Fermionic corrections to quark and gluon form factors in four-loop QCD

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We analytically compute all four-loop QCD corrections to the photon-quark and Higgs-gluon form factors involving a closed massless fermion loop. Our calculation of nonplanar vertex integrals confirms a previous conjecture for the analytical form of the nonfermionic contributions to the collinear anomalous dimensions of quarks and gluons.

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I. INTRODUCTION

Two of the most important processes which are studied in great detail at the CERN LHC are the production of lepton pairs and Higgs bosons. The total cross sections for Drell-Yan lepton pair production through virtual photons and Higgs boson production through the dominant gluon fusion channel are known to next-to-next-to-next-to-leading order ($\mathrm{N^3LO}$) in perturbation theory [1–3] in the limit of an infinitely heavy top quark. Historically, first the virtual corrections have been computed and the real radiation contributions have been added later. In this paper we take an important step towards the $\mathrm{N^4LO}$ corrections and provide analytic results for the fermionic contribution to the virtual corrections, both to the Drell-Yan and Higgs boson production processes.

The virtual corrections are conveniently expressed in terms of form factors of the photon-quark vertex and the effective gluon-Higgs boson vertex. Let us denote the corresponding bare vertex functions by Γ_q^μ and $\Gamma_g^{\mu\nu}$, respectively. Then the bare form factors are obtained from

$$F_q(q^2) = -\frac{1}{4(1-\epsilon)q^2}\mathrm{Tr}(\not{q}_2\Gamma^\mu\not{q}_1\gamma_\mu),$$
$$F_g(q^2) = \frac{(q_1\cdot q_2 g_{\mu\nu} - q_{1,\mu}q_{2,\nu} - q_{1,\nu}q_{2,\mu})}{2(1-\epsilon)}\Gamma_g^{\mu\nu}, \quad (1)$$

where our overall normalization is such that both form factors are 1 at leading order. Further, we work in conventional dimensional regularization and use $d = 4 - 2\epsilon$ for the space-time dimension. The external momentum of the photon and Higgs is $q = q_1 + q_2$ and q_1 and q_2 are the incoming momenta of the quark and antiquark in the case of F_q and of the gluons in the case of F_g. Some sample Feynman diagrams contributing to the fermionic part of F_q and F_g are shown in Fig. 1. We define the perturbative expansion of F_q and F_g in terms of the bare strong coupling constant and write

$$F_x = 1 + \sum_{n\geq 1}\left(\frac{\alpha_s^0}{4\pi}\right)^n\left(\frac{4\pi}{e^{\gamma_E}}\right)^{n\epsilon}\left(\frac{\mu^2}{-q^2-i0}\right)^{n\epsilon}F_x^{(n)}, \quad (2)$$

with $x \in \{q, g\}$.

Two-loop corrections to F_q have been computed in Refs. [4–7] and the first two-loop calculation for F_g has been performed in [8]. In the first three-loop calculation of F_q and F_g [9] the coefficients of the highest ϵ expansion terms of three master integrals were only known numerically. These coefficients have been computed in [10]. The results of [9] have been confirmed in Refs. [11–13]. For the computation of three-loop master integrals we also refer to [14].

At four-loop order there are only partial results. For F_q, the large-N_c limit, which only involves planar diagrams, has been considered in Refs. [15,16], the n_f^2 terms are available from [17], the complete contribution from color structure $(d_F^{abcd})^2$ has been computed in [18] and confirmed in [19]. For F_q and F_g, all corrections with three or two

