$, $\tilde{\omega}_{\tilde{x},m}(i) \approx \tilde{0}$ for all $i \in T$.

Obviously, convexity of $(T, \tilde{v})$ implies convexity of $(T, \tilde{\omega}_{\tilde{x}})$. As a result, the monotonic fuzzy cover $(T, \tilde{\omega}_{\tilde{x},m})$ is also convex by Proposition 4.15 (iii). It follows from Theorem 4.5 that $\mathscr{C}(T, \tilde{\omega}_{\tilde{x},m}) \neq \emptyset$. Then, there exists $\tilde{z} = (\tilde{z}_k)_{k \in T}$ such that $\tilde{z} \in \mathscr{C}(T, \tilde{\omega}_{\tilde{x},m})$. In particular, $\tilde{z}(T) \approx \tilde{\omega}_{\tilde{x},m}(T) \succ \tilde{0}$ and $\tilde{z}_i \succcurlyeq \tilde{\omega}_{\tilde{x},m}(i) \succcurlyeq \tilde{0}$ for all $i \in T$, from this, it is clear that there exists $i^* \in T$

with $\tilde{z}_{i^*} \succ \tilde{0}$. Define $\tilde{y}^* = (\tilde{y}_k)_{k \in T}$ by

$$\tilde{y}_i = \begin{cases} \tilde{x}_i \tilde{+} \tilde{z}_i \tilde{+} \tilde{\beta} & \text{if } i \in T, i \neq i^*, \\ \tilde{x}_i \tilde{+} \tilde{z}_i \tilde{-} (|T| - 1)\tilde{\beta} & \text{if } i = i^*, \end{cases}$$

where $\tilde{\beta}$ is a fuzzy number satisfying $\tilde{0} \prec \tilde{\beta} \prec (|T| - 1)^{-1} \tilde{z}_{i^*}$, and then,

$$E(\tilde{y}_i) = \begin{cases} E(\tilde{x}_i) + E(\tilde{z}_i) + E(\tilde{\beta}) & \text{if } i \in T, i \neq i^*, \\ E(\tilde{x}_i) + E(\tilde{z}_i) - (|T| - 1)E(\tilde{\beta}) & \text{if } i = i^*, \end{cases}$$

where $E(\tilde{\beta})$ is a real number satisfying $0 < E(\tilde{\beta}) < (|T| - 1)^{-1} E(\tilde{z}_{i^*})$. For the selected coalition T,

$$\begin{aligned} &E(\tilde{y}(T)) = E(\tilde{x}(T)) + E(\tilde{\omega}_{\tilde{x},m}(T)) = E(\tilde{v}(T)),\ T \subseteq N, \\ &E(\tilde{y}_i) > E(\tilde{x}_i) \text{ if } i \in T, \end{aligned}$$

which implies

$$\begin{aligned} &\tilde{y}(T) \approx \tilde{v}(T),\ T \subseteq N, \\ &\tilde{y}_i \succ \tilde{x}_i \text{ if } i \in T. \end{aligned}$$

From this, we deduce that $(T, \tilde{y}^*)$ is an objection of player i^* against any player in $N \backslash T$ at the indifference fuzzy imputation $\tilde{x}$ in the cooperative fuzzy game $\tilde{v}$. Next, we show that there exists no counter-objection to the above objection $(T, \tilde{y}^*)$. Consider any coalition $R \subseteq N \backslash i^*$ satisfying $R \cap (N \backslash T) \neq \emptyset$. Because $R \cup T \neq T$, the coalition T specified above yields

$$\tilde{e}(R \cup T, \tilde{x}) \prec \tilde{e}(T, \tilde{x}).$$

The strict inequality and convexity of $\tilde{v} \in \mathcal{F}G^N$ imply

$$\tilde{e}(R, \tilde{x}) \preccurlyeq \tilde{e}(R \cap T, \tilde{x}) \tilde{+} \tilde{e}(R \cup T, \tilde{x}) \tilde{-} \tilde{e}(T, \tilde{x}) \prec \tilde{e}(R \cap T, \tilde{x}),$$

and

$$\tilde{e}(R \cap T, \tilde{x}) = \tilde{\omega}_{\tilde{x}}(R \cap T) \preccurlyeq \tilde{\omega}_{\tilde{x},m}(R \cap T) \preccurlyeq \tilde{z}(R \cap T).$$

Then,

$$\begin{aligned}\tilde{e}(R,\tilde{x}) &\prec \tilde{e}(R\cap T,\tilde{x}) \preccurlyeq \tilde{z}(R\cap T)\\ &= \tilde{y}(R\cap T)\tilde{-}\tilde{x}(R\cap T)\tilde{-}|R\cap T|\tilde{\beta}\\ &\preccurlyeq \tilde{y}(R\cap T)\tilde{-}\tilde{x}(R\cap T).\end{aligned}$$

It follows that

$$\begin{aligned}\tilde{v}(R) &= \tilde{e}(R,\tilde{x})\tilde{+}\tilde{x}(R)\\ &\prec \tilde{y}(R\cap T)\tilde{-}\tilde{x}(R\cap T)\tilde{+}\tilde{x}(R)\\ &= \tilde{y}(R\cap T)\tilde{+}\tilde{x}(R\backslash T).\end{aligned}$$

The strict inequality expresses that the coalition R cannot be used for a counter-objection. Since R is an arbitrary coalition including at least one player in $N\backslash T$ except for player i^*, we may conclude that there exists no counter-objection to the above objection $(T,\tilde{y}^*)$ of player i^* against any player in $N\backslash T$ with respect to the indifference fuzzy imputation $\tilde{x}$ in the cooperative fuzzy game $\tilde{v}$. Therefore, $\tilde{x}\notin \mathscr{M}_1^{ind}(N,\tilde{v})$. $\square$

Following the same lines as in the proof of Theorem 4.16, we have the following result for the Mas-Colell indifference fuzzy bargaining set.

Theorem 4.17. $\mathscr{C}(N,\tilde{v})=\mathscr{M}\mathscr{B}(N,\tilde{v})$ *for every* $\tilde{v}\in\mathcal{C}^N$.

From Theorem 4.16 and Theorem 4.17, we have the following immediate consequence.

Corollary 4.18. $\mathscr{C}(N,\tilde{v})=\mathscr{M}_1^{ind}(N,\tilde{v})=\mathscr{M}\mathscr{B}(N,\tilde{v})$ *for every* $\tilde{v}\in\mathcal{C}^N$.

4.5 Characterizing the class of superadditive cooperative fuzzy games for which the indifference fuzzy core and the indifference fuzzy bargaining sets coincide

In Subsection 4.5.2, we have shown that the indifference fuzzy core and the indifference fuzzy bargaining sets of convex cooperative fuzzy games coincide. Convexity is stronger than superadditivity for cooperative fuzzy games. In this section, we characterize the class of superadditive cooperative fuzzy games for which all solution concepts in this chapter coincide in a similar way as for convex cooperative fuzzy games. The characterization generalizes the results in Solymosi (1999, [78]) for TU-games. First, we provide the definition of a superadditive cooperative fuzzy game.

We say that $\tilde{v} \in \mathcal{F}G^N$ is superadditive if

$$\tilde{v}(S)\tilde{+}\tilde{v}(T) \preccurlyeq \tilde{v}(S \cup T),$$

for all $S, T \in 2^N$ with $S \cap T = \emptyset$. Denote the set of all *superadditive cooperative fuzzy games* by $\mathcal{S}^N$.

Notice that if $\tilde{v}$ is superadditive, then $(N, \tilde{\omega}_{\tilde{x}})$ is superadditive for every $\tilde{x} \in \mathbb{FR}^n$.

Theorem 4.19. *Let $\tilde{v} \in \mathcal{S}^N$ and $\tilde{x} \in \mathscr{M}_1^{ind}(N, \tilde{v})$. $\tilde{x} \in \mathscr{C}(N, \tilde{v})$ if and only if the maximal excess fuzzy game $(N, \tilde{\omega}_{\tilde{x},m})$ induced by $\tilde{x}$ is balanced.*

Proof. For each $\tilde{v} \in \mathcal{S}^N$, it is easily seen that $\mathscr{I}(N, \tilde{v}) \neq \emptyset$ and $\mathscr{M}_1^{ind}(N, \tilde{v}) \neq \emptyset$.

'Only if': If $\tilde{x} \in \mathscr{C}(N, \tilde{v})$, then $\tilde{\omega}_{\tilde{x},m}(S) \approx \tilde{0}$ for all $S \subseteq N$ and the induced game is trivially balanced.

'If': we proceed by contraposition and contradiction. Let $\tilde{x} \notin \mathscr{C}(N, \tilde{v})$. Suppose $(N, \tilde{\omega}_{\tilde{x},m})$ is balanced, in which case $\tilde{\omega}_{\tilde{x},m}(N) \succ \tilde{0}$. Let $\tilde{u} \in \mathscr{C}(N, \tilde{\omega}_{\tilde{x},m})$. From $\tilde{u}_i \succcurlyeq \tilde{0}$ and $\tilde{u}(N) \approx \tilde{\omega}_{\tilde{x},m}(N) \succ \tilde{0}$, it follows that the set $P = \{i \in N \mid \tilde{u}_i \succ \tilde{0}\}$ is nonempty. For every $S \subseteq N$ with $\tilde{e}(S, \tilde{x}) \approx \tilde{\omega}_{\tilde{x},m}(N)$, it should be that $P \subseteq S$. In fact, $\tilde{e}(S, \tilde{x}) \preccurlyeq \tilde{\omega}_{\tilde{x},m}(S) \preccurlyeq$

$\tilde{u}(S) \preccurlyeq \tilde{u}(S)\tilde{+}\tilde{u}(N\backslash S) = \tilde{u}(N) \approx \tilde{\omega}_{\tilde{x},m}(N) \approx \tilde{e}(S,\tilde{x})$ implies $\tilde{u}_j \approx \tilde{0}$ for all $j \in N\backslash S$.

Let $\hat{S}$ be a maximal coalition for which $\tilde{e}(\hat{S},\tilde{x}) \approx \tilde{\omega}_{\tilde{x},m}(N)$. Clearly, $\emptyset \neq \hat{S} \neq N$ and $\tilde{e}(\hat{S},\tilde{x}) \approx \tilde{u}(\hat{S})$. Moreover, $\tilde{v}$ superadditive implies that for $T \neq \emptyset$ with $T \cap \hat{S} = \emptyset$, $\tilde{e}(\hat{S},\tilde{x}) \succ \tilde{e}(\hat{S} \cup T,\tilde{x}) \succcurlyeq \tilde{e}(\hat{S},\tilde{x})\tilde{+}\tilde{e}(T,\tilde{x})$. Thus, $\tilde{e}(T,\tilde{x}) \prec \tilde{0}$ for every nonempty $T \subseteq N$ with $T \cap \hat{S} = \emptyset$. Fix $i \in P \subseteq \hat{S}$, and define the vector $\tilde{y} = (\tilde{y}_k)_{k\in\hat{S}}$ by

$$\begin{aligned}
\tilde{y}_i &= \tilde{x}_i\tilde{+}\frac{\tilde{u}_i}{|\hat{S}|},\\
\tilde{y}_k &= \tilde{x}_k\tilde{+}\tilde{u}_k\tilde{+}\frac{\tilde{u}_i}{|\hat{S}|} \text{ if } k \neq i, k \in \hat{S}.
\end{aligned}$$

Since $\tilde{u}_k \succcurlyeq 0$ for every $k \in \hat{S}$ and by the above definition, it is clear that

$$\begin{aligned}
&\tilde{y}(\hat{S}) \approx \tilde{v}(\hat{S}),\\
&\tilde{y}_k \succ \tilde{x}_k \text{ for all } k \in \hat{S}.
\end{aligned}$$

Namely, $(\hat{S},\tilde{y})$ is an objection of i against j at $\tilde{x}$.

