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Abstract: This study investigates 48 preservice middle school teachers' formal textbook definitions, formulas, and graphs of the directly and inversely proportional relationships. The connections among these three types of representations and how preservice teachers' representations differ according to the two relationships are also examined. The data of the study included the preservice teachers' responses to a paper-and-pencil test with two items. An explanatory research design model was followed when developing this study. The study findings indicated the preservice teachers' over-attention to the simultaneous increases and/or decreases when representing the directly and inversely proportional relationships. Only four preservice teachers stated in their definitions that the ratio was constant in the directly proportional relationship. Whereas none of them stated constant product in their inversely proportional definition. Although the preservice teachers were slightly better at writing the direct proportion formula than the inverse proportion formula, this difference was much greater in terms of drawing the direct and inverse proportion graphs. Sixty percent of the preservice teachers drew their inverse proportion graphs as linear with negative slopes. Moreover, the analysis showed that the preservice teachers' definitions were not well-linked with their graphs and formulas. On the other hand, the preservice teachers who were better at providing correct formulas were also better at drawing the correct graphs. Implications for teaching and future study suggestions are discussed.

Keywords: Directly proportional relationships, external representations, inversely proportional relationships, preservice mathematics teachers, proportional reasoning

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Introduction

There has been a shift in mathematics education from a traditional instruction centered perspective, which emphasizes rote computations and rule memorizations, to a contemporary instruction perspective that highlights the importance of developing meaningful understanding of mathematics. Along with this shift, the traditional instruction of concepts such as ratio, proportion, and proportional relationships have been modified to meet the needs of students. Understanding these concepts forms a very important part of school mathematics (Lamon, 2007; Lobato & Ellis, 2010) and are also essential for students' achievements in their future endeavors (Kilpatrick et al., 2001; National Council of Teachers of Mathematics [NCTM], 2000). While traditional instruction on these concepts is primarily based on mechanic operations such as the cross-multiplication and across-multiplication algorithms (e.g., Arican, 2018, 2019; Fisher, 1988; Harel & Behr, 1995; Orrill & Brown, 2012), contemporary instruction focuses more on making sense of multiplicative relationships presented in ratios, proportions, and proportional relationships. In the literature, the ratio is defined as a multiplicative comparison of two quantities with the same or different units (Lobato & Ellis, 2010). Hence, a meaningful understanding of the ratio concept requires detecting multiplicative relationships between quantities that form the ratio. A proportion then is a mathematical expression showing the equivalence of the two ratios (Fisher, 1988; Lobato & Ellis, 2010).

There are two types of proportional relationships between quantities: directly proportional and inversely proportional. The formal textbook definitions of these relationships are presented as follows: If x and y are two variables and a constant real number, k, exists such that $y = k*x$, then we call that the variable y is directly proportional to the variable x (Argun et al., 2014). On the other hand, if x and y are two variables and a constant real number, k, exists such that $y*x = k$, then the variable y is inversely proportional to the variable x (Argun et al., 2014). Therefore, directly and inversely proportional relationships consist of multiplicative relationships between quantities, a constant ratio relationship and a constant product relationship, respectively. Although the cross-multiplication and across-multiplication strategies are the direct results of these constant ratio and product relationships, very few students recognize these multiplicative relationships when working on proportion problems.

Besides definitions, which are regarded as verbal representations, graphical and symbolic representations are also used in expressing directly and inversely proportional relationships. The formula $y = k*x$ is a symbolic representation of a directly proportional relationship. Hence, the directly proportional relationship is a special form of linear relationships that are expressed by $y = k*x + b$ formula. The directly proportional formula is obtained when b is equal to zero. Therefore, the directly proportional graph, which is graphical representation of the directly proportional relationship, always passes through the origin. However, attending to simultaneous increases of the values of quantities, students, preservice teachers (PSTs), and even in-service teachers tend to infer a directly proportional relationship between quantities even if their graphs do not pass through the origin, which occurs when $a > 0$ and $b \neq 0$ (i.e., affine model with positive slope) (Arican, 2019; De Bock et al., 2015). On the other hand, the formula $y*x = k$ is a symbolic representation of an inversely proportional relationship. The graph of this inverse relationship is asymptotic to the x and y-axis at zero. Like direct proportion graph, attending to simultaneous increases and decreases of the values of corresponding quantities, students, PSTs, and in-service teachers tend to draw it as a linear decreasing graph that can be best expressed by $y = a*x + b$ formula, which occurs when $a < 0$ and $b \neq 0$ (i.e., affine model with negative slope) (Arican, 2019; De Bock et al., 2015).

Although multiplicative relationships present between two directly proportional quantities and two inversely proportional quantities, which are also expressed by their definitions and formulas, as stated above, many students, PSTs, and in-service teachers may incline to infer additive affine relationships as directly proportional or inversely relationship. Students' difficulties with understanding multiplicative relationships presented between proportional quantities indicate issues with teachers' proportional reasoning and knowledge of directly and inversely proportional relationships. Proportional reasoning necessitates the ability to represent proportional relationships using varying mathematical models (i.e., tables, graphs, equations, diagrams, and verbal descriptions) (Common Core State Standards Initiative [CCSSI], 2010). Hence, to develop students' robust understanding of proportional relationships, teachers should know definitions, formulas, and graphs of directly and inversely proportional relationships and connections among these three representations. By the term *connection*, we imply relations among these three types of representations. Therefore, a strong understanding of connections among representations requires the knowledge of each representation and relations among them, their roles in expressing directly and proportional relationships, and being able to transition between representations when solving direct and inverse proportion problems.

As state above, PSTs may have difficulties with using appropriate external representations to express the directly and inversely proportional relationships and understanding connections among these representations. PSTs' difficulties suggest lack of their knowledge on representations used in expressing the directly and inversely proportional relationship concepts. However, in mathematics education literature, PSTs' knowledge of these three representations and connections among them have not been given enough attention. Hence, this study investigated 48 PSTs' definitions, formulas, and graphs of the directly and inversely proportional relationships, connections among these three representations, and how their representations vary according to the two relationships. Examining PSTs' knowledge on these representations can assist university educators for determining their understanding and difficulties with the directly and inversely proportional relationships. The following questions are investigated in this study:

(1) How do preservice middle school mathematics teachers define the directly and inversely proportional relationships and represent these relationships using graphs and formulas?
(2) How connected are the definitions, formulas, and graphs of preservice middle school mathematics teachers with each other?
(3) How do preservice middle school mathematics teachers' representations vary according to the directly and inversely proportional relationships?

