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A new long-step interior point algorithm for linear programming based on the algebraic equivalent transformation

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Abstract In this paper, we investigate a new primal-dual long-step interior point algorithm for linear optimization. Based on the step-size, interior point algorithms can be divided into two main groups, short-step and long-step methods. In practice, long-step variants perform better, but usually, a better theoretical complexity can be achieved for the short-step methods. One of the exceptions is the large-update algorithm of Ai and Zhang. The new wide neighbourhood and the main characteristics of the presented algorithm are based on their approach. In addition, we use the algebraic equivalent transformation technique by Darvay to determine the search directions of the method.

Keywords Mathematical programming · Linear optimization · Interior point algorithms · Algebraic equivalent transformation technique

JEL Classification Number: C61

1 Introduction

In this paper, we propose a new long-step interior point algorithm for linear optimization. We consider the primal-dual linear programming (LP) problem pair in the following standard form:

$$\left.\begin{aligned} \min\ \mathbf{c}^T\mathbf{x} \\ A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}\right\} \qquad\qquad \left.\begin{aligned} \max\ \mathbf{b}^T\mathbf{y} \\ A^T\mathbf{y} + \mathbf{s} = \mathbf{c} \\ \mathbf{s} \geq \mathbf{0} \end{aligned}\right\} \tag{1}$$

where $A \in \mathbb{R}^{m\times n}$ with full row rank, $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$ are given.

The first practical polynomial time interior point algorithm for solving linear programming problems has been published by Karmarkar in 1984 [13]. Since then, this approach received much attention and numerous new interior point methods have been introduced not just for linear optimization, but for many other problem classes as well, such as linear complementarity problems (LCPs), convex optimization, symmetric optimization, second order cone optimization etc.

Based on the step-length, interior point algorithms can be divided into two main groups, short-step and long-step methods. Long-step methods perform better in practice, but generally, short-step variants have better theoretical complexity $O(\sqrt{n}L)$. In the last twenty years, different attempts have been made to overcome this issue, e.g., [3, 18, 21].

The wide neighbourhood $\mathcal{N}_\infty^-$ has been proposed by Kojima et al. [15]. Their algorithm turned out to be efficient in practice, and its complexity was $O(nL)$. In 2005, Ai and Zhang [2] introduced an interior point algorithm that works in a new wide neighbourhood of the central path and it is a long-step method. They proved that the method has the same theoretical complexity as the best-known short-step variants.

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Using the wide neighbourhood applied by Ai and Zhang, several authors proposed new long-step methods with the best known theoretical complexity. There are related results for linear programming [9,17,24], for horizontal linear complementarity problems [22] and also for semidefinite optimization [10,16,19].
To be able to determine new search directions in interior point algorithms, Darvay introduced the method of algebraic equivalent transformation [4]. His main idea was to apply a strictly increasing, continuously differentiable function φ to the centering equation of the central path system, then apply Newton's method to determine the new search directions. In his paper, Darvay applied the function $\varphi(t) = \sqrt{t}$, and introduced a new, short-step algorithm for linear optimization. Most algorithms in the literature can be considered as a special case of this technique, where $\varphi(t) = t$, i.e., the identity map. The function $\varphi(t) = t - \sqrt{t}$ has been introduced by Darvay et al. [8], also in context with linear optimization and has recently been investigated in several papers of Darvay and his coauthors. In 2020, they presented a corrector predictor IPA for linear optimization [5], and proposed another corrector predictor IPA for sufficient LCPs [7], while in 2021, they introduced a short-step IPA for sufficient LCPs [6]. In this paper, we investigate a new, long-step interior point algorithm for linear optimization, also based on the function $\varphi(t) = t - \sqrt{t}$. Furthermore, the function $\varphi(t) = \frac{\sqrt{t}}{2(1+\sqrt{t})}$ has been proposed by Kheirfam and Haghighi [14], to solve $\mathcal{P}^*(\kappa)$ linear complementarity problems.
Most of the algorithms based on the algebraic equivalent transformation technique are short-step variants, except for the method of Darvay et al. [9], which is based on the function $\varphi(t) = \sqrt{t}$ and applies an Ai-Zhang type wide neighbourhood.

Throughout this paper, we use the following notations. Scalars and indices are denoted by lowercase Latin letters. Vectors are denoted by bold lowercase Latin letters and we use uppercase Latin letters to denote matrices. Sets are denoted by capital calligraphic letters. Let $\mathbf{x}, \mathbf{s} \in \mathbb{R}^n$ be two vectors, then $\mathbf{xs}$ is the componentwise, namely Hadamard product of $\mathbf{x}$ and $\mathbf{s}$. $\mathbf{x}^+$ and $\mathbf{x}^-$ stand for the positive and negative part of the vector $\mathbf{x}$, i.e.,

$$\mathbf{x}^+ = \max\{\mathbf{x}, \mathbf{0}\} \in \mathbb{R}^n \text{ and } \mathbf{x}^- = \min\{\mathbf{x}, \mathbf{0}\} \in \mathbb{R}^n,$$

where the maximum and minimum are taken componentwise.
If $\alpha \in \mathbb{R}$, $\mathbf{x}^\alpha = [x_1^\alpha, x_2^\alpha, \dots, x_n^\alpha]^T$, and if $s_i \neq 0$ holds for all $i \in \{1, \dots, n\}$, then the fraction of $\mathbf{x}$ and $\mathbf{s}$ is the vector $\mathbf{x}/\mathbf{s} = [x_1/s_1, x_2/s_2 \dots, x_n/s_n]^T$. The vector of ones is denoted by $\mathbf{e}$. $\|\mathbf{x}\|$ is the Euclidean norm of $\mathbf{x}$, $\|\mathbf{x}\|_1 = \sum_{i=1}^n |x_i|$ denotes the L^1 (Manhattan) norm of $\mathbf{x}$, and $\|\mathbf{x}\|_\infty = \max_{i=1}^n |x_i|$ is the infinity norm of $\mathbf{x}$. $\operatorname{diag}(\mathbf{x})$ is the diagonal matrix with the elements of the vector $\mathbf{x}$ in its diagonal. Finally, $\mathcal{I}$ denotes the index set $\mathcal{I} = \{1, \dots, n\}$.
The paper is organized as follows. In Section 2 we give an overview of Darvay's algebraic equivalent transformation technique. In Section 3 we define a new wide neighbourhood, introduce a large-update interior point algorithm and examine its correctness. In the last subsection, we prove that the complexity of the new method is $O(\sqrt{n}L)$. In Section 4 we present our preliminary numerical results. Section 5 summarizes our conclusions.

2 The algebraic equivalent transformation technique

The optimality criteria of the primal-dual pair (1) can be formulated as:

$$\left.\begin{aligned} A\mathbf{x} = \mathbf{b},&\quad \mathbf{x} \geq \mathbf{0} \\ A^T\mathbf{y} + \mathbf{s} = \mathbf{c},&\quad \mathbf{s} \geq \mathbf{0} \\ \mathbf{xs} = \mathbf{0}.& \end{aligned}\right\}$$

In the case of interior point algorithms, instead of the third equation of the optimality criteria (the complementarity condition), we consider a relaxed version

$$\left.\begin{aligned} A\mathbf{x} = \mathbf{b},&\mathbf{x} \geq \mathbf{0} \\ A^T\mathbf{y} + \mathbf{s} = \mathbf{c},&\ \mathbf{s} \geq \mathbf{0} \\ \mathbf{xs} = \nu\mathbf{e},& \end{aligned}\right\} \tag{2}$$

where ν is a given positive parameter. This system is the central path problem belonging to the given primal-dual LP pair.