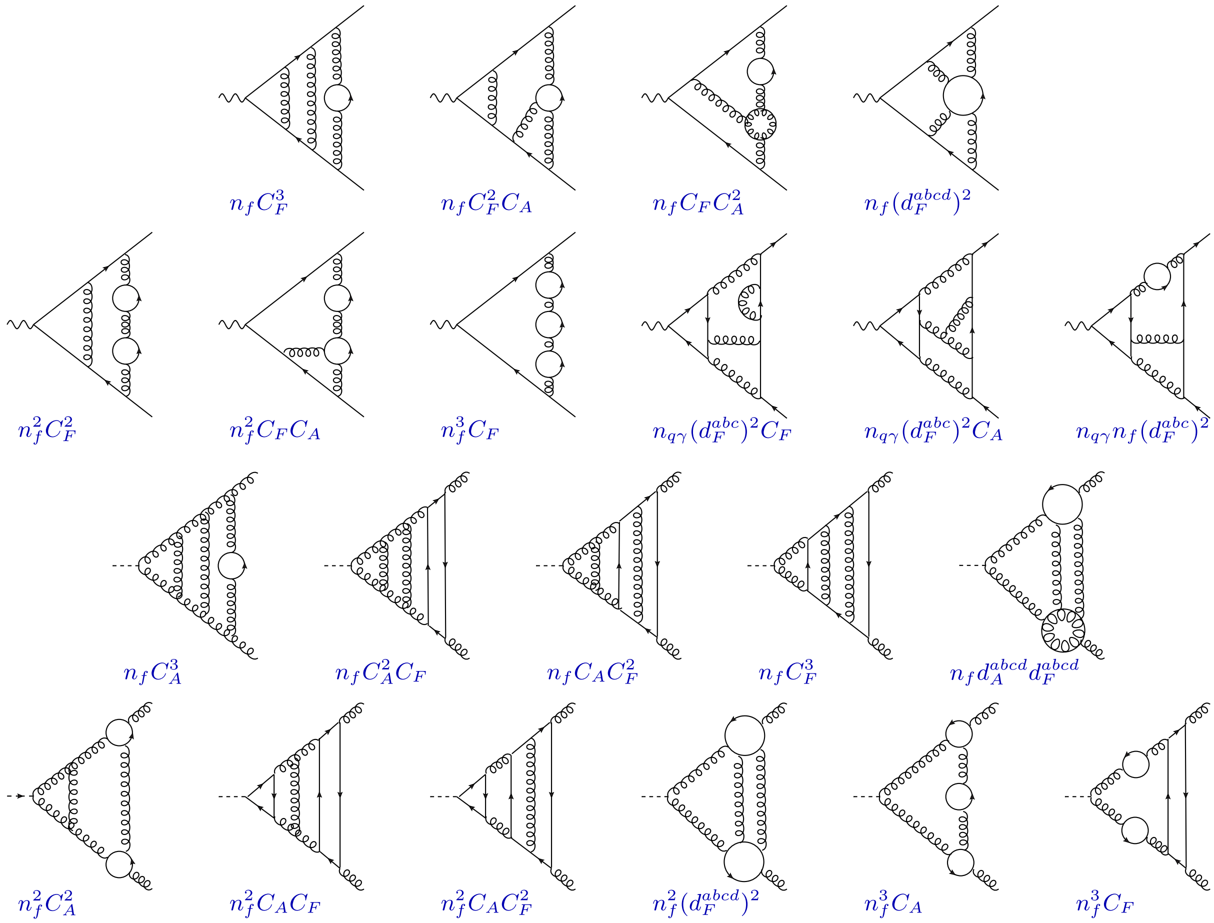
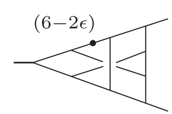


FIG. 1. Sample Feynman diagrams contributing to the fermionic corrections to F_q and F_g at four-loop order. Straight and curly lines denote quarks and gluons, respectively. Both planar and nonplanar diagrams contribute.

closed fermion loops were calculated in [20,21], respectively, including also the singlet contributions.

There are a number of works where pole parts of the form factors have been computed. In fact, from the $1/\epsilon^2$ pole it is possible to extract the so-called cusp anomalous dimension. A complete calculation based on the form factors can be found in Ref. [19]. In that work a basis of finite integrals has been chosen and expanded only to lower orders in ϵ in order to obtain the required weight six information. Reference [19] confirmed the expression presented in Ref. [22], which is based on a calculation in $\mathcal{N} = 4$ super Yang-Mills, other known QCD results, and conjectural input for one term in the matter contributions [23]. Partial contributions to the QCD cusp anomalous dimension are available from [15–17,20,24–28] and numerical results are presented in Refs. [29,30].

