[The Mathematics Enthusiast](https://scholarworks.umt.edu/tme)

Volume 19
Number 2 *Number 2*

Article 15

6-2022

About the Teaching and Learning of Differentiability for Piecewise Functions in Science Degrees' First-Year Calculus Courses

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ABSTRACT: In this work, we present an issue we have observed over the last several years in the first year of Spanish science degrees' (such as mathematics, physics or engineering) calculus courses. It is related to the study of differentiability of piecewise one real variable real functions at a point. We have analyzed how students study the differentiability of piecewise functions at a point, explaining the students' work and reasoning and commenting about the common misunderstandings we have found. Then, we have researched about how to help the students in their learning process related to this calculus concept, and we have also used several activities or ways to work: student meeting groups to talk about their worries and misunderstandings, mathematical definitions, explanations and theory and also working out specific examples.

Keywords: One-variable calculus, differentiability, derivative, $\mathscr{C}^1$ class of functions, assemblies, mathematics education.

The Mathematics Enthusiast, **ISSN 1551-3440, vol. 19, no. 2**, pp. 677–686

1 Introduction

Nowadays, the concept of differentiability is something that the Spanish students usually learn in the last grades of high school and go deeper into it in the first year of different science degrees (for example, mathematics, physics or engineering degrees). The importance of learning this concept well and also using it appropriately in many fields of science is paramount. That's why it is included in the obligatory high school syllabi and also in many calculus or general mathematic courses of different college degrees.

Differentiability could be considered one of the concepts, among others, that is generally studied or applied to simpler or more basic functions in high school. But then, when students arrive to university and have to use and apply it to a general type of more peculiar or complex functions, where any kind of function could be considered, the confusion arises. This is because in Spanish high schools precise definitions, theorems and mathematical reasonings are not worked on in depth nor precisely[1]. Those are the things that give us the strength and knowledge to face the difficulties and solve the problems based on pure mathematics.

After several years of experience teaching in the first year of mathematics, physics and engineering degrees, we have realized that there are some affirmations or results that are used in high school that are not really true in general. That is, there are some affirmations that might not be true for every element of a set; for example, functions, matrixes or numbers. So, when we teach in first-year courses, we know that we have to open our minds, be patient and do a lot of work clarifying and helping students unlearn in order to better establish the main concepts of the topics that they have already learnt a bit in high school.

We think that the main reason for using these affirmations that only work for some specific elements of a set in high school is to simplify the mathematics and the work required. And it could be, to have more time to teach other topics that are in the syllabus of the corresponding course.

On the other hand, there is another matter that causes difficulties to freshman year students on their way to understanding and work out the exercises or problems we deal with in the first year of calculus courses in college. As we already anticipated a bit a few paragraphs before, the way mathematics is taught in general in Spanish high schools is usually not in a very deep way nor very precise mathematically. In different meetings we have had with these students and also based on what we have observed in their calculus' reasonings and work, we have concluded the following. In Spanish high schools, it is very common to not teach or work with the exact definitions of different concepts (for example, the derivative of a function), mathematical language, notation or reasoning and not even a bit of theory behind the concepts they study in each chapter of the corresponding syllabus. When the students arrive to university and receive a calculus course where the concepts are defined more precisely and corresponding theoretical results are taught, some of them with their proofs, they receive a big shock. The change of approach to teaching and working out mathematical concepts (from a more mechanical way, in the high school, to a bit more theoretical or profound way, in the university) is not easy for them. In the university, the students find difficulties to reason and write more detailed explanations for the steps they are working out in the problems using the theory taught in the calculus course. This problem is usually very common in different countries and several research articles that deal with it can be found in the literature, see for example [Bre], [BGMLT] or [GL].

There are mainly two things that we want to analyze and show in this research work. First, the misunderstanding we have found in the students' reasoning when studying the differentiability for piecewise one variable functions. Secondly, the way we have worked to understand and improve this misunderstanding in collaboration with the students.

As far as we know, there has not been any similar research that has been done in the recent last years. But we can find some interesting research related to calculus courses in the university and also the teaching and learning of the concepts of derivative and differentiability. For example, in [DCVM] we can find some didactic strategies to guide the teaching and learning process of the derivative concept for real functions of one real variable. On the other hand, in [Jut1] and [Jut2] we can see some research about how the students understand continuity and differentiability in calculus courses and also the students' choices of representations and their strategies to justify their reasonings. More precisely, in [BGMLT], [FT], [Ort] and [Tal2] we can find different research, opinions and thoughts related to the difficulties in teaching and learning the concepts related to differentiability, such as limits, notations and the concept and meaning of the derivative.