Since $\tilde{x} \in \mathscr{M}_1^{ind}(N,\tilde{v})$, there must exist $j \in N \setminus S$ with a counter-objection $(T,\tilde{z})$ for the objection $(\hat{S},\tilde{y})$ of player i. By Remark 4.7, a pair $(T,\tilde{z})$ can be used for a counter-objection if and only if $\tilde{e}(T,\tilde{z}) \succcurlyeq \tilde{0}$, which means that $T \cap \hat{S} \neq \emptyset$. It holds that

$$\begin{aligned}
\tilde{z}(T) &\succcurlyeq \tilde{x}(T\backslash\hat{S})\tilde{+}\tilde{y}(T\cap\hat{S})\\
&= \tilde{x}(T\backslash\hat{S})\tilde{+}\tilde{x}(T\cap\hat{S})\tilde{+}\tilde{u}(T\cap\hat{S})\tilde{+}\frac{\tilde{u}_i}{|\hat{S}|}|T\cap\hat{S}|\\
&\succ \tilde{x}(T)\tilde{+}\tilde{u}(T) \succcurlyeq \tilde{x}(T)\tilde{+}\tilde{e}(T,\tilde{x}) = \tilde{v}(T),
\end{aligned}$$

where the strict inequality is a direct consequence of $\tilde{u}_k \approx \tilde{0}$ for all $k \in N\backslash\hat{S}$ and $\tilde{u}_i \succ \tilde{0}$. We establish a contradiction to the premise $\tilde{z}(T) \approx \tilde{v}(T)$. Therefore, we conclude that every $\tilde{x} \in \mathscr{M}_1^{ind}(N,\tilde{v})$ for which $(N,\tilde{\omega}_{\tilde{x},m})$ is balanced, belongs to $\mathscr{C}(N,\tilde{v})$. □

Corollary 4.20. *For every $\tilde{v} \in \mathcal{S}^N$,* *(i)* $\mathscr{C}(N, \tilde{v}) \neq \emptyset$ *if and only if* $(N, \tilde{\omega}_{\tilde{x},m})$ *is balanced for some* $\tilde{x} \in \mathscr{M}_1^{ind}(N, \tilde{v})$*;*

(ii) $\mathscr{M}_1^{ind}(N, \tilde{v}) = \mathscr{C}(N, \tilde{v}) \neq \emptyset$ *if and only if* $(N, \tilde{\omega}_{\tilde{x},m})$ *is balanced for every* $\tilde{x} \in \mathscr{M}_1^{ind}(N, \tilde{v})$*.*

For the Mas-Colell indifference fuzzy bargaining set, similar conclusions hold.

Theorem 4.21. *Let* $\tilde{v} \in \mathcal{S}^N$ *and* $\tilde{x} \in \mathscr{M}\mathscr{B}(N, \tilde{v})$*.* $\tilde{x} \in \mathscr{C}(N, \tilde{v})$ *if and only if the maximal excess fuzzy game* $(N, \tilde{\omega}_{\tilde{x},m})$ *induced by* $\tilde{x}$ *is balanced.*

Corollary 4.22. *For every* $\tilde{v} \in \mathcal{S}^N$*,*

(i) $\mathscr{C}(N, \tilde{v}) \neq \emptyset$ *if and only if* $(N, \tilde{\omega}_{\tilde{x},m})$ *is balanced for some* $\tilde{x} \in \mathscr{M}\mathscr{B}(N, \tilde{v})$*;*

(ii) $\mathscr{M}\mathscr{B}(N, \tilde{v}) = \mathscr{C}(N, \tilde{v}) \neq \emptyset$ *if and only if* $(N, \tilde{\omega}_{\tilde{x},m})$ *is balanced for every* $\tilde{x} \in \mathscr{M}\mathscr{B}(N, \tilde{v})$*.*

Holzman (2001, [41]) proved that the Davis-Maschler bargaining set of superadditive classical games is a subset of the Mas-Colell bargaining set. Here, we obtain the following theorem which extends this result to cooperative fuzzy games.

Theorem 4.23. $\mathscr{M}_1^{ind}(N, \tilde{v}) \subseteq \mathscr{M}\mathscr{B}(N, \tilde{v})$ *for every* $\tilde{v} \in \mathcal{S}^N$*.*

Proof. We proceed by contradiction. Assume that $\tilde{x} \in \mathscr{M}_1^{ind}(N, \tilde{v}) \backslash \mathscr{M}\mathscr{B}(N, \tilde{v})$. Let $(S, \tilde{y})$ be a justified objection (for details we refer to Definition 4.14) at $\tilde{x}$. Choose S as a maximal coalition among all justified objections (i.e., if $(U, \tilde{y})$ is a justified objection with $S \subseteq U$, then, $U = S$). Let $k \in S$ be such that the corresponding inequality in (4.6) is strict. According to the definition of an objection, it holds that $S \neq N$. Let $l \in N \backslash S$. We can modify $\tilde{y}$ such that its k-component decreases but the inequalities in (4.6) for the elements in $S \setminus \{k\}$ are strict while ensuring that the total payoff does not change. Then, we obtain $\tilde{y}'$ as follows

$$\tilde{\sum}_{j \in S} \tilde{y}_j' = \tilde{\sum}_{j \in S} \tilde{y}_j,$$
$$\tilde{y}_j' \succ \tilde{y}_j \succcurlyeq \tilde{x}_j \text{ for all } j \in S \backslash \{k\},$$

$$\tilde{y}_k^{'} \succ \tilde{x}_k.$$

By dividing $(\tilde{y}_k \tilde{-} \tilde{y}_k^{'})$ into $|S|-1$ equal amounts, i.e., $\tilde{q} = \frac{(\tilde{y}_k \tilde{-} \tilde{y}_k^{'})}{|S|-1}$ and setting $\tilde{y}_j^{'} = \tilde{y}_j \tilde{+} \tilde{q}$ for every $j \in S \backslash \{k\}$, it can be easily seen that $(S, \tilde{y}^{'})$ is an objection of player k against l at $\tilde{x}$ in the sense of Davis-Maschler (for details we refer to Definition 4.13). Since $\tilde{x} \in \mathscr{M}_1^{ind}(N, \tilde{v})$, there exists a counter-objection $(T, \tilde{z})$ to the objection $(S, \tilde{y}^{'})$ for player l against k at $\tilde{y}^{'}$, where $(T, \tilde{z})$ satisfies

$$\begin{aligned} &\tilde{z}(T) \approx \tilde{v}(T) \text{ and } T \in \Gamma_{lk}, \\ &\tilde{z}_j \succcurlyeq \tilde{y}_j^{'} \succ \tilde{y}_j \text{ for all } j \in T \cap S, \\ &\tilde{z}_j \succcurlyeq \tilde{x}_j \text{ for all } j \in T \backslash S. \end{aligned}$$

If $T \cap S \neq \emptyset$, then one of the inequalities above is strict and, therefore, $(T, \tilde{z})$ is also a counter-objection to $(S, \tilde{y})$ in the sense of Mas-Colell, establishing a contradiction with the choice of $(S, \tilde{y})$. If $T \cap S = \emptyset$, by superadditivity, it should be that

$$\tilde{y}(S) \tilde{+} \tilde{z}(T) \approx \tilde{v}(S) \tilde{+} \tilde{v}(T) \preccurlyeq \tilde{v}(S \cup T).$$

Then, we can find $\tilde{y}^*$ and $\tilde{z}^*$ with $\tilde{y}^* \succcurlyeq \tilde{y}$ and $\tilde{z}^* \succcurlyeq \tilde{z}$ such that $((\tilde{y}^*, \tilde{z}^*), S \cup T)$ is a justified objection in the sense of Mas-Colell, establishing a contradiction with the choice of $(S, \tilde{y})$ with S being maximal. Consequently, there exists no $\tilde{x} \in \mathscr{M}_1^{ind}(N, \tilde{v}) \backslash \mathscr{MB}(N, \tilde{v})$. Therefore, $\mathscr{M}_1^{ind}(N, \tilde{v}) \subseteq \mathscr{MB}(N, \tilde{v})$. □

The inclusion in Theorem 4.23 may be strict, as is illustrated in the following example.

Example 4.4. Let $\tilde{v} \in \mathcal{FG}^N$ with $N = \{1, 2, 3\}$. For $S \subseteq N$,

$$\tilde{v}(S) = \begin{cases} \lfloor -1, 0, 0, 1 \rfloor & \text{if } |S| = 1, \\ \lfloor \frac{1}{2}, 1, 1, \frac{3}{2} \rfloor & \text{if } |S| \geq 2. \end{cases}$$

Notice that these coalitional values are trapezoidal fuzzy numbers. One can see that $\mathscr{M}_1^{ind}(N, \tilde{v}) = \{(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)\}$, where $\tilde{x}_1 \approx \tilde{x}_2 \approx \tilde{x}_3 = \lfloor 0, \frac{1}{3}, \frac{1}{3}, \frac{2}{3} \rfloor$, while

$$\mathscr{M}\mathscr{B}(N,\tilde{v}) = \{(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3)|\tilde{x}_1\tilde{+}\tilde{x}_2\tilde{+}\tilde{x}_3 \approx \tilde{v}(123),$$
$$\tilde{x}_i \prec \tilde{y}, i=1,2,3\},$$

with $\tilde{y} = \lfloor \frac{1}{6}, \frac{1}{2}, \frac{1}{2}, \frac{5}{6} \rfloor$. Therefore, $\mathscr{M}_1^{ind}(N,\tilde{v}) \subset \mathscr{M}\mathscr{B}(N,\tilde{v})$.

4.6 Conclusion

The order relation defined in this chapter can compare fuzzy numbers that the partial order relations cannot. The relation established by the partial order relation in Mallozzi et al. (2011, [49]) for two comparable fuzzy numbers is inherited by the total order relation defined in this chapter.

To improve the $\mathcal{F}$-core of cooperative fuzzy games, we define the indifference fuzzy core in this chapter using the total order relation of fuzzy numbers. The indifference fuzzy core may be empty. We define the indifference fuzzy nucleolus and the indifference fuzzy bargaining sets based on the total order relation of fuzzy numbers. Actually, these three concepts of solutions for cooperative fuzzy games rely on the expected values of fuzzy numbers. Alternatively, we can select payoff vectors for cooperative fuzzy games by selecting payoff vectors of associated expected games and translating them back to fuzzy numbers.

TU-games and interval games are special cases of cooperative fuzzy games. The results in this chapter are generalizations of results for TU-games. If the cooperative fuzzy game degenerates into a TU-game, the introduced solution concepts can be easily identified with the corresponding solution concepts for TU-games. Moreover, when a cooperative fuzzy game degenerates into an interval game, we can define the nucleolus, bargaining sets for the interval games similarly to Definitions 4.9, 4.13, 4.14. Besides, the results obtained in this chapter can be translated to interval games.

Chapter 5

Existence of an equilibrium for pure exchange economies with fuzzy preferences

5.1 Introduction

The theory of competitive equilibrium was set up by Walras (1874, [83]). He established a system of simultaneous equations that described an economy, and also derived the equilibrium prices and equilibrium allocation. The first rigorous result on the existence of an equilibrium was reached by Wald (1951, [82]). With advances in linear programming, nonlinear analysis and game theory, some discoveries about the existence of an equilibrium were made by other researchers, like Gale (1955, [34]), McKenzie (1959, [53]), Aumann (1964, [9]), Aumann (1966, [10]), Hildenbrand (1970, [40]), Bewley (1972, [13]), Liu (2017, [48]), and Alon and Lehrer (2019, [3]). In particular, Arrow and Debreu (1954, [8]) considered the application of fixed point theory to equilibrium problems, generalizing Nash's theorem on the existence of equilibrium points for non-cooperative games (1950, [60]), and derived the existence of an equilibrium in an abstract economy, which is a variation on the notion of a non-cooperative game. Recently, the use of variational inequalities has allowed an alternative approach to the study of equilibria, c.f. Donato et al. (2008, 2016, [28–31]), Anello et al. (2010, 2012, [6, 7]), Joré et al. (2007, [42]), Milasi et al. (2019, [55]), Bianchi et al. (2019, [14]) and Milasi (2014, [54]).

In a pure exchange economy (PXE), there is a set of agents, each with an initial commodity bundle for trading. Each agent has a definite preference order (shortly preference) on the set of all commodity bundles. Moreover, it is often assumed that each agent's preference is measured by a real utility function. Thus, the agent's goal is to select the consumption vector that maximizes his utility. The concept of competitive equilibrium, as introduced by Aumann (1964, [9]), is a state of the market abiding by "the law of supply and demand". It consists of a price structure, where the total supply of each good exactly balances the total demand, and an allocation that results from trading at these prices.