Recently, the collinear anomalous dimensions have been computed in Ref. [31] by extracting the $1/\epsilon$ poles of the quark and gluon form factors. All contributions could be computed analytically except one contribution from a nonplanar four-loop integral defined in $6-2\epsilon$ dimensions, which is parametrized by $\mathcal{H}$ (cf. Eq. (10) of [31]),



$$\equiv \mathcal{H} + \mathcal{O}(\epsilon). \tag{3}$$

A numerical evaluation of $\mathcal{H}$ to 10 significant digits allowed not only to significantly improve upon the numerical results of Refs. [32,33]; together with an assumption on the multiple zeta values present in $\mathcal{H}$, an analytic expression could be conjectured. Using the results obtained in the present paper we confirm this expression; details are presented in the next section where we outline some of the techniques used for our calculation. Our analytic results are presented in Sec. III, and Sec. IV contains our conclusions and an outlook to the full result.

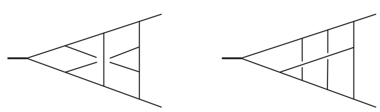
II. CALCULATION

We employ `Qgraf` [34] to generate the required Feynman diagrams with closed fermion loops, 2464 diagrams for $F_q^{(4)}$ and 18642 diagrams for $F_g^{(4)}$. After applying the projectors and performing the numerator algebra with `Form 4` [35], we obtain the form factors F_q and F_g

expressed as a linear combination of scalar functions belonging to properly defined integral families. Each function has 18 indices where up to twelve correspond to the different propagators of the diagrams. In addition to planar diagrams (see, e.g., Refs. [15,16,36]) there are nonplanar diagrams; it is the latter which pose challenges. We perform the calculation in a general R_ξ gauge and check explicitly that terms proportional to ξ cancel in our results.

From the computational point of view there are two challenges one has to deal with. The first one is the integration-by-parts reduction [37,38] of the scalar integrals, which appear in the amplitudes, to so-called master integrals. For this task, we employ the setup described in Ref. [19], which is based on the codes `Reduze 2` [39] and `Finred`, implementing techniques from [40–46].

The second challenge is the computation of the master integrals as a Laurent series in ϵ. Here we have two approaches at hand. The first one is based on the construction of a basis of finite master integrals [13,47,48], partly in $6-2\epsilon$ dimensions. Subsequently the program `HyperInt` [49] is used to compute the ϵ expansion of the master integrals. This approach allowed us to compute all ϵ coefficients of the master integrals required for the fermionic four-loop corrections except for $\mathcal{H}$. We wish to note, however, that it remains unclear whether the evaluation of the constants of transcendental weight eight (or even higher) of some of the nonplanar twelve-line master integrals is possible in this approach. In particular, for the two Feynman integrals corresponding to nonplanar graphs with twelve edges,

 (4)

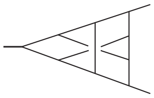
it is not known whether a linearly reducible [50,51] Feynman parametric representation exists.

In this paper, we show that both remaining not linearly reducible topologies (4) can be solved using a second method, which is based on differential equations. While the method of differential equations [52–54] is not directly applicable to one-scale Feynman integrals, we can introduce an additional scale parameter, as was suggested in Ref. [55]. On the one hand, we complicate the situation. On the other hand, we obtain the possibility to apply the full power of the method of differential equations (see, e.g., Refs. [56,57] and Ref. [58] [Sec. E.8]). In the context of massless four-loop form factors this approach has been applied successfully in Refs. [15–18]. In a first step one introduces a second mass scale by imposing a virtuality $q_2^2 \neq 0$ on one of the external partons, which apparently makes the problem more complicated. However, we are now in a position to establish differential equations for the master integrals in the variable $x = q_2^2/q^2$ which determine the connection between the points $x = 1$ and $x = 0$. The boundary conditions are then easy to fix at $x = 1$ as in this point our integrals turn into massless propagator integrals for which analytic results are known at least up to weight twelve [59,60]. A detailed description of the procedure can be found in Ref. [18].