To achieve our purposes, we have divided our article as follows. First, in Section 2, we will remember the definitions of continuity, differentiability and $\mathscr{C}^1$ class of functions and remark about them. In Section 3, we will present the piecewise functions where we want to study differentiability and explain how many students do this kind of exercises. Later, in the same section, we will explain the main error or misunderstanding we find in the

[1] This situation seems to be common in other countries too, and many researchers have a similar opinion, see for example [BGMLT], [Mor], [SS] or [Tal1].

students' reasoning and also give some theoretical foundations to understand better the mathematics behind this issue. We will finish this section by working out a piecewise function's continuity, differentiability and $\mathscr{C}^1$ class property analysis completely and in detail.

Section 4 has been dedicated to explaining our experience finding the best manner in which to help the students in their learning process. We have explained the procedure we have followed and also the results we have seen. Finally, in Section 5, we have summarized the observations and conclusions we have reached after this useful and enriching research. We have also proposed five exercises where the students can practice the study of continuity, differentiability and $\mathscr{C}^1$ class of functions (by themselves or with the help of their teachers or professors).

2 Definitions of continuity, differentiability and $\mathscr{C}^1$ class of functions

In this section, we will remember the definitions of continuity, differentiability and $\mathscr{C}^1$ class of functions.

Definition 2.1 (Continuity at a point)**.** Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function which domain is D and let $x = a$ be a point in D. f is said to be continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists and

$$\lim_{x \to a} f(x) = f(a).$$

We will also remember the discontinuity types.

Definition 2.2 (Discontinuity types)**.**

1. If the function has equal finite lateral limits at a point $x = a$ but $f(a)$ doesn't exist or is not equal to the lateral limits, then, the discontinuity is called removable discontinuity.

2. If the lateral limits at a point $x = a$ are finite but not equal, one of the lateral limits is finite and the other is infinite or both lateral limits are infinite, then, the discontinuity is called of the first kind.

3. If at least one of the lateral limits doesn't exist at a point $x = a$, then, the discontinuity is called of the second kind or essential.

Remark 2.3. Here, we remember that if a limit is equal to infinity, it is considered that it doesn't exist.

Definition 2.4 (Differentiability at a point)**.** Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function which domain is D and let $x = a$ be a point in D. f is said to be differentiable in $x = a$ if the following limit exists,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

In that case, we will define the derivative of f at the point $x = a$ and denote it by $f'(a)$ as that limit.

Remark 2.5. The limit of the above definition could be replaced by this one,

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

The quotient in this limit is also called Newton's quotient.

There is another definition we will need in order to explain and develop our ideas in this article, and one that Spanish high-school students usually don't learn in their mathematics courses. The $\mathscr{C}^1$ class of functions set we will define now is usually included in the first or second year college calculus courses' syllabus.

Definition 2.6 ($\mathscr{C}^1$ class at a point)**.** Let $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function which domain is D and let $x = a$ be a point in D. f is said to be of differentiability class $\mathscr{C}^1$ at $x = a$ if it is differentiable in $x = a$ and if its derivative is continuous at the same point.

3 How some students analyze the differentiability of piecewise functions

The first time Spanish high-school students learn about differentiability is in 11th grade (in Spain we call it *primero de bachiller*)[2]. As far as we know, in high school, students learn that they have to use the definition to study the differentiability of a function at a point, above all if it's an uncertain point. The problem or issue we want to analyze here is when the students want to study the differentiability of a piecewise function at a changing point. That is, imaging that we have the following piecewise function, for example,

$$f(x) = \begin{cases} f_1(x), & x \in D, \\ f_2(x), & x \notin D, \end{cases}$$

where D is a subset of $\mathbb{R}$. We will suppose that f is continuous in ∂D (the boundary of D), f_1 is differentiable in $\mathring{D}$ (the open set of D) and f_2 is differentiable in $\mathring{D^c}$ (the open set of the complement of D).