It is worth mentioning that preference can be seen as an individual's attitude toward a set of consumption vectors in the economy. In classical economic theory, an agent's satisfaction degree of one consumption vector relative to another is either 0 or 1. According to a conclusion in Debreu (1954, [26]), under certain conditions, there exists a real utility function over the set of consumption vectors which is order-preserving with respect to the binary relation. Therefore, we can associate to each consumption vector a real number a that reflects the satisfaction experienced by the agent when in possession of this consumption vector. Following Nakamura (1986, [59]), the comparision of two consumption vectors may be ambiguous. For that reason, Nakamura (1986, [59]) introduced a fuzzy binary relation and showed that, under mild assumptions, there is an order-preserving fuzzy utility function. Notice that an agent's attitude is not necessarily clear or coherent when facing a variety of consumption vectors. In such a situation, a real utility function is no longer reasonable. Agent i's utility of a consumption vector is better represented by an interval $[\underline{a}, \overline{a}]$.

When agent i's utility of a consumption vector is given by lower and upper bounds, it is natural to suppose that i's utility value falls into the interval $[\underline{a}, \overline{a}]$. Hence, each agent's utility of a consumption vector becomes a fuzzy number $\tilde{a}$. This immediately leads us to a challenging problem of determining the agent's preference if his utility for a consumption vector is a fuzzy number. Actually, under the above assumption about an agent's satisfaction degree of this consumption vector relative to another, the satisfaction degree is not a constant value in $[0, 1]$, but varies continuously in $[0, 1]$. For the case where the agent's satisfaction degree of this consumption vector relative to another monotonically increases with respect to his utility from lower bound to upper bound, we propose to evaluate the degrees of relative satisfaction for every pair of consumption vectors using fuzzy preferences. Thus, we mainly study some relevant issues derived from fuzzy preferences in this chapter.

We put forward the pure exchange economy with fuzzy preferences (PXE-FP) model, where each agent has an initial commodity bundle for trading and a fuzzy preference on the set of all commodity bundles. On the basis of this model, there are three key problems: (1) how to evaluate the utility of different consumption vectors while taking account of fuzzy preferences; (2) how to determine whether a fuzzy competitive equilibrium exists; and (3) how to compute a fuzzy competitive equilibrium.

The main problem with which an agent is confronted in a PXE-FP is choosing one or more consumption vectors from his budget set. The budget set is the set of admissible commodity vectors that an agent can afford at given prices with the value of his initial endowment. Thus, a selection criterion is necessary for the agent. One approach to formalize this criterion is to suppose that the agent has a fuzzy utility index, that is, to define a fuzzy-valued function on the set of consumption vectors. It is assumed that the agent would fuzzily prefer one consumption vector to another if his fuzzy utility of one is greater than that of the other, and would be fuzzily indifferent if the fuzzy utilities of the two vectors are equal. A total order relation of fuzzy numbers is needed to compare the fuzzy utilities of different consumption vectors. This allows to address the agent's problem by finding all the consumption vectors that maximize the fuzzy utility on his budget set.

We provide solutions to the three problems mentioned above when considering fuzzy preference orders (shortly fuzzy preferences). Firstly, as mentioned earlier, it is essential to define a fuzzy order that allows to represent an agent's fuzzy preference by a fuzzy utility function that maps the consumption set to the set of fuzzy numbers. This enables each agent to choose a consumption vector based on the value of his fuzzy utility. For this, we formulate a fuzzy binary relation for every two elements in a reference set (usually a consumption set in this thesis). A total order relation defined by Zhang et al. (2019, [87]) using expected values of fuzzy numbers plays an important role in searching for the best consumption vector, i.e., finding the maximal fuzzy utility.

Secondly, after developing a link between the agent's fuzzy preferences and the fuzzy utility function, we establish the existence of a fuzzy competitive equilibrium that provides market prices and redistribution of goods for the PXE-FP. Based on the total order relation of fuzzy numbers and the expected mapping of the fuzzy utility function, we apply Kakutani's theorem (1941, [44]) to prove the existence of a fuzzy Nash equilibrium for fuzzy non-cooperative games, in which the payoffs of all strategy profiles for each agent are fuzzy numbers. Then, we generalize fuzzy Nash equilibria and prove that a fuzzy competitive equilibrium exists under some mild assumptions.

Thirdly, variational inequalities allow an alternative approach to explain economic equilibria, whose relevance lies in the analysis of the properties for the equilibrium price and allocation. We define the expected utility function according to the expected value of the fuzzy utility for every consumption vector. Finally, by maximizing the expected utility of each agent, we can characterize a fuzzy competitive equilibrium as a solution to a related quasi-variational inequality, which results in an alternative existence proof of the fuzzy competitive equilibrium. As an application, an example of the PXE-FP with two goods and two agents is provided.

The main contributions of this chapter are (1) to propose a fuzzy preference order; (2) to prove the existence of a continuous fuzzy order-preserving function (utility) on the consumption set under certain conditions; and (3) to show the existence of a fuzzy competitive equilibrium.

The chapter, which is based on Zhang et al. (2020, [88]), is organized as follows. Section 5.2 proposes the fuzzy preference relation and illustrates the link between the fuzzy preference relation and the fuzzy order-preserving function. The PXE-FP is introduced and the existence of a fuzzy competitive equilibrium is proved in Section 5.3. Section 5.4 concludes with a brief summary.

5.2 Fuzzy preference relation and fuzzy utility function

In this section, we define a fuzzy preference relation and conclude that there exists an order-preserving fuzzy utility function for a given fuzzy preference relation on a reference set, following the same lines as the classical $0-1$ binary relation. For definitions and notations used in this chapter, we refer to Subsections 1.2.1 and 1.3.1.

Based on the analysis of preference relations in Subsection 1.3.1, it is natural to assume that an agent's satisfaction degree of a consumption vector relative to another monotonically increases as his utility changes from lower bound to upper bound. Consequently, the agent's satisfaction degree of this consumption vector relative to another is not a constant value in $[0,1]$, but varies continuously in $[0,1]$. Therefore, we define the following fuzzy binary relation $\mathfrak{R}$ of a reference set X (usually in the finite vector space of commodity bundles in this thesis).

Definition 5.1. [c.f. page 13-14] Let X be a reference set. A *fuzzy binary relation* $\mathfrak{R}$ in $X \times X$ is a pair $(X, \mu_{\mathfrak{R}})$, where $\mu_{\mathfrak{R}} : X \times X \to \mathbb{FR}$ is the satisfaction function of $\mathfrak{R}$, and for $x, y \in X$, $\mu_{\mathfrak{R}}(x,y)$ represents the satisfaction degree of x relative to y.

Based on the fuzzy binary relation $\mathfrak{R}$ and the total order relation $\succcurlyeq$ as in Definition 4.2, we define a fuzzy preference relation $\succsim_{\mathfrak{R}}$ on a reference set X.

Definition 5.2. Let $x, y \in X$.

x is *fuzzily weakly preferred* to y, denoted by $x \succsim_{\mathfrak{R}} y$, if $\mu_{\mathfrak{R}}(x,y) \succcurlyeq \mu_{\mathfrak{R}}(y,x)$.

x is *fuzzily indifferent* to y, denoted by $x \sim_{\mathfrak{R}} y$, if $\mu_{\mathfrak{R}}(x, y) \approx \mu_{\mathfrak{R}}(y, x)$.

x is *fuzzily strongly preferred* to y, denoted by $x \succ_{\mathfrak{R}} y$, if $\mu_{\mathfrak{R}}(x, y) \succ \mu_{\mathfrak{R}}(y, x)$.

The fuzzy preference relation $\succsim_{\mathfrak{R}}$ is considered to be "consistent" if $\succsim_{\mathfrak{R}}$ is transitive, i.e., $\mu_{\mathfrak{R}}(x, y) \succcurlyeq \mu_{\mathfrak{R}}(y, x)$ and $\mu_{\mathfrak{R}}(y, z) \succcurlyeq \mu_{\mathfrak{R}}(z, y)$ imply that $\mu_{\mathfrak{R}}(x, z) \succcurlyeq \mu_{\mathfrak{R}}(z, x)$.

We assume the fuzzy preference relation $\succsim_{\mathfrak{R}}$ is "consistent". Owing to the total order relation of fuzzy numbers defined by Zhang et al. (2019, [87]), the fuzzy preference relation $\succsim_{\mathfrak{R}}$ of a reference set X satisfies the following properties:

(1) For each $x \in X$, $x \succsim_{\mathfrak{R}} x$ (reflexivity);

(2) For each $x, y, z \in X$, $x \succsim_{\mathfrak{R}} y$ and $y \succsim_{\mathfrak{R}} z$ implies $x \succsim_{\mathfrak{R}} z$ (transitivity);

(3) For each $x, y \in X$, $x \succsim_{\mathfrak{R}} y$ and/or $y \succsim_{\mathfrak{R}} x$ (completeness);

(4) For each $x, y \in X$, if $x \succsim_{\mathfrak{R}} y$ and $y \succsim_{\mathfrak{R}} x$, then $x \sim_{\mathfrak{R}} y$ (antisymmetry).

We say that the fuzzy preference relation $\succsim_{\mathfrak{R}}$ is a completely ordered relation and $(X, \succsim_{\mathfrak{R}})$ is a completely ordered space. The associated completely ordered topology on X is generated by the sets $\{z \in X \mid x \precsim_{\mathfrak{R}} z\}$ and $\{z \in X \mid z \precsim_{\mathfrak{R}} x\}$ for all $x \in X$.

Definition 5.3. For each $x, y, z \in X$, if z satisfies

$$x \precsim_{\mathfrak{R}} z \precsim_{\mathfrak{R}} y,$$

then we say that z belongs to the fuzzy interval $[x, y]$. If

$$x \prec_{\mathfrak{R}} z \prec_{\mathfrak{R}} y,$$

then we say that z belongs to the fuzzy interval (x, y).

A *natural topology* on a reference set X is a completely ordered topology for which the sets $\{z \in X \mid z \precsim_{\mathfrak{R}} y\}$ and $\{z \in X \mid x \precsim_{\mathfrak{R}} z\}$ are closed for all $x, y \in X$. Recall that a set $Y \subseteq X$ is closed if each convergent sequence $\{x^{(k)}\}$ of points in Y converges in Y.

Definition 5.4. For each $\tilde{a}, \tilde{b}, \tilde{c} \in \mathbb{FR}$, if $\tilde{c}$ satisfies

$$\tilde{a} \preccurlyeq \tilde{c} \preccurlyeq \tilde{b},$$

then we say that $\tilde{c}$ belongs to the fuzzy number interval $[\tilde{a}, \tilde{b}]$. If

$$\tilde{a} \prec \tilde{c} \prec \tilde{b},$$

then we say that $\tilde{c}$ belongs to the fuzzy number interval $(\tilde{a}, \tilde{b})$.

A fuzzy function $\tilde{f} : X \to \mathbb{FR}$ defined on a reference set X is said to be *order-preserving* if $x \precsim_{\mathfrak{R}} y$ is equivalent to $\tilde{f}(x) \preccurlyeq \tilde{f}(y)$. The image of the function $\tilde{f}$ is denoted by $\tilde{f}(X)$.

For $x \in X$, let $q_x = \{y \in X \mid y \sim_{\mathfrak{R}} x\}$ be the collection of all elements fuzzily indifferent to x in X. The *quotient set* $X/_\sim$ is the set of all *fuzzy indifference classes* in a reference set X. Formally, $X/_\sim = \{q_x \mid x \in X\}$ and is often denoted by Q. For each $q \in Q$, q is a fuzzy indifference class in X.

Following the same lines as in the proofs by Debreu (1954, [26]), we get the existence of a fuzzy utility function for a given reference set X with a fuzzy preference relation $\succsim_{\mathfrak{R}}$.

Lemma 5.5. *Let X be a reference set with a fuzzy preference relation $\succsim_{\mathfrak{R}}$. Let the quotient set Q of X be countable. There exists a fuzzy order-preserving function which is continuous in any natural topology on X.*

Lemma 5.6. *Let X be a reference set with a fuzzy preference relation $\succsim_{\mathfrak{R}}$. Let $\mathcal{R}$ be a countable subset of X satisfying that for every pair $x, y \in X$, $x \precsim_{\mathfrak{R}} y$, there is an element $r \in \mathcal{R}$ such that $x \precsim_{\mathfrak{R}} r \precsim_{\mathfrak{R}} y$. Then there exists a continuous fuzzy order-preserving function in any natural topology on X.*

A completely ordered topological space $(X, \succsim_{\mathfrak{R}})$ is perfectly separable if there exists a countable class of open sets such that every open set in $(X, \succsim_{\mathfrak{R}})$ is the union of sets of the countable class.