Let $\mathcal{F} = \{(\mathbf{x}, \mathbf{y}, \mathbf{s}) : A\mathbf{x} = \mathbf{b},\ A^T\mathbf{y} + \mathbf{s} = \mathbf{c},\ \mathbf{x} \geq \mathbf{0},\ \mathbf{s} \geq \mathbf{0}\}$ denote the set of primal-dual feasible solutions and $\mathcal{F}_+ = \{(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{F} : \mathbf{x} > \mathbf{0},\ \mathbf{s} > \mathbf{0}\}$ the set of strictly feasible solutions.
If $\mathcal{F}_+ \neq \emptyset$, then for each $\nu > 0$ system (2) has a unique solution [23], it is called the ν-center. Furthermore, as ν tends to 0, the ν-centers converge to a solution of the linear programming problem (1).
To be able to find new search directions, Darvay introduced the algebraic equivalent transformation technique (AET) [4]. His main idea was to transform the central path problem (2) to an equivalent form:

$$\left.\begin{aligned} A\mathbf{x} = \mathbf{b}, &\quad \mathbf{x} \geq \mathbf{0} \\ A^T\mathbf{y} + \mathbf{s} = \mathbf{c}, &\quad \mathbf{s} \geq \mathbf{0} \\ \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\nu}\right) = \varphi\left(\mathbf{e}\right), & \end{aligned}\right\} \tag{3}$$

where $\varphi : (\xi, \infty) \to \mathbb{R}$ is a continuously differentiable function with $\varphi'(t) > 0$ for all $t \in (\xi, \infty)$, $\xi \in [0, 1)$. However, the transformed system (3) does not modify the central path, it determines different search directions depending on the function φ. More precisely, if we are in the point $(\mathbf{x}, \mathbf{s}) \in \mathcal{F}_+ \subset \mathbb{R}^n$ and we take a step toward the $\nu = \tau\mu$-center, where $\mu = \mathbf{x}^T\mathbf{s}/n$ and $\tau \in (0, 1)$ is a given update parameter, then applying Newton's method to (3), the search direction $(\Delta\mathbf{x}, \Delta\mathbf{s})$ is the solution of the following system:

$$\left.\begin{aligned} A\Delta\mathbf{x} &= \mathbf{0} \\ A^T\Delta\mathbf{y} + \Delta\mathbf{s} &= \mathbf{0} \\ \mathbf{s}\Delta\mathbf{x} + \mathbf{x}\Delta\mathbf{s} &= \tau\mu\frac{\varphi(\mathbf{e}) - \varphi\left(\frac{\mathbf{x}\mathbf{s}}{\tau\mu}\right)}{\varphi'\left(\frac{\mathbf{x}\mathbf{s}}{\tau\mu}\right)}. \end{aligned}\right\} \tag{4}$$

To make the analysis of interior point algorithms easier, we usually consider a scaled version of (4). Let

$$\mathbf{v} = \sqrt{\frac{\mathbf{x}\mathbf{s}}{\tau\mu}}, \quad \mathbf{dx} = \frac{\mathbf{v}\Delta\mathbf{x}}{\mathbf{x}}, \quad \mathbf{ds} = \frac{\mathbf{v}\Delta\mathbf{s}}{\mathbf{s}}, \text{ and } \bar{A} = A \operatorname{diag}\left(\frac{\mathbf{v}}{\mathbf{s}}\right).$$

With these notations, the scaled Newton-system can be written as:

$$\left.\begin{aligned} \bar{A}\mathbf{dx} &= \mathbf{0} \\ \bar{A}^T\Delta\mathbf{y} + \mathbf{ds} &= \mathbf{0} \\ \mathbf{dx} + \mathbf{ds} &= \mathbf{p}_\varphi, \end{aligned}\right\}$$

where

$$\mathbf{p}_\varphi = \frac{\varphi(\mathbf{e}) - \varphi(\mathbf{v}^2)}{\mathbf{v}\varphi'(\mathbf{v}^2)}.$$

In this paper, we investigate the function $\varphi(t) = t - \sqrt{t}$, $t > 1/2$ (i.e., $\xi = 1/2$) introduced by Darvay et al. [8]. Since we fixed the function φ, from now on, we omit the subscript φ and simply write

$$\mathbf{p} = \frac{2(\mathbf{v} - \mathbf{v}^2)}{2\mathbf{v} - \mathbf{e}}.$$

Our goal is to introduce a new long step interior point algorithm based on this function. To be able to prove the correctness of this method, we need to ensure that $\mathbf{p}$ is well-defined. Therefore we assume that $v_i > 1/2$ is satisfied for all $i \in \mathcal{I}$.
Let p be the function for which $p(v_i) = p_i$ holds for all $v_i \in (1/2, \infty)$, i.e.,

$$p : \left(\frac{1}{2}, \infty\right) \to \mathbb{R}, \quad p(t) = \frac{2(t - t^2)}{2t - 1}.$$

Throughout the analysis, we will also investigate different estimations of the function $p(t)$.

3 The new algorithm

The main idea of Ai and Zhang was to decompose the Newton-directions into positive and negative parts and use different step-lengths with the two components [2]. If we apply this approach to the system (4), we get the following two systems:

$$\left.\begin{aligned} A\Delta\mathbf{x}_- &= \mathbf{0}\\ A^T\Delta\mathbf{y}_- + \Delta\mathbf{s}_- &= \mathbf{0}\\ \mathbf{s}\Delta\mathbf{x}_- + \mathbf{x}\Delta\mathbf{s}_- &= \tau\mu\mathbf{v}\mathbf{p}^- \end{aligned}\right\} \qquad \left.\begin{aligned} A\Delta\mathbf{x}_+ &= \mathbf{0}\\ A^T\Delta\mathbf{y}_+ + \Delta\mathbf{s}_+ &= \mathbf{0}\\ \mathbf{s}\Delta\mathbf{x}_+ + \mathbf{x}\Delta\mathbf{s}_+ &= \tau\mu\mathbf{v}\mathbf{p}^+, \end{aligned}\right\} \tag{5}$$

and the new point with step length $\alpha = (\alpha_1, \alpha_2)$ will be $\mathbf{x}(\alpha) = \mathbf{x} + \alpha_1\Delta\mathbf{x}_- + \alpha_2\Delta\mathbf{x}_+$ and $\mathbf{s}(\alpha) = \mathbf{s} + \alpha_1\Delta\mathbf{s}_- + \alpha_2\Delta\mathbf{s}_+$.
It is important to notice that $\Delta\mathbf{x}_+$ is not the positive part of $\Delta\mathbf{x}$ (in this case the sign $+$ is a subscript instead of a superscript), it is the solution of the system with $\mathbf{p}^+$ on its right hand side. The notation is similar for the other solutions of these systems.
We introduce the index sets $\mathcal{I}_+ = \{i \in \mathcal{I} : x_i s_i \le \tau\mu\} = \{i \in \mathcal{I} : v_i \le 1\}$, and $\mathcal{I}_- = \mathcal{I} \setminus \mathcal{I}_+$. Under the technical assumption $v_i > \frac{1}{2}$, the nonnegativity of a coordinate p_i is equivalent to $i \in \mathcal{I}_+$.
To be able to formulate the scaled version of the two systems, we introduce the following notations:

$$\mathbf{dx}_- = \frac{\mathbf{v}\Delta\mathbf{x}_-}{\mathbf{x}},\ \mathbf{ds}_- = \frac{\mathbf{v}\Delta\mathbf{s}_-}{\mathbf{s}},\ \mathbf{dx}_+ = \frac{\mathbf{v}\Delta\mathbf{x}_+}{\mathbf{x}},\ \mathbf{ds}_+ = \frac{\mathbf{v}\Delta\mathbf{s}_+}{\mathbf{s}},$$

and the scaled systems are

$$\left.\begin{aligned} \bar{A}\mathbf{dx}_- &= \mathbf{0}\\ \bar{A}^T\Delta\mathbf{y}_- + \mathbf{ds}_- &= \mathbf{0}\\ \mathbf{dx}_- + \mathbf{ds}_- &= \mathbf{p}^-, \end{aligned}\right\} \qquad \left.\begin{aligned} \bar{A}\mathbf{dx}_+ &= \mathbf{0}\\ \bar{A}^T\Delta\mathbf{y}_+ + \mathbf{ds}_+ &= \mathbf{0}\\ \mathbf{dx}_+ + \mathbf{ds}_+ &= \mathbf{p}^+. \end{aligned}\right\} \tag{6}$$

3.1 Wide neighbourhood

The wide neighbourhood $\mathcal{N}_\infty^-$ has been introduced by Kojima et al. [15]. It is defined as follows:

$$\mathcal{N}_\infty^-(1-\tau) = \{(\mathbf{x},\mathbf{y},\mathbf{s}) \in \mathcal{F}_+ : \mathbf{xs} \ge \tau\mu\mathbf{e}\} = \{(\mathbf{x},\mathbf{y},\mathbf{s}) \in \mathcal{F}_+ : \mathbf{v} \ge \mathbf{e}\}.$$

Notice that this means that a point is in the neighbourhood $\mathcal{N}_\infty^-(1-\tau)$ if and only if the corresponding index set $\mathcal{I}_+$ is empty, namely $\mathbf{p}^+ = \mathbf{0}$. In the analysis, we are going to use a new neighbourhood that depends only on the positive part of vector $\mathbf{p}$:

$$\mathcal{W}(\tau,\beta) = \left\{(\mathbf{x},\mathbf{y},\mathbf{s}) \in \mathcal{F}_+ : \|\mathbf{p}^+\| \le \beta \text{ and } \mathbf{v} > \frac{1}{2}\mathbf{e}\right\},$$

where $0 < \beta < 1/2$ is a given parameter value. The role of the technical condition $\mathbf{v} > \mathbf{e}/2$ has been discussed at the end of Section 2. This neighbourhood is a modification of the one introduced by Ai and Zhang [2] (since they require $\|\mathbf{v}\mathbf{p}^+\| \le \beta$) and it is equivalent to the one used by Darvay and Takcs for the function $\varphi(t) = \sqrt{t}$ in [9].
Following the idea of Ai and Zhang [2], the next lemma verifies that $\mathcal{W}(\tau,\beta)$ is indeed a wide neighbourhood:

Lemma 1 *Let $0 < \beta < 1/2$ and $0 < \tau < 1$ be given parameters, and let $\gamma = 1/4\ (1 + \sqrt{1-2\beta})^2\tau$. Then*

$$\mathcal{N}_\infty^-(1-\tau) \subseteq \mathcal{W}(\tau,\beta) \subseteq \mathcal{N}_\infty^-(1-\gamma).$$

Proof If $(\mathbf{x},\mathbf{y},\mathbf{s}) \in \mathcal{N}_\infty^-(1-\tau)$, then $\|\mathbf{p}^+\| = 0 < \beta$ and $\mathbf{v} \ge \mathbf{e} > 1/2\mathbf{e}$.
For the second inclusion, let $(\mathbf{x},\mathbf{y},\mathbf{s}) \in \mathcal{W}(\tau,\beta)$ and assume indirectly that there exists an index $i \in \mathcal{I}$ for which $x_i s_i < \gamma\mu$, i.e., $v_i^2 < \gamma/\tau = 1/4\ (1+\sqrt{1-2\beta})^2$.
Since $p(t)$ is a strictly decreasing function,

$$p_i = p(v_i) > \frac{2\left(\sqrt{\frac{\gamma}{\tau}} - \frac{\gamma}{\tau}\right)}{2\sqrt{\frac{\gamma}{\tau}} - 1} = \frac{\beta}{\sqrt{1-2\beta}} > \beta,$$

which is a contradiction.

The following lower and upper bounds on the coordinates of the vector $\mathbf{v}$ will be useful for different estimations during the analysis.

Corollary 1 *Let* $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{W}(\tau, \beta)$ *then*

$$\frac{1+\sqrt{1-2\beta}}{2} \leq v_i \leq 1 \qquad \forall i \in \mathcal{I}_+,$$
$$1 < v_i \leq \sqrt{n/\tau} \qquad \forall i \in \mathcal{I}_-.$$

Proof The first statement directly follows from Lemma 1. The upper bound $v_i \leq \sqrt{n/\tau}$ holds for all $i \in \mathcal{I}$ since

$$\sum_{i\in\mathcal{I}} v_i^2 = \sum_{i\in\mathcal{I}} \frac{x_i s_i}{\tau\mu} = \frac{1}{\tau\mu}\mathbf{x}^T\mathbf{s} = \frac{n}{\tau}. \tag{7}$$

Before presenting the analysis, we give the pseudocode of the interior point algorithm.

Input: $A \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c} \in \mathbb{R}^n$
the update parameter $0 < \tau < 1$,
the neighbourhood parameter $0 < \beta < 1$,
the accuracy parameter $\varepsilon > 0$,
an initial point $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{s_0}) \in \mathcal{W}(\tau, \beta)$.