When the students arrive to the first year of science degrees and want to analyze the differentiability of only one function at a point, they use the definition of differentiability. But, when they want to analyze these kind of piecewise functions, many students show in their work that instead of using the differentiability definition, they use the continuity of the derivative of the function f. That is, they calculate the derivative of f in this way,

$$f'(x) = \begin{cases} f_1'(x), & x \in \mathring{D}, \\ f_2'(x), & x \in \mathring{D^c}. \end{cases}$$

And then, if $a \in \partial D$, they study the continuity of f' at $x = a$ following these arguments:

1) If $\lim\limits_{x\to a^+} f_1'(x)$ and $\lim\limits_{x\to a^-} f_2'(x)$ exist and are equal, then, f is differentiable at $x = a$ and $f'(a) = \lim\limits_{x\to a^+} f_1'(x) = \lim\limits_{x\to a^-} f_2'(x)$.

2) If $\lim\limits_{x\to a^+} f_1'(x)$ or $\lim\limits_{x\to a^-} f_2'(x)$ don't exist or both exist but are not equal, then, f is not differentiable at $x = a$.

So, this is the argument that is behind the students' above reasoning, respectively:

1) If f' is continuous at $x = a$, then, f will be differentiable at $x = a$.

2) If f' is not continuous at $x = a$, then, f will not be differentiable at $x = a$.

Reviewing the theory of continuity, differentiability and $\mathscr{C}^1$ class of functions and formulating some theorems in the next section, we will see that the above first argument is correct but not the second one.

3.1 What is the error in the students' work?

We will start remembering the main theory and results[3] that relate continuity to differentiability.

Theorem 3.1. *If $f : D \subset \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable at a point $a \in D$, then, f is continuous at a as well.*

After seeing this theorem, we will know that all differentiable functions at a point will be continuous at the same point. Moreover, we can also conclude the contrapositive of what the theorem assures, if a function is not continuous at a point, then, it will not be differentiable at the same point.

On the other hand, looking at the definitions we gave before, in Section 2, we can recognize that what the students want to study by doing their procedure is if a function is of $\mathscr{C}^1$ class. Although they are not really conscious of this, because in most of the science degrees' first-year calculus courses, the students don't even know what $\mathscr{C}^1$ functions are or their relation to differentiability.

So, looking at the definitions of continuity, differentiability and $\mathscr{C}^1$ class functions at a point, we observe that this relation holds,

$$\mathscr{C}^1 \subsetneq \{\text{differentiable functions}\} \subsetneq \{\text{continuous functions}\}. \tag{3.1}$$

[2]We have to mention that, sometimes, we have found students that didn't learn it in high school or hardly worked with the definition. This will cause more disadvantages and difficulties in the learning process of a course like Calculus in College.

[3]In order to not extend too long or get too technical, as that is not the purpose of this work, we will skip the proofs of the results we see here. However, if the reader is interested in them, the proofs and technical details can be found in [Abb], [Bur] and [Spi], for example.

Therefore, when students do what we explained before, as they are analyzing if a function is of $\mathscr{C}^1$ class at a point, due to relation (3.1), if the result is positive, they are not making any error. Because if a function is $\mathscr{C}^1$ at a point, it will also be differentiable at the same point. But the problem comes when they study if a function is $\mathscr{C}^1$ at a point and the result is negative. What happens if a function is not $\mathscr{C}^1$ at a point, will it be differentiable or not? That is the key or the main mathematical and learning problem we want to show in this article. We think that when we ask or explain the students this question, we are forcing them to an area of blind spot in their knowledge. As they previously haven't needed or haven't studied the basic analysis theory of this topic, they feel like they don't have foundations to base their arguments on. So, this could be a point where a reeducation could start to happen little by little.

Next, we will see two interesting and significant results in order to understand better what we are dealing with in this article (they could be found in [Bur]). In Proposition 3.2, we will see what mathematical result we can obtain with the students' procedure. On the other hand, in Proposition 3.3, we will see what the characteristic of the functions that are differentiable at a point but are not of $\mathscr{C}^1$ class at that point is, that is, their derivative is not continuous at that point. These are the kind of functions that cause the students' procedure to fail or escape their understanding in that area of blind spot we have mentioned before.