Theorem 5.7. *Let X be a reference set with a fuzzy preference relation $\succsim_{\mathfrak{R}}$. Let $(X, \succsim_{\mathfrak{R}})$ be a perfectly separable space. If for every $x' \in X$, the sets $\{x \in X \mid x \precsim_{\mathfrak{R}} x'\}$ and $\{x \in X \mid x' \precsim_{\mathfrak{R}} x\}$ are closed, then there exists a* *fuzzy order-preserving function which is continuous in any natural topology on X.*

Finally, for a given reference set X with fuzzy preference relation $\succsim_{\mathfrak{R}}$, a *fuzzy utility function* is an order-preserving function $\tilde{u}(x)$ that maps X into the set of all fuzzy numbers.

Corollary 5.8. *Let X be a reference set with a fuzzy preference relation $\succsim_{\mathfrak{R}}$. If for every $x' \in X$ the sets $\{x \in X \mid x \precsim_{\mathfrak{R}} x'\}$ and $\{x \in X \mid x' \precsim_{\mathfrak{R}} x\}$ are closed, then the fuzzy utility function $\tilde{u}(x)$ on X is continuous.*

5.3 Existence of fuzzy competitive equilibria

In this section, we establish a pure exchange economy with fuzzy preferences (PXE-FP) model and prove the existence of a fuzzy competitive equilibrium.

5.3.1 Pure exchange economy with fuzzy preferences

Given several possible alternative consumption vectors, the agent's choice may be uncertain. In this Subsection, we define pure exchange economy with fuzzy preferences (PXE-FP) and fuzzy competitive equilibria. In the remaining of this chapter, an agent i's fuzzy preference relation on a consumption set X_i is denoted by $\succsim^i_{\mathfrak{R}}$.

Definition 5.9. A *Pure exchange economy with fuzzy preferences* (PXE-FP) consists of n agents, indexed by $i \in N = \{1, \cdots, n\}$, who trade on l goods, indexed by $h \in H = \{1, \cdots, l\}$. Each agent i has a fuzzy preference $\succsim^i_{\mathfrak{R}}$, an initial endowment vector $\mathbf{w}_i = (w_{i1}, \cdots, w_{il}) \in \mathbb{R}^l_+$, and a consumption set $X_i \subseteq \mathbb{R}^l_+$. Each element $\mathbf{x}_i = (x_{i1}, \cdots, x_{il})$ is called a consumption vector of i. Formally, we denote a PXE-FP as $\tilde{\mathcal{E}} = (H, N, (X_i, \succsim^i_{\mathfrak{R}}, \mathbf{w}_i)_{i \in N})$.

Remark 5.1. (i) By Theorem 5.7 and Corollary 5.8, the fuzzy preference $\succsim^i_{\mathfrak{R}}$ as in Definition 5.9 can be measured by a fuzzy utility function $\tilde{u}_i$.

(ii) The classical pure exchange economy is a special case of the PXE-FP when the agent's satisfaction degrees for each pair of consumption vectors is only 0 or 1.

An agent wants to maximize his fuzzy utility among all consumption vectors that belong to his budget set. The budget set of agent i is the set of admissible consumption vectors that are affordable for the agent at price vector $\mathbf{p} = (p_1, \cdots, p_l)$ with the value generated by his initial endowment vector $\mathbf{w}_i$, i.e.,

$$B_i(\mathbf{p}) = \{\mathbf{x}_i \mid \mathbf{x}_i \in X_i, \langle \mathbf{p}, \mathbf{x}_i \rangle \leq \langle \mathbf{p}, \mathbf{w}_i \rangle\}.$$

This leads to the following optimization problem. For all $i \in N$ and $\mathbf{p} \in P$,

$$\max_{\mathbf{x}_i \in B_i(\mathbf{p})} \tilde{u}_i(\mathbf{x}_i), \tag{5.1}$$

In turn, the agent's income can be regarded as the receipts from sales of the initial endowments. In the literature, the market for each good is usually considered to be in equilibrium if the supply of a good equals its demand. However, the price of some good may be zero, which means that supply will exceed demand. The aggregate excess demand is $\mathbf{z} = (z_1, \cdots, z_l) \in \mathbb{R}^l$, where $z_h = \sum\limits_{i \in N} (x_{ih} - w_{ih})$ and $x_{ih} - w_{ih}$ is agent i's excess demand of good $h \in H$.

Definition 5.10. For a PXE-FP $\tilde{\mathcal{E}}$, a pair $(\bar{\mathbf{p}}, \bar{\mathbf{x}})$ is said to be a *fuzzy competitive equilibrium* of $\tilde{\mathcal{E}}$ if it satisfies the following conditions:

(1) $\bar{\mathbf{x}}_i \in \arg \max\limits_{\mathbf{x}_i \in B_i(\bar{\mathbf{p}})} \tilde{u}_i(\mathbf{x}_i)$.

(2) $\bar{\mathbf{p}} \in P = \{\mathbf{p} \mid \mathbf{p} \in \mathbb{R}^l, \mathbf{p} \geqq \mathbf{0}, \sum\limits_{h \in H} p_h = 1\}$.

(3) $\bar{\mathbf{z}} \leqq \mathbf{0}, \langle \bar{\mathbf{p}}, \bar{\mathbf{z}} \rangle = 0$, where $\bar{z}_h = \sum\limits_{i \in N} (\bar{x}_{ih} - w_{ih})$

Notice that in Condition (1), $\tilde{u}_i(\mathbf{x}_i)$ depends on the value of $\bar{\mathbf{p}}$ and represents the fuzzy utility function of agent i. Condition (2) states that prices should be non-negative and not all zero. Without any loss of generality, we can normalize the vector $\bar{\mathbf{p}}$ by restricting the sum of its coordinates to be 1. The first part of Condition (3), i.e., $\bar{\mathbf{z}} \leqq \mathbf{0}$, indicates that agents in the economy cannot consume more than their initial endowments. The second part of Condition (3), i.e., $\langle \bar{\mathbf{p}}, \bar{\mathbf{z}} \rangle = 0$, implies that the net value of trade is zero. All money that is paid for demanded goods by consumers, is received by consumers who have the initial endowments.

A fuzzy competitive equilibrium is a state of the market that emerges by "the law of supply and demand". It consists of a competitive equilibrium price $\bar{\mathbf{p}}$ and a competitive equilibrium allocation $\bar{\mathbf{x}}$ such that for each agent i, the fuzzy utility of $\bar{\mathbf{x}}_i$ is maximal in his budget set.

Based on the total order relation of fuzzy numbers and the expected function of a fuzzy mapping, we show the existence of a fuzzy competitive equilibrium by two methods. The first method mainly generalizes the existence of a fuzzy Nash equilibrium using the fixed point theorem in Nash (1950, [60]). However, uniqueness of the fuzzy competitive equilibrium cannot be shown in this way. Therefore, we propose a second method which applies an associated quasi-variational inequality. Under some specific conditions, the solution of the corresponding quasi-variational inequality is unique and, therefore, there exists only one fuzzy competitive equilibrium of the PXE-FP. Furthermore, this unique fuzzy competitive equilibrium can be characterized by the solution of the corresponding quasi-variational inequality.

Before substantiating the existence of a fuzzy competitive equilibrium for a PXE-FP, we obtain the existence of fuzzy Nash equilibria for fuzzy non-cooperative games.

5.3.2 Fuzzy non-cooperative games

A *non-cooperative game* consists of a set of n players $N = \{1, \cdots, n\}$, each of whom has a strategy set S_i and a payoff function. The set of strategy profiles of the game is denoted by $S = \prod_{i=1}^{n} S_i$. The set of all strategy profiles concerning players except i is denoted by $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$. Given a strategy profile $s_{-i} \in S_{-i}$ and a strategy $s_i \in S_i$, we denote $s = (s_i, s_{-i}) \in S$ the strategy profile selected by all players. The strategy profile $s \in S$ chosen by the players determines the payoff for each player. Generally, $u_i : S \to \mathbb{R}$ is used to denote the payoff function of the i-th player and $G = (N, (S_i, u_i)_{i \in N})$ denotes a non-cooperative game.

Imprecise information in the decision-making process results in the imprecision of payoffs. Thus, we define a fuzzy non-cooperative game by formulating the imprecise payoff by a fuzzy number.

Definition 5.11. A *fuzzy non-cooperative game* consists of a set of players $i \in N = \{1, \cdots, n\}$, each of whom has a strategy set S_i and a fuzzy payoff function $\tilde{u}_i : S \to \mathbb{FR}$, where $S = \prod_{i=1}^{n} S_i$ is the set of strategy profiles. Formally, we denote a fuzzy non-cooperative game as $G_{\mathcal{F}} = (N, (S_i, \tilde{u}_i)_{i \in N})$.

Definition 5.12. Let $G_{\mathcal{F}} = (N, S_i, \tilde{u}_i)$ be a fuzzy non-cooperative game. The strategy profile $(s_i^\star, s_{-i}^\star)$ is said to be a *fuzzy Nash equilibrium* if

$$\tilde{u}_i(s_i^\star, s_{-i}^\star) \succcurlyeq \tilde{u}_i(s_i, s_{-i}^\star) \text{ for each } s_i \in S_i, i \in N,$$

where $s_{-i}^\star = \{s_1^\star, \cdots, s_{i-1}^\star, s_{i+1}^\star, \cdots, s_n^\star\}$.

To improve the players' choices, we formally define a randomized strategy, called a mixed strategy. In a mixed strategy, players can choose a probability distribution over their sets of possible strategies and evaluate the random payoff using the expected fuzzy payoff of the mixed strategy. Using fuzzy addition and scalar multiplication (see Subsection 1.2.1), the expected fuzzy payoff of the mixed strategy is still a fuzzy payoff. Thus, a fuzzy Nash equilibrium in mixed strategies can be similarly defined as in Definition 5.12.

5.3.3 Fixed point method in an abstract economy

The following result on existence of fuzzy Nash equilibria follows from the total order relation of fuzzy numbers as in Definition 4.2 (see page 80), the continuity of the fuzzy payoff functions, and the fixed point theorem in Nash (1950, [60]). The proof is, therefore, omitted.

Theorem 5.13. *Let $G_{\mathcal{F}} = (N, (S_i, \tilde{u}_i)_{i \in N})$ be a fuzzy non-cooperative game. If S_i is a nonempty, compact, and convex set for all $i \in N$, then there exists a fuzzy Nash equilibrium in mixed strategies.*

Next, we introduce a generalization of fuzzy non-cooperative games named fuzzy abstract economy and define a fuzzy equilibrium of a fuzzy abstract economy.

Definition 5.14. A *fuzzy abstract economy* consists of n agents $N = \{1, \cdots, n\}$, each of whom has an action set $\mathcal{H}_i \subseteq \mathbb{R}^l$ and a fuzzy payoff function $\tilde{f}_i$ defined over $\mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n$. For each point $\mathbf{a}_{-i} \in \mathcal{H}_{-i} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_{i-1} \times \mathcal{H}_{i+1} \times \cdots \times \mathcal{H}_n$, the choice of agent i is restricted to a set-valued function $A_i(\mathbf{a}_{-i}) \subseteq \mathcal{H}_i$. Formally, we denote a fuzzy abstract economy as $(N, (\mathcal{H}_i, \tilde{f}_i, A_i(\mathbf{a}_{-i}))_{i \in N})$.

To understand the above definition, we consider the special case where the functions $A_i(\mathbf{a}_{-i})$ are constant, i.e., $A_i(\mathbf{a}_{-i})$ is a fixed subset of $\mathcal{H}_i$, independent of $\mathbf{a}_{-i}$. Let $A_i(\mathbf{a}_{-i}) = \mathcal{H}_i$. The following interpretation may be given: There are n agents. Agent i can choose each element $\mathbf{a}_i \in \mathcal{H}_i$. After all agents have made their choices, agent i receives $\tilde{f}_i(\mathbf{a})$. In this case, the fuzzy abstract economy degenerates into a fuzzy non-cooperative game.

The choice of an action by one agent in a fuzzy abstract economy can affect both the fuzzy payoff and the domain of actions of other agents. In economic models, an agent i's actions can be regarded as alternative consumption vectors, which are restricted by the budget constraint. The budget constraint of i is that the cost of the goods chosen at current prices does not exceed his income determined by choices made by other agents. Hence, for an agent in economic models, the function $A_i(\mathbf{a}_{-i})$ must not be regarded as a constant.