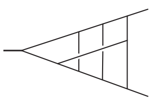
In our calculation we employ `Fire 6` [61] in combination with `LiteRed` [44,62] to compute the reductions for the differential equations and closely follow the algorithm of Refs. [63,64] as implemented in `Libra` [65] to bring our system in ϵ form. The complexity of the two topologies (4) is somewhat reflected also in the properties of the differential equations. First, it appears that differential equations for these two topologies, in addition to singularities at $x = 0$ and 1, have singularities at $x = -1, 4, 1/4$ (at $x = 4$) for the first (second) topology, respectively. Among those, the singularity at $x = 1/4$ is especially troublesome as it lies inside the segment (0,1) connecting the points of interest. Moreover, it appears that in order to reduce the system to ϵ form, we need to introduce the algebraic extensions $x_1 = \sqrt{x}$, $x_2 = \sqrt{x-1/4}$, and $x_3 = \sqrt{1/x - 1/4}$. Fortunately, for each specific iterated integral which appears in the ϵ expansion of the master integrals of these two topologies it is possible to find the rationalizing variable change. As a result, the master integrals of the two topologies are not directly expressed via nonalternating multiple zeta values, but rather via Goncharov polylogarithms with letters in the alphabet $\{0, \pm 1, \pm i\sqrt{3}, e^{\pm i\pi/3}, e^{\pm 2i\pi/3}, e^{\pm i\pi/3}/2\}$. Using the PSLQ algorithm, we were able to express our final result for the master integrals with two massless legs and their subtopologies through to weight nine in terms of regular zeta values, ζ_n ($n = 2, \ldots, 9$), and the multiple zeta value

$$\zeta_{5,3} = \sum_{m=1}^{\infty} \sum_{n=1}^{m-1} \frac{1}{m^5 n^3} \approx 0.0377076729848. \quad (5)$$

Our results for the corner integrals of the two nonplanar topologies through to the finite parts are



$$
\begin{aligned}
&= \frac{1}{\epsilon^8}\left(\frac{7}{18}\right) + \frac{1}{\epsilon^7}\left(\frac{55}{24}\right) + \frac{1}{\epsilon^6}\left(-\frac{67}{9}\zeta_2 - \frac{797}{144}\right) + \frac{1}{\epsilon^5}\left(-\frac{442}{9}\zeta_3 - \frac{643}{18}\zeta_2 + \frac{1193}{144}\right) + \frac{1}{\epsilon^4}\left(-\frac{9199}{360}\zeta_2^2 - \frac{3547}{18}\zeta_3\right.\\
&\left. + \frac{7793}{72}\zeta_2 + \frac{1013}{48}\right) + \frac{1}{\epsilon^3}\left(-\frac{2858}{3}\zeta_5 + \frac{27617}{36}\zeta_3\zeta_2 - \frac{3439}{180}\zeta_2^2 + \frac{60893}{72}\zeta_3 - \frac{1897}{8}\zeta_2 - \frac{43895}{144}\right) + \frac{1}{\epsilon^2}\left(\frac{179927}{72}\zeta_3^2 - \frac{40853}{252}\zeta_2^3\right.\\
&\left. - 2780\zeta_5 + \frac{23467}{9}\zeta_3\zeta_2 + \frac{132359}{180}\zeta_2^2 - \frac{66607}{24}\zeta_3 - \frac{5423}{72}\zeta_2 + \frac{311383}{144}\right) + \frac{1}{\epsilon}\left(-\frac{1015395}{32}\zeta_7 + \frac{30493}{2}\zeta_5\zeta_2 + \frac{274199}{90}\zeta_3\zeta_2^2\right.\\
&\left. + \frac{44984}{9}\zeta_3^2 - \frac{540823}{420}\zeta_2^3 + \frac{477281}{24}\zeta_5 - \frac{412181}{36}\zeta_3\zeta_2 - \frac{117101}{30}\zeta_2^2 + \frac{410629}{72}\zeta_3 + \frac{400999}{72}\zeta_2 - \frac{622069}{48}\right) + \frac{122261}{15}\zeta_{5,3}\\
&+ \frac{1298525}{12}\zeta_5\zeta_3 - \frac{942899}{36}\zeta_3^2\zeta_2 - \frac{121150681}{9000}\zeta_2^4 - \frac{2558101}{16}\zeta_7 + \frac{360793}{6}\zeta_5\zeta_2 - \frac{53821}{18}\zeta_3\zeta_2^2 - \frac{1428953}{72}\zeta_3^2 + \frac{2037031}{168}\zeta_2^3\\
&- \frac{1989461}{24}\zeta_5 + \frac{526387}{12}\zeta_3\zeta_2 + \frac{245017}{18}\zeta_2^2 + \frac{738547}{72}\zeta_3 - \frac{1198061}{24}\zeta_2 + \frac{10519199}{144} + \mathcal{O}(\epsilon),
\end{aligned}
\tag{6}
$$



