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Jakob Bernoulli Finds Exact Sums of Infinite Series (Calculus Version)

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1 On the Convergence and Summation of Infinite Series

Consider this infinite series:

$$\frac{1}{2}+\frac{8}{10}+\frac{27}{50}+\frac{64}{250}+\cdots.$$

Task 1

(a) Prove that the series converges. Be sure to identify which convergence test you applied and how it justified this claim.

(b) Can you find the exact sum of the series? If so, explain how you determined it. If you find you can't, what does your answer in (a) tell you that helps out with this question?

The questions posed in the above Task should clarify for the reader that *knowing that a series converges is independent from discovering its sum.* Consequently, the mastery of tests for convergence of infinite series, while important for determining whether a given series *has* a sum, does not help the mathematician to learn what the series sums to exactly.

As it happens, in standard presentations of the theory of infinite series, we often stop short of finding the *exact* values of most of the convergent series we encounter. The focus of the theory of series is placed squarely on developing tests for determining convergence, and there is a long list of such criteria that we are asked to practice and apply (the comparison test, the ratio test, the root test, the integral test, etc.). To be sure, there are justifiable reasons for being concerned with the convergence of series — there is no hope to determine the sum of a series if we can show that it diverges! — but if we demonstrate that a series does converge, we are generally no closer to determining its sum. Indeed, the student introduced to series comes across few examples of these objects whose sums are precisely determinable. Notable exceptions to this include geometric series, such as $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=2$, and telescoping series like

$$\begin{aligned}\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\cdots &= \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\cdots\\ &= 1+\left(-\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\left(-\frac{1}{5}+\frac{1}{5}\right)+\cdots\\ &= 1,\end{aligned}$$

whose sums are straightforward to discover.

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This project will introduce you to work of Jakob Bernoulli (1655–1705), who, in a time before the full development of the theory of convergent series, discovered methods for determining the sums of a wide variety of infinite series beyond the few types listed above. Since producing the sum of a series is a guarantee of its convergence, what Bernoulli offered us is a straightforward and accessible method for evaluating sums of these convergent series, providing a very satisfying result — a series which converges because we know its sum, as compared with a series which we know converges although its sum is largely a mystery to us.

2 Jakob Bernoulli's *Tractatus de Seriebus Infinitis*

Jakob Bernoulli was born in Basel, Switzerland, into a well-to-do Protestant family, the eldest son of a town Councillor and Master of the local artist's guild. Jakob's father sent him off to university in Basel to prepare him for a career in theology, but while at school, the young Bernoulli became enthralled by the exciting developments being made in the seventeenth century in astronomy and mathematics. He eventually became a leading member of an active community of mathematicians across Europe working on experimental science and analytic mathematics, and he published frequently in the newly established academic journals of the day. He took a position as Professor of mathematics in Basel in 1683, and trained his younger brother Johann (1667–1748) in mathematics, famously becoming his professional rival in later years. Both Bernoulli brothers mastered the new analytic calculus that was being developed by Gottfried Leibniz (1646–1716) on the Continent and Isaac Newton (1643–1727) in England, and the Bernoulli brothers were instrumental in helping to develop its principles.

Over a period of almost twenty years, from the late 1680s to the early 1700s, Jakob Bernoulli wrote five treatises on the theory of infinite series. These works were collected, combined and published in the decade after Jakob's death, as *Tractatus de Seriebus Infinitis* (*A Treatise on Infinite Series*) [Bernoulli, 1713]. The work was bundled together in the same publication with an even more influential treatise named *Ars Conjectandi* (*The Art of Conjecturing*), which represented the most comprehensive work to date in the theory of probability.

In the pages below, we will read a portion of the first part of this treatise on series, which Jakob Bernoulli wrote in 1689.[1] It represented the first systematic treatment of series on their own terms. In Proposition XIV of the *Tractatus*, Bernoulli set forward a method for determining the sums of certain kinds of infinite series by means of a strategic rewriting of their terms.

Before we take up Bernoulli's work in detail, let's review first some of the terminology that he employed, some of which has evolved over the intervening years, and the central ideas regarding series. Infinite sums have been investigated since ancient times; indeed, Greek mathematicians like Archimedes of Syracuse (ca. 287–212 BCE) famously used such sums to solve problems in geometry like the *quadrature of the circle and the parabola*.[2] These sums always involved certain patterns in the terms that were to be added together, patterns which, in the centuries before even Archimedes did his work, mathematicians in the school attributed to Pythagoras of Elea (6th century, BCE) found to be worthy of study.

Today, we call any ordered list of numbers $a_1, a_2, a_3, \ldots$ a *sequence*. For mathematicians in the ancient world, a different term, *progression*, was often favored. And three types of progression were the primary objects of study:

- an *arithmetic* (pronounced "ar-ith-**met**-ik") progression is a sequence of numbers $a_1, a_2, a_3, \ldots$ such that the difference between any consecutive terms is a fixed constant. That is, for all $n \geq 1$, $a_{n+1} - a_n = c$ for some fixed c, often called the *common difference* of the progression.
  - The *natural numbers*, $1, 2, 3, 4, 5, \ldots$, provide a clear example of an arithmetic sequence. Here, $c = 1$.
  - The sequence $1, 3, 5, 7, \ldots$ of odd positive integers is also arithmetic. Here, $c = 2$.

[1] All translations of selections from [Bernoulli, 1713] were prepared by the first author of this project.
[2] *Quadrature* is an old term derived from Latin used to denote the determination of areas.

- The sequence $1, 4, 9, 16, 25 \ldots$ of squares is not arithmetic. Notice that $16 - 9 = 7$ while $4 - 1 = 3$, and $7 \neq 3$, so the sequence has no common difference.

- A *geometric* progression is a sequence of numbers $a_1, a_2, a_3, \ldots$ such that the ratio between any two consecutive terms is a fixed nonzero constant. That is, for all $n \geq 1$, $\frac{a_{n+1}}{a_n} = r$ for some fixed $r > 0$. This value r is often referred to as the *common ratio* of the sequence.
  - The sequence $1, 2, 4, 8, 16, \ldots,$ is an example of a geometric progression. Here, $r = 2$.
  - The sequence $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots,$ is also geometric. Here, $r = \frac{1}{3}$.

- A *harmonic* progression is one whose reciprocals form an arithmetic progression. That is, for all $n \geq 1$, $\frac{1}{a_{n+1}} - \frac{1}{a_n} = c$ for some constant value c. The quintessential example of a harmonic progression is the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$; a second example is afforded by the reciprocals of the positive even numbers: $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots$.[3]

This is by no means an exhaustive list of all the kinds of progressions known to the ancients, and to Jakob Bernoulli in 1689, when he began his systematic study of these objects. In the pages below, you will see how Bernoulli described more of these types of number sequences.