Definition 5.15. Let $(N, (\mathcal{H}_i, \tilde{f}_i, A_i(\mathbf{a}_{-i}))_{i \in N})$ be a fuzzy abstract economy. $\bar{\mathbf{a}}$ is said to be a *fuzzy equilibrium point* if

$$\tilde{f}_i(\bar{\mathbf{a}}_{-i}, \bar{\mathbf{a}}_i) \approx \max_{\mathbf{a}_i \in A_i(\bar{\mathbf{a}}_{-i})} \tilde{f}_i(\bar{\mathbf{a}}_{-i}, \mathbf{a}_i) \text{ for all } i \in N, \bar{\mathbf{a}}_i \in A_i(\bar{\mathbf{a}}_{-i}).$$

We recall some definitions in Debreu (1952, [25]). The *graph* of $A_i(\mathbf{a}_{-i})$ is the set $\{\mathbf{a} \mid \mathbf{a}_i \in A_i(\mathbf{a}_{-i})\}$. The set-valued function $A_i(\mathbf{a}_{-i})$ is said to be *continuous* at $\mathbf{a}^0_{-i}$ if for each sequence $\{\mathbf{a}^{(k)}_{-i}\}$ converging to $\mathbf{a}^0_{-i}$, there exists a sequence $\{\mathbf{a}^{(k)}_i\}$ converging to $\mathbf{a}^0_i$ such that $\mathbf{a}^{(k)}_i \in A_i(\mathbf{a}^{(k)}_{-i})$ for all k.

The following result generalizes Theorem 5.13, giving conditions for the existence of a fuzzy equilibrium of a fuzzy abstract economy. The proof is based on the total order relation of fuzzy numbers and the expected function of a fuzzy mapping. The proof is, therefore, omitted.

Lemma 5.16. *A fuzzy abstract economy $(N, (\mathcal{H}_i, \tilde{f}_i, A_i(\mathbf{a}_{-i}))_{i\in N})$ has a fuzzy equilibrium point if*

(i) for each $i \in N$, $\mathcal{H}_i$ is compact and convex, $\tilde{f}_i(\mathbf{a}_{-i}, \mathbf{a}_i)$ is fuzzy continuous on $\mathcal{H}$ and fuzzy quasi-concave in $\mathbf{a}_i$;

(ii) for every $\mathbf{a}_{-i}$, $A_i(\mathbf{a}_{-i})$ is a continuous function whose graph is a closed set; and

(iii) for every $\mathbf{a}_{-i}$, the set $A_i(\mathbf{a}_{-i})$ is convex and nonempty.

We make the following assumptions about the consumption units in a PXE-FP. Afterwards, we show existence of a fuzzy competitive equilibrium for PXE-FP.

Let $\tilde{\mathcal{E}} = (H, N, (X_i, \succsim^i_{\mathfrak{R}}, \mathbf{w}_i)_{i\in N})$ be a PXE-FP. For each good $h \in H$, the consumption of every agent $i \in N$ is necessarily non-negative , i.e., $x_{ih} \geq 0$. For each $i \in N$ and $h \in H$:

Assumption I The set of consumption vectors X_i available to i is a closed convex subset of $\mathbb{R}^l_+$.

Assumption II The sets $\{\mathbf{x}_i \in X_i \mid \mathbf{x}_i \precsim^i_{\mathfrak{R}} \mathbf{x}'_i\}$ and $\{\mathbf{x}_i \in X_i \mid \mathbf{x}'_i \precsim^i_{\mathfrak{R}} \mathbf{x}_i\}$ are closed for all $\mathbf{x}'_i \in X_i$.

Assumption II ensures the continuity of $\tilde{u}_i(\mathbf{x}_i)$ as shown in Theorem 5.7.

Assumption III For each $\mathbf{x}_i \in X_i$, there is $\mathbf{x}'_i \in X_i$ such that $\tilde{u}_i(\mathbf{x}'_i) \succ \tilde{u}_i(\mathbf{x}_i)$.

Assumption III assumes that there is no saturation point, i.e., no consumption vector that an individual would fuzzily prefer to all others.

Assumption IV If $\tilde{u}_i(\mathbf{x}_i) \succ \tilde{u}_i(\mathbf{x}'_i)$ and $0 < \lambda < 1$, then $\tilde{u}_i[\lambda \mathbf{x}_i + (1 - \lambda)\mathbf{x}'_i] \succ \tilde{u}_i(\mathbf{x}'_i)$.

Assumption IV corresponds to the usual assumption on convexity of the fuzzy indifference surfaces, in the sense that the set

$$\{\mathbf{x}_i \in X_i \mid u^i_E(\mathbf{x}_i) \geq a\}$$

is a convex set for each fixed real number a, where $u^i_E(\mathbf{x}_i)$ is the expected utility function of the fuzzy utility function $\tilde{u}_i(\mathbf{x}_i)$.

Recall that agent i has an initial endowment vector $\mathbf{w}_i$ of different goods available.

Assumption V For some $\mathbf{x}_i \in X_i$, $\mathbf{x}_i < \mathbf{w}_i$.

Assumption V states that any agent could exhaust his initial endowments in some feasible way and still have a positive amount of each good available for trading in the PXE-FP.

Following the same lines as in the proofs by Arrow and Debreu (1954, [8]), we get the existence of a fuzzy competitive equilibrium of a PXE-FP $\tilde{\mathcal{E}}$.

Theorem 5.17. *For a PXE-FP $\tilde{\mathcal{E}}$, if $\tilde{\mathcal{E}}$ satisfies Assumptions I-V, then there is a fuzzy competitive equilibrium of $\tilde{\mathcal{E}}$.*

Proof. We prove the theorem in five steps.

(1) We establish the following fuzzy abstract economy $\tilde{E}$.

For $\mathbf{x} \in X$ and $\mathbf{p} \in P$, let $\mathbf{y} = (\mathbf{x}, \mathbf{p})$. For each agent i, $\mathbf{y}_{-i} = (\mathbf{x}_{-i}, \mathbf{p})$ denotes a point in $X_1 \times \cdots \times X_{i-1} \times X_{i+1} \times \cdots \times X_n \times P$ and $\mathbf{y}_{-(n+1)} = \mathbf{x}$. Define

$$A_i(\mathbf{y}_{-i}) = \{\mathbf{x}_i \in X_i \mid \langle \mathbf{p}, \mathbf{x}_i \rangle \leq \langle \mathbf{p}, \mathbf{w}_i \rangle\}.$$

Then, we consider the fuzzy abstract economy

$$\tilde{E} = (N \cup \{n+1\}, (X_i, \tilde{u}_i, A_i(\mathbf{y}_{-i}))_{i \in N \cup \{n+1\}}),$$

where $n+1$ represents a fictitious agent that chooses the market price of the goods, $X_{n+1} = A_{n+1}(\mathbf{y}_{-(n+1)}) = P$, $\tilde{u}_{n+1}(\mathbf{y}) = \langle \mathbf{p}, \mathbf{z} \rangle$, $\mathbf{z} = (z_1, \cdots, z_l)$, and

$z_h = \sum\limits_{i \in N} (x_{ih} - w_{ih})$. Each of the n consumption agents chooses a vector $\mathbf{x}_i$ from X_i, subject to $\mathbf{x}_i \in A_i(\mathbf{y}_{-i})$, and receives $\tilde{u}_i(\mathbf{x}_i)$; the $(n+1)$-th fictitious agent, i.e., the market participant, chooses $\mathbf{p}$ from P and obtains $\langle \mathbf{p}, \mathbf{z} \rangle$.

(2) We show that if there exists a fuzzy equilibrium point of the fuzzy abstract economy $\tilde{E}$, then this fuzzy equilibrium point is also a fuzzy competitive equilibrium of the PXE-FP $\tilde{\mathcal{E}}$ as described in Definition 5.10.

Let $(\bar{\mathbf{p}}, \bar{\mathbf{x}})$ be a fuzzy equilibrium point for the fuzzy abstract economy $\tilde{E}$. Obviously, Conditions (1) and (2) follow immediately from the definition of a fuzzy equilibrium point of $\tilde{E}$.

It remains to verify Condition (3). Obviously, each agent spends his entire income because of the absence of saturation. To be more precise, Assumption III ensures that there exists at least one consumption vector $\mathbf{x}'_i \in X_i$ such that

$$\tilde{u}_i(\mathbf{x}'_i) \succ \tilde{u}_i(\bar{\mathbf{x}}_i).$$

Let $\lambda \in [0, 1]$. By Assumption IV,

$$\tilde{u}_i[\lambda \mathbf{x}'_i + (1 - \lambda)\bar{\mathbf{x}}_i] \succ \tilde{u}_i(\bar{\mathbf{x}}_i).$$

In other words, in every neighbourhood of $\bar{\mathbf{x}}_i$, there is at least one point of X_i fuzzily preferred to $\bar{\mathbf{x}}_i$. Due to Condition (1), $\langle \bar{\mathbf{p}}, \bar{\mathbf{x}}_i \rangle \leq \langle \bar{\mathbf{p}}, \mathbf{w}_i \rangle$. Assume that the strict inequality holds. We can choose a point of X_i for which the inequality still holds and which is fuzzily preferred to $\bar{\mathbf{x}}_i$, establishing a contradiction to Condition (1). Hence, $\langle \bar{\mathbf{p}}, \bar{\mathbf{x}}_i \rangle = \langle \bar{\mathbf{p}}, \mathbf{w}_i \rangle$. In order to attain his equilibrium consumption plan $\bar{\mathbf{x}}_i$, agent i must actually receive the total income given by the initial endowments. Thus, he cannot withhold any initial holdings of any good h from the market if $p_h > 0$. Since $\bar{\mathbf{z}} = (\bar{z}_1, \cdots, \bar{z}_l)$, where $\bar{z}_h = \sum\limits_{i \in N} (x_{ih} - w_{ih})$, it follows that

$$\langle \bar{\mathbf{p}}, \bar{\mathbf{z}} \rangle = 0. \tag{5.2}$$

Let $\mathbf{e}^h$ denote the vector with all coordinates equal to 0, except the h-th which is 1. Obviously, $\mathbf{e}^h \in P$. Also, by Definition 5.15, $0 = \langle \bar{\mathbf{p}}, \bar{\mathbf{z}} \rangle \geq$

$\langle \mathbf{e}^h, \bar{\mathbf{z}} \rangle = \bar{z}_h$. Since this holds for any $h \in H$,

$$\bar{\mathbf{z}} \leqq \mathbf{0}. \tag{5.3}$$

By (5.2) and (5.3), we get that Condition (3) holds. It has been shown that any fuzzy equilibrium point of $\tilde{E}$ satisfies Conditions (1)-(3), so it is a fuzzy competitive equilibrium of $\tilde{\mathcal{E}}$. The converse is obviously also true.

(3) We establish a new abstract economy $\tilde{E}'$, which is a modification of $\tilde{E}$. Then, we show the existence of a fuzzy equilibrium point of this abstract economy $\tilde{E}'$.

Unfortunately, Lemma 5.16 is not directly applicable to $\tilde{E}$, because the action space is not compact. Let X_i' be the set of consumption vectors available to agent i, taking the resource limitations into account. Formally,

$$X_i' = \{\mathbf{x}_i \in X_i \subseteq \mathbb{R}^l_+ \mid \text{for each } j \neq i \text{ there exists } \mathbf{x}_j \in X_j \text{ such that } \mathbf{z} \leqq \mathbf{0}\}.$$

We show that X_i' is bounded. Clearly, if there exists a fuzzy equilibrium point $\bar{\mathbf{x}}_i$ of $\tilde{E}$, then the fuzzy equilibrium point $\bar{\mathbf{x}}_i$ belongs to X_i'.

Let $\mathbf{x}_i \in X_i'$. By the definition of X_i', there exists $\mathbf{x}_j \in X_j$, $j \in N \setminus \{i\}$, such that

$$\sum_{j \in N} \mathbf{x}_j - \sum_{j \in N} \mathbf{w}_j \leq 0.$$

Since $\mathbf{x}_j \geqq 0$ for all $j \in N$ by Definition 5.9, we have

$$\mathbf{0} \leqq \mathbf{x}_i \leqq \sum_{j \in N} \mathbf{w}_j - \sum_{j \in N \setminus \{i\}} \mathbf{x}_j \leqq \sum_{j \in N} \mathbf{w}_j.$$

Thus, X_i' is bounded for all i.