In the following section, we report some literature on external representations and their applications in educational settings.

Literature Review on Representations of Proportional Relationships

Goldin (2003) explains external representations as objectively defined and accepted notational systems used in representing and communicating mathematical ideas such as diagrams, number lines, graphs, formulas, psychical models, written or spoken language, equations, etc. In classrooms, external representations facilitate students' learning by supporting their development of conceptual understanding and helping them in grasping connections between mathematical concepts (NCTM, 2000). Researchers (e.g., Duval, 2002; Gagatsis & Shiakalli, 2004; Yerushalmy, 2006) claim that the ability to use multiple external representations and being able to switch between these representations facilitates problem solving and learning. According to Spiro and Jehng (1990), multiple external representations can support the development of knowledge since the same idea can be expressed by using different forms of representations.

Examining mathematics education literature shows that most research on the role of external representations in mathematics learning has been conducted on the concept of function (e.g., Even, 1998; Gagatsis & Shiakalli, 2004; İlhan, 2019; McGowan, & Tall, 2001) especially on linear functions and representational flexibility (e.g., Acevedo Nistal et al., 2012a and 2012b). This is partly because functions can be expressed using different types of representations. In the domain of ratio and proportions, most studies (e.g., Aboul Hosn, 2015; De Bock et al., 2015) focused on the role of using external representations on students' understanding of proportional relationships. Hence, there is lack of research on PSTs' knowledge of external representations and connections among them.

Regarding with the existing studies on PSTs' knowledge on external representations, Akkus-Cıkla and Duatepe (2002) and Dogruel (2019) reported PSTs' lack of knowledge on definitions of ratio and proportion concepts. Moreover, Arican (2019), Johnson (2017), and Lo (2004) stated PSTs' difficulties with using appropriate external representations to present directly and inversely proportional relationships. Conducting a study with 40 PSTs, Arican (2019) noted their over attention to simultaneous increases and/or decreases (i.e., x increases and y increases or x increases and y decreases) when defining directly and inversely relationships. Many of these PSTs did not recognize constant ratio and constant product relationships occurring between corresponding values of quantities. Furthermore, Arican (2019) reported the PSTs' hesitations with starting the directly proportional graph from the origin in which some of them drew their graphs by excluding the origin. In addition, many PSTs drew the inversely proportional graph using an affine model with negative slope. Finally, when the PSTs were provided with the graphs of two affine functions with positive slope and negative slope, under the *illusion of linearity* (e.g., Modestou, & Gagatsis, 2007; Van Dooren et al., 2003), many of them inferred that those two graphs were expressing directly proportional relationship and inversely proportional relationship, respectively.

Johnson (2017) investigated representations used by 25 elementary and secondary mathematics education PSTs when solving nine proportional reasoning problems. The PSTs' responses suggested three common difficulties that were prevalent among them: epistemological obstacle of linearity, confusion between ratio and fraction, and inability to reason quantitatively. The PSTs who had high-level of proportional reasoning were better at creating multiple external representations to solve given tasks than the PSTs with moderate and low-level of proportional reasoning. Johnson (2017) also observed that the PSTs with high-level of proportional reasoning were able to overcome three common difficulties by using multiple external representations in their solutions. Furthermore, Lo (2004) provided a missing-value proportion problem to 22 PSTs in an elementary mathematics program. The PSTs' solutions suggested their difficulties with explaining the meaning behind their calculations using appropriate pictures. In addition, the PSTs who were able to draw pictures could not provide written explanations to justify the connection between their calculations and pictures. Hence, she suggested encouraging PSTs in producing multiple representations of proportional relationships to develop their meaningful understanding of these relationships.

The PSTs' difficulties with using appropriate representations to express proportional relationships suggest issues related with the quality of mathematics instruction that they received in middle and high school. Agreeing with this deduction, Shield and Dole (2002, 2013) reported limitations in the definitions of ratio, rate, and proportion concepts in the lower secondary school textbooks. Shield and Dole (2002, 2013) stated incapability of secondary school textbooks on promoting students' meaningful understanding of proportional reasoning. Similarly, Ahl (2016) noted the impact of research findings being low on the representations of proportional relationships in mathematics textbooks. On the contrary, examining mathematics textbooks used in Turkish schools, Bayazit (2013) reported that 75% of the tasks used for teaching proportion concept demanded cognitively high-level student responses. In addition, he noted that most of the tasks were presented using multiple representations and had the capacity to promote students' proportional reasoning.

As stated above, there is a need for investigating PSTs' knowledge on external representations used in expressing directly and inversely proportional relationships. Although there are many types of external representations, this study only focused on the PSTs' definitions, formulas, and graphs because these three forms of representations are the ones mostly used in classrooms to express directly and inversely proportional relationships. In the following section, we report research design, participant recruitment procedure, data collection tools, and data analysis procedure.

Methods

Research Design

An explanatory research design model (e.g., Fraenkel & Wallen, 2006) was followed when developing this study. Explanatory research design assists researchers in understanding some phenomena more efficiently by allowing both quantitative and qualitative methods in analyzing the data. Hence, this research design well-suited with the purpose of this study that was conducting in-depth analysis on the PSTs' definitions, formulas, and graphs of the directly and inversely proportional relationships and connections among them. Moreover, both qualitative and quantitative methods were used in analyzing and reporting the PSTs' responses.

Participants and Recruitment Procedure

The participants of this study include 48 first-year PSTs (13 male and 35 female) who were enrolled in the middle school mathematics program of a Turkish university. The PSTs were attending a mathematics course on general geometry topics (e.g., angles, triangles, rectangles, polygons, and solid geometry) in two separate groups, both included 24 PSTs, during the spring semester of 2019. The PSTs did not have any university level instruction on the directly and inversely proportional relationships before their participation in the study. Hence, they provided responses to the paper-and-pencil items using the knowledge that they acquired in middle and high school. In Turkey, the instruction on these two relationships, which are provided in middle and high school, usually focuses on rule memorization and routine computations (e.g., cross-multiplication and across-multiplication strategies). Hence, following a purposive sampling method (e.g., Patton, 2005), we aimed at examining the PSTs' understanding of the directly and inversely proportional relationships before any intervention on these two relationships.