$$
\begin{aligned}
&= \frac{1}{\epsilon^8}\left(\frac{1}{18}\right) + \frac{1}{\epsilon^7}\left(\frac{13}{48}\right) + \frac{1}{\epsilon^6}\left(-\frac{4}{9}\zeta_2 - \frac{193}{192}\right) + \frac{1}{\epsilon^5}\left(-\frac{395}{72}\zeta_3 - \frac{175}{144}\zeta_2 + \frac{167}{48}\right) + \frac{1}{\epsilon^4}\left(-\frac{13577}{720}\zeta_2^2 - \frac{347}{18}\zeta_3\right.\\
&\left. + \frac{373}{96}\zeta_2 - \frac{385}{36}\right) + \frac{1}{\epsilon^3}\left(-\frac{5941}{24}\zeta_5 - \frac{799}{36}\zeta_3\zeta_2 - \frac{130661}{1440}\zeta_2^2 + \frac{1043}{32}\zeta_3 - \frac{1225}{72}\zeta_2 + \frac{1771}{72}\right) + \frac{1}{\epsilon^2}\left(-\frac{1687}{72}\zeta_3^2 - \frac{1495399}{5040}\zeta_2^3\right.\\
&\left. - \frac{23419}{24}\zeta_5 - \frac{6953}{24}\zeta_3\zeta_2 + \frac{13921}{90}\zeta_2^2 - \frac{6655}{72}\zeta_3 + \frac{730}{9}\zeta_2 + \frac{11}{4}\right) + \frac{1}{\epsilon^1}\left(-\frac{433869}{64}\zeta_7 - \frac{21667}{24}\zeta_5\zeta_2 + \frac{101723}{90}\zeta_3\zeta_2^2 - \frac{5267}{12}\zeta_3^2\right.\\
&\left. - \frac{14629019}{10080}\zeta_2^3 + \frac{257675}{96}\zeta_5 + \frac{30845}{144}\zeta_3\zeta_2 - \frac{981}{8}\zeta_2^2 + \frac{45509}{72}\zeta_3 - \frac{8633}{24}\zeta_2 - \frac{41461}{72}\right) + \frac{43916}{15}\zeta_{5,3} + \frac{44957}{4}\zeta_5\zeta_3\\
&+ \frac{133033}{36}\zeta_3^2\zeta_2 - \frac{257784979}{56000}\zeta_2^4 - \frac{1611457}{64}\zeta_7 - \frac{207481}{24}\zeta_5\zeta_2 + \frac{4033391}{720}\zeta_3\zeta_2^2 + \frac{846605}{288}\zeta_3^2 + \frac{4194707}{1440}\zeta_2^3 - \frac{76189}{12}\zeta_5\\
&+ \frac{61787}{36}\zeta_3\zeta_2 - \frac{168493}{360}\zeta_2^2 - \frac{59869}{12}\zeta_3 + \frac{48311}{36}\zeta_2 + \frac{197863}{36} + \mathcal{O}(\epsilon).
\end{aligned}
\tag{7}
$$

in the conventions of [36]. Combining the integral solutions obtained by direct integration with the result (6) allows us to determine

$$
\begin{aligned}
\mathcal{H} = {}& \frac{161}{16}\zeta_7 + \frac{5}{2}\zeta_5\zeta_2 - \frac{5}{2}\zeta_3\zeta_2^2 + 10\zeta_3^2 + \frac{223}{210}\zeta_2^3\\
&- 25\zeta_5 - 6\zeta_3\zeta_2 + \frac{3}{10}\zeta_2^2 + 9\zeta_3.
\end{aligned}
\tag{8}
$$

This value for $\mathcal{H}$ agrees with the expression conjectured in [31] and thus confirms the nonfermionic contributions to the collinear anomalous dimensions in that reference analytically. Moreover, this result provides the last remaining master integral coefficient required for the present calculation.