The sum $a_1 + a_2 + a_3 + \cdots$ of the numbers in any sequence $a_1, a_2, a_3, \ldots$ is what we call a *series*, and a series can be either finite or infinite depending on whether there are finitely many or infinitely many numbers being summed. The most obvious conclusion to be made regarding these mathematical objects is the simple fact that *any finite series will necessarily have a finite sum, while an infinite series may or may not.* So while there are many interesting things to say about finite series, we are chiefly interested in this project with the more fascinating world of infinite series. When an infinite series can be assigned a definite sum, we say that the series *converges*, and when this is impossible, we say it *diverges*. Indeed, the truly fascinating thing about infinite series, however, is that many are such that we *can* logically and definitively assign them finite sums!

Task 2 Consult your favorite reference work or calculus textbook for the answer to this question; what does it really mean to say that we can "assign an infinite series a finite sum"? In other words, what is the formal definition of a convergent series?[4]

3 On Progressions and Figurate Numbers

In the early pages of his *Tractatus*, after stating some simple axioms to govern the behavior of the quantities which will be involved in the series he studied,[5] Bernoulli discussed what it meant for a sequence of numbers to continue without end to infinity. He then turned his attention to the first type of infinite series.

VIII. *To find the sum S of any geometric progression $A, B, C, D, E.$ ...*

Corollary If a decreasing geometric progression continues to infinitely many terms, then the final term vanishes ... and the sum of all [terms] equals $\frac{A^2}{A-B}$; whence, moreover, it is made clear by this stipulation that infinitely many terms can produce a finite sum.

[3] The reader should be alerted that while arithmetic and geometric progressions will be found in Bernoulli's work below, we will not see a harmonic progression (although he does consider harmonic progressions elsewhere in his *Tractatus*); we include this definition here for the sake of providing a full picture of the landscape in which Bernoulli worked.

[4] This standard definition is largely credited to Augustin-Louis Cauchy (1789–1857) and Niels Henrik Abel (1802–1829); other different formulations came later in the nineteenth and twentieth centuries. but none of these notions was established until more than 100 years after Bernoulli wrote the treatise we will consider here!

[5] Bernoulli called these quantities "magnitudes," evoking the terminology found in the classical *Elements* of Euclid (ca. 300 BCE), a paradigm of the kind of systematic and rigorous theoretical approach that Bernoulli attempted to reproduce in his treatise on series.

Task 3 Bernoulli uses the term "progression" here, a common synonym for *sequence*. What conditions must be satisfied by the (positive) numbers $A, B, C, D, E, \ldots$, if this sequence is to be a *geometric* progression? (Hint: Let $r = \frac{B}{A}$.)

Task 4 We know that the exact value of a geometric series

$$a + ar + ar^2 + ar^3 + \cdots$$

is given by

$$\frac{a}{1-r},$$

provided the ratio r satisfies the inequality $|r| < 1$.

(a) Use the formula above to determine the exact value of the geometric series

$$1 + \frac{1}{7} + \frac{1}{49} + \frac{1}{343} + \cdots + \frac{1}{7^n} + \cdots$$

(b) Determine the exact value of this geometric series:

$$5 + \frac{10}{3} + \frac{20}{9} + \frac{40}{27} + \cdots .$$

(c) Explain why the geometric series

$$1 + \frac{10}{9} + \frac{10^2}{9^2} + \frac{10^3}{9^3} + \cdots$$

does not converge.

(d) In the text above, Bernoulli gives the formula $\frac{A^2}{A-B}$ for "the sum S of any geometric progression". Show that his formula agrees with the one given at the beginning of this Task.

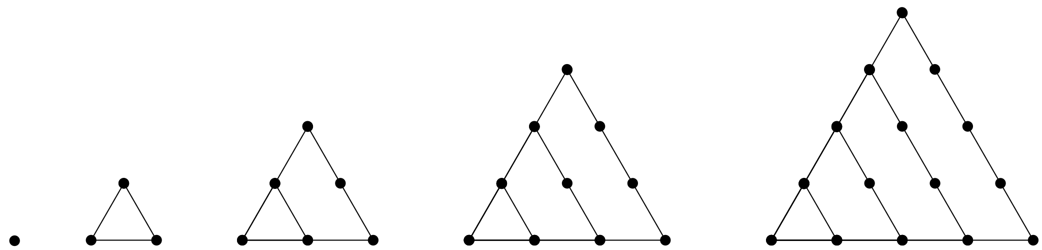
Task 5 What do you think Bernoulli meant in the excerpt above by "the final term" of a sequence that "continues to infinitely many terms"? And why did he say that it "vanishes"?

Before we look further into Bernoulli's work on infinite series, we next note his consideration of other "progressions" of numbers, of interest to him and his contemporaries because of their historical importance to mathematicians, dating back to the ancient Greeks. These sequences of positive integers, called *figurate numbers*, are the subject of the next portion of text we will study.

XIV. *To find the sum of an infinite series of fractions whose denominators grow howsoever by a geometric progression, and whose numerators proceed in like manner as the natural numbers* $1, 2, 3, 4,$ *etc., or triangular numbers* $1, 3, 6, 10,$ *etc., or pyramidal numbers* $1, 4, 10, 20,$ *etc.,* $\ldots$

By longstanding tradition dating back to the Greeks, mathematicians viewed the sequences that Bernoulli presented here in terms of very simple figures of increasing dimensions. As a starting point, the *natural numbers* $1, 2, 3, 4, \ldots$ counted the dots in an (indefinitely) extendable pattern of figures consisting of one-dimensional lines of points:

The *triangular numbers* $1, 3, 6, 10, \ldots$ counted the dots in this pattern of figures:



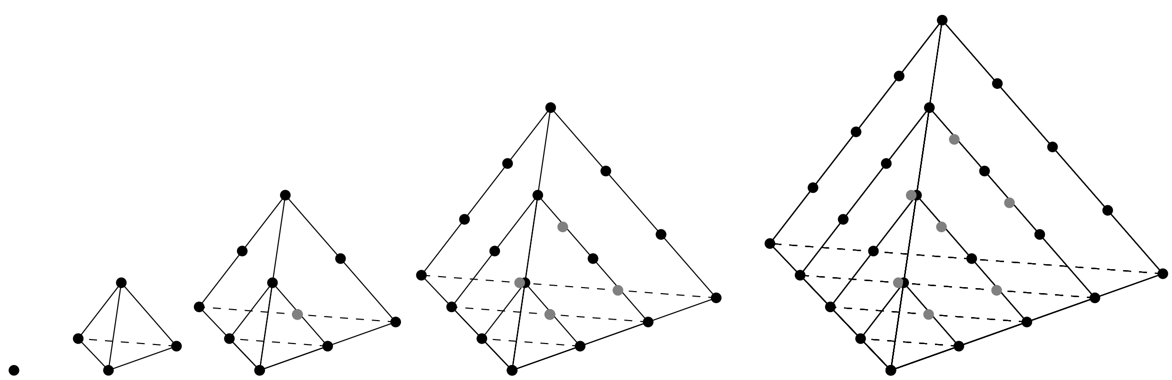
Indeed, the array pictured here suggests a simple pattern for the triangular numbers:

$$\begin{aligned}
T_1 &= 1 = 1 \\
T_2 &= 3 = 1 + 2 \\
T_3 &= 6 = 1 + 2 + 3 \\
T_4 &= 10 = 1 + 2 + 3 + 4 \\
T_5 &= 15 = 1 + 2 + 3 + 4 + 5 \\
&\vdots
\end{aligned}$$

In particular, the tenth triangular number can be identified as

$$T_{10} = 55 = 1 + 2 + 3 + \cdots + 10.$$

By raising the dimensionality of the figures one more time, we obtain this sequence of figures:



Counting the dots in each of these clusters, we obtain the sequence of *pyramidal numbers*, $1, 4, 10, 20, 35, \ldots$.

Notice that the diagrams indicate how the sequence is built from summing triangular numbers:

$$\begin{aligned} 1 &= 1 \\ 4 &= 1+3 \\ 10 &= 1+3+6 \\ 20 &= 1+3+6+10 \\ 35 &= 1+3+6+10+15 \\ &\vdots \end{aligned}$$

Following the patterns we have considered above, the next step in this development is to build up a new sequence by summing the terms of the pyramidal number sequence. This is illustrated as follows:

$$\begin{aligned} 1 &= 1 \\ 5 &= 1+4 \\ 15 &= 1+4+10 \\ 35 &= 1+4+10+20 \\ 70 &= 1+4+10+20+35 \\ &\vdots \end{aligned}$$

This produces the sequence $1, 5, 15, 35, 70, \ldots$, which Bernoulli and his contemporaries called the *pyramido-pyramidal numbers*. He did not attempt to visualize the pyramido-pyramidal numbers with dot patterns, as it would have required some way to represent four-dimensional objects, a serious challenge for a geometer (both then and now!).

With this brief introduction to figurate numbers in hand, we now consider how Bernoulli worked to sum certain kinds of series, which according to him, consisted of "fractions whose denominators grow howsoever by a geometric progression, and whose numerators" are represented as figurate numbers.

4 Bernoulli's Summation Techniques

Recall that in section XIV of his treatise Bernoulli set out to pose a problem about summing certain types of series which he categorized into multiple subtypes. In this project, we will only consider the first three of these subtypes.[6] Bernoulli was able to determine the exact sums of these series by a method wherein he "split" the series into a collection of simpler series, a technique that then allowed him to re-express the original series by another which was much easier to sum exactly.

4.1 "If the numerators grow as the natural numbers"

XIV. *To find the sum of an infinite series of fractions whose denominators grow howsoever by a geometric progression, ...*

1. *If the numerators grow as the natural numbers:*

The sum may be found by resolving the proposed series A into infinitely many series B, C, D, E, etc., each of which proceeds geometrically, whose sums are found, and (if you exclude the first) constitute

[6] If you're interested in seeing how Bernoulli summed all five subtypes of series, check out the longer (capstone) version of this project, available at `https://digitalcommons.ursinus.edu/triumphs_calculus`.

...a new geometric progression F, whose sum is found in the same way as the others by Corollary VIII. Showing the work in detail:

$$A = \tfrac{a}{b} + \tfrac{a+c}{bd} + \tfrac{a+2c}{bd^2} + \tfrac{a+3c}{bd^3} + \cdots = B + C + D + E + \cdots$$

$$\begin{aligned}
B &= \tfrac{a}{b} + \tfrac{a}{bd} + \tfrac{a}{bd^2} + \tfrac{a}{bd^3} + \cdots = \tfrac{ad}{bd-b} \\
C &= .\ + \tfrac{c}{bd} + \tfrac{c}{bd^2} + \tfrac{c}{bd^3} + \cdots = \tfrac{c}{bd-b} \\
D &= \ \cdots\cdots\ \tfrac{c}{bd^2} + \tfrac{c}{bd^3} + \cdots = \tfrac{c}{bd^2-bd} \\
E &= \ \cdots\cdots\cdots\cdots\ \tfrac{c}{bd^3} + \cdots = \tfrac{c}{bd^3-bd^2} \\
\ldots &= \ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\quad \left.\right\} \ F = \tfrac{cd}{b(d-1)^2}, \text{ which when added to the first term } \tfrac{ad}{bd-b} \text{ produces the total of the proposed series } A$$

$$\frac{ad}{b(d-1)} + \frac{cd}{b(d-1)^2} = \text{the sum.}$$

Let's begin to make sense of what Bernoulli did here by considering a specific example of the series A.

Task 6

(a) Using $a = 0$, $b = 5$, $c = 1$, and $d = 3$, write the first four nonzero terms of A.

(b) What is the general term of series A using the values for a, b, c, and d specified above? What convergence test would you use to determine whether this series A converges? Use the test to determine whether A converges or diverges.

(c) Given that $a = 0$, what does the sum B (which appears in Bernoulli's table above) equal?

Task 7

(a) In words, describe what Bernoulli did to build the individual terms of the series $C, D, E, \ldots$ using the original series A.

(b) Write the first four terms of C using the values a, b, c, and d that were used above. What kind of series is C? Find the exact value of C using a known formula for this type of series. Does your answer for C match what you get by using Bernoulli's general formula for C which appears in the table above?

(c) Write the first four terms of D using the values a, b, c, and d that were used above. What kind of series is D? Find the exact value of D using a known formula for this type of series. Does your answer for D match what you get by using Bernoulli's general formula for D which appears in the table above?

(d) Write out the first four terms of E using the values a, b, c, and d that were used above. How do the terms of this series differ from those in series C and D above? How are they similar? What is the value of the sum of series E?

(e) Do you see a pattern in the values of the series C, D, and E? What do you think the value of the next sum would be?