The Data Collection Procedure and Tools

The data of this study included the PSTs' written responses to a paper-and-pencil test with two items. The PSTs were given 40 minutes to complete the paper-and-pencil test. In the first and second items, the PSTs were asked to provide formal textbook definitions, graphs, and formulas of directly and inversely proportional relationships, respectively (Table 1). Hence, these two items aimed at investigating the PSTs' knowledge of these three representations that they learned during the formal education in middle and high school.

Table 1
The paper-and-pencil items

Item	Description
Direct Proportion	Please write the definition of a directly proportional relationship, draw the graph of it, and if *x* and *y* are two directly proportional quantities, then write the formula that represents this relationship beside your graph.
Inverse Proportion	Please write the definition of an inversely proportional relationship, draw the graph of it, and if *x* and *y* are two inversely proportional quantities, then write the formula that represents this relationship beside your graph.

The Data Analysis

After collecting the PSTs' written responses, we used a conventional content analysis method (e.g., Hsieh & Shannon, 2005) to analyze their responses. This type of analysis reported to be an appropriate method when an "existing theory or research literature on a phenomenon is limited," and "researchers avoid using preconceived categories" rather allow the categories emanate from the data (Hsieh & Shannon, 2005, p. 1279). To conduct the content analysis, we recorded the PSTs' definitions in an Excel file and categorized these definitions by providing codes for each definition. Similarly, we coded the PSTs' formulas and graphs for the two items, respectively. Next, considering our research questions, we generated tables using the codes in our Excel file. We provided these codes in the results section when presenting our findings. In these tables, the findings were reported using descriptive statistics (i.e., frequencies and percentages). The PSTs' responses to the paper-and-pencil test and findings obtained from the analysis of these responses are presented in the following pages.

Results

The PSTs' definitions of the directly proportional relationship are presented in Table 2. Table 2 shows that 22 PSTs (45.8%) defined the relationship by just attending to the simultaneous increases or decreases. Moreover, besides attending to the simultaneous increases or decreases, 17 PSTs (35.42%) also pointed out in their definitions that quantities were increasing (or decreasing) at the same ratio. However, only four PSTs (8.33%) in some extent stated that in a directly proportional relationship, the corresponding values of quantities were forming a constant ratio. Two PSTs defined the relationship by attending to this constant ratio and simultaneous increases or decreases, and two PSTs defined it by also including the increments (or decrements) being at the same ratio. In addition, three definitions were classified as *other* because these definitions did not fit within the remaining four categories. For instance, one of these three PSTs defined the directly proportional relationship as an equivalence relationship, and the remaining two PSTs defined it as follows: "It is the relation of a quantity with a related quantity" and "If there is a uniform increase in a certain proportion, then there is a direct proportion." Finally, two PSTs did not provide a response.

Table 2
The PSTs' definitions of directly proportional relationship and their frequencies

	Definition Examples	f
Increase-Increase (II)	When x increases, y also increases, then they are directly proportional.	22
Increase-Increase + At the Same Ratio (II+SR)	If one of the two interconnected situations increases, the other increases at the same ratio or both decreases. It is the increase or decrease of two terms in proportion to each other. It is when the x value increases, y value increases at the same ratio.	17
II + Constancy of the Ratio (II+CR)	It is having a constant ratio between x and y.	2
II+SR + Constancy of the Ratio (II+SR+CR)	It is a form of proportion in which one value increases while the other increases at the same ratio, and there is a certain ratio between them.	2
Other	Equivalences that increase or decrease regularly are called direct proportion.	3
No Answer		2

The PSTs' definitions of the inversely proportional relationship are presented in Table 3. Like directly proportional relationship, most PSTs (58.3%) defined the relationship by just attending to the simultaneous increases and decreases. Moreover, 12 PSTs (25%) attended to both simultaneous increases and decreases and these increments and decrements being at the same ratio. In addition, five PSTs' definitions were classified as other, and three PSTs did not provide a response. Comparing with the directly proportional relationship, Table 3 suggested an increase in the number of PSTs who just relied on the simultaneous increases and decreases and a drop in the number of PSTs who also attended to these increments and decrements being at the same ratio. Furthermore, Table 3 showed that none of the PSTs mentioned the product of corresponding values were yielding a constant number.

The PSTs' formulas of the directly and inversely proportional relationships and their frequencies are presented in Table 4. Table 4a shows that 30 PSTs (62.5%) were able to provide the direct proportion formula. However, there were also 12 partially correct formulas, x/y, x/y = 1, x/y = 2, k = x/2y, and ad = bc are considered as partially correct. Furthermore, six PSTs did not provide a response for the direct proportion formula. Hence, considering there were not any incorrect formulas, the PSTs were good at providing the direct proportion formula. On the other hand, Table 4b shows that 26 PSTs (54.16%) were able to provide the inversely proportional relationship formula. The formulas, x*y, x*y = 1, x*y = 4, and ab =cd are considered as partially correct. Moreover, seven PSTs did not provide a response, and five PSTs provided incorrect formulas. In Table 4b, the total number of formulas is 49 because one PST provided two formulas. When the formulas in Table 4 are compared, the direct proportion formulas included less variation than the inverse proportion formula. In addition, while there are only correct and partially correct formulas for the direct proportion, incorrect formulas are also presented for the inverse proportion.