Proposition 3.2. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function where $I = [a, a+\delta)$ or $I = (a-\delta, a]$, $\delta > 0$ and $a \in \mathbb{R}$. If f is continuous in the interval I, f is differentiable in $I - \{a\}$ and $\lim\limits_{x\to a^+} f'(x)$ or $\lim\limits_{x\to a^-} f'(x)$ exist, then, f is differentiable from the right or from the left and that derivative is $f'(a^+) = \lim\limits_{x\to a^+} f'(x)$ or $f'(a^-) = \lim\limits_{x\to a^+} f'(x)$.*

Proposition 3.3. *Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a function. If f is differentiable in I, then, the only possible discontinuities of f' have to be essential discontinuities.*

As we can see in (3.1), there are functions that are differentiable but are not in $\mathscr{C}^1$ class, that is, their derivative is not continuous. We can find several functions of this type, for example, we will analyze the next one completely working it out step by step.

Example 1.

$$f(x) = \begin{cases} x^3 \cdot \cos\left(\frac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We will study the continuity in $\mathbb{R}$ of the above f function, then, the differentiability in $\mathbb{R}$ and, lastly, $\mathscr{C}^1$ class in $\mathbb{R}$. First, we observe that the domain of f is $\mathbb{R}$.

a) Continuity in $\mathbb{R}$.

f is a piecewise function. In $\mathbb{R}\backslash\{0\}$, $x \mapsto \cos(1/x^2)$ is a composition of continuous functions and $x \mapsto x^3\cos(1/x^2)$ is a product of two continuous functions. So, f will be continuous in $\mathbb{R}\backslash\{0\}$.

We will now analyze what happens when $x = 0$. For that purpose, we will calculate the limit when x tends to 0.

$$\lim_{x\to 0} x^3 \cdot \cos\left(\frac{1}{x^2}\right) = 0 \cdot (\text{bounded}) = 0.$$

Thus, as $f(0) = 0$ and $\lim\limits_{x\to 0} f(x) = 0$, $\lim\limits_{x\to 0} f(x) = f(0)$ and f is continuous at $x = 0$. Therefore, f is continuous in all $\mathbb{R}$.

b) Differentiability in $\mathbb{R}$.

In $\mathbb{R}\backslash\{0\}$, $x \mapsto \cos(1/x^2)$ is a composition of two differentiable functions and $x \mapsto x^3\cos(1/x^2)$ is a multiplication of two differentiable functions. So, f will be differentiable when $x \neq 0$.

We will now analyze what happens when $x = 0$. For that purpose, we will use Definition 2.4 and calculate the following limit,

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{h^3 \cos\left(\frac{1}{h^2}\right) - 0}{h} = \lim_{h\to 0} h^2 \cos\left(\frac{1}{h^2}\right)$$
$$= 0 \cdot (\text{bounded}) = 0.$$

Therefore, f is differentiable at $x = 0$ and its derivative is $f'(0) = 0$.
We can now conclude that f is differentiable in $\mathbb{R}$.

c) $\underline{\mathscr{C}^1 \text{ in } \mathbb{R}}$.

First, remember by Definition 2.6, that, as we have already proved that f is differentiable in $\mathbb{R}$, if we want to know if f is of $\mathscr{C}^1$ class, we will have to study the continuity of f'.

Using the derivative rules and the information we obtained in the previous part, we obtain that

$$f'(x) = \begin{cases} 3x^2 \cdot \cos\left(\frac{1}{x^2}\right) + 2\sin\left(\frac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We can observe that, when $x \neq 0$, f' is continuous because it is built by composition of continuous functions and the sum and multiplication of continuous functions. Hence, f' is continuous when $x \neq 0$.
Now, we will study what happens at $x = 0$. Let we calculate the limit of f' at $x = 0$,

$$\lim_{x\to 0} 3x^2 \cdot \cos\left(\frac{1}{x^2}\right) + 2\sin\left(\frac{1}{x^2}\right) = 0 \cdot (\text{bounded}) + \nexists.$$

Therefore, the above limit doesn't exist and f' is not continuous at $x = 0$. Thus, f is not of $\mathscr{C}^1$ class at $x = 0$.