For each i, we can select a positive real number c_i so that the hypercube $C_i = \{\mathbf{x} \in \mathbb{R}^l \mid |x_h| \leq c_i \text{ for all } h\}$ is contained in the interior of X_i'. Let $X_i'' = X_i \cap C_i$. We propose a new abstract economy $\tilde{E}'$, which is a modification of $\tilde{E}$ by replacing X_i by X_i''. Let $A_i'(\mathbf{y}_{-i})$ be the corresponding modification of $A_i(\mathbf{y}_{-i})$, and thus $\tilde{E}' = (N \cup \{n+1\}, (X_i'', \tilde{u}_i, A_i'(\mathbf{y}_{-i}))_{i \in N \cup \{n+1\}})$, where $X_{n+1}'' = A_{n+1}'(\mathbf{y}_{-(n+1)}) = P$. We now show that all the conditions of Lemma 5.16 are satisfied for $\tilde{E}'$.

(i) For each $i \in N \cup \{n+1\}$, X''_i is compact and convex and $\tilde{u}_i$ is fuzzy continuous and fuzzy quasi-concave:

For $i \in N$, X''_i is compact and convex because it is the nonempty intersection of C_i and X_i which are compact and convex (c.f. Assumption I). Moreover, $\tilde{u}_i$ is fuzzy continuous and fuzzy quasi-concave in $\mathbf{x}_i$ because $\tilde{E}'$ is a fuzzy abstract economy.

For $i = n+1$, P is evidently compact and convex and $\tilde{u}_{n+1}(\mathbf{y}) = \langle \bar{\mathbf{p}}, \bar{\mathbf{z}} \rangle$ is clearly fuzzy continuous and fuzzy quasi-concave.

(ii) For each $i \in N \cup \{n+1\}$ and each $\mathbf{y}_{-i}$, $A'_i(\mathbf{y}_{-i})$ is a continuous function whose graph is a closed set:

For $i \in N$, since the budget constraint is a weak inequality between two continuous functions of $\mathbf{p}$, it is easily seen that the graph of $A'_i(\mathbf{y}_{-i})$ is closed.

For $i = n+1$, $A'_{n+1}(\mathbf{y}_{-(n+1)}) = P$ and it is easily seen that the graph of $A'_{n+1}(\mathbf{y}_{-(n+1)}) = P$ is closed.

Furthermore, from the Remark in Section 3.3.5 in Arrow and Debreu (1954, [8]), if Assumption V holds, then for any $i \in N \cup \{n+1\}$, $A'_i(\mathbf{y}_{-i})$ is continuous at the point $\mathbf{y}_{-i} = (\mathbf{x}_1, \cdots, \mathbf{x}_{i-1}, \mathbf{x}_{i+1}, \cdots, \mathbf{x}_n, \mathbf{p})$.

(iii) For each $i \in N \cup \{n+1\}$ and each $\mathbf{y}_{-i}$, the set $A'_i(\mathbf{y}_{-i})$ is convex and nonempty:

For agent $i \in N$, the set $A'_i(\mathbf{y}_{-i})$ is defined by a linear inequality in $\mathbf{x}_i$ and hence is convex. For each i, let $\mathbf{x}'_i \in X_i$ satisfy Assumption V, i.e., $\mathbf{x}'_i \leqq \mathbf{w}_i$. Since $\sum_{i \in N} (\mathbf{x}'_i - \mathbf{w}_i) \leqq \mathbf{0}, \mathbf{x}'_i \in X'_i$ for each i, it holds that $\mathbf{x}'_i \in C_i$. Therefore, $\mathbf{x}'_i \in A_i(\mathbf{y}_{-i})$ for all $\mathbf{y}_{-i}$. Since $A'_i(\mathbf{y}_{-i}) = A_i(\mathbf{y}_{-i}) \cap C_i$, $A'_i(\mathbf{y}_{-i})$ contains $\mathbf{x}'_i$ and therefore is nonempty.

For $i = n+1$, P is convex and nonempty.

Following from Lemma 5.16, we obtain the existence of a fuzzy equilibrium point $\bar{\mathbf{y}} = (\bar{\mathbf{x}}_1, \cdots, \bar{\mathbf{x}}_n, \bar{\mathbf{p}})$ for the fuzzy abstract economy $\tilde{E}'$.

(4) We show that the fuzzy equilibrium point $\bar{\mathbf{y}} = (\bar{\mathbf{x}}_1, \cdots, \bar{\mathbf{x}}_n, \bar{\mathbf{p}})$ of the fuzzy abstract economy $\tilde{E}'$ is also a fuzzy equilibrium point of the fuzzy abstract economy $\tilde{E}$.

Let $\bar{\mathbf{y}} = (\bar{\mathbf{x}}_1, \cdots, \bar{\mathbf{x}}_n, \bar{\mathbf{p}})$ be a fuzzy equilibrium point of the fuzzy abstract economy $\tilde{E}'$. From the definition of $A_i'(\mathbf{y}_{-i})$, it follows that $\langle \bar{\mathbf{p}}, \bar{\mathbf{x}}_i \rangle \leq \langle \bar{\mathbf{p}}, \mathbf{w}_i \rangle$. If we sum over i, then $\langle \bar{\mathbf{p}}, \sum_{i\in N} \bar{\mathbf{x}}_i \rangle \leq \langle \bar{\mathbf{p}}, \sum_{i\in N} \mathbf{w}_i \rangle$ or $\langle \bar{\mathbf{p}}, \bar{\mathbf{z}} \rangle \leq 0$. For a fixed $\bar{\mathbf{z}}$, $\bar{\mathbf{p}}$ is the optimal value of the maximization problem $\langle \mathbf{p}, \bar{\mathbf{z}} \rangle$ for $\mathbf{p} \in P$. Following the same lines as in the proof of formula (5.3) in (2), it follows that formula (5.3) holds. From (5.3) and the definition of X_i' and C_i, $\bar{\mathbf{x}}_i \in X_i'$ and $\bar{\mathbf{x}}_i$ is an interior point of C_i for all i. We proceed by contradiction. Assume that for some $\mathbf{x}_i' \in A_i(\bar{\mathbf{y}}_{-i})$, $\tilde{u}_i(\mathbf{x}_i') \succ \tilde{u}_i(\bar{\mathbf{x}}_i)$. By Assumption IV,

$$\tilde{u}_i[\lambda \mathbf{x}_i' + (1-\lambda)\bar{\mathbf{x}}_i] \succ \tilde{u}_i(\bar{\mathbf{x}}_i) \text{ if } 0 < \lambda < 1.$$

However, for $\lambda \in [0, 1]$ small enough, $\lambda \mathbf{x}_i' + (1-\lambda)\bar{\mathbf{x}}_i \in C_i$. By convexity of $A_i(\bar{\mathbf{y}}_{-i})$, it holds that $\lambda \mathbf{x}_i' + (1-\lambda)\bar{\mathbf{x}}_i \in A_i(\bar{\mathbf{y}}_{-i})$. Consequently, $\lambda \mathbf{x}_i' + (1-\lambda)\bar{\mathbf{x}}_i \in A_i'(\bar{\mathbf{y}}_{-i})$, establishing a contradiction to the definition of $\bar{\mathbf{x}}_i$ as a fuzzy equilibrium point of $\tilde{E}'$. Thus,

$$\bar{\mathbf{x}}_i \approx \arg \max_{\mathbf{x}_i \in A_i(\bar{\mathbf{y}}_{-i})} \tilde{u}_i(\mathbf{x}_i).$$

That $\bar{\mathbf{p}}$ maximizes $\langle \mathbf{p}, \bar{\mathbf{z}} \rangle$ for $\mathbf{p} \in P$ is directly obtained by the definition of a fuzzy equilibrium point for $\tilde{E}'$, since the domain of $\mathbf{p}$ is the same in both fuzzy abstract economies $\tilde{E}$ and $\tilde{E}'$. Hence, the point $\bar{\mathbf{y}} = (\bar{\mathbf{x}}_1, \cdots, \bar{\mathbf{x}}_n, \bar{\mathbf{p}})$ is also a fuzzy equilibrium point for $\tilde{E}$. The converse is obvious.

(5) From (2) and (4), it follows that the fuzzy equilibrium point of $\tilde{E}'$ is a fuzzy competitive equilibrium of $\tilde{\mathcal{E}}$. Also, by (3), it follows that there is a fuzzy competitive equilibrium of a PXE-FP $\tilde{\mathcal{E}}$. □

5.3.4 Variational approach

From the total order relation of fuzzy numbers given in Definition 4.2 and the expected utility function $u_E^i(\mathbf{x}_i)$ (see page 16) of the fuzzy utility function $\tilde{u}_i(\mathbf{x}_i)$, we get that (5.1) is equivalent to

$$u_E^i(\bar{\mathbf{x}}_i) = \max_{\mathbf{x}_i \in B_i(\mathbf{p})} u_E^i(\mathbf{x}_i). \tag{5.4}$$

We assume for $i \in N$:

(i) u_E^i is continuous and strictly concave on X_i.

(ii) For each $\mathbf{p} \in P$ and $\mathbf{x}_i \in B_i(\mathbf{p})$, $\nabla u_E^i(\mathbf{x}_i) \neq 0$.

(iii) For each $\mathbf{p} \in P$ and $\mathbf{x}_i \in \partial B_i(\mathbf{p})$, if $x_{ih} = 0$, $h \in H$, then $\frac{\partial u_E^i(\mathbf{x}_i)}{\partial x_{ih}} > 0$.

(iv) $\lim\limits_{\substack{\|\mathbf{x}_i\| \to +\infty \\ \mathbf{x}_i \in B_i(\mathbf{p})}} u_E^i(\mathbf{x}_i) = -\infty$.

(v) Any agent is endowed with a positive quantity of at least one good, i.e.,

$$\forall i \in N,\ \exists h : w_{ih} > 0.$$

Under Assumptions (i-v), for all $i \in N$, the maximization problem (5.4), i.e., (5.1), has a unique solution $\bar{\mathbf{x}}_i(\mathbf{p})$ for each $\mathbf{p} \in P$, denoted by $\bar{\mathbf{x}}_i$.

Therefore, the fuzzy competitive equilibrium of Definition 5.10 is equivalent to the following statement:

Proposition 5.18. *For a PXE-FP $\tilde{\mathcal{E}}$, let $\bar{\mathbf{p}} \in P$ and $\bar{\mathbf{x}} \in B(\bar{\mathbf{p}}) = \prod_{i \in N} B_i(\bar{\mathbf{p}})$. The pair $(\bar{\mathbf{p}}, \bar{\mathbf{x}}) \in P \times B(\bar{\mathbf{p}})$ is a* fuzzy competitive equilibrium *of $\tilde{\mathcal{E}}$ if and only if*

$$\tilde{u}_i(\bar{\mathbf{x}}_i) \approx \max_{\mathbf{x}_i \in B_i(\bar{\mathbf{p}})} \tilde{u}_i(\mathbf{x}_i) \text{ for all } i \in N, \tag{5.5}$$

and

$$z_h = \sum_{i \in N} (\bar{x}_{ih} - w_{ih}) \leq 0 \text{ for each } h \in H.$$

Based on the expected utility function (see page 16), (5.5) is equivalent to

$$u_E^i(\bar{\mathbf{x}}_i) = \max_{\mathbf{x}_i \in B_i(\bar{\mathbf{p}})} u_E^i(\mathbf{x}_i).$$

Notice that $\bar{\mathbf{x}}_i$ depends on $\bar{\mathbf{p}}$ through the budget set $B_i(\bar{\mathbf{p}})$. We write $\bar{\mathbf{x}}_i$ instead of $\bar{\mathbf{x}}_i(\bar{\mathbf{p}})$ for easiness of notation.

By Theorem 1 in Anello et al. (2010, [6]), it is obvious that the pair $(\bar{\mathbf{p}}, \bar{\mathbf{x}}) \in P \times B(\bar{\mathbf{p}})$ is a fuzzy competitive equilibrium of a PXE-FP if and only if it is a solution to the following quasi-variational inequality:

$$\sum_{i \in N} \langle -\nabla u_E^i(\bar{\mathbf{x}}_i), (\mathbf{x}_i - \bar{\mathbf{x}}_i) \rangle - \langle \sum_{i \in N} (\bar{\mathbf{x}}_i - \mathbf{w}_i), (\mathbf{p} - \bar{\mathbf{p}}) \rangle \geq 0, \tag{5.6}$$

for each $(\mathbf{p}, \mathbf{x}) \in P \times B(\bar{\mathbf{p}})$.