Table 3
The PSTs' definitions of inversely proportional relationship and their frequencies

	Definition Examples	f
Increase-Decrease (ID)	When *x* increases *y* decreases or when *x* decreases *y* increases.	28
Increase-Decrease + At the Same Ratio (ID+SR)	It is when the *x* value increases, *y* value decreases at the same ratio.	12
Other	In a two-variable relation, if the *y* value decreases as the *x* value increases, or if the *y* value increases as the *x* value decreases, this type of proportion is the inverse proportion. It is a form of proportion in which one value increases while the other decreases, and there is a constant ratio between them. Equivalences that do not increase or decrease on a regular basis are called the inverse proportion.	5
No Answer		3

The PSTs' directly proportional graphs and their frequencies are presented in Table 5. Twenty-one PSTs (43.75%) drew their graphs as the one presented in Table 5a. These PSTs appeared to attend to the linearity and the fact that graph starts from the origin. Furthermore, 23 PSTs (47.92%) drew their graphs as in Table 5b. These PSTs attended to the multiplicative relationship between numbers besides attending to the linearity and starting the graph from the origin. The graph in Table 5c is an interesting one because although it is mathematically a correct representation of the directly proportional relationship, the instruction usually focuses on drawing the graph on the first quadrant. On the other hand, one PST drew a nonlinear graph (5d), and two PSTs hesitated to start their graphs from the origin (5e). Although these three PSTs recognized the multiplicative relationship between x and y values, they had difficulty obtaining appropriate graphs. Therefore, the PSTs were good at providing the directly proportional relationship graphs because graphs in 5d and 5e could be considered as partially correct, and there were no incorrectly represented graphs.

Table 4

The PSTs' directly and inversely proportional relationship formulas and their frequencies

(a) Formula of Direct Proportion	f	(b) Formula of Inverse Proportion	f
x/y = k (y/x =k)	30	**x*y = k**	26
x/y	4	**x*y**	5
x/y = 1, x/y = 2 (*)	4	**x*y = 1, x*y = 4 (**)**	3
ad = bc	2	**ab = cd**	3
5x = y = k	1	**x = -y**	1
k = x/2y = 2x/4y	1	**x/y = 1/k**	1
No Answer	6	**x/5 = y/1 = k**	1
Total	48	**x = y = k**	2
		No Answer	7
		Total	49

Note. (*) Three PSTs wrote x/y = 1, and one wrote x/y = 2; (**) Two PSTs wrote x*y = 1, and one wrote x*y = 4.

Table 5

The PSTs' directly proportional relationship graphs and their frequencies

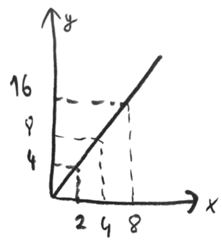
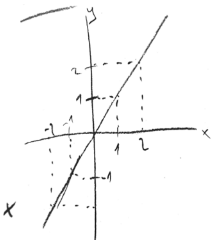
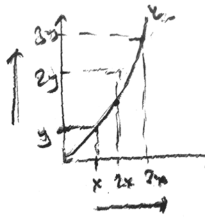
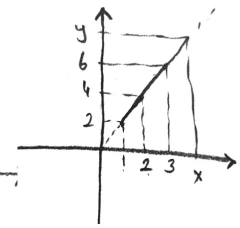
Graphs	**a**	**b**	**c**	**d**	**e**
					
f	21	23	1	1	2

Table 6 shows the PSTs' inversely proportional relationship graphs. The PSTs' inversely proportional graphs can be gathered into two categories based on their shapes. The graphs of the first category included linear decreasing graphs with negative slopes (L1, L2, L3, L4, L5, and L6). On the other hand, the graphs in the second category included curved graphs (C1, C2, and C3). Furthermore, the graphs can be divided into three categories in terms of values and relationships among them: Graphs without values, graphs containing a multiplicative relationship, graphs with values and without a multiplicative relationship.

Table 6 shows that eight PSTs drew their graphs as in L1. These PSTs appeared to understand the problematic side of intersecting the line with the axes but did not recognize that the product of corresponding values was equal to a constant. Moreover, by intersecting the line with the axes, 12 PSTs drew their inversely proportional graphs as in L2. Moreover, four PSTs drew their graphs as in L3, and two PSTs drew their graphs as in L4. Although these PSTs recognized the constant product relationship between the values, they drew the inversely proportional graphs in such they represent the graph of a linear (affine) function with a negative slope. Furthermore, three PSTs drew their graphs like L2 but included values for the x and y axes (see L5). These PSTs appeared to think additively rather than multiplicatively because they decreased and increased the y and x values by the same number, respectively. Finally, one PST drew the graph of y = -x equation to represent the inversely proportional graph (see L6). As a result, 30 PSTs (62.5%) drew linear decreasing graphs, and only six of them (i.e., L3 and L4, graphs with multiplicative relationship) can be considered as partially correct graphs.

In Table 6, the curved category shows that only 18 PSTs (% 37.5) were able to provide the correct representation of the inversely proportional graph. Eleven, five, and two PSTs drew their graphs as in C1, C2, and C3. The PSTs who drew their graphs as C1 seemed to know the shape of the inversely proportional graph; however, they drew their graphs without providing the x and y values on the axes. On the other hand, the PSTs who drew their graphs as in C2 and C3 showed the constancy of products. The only difference between C2 and C3 was that the PSTs who drew their graphs as in C2 used letters instead of numbers. When the information in Table 5 and Table 6 were compared, like formulas, the variation in the directly proportional graphs was less than the variation in the inversely proportional graphs. In terms of accuracy of graphs, we encountered the same situation as in the formulas. While the directly proportional graphs were either correct or partially correct, there were also incorrect inversely proportional graphs.

Table 6
The PSTs' inversely proportional relationship graphs and their frequencies