$\mathbf{x} := \mathbf{x}_0$, $\mathbf{y} := \mathbf{y}_0$, $\mathbf{s} := \mathbf{s}_0$ and $\mu := \mu_0 = \mathbf{x}_0^T\mathbf{s}_0/n$
while $\mathbf{x}^T\mathbf{s} > \epsilon$ **do**
  Determine $\Delta\mathbf{x}_+, \Delta\mathbf{s}_+, \Delta\mathbf{y}_+$ and $\Delta\mathbf{x}_-, \Delta\mathbf{s}_-, \Delta\mathbf{y}_-$ according to (5);
  Set $\alpha_2 = 1$ and $\alpha_1 = \max\{\alpha_1 \in [0,1] : (\mathbf{x}(\alpha), \mathbf{y}(\alpha), \mathbf{s}(\alpha)) \in \mathcal{W}(\tau, \beta)\}$,
  where $\mathbf{x}(\alpha) = \mathbf{x} + \alpha_1\Delta\mathbf{x}_- + \alpha_2\Delta\mathbf{x}_+$, $\mathbf{y}(\alpha) = \mathbf{y} + \alpha_1\Delta\mathbf{y}_- + \alpha_2\Delta\mathbf{y}_+$ and
  $\mathbf{s}(\alpha) = \mathbf{s} + \alpha_1\Delta\mathbf{s}_- + \alpha_2\Delta\mathbf{s}_+$;
  $(\mathbf{x}, \mathbf{y}, \mathbf{s}) := (\mathbf{x}(\alpha), \mathbf{y}(\alpha), \mathbf{s}(\alpha))$;
  $\mu := \mathbf{x}^T\mathbf{s}/n$;
end

Algorithm 1: Outline of the algorithm

During the analysis, we consider the case of $\alpha_2 = 1$, i.e., we take a full Newton-step in the direction $(\Delta\mathbf{x}_+, \Delta\mathbf{s}_+)$, and determine a value of α_1 so that the desired complexity of the algorithm can be achieved. From now on, we assume that a point $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{W}(\tau, \beta)$ is given, and in the next section, we prove the correctness of the algorithm.

3.2 Analysis of the algorithm

Let us introduce the following notations:

$$\mathbf{dx}(\alpha) = \alpha_1\mathbf{dx}_- + \alpha_2\mathbf{dx}_+, \quad \mathbf{ds}(\alpha) = \alpha_1\mathbf{ds}_- + \alpha_2\mathbf{ds}_+,$$

$$\mathbf{h}(\alpha) = \tau\mu\mathbf{v}^2 + \alpha_1\tau\mu\mathbf{v}\mathbf{p}^- + \alpha_2\tau\mu\mathbf{v}\mathbf{p}^+,$$

where $\alpha_1, \alpha_2 \in [0,1]$ are given step-lengths, their values will be specified later. With these notations, $\mathbf{x}(\alpha)\mathbf{s}(\alpha) = (\mathbf{x} + \alpha_1\Delta\mathbf{x}_- + \alpha_2\Delta\mathbf{x}_+)(\mathbf{s} + \alpha_1\Delta\mathbf{s}_- + \alpha_2\Delta\mathbf{s}_+)$ can be written as

$$\mathbf{x}(\alpha)\mathbf{s}(\alpha) = \mathbf{h}(\alpha) + \tau\mu\mathbf{dx}(\alpha)\mathbf{ds}(\alpha).$$

It is important to notice that the search directions are orthogonal as usually for LP problems, since

$$\mathbf{dx}(\alpha)^T\mathbf{ds}(\alpha) = \alpha_1^2\mathbf{dx}_-^T\mathbf{ds}_- + \alpha_1\alpha_2(\mathbf{dx}_-^T\mathbf{ds}_+ + \mathbf{dx}_+^T\mathbf{ds}_-) + \alpha_2^2\mathbf{dx}_+^T\mathbf{ds}_+,$$

furthermore $\mathbf{dx}_+$ and $\mathbf{dx}_-$ are in the kernel of the matrix $\bar{A}$, while $\mathbf{ds}_+$ and $\mathbf{ds}_-$ are in the rowspace of $\bar{A}$ (see system (6)), therefore all four scalar products are 0 in the previous expression.
The next two lemmas give lower bounds on the value of $\mathbf{h}(\alpha)$.

Lemma 2 *Let* $\alpha \in [0,1]^2$, *then* $h_i(\alpha) \geq \tau\mu$ *for all* $i \in \mathcal{I}_-$.

Proof In the case of $i \in \mathcal{I}_-$, $v_i > 1$ and $h_i(\alpha) = \tau\mu v_i(v_i + \alpha_1 p_i)$. We need to prove that $v_i(v_i + \alpha_1 p_i) \geq 1$, i.e., $\alpha_1 \leq \frac{1-v_i^2}{v_i p_i}$ holds.
Let us examine the expression $\frac{1-t^2}{tp(t)}$ over the interval $(1, \infty)$:

$$\frac{1-t^2}{tp(t)} = \frac{1-t^2}{t}\frac{2t-1}{2t(1-t)} = \frac{2t^2+t-1}{2t^2} = 1 + \frac{t-1}{2t^2} > 1.$$

On the other hand, $\alpha_1 \leq 1$ by definition. Thus $h_i(\alpha) \geq \tau\mu$ holds for all $i \in \mathcal{I}_-$.

We show that $\mathbf{h}(\alpha)$ is a componentwise strictly positive vector.

Lemma 3 *Let* $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{W}(\tau, \beta)$ *and* $\alpha \in [0, 1]^2$, *then* $\mathbf{h}(\alpha) \geq \gamma\mu\mathbf{e}$, *and consequently* $\mathbf{h}(\alpha) > \mathbf{0}$.

Proof By Lemma 1, $\tau\mu v_i^2 = \mathbf{x}_i\mathbf{s}_i \geq \gamma\mu$ for all $i \in \mathcal{I}$. Furthermore, if $i \in \mathcal{I}_+$, then $v_i p_i > 0$, so $h_i(\alpha) \geq \tau\mu v_i^2 \geq \gamma\mu$.
In the case of $i \in \mathcal{I}_-$, the statement is a consequence of Lemma 2, since $h_i(\alpha) \geq \tau\mu \geq \gamma\mu$.

To be able to prove the feasibility of the new iterates and to ensure that they stay in the neighbourhood $\mathcal{W}(\tau, \beta)$, we need the following technical lemma:

Lemma 4 *Let* $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{W}(\tau, \beta)$, $\alpha_1 = \sqrt{\frac{\beta\tau}{n}}$ *and* $\alpha_2 = 1$. *Then*

$$\| \left[\mathbf{dx}(\alpha)\mathbf{ds}(\alpha)\right]^- \|_1 = \| \left[\mathbf{dx}(\alpha)\mathbf{ds}(\alpha)\right]^+ \|_1 \leq \frac{1}{2}\beta.$$

Proof According to Lemma 3.5 of Ai and Zhang [2] and using the orthogonality of $\mathbf{dx}(\alpha)$ and $\mathbf{ds}(\alpha)$, we have

$$\begin{aligned} &\|[\mathbf{dx}(\alpha)\mathbf{ds}(\alpha)]^-\|_1 = \|[\mathbf{dx}(\alpha)\mathbf{ds}(\alpha)]^+\|_1 \leq \frac{1}{4}\|\mathbf{dx}(\alpha) + \mathbf{ds}(\alpha)\|^2 \\ =&\frac{1}{4}\|\alpha_1(\mathbf{dx}_- + \mathbf{ds}_-) + \alpha_2(\mathbf{dx}_+ + \mathbf{ds}_+)\|^2 = \frac{1}{4}\left(\alpha_1^2\|\mathbf{p}^-\|^2 + \alpha_2^2\|\mathbf{p}^+\|^2\right). \end{aligned}$$

By the definition of $\mathcal{W}(\tau, \beta)$, we have $\|\mathbf{p}^+\| \leq \beta$. We need to estimate the term $\|\mathbf{p}^-\|^2$:

$$\|\mathbf{p}^-\|^2 = \sum_{i \in \mathcal{I}_-}\left(v_i - \frac{v_i}{2v_i - 1}\right)^2 \leq \sum_{i \in \mathcal{I}_-} v_i^2 \leq \sum_{i \in \mathcal{I}} v_i^2 = \frac{n}{\tau},$$

according to (7).
Using these two estimations and substituting the values of α_1 and α_2, we can write

$$\frac{1}{4}\left(\alpha_1^2\|\mathbf{p}^-\|^2 + \alpha_2^2\|\mathbf{p}^+\|^2\right) \leq \frac{1}{4}\frac{\beta\tau}{n}\frac{n}{\tau} + \frac{1}{4}\beta^2 = \frac{1}{4}\beta + \frac{1}{4}\beta^2 \leq \frac{1}{2}\beta.$$

The next lemma gives a positive lower bound on the vector $\mathbf{x}(\alpha)\mathbf{s}(\alpha)$, which is the first step to prove the strict feasibility of the new point.