III. RESULTS FOR FORM FACTORS

In this section we present the complete fermionic four-loop corrections to the form factors F_q and F_g in massless QCD. We express the results in terms of $SU(N_c)$ color factors and use

$$
\begin{aligned}
&C_F = \frac{N_c^2-1}{2N_c}, \quad C_A = N_c, \quad N_A = N_c^2 - 1,\\
&N_F = N_c, \quad n_{q\gamma} = \sum_{q'}\frac{Q_{q'}}{Q_q}, \quad \frac{d_F^{abc}d_F^{abc}}{N_A} = \frac{N_c^2-4}{16N_c},\\
&\frac{d_F^{abcd}d_F^{abcd}}{N_A} = \frac{N_c^4 - 6N_c^2 + 18}{96N_c^2},\\
&\frac{d_A^{abcd}d_F^{abcd}}{N_A} = \frac{N_c(N_c^2+6)}{48},
\end{aligned}
\tag{9}
$$

where Q_q is the fractional charge of the quark q and n_f is the number of active quark flavors. Without loss of generality we have used for the trace normalization $T_F = 1/2$.

The ϵ expansion of the fermionic corrections to both form factors starts with seventh-order poles in $1/\epsilon$, reflecting the fact that fermionic corrections start to contribute only at two loops. Similarly, four-loop contributions with more than two closed fermion loops or specific color factors start even later in the ϵ expansion. The corresponding poles through to order $1/\epsilon$ can be obtained from [19]; they consist of zeta values with transcendental weight up to six.

Here, we calculate the complete finite part of the fermionic four-loop contributions to F_q and F_g and obtain an analytical result in terms of zeta values with transcendental weight up to seven. Our result for the finite part of $F_q^{(4)}$ reads

$$
\begin{aligned}
F_q^{(4)}|_{\epsilon^0} = {} & n_f C_F^3\left(\frac{1153615}{126}\zeta_7+\frac{6316}{9}\zeta_5\zeta_2+\frac{229468}{135}\zeta_3\zeta_2^2+\frac{547270}{81}\zeta_3^2+\frac{1341628}{2835}\zeta_2^3-\frac{3467995}{162}\zeta_5-\frac{192737}{81}\zeta_3\zeta_2-\frac{1420133}{1215}\zeta_2^2\right.\\
& \left.-\frac{38482147}{972}\zeta_3+\frac{12734681}{648}\zeta_2+\frac{7837713013}{419904}\right)+n_f C_A C_F^2\left(\frac{5669}{4}\zeta_7-\frac{249194}{135}\zeta_5\zeta_2-\frac{417244}{405}\zeta_3\zeta_2^2-\frac{11438080}{729}\zeta_3^2\right.\\
& \left.-\frac{38510}{63}\zeta_2^3+\frac{4959127}{243}\zeta_5+\frac{408107}{81}\zeta_3\zeta_2+\frac{8388679}{2916}\zeta_2^2+\frac{2642551543}{26244}\zeta_3-\frac{111491363}{2187}\zeta_2-\frac{326984889779}{3779136}\right)\\