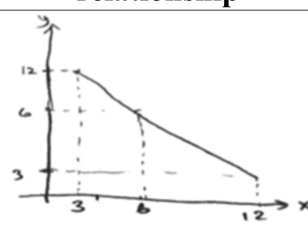
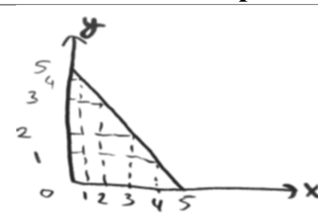
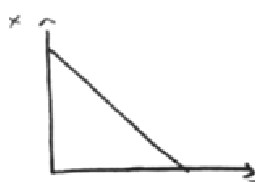
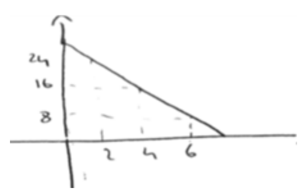
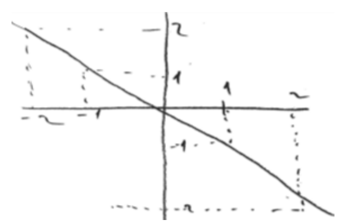
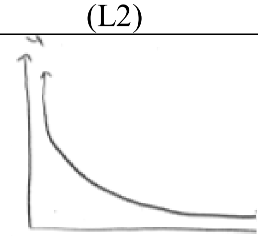
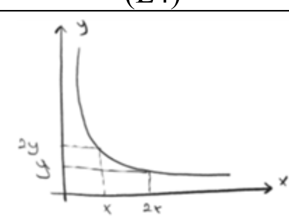
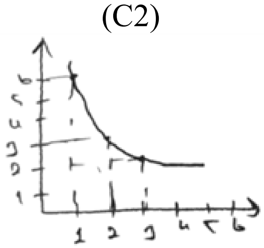
	Graphs without values	**f**	**Graphs with values and a multiplicative relationship**	**f**	**Graphs with values and without a multiplicative relationship**	**f**
Linear Graphs	 (L1)	8	 (L3)	4	 (L5)	3
	 (L2)	12	 (L4)	2	 (L6)	1
Curved Graphs	 (C1)	11	 (C2)	5		
			 (C3)	2		

Table 7 presents the distribution of the PSTs' directly proportional definitions and formulas in terms of their graphs. According to Table 7, 12 out of the 21 PSTs (57.14%) who drew their directly proportional graphs as in 5a stated the multiplicative relationship (i.e., the ratio of corresponding values is equal to a constant number or the increments [or decrements] being at the same ratio) between quantities only in their formulas (i.e., bold numbers); seven of them (33.3%) stated the multiplicative relationship both in their formulas and definitions (i.e., orange number); and two PSTs (9.5%) did not state the multiplicative relationship in any of the representations. Ten out of the 23 PSTs (43.47%) who drew their graphs as in 5b stated the multiplicative relationship in all three representations (i.e., green numbers); nine (39.1%) stated the multiplicative relationship in their formulas and graphs (i.e., red numbers); three stated it in their definitions and graphs (i.e., blue numbers); and one PST only pointed out the relationship in her graph. The PSTs who drew their graphs as in 5c and 5d stated the multiplicative relationship in their formulas and graphs. Finally, one of the two PSTs who drew their graphs as in 5e stated the multiplicative relationship in all three representations, and the second one stated it in his formula and graph. In sum, Table 7 shows that 11 PSTs (22.9%) stated multiplicative relationship between quantities in all three representations, 22 PSTs (45.8%) in two representations, 13 PSTs (27.1%) in only one representation, and two PSTs did not state the multiplicative relationship in any representations.

Table 7
The distribution of the PSTS' directly proportional definitions and formulas in terms of their graphs

Graphs	Formula / Definition	II	IISR	II+CR	IISR+ CR	Other	No Answer	Total
5a	**x/y = k (y/x =k)**	**9**	7					16
	x/y	**2**						2
	x/y = 1, x/y = 2					**1**		1
	No Answer	2						2
	Total	13	7			1		21
5b	**x/y = k (y/x =k)**	5	**3**	**1**	**1**	1	2	13
	x/y		**1**		**1**			2
	ad = bc	1	**1**					2
	5x = y = k		**1**					1
	k = x/2y = 2x/4y		**1**					1
	No Answer		2	1		1		4
	Total	6	9	2	2	2	2	23

5c	**x/y=1**				1	1
5d	**x/y = k (y/x =k)**	1				1
5e	**x/y = 1, x/y = 2**	1	1			2

Table 8
The distribution of the PSTS' inversely proportional definitions and formulas in terms of their graphs

Graphs	**Formula / Definition**	**ID**	**IDSR**	**Other**	**No Answer**	**Total**
C1	**x*y = k**	7	2	1	1	11
C2	**x*y = k**	2				2
	x*y	1	2			3
	Total	3	2			5
C3	**x*y = k**			1		1
	No Answer			1		1
	Total			2		2
	x*y = k	2	1		1	4
	x*y	1				1
L1	**x*y = 1, x*y = 4**	1				1
	ab = cd	1				1
	No Answer	1				1
	Total	6	1		1	8
	x*y = k	4	1		1 (*)	6
	x*y	1				1
L2	**x*y = 1, x*y = 4**			1		1
	x/y = 1/k		1			1
	x = y = k				1	1
	No Answer	3				3
	Total	8	2	1	2	13
	x*y = k	1	1			2
L3	**x*y = 1, x*y = 4**	1				1
	ab = cd		1			1
	Total	2	2			4
	x = y = k	1				1
L4	**No Answer**		1			1
	Total	1	1			2
	ab = cd		1			1
L5	**x/5 = y/1 = k**		1			1
	No Answer	1				1
	Total	1	2			3
L6	**x = -y**			1		1
Total		28	12	5	4	49/49

Note. (*) A PST provided two formulas, so his response for the definition (he did not provide an answer) counted twice.

Table 8 shows the distribution of the PSTs' inversely proportional definitions and formulas in terms of their graphs. In term of definitions of 18 PSTs with curved graphs (i.e., C1, C2, and C3), there was a concentration on the definitions coded as ID and IDSR, in which 10 (55.55%) and four (22.22%) definitions were categorized in these two groups, respectively. Regarding 18 PSTs' formulas, 14 formulas (77.7%) were coded as correct, three were coded as partially correct, and one PST did not provide a formula. In terms of PSTs with linear graphs, 18 out of 30 PSTs' (60%) definitions were coded as ID, eight as IDSR (26.66%), two as other, and three did not provide an answer. Investigating their formulas, the analysis showed that 12 out of 30 PSTs (40%) provided correct formulas, eight provided partially correct formulas (26.66%), six did not provide formulas, and four provided incorrect formulas. Finally, in sum, none of the PSTs stated the multiplicative relationship (i.e., the product of corresponding values is equal to a constant number or the increments/decrements being at the same ratio) in all three representations, 10 PSTs (20.8%) stated it in two representations, 28 PSTs (58.3%) only stated it in one representation, and 10 PSTs (20.8%) did not state it in any representation.