(f) Given the pattern that you just observed, what kind of series is $C + D + E + \ldots$? Given this, compute the sum that Bernoulli labels $F = C + D + E + \ldots$.

(g) Now let's put everything together. We have determined the values of B and $F = C + D + E + \ldots$ for the specific values $a = 0$, $b = 5$, $c = 1$, and $d = 3$. Given that $A = B + F$, determine the value of A in this particular case.

Task 8 Now repeat Tasks 6 and 7 using these values: $a = 5$, $b = 7$, $c = 1$, and $d = 3$.

Let's reflect on what Bernoulli accomplished here. He artfully pulled apart the terms of the series A to create a list of other geometric series, thereby decomposing the original series into component parts, each of which is a series whose sum was straightforward to evaluate.[7] It just so happened that the resulting sum of sums was itself a series which was straightforward to sum. *Et voilá*! He now had a formula that itself applies to an infinite number of different series (as we vary the values of the parameters a, b, c, d):

$$A = \frac{a}{b} + \frac{a+c}{bd} + \frac{a+2c}{bd^2} + \frac{a+3c}{bd^3} + \cdots = \frac{ad}{b(d-1)} + \frac{cd}{b(d-1)^2}. \tag{1}$$

In the case where $a = 0$, this reduces to the even simpler, but still useful, formula

$$A_0 = \frac{c}{bd} + \frac{2c}{bd^2} + \frac{3c}{bd^3} + \cdots = \frac{cd}{b(d-1)^2}. \tag{2}$$

4.2 "If the numerators are as the triangular numbers"

Bernoulli was not satisfied with only one successful summation. He recognized that this splitting and recombining of terms could apply to sum other series.

2. *If the numerators are as the triangular numbers:*

The given series G is resolvable into another H, whose numerators are as in the preceding hypothesis, as such:

$$\begin{array}{l} G = \frac{c}{b}+\frac{3c}{bd}+\frac{6c}{bd^2}+\frac{10c}{bd^3}+\cdots \\ \hline \frac{c}{b}+\frac{c}{bd}+\frac{c}{bd^2}+\frac{c}{bd^3}+\cdots = \frac{cd}{bd-b} \\ \quad +\frac{2c}{bd}+\frac{2c}{bd^2}+\frac{2c}{bd^3}+\cdots = \frac{2c}{bd-b} \\ \qquad +\frac{3c}{bd^2}+\frac{3c}{bd^3}+\cdots = \frac{3c}{bd^2-bd} \\ \qquad\quad +\frac{4c}{bd^3}+\cdots = \frac{4c}{bd^3-bd^2} \\ \qquad\qquad +\cdots = \cdots \end{array}$$

$H = \frac{cd^3}{b(d-1)^3}$, since this series is to the preceding $\frac{c}{bd} + \frac{2c}{bd^2} + \frac{3c}{bd^3} + \cdots = \frac{cd}{b(d-1)^2}$ as d^2 is to $d-1$.

Bernoulli seemed less interested here in paragraph 2 in explaining himself as carefully as he did in paragraph 1 above. Still, the method used to sum the series G is essentially of the same type as the one employed earlier: a splitting of the individual terms to lay out a sequence of geometric series whose sums themselves formed a new series which itself could be summed because it possessed a familiar structure. Let us now see how this was done.

Task 9 Here is the splitting that Bernoulli performed in paragraph 2 on the initial series G:

$$\begin{array}{rl} G = & \frac{c}{b}+\frac{3c}{bd}+\frac{6c}{bd^2}+\frac{10c}{bd^3}+\cdots \\ \hline G_1 = & \frac{c}{b}+\frac{c}{bd}+\frac{c}{bd^2}+\frac{c}{bd^3}+\cdots = \frac{cd}{bd-b} \\ G_2 = & \quad +\frac{2c}{bd}+\frac{2c}{bd^2}+\frac{2c}{bd^3}+\cdots = \frac{2c}{bd-b} \\ G_3 = & \qquad +\frac{3c}{bd^2}+\frac{3c}{bd^3}+\cdots = \frac{3c}{bd^2-bd} \\ G_4 = & \qquad\quad +\frac{4c}{bd^3}+\cdots = \frac{4c}{bd^3-bd^2} \\ \vdots = & \qquad \vdots \qquad\qquad = \vdots \end{array}$$

[7] Comment about infinitary use of the commutative and associative properties of addition!

We have labeled the individual component series $G_1, G_2, \ldots$, noting that the splitting required infinitely many new series, each beginning one term further forward in the array than the one above it.

(a) Bernoulli could have split series G into component parts in a number of different ways. What pattern did he exploit to create the different subscripted series G_1 and G_2 and G_3 and $\ldots$? For instance, why was the term $\frac{6c}{bd^2}$ split into the three terms $\frac{c}{bd^2}, \frac{2c}{bd^2}, \frac{3c}{bd^2}$? Similarly, why was the term $\frac{10c}{bd^3}$ split into the four terms $\frac{c}{bd^3}, \frac{2c}{bd^3}, \frac{3c}{bd^3}, \frac{4c}{bd^3}$? As you formulate your answer, consider the work we did on p. 5, where we laid out the sequence of triangular numbers.

(b) Are each of the series $G_1, G_2, \ldots$ geometric? How can you tell?

(c) Verify the sums that Bernoulli gave for the series G_1, G_2, G_3.

(d) Recall that in paragraph 2, Bernoulli defined the series H to be

$$H = G_1 + G_2 + G_3 + G_4 + \cdots = \frac{cd}{bd-b} + \frac{2c}{bd-b} + \frac{3c}{bd^2-bd} + \frac{4c}{bd^3-bd^2} + \cdots$$

and concluded that paragraph by remarking that "this series is to the preceding $\frac{c}{bd} + \frac{2c}{bd^2} + \frac{3c}{bd^3} + \cdots$ as d^2 is to $d-1$." Compare the series identified here by Bernoulli, $\frac{c}{bd} + \frac{2c}{bd^2} + \frac{3c}{bd^3} + \cdots$, which is also equal to the series we called A_0 in equation (2) above, with this series for H; show that the comparison leads to the proportion

$$\frac{H}{A_0} = \frac{d^2}{d-1}.$$

(e) Now use the known closed formula (2) for the sum of A_0 to conclude (as did Bernoulli) that

$$G = H = \frac{cd^3}{b(d-1)^3}. \tag{3}$$

Task 10 Identify the values of the parameters that Bernoulli called b, c and d in his expression for the series G which lead to the series

$$\frac{1}{5} + \frac{3}{10} + \frac{6}{20} + \frac{10}{40} + \cdots.$$

Then use Bernoulli's results in paragraph 2 as formulated in Task 9(e) to find the exact value of this series.