Donato et al. (2008, [28]) proved that $(\bar{\mathbf{p}}, \bar{\mathbf{x}}) \in P \times B(\bar{\mathbf{p}})$ is a solution of (5.6) if and only if for all $i \in N$, $\bar{\mathbf{x}}_i(\mathbf{p})$ is a solution to

$$\langle -\nabla u_E^i(\bar{\mathbf{x}}_i), (\mathbf{x}_i - \bar{\mathbf{x}}_i) \rangle \geq 0 \text{ for all } \mathbf{x}_i \in B_i(\mathbf{p}), \tag{5.7}$$

and $\bar{\mathbf{p}}$ is the solution to

$$\langle -\sum_{i \in N} (\bar{\mathbf{x}}_i - \mathbf{w}_i), (\mathbf{p} - \bar{\mathbf{p}}) \rangle \geq 0 \text{ for all } \mathbf{p} \in P. \tag{5.8}$$

Notice that when the operator $-\nabla u_E^i(\bar{\mathbf{x}}_i)$ is strongly monotone, variational inequality (5.7) has a unique solution. If $\bar{\mathbf{x}}_i$ is a continuous function (on $\mathbf{p}$), then the variational inequality problem (5.8) admits a solution $\bar{\mathbf{p}} \in P$, since P is closed, convex, and bounded.

Therefore, we get the following theorem about the existence of fuzzy equilibrium solutions immediately by an associated quasi-variational inequality.

Theorem 5.19. *For a PXE-FP $\tilde{\mathcal{E}}$, the pair $(\bar{\mathbf{p}}, \bar{\mathbf{x}}) \in P \times B(\bar{\mathbf{p}})$ is a fuzzy competitive equilibrium of $\tilde{\mathcal{E}}$ if and only if $(\bar{\mathbf{p}}, \bar{\mathbf{x}})$ is a solution to the quasi-variational inequality (5.6).*

The following example will illustrate how to obtain fuzzy competitive equilibria by a related quasi-variational inequality. Due to the fuzzy utilities of different consumption vectors, the PXE-FP in the example has a unique fuzzy competitive equilibrium.

Example 5.1. We consider a market consisting of two different goods, denoted by good $h = 1, 2$, and two agents, i.e., agent $i = 1, 2$. Each agent is endowed with an initial vector $\mathbf{w}_i = (w_{i1}, w_{i2})$. The consumption vector of each agent $i = 1, 2$ is $\mathbf{x}_i = (x_{i1}, x_{i2})$. It is assumed that each commodity is sold and purchased at only one price and the price vector is $\mathbf{p} = (p_1, p_2)$ satisfying $p_1 + p_2 = 1$. Following from the existence of an order-preserving fuzzy utility function given in Theorem 5.7, we assume that each agent has a fuzzy utility function defined as follows:

$$\begin{aligned}
&\tilde{u}_i(x_{i1}, x_{i2}) = \tilde{-}\lfloor 0, \frac{1}{2}, \frac{1}{2}, 1\rfloor (x_{i1})^2 \tilde{-} \lfloor 0, \frac{1}{3}, \frac{2}{3}, 1\rfloor (x_{i2})^2 \\
&\tilde{-}\lfloor 2b_{i1}, b_{i1}, b_{i1}, 0\rfloor x_{i1} \tilde{-} \lfloor 2b_{i2}, \frac{3}{2}b_{i2}, \frac{1}{2}b_{i2}, 0\rfloor x_{i2} \\
&\tilde{+}\lfloor 2c_i, c_i, c_i, 0\rfloor,
\end{aligned}$$

where $b_{i1}, b_{i2}, c_i \in \mathbb{R}$.

It is easily shown that

$$u_E^i(x_{i1}, x_{i2}) = -\frac{1}{2}(x_{i1})^2 - \frac{1}{2}(x_{i2})^2 - b_{i1}x_{i1} - b_{i2}x_{i2} + c_i.$$

For agent $i = 1, 2$, $-\nabla u_E^i(\mathbf{x}_i) = (x_{i1} + b_{i1}, x_{i2} + b_{i2})$, we fix $\mathbf{p} \in P$ and find $\bar{\mathbf{x}}_i \in B_i(\mathbf{p}) = \{\mathbf{x}_i = (x_{i1}, x_{i2}) \mid p_1(x_{i1} - w_{i1}) + p_2(x_{i2} - w_{i2}) \leq 0\}$ such that for all $\mathbf{x}_i \in B_i(\mathbf{p})$,

$$(\bar{x}_{i1} + b_{i1})(x_{i1} - \bar{x}_{i1}) + (\bar{x}_{i2} + b_{i2})(x_{i2} - \bar{x}_{i2}) \geq 0. \tag{5.9}$$

Notice that x_{ih} is a function of $\mathbf{p}$. Since the operator $-\nabla u_E^i(\mathbf{x}_i)$ is strongly monotone, there exists a unique solution $\bar{\mathbf{x}}_i$ to the variational inequality. Assumption (ii) is satisfied if we assume $-b_{ih} > w_{ih}$. Obviously, the solution to (5.9) lies in the following set:

$$\{\mathbf{x}_i \in \mathbb{R}_+^2 \mid p_1(x_{i1} - w_{i1}) + p_2(x_{i2} - w_{i2}) = 0\}. \tag{5.10}$$

Moreover, we need to find the solution $\bar{\mathbf{p}} = (\bar{p}_1, \bar{p}_2) \in P$ such that for all $\mathbf{p} = (p_1, p_2) \in P$,

$$-z_1(p_1 - \bar{p}_1) - z_2(p_2 - \bar{p}_2) \geq 0, \tag{5.11}$$

where $z_h = (\bar{x}_{1h} - w_{1h}) + (\bar{x}_{2h} - w_{2h})$ is the aggregate excess demand function of each good $h = 1, 2$. Since $p_1 + p_2 = 1$, from (5.11), it holds that

$$(z_1 - z_2)(p_2 - \bar{p}_2) \geq 0 \text{ for } p_2 \in [0, 1]. \tag{5.12}$$

Notice that when $p_2 = 0$, $(z_1 - z_2)(-\bar{p}_2) \geq 0$ implies $z_1 - z_2 \leq 0$. Similarly, when $p_2 = 1$, $(z_1 - z_2)(1 - \bar{p}_2) \geq 0$ implies $z_1 - z_2 \geq 0$, since $p_2 \in [0, 1]$. Besides, since inequality (5.12) has to hold for each $p_2 \in [0, 1]$, and since $\bar{\mathbf{p}} \in P$, solving (5.12) is equivalent to solving

$$z_1 - z_2 = 0. \tag{5.13}$$

Recall that $\bar{\mathbf{p}}$ needs to be in P. Then,

$$\bar{p}_1 = 1 - \bar{p}_2, \bar{p}_2 \in [0, 1]. \tag{5.14}$$

Next, we discuss the solution to (5.9) and (5.12). First, we show that the prices of these two goods satisfy $p_1 > 0$ and $p_2 > 0$.

(i) If $p_1 = 0$ and $p_2 = 1$, the budget set of agent $i = 1, 2$ is $B_i(0, 1) = \{\mathbf{x}_i \in \mathbb{R}^2_+ \mid x_{i2} \leq w_{i2}\}$. Hence, the solution to (5.9) is $(\bar{x}_{i1}(0, 1), \bar{x}_{i2}(0, 1)) = (-b_{i1}, w_{i2})$. Moreover, $\mathbf{p} = (0, 1)$ is the solution to (5.12) if and only if $z_1 - z_2 < 0$. But from $(\bar{x}_{i1}(0, 1), \bar{x}_{i2}(0, 1)) = (-b_{i1}, w_{i2})$, we get that $z_1 - z_2 = -b_{11} - w_{11} - b_{21} - w_{21} > 0$, establishing a contradiction to $z_1 - z_2 < 0$. Therefore, $\mathbf{p} = (0, 1)$ is not the solution to (5.12).

(ii) If $p_1 = 1$ and $p_2 = 0$, the budget set of agent $i = 1, 2$ is $B_i(1, 0) = \{\mathbf{x}_i \in \mathbb{R}^2_+ \mid x_{i1} \leq w_{i1}\}$. Thus, the solution to (5.9) is $(\bar{x}_{i1}(1, 0), \bar{x}_{i2}(1, 0)) = (w_{i1}, -b_{i2})$. Furthermore, $\mathbf{p} = (1, 0)$ is the solution to (5.12) if and only if $z_1 - z_2 > 0$. However, from $(\bar{x}_{i1}(1, 0), \bar{x}_{i2}(1, 0)) = (w_{i1}, -b_{i2})$, we show that $z_1 - z_2 = b_{12} + w_{12} + b_{22} + w_{22} < 0$, establishing a contradiction to $z_1 - z_2 > 0$. Thus, $\mathbf{p} = (1, 0)$ is not the solution to (5.12).

Then, we can assume $p_1 > 0$, $p_2 > 0$. For agent $i = 1, 2$, by (5.10), it follows that

$$x_{i2} = w_{i2} - \frac{p_1}{p_2}(x_{i1} - w_{i1}), \tag{5.15}$$

and $x_i \in \mathbb{R}^2_+$ implies

$$0 \leq x_{i1} \leq w_{i1} + w_{i2}\frac{p_2}{p_1} \tag{5.16}$$

Furthermore, by substituting (5.15) for x_{i2} and $\bar{x}_{i2}$ in (5.9), it can be seen that

$$\left[\frac{p_1^2 + p_2^2}{p_2^2}\bar{x}_{i1} - (w_{i2} + b_{i2})\frac{p_1}{p_2} - w_{i1}(\frac{p_1}{p_2})^2 + b_{i1}\right](x_{i1} - \bar{x}_{i1}) \geq 0. \tag{5.17}$$

Since (5.16) has to hold for any $x_i \in B_i(\mathbf{p})$, by taking $x_{i1}^1, x_{i1}^2 \in B_i(\mathbf{p})$ with $x_{i1}^1 \leq \bar{x}_{i1} \leq x_{i1}^2$, (5.16) is equivalent to solving the equality

$$\frac{p_1^2 + p_2^2}{p_2^2}\bar{x}_{i1} - (w_{i2} + b_{i2})\frac{p_1}{p_2} - w_{i1}(\frac{p_1}{p_2})^2 + b_{i1} = 0, \tag{5.18}$$

while (5.16) and (5.17) applied to $\bar{x}_i$ becomes

$$0 \leq \bar{x}_{i1} \leq w_{i1} + w_{i2}\frac{p_2}{p_1}, \bar{x}_{i2} = w_{i2} - \frac{p_1}{p_2}(x_{i1} - w_{i1}). \tag{5.19}$$

The solution to (5.18) and (5.19) is

$$\left\{\begin{array}{ll} \bar{x}_{i1} = & \frac{p_2^2}{p_1^2+p_2^2}\left[w_{i1}(\frac{p_1}{p_2})^2 + (w_{i2} + b_{i2})\frac{p_1}{p_2} - b_{i1}\right], \\ \\ \bar{x}_{i2} = & \frac{p_1^2}{p_1^2+p_2^2}\left[w_{i2}(\frac{p_2}{p_1})^2 + (w_{i1} + b_{i1})\frac{p_2}{p_1} - b_{i2}\right], \end{array}\right. \tag{5.20}$$

under the condition that $(p_1, p_2) \in P$, and since $\bar{x}_i \geq 0$, we have

$$\left\{\begin{array}{l} w_{i1}(\frac{p_1}{p_2})^2 + (w_{i2} + b_{i2})\frac{p_1}{p_2} - b_{i1} \geq 0, \\ w_{i2}(\frac{p_2}{p_1})^2 + (w_{i1} + b_{i1})\frac{p_2}{p_1} - b_{i2} \geq 0. \end{array}\right. \tag{5.21}$$

Case 1. If system (5.18)-(5.19) has a solution for each agent $i = 1, 2$, i.e., condition (5.21) holds for each agent $i = 1, 2$, then the solution to the variational inequality (5.9) is (5.20). Combining formulas (5.13), (5.14) and (5.20), one can see that

$$\begin{cases} \frac{Bp_1^2+(B-A)p_1p_2-Ap_2^2}{p_1^2+p_2^2} = 0, \\ p_2 = 1 - p_1. \end{cases} \tag{5.22}$$

Consequently, the solution to (5.22), i.e., (5.12) is

$$\begin{cases} \bar{p}_1 = \frac{A}{A+B}, \\ \\ \bar{p}_2 = \frac{B}{A+B}, \end{cases}$$

where $A = w_{11} + b_{11} + w_{21} + b_{21}, B = w_{12} + b_{12} + w_{22} + b_{22}$.