& +n_f C_A^2 C_F\left(\frac{6943}{24}\zeta_7+\frac{2755}{9}\zeta_5\zeta_2+\frac{1912}{45}\zeta_3\zeta_2^2+\frac{202210}{27}\zeta_3^2+\frac{128953}{1620}\zeta_2^3-\frac{34844257}{3240}\zeta_5-\frac{119489}{108}\zeta_3\zeta_2-\frac{1251893}{540}\zeta_2^2\right.\\
& \left.-\frac{111467677}{1944}\zeta_3+\frac{123861583}{3888}\zeta_2+\frac{21075909203}{279936}\right)+n_f\frac{d_F^{abcd}d_F^{abcd}}{N_F}\left(-1240\zeta_7+\frac{992}{3}\zeta_5\zeta_2-\frac{3952}{15}\zeta_3\zeta_2^2+\frac{680}{9}\zeta_3^2\right.\\
& \left.+\frac{41620}{189}\zeta_2^3+\frac{95098}{27}\zeta_5+\frac{92}{3}\zeta_3\zeta_2+\frac{7552}{45}\zeta_2^2-\frac{21566}{9}\zeta_2-\frac{13414}{27}\zeta_3+\frac{3190}{3}\right)+n_f^2 C_F^2\left(\frac{689582}{729}\zeta_3^2+\frac{191252}{945}\zeta_2^3\right.\\
& \left.+\frac{187364}{1215}\zeta_5+\frac{177800}{729}\zeta_3\zeta_2+\frac{4777}{135}\zeta_2^2-\frac{90719803}{13122}\zeta_3+\frac{44208841}{13122}\zeta_2+\frac{5325319081}{944784}\right)+n_f^2 C_A C_F\left(-\frac{1714}{3}\zeta_3^2-\frac{2836}{315}\zeta_2^3\right.\\
& \left.+\frac{150886}{135}\zeta_5-\frac{436}{3}\zeta_3\zeta_2-\frac{3722}{135}\zeta_2^2+\frac{2897315}{486}\zeta_3-\frac{5825827}{1296}\zeta_2-\frac{3325501813}{279936}\right)+n_f^3 C_F\left(-\frac{2194}{135}\zeta_5-\frac{820}{81}\zeta_3\zeta_2+\frac{3322}{135}\zeta_2^2\right.\\
& \left.-\frac{20828}{243}\zeta_3+\frac{145115}{729}\zeta_2+\frac{10739263}{17496}\right)+n_{q\gamma}\frac{d_F^{abc}d_F^{abc}C_F}{N_F}\left(26624\zeta_7+1792\zeta_5\zeta_2+\frac{35584}{15}\zeta_3\zeta_2^2+\frac{30688}{9}\zeta_3^2-\frac{179152}{189}\zeta_2^3\right.\\
& \left.-\frac{170224}{27}\zeta_5-\frac{8656}{3}\zeta_3\zeta_2+\frac{60416}{45}\zeta_2^2-\frac{100624}{9}\zeta_2-\frac{71552}{27}\zeta_3-\frac{89360}{9}\right)+n_{q\gamma}\frac{d_F^{abc}d_F^{abc}C_A}{N_F}\left(-\frac{13972}{3}\zeta_7-1840\zeta_5\zeta_2\right.\\
& \left.-\frac{784}{5}\zeta_3\zeta_2^2-13008\zeta_3^2-\frac{618328}{189}\zeta_2^3+\frac{54436}{9}\zeta_5+\frac{15112}{3}\zeta_3\zeta_2-\frac{95692}{45}\zeta_2^2+\frac{45976}{9}\zeta_3+\frac{107984}{9}\zeta_2+\frac{65264}{9}\right)\\
& +n_{q\gamma}n_f\frac{d_F^{abc}d_F^{abc}}{N_F}\left(2304\zeta_3^2+\frac{545792}{945}\zeta_2^3-\frac{8512}{9}\zeta_5-\frac{1888}{3}\zeta_3\zeta_2+\frac{12448}{45}\zeta_2^2-\frac{3520}{9}\zeta_3-\frac{16928}{9}\zeta_2-\frac{11296}{9}\right)\\
& +\text{contributions without closed fermion loop.}
\end{aligned}
\tag{10}
$$