4.3 "If the numerators are as the pyramidal numbers"

3. *If the numerators are as the pyramidal numbers:*

The series may be resolved into another whose numerators grow as the triangular numbers, which has a ratio to the preceding series as d to $d-1$; whence its sum is found to be $= \frac{cd^4}{b(d-1)^4}$. More generally, if the numerators of the given series are as the figurate numbers of any degree, their sum will have [a ratio] to the sum of a similar series with the previous degree as d is to $d-1$; whence the sum of all remaining terms is quite easily found.

The reader has no doubt noticed that in paragraph 3, Bernoulli has relied on carrying out a pattern of analysis which he had established in paragraphs 1 and 2, and which he then expected his readers to follow here, providing little in the way of details himself. The object of concern in paragraph 3 is therefore a series similar in form to series

$$G = \frac{c}{b} + \frac{3c}{bd} + \frac{6c}{bd^2} + \frac{10c}{bd^3} + \cdots$$

from paragraph 2, but modified so that "the numerators are as the pyramidal numbers." (Recall what this sequence is; see p. 6.) In other words, Bernoulli has devoted paragraph 3 to determining the sum of the series

$$\frac{c}{b} + \frac{4c}{bd} + \frac{10c}{bd^2} + \frac{20c}{bd^3} + \cdots .$$

To carry forward Bernoulli's own alphabetical conventions, we will label this series I. Let us turn our attention now to figuring out how Bernoulli was able to determine that "its sum is found to be $= \frac{cd^4}{b(d-1)^4}$."

Task 11 We mimic Bernoulli's analysis of the series G to sum the series I by decomposing it into an infinite sequence of geometric series.

(a) Keeping in mind that the pyramidal numbers are generated as cumulative sums of the triangular numbers, that is,

$$1 = 1, \quad 4 = 1 + 3, \quad 10 = 1 + 3 + 6, \quad \ldots,$$

write out the array given below and fill in the missing (underlined) terms in the decomposition of I into the series I_1, I_2, I_3, I_4. (Note that the first term of the series I_n has the nth triangular number in its numerator.)

$$\begin{array}{rl} I = & \frac{c}{b} + \frac{4c}{bd} + \frac{10c}{bd^2} + \frac{20c}{bd^3} + \cdots \\ \hline I_1 = & \frac{c}{b} + \underline{\quad} + \underline{\quad} + \underline{\quad} + \cdots = \square \\ I_2 = & \quad + \frac{3c}{bd} + \underline{\quad} + \underline{\quad} + \cdots = \square \\ I_3 = & \quad\quad + \frac{6c}{bd^2} + \underline{\quad} + \cdots = \square \\ I_4 = & \quad\quad\quad + \frac{10c}{bd^3} + \cdots = \square \\ \vdots = & \quad \vdots \quad\quad\quad = \vdots \end{array}$$

Then fill in the (boxed) expressions on the right by determining the exact values of each of the (geometric) sums I_1, I_2, I_3, and I_4.

(b) Now let J be the series whose terms are the sums of the series $I_1, I_2, I_3, I_4, \ldots$, that is,

$$J = I_1 + I_2 + I_3 + I_4 + \cdots$$

Using your work in (a) above, write out J as an infinite series of terms, and give the details to verify that

$$J = \frac{d}{d-1}\left(\frac{c}{b} + \frac{3c}{bd} + \frac{6c}{bd^2} + \frac{10c}{bd^3} + \cdots\right).$$

(c) In light of your work in (b) above and in Task 9, explain what Bernoulli meant when he said that series I "may be resolved into another whose numerators grow as the triangular numbers, which has a ratio to the preceding series as d to $d-1$." In particular, to what did he refer as "the preceding series"?

(d) Finally, verify Bernoulli's claim that "its sum is found to be $= \frac{cd^4}{b(d-1)^4}$." Of course, you must also identify which series he was referring to here!

Task 12 Find the exact value of the series

$$\frac{1}{5}+\frac{4}{10}+\frac{10}{20}+\frac{20}{40}+\ldots,$$

using Bernoulli's results in paragraph 3 above.

As we have seen, Bernoulli successfully discovered formulas for the sums of a number of "infinite series of fractions whose denominators grow howsoever by a geometric progression" and whose numerators "grow as the natural numbers" $1, 2, 3, \ldots$ (par. 1); "as the triangular numbers" $1, 3, 6, \ldots$ (par. 2); and "as the pyramidal numbers" $1, 4, 10, \ldots$ (par. 3). But at the end of paragraph 3, he made a final sweeping claim that indicated his awareness that this pattern of discovery could be indefinitely extended:

More generally, if the numerators of the given series are as the figurate numbers of any degree, their sum will have [a ratio] to the sum of the similar series with the previous degree as d is to $d-1$; whence the sum of all remaining terms is quite easily found.

This begs for us a natural question: What is the next series that Bernoulli may have had in mind when he made the statement above?

Task 13 (a) When we set $a = 0$ in the series A from paragraph 1, and multiply through by d, we get the first series A^* below. Copy down this series. Then, as shown here, write out the similar series G from paragraph 2 underneath A^*, followed by series I from paragraph 3.

$$A^* = \frac{c}{b}+\frac{2c}{bd}+\frac{3c}{bd^2}+\frac{4c}{bd^3}+\cdots$$
$$G = \frac{c}{b}+\frac{3c}{bd}+\frac{6c}{bd^2}+\frac{10c}{bd^3}+\cdots$$
$$I = \frac{c}{b}+\frac{4c}{bd}+\frac{10c}{bd^2}+\frac{20c}{bd^3}+\cdots$$

What should the next series in this sequence look like? Write it down also, label it K, and identify the *name* of the sequence of numbers that appears in the numerators of its terms. You can recognize it from work we did back in Section 3.

(b) From the work we've done above, determine the values of the sums of A^*, G, and I in terms of b, c, and d. Using these facts, infer what the sum of the series K "must" be.

(c) Confirm your guess for the formula for K by performing an analysis similar to the one in Task 11. Now isn't that satisfying?

4.4 Taking Stock

Bernoulli summarized the results above in the next brief section of text by presenting some examples of what he had accomplished.