We present a comparison of the PSTs' directly and inversely proportional definitions in Table 9. The frequencies in Table 9 having a diagonal distribution indicates that the PSTs provided parallel definitions of the two relationships. In other words, the PSTs defined these relationships following a similar structure. For instance, 20 PSTs (41.66%) described the directly and inversely proportional relationships by attending to simultaneous increases and/or decreases. An important point here is that eight out of 19 PSTs (42.1%) who referred to the "same ratio" or "constancy of ratio" in their directly proportional relationship definitions preferred defining the inversely proportional relationship by only attending to the simultaneous increases and decreases. This finding and the fact that none of the PSTs stated the constant product of values suggest that the PSTs had more difficulty with defining the inversely proportional relationship than the directly proportional relationship.

Table 9
A comparison of the PSTS' directly and inversely proportional relationship definitions

	II	**IISR**	**II+CR**	**IISR+CR**	**Other**	**No Answer**	**Total**
ID	20	6	2				28
IDSR		11		1			12
Other				1	4		5
No Answer	1					2	3

Total	21	17	2	2	4	2	48

Table 10 presents a comparison of the PSTs' directly and inversely proportional formulas. Like Table 9, the frequencies are almost diagonally distributed. This situation indicates that the PSTs wrote the formulas of directly and inversely proportional relationships in a similar manner. Table 10 shows that 26 out of 31 PSTs (83.87%) who provided the correct proportion formula also provided the correct inverse proportion formula. Two PSTs did not provide the inverse proportion formula, and three provided the wrong formula. Furthermore, five PSTs did not provide both formulas. It was noteworthy that one PST provided x/y = k formula for the direct proportion and x/y = 1/k formula for the inverse proportion. Similarly, another PST provided 5x = y = k formula for the direct proportion and x/5 = y/1 = k formula for the inverse proportion.

Table 10

A comparison of the PSTS' directly and inversely proportional relationship formulas

	x/y = k	**x/y**	**x/y=1, x/y=4**	**ad=bc**	**5x=y=k**	**k=x/2y=2x/4y**	**No Answer**	**Total**
xy=k	26							26
xy		4				1		5
xy=1, 4			3					3
ab=cd				2			1	3
x=-y			1					1
x/y=1/k	1							1
x/5=y/1=k					1			1
x=y=k	2							2
No Answer	2						5	7
Total	31	4	4	2	1	1	6	49

Table 11 presents a comparison of the PSTs' directly and inversely proportional graphs. Table 11 shows that 33.3% and 43.47% of those who drew their graphs as in 5a and 5b drew the inversely proportional relationship graph correctly, respectively. Furthermore, the PST who drew the directly proportional graph as in 5c (i.e., the graph of y = x) drew the inverse proportion graph as in L6 (i.e., the graph of y = -x). Moreover, a PST provided curved graphs for both relationships (graphs in 5d and C1). In addition, two PSTs who drew their directly proportional relationship graph without passing through the origin (i.e., 5e) provided their inverse proportion graphs as in L1 and L3, respectively, by not intersecting the line with the axes.

Table 11

A comparison of the PSTS' directly and inversely proportional relationship graphs

	5a	5b	5c	5d	5e	Total
L1	5	2			1	8
L2	8	4				12
L3	1	2			1	4
L4		2				2
L5		3				3
L6			1			1
C1	7	3		1		11
C2		5				5
C3		2				2
Total	21	23	1	1	2	48

Discussion and Conclusions

This study investigated 48 PSTs' definitions, formulas, and graphs of the directly and inversely proportional relationships, connections among these three representations, and how their representations varied according to the two relationships. As stated previously, there is limited number of studies on PSTs' knowledge of these three representations and connections among them. By conducting this study, we aimed at understanding the PSTs' knowledge of these three representations and whether were able to provide them in accordance with each other.

We investigated three research questions. Regarding the first research question, the findings showed that many PSTs only attended to the simultaneous increases and decreases in the values of quantities when defining directly (45.8%) and inversely proportional relationships (58.3%). Moreover, while only four PSTs stated that corresponding values of x and y were yielding a constant ratio, none of them defined inversely proportional relationship by paying attention to the product of corresponding values yielding a constant number. Arican (2019) also reported PSTs' over-attention to these simultaneous increases and/or decreases of quantities when determining directly and inversely proportional relationships. For Arican (2019), the PSTs' over-attention is a result of rote computations and rule memorization that they were taught in middle and high school.

The findings showed the PSTs' overall success in providing direct and inverse proportion formulas and drawing their graphs. However, they were more successful in providing the direct proportion formula than the inverse proportion formula in which 62.5% and 54.16% of the PSTs provided correct formulas and 25% and 22.9% of the PSTs provided partially correct formulas, respectively. Similarly, the PSTs were better at drawing the directly proportional graph (45 correct [93.75%] and three partially correct) than the inversely proportional graph (18 correct [37.5%], six partially correct, and 24 incorrect [50%]). Sixty percent of the PSTs drew their inversely proportional graphs in the form of an affine function with negative slope. Arican (2019) also reported this inappropriate tendency of PSTs when drawing the inversely proportional graph. In the current study, although six PSTs drew their inversely proportional graphs in the forms of L3 and L4, which suggested their understanding of constant product relationship between corresponding values of x and y, they still drew linear graphs. Therefore, the PSTs' over-attention to the linearity (e.g., Johnson, 2017; Van Dooren et al., 2003) and simultaneous increases and decreases (Arican, 2019) appeared to hinder many PSTs' abilities in drawing the inversely proportional relationship graph.

Regarding the second research question, although many PSTs were able represent the constant ratio and constant product formed by the corresponding x and y values either in their formulas or graphs, only four PSTs referred to the constant ratio in their definitions and none of them referred to the constant product in their definitions. Hence, these findings suggested a poor connection between the PSTs' definitions of these two relationships and formulas and graphs that they provided for them. On the other hand, the PSTs who were good at providing correct formulas of the two relationships were also more successful in drawing the correct graphs and vice versa. Table 7 shows that 29 out of 30 PSTs (96.66%) who provided correct direct proportion formula drew the correct direct proportion graph. Similarly, 18 out of 26 PSTs (69.2%) who provided correct inverse proportion formula drew the correct inverse proportion graph (see Table 8). Therefore, these findings suggested a positive association between the PSTs' knowledge of the direct and inverse proportion formulas and knowledge of their graphs. However, overall, only 11 PSTs (22.9%) attended to the multiplicative relationship between two directly proportional quantities in all three representations, and none of them attended to the multiplicative relationship between two inversely proportional quantities in all three representations. These findings showed a poor association among the PSTs' knowledge on the three representations.