Lemma 5 *Let* $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{W}(\tau, \beta)$, $\alpha_1 = \sqrt{\frac{\beta\tau}{n}}$ *and* $\alpha_2 = 1$. *Then*

$$\mathbf{x}(\alpha)\mathbf{s}(\alpha) \geq \frac{1 - 2\beta + \sqrt{1 - 2\beta}}{2}\tau\mu\mathbf{e}$$

holds.

Proof By Lemma 3, we have $\mathbf{h}(\alpha) \geq \gamma\mu\mathbf{e}$. Using Lemma 4 and substituting the value of γ, we get

$$\begin{aligned} \mathbf{x}(\alpha)\mathbf{s}(\alpha) &= \mathbf{h}(\alpha) + \tau\mu\mathbf{dx}(\alpha)\mathbf{ds}(\alpha) \geq \gamma\mu\mathbf{e} - \tau\mu\|[\mathbf{dx}(\alpha)\mathbf{ds}(\alpha)]^-\|_1\mathbf{e} \\ &\geq \gamma\mu\mathbf{e} - \tau\mu\frac{1}{2}\beta\mathbf{e} = \tau\mu\left(\frac{\gamma}{\tau} - \frac{\beta}{2}\right)\mathbf{e} = \frac{1 - 2\beta + \sqrt{1 - 2\beta}}{2}\tau\mu\mathbf{e}. \end{aligned}$$

The following statement is the linear programming analogue of Proposition 3.2 by Ai and Zhang [2] (they proposed it for monotone linear complementarity problems). The proof remains the same.

Lemma 6 *Let* $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{F}^+$ *and* $(\Delta\mathbf{x}, \Delta\mathbf{y}, \Delta\mathbf{s})$ *be the solution of the system*

$$\begin{aligned} A\Delta\mathbf{x} &= \mathbf{0} \\ A^T\Delta\mathbf{y} + \Delta\mathbf{s} &= \mathbf{0} \\ \mathbf{s}\Delta\mathbf{x} + \mathbf{x}\Delta\mathbf{s} &= \mathbf{z}. \end{aligned}$$

If $\mathbf{z} + \mathbf{x}\mathbf{s} > 0$ *and* $(\mathbf{x} + t_0\Delta\mathbf{x})(\mathbf{s} + t_0\Delta\mathbf{s}) > 0$ *holds for some* $t_0 \in (0, 1]$*, then* $\mathbf{x} + t\Delta\mathbf{x} > 0$ *and* $\mathbf{s} + t\Delta\mathbf{s} > 0$ *for all* $t \in (0, t_0]$*.*

We have already proved that $h(\alpha) > 0$ for all $\alpha \in [0, 1]$ (see Lemma 3), and $\mathbf{x}(\alpha)\mathbf{s}(\alpha) > 0$ for $\alpha_1 = \sqrt{\beta\tau/n}$ and $\alpha_2 = 1$ (see Lemma 5), therefore by Lemma 6, we have that the new points are also strictly positive, namely $\mathbf{x}(\alpha) > \mathbf{0}$ and $\mathbf{s}(\alpha) > \mathbf{0}$.
The following two statements propose bounds on the duality gap of the new point: $\mu(\alpha) = \mathbf{x}(\alpha)^T\mathbf{s}(\alpha)/n$.

Lemma 7 *Let* $\alpha_1 = \sqrt{\frac{\beta\tau}{n}}$ *and* $\alpha_2 = 1$*, then* $\mu(\alpha) \geq (1 - \alpha_1)\,\mu$*.*

Proof Since $\mathbf{v}^T\mathbf{p}^+ \geq 0$, moreover $v_ip_i = v_i^2 - \frac{v_i^2}{2v_i-1} \leq v_i^2$ for all $i \in \mathcal{I}_-$ and by (7),

$$\begin{aligned} \mu(\alpha) = \frac{\mathbf{x}(\alpha)^T\mathbf{s}(\alpha)}{n} &= \mu + \frac{\alpha_1\tau\mu}{n}\mathbf{v}^T\mathbf{p}^- + \frac{\alpha_2\tau\mu}{n}\mathbf{v}^T\mathbf{p}^+ \geq \mu + \frac{\alpha_1\tau\mu}{n}\mathbf{v}^T\mathbf{p}^- \\ &= \mu - \frac{\alpha_1\tau\mu}{n}\sum_{i\in\mathcal{I}_-}\frac{2v_i(v_i^2 - v_i)}{2v_i - 1} \geq \mu - \frac{\alpha_1\tau\mu}{n}\sum_{i\in\mathcal{I}} v_i^2 = (1 - \alpha_1)\,\mu. \end{aligned}$$

The following theorem guarantees the proper reduction of the duality gap after an iteration:

Lemma 8 *Assume that* $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{W}(\tau, \beta)$*,* $\alpha_1 = \sqrt{\frac{\beta\tau}{n}}$ *and* $\alpha_2 = 1$*. Then*

$$\mu(\alpha) \leq \left(1 - \sqrt{\frac{\beta\tau}{n}}\left[\frac{8}{9}(1-\tau) - \sqrt{\beta\tau}\right]\right)\mu. \tag{8}$$

Proof

$$\mu(\alpha) = \frac{\mathbf{x}(\alpha)^T\mathbf{s}(\alpha)}{n} = \mu + \frac{\alpha_1\tau\mu}{n}\mathbf{v}^T\mathbf{p}^- + \frac{\alpha_2\tau\mu}{n}\mathbf{v}^T\mathbf{p}^+.$$

First, let us estimate the term $\mathbf{v}^T\mathbf{p}^+$:

$$\mathbf{e}^T\left(\mathbf{v}\mathbf{p}^+\right) = \left\|\mathbf{v}\mathbf{p}^+\right\|_1 \leq \sqrt{n}\left\|\mathbf{v}\mathbf{p}^+\right\| \leq \sqrt{n}\beta. \tag{9}$$