In this vein, let examples be provided by the following series, whose numerators are the

Natural nos.	$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots=2$
Triangular nos.	$\frac{1}{2}+\frac{3}{4}+\frac{6}{8}+\frac{10}{16}+\frac{15}{32}+\cdots=4$
Pyramidal nos.	$\frac{1}{2}+\frac{4}{4}+\frac{10}{8}+\frac{20}{16}+\frac{35}{32}+\cdots=8$

Task 14 Identify the three series given here by Bernoulli (labeled as examples with natural numbers, triangular numbers and pyramidal numbers in the numerators of their terms) with the appropriate series from Task 13. Give the values of the parameters b, c, and d in each of these three series. Then, using these values and Bernoulli's results, verify the sums that Bernoulli offered above.

Task 15 Using the various results of Bernoulli that we have considered above, find exact values of the sums of the following series:

(a) $\dfrac{2}{1}+\dfrac{6}{7}+\dfrac{12}{49}+\dfrac{20}{343}+\dots$

(b) $\dfrac{1}{2}+\dfrac{8}{10}+\dfrac{27}{50}+\dfrac{64}{250}+\dots$ [We return to where we began in Task 1. Thanks, Master Bernoulli!]

(c) $\dfrac{2}{5}+\dfrac{16}{25}+\dfrac{72}{125}+\dfrac{256}{625}+\dots$ [Hint: take $d=\frac{5}{2}$.]

(d) $\dfrac{3}{2}+1+\dfrac{5}{12}+\dfrac{5}{36}+\dots$ [Hint: take $d=6$.]

5 A Modern Approach to Bernoulli's Series Involving Triangular Numbers and Higher-Dimensional Analogues

In this section, our goal is to extend a pattern that Bernoulli mentioned in the final excerpt we encountered at the end of Section 4. The first three examples in his list were the following:

Natural nos.	$\frac{1}{2}+\frac{2}{4}+\frac{3}{8}+\frac{4}{16}+\frac{5}{32}+\cdots=2$
Triangular nos.	$\frac{1}{2}+\frac{3}{4}+\frac{6}{8}+\frac{10}{16}+\frac{15}{32}+\cdots=4$
Pyramidal nos.	$\frac{1}{2}+\frac{4}{4}+\frac{10}{8}+\frac{20}{16}+\frac{35}{32}+\cdots=8$

It's interesting to note that the values of these three sums are all powers of 2, namely, 2, 4, and 8. Might it be possible to demonstrate these three results relatively easily using modern calculus tools? And, if so, is there a way to naturally extend this list so that the sums in question continue the pattern of summing to powers of 2? The answer to both of these questions is a resounding YES!

Our starting point here is the geometric series formula

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots, \tag{4}$$

which holds whenever $|x| < 1$. Our first step is then to differentiate both sides of (4) with respect to x. This yields

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots. \tag{5}$$

Note that the left–hand side of the above simply requires the quotient rule, while the right–hand side is simply generated by applying the power rule to each of the terms in the power series. Next, we multiply both sides of (5) by x to yield

$$\frac{x}{(1-x)^2} = x + 2x^2 + 3x^3 + 4x^4 + \cdots \tag{6}$$

Task 16 Write out the first five terms of the series in (6) obtained by setting $x = \frac{1}{2}$. Then set $x = \frac{1}{2}$ on the left-hand side of equation (6); what does this give us for the sum of this series? How does this relate to what Bernoulli stated in the excerpt at the beginning of this section?

Next, we want to find a way to verify the sum of the second series above,

$$\frac{1}{2} + \frac{3}{4} + \frac{6}{8} + \frac{10}{16} + \frac{15}{32} + \cdots = 4, \tag{7}$$

using the same kind of technique. To do so, we immediately return to (5) and differentiate again with respect to x. This yields

$$\frac{2}{(1-x)^3} = 2 + 6x + 12x^2 + 20x^3 + \cdots. \tag{8}$$

As before, the left–hand side of (8) arises from the quotient rule (and some algebraic simplification) while the right–hand side follows from term–by–term differentiation of the power series in question.

Now return to the series (7) we are trying to study. The numerators of the terms in that series are $1, 3, 6, 10, \ldots$ Unfortunately, we don't have $1, 3, 6, 10, \ldots$ as the coefficients of the terms in our power series; instead, we have $2, 6, 12, 20, \ldots$. However, if we pause for a moment, we quickly notice that these are the doubles of the triangular numbers that we expect! So, we return to (8) and divide through by 2 to get

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^3 + \cdots. \tag{9}$$

We *now* multiply both sides of (9) by x to get

$$\frac{x}{(1-x)^3} = x + 3x^2 + 6x^3 + 10x^4 + \cdots. \tag{10}$$

Task 17 Write out the first five terms of the series in (10) obtained by setting $x = \frac{1}{2}$. Then set $x = \frac{1}{2}$ on the left-hand side of equation (10); what does this give us for the sum of this series? How does it relate to what Bernoulli stated in the excerpt at the beginning of this section?

Now consider the third series in Bernoulli's list above:

$$\frac{1}{2}+\frac{4}{4}+\frac{10}{8}+\frac{20}{16}+\frac{35}{32}+\cdots=8.$$

The nature of the relationship between the triangular numbers and the pyramidal numbers (which we discussed in Section 3) suggests that we can return to one of the power series equalities that we constructed earlier and differentiate it again. Of course, the question is: with which equation should we start?

It turns out that equation (9) is our prize.

Task 18

(a) Differentiate both sides of equation (9). Then divide through the resulting equation by 3 and multiply through by x. This will produce an equation similar in form to (10) with a series on the right side whose coefficients are *not* the triangular numbers. Which familiar sequence of numbers appears as coefficients of this series?

(b) Set $x=\frac{1}{2}$ on both sides of the resulting equation you obtained in part (a); what does this give you? How does this relate to what Bernoulli stated in the excerpt at the beginning of this section?

Based on the work we've performed here, and the patterns that have emerged, we now consider a few tasks which naturally extend these results.

Task 19

Let us first identify the next series that should be considered. We learned in Section 3 that the triangular numbers are formed as partial sums of the natural numbers. And we also learned there that the pyramidal numbers are formed as partial sums of the triangular numbers. It is natural to then consider partial sums of the pyramidal numbers in order to obtain the list of numerators for the next series to consider.

(a) With the above in mind, confirm that the next series to consider is of the form

$$\frac{1}{2}+\frac{5}{4}+\frac{15}{8}+\frac{35}{16}+\frac{70}{32}+\cdots$$

(b) If the above series is the "correct" next series in this list, then what do you conjecture is the exact value of this series (in order for the pattern of the exact values to continue)?

(c) Find the exact value of the sum of the series in part (a) by mimicking the method of Task 18.