In the last research question, we investigated how PSTs' representations varied according to the relationship provided. Table 9 showed that the PSTs provided parallel definitions of the two relationships. Hence, they usually tended to follow a similar structure when defining these two relationships. Like definitions, the frequencies of the PSTs' formulas were almost diagonally distributed that indicated the PSTs' tendency to write direct and inverse proportion formulas in a similar manner. Table 10 showed that 83.87% of the PSTs who provided the correct direct proportion formula also provided the correct inverse proportion formula. Furthermore, Table 11 showed that 33.3% and 43.47% of the PSTs who drew their direct proportion graphs as in 5a and 5b were able to draw the correct inverse proportion graph, respectively. Therefore, the PSTs who attended to the multiplicative relationship between numbers in their direct proportion graphs were more likely to draw the inverse proportion graph.

Implications for Teaching and Suggestions for Future Studies

In the current study, many PSTs relied on the simultaneous increases and/or decreases when representing directly and inversely proportional relationships. While attention to these simultaneous increases and/or decreases might have helped some PSTs to obtain the directly proportional graph, it hindered their successes in obtaining the inversely proportional graph. Hence, 60% of the PSTs drew incorrect linear graphs to express an inversely proportional relationship. Following this finding, Arican (2019) and Izsák and Jacobson (2017) noted PSTs' over-attention to these simultaneous increases and/or decreases being not very helpful in distinguishing proportional relationships from nonproportional relationships. Furthermore, the findings showed a poor connection among the three types of external representations. Therefore, by disclosing the PSTs' difficulties with three types of representations and connections among them, this study aims at assisting university educators in developing PSTs' meaningful understanding of directly and inversely proportional relationships.

The PSTs' difficulties discussed in this study appeared to be rooted in the instruction that they had received in middle and high school. Hence, as noted by Shield and Dole (2002, 2013) and Ahl (2016), the PSTs' difficulties suggested issues with the capacity of teachers and middle and high school textbooks on promoting students' meaningful understanding of directly and inversely proportional relationships. Therefore, the PSTs' difficulties in the current study contradict with the findings of Bayazit (2013), who reported that Turkish mathematics textbooks presented proportion tasks using multiple representations and had the capacity to promote students' proportional reasoning. The findings of the current study suggest a careful examination of the quality of mathematics textbooks in terms of their capacities for using multiple external representations in expressing directly and inversely proportional relationships and developing students' meaningful understanding of these two relationships. Thus, we suggest future studies to investigate reasons behind the PSTs' difficulties by conducting research with middle and high school students and examining the capacity of mathematics textbooks on promoting the development of students' proportional reasoning.

References

Aboul Hosn, R. (2015). Representations and proportional reasoning in a problem solving context (Unpublished doctoral dissertation). Lebanese American University, Beirut.

Acevedo Nistal, A., Van Dooren, W. & Verschaffel, L. (2012a). What counts as a flexible representational choice? An evaluation of students' representational choices to solve linear function problems. *Instructional Science, 40*(6), 999–1019. DOI 10.1007/s11251-011-9199-9

Acevedo Nistal, A., Van Dooren, W. & Verschaffel, L. (2012b). Students' reported justifications for their representational choices in linear function problems: An interview study. *Educational Studies, 39*(1), 104–117. Retrieved from https://www.tandfonline.com/doi/full/10.1080/03055698.2012.674636

Ahl, L. M. (2016). Research findings' impact on the representation of proportional reasoning in Swedish mathematics textbooks. *REDIMAT, 5*(2), 180–204. Retrieved from http://dx.doi.org/10.4471/redimat.2016.1987

Akkus-Cıkla, O., & Duatepe, A. (2002). A qualitative study on the proportional reasoning skills of the preservice elementary mathematics teachers. *Hacettepe Journal of Education Faculty,* 23, 32–40.

Argun, Z., Arikan, A., Bulut, S., & Halıcıoglu, S. (2014). *Temel matematik kavramların künyesi* [Curriculum of basic mathematics concepts]. Ankara: Gazi Kitabevi.

Arican, M. (2018). Preservice middle and high school mathematics teachers' strategies when solving proportion problems. *International Journal of Science and Mathematics Education, 16*(2), 315–335.

Arican, M. (2019). Preservice mathematics teachers' understanding of and abilities to differentiate proportional relationships from nonproportional relationships. *International Journal of Science and Mathematics Education, 17*(7), 1423–1443.

Bayazit, I. (2013). Quality of the tasks in the new Turkish elementary mathematics textbooks: The case of proportional reasoning. *International Journal of Science and Mathematics Education, 11*(3), 651–682. Retrieved from https://link.springer.com/article/10.1007/s10763-012-9358-8

Ben-Chaim, D., Keret, Y., Ilany, B. (2007). Designing and implementing authentic investigative proportional reasoning tasks: The impact on preservice mathematics teachers' content and pedagogical knowledge and attitudes. *Journal of Mathematics Teacher Education, 10*(4-6), 333–340. DOI 10.1007/s10857-007-9052-x

Common Core State Standards Initiative (2010). *The common core state standards for mathematics*. Washington, D.C.: Author.

De Bock, D., Van Dooren, W., & Verschaffel, L. (2015). Students' understanding of proportional, inverse proportional, and affine functions: Two studies on the role of external representations. *International Journal of Science and Mathematics Education, 13*(1), 47–69. Retrieved from https://link.springer.com/article/10.1007/s10763-013-9475-z

Dogruel, A. B. (2019). An examination of elementary school mathematics teachers' pedagogical content knowledge on ratio and proportion (Unpublished Master's Thesis). Afyon Kocatepe Universtiy, Afyon.