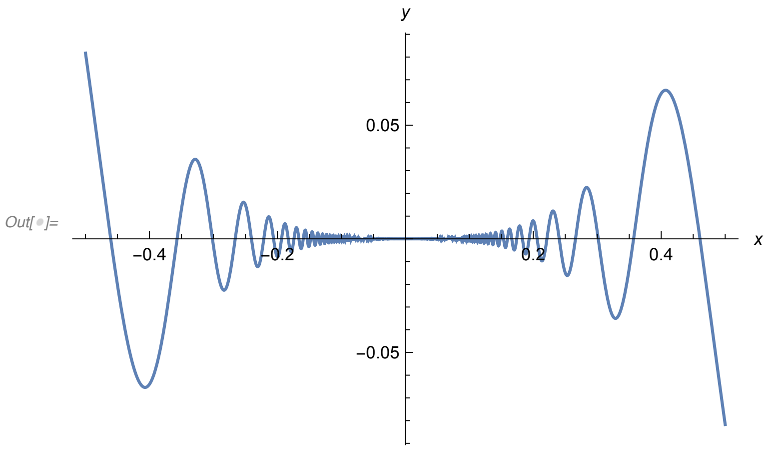
With this example, we have confirmed that we can't write an equal in the following relation

$$\mathscr{C}^1 \subsetneq \{\text{differentiable functions}\},$$

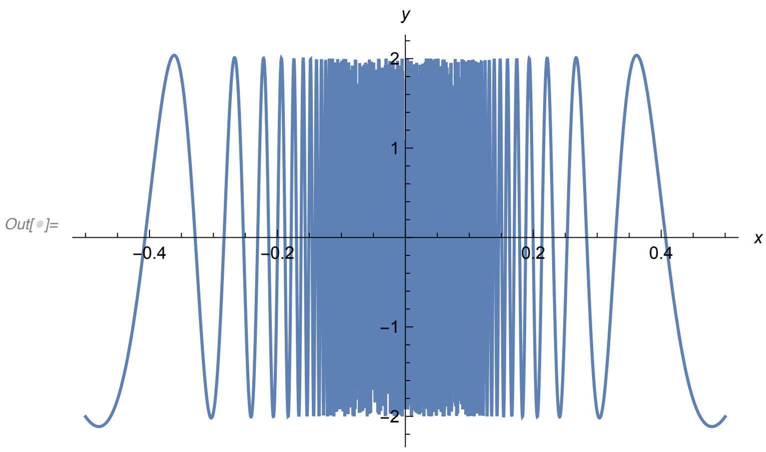
and that $\mathscr{C}^1$ is a proper subset of differentiable functions.

If we had followed the students incorrect procedure, as f' is not continuous at $x = 0$, we would have concluded that f is not differentiable at $x = 0$, and we would be wrong.

We can also see how our f function's graphic is in the $[-0.5, 0.5]$ interval, for example.



And this is f' function's graphic in the same interval.



In the latter graphic, we can intuit that there is an essential discontinuity for f' at the point $x = 0$. We have proved that analytically before, but, as we mentioned in Proposition 3.3, we already knew that if a function is differentiable at a point (in this case at $x = 0$) and if f' is not continuous at that point, then, the discontinuity must be essential. And that's what we can observe in the above graphic of f'.

4 Our experience finding the way to help the students in their learning process

In this school year (2020-2021), as a way to involve the students themselves in their educational process, we have organized different meetings with them in order to talk about their learning process and what the difficulties they find in the calculus course are. In some of these meetings we have talked about the concept of differentiability. The ideas, worries and results the students have expressed have been included in Sections 1 and 3 of this article and also in the next paragraphs.

In our university, the freshman year calculus course is a complete year course, from September to May, and we usually do a midterm exam in January. Below, in Figure 1, we have collected the results from an exercise related to studying differentiability of piecewise functions that we have tested students on in this midterm exam during the last three school years (2018-2019, 2019-2020 and 2020-2021). We have considered four result options to evaluate and compare the students' work we are interested in: to do the corresponding exercise correctly, to do it wrong, to not answer that exercise or to not do the complete exam (the exam consist of several exercises, each one related to each chapter of the course).