For the finite part of $F_g^{(4)}$ we obtain

$$
\begin{aligned}
F_g^{(4)}|_{\epsilon^0} = {}& n_f C_A^3\Big(\frac{365579}{63}\zeta_7-\frac{64151}{135}\zeta_5\zeta_2+\frac{410228}{405}\zeta_3\zeta_2^2-\frac{11278261}{2916}\zeta_3^2-\frac{1672838}{2835}\zeta_2^3-\frac{24219919}{4860}\zeta_5+\frac{638345}{486}\zeta_3\zeta_2\\
&-\frac{1464209}{29160}\zeta_2^2+\frac{483257171}{52488}\zeta_3-\frac{55297501}{34992}\zeta_2-\frac{322439904151}{7558272}\Big)+n_f C_A^2 C_F\Big(-\frac{66967}{36}\zeta_7+\frac{10022}{9}\zeta_5\zeta_2-\frac{8824}{9}\zeta_3\zeta_2^2\\
&+\frac{129887}{81}\zeta_3^2-\frac{163538}{945}\zeta_2^3+\frac{2166853}{1620}\zeta_5+\frac{91172}{81}\zeta_3\zeta_2-\frac{10414}{27}\zeta_2^2+\frac{25799269}{5832}\zeta_3-\frac{6154445}{3888}\zeta_2-\frac{821501405}{139968}\Big)\\
&+n_f C_A C_F^2\Big(2692\zeta_7-1420\zeta_5\zeta_2+\frac{2500}{3}\zeta_3\zeta_2^2+\frac{2720}{3}\zeta_3^2+\frac{376624}{945}\zeta_2^3-\frac{50591}{6}\zeta_5+\frac{1321}{3}\zeta_3\zeta_2-\frac{4057}{5}\zeta_2^2+\frac{85357}{36}\zeta_3\\
&+\frac{6451}{36}\zeta_2-\frac{24275}{432}\Big)+n_f C_F^3\Big(3360\zeta_7-2940\zeta_5-156\zeta_3+\frac{169}{2}\Big)+n_f\frac{d_A^{abcd}d_F^{abcd}}{N_A}\Big(\frac{2464}{3}\zeta_7+1824\zeta_5\zeta_2-\frac{1088}{3}\zeta_3\zeta_2^2\\
&-\frac{15700}{3}\zeta_3^2-\frac{245536}{945}\zeta_2^3+\frac{108692}{9}\zeta_5+\frac{1544}{9}\zeta_3\zeta_2-\frac{35108}{45}\zeta_2^2-\frac{89932}{9}\zeta_3+\frac{9580}{27}\zeta_2+\frac{6944}{9}\Big)+n_f^2 C_A^2\Big(\frac{717266}{729}\zeta_3^2\\
&+\frac{58858}{945}\zeta_2^3-\frac{62671}{1215}\zeta_5-\frac{87329}{1458}\zeta_3\zeta_2+\frac{380701}{3240}\zeta_2^2+\frac{27036815}{52488}\zeta_3-\frac{53253361}{104976}\zeta_2+\frac{110016540845}{7558272}\Big)\\
&+n_f^2 C_A C_F\Big(-\frac{4354}{3}\zeta_3^2-\frac{143524}{945}\zeta_2^3-\frac{47948}{27}\zeta_5-\frac{3688}{9}\zeta_3\zeta_2-\frac{13820}{81}\zeta_2^2-\frac{538853}{162}\zeta_3-\frac{27629}{81}\zeta_2+\frac{3697777}{324}\Big)\\
&+n_f^2 C_F^2\Big(\frac{2432}{3}\zeta_3^2+\frac{32512}{135}\zeta_2^3+3360\zeta_5+\frac{128}{3}\zeta_3\zeta_2-\frac{484}{3}\zeta_2^2-\frac{36260}{9}\zeta_3+\frac{439}{9}\zeta_2-\frac{80281}{216}\Big)\\
&+n_f^2\frac{d_F^{abcd}d_F^{abcd}}{N_A}\Big(512\zeta_3^2-960\zeta_5+\frac{384}{5}\zeta_2^2+1520\zeta_3-\frac{9008}{9}\Big)+n_f^3 C_A\Big(-\frac{14474}{135}\zeta_5+\frac{4556}{81}\zeta_3\zeta_2-\frac{1418}{27}\zeta_2^2\\
&-\frac{99890}{243}\zeta_3+\frac{38489}{729}\zeta_2-\frac{20832641}{17496}\Big)+n_f^3 C_F\Big(-\frac{508069}{324}+\frac{4288}{27}\zeta_5-64\zeta_3\zeta_2+\frac{2288}{27}\zeta_2^2+\frac{24812}{27}\zeta_3+\frac{3074}{27}\zeta_2\Big)\\
&+\text{contributions without closed fermion loop.} \qquad (11)
\end{aligned}
$$

The n_f^3, n_f^2, and $n_{q\gamma}n_f$ terms agree with the results presented in Refs. [20,21]. Our expression for F_q reproduces the result of Ref. [15] in the large-N_c limit. The remaining, subleading color terms for $F_q^{(4)}$ and all of the terms linear in n_f for $F_g^{(4)}$ are new.

IV. CONCLUSIONS

In this paper, we calculated the complete fermionic corrections to the photon-quark and Higgs-gluon vertices in massless four-loop QCD. We solved two nonplanar vertex topologies using the method of differential equations and found a result in terms of multiple zeta values. This renders the only two topologies which were not known to be linearly reducible accessible, such that the main obstacle for the remaining four-loop corrections has been removed. Our calculation confirms a previous conjecture for the analytical solution of one of the integrals in this topology, which fully establishes the pole terms of all nonfermionic four-loop corrections analytically.

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