Duval, R. (2002). The cognitive analysis of problems of comprehension in the learning of mathematics. *Mediterranean Journal for Research in Mathematics Education, 1*(2), 1–16. Retrieved from https://link.springer.com/article/10.1007/s10649-006-0400-z

Even, R. (1998). Factors involved in linking representations of functions. *Journal of Mathematical Behavior, 17*(1), 105–121. Retrieved from https://www.sciencedirect.com/science/article/abs/pii/S0732312399800637

Fisher, L. C. (1988). Strategies used by secondary mathematics teachers to solve proportion problems. *Journal for Research in Mathematics Education, 19*(2), 157–168. Retrieved from https://pubs.nctm.org/view/journals/jrme/19/2/article-p157.xml

Fraenkel, J. R., & Wallen, N. E. (2006). *How to design and evaluate research in education* (6th ed.). New York: NY, McGraw-Hill.

Gagatsis, A. & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology, 24*, 645–657. Retrieved from https://doi.org/10.1080/0144341042000262953

Goldin, G. A. (2003). Representation in school mathematics: A unifying research perspective. In J. Kilpatrick, W. Gary Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 275–285). Reston, NJ: National Council of Teachers of Mathematics.

Harel, G., & Behr, M. (1995). Teachers' solutions for multiplicative problems. *Hiroshima Journal of Mathematics Education*, 3, 31–51.

Hsieh, H. F., & Shannon, S. E. (2005). Three approaches to qualitative content analysis. *Qualitative Health Research, 15*(9), 1277–1288. Retrieved from https://doi.org/10.1177/1049732305276687

Hull, L. S. H. (2000). *Teachers' mathematical understanding of proportionality: Links to curriculum, professional development, and support* (Unpublished doctoral dissertation). The University of Texas at Austin, Austin, TX.

İlhan, A. (2019). Dokuzuncu sınıf öğrencilerinin farklı temsiller bağlamında fonksiyon kavramı bilgisi oluşturma süreçleri [The processes of creating function concept of ninth grade students in the context of different representations] (Unpublished master's thesis). Kastamonu University, Kastamonu.

Izsák, A., & Jacobson, E. (2017). Preservice teachers' reasoning about relationships that are and are not proportional: A knowledge-in-pieces account. *Journal for Research in Mathematics Education, 48(*3), 300–339. Retrieved from https://pubs.nctm.org/view/journals/jrme/48/3/article-p300.xml

Johnson, K. (2017). A study of pre-service teachers use of representations in their proportional reasoning. In Galindo, E., & Newton, J., (Eds.), *Proceedings of the 39th North American Chapter of the International Group for the Psychology of Mathematics Education conference* (pp. 551–558). Indianapolis, IN.

Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up: Helping children learn mathematics.* Washington, DC: National Academy Press.

Lamon, S. (2007). Rational numbers and proportional reasoning: Toward a theoretical framework for research. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol 1, pp. 629–667). Charlotte, NC: Information Age Publishing.

Lim, K. (2009). Burning the candle at just one end: Using nonproportional examples helps students determine when proportional strategies apply. *Mathematics Teaching in the Middle School, 14*(8), 492–500. Retrieved from https://pubs.nctm.org/view/journals/mtms/14/8/article-p492.xml

Livy, S., & Herbert, S. (2013). Second-year pre-service teachers' responses to proportional reasoning test items. *Australian Journal of Teacher Education, 38*(11), 17–32. Retrieved from https://search.informit.org/doi/10.3316/INFORMIT.680187799415198

Lobato, J., & Ellis, A. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics: Grades 6-8.* National Council of Teachers of Mathematics. 1906 Association Drive, Reston, VA 20191-1502.

Lo, J. J. (2004). Prospective elementary school teachers' solution strategies and reasoning for a missing value proportion task. In M. J. Høines & A. B. Fuglestad (Eds.), *Proceedings of the 28th International Group for the Psychology of Mathematics Education Conference* (pp. 265–272). Bergen, Norway.

McGowan, M., & Tall, D. (2001). Flexible thinking, consistency, and stability of responses: A study of divergence. Retrieved from http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.411.7057&rep=rep1&type=pdf

Modestou, M., & Gagatsis, A. (2007). Students' improper proportional reasoning: A result of the epistemological obstacle of "linearity". *Educational Psychology, 27*(1), 75–92. Retrieved from https://doi.org/10.1080/01443410601061462

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Orrill, C. H., & Brown, R. E. (2012). Making sense of double number lines in professional development: Exploring teachers' understandings of proportional relationships. *Journal of Mathematics Teacher Education, 15*(5), 381–403. Retrieved from https://doi.org/10.1007/s10857-012-9218-z

Patton, M. Q. (2005). *Qualitative research*. John Wiley & Sons, Ltd.

Shield, M. J, & Dole, S. (2002). Investigating textbook presentations of ratio and proportion. In B. Barton, K.C. Irwin, M. Pfannkuch, & M. O. J. Thomas (Eds.), *Proceedings of the Twenty-fifth Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 608–616). Auckland, N.Z.

Shield, M. J, & Dole, S. (2013). Assessing the potential of mathematics textbooks to promote deep learning. *Educational Studies in Mathematics, 82*(2), 183–199. Retrieved from https://doi.org/10.1007/s10649-012-9415-9

Spiro, R. J., & Jehng, J. C. (1990). Cognitive flexibility and hypertext: Theory and technology for the nonlinear and multidimensional traversal of complex subject matter. In D. Nix & R. Spiro (Eds.), *Cognition, education, and multimedia: Exploring ideas in high technology* (pp. 163–205). Hillsdale, NJ: Lawrence Erlbaum Associates.

Van Dooren, W., De Bock, D., Depaepe, F., Janssens, D., & Verschaffel, L. (2003). The illusion of linearity: Expanding the evidence towards probabilistic reasoning. *Educational Studies in Mathematics, 53*(2), 113–138. Retrieved from https://doi.org/10.1023/A:1025516816886

Yerushalmy, M. (2006). Slower algebra students meet faster tools: Solving algebra word problems with graphing software. *Journal for Research in Mathematics Education, 37*(5), 356–387. Retrieved from https://doi.org/10.2307/30034859

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