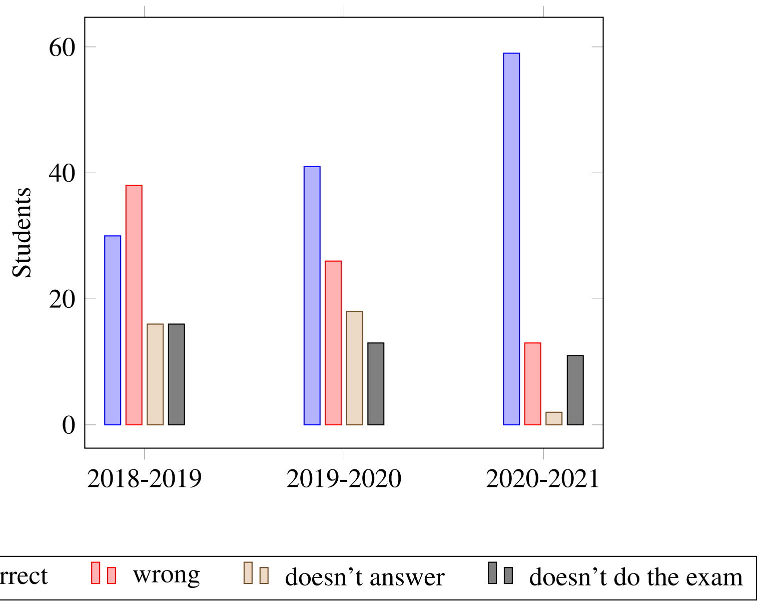


Figure 1: Results of the midterm exam.

In the 2018-2019 school year, we hardly had time to talk with the students calmly and thoroughly about the study of differentiability of piecewise functions nor about the difficulty they usually have with it. So, when the midterm exam arrived, we can see in the results of Figure 1 that many students did the corresponding exercise wrong or didn't even answer it. During the 2019-2020 school year we realized about the students' problem and tried to explain it better in the exercise classes, highlighting the mistake the students usually make in this type of exercises. In the above bar chart we can appreciate the students' improvement in their results. This school year (2020-2021) we caught the problem early and tried to talk with the students more about it, making them aware of it (the clear improvements in the results can be seen in Figure 1). Apart from explaining the concept better, we also did these two activities:

(i) Before explaining differentiability very deeply in class and without mentioning anything about the main problem they usually have, in November, we sent them Example 1 as an exercise for homework. We asked them to do that exercise with the ideas they had from high school. We can observe the results in the next bar chart, Figure 2, and see that many students made their common error or another type of mistake when working out the exercise[4].

The bad results obtained in November's column of Figure 2 are also the reflection and consequence of the mathematical difficulties and misunderstandings that the students bring to the university, those that we mentioned in Section 1.

(ii) We started having meetings[5] every fifteen days in order to talk about their learning process and their main difficulties in the first months of the calculus course. We have talked about many interesting things related to their education, and we continue doing it. We think that giving the students this possibility to express their ideas and feelings, has made them more conscious, more aware of what they do when working out a math problem, more confident with the professors and also among themselves.

We can see clearly in the next bar chart, Figure 2, the difference between the results when doing activity (i) in November's columns and the results obtained after working as in (ii) in January's columns. A significant improvement has being obtained, and we think that the students feel more confident and sure about their learning process now.

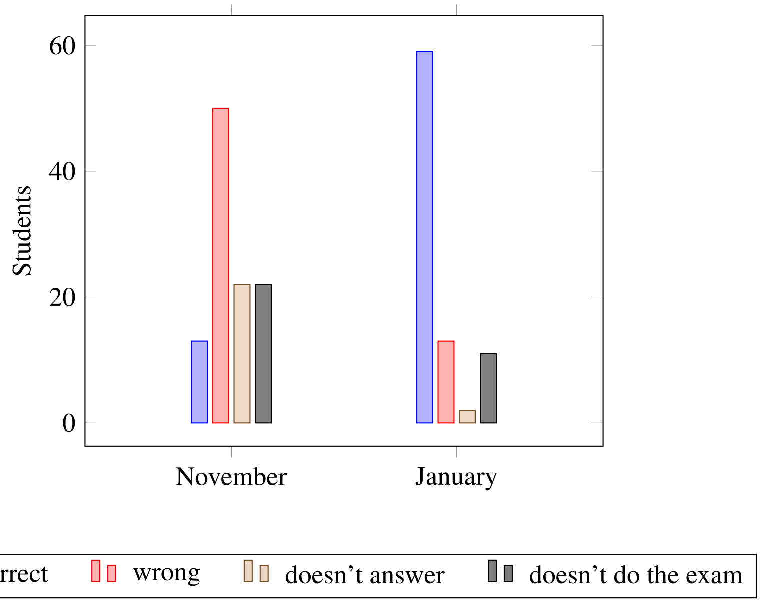


Figure 2: Results of 2020-2021 school year: November's homework and January's midterm exam.