6 Conclusion

There is much more in Bernoulli's *Tractatus* about the summation of infinite series than we can discuss here. For instance, in the proposition following the one we studied in this project, he solved this problem:

XV. To find the sum of the infinite series of fractions whose numerators make up a series of equal numbers, and whose denominators are either the triangular numbers or equimultiples of these.

The simplest example of such a series is the sum of the reciprocals of the triangular numbers,

$$\frac{1}{1}+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\cdots,$$

a series which you will almost certainly find presented in any standard textbook introduction to the theory of infinite series. Rather than give away the answer here, we urge all our readers to look up the very simple "telescoping" method presented in such books for summing this series.

Later in the *Tractatus*, Bernoulli considered what turned out be the deceptively thorny problem of summing the series of the reciprocals of the squares,

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots .$$

He himself joined a long line of his contemporaries who tried without success to resolve the exact sum of this series.

. . . it is more difficult than one might expect to seek out this sum, even though we learn that it is finite by [comparison with] other [series], as [this one] is clearly smaller: If someone should discover this [sum], communicate with us, for which we would be greatly appreciative, for it has eluded our diligence up to now.

This problem awaited the mathematical skills of the great Leonhard Euler (1707–1783), a fellow Swiss countryman of Bernoulli, but who was born shortly after Jakob Bernoulli's death.[8] Indeed, it was the resolution of this problem, in which Euler determined that

$$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6},$$

that essentially launched his career as one of the most prolific and accomplished mathematicians in history.[9]

References

Jakob Bernoulli. *Ars conjectandi, opus posthumum; accedit Tractatus de seriebus infinitis et epistola gallice scripta; De ludo pilae reticularis.* Thurnisiorum Fratrum, Basel, 1713. Reprinted by Culture et civilisation, Bruxelles, 1968.

[8] Jakob Bernoulli died at age 50 in 1705, of tuberculosis, a common (and unfortunately, often fatal) ailment of the times.

[9] For more about this series, see the project *Euler's Calculation of the Sum of the Reciprocals of the Squares* by Kenneth M. Monks, available for download at `https://blogs.ursinus.edu/triumphs/`.

Notes to Instructors

PSP Content: Topics and Goals

This Primary Source Project (PSP) is designed to provide students an immersive experience in determining the sums of infinite series by following the work of Jakob Bernoulli (1655-1705) in his *Tractatus de Seriebus Infinitis*. Students typically encounter the theory of infinite series in their second semester course in calculus, in which the focus is the determination of the *convergence* of series. While this topic is vital to the study and use of series in mathematical analysis, it turns out that, with the notable exceptions of geometric series and telescoping series, whose sums are easy to determine, students generally conclude this course realizing that very few of the infinite series which they can determine to be convergent can also be exactly summed. There are a few clear motivations for this approach to infinite series, characterized by a focus on the convergence tests and approximation methods. One such motivation is the ease with which many can now calculate the sum of a very large number of terms of a series thanks to advances in technology (calculators and computers) and the advent of computer algebra systems. Another motivation is that the idea of approximation is "natural" or important among calculus students who are intending to complete degrees in a variety of STEM fields, including a multitude of students in engineering programs.

The transition from a study of convergence tests for infinite series to an introduction to power series and their representations of the classical analytic functions can be rather jarring for many students. Having spent considerable time and energy on the mechanics of determining whether a series converges, there are few opportunities for them to realize exact values for the series which they find to converge. One should be drawn to the question of finding the sum of a series once its convergence is determined, but this question is so little considered. Of course, students must become familiar with the ideas of series convergence to appreciate the more sophisticated notion of power series and the associated intervals on which they converge. Even so, we believe that this transition (from series of numbers to power series) can be strengthened pedagogically by providing ways for students to see first hand, once they appreciate the difference between convergence and divergence, how to find the exact sums of certain convergent infinite series before they encounter power series, and Jakob Bernoulli's work offers an ideal opportunity to do this.

Bernoulli's work in his *Tractatus* demonstrates an accessible approach to determining the exact sums of certain kinds of infinite series, including those of the form

$$\sum_{n=1}^{\infty} \frac{c}{bd^{n-1}} \binom{n+k-1}{k},$$

where $b \neq 0$ and c are arbitrary, $|d| > 1$ and $k = 1, 2, 3$ and 4. Completing this PSP should provide the student with a wealth of examples of series whose sums can be found *exactly*.

While the intended audience is precisely those students who have just studied the standard tests for convergence of infinite series, this project is also suitable as an enrichment experience for any student who has completed the traditional introduction to the theory of infinite series, or for students in a history of mathematics course as an opportunity to consider the work of a prominent seventeenth century mathematician at the cusp of the many innovations that heralded the "invention" of calculus in this generation.

Student Prerequisites

Little is required of students to work on this project besides an introductory knowledge of infinite series. They should already be familiar with summing geometric series, with the notion of convergence versus divergence of a series, and with the more common convergence tests (for instance, Task 1 expects the student to employ a ratio test to check the convergence of a series). While a comprehensive working knowledge of all the standard convergence tests is not required or used in this project, the more students are aware of the contingent nature of convergence, the more satisfying the impact should be of coming to know methods for computing the sums of series which do converge. In Section 5, students are asked to differentiate the terms of a power series and some rational expressions; we trust that the typical second-semester calculus student will be comfortable with this. We recognize that differentiation of power series can, in some circumstances, lead to problems with convergence of the resulting series (even though this is not the case with any of the series we encounter in this PSP). Nonetheless, we believe that discussing the validity of such operations with beginning calculus students is counterproductive and better left for another time and place!

The authors of this project have endeavored to minimize the use of summation notation in this PSP so as to remove as many obstacles as possible for students for whom this symbolism is new and challenging. We would rather that students focus on grappling with Bernoulli's clever methods for manipulating the terms of his series to discover the values of their sums. Of course, this does not prevent instructors from using — and asking their students to employ — this standard notation for infinite series.

PSP Design, and Task Commentary

The project begins with a simple task designed to alert students that knowing that a series converges tells one nothing about the sum of that series. This establishes a focus for students' work with Bernoulli on the summation (rather than the convergence) of certain infinite series. The types of series that Bernoulli summed in his treatise are organized by him through the classical language of arithmetic and geometric progressions, and figurate numbers: natural numbers, triangular numbers, pyramidal numbers, etc. These objects are defined and explored in Section 3 of the project after a brief historical account in Section 2 that provides the context for what students will be reading. Instructors who wish to explore figurate numbers more deeply with their students can do no better than turn to the PSP *Construction of the Figurate Numbers* by Jerry Lodder (available for download at the TRIUMPHS website printed at the end of this project).