Case 2. If system (5.18)-(5.19) does not have any solution, then we find the solution to (5.9) on the boundary of the set $\{\mathbf{x}_i \in \mathbb{R}_+^2 \mid p_1(x_{i1} - w_{i1}) + p_2(x_{i2} - w_{i2}) = 0\}$, which is either $\bar{x}_{i1} = 0$ or $\bar{x}_{i2} = 0$. The pair

$$\begin{cases} \bar{x}_{i1} = 0, \\ \bar{x}_{i2} = w_{i2} + w_{i1}\frac{p_1}{p_2}, \end{cases} \tag{5.23}$$

is the solution to variational inequality (5.9) when (5.18) turns out to have $\bar{x}_{i1} < 0$, which happens if and only if

$$w_{i1}(\frac{p_1}{p_2})^2 + (w_{i2} + b_{i2})\frac{p_1}{p_2} - b_{i1} < 0, \tag{5.24}$$

for $(p_1, p_2) \in P$.

If (5.24) does not hold, it follows that

$$\begin{cases} \bar{x}_{i1} = w_{i1} + w_{i2}\frac{p_2}{p_1}, \\ \bar{x}_{i2} = 0, \end{cases} \tag{5.25}$$

is the solution to variational inequality (5.9) when (5.20) turns out to have $\bar{x}_{i2} < 0$, which happens if and only if

$$w_{i2}(\frac{p_2}{p_1})^2 + (w_{i1} + b_{i1})\frac{p_2}{p_1} - b_{i2} < 0, \tag{5.26}$$

for $(p_1, p_2) \in P$.

Notice that solution (5.23) or (5.25) in this case is continuous in P.

If condition (5.24) holds, for each agent $i = 1, 2$, the solution to (5.18) is (5.23). In this situation, solving (5.12) is equivalent to solving the system

$$1 + \frac{p_1}{p_2} = 0, \tag{5.27}$$

Since $\bar{\mathbf{p}} \in P$, we have

$$\bar{p}_1 = 1 - \bar{p}_2 \geq 0, \bar{p}_2 > 0. \tag{5.28}$$

It is found that (5.27)-(5.28) have no solution, which shows the solution to (5.12) in the boundary of P, i.e., $\bar{\mathbf{p}} = (0, 1)$.

If (5.26) holds for every agent $i = 1, 2$, the solution to (5.18) is (5.25). In this situation, the solution of (5.12) is the same as the solution to the following system

$$1 + \frac{p_2}{p_1} = 0, \tag{5.29}$$

Recall that $\bar{\mathbf{p}}$ needs to be in P. Then,

$$\bar{p}_1 = 1 - \bar{p}_2 > 0, \bar{p}_2 \geq 0. \tag{5.30}$$

It follows that (5.29)-(5.30) have no solution, which implies that the solution to (5.12) lies in the boundary of P, i.e., $\bar{\mathbf{p}} = (1, 0)$.

Therefore, $(\bar{p}_1, \bar{p}_2) = (\frac{A}{A+B}, \frac{B}{A+B})$ is the unique solution to (5.12). As a consequence, the fuzzy competitive equilibrium of the PXE-FP is $(\bar{\mathbf{p}}, \bar{\mathbf{x}})$, where

$$\bar{\mathbf{p}} = (\frac{A}{A+B}, \frac{B}{A+B}),$$

$$\bar{\mathbf{x}} = \begin{pmatrix} \frac{w_{11}A^2 + (w_{12}+b_{12})AB - b_{11}B^2}{A^2+B^2} & \frac{w_{12}B^2 + (w_{11}+b_{11})AB - b_{12}A^2}{A^2+B^2} \\ \\ \frac{w_{21}A^2 + (w_{22}+b_{22})AB - b_{21}B^2}{A^2+B^2} & \frac{w_{22}B^2 + (w_{21}+b_{21})AB - b_{22}A^2}{A^2+B^2} \end{pmatrix}.$$

We explain the solution of the quasi-variational inequality from the definition of a fuzzy competitive equilibrium as follows:

(i) The supply of good $h = 1, 2$ equals the demand following from

$$\begin{aligned}\bar{x}_{11} + \bar{x}_{21} &= \frac{(w_{11} + w_{21})A^2 + AB^2 - (b_{21} + b_{11})B^2}{A^2 + B^2} \\ &= \frac{(w_{11} + w_{21})A^2 + (w_{11} + w_{21})B^2}{A^2 + B^2} \\ &= w_{11} + w_{21}.\end{aligned}$$

Similarly, it holds that $\bar{x}_{12} + \bar{x}_{22} = w_{12} + w_{22}$.

(ii) At the equilibrium price, both agents can afford their allocation for their given initial endowments, i.e.,

$$\begin{aligned}&\bar{p}_1(\bar{x}_{11} - w_{11}) + \bar{p}_2(\bar{x}_{12} - w_{12}) = \\ &\frac{A}{A+B}\left[\frac{w_{11}A^2 + (w_{12} + b_{12})AB - b_{11}B^2}{A^2 + B^2} - w_{11}\right] \\ &+ \frac{B}{A+B}\left[\frac{w_{12}B^2 + (w_{11} + b_{11})AB - b_{12}A^2}{A^2 + B^2} - w_{12}\right] \\ &= \frac{A}{A+B}\left[\frac{(w_{12} + b_{12})AB - (b_{11} + w_{11})B^2}{A^2 + B^2}\right] \\ &+ \frac{B}{A+B}\left[\frac{(w_{11} + b_{11})AB - (b_{12} + w_{12})A^2}{A^2 + B^2}\right] = 0.\end{aligned}$$

Similarly, $\bar{p}_1(\bar{x}_{21} - w_{21}) + \bar{p}_2(\bar{x}_{22} - w_{22}) = 0$.

(iii) Agent $i = 1, 2$ fuzzily weakly prefers the consumption bundle $(\bar{x}_{i1}, \bar{x}_{i2})$ to the initial endowment vector (w_{i1}, w_{i2}), i.e.,

$$\begin{aligned}&E(\tilde{u}_1(\bar{x}_{11}, \bar{x}_{12})) - E(\tilde{u}_1(w_{11}, w_{12})) = \\ &\frac{w_{11}^2(A^2 + B^2)^2 - [w_{11}A^2 + (w_{12} + b_{12})AB - b_{11}B^2]^2}{2(A^2 + B^2)^2} \\ &- \frac{w_{12}^2(A^2 + B^2)^2[w_{12}B^2 + (w_{11} + b_{11})AB - b_{12}A^2]^2}{2(A^2 + B^2)^2} \\ &- \frac{b_{11}[w_{11}A^2 + (w_{12} + b_{12})AB - b_{11}B^2]}{A^2 + B^2} \\ &- \frac{b_{12}[w_{12}B^2 + (w_{11} + b_{11})AB - b_{12}A^2]}{A^2 + B^2}\end{aligned}$$

$$+ b_{11}w_{11}^2 + b_{12}w_{12}^2$$
$$= \frac{[(w_{11}+b_{11})B - (w_{12}+b_{12})A]^2}{2(A^2+B^2)} \geq 0.$$

Hence, $\tilde{u}_1(\bar{x}_{11}, \bar{x}_{12}) \succcurlyeq \tilde{u}_1(w_{11}, w_{12})$. If the equality holds, $(w_{11}+b_{11})B = (w_{12}+b_{12})A$. In this case, $\bar{x}_{1h} = w_{1h}$, $h = 1, 2$, which means that the initial endowment vector for agent 1 is optimal. Similarly, we can get that $\tilde{u}_2(\bar{x}_{21}, \bar{x}_{22}) \succcurlyeq \tilde{u}_2(w_{21}, w_{22})$. The two goods are distributed efficiently between the two agents after the exchange of goods.

5.4 Conclusion

In this chapter, we build a fuzzy binary relation to evaluate the fuzzy preference relation of various alternative consumption vectors. Following the same lines as in classical theory for the $0-1$ binary relation, we conclude that there exists a continuous fuzzy order-preserving function on the consumption set under some assumptions. Furthermore, the existence result of a fuzzy competitive equilibrium for the pure exchange economy with fuzzy preferences (PXE-FP) is obtained in two different ways. When each agent's attitude is uncertain to different consumption vectors, we can obtain the redistribution and price vector of goods for pure exchange economies with fuzzy preferences as proposed in this chapter.

We show the existence of fuzzy competitive equilibria for the PXE-FP under some assumptions. Future research on the PXE-FP is needed to determine uniqueness and stability of the fuzzy competitive equilibrium. The latter study would require the specification of a dynamic competitive market with fuzzy preferences. Finally, the existence and stability of an equilibrium could be shown by applying the generalized linear discrete-time system (see [22], [23], [62], [1], [57]) with fuzzy dynamic PXE-FP. A concrete dynamic PXE-FP simulation model could also be provided to confirm the results. Besides, further investigation could be on generalizing different economic models to fuzzy preferences as the abstract fuzzy economies studied by Patriche (2014, [64]) and the restricted participation on financial markets proposed by Donato et al. (2020, [31]).

Summary

This thesis considers solutions of cooperative games based on different definitions of excess, also for cooperative fuzzy games. Moreover, we propose the exchange economy model with fuzzy preferences based on a fuzzy binary relation. This fuzzy binary relation means that each agent's satisfaction degree for either one of each pair of consumption vectors is not a constant value in $[0, 1]$, but varies continuously in $[0, 1]$.

In Chapter 2, we define weighted excesses of players in TU-games which are obtained by summing up all the weighted excesses of all coalitions to which they belong. Next, we give three characterizations of the least square values for TU-games: by lexicographically minimizing the individual weighted excesses of players, by minimizing the variance of the players' weighted excesses on the preimputation set, and by showing that they are the center of the weighted super core defined by certain lower and upper bounds for the core payoff vectors. Finally, these lower and upper bounds for the core inspired us to introduce a new solution for cooperative TU-games that has a strong similarity with the Shapley value.

In Chapter 3, we consider a more general definition of excess to measure the dissatisfaction for coalitions of players in cooperative games. This is done by considering affine combinations (specially convex combinations) of the classical excess and the proportional excess. In view of this so-called α-excess, we define new solution concepts for cooperative games, such as the $\alpha\varepsilon$-core, α least core, α-prenucleolus, and α-prekernel. We illustrate that the $\alpha\varepsilon$-core, α-least core and α-prekernel are different for different values of α with the specific examples. We characterize the α-least core

in terms of α-strong stability and the α-prekernel by strong stability and the α-balanced surplus property. Further, we show that the payoff vector originated by the α-prenucleolus belongs to the α-prekernel. This shows that the α-prekernel is nonempty.

In Chaper 4, we define a total order relation of fuzzy numbers based on expected values of fuzzy numbers. In view of the total order relation of fuzzy numbers, we show that the introduced concepts of the indifference fuzzy core, nucleolus and bargaining sets of cooperative games with fuzzy payoffs are well-defined. Moreover, we obtain a necessary and sufficient condition for non-emptiness of the indifference fuzzy core. It is shown that there is at least one fuzzy payoff vector in the indifference fuzzy nucleolus. We show that the indifference fuzzy core and the two indifference fuzzy bargaining sets of convex cooperative fuzzy games coincide. Moreover, we characterize the class of superadditive cooperative fuzzy games for which the two indifference fuzzy bargaining sets and the indifference fuzzy core coincide.

Finally, in Chapter 5, we focus on a new model of pure exchange economy with fuzzy preferences (PXE-FP) and analyze the existence of equilibria. The proposed model integrates exchange, consumption and the agent's fuzzy preference on the consumption set. We set up a new fuzzy binary relation on the consumption set to evaluate fuzzy preferences. For a given fuzzy preference relation, we prove that there exists a continuous fuzzy order-preserving function on the consumption set under certain mild conditions. Existence of a fuzzy competitive equilibrium for the PXE-FP is proved through a new result on the existence of fuzzy Nash equilibra for fuzzy noncooperative games. Finally, we show that fuzzy competitive equilibria can be characterized as a solution to an associated quasi-variational inequality, giving rise to equilibria.

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