5 Observations, conclusions and suggested work for students

Apart from taking the problem with time and bringing the students' attention to their typical error, we have learnt several things, above all with the activity (ii) we explained in Section 4. Our experience this school year has taught us that creating groups to talk with the students about their difficulties in learning this concept and in general in the calculus course has been a tremendous help for everyone. We have observed that asking the students what they think and making them responsible for their learning process is a good way to learn this kind of concepts. Because

[4]As this test or homework was only one exercise about differentiability of piecewise functions, the students that didn't do the exercise were the same as the students that didn't do the exam.

[5]The original idea for this kind of meetings could come, among others, from Celestine Freinet, [Fre1] and [Fre2], who was a French pedagogue and teacher. In his work these meetings were called as assemblies.

we give them the possibility to express the feelings and thoughts they have when they face these learning difficulties or misunderstandings. This makes them more connected to everything, such as their education, themselves, their classmates and professors. We can say also that we have seen and felt the students better and more comfortable and involved in their education. Moreover, this big improvement has been also reflected in their academic results.

We think that a good way to introduce the students to the understanding of a further study of piecewise functions' differentiability is to explain it like in Example 1. First, working out the problem as they usually do with the knowledge they bring from high school and analyzing the mistake they make. And then, doing it correctly with the differentiability definition.

We also include some exercises below, with the help of [Ban], in order for the students to practice by themselves and consolidate the learning of this issue. There, the students can practice the study of continuity, differentiability and $\mathscr{C}^1$ class of functions. Below the statement of each exercise, its solution can be found.

Exercise 1. Given the following function, study its continuity, differentiability and $\mathscr{C}^1$ class in $\mathbb{R}$.

$$f(x) = \begin{cases} x^2 \cdot \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(Solution: f is continuous and differentiable in $\mathbb{R}$, but it is not of $\mathscr{C}^1$ class at $x = 0$.)

Exercise 2. Given the following function, study its continuity, differentiability and $\mathscr{C}^1$ class in $\mathbb{R}$ ($q > 0$ and $p > 1$).

$$f(x) = \begin{cases} x^p \cdot \sin\left(\frac{1}{x^q}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(Solution: If $0 < q < p-1$, f is continuous, differentiable and of $\mathscr{C}^1$ class in $\mathbb{R}$. If $0 < p-1 \leqslant q$, f is continuous and differentiable in $\mathbb{R}$, but it is not of $\mathscr{C}^1$ class at $x = 0$.)

Exercise 3. Given the following function, study its continuity, differentiability and $\mathscr{C}^1$ class in $\mathbb{R}$.

$$f(x) = \begin{cases} x^2 \cdot (1-x)^2 \sin\left(\frac{1}{\pi x(1-x)}\right), & x \in (0,1), \\ 0, & \text{else}. \end{cases}$$

(Solution: f is continuous and differentiable in $\mathbb{R}$, but it is not of $\mathscr{C}^1$ class at $x = 0$ and at $x = 1$.)

Exercise 4. Given the following function, study its continuity, differentiability and $\mathscr{C}^1$ class in $\mathbb{R}$.

$$f(x) = \begin{cases} \sin x \cdot \sin\left(\frac{1}{\sin x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(Solution: f is continuous in $\mathbb{R}$ but not differentiable at $x = 0$. Therefore, by (3.1), f is not of $\mathscr{C}^1$ class at $x = 0$.)

Exercise 5. Given the following function, study its continuity, differentiability and $\mathscr{C}^1$ class in $\mathbb{R}$.

$$f(x) = \begin{cases} x \cdot \left[1 + \frac{1}{3}\sin\left(\log x^2\right)\right], & x \neq 0, \\ 0, & x = 0. \end{cases}$$

(Solution: f is continuous in $\mathbb{R}$ but not differentiable at $x = 0$. Therefore, by (3.1), f is not of $\mathscr{C}^1$ class at $x = 0$.)

Acknowledgment. *I would like to thank the 2020-2021 Physics and Electronical Engineering freshman year students for their help and interest in developing this project and taking part in our subject during this difficult pandemic year.*

I would also like to thank Professor Martín Blas Pérez-Pinilla for giving me some interesting bibliographies to carry out this work and also for giving me advice and recommending several ideas. Professor Pedro Alegría also helped me with some observations and recommendations.

The author was supported by the Spanish Government's Ministry of Science, Innovation and Universities' grant, PGC2018-094522-B-I00, and the Basque Government's grant, IT12247-19.

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