The first inequality holds since $\mathbf{e}^T\mathbf{u} \leq \|\mathbf{u}\|_1$, and applying the Cauchy-Schwartz inequality we get the second estimation. Using the property $v_i < 1$ when $i \in \mathcal{I}_+$, and the definition of the neighbourhood $\mathcal{W}(\tau, \beta)$, the last inequality can also be verified.
To get an upper bound on the expression $\mathbf{v}^T\mathbf{p}^-$, consider the inequalities $2\mathbf{v} - \mathbf{e} > \mathbf{0}$ and $v_i > 1$ for all $i \in \mathcal{I}_-$:

$$\begin{aligned} \mathbf{v}^T\mathbf{p}^- = \mathbf{e}^T\left(\mathbf{v}\frac{2(\mathbf{v} - \mathbf{v}^2)^-}{2\mathbf{v} - \mathbf{e}}\right) &= \sum_{i\in\mathcal{I}_-}\frac{2v_i(v_i - v_i^2)}{2v_i - 1} \\ &= \sum_{i\in\mathcal{I}_-}\frac{2v_i^2}{(1 + v_i)(2v_i - 1)}(1 - v_i^2) \\ &\leq \sum_{i\in\mathcal{I}_-}\frac{8}{9}(1 - v_i^2) \leq \sum_{i\in\mathcal{I}}\frac{8}{9}(1 - v_i^2) = \frac{8}{9}n\left(1 - \frac{1}{\tau}\right). \end{aligned} \tag{10}$$

Using (9) and (10) we obtain

$$\mu(\alpha) \leq \mu + \frac{\alpha_1\tau\mu}{n}\frac{8}{9}n\left(1 - \frac{1}{\tau}\right) + \frac{\alpha_2\tau\mu}{n}\sqrt{n}\beta = \left(1 - \alpha_1\left[\frac{8}{9}(1-\tau) - \sqrt{\beta\tau}\right]\right)\mu.$$

Notice, that the upper bound on $\mu(\alpha)$ in (8) is positive for all $\beta, \tau \in (0,1)$. Indeed,

$$1 - \sqrt{\frac{\beta\tau}{n}} \left[\frac{8}{9} (1-\tau) - \sqrt{\beta\tau} \right] \geq 1 - \frac{8}{9}(1-\tau) > \frac{1}{9}.$$

With a suitable parameter setting, we can ensure that the duality gap decreases strictly monotonically, i.e., $\mu(\alpha) < \mu$.

Corollary 2 *Let $\tau \leq 1/2$ and $\beta \leq 1/4$. If $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{W}(\tau, \beta)$, $\alpha_1 = \sqrt{\frac{\beta\tau}{n}}$ and $\alpha_2 = 1$, then $\mu(\alpha) < \mu$ holds.*

Proof We need to check whether the multiplier of μ in inequality (8) is less than 1. This means that $8/9(1-\tau) - \sqrt{\beta\tau} > 0$ and this holds when $\beta < 64/81(1-\tau)^2/\tau$, which is satisfied for our choice of parameter values.

In addition to strict feasibility, we also need to prove the fulfilment of the technical condition $\mathbf{v}(\alpha) = \sqrt{\frac{\mathbf{x}(\alpha)\mathbf{s}(\alpha)}{\tau\mu(\alpha)}} > \frac{1}{2}\mathbf{e}$.

Lemma 9 *Let $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{W}(\tau, \beta)$, $\alpha_1 = \sqrt{\frac{\beta\tau}{n}}$ and $\alpha_2 = 1$. If $\beta < \frac{\sqrt{3}}{4}$, then $\mathbf{v}(\alpha) > \frac{1}{2}\mathbf{e}$ holds.*

Proof From Lemma 5 and Corollary 2, we have

$$\mathbf{v}^2(\alpha) = \frac{\mathbf{x}(\alpha)\mathbf{s}(\alpha)}{\tau\mu(\alpha)} \geq \frac{1 - 2\beta + \sqrt{1-2\beta}}{2}\mathbf{e}. \tag{11}$$

Since $\frac{1-2\beta+\sqrt{1-2\beta}}{2} > \frac{1}{4}$ if $\beta < \frac{\sqrt{3}}{4}$, we have proved the statement.

To show that the new iterates remain in the neighbourhood $\mathcal{W}(\tau, \beta)$, we need another technical lemma:

Lemma 10 *Let $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{W}(\tau, \beta)$, $\alpha_1 = \sqrt{\frac{\beta\tau}{n}}$ and $\alpha_2 = 1$. Then*

$$\|[\tau\mu(\alpha)\mathbf{e} - \mathbf{h}(\alpha)]^+\| \leq \beta\tau\mu(\alpha) \left(1 - \frac{1 + \sqrt{1-2\beta}}{2} \right).$$

Proof Based on Lemma 2, $\tau\mu(\alpha) - h_i(\alpha) \leq 0$ for all $i \in \mathcal{I}_-$, therefore we need to examine indices only from $\mathcal{I}_+$.
Since $1/2 < v_i \leq 1$ for all $i \in \mathcal{I}_+$, we have

$$\frac{1-v_i^2}{p_i} = \frac{(1-v_i^2)(2v_i - 1)}{2v_i(1-v_i)} = \frac{2v_i^2 + v_i - 1}{2v_i} = v_i + \frac{1}{2} - \frac{1}{2v_i} \leq v_i \leq 1 \quad \forall i \in \mathcal{I}_+. \tag{12}$$

Using Corollary 2 and (12), we obtain for $i \in \mathcal{I}_+$ that

$$\begin{aligned} \tau\mu(\alpha) - h_i(\alpha) &= \tau\mu(\alpha) - \tau\mu\left(v_i^2 + v_i p_i\right) \leq \tau\mu(\alpha)\left(1 - v_i^2 - v_i p_i\right) \\ &= \tau\mu(\alpha) p_i (1 - v_i) \leq \tau\mu(\alpha) p_i \left(1 - \frac{1+\sqrt{1-2\beta}}{2}\right), \end{aligned}$$

where in the last estimation, we used the first statement of Corollary 1.
This, together with the definition of $\mathcal{W}(\tau, \beta)$ concludes the proof,

$$\|(\tau\mu(\alpha)\mathbf{e} - \mathbf{h}(\alpha))^+\| \leq \tau\mu(\alpha)\|\mathbf{p}^+\| \left(1 - \frac{1+\sqrt{1-2\beta}}{2}\right) \leq \beta\tau\mu(\alpha)\left(1 - \frac{1+\sqrt{1-2\beta}}{2}\right).$$

Now we are ready to prove that after an iteration, if the right hand side of the third equation in the Newton system (6) is denoted by $\mathbf{p}(\alpha)$, then $\|\mathbf{p}(\alpha)^+\| \leq \beta$ holds. Together with Lemma 9, this means that the new iterates after the Newton-step remain in the neighbourhood $\mathcal{W}(\tau, \beta)$.