Section 4 is the heart of the project. It assists students as they work through Bernoulli's Proposition XIV, using variations on a technique that splits the given infinite series to be summed into an infinite number of simpler series (all geometric), each of whose sums then produce a reorganized representation of the given series whose sum was handled in an earlier paragraph of the treatise! The series for which Bernoulli found exact sums in this Proposition are those of the form

$$\sum_{n=1}^{\infty} \frac{p_n}{q_n},$$

where q_n is a geometric progression (of positive values with common ratio greater than 1 to ensure convergence) and

- in subsection 4.1 (and Tasks 6-8), p_n is an arithmetic progression;
- in subsection 4.2 (and Tasks 9 and 10), p_n is proportional to the sequence of triangular numbers;
- in subsection 4.3 (and Tasks 11 and 12), p_n is proportional to the sequence of pyramidal numbers (with a mention — see Task 13 — of how to proceed for higher-order pyramidal numbers).

Finally, in Section 5, students are shown a more modern approach generalizing the results that Bernoulli presented in his treatise (with Tasks 16-19) where series in which p_n is proportional to a higher-order pyramidal number $\binom{n+k}{k}$ are handled.

Suggestions for Classroom Implementation

This project was designed for the student who has recently completed a unit on infinite series and criteria for their convergence and have been introduced to power series.

We expect that students will be doing preparatory work before each class day, that they will work (at least some of the time) in small groups with each other in the classroom, and that they will complete homework and prepare formal write-ups of their classroom work outside of class. See the implementation schedule below for details on our specific suggestions about this.

LaTeX code of this entire PSP is available from the author by request to facilitate preparation of advanced preparation / reading guides or 'in-class worksheets' based on tasks included in the project. The PSP itself can also be modified by instructors as desired to better suit their goals for the course.

Possible Modifications of the PSP

Expect about a week of class time to work through this project. For instructors with more serious time restrictions, it should be possible to omit Section 5 altogether and save a full class day. We should alert instructors that there is a longer version of this PSP (with the same title) that takes these topics further than they are taken here. This longer version might be used by students as a capstone experience or for prospective high school teachers. It is also available for free download at the TRIUMPHS Digital Commons website given below.

Sample Implementation Schedule (based on a 50-minute class period)

Instructors planning to implement this project in a course that meets twice a week in 75-minute periods will need to adapt the schedule given here appropriately. Of course, the actual number of class periods spent on each section naturally depends on the instructor's goals; what follows is merely one suggestion.

Day 1

- Ask students to read Sections 1 and 2 of the project, to jot down any questions they may have as a result of their reading, and to write up Tasks 1 and 2 in advance of the first meeting.
- Address any questions about the reading at the opening of the period. Have a student read aloud the opening of Section 3, including the first brief excerpt of Bernoulli's writing. Set students to work in small groups on Tasks 3–5 (whole class discussions may be needed for students to agree on how to answer the questions in Task 3 and 5). Repeat with the common reading of the statement of Proposition XIV, and have students work in groups on Tasks 6 and 7. Should time remain, have them finish with Task 8.
- Assign for homework what is not completed in class to work through Section 4.1. Also have them write up clean solutions for any of the Tasks they have completed that you believe will serve them well by producing a written record of their thinking.

Day 2

- Field questions from their earlier work, then begin a common reading of Sections 4.2 and 4.3.
- The goal of the day should be to make sense of Bernoulli's paragraphs 2 and 3 by working in groups through Tasks 9, 11 and 13. These are the most technically demanding of the project, so it is likely that the entire period may be required to complete them. Tasks 10 and 12 can be left for homework, should time be at a premium.
- Again, decide what you believe is useful for students to formally write up as homework from this day's material. We recommend including Tasks 14 and 15 in this assignment.

Day 3

- Section 5 extends the discussion that concludes Bernoulli's Proposition XIV. Ask students to prepare for this class by reading the analysis presented at the beginning of Section 5, before Task 16. Students who have any questions should make notes of what the issues are and bring them to class to resolve with discussion at the beginning of the period.
- After resolving the students' questions, complete Tasks 16-19.
- Students should write up their answers to anything needed to complete their work.

Connections to other Primary Source Projects

The following TRIUMPHS PSPs are also freely available for use in teaching standard topics in the calculus sequence. The PSP author name is listed (together with the general content focus, if this is not explicitly given in the project title). With the exception of the last project in the list (which should require up to

2 full weeks for implementation), each of these is designed to be completed in 1–2 class periods. It should be noted that the penultimate and antepenultimate in the list are also devoted to the topic of infinite series. Classroom-ready versions of these projects can be downloaded from `https://digitalcommons.ursinus.edu/triumphs_calculus`.

- *The Derivatives of the Sine and Cosine Functions*, Dominic Klyve
- *L'Hôpital's Rule*, Daniel E. Otero
- *Fermat's Method for Finding Maxima and Minima*, Kenneth M. Monks
- *Beyond Riemann Sums: Fermat's Method of Integration*, Dominic Klyve
- *How to Calculate π: Buffon's Needle (Calculus Version)*, Dominic Klyve (integration by parts)
- *Gaussian Guesswork: Elliptic Integrals and Integration by Substitution*, Janet Heine Barnett
- *Gaussian Guesswork: Polar Coordinates, Arc Length and the Lemniscate Curve*, Janet Heine Barnett
- *Gaussian Guesswork: Infinite Sequences and the Arithmetic-Geometric Mean*, Janet Heine Barnett
- *Investigations Into d'Alembert's Definition of Limit (Calculus Version)*, Dave Ruch (definition of limit)
- *How to Calculate π: Machin's Inverse Tangents*, Dominic Klyve (infinite series)
- *Euler's Calculation of the Sum of the Reciprocals of Squares*, Kenneth M. Monks (infinite series)
- *Fourier's Proof of the Irrationality of e*, Kenneth M. Monks (infinite series)
- *Bhāskara's Approximation to and Mādhava's Series for Sine*, Kenneth M. Monks (approximation, power series)
- *Braess' Paradox in City Planning: An Application of Multivariable Optimization*, Kenneth M. Monks
- *Stained Glass, Windmills and the Edge of the Universe: An Exploration of Green's Theorem*, Abe Edwards
- *The Fermat-Torricelli Point and Cauchy's Method of Gradient Descent*, Kenneth M. Monks (partial derivatives, multivariable optimization, gradients of surfaces)
- *The Radius of Curvature According to Christiaan Huygens*, Jerry Lodder

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For more information about TRIUMPHS, visit `https://blogs.ursinus.edu/triumphs/`.