Lemma 11 *Let $\beta \leq \frac{1}{8}$, $\tau \leq \frac{1}{8}$. If $(\mathbf{x}, \mathbf{y}, \mathbf{s}) \in \mathcal{W}(\tau, \beta)$, $\alpha_1 = \sqrt{\frac{\beta\tau}{n}}$ and $\alpha_2 = 1$, then the new point stays in the neighbourhood, namely $(\mathbf{x}(\alpha), \mathbf{y}(\alpha), \mathbf{s}(\alpha)) \in \mathcal{W}(\tau, \beta)$.*

Proof By definition and Lemma 9, we need to prove

$$\left\|\mathbf{p}(\alpha)^+\right\| = \left\|\left[\frac{2\mathbf{v}(\alpha)(\mathbf{e}-\mathbf{v}(\alpha))}{2\mathbf{v}(\alpha)-\mathbf{e}}\right]^+\right\| \le \beta.$$

Since $\frac{2\mathbf{v}(\alpha)}{2\mathbf{v}^2(\alpha)+\mathbf{v}(\alpha)-\mathbf{e}} > \mathbf{0}$ when $\mathbf{v}(\alpha) > 1/2\mathbf{e}$,

$$\begin{aligned}\left\|\mathbf{p}(\alpha)^+\right\| &= \left\|\frac{2\mathbf{v}(\alpha)}{(2\mathbf{v}(\alpha)-\mathbf{e})\,(\mathbf{e}+\mathbf{v}(\alpha))}\left[\mathbf{e}-\mathbf{v}^2(\alpha)\right]^+\right\| \\ &\le \left\|\frac{2\mathbf{v}(\alpha)}{2\mathbf{v}^2(\alpha)+\mathbf{v}(\alpha)-\mathbf{e}}\right\|_\infty \left\|\left[\mathbf{e}-\mathbf{v}^2(\alpha)\right]^+\right\|. \qquad (13)\end{aligned}$$

Let $q : \left(\frac{1}{2},\infty\right) \to \mathbb{R}$ and $q(t) = \frac{2t}{2t^2+t-1}$. This function is strictly decreasing on its domain, therefore using (11), the first term in (13) can be estimated as

$$\left\|\frac{2\mathbf{v}(\alpha)}{2\mathbf{v}^2(\alpha)+\mathbf{v}(\alpha)-\mathbf{e}}\right\|_\infty \le q\left(\sqrt{\frac{1-2\beta+\sqrt{1-2\beta}}{2}}\right), \qquad (14)$$

where the expression $\sqrt{(1-2\beta+\sqrt{1-2\beta})/2}$ is strictly decreasing in β, therefore the upper bound is strictly increasing in β.
To give an upper bound on $\left\|\left[\mathbf{e}-\mathbf{v}^2(\alpha)\right]^+\right\|$, we use Lemmas 10, 4 and then 7:

$$\begin{aligned}\left\|\left[\mathbf{e}-\mathbf{v}^2(\alpha)\right]^+\right\| &= \frac{1}{\tau\mu(\alpha)}\left\|\left[\tau\mu(\alpha)\mathbf{e}-\mathbf{x}(\alpha)\mathbf{s}(\alpha)\right]^+\right\| \\ &\le \frac{1}{\tau\mu(\alpha)}\left(\left\|\left[\tau\mu(\alpha)\mathbf{e}-\mathbf{h}(\alpha)\right]^+\right\| + \tau\mu\left\|\left[\mathbf{dx}(\alpha)\mathbf{ds}(\alpha)\right]^-\right\|\right) \\ &\le \frac{1}{\tau\mu(\alpha)}\left(\beta\tau\mu(\alpha)\left(1-\frac{1+\sqrt{1-2\beta}}{2}\right)+\tau\mu\frac{\beta}{2}\right) \\ &\le \beta\left(\frac{1-\sqrt{1-2\beta}}{2}+\frac{1}{2-2\sqrt{\beta\tau}}\right), \qquad (15)\end{aligned}$$

where the last term is strictly increasing both in β and τ.
Using the just proved inequalities (13), (14) and (15), we obtain

$$\left\|\left(\frac{2\mathbf{v}(\alpha)(\mathbf{e}-\mathbf{v}(\alpha))}{2\mathbf{v}(\alpha)-\mathbf{e}}\right)^+\right\| \le \beta\left[q\left(\sqrt{\frac{1-2\beta+\sqrt{1-2\beta}}{2}}\right)\left(\frac{1-\sqrt{1-2\beta}}{2}+\frac{1}{2-2\sqrt{\beta\tau}}\right)\right]. \qquad (16)$$

To prove that this expression is less than or equal to β, we need to ensure that the value of the term in square brackets is at most 1. Notice, that by the monotonicity of the estimations (14) and (15), their product is also strictly increasing both in β and τ. Moreover, substituting $\beta = \tau = 1/8$, the coefficient of β on the right hand side of (16) is less than 0.77, which concludes the proof.

3.3 Complexity of the new algorithm

Theorem 1 *Let $\beta = \tau = \frac{1}{8}$, $\alpha_1 = \sqrt{\frac{\beta\tau}{n}}$, $\alpha_2 = 1$, and suppose that a starting point $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{s}_0) \in \mathcal{W}(\tau,\beta)$ is given. Then the algorithm provides an ε-optimal solution in*

$$O\left(\sqrt{n}\log\frac{\mathbf{x}_0^T\mathbf{s}_0}{\varepsilon}\right)$$

iterations.

Proof Let $(\mathbf{x}_k, \mathbf{y}_k, \mathbf{s}_k)$ denote the point given by the algorithm in the k^{th} iteration. According to Lemma 8, the following inequality holds for the duality gap in the k^{th} iteration:

$$\frac{\mathbf{x}_k^T\mathbf{s}_k}{n} = \mu_k \le \mu_{k-1}\left(1-\sqrt{\frac{\beta\tau}{n}}\left[\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right]\right) \le \mu_0\left(1-\sqrt{\frac{\beta\tau}{n}}\left[\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right]\right)^k.$$

From this, we get that $\mathbf{x}_k^T\mathbf{s}_k \le \varepsilon$ holds if

$$\left(1-\sqrt{\frac{\beta\tau}{n}}\left[\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right]\right)^k \mu_0 n \le \varepsilon$$

is satisfied. Taking the logarithm of both sides, we obtain

$$k\log\left[1-\sqrt{\frac{\beta\tau}{n}}\left(\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right)\right]+\log(\mu_0 n)\le\log\varepsilon.$$

Using the inequality $-\log(1-\vartheta)\ge\vartheta$, we can require the fulfilment of the stronger inequality

$$-k\sqrt{\frac{\beta\tau}{n}}\left(\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}\right)+\log(\mu_0 n)\le\log\varepsilon.$$

The last inequality is satisfied when

$$k\ge\sqrt{\frac{n}{\beta\tau}}\frac{1}{\frac{8}{9}(1-\tau)-\sqrt{\tau\beta}}\log\left(\frac{\mathbf{x}_0^T\mathbf{s}_0}{\varepsilon}\right),$$

and this proves the statement.

4 Numerical results

To test the efficiency of the algorithm, we implemented it in Matlab and solved 60 linear programming problem instances from the Netlib library [11]. The numerical experiments were carried out on a Dell laptop with Intel i7 processor and 16 GB RAM.
First, we transformed the problems to standard form, then eliminated the redundant constraints using the procedure `eliminateRedundantRows.m` from [20]. After these reformulations, we applied a similar method to procedure CLEAN of Adler [1] to eliminate fix-valued variables from the linear programming problems.
To be able to give strictly feasible initial points in the neighbourhood $\mathcal{W}(\tau,\beta)$, we first transformed the problems to symmetric form, then applied the self-dual embedding technique [25]. To avoid doubling the number of constraints in the first case, we carried out this reformulation according to the last Remark of Jansen et al. [12, p. 232]. For the embedded problem, we may choose $\mathbf{x}=\mathbf{e}$ and $\mathbf{s}=\mathbf{e}$ as proper initial points since they are strictly feasible and included in the neighbourhood $\mathcal{W}(\tau,\beta)$.
The number of rows and columns after the reformulations and the embedding procedure are denoted by m and n, respectively, and are shown in Table 1.
The step-lengths α_1 and α_2 were calculated in the following greedy way. We fixed the value of α_2 as 1 and determined the largest α_1 value so that the new point $(\mathbf{x}(\alpha),\mathbf{y}(\alpha),\mathbf{s}(\alpha))$ remains in the neighbourhood $\mathcal{W}(\tau,\beta)$.
To compare this algorithm to the methods introduced by Ai and Zhang [2] based on the function $\varphi(t)=t$ (but we used the slightly different neighbourhood $\mathcal{W}(\tau,\beta)$, see the beginning of subsection 3.1), and Darvay and Takcs [9] using the function $\varphi(t)=\sqrt{t}$, we also implemented these versions and compared the results.
The value of the precision parameter ε was 10^{-6}. The number of iterations and the running time (in seconds) required to achieve this precision (i.e., to find a point such that the duality gap is less than ε) for the different algorithm variants are shown in Table 1.

			t		$\sqrt{t}$		$t-\sqrt{t}$	
	m	**n**	**Iterations**	**Time (s)**	**Iterations**	**Time (s)**	**Iterations**	**Time (s)**
25fv47	1487	2974	22	131.7569	21	136.6264	22	144.4022
adlittle	135	270	27	0.3356	26	0.3584	27	0.3673
afiro	39	78	10	0.0223	10	0.0232	10	0.0294
agg	489	978	13	3.4002	13	3.4435	13	3.2791
agg2	760	1520	36	28.9524	35	27.8367	36	28.7152
agg3	760	1520	45	39.4284	41	35.4031	44	38.0878
bandm	431	862	17	3.2326	17	3.5622	17	3.2464
beaconfd	219	438	19	0.7566	18	0.6311	19	0.7159
blend	50	100	9	0.0246	9	0.0272	9	0.0280
bnl1	1234	2468	13	42.6191	13	43.5853	13	43.2657
bnl2	2094	4188	15	238.8595	16	255.0724	15	237.5986
bore3d	137	274	12	0.3919	13	0.2259	13	0.1838
brandy	264	528	44	2.7736	41	2.4552	44	2.6572
degen2	759	1518	25	19.7824	24	19.2664	25	20.1494
e226	412	824	35	6.2382	33	5.9164	35	6.2463
etamacro	575	1150	18	7.5531	18	7.5368	18	7.8146
fffff800	681	1362	17	12.2472	17	11.8787	17	11.8125
finnis	988	1976	63	114.2177	59	108.0446	64	115.3623
fit1d	2077	4154	37	709.9261	35	724.4431	37	729.4192
fit1p	2078	4156	39	643.8464	36	595.2378	38	630.1816
ganges	1888	3776	29	321.2697	29	332.4219	29	331.8121
gfrd_pnc	1026	2052	25	54.6242	24	51.8854	25	53.3827
grow15	1247	2494	31	112.2873	28	101.0702	32	115.7778
grow7	583	1166	30	13.5660	28	13.0630	29	13.1419
israel	318	636	49	4.2383	45	3.8616	48	4.1214
kb2	47	94	10	0.0276	10	0.0337	9	0.0332
lotfi	232	464	23	0.9472	23	1.0001	23	1.0383
maros	1339	2678	18	88.2226	19	92.7842	19	91.5384
nug05	227	454	12	0.4838	13	0.5307	12	0.4927
nug06	488	976	14	3.5828	14	3.7011	14	3.5186
nug07	933	1866	21	32.3303	21	36.8220	21	35.2970
nug08	1634	3268	17	139.0829	17	138.9054	16	133.7236
osa_07	1083	2166	11	30.0147	12	33.6809	11	31.1808
osa_14	2302	4604	11	270.7508	12	287.0070	11	265.6473
pilotnov	2356	4712	31	901.6259	32	927.0451	31	906.7585
recipe	142	284	18	0.2314	18	0.2413	18	0.2439
sc105	91	182	8	0.0536	9	0.0696	8	0.0606
sc205	172	344	8	0.1730	9	0.1886	8	0.1779
sc50a	46	92	8	0.0262	8	0.0238	8	0.0233
sc50b	45	90	8	0.0262	9	0.0283	8	0.0286
scagr25	517	1034	15	4.6002	14	4.1373	15	4.4784
scagr7	121	242	11	0.1129	11	0.1093	11	0.1133
scfxm1	489	978	22	6.0823	22	6.1991	22	6.0934
scfxm2	981	1962	23	44.7811	23	43.1529	23	42.8958
scfxm3	1473	2946	23	127.3734	23	128.9046	23	127.4328
scorpion	135	270	15	0.1852	14	0.1712	15	0.1835
scrs8	745	1490	12	10.9358	12	11.1499	12	11.0733
sctap1	156	312	7	0.1273	7	0.1391	7	0.1326
scsd1	762	1524	17	15.1827	17	14.2434	17	14.4794
scsd6	1352	2704	22	101.3877	23	105.6440	23	107.6813
scsd8	2752	5504	19	729.4550	20	770.2499	20	767.5080
share2b	161	322	21	0.3521	20	0.3750	21	0.3929
ship04l	1956	3912	30	377.0082	29	361.8736	30	372.6752
ship08s	1900	3800	43	653.7719	41	528.1463	42	537.2088
standata	1286	2572	38	137.0147	36	131.0305	38	145.1998
standgub	1287	2574	38	150.8201	36	140.6808	38	147.5780
standmps	806	1612	24	26.0594	24	25.7239	24	25.7260
stocfor1	125	250	11	0.1191	12	0.1745	11	0.1518
stocfor2	2285	4570	13	363.8573	14	388.0362	13	370.9294
wood1p	1805	3610	62	849.9000	59	811.0538	61	844.0545
		Average	22.7333	126.3176	22.2000	124.6189	22.7000	125.6258

Table 1: Numerical results for the Netlib test problems

According to our numerical results, there is no significant difference in the performance of the three algorithms for linear programming problems. The number of iterations is exactly the same for the three variants in 21 cases, and the difference is only one iteration for 26 test problems (out of the 60).
In terms of running time, the three variants also perform similarly, although the average running time of the second algorithm is slightly better.
To further our research, we are planning to generalize this method to sufficient linear complementarity problems and we expect that the choice of the function φ will cause a much more significant difference in the performance of the different variants.

5 Conclusion

We investigated a new long-step interior point algorithm based on the algebraic equivalent transformation technique, using the function $\varphi(t) = t - \sqrt{t}$ and a new Ai-Zhang-type wide neighbourhood.
We proved that the algorithm is well-defined and provides an ε-optimal solution in $O(\sqrt{n}L)$ steps, therefore it has the same theoretical complexity as the best short-step variants. According to our preliminary numerical results, the new algorithm performs well in practice.
To extend our results, we would like to propose a similar long-step algorithm for $\mathcal{P}^*(\kappa)$ linear complementarity problems, based on the function $\varphi(t) = t - \sqrt{t